General solution of Linear Homogeneous Difference Equations with variable coefficients.

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Solutions of kth-order Linear Homogeneous Difference Equation (1).

$$x^{(k)}(n+1)=\sum_{i=0}^{k-1} p_{i+1}(n)x^{(k)}(n-i)$$

Any initial condition $x_{-1}, x_0, \ldots, x_{k-2}$

z_0=-1 it is taken, without loss of generality. Representation for any $z_0 \in Z$ follows.

Analytical (Asymptotic) Theory

Eigenvalue methods, (Poincaré, Perron). Complex analysis (Birkhoff, Adams).
Change of variables and Minimal solution, (Kreuser, Wimp).
See Elaydi's book, *An Introduction to Difference Equations* . Springer 2005.

Operational Calculus

With the approach of Mikusinski Convolution Operator
(second order, Takaci, and third order, Zhou).

Constructive Theory

J. Popenda, ***One expression for the solutions of 2th-order D.E.***, Proceedings of A.M.S Vol 100 nº 1 (1987).

Ranjan Mallik. ***Solutions of linear difference equations with variable coefficients,*** Journal of Mathematical Analysis and Applications,vol. 222, no. 1, pp. 79-91, (1998).

On the solution of a linear homogeneous difference equation with variable coefficients*, SIAM J Mathematical Analysis,* vol. 31, no. 2, pp. 375-385, (2000).

R. Mallik have found a closed solution of kth-order linear homogeneous difference equation by a method based in combinatorial properties of the companion matrix.
The compact representation is not explicit. It depends on combinatorial conditions.
Expression for the general solution is an open problem.

In this work, the general solution has been obtained independently and with a constructive approach.
This produces the general solution by means of a linear combination of similar nested sums, fundamental solutions.
The induction proof is less complex than Mallik's combinatorial proof.
The knack of the method is to begin the nested sums by the upper factors.

2th-order general solution

First Step: A suitable change of variable (1). It is usual in assymptotic methods.

$$x^{(2)}(n) = y^{(2)}(n)\prod_{i=0}^{n-1} p_1(i)$$

Equivalent 2th-order equation (2), $y^{(2)}(n+1) = y^{(2)}(n) + \alpha_2(n)\, y^{(2)}(n-1)$

With the same initial conditions, and $\alpha_2(n) = \dfrac{p_2(n)}{p_1(n-1)\, p_1(n)},\ n \ge 1,\ \ \alpha_2(0) = \dfrac{p_2(0)}{p_1(0)}.$

Second Step: **Proposition:** The solution of equation (2) is of the form: (**Proof: Induction on n.)**

$$y^{(2)}(n) = y_0 \Phi_n^{(2,0)}\left(\alpha_2(1), \alpha_2(2), \ldots, \alpha_2(n-1)\right) + y_{-1}\alpha_2(0)\,\Phi_{n-1}^{(2,1)}\left(\alpha_2(2), \alpha_2(3), \ldots, \alpha_2(n-1)\right)$$

$$\text{with } \Phi_{n-j}^{(2,j)}\left(\vec{\alpha}_2\right) = \sum_{l=0}^{\left\lfloor \frac{n-j}{2} \right\rfloor} \left(\sum_{k_1=2l+j-1}^{n-1} \alpha_2(k_1) \left(\sum_{k_2=2(l-1)+j-1}^{k_1-2} \alpha_2(k_2) \left(\ldots \left(\sum_{k_l=1+j}^{k_{l-1}-2} \alpha_2(k_l) \right)\right)\right)\right)$$

When *l* = 0 the sum adds 1 by convention.

From (1) and (2) and by the form of $\Phi^{(2,j)}_{n-j}$ the general solution is:

$$x^{(2)}(n) = c_0\left(\prod_{i=0}^{n-1} p_1(i)\right)\Phi_n^{(2,0)} + c_{-1}p_2(0)\left(\prod_{i=1}^{n-1} p_1(i)\right)\Phi_{n-1}^{(2,1)} = c_0\Psi_n^{(2,0)} + c_{-1}p_2(0)\,\Psi_{n-1}^{(2,1)}$$

Solution of 3th-order linear D.E.

<u>First Step</u>: The same change of variable (1).

$$x^{(3)}(n) = y^{(3)}(n)\prod_{i=1}^{n-1} p_1(i)$$

Equivalent 3th-order equation (2),

$$y^{(3)}(n+1) = y^{(3)}(n) + \alpha_2(n)\, y^{(3)}(n-1) + \alpha_3(n)\, y^{(3)}(n-2)$$

With the same initial conditions, and

$$\alpha_2(n) = \frac{p_2(n)}{p_1(n-1)\,p_1(n)},\ \text{n} \geq 2,\ \alpha_2(1) = \frac{p_2(1)}{p_1(1)}.$$

$$\alpha_3(n) = \frac{p_3(n)}{p_1(n-2)\,p_1(n-1)\,p_1(n)},\ \text{n} \geq 3,\ \alpha_3(1) = \frac{p_3(1)}{p_1(1)},\ \ \alpha_3(2) = \frac{p_3(2)}{p_1(1)\,p_1(2)}..$$

<u>Second Step</u>: **Proposition:** The solution of equation (2) is of the form: **(Proof: Induction on n.)**

$$y^{(3)}(n) = y_1 \Phi_n^{(3,0)} + \sum_{i=1}^{2} y_{1-i}\left(\sum_{j=1}^{3-i} \alpha_{i+j}(j)\,\Phi_{n-j}^{(3,j)}\right)$$

$$\Phi_{n-j}^{(3,j)}\left(\vec{\alpha}_2, \vec{\alpha}_3\right) = \sum_{l=0}^{\left\lfloor \frac{n-1-j}{2} \right\rfloor}\left(\sum_{i_1=2}^{3}\left(\sum_{k_1=2l+(i_1-2)+j}^{n-1} \alpha_{i_1}(k_1)\left(\sum_{i_2=2}^{3}\left(\sum_{k_2=2(l-1)+(i_2-2)+j}^{k_1-i_1} \alpha_{i_2}(k_2)\left(\cdots\left(\sum_{i_l=2}^{3}\left(\sum_{k_l=i_l+j}^{k_{l-1}-i_{l-1}} \alpha_{i_l}(k_l)\right)\right)\right)\right)\right)\right)\right)$$

From (1) and (2) and by the form of $\Phi^{(2,j)}_{n-j}$ the general solution is, n >1:

$$x^{(3)}(n)=\left(\prod_{i=1}^{n-1}p_1(i)\right)\left(c_1\Phi_n^{(3,0)}+\sum_{j=1}^{2}c_{1-j}\left(\sum_{l=1}^{3-j}\alpha_{j+l}(l)\Phi_{n-l}^{(3,l)}\right)\right)$$

$$x^{(3)}(n)=c_1\Psi_n^{(3,0)}+\left(c_0p_2(1)+c_{-1}p_3(1)\right)\Psi_{n-1}^{(3,1)}+c_0p_3(2)\Psi_{n-2}^{(3,2)}=t_1\Psi_n^{(3,0)}+t_2\Psi_{n-1}^{(3,1)}+t_3\Psi_{n-2}^{(3,2)}.$$

Solution of kth-order equation

The method is a natural continuation of previous cases. General solution is, *n > k-2*.

$$x^{(k)}(n)=\left(\prod_{i=k-2}^{n-1}p_1(i)\right)\left(c_{k-2}\Phi_n^{(k,0)}+\sum_{j=1}^{k-1}c_{k-j-2}\left(\sum_{l=1}^{k-j}\alpha_{j+l}(k-3+l)\Phi_{n-l}^{(k,l)}\right)\right)$$

Now the expression for $\Phi^{(k,l)}_{n-l}$ is given:

$$\Phi_{n-l}^{(k,l)}=\sum_{m=0}^{\left\lfloor\frac{n-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+(k-3)+(i_1-2)+l}^{n-1}\alpha_{i_1}(k_1)\left(\sum_{i_2=2}^{k}\left(\sum_{k_2=2(m-1)+(k-3)+(i_2-2)+l}^{k_1-i_1}\alpha_{i_2}(k_2)\left(\dots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+l+k-3}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)\right)$$

Solution of kth-order equation

Similar expression for the α_i, $i = 2,\ldots,k$.

$$\alpha_2(n) = \frac{p_2(n)}{p_1(n-1)\,p_1(n)},\ \mathrm{n} \geq k-1,\ \alpha_2(k-2) = \frac{p_2(k-2)}{p_1(k-2)}.$$

$$\alpha_3(n) = \frac{p_3(n)}{p_1(n-2)\,p_1(n-1)\,p_1(n)},\ \mathrm{n} \geq k,\ \alpha_3(k-2) = \frac{p_3(k-2)}{p_1(k-2)},\ \ \alpha_3(k-1) = \frac{p_3(k-1)}{p_1(k-2)\,p_1(k-1)}.$$

$$\alpha_4(n) = \frac{p_4(n)}{p_1(n-3)\,p_1(n-2)\,p_1(n-1)\,p_1(n)},\ \mathrm{n} \geq k+1,\ \alpha_4(k-2) = \frac{p_4(k-2)}{p_1(k-2)},\ \ \alpha_4(k-1) = \frac{p_4(k-1)}{p_1(k-2)\,p_1(k-1)},$$

$$\alpha_4(k) = \frac{p_4(k)}{p_1(k-2)\,p_1(k-1)\,p_1(k)}.$$

....

$$\alpha_k(n) = \frac{p_k(n)}{\prod_{i=1}^{k} p_1(n-k+i)},\ \mathrm{n} \geq 2k-3,\ \alpha_k(k-2) = \frac{p_k(k-2)}{p_1(k-2)},\ \ \alpha_k(k-1) = \frac{p_k(k-1)}{p_1(k-2)\,p_1(k-1)},\ldots.$$

$$\ldots,\ \alpha_k(2k-4) = \frac{p_k(2k-4)}{\prod_{i=1}^{k-1} p_1(k-3+i)}.$$

Proposition: The general solution for *n > k-2* of equation

$$x^{(k)}(n+1) = \sum_{i=0}^{k-1} p_{i+1}(n) x^{(k)}(n-i) \quad (1)$$

is of the form:

$$x^{(k)}(n) = \left(\prod_{i=k-2}^{n-1} p_1(i)\right)\left(c_{k-2}\Phi_n^{(k,0)} + \sum_{j=1}^{k-1} c_{k-j-2}\left(\sum_{l=1}^{k-j} \alpha_{j+l}(k-3+l)\Phi_{n-l}^{(k,l)}\right)\right) \quad (2)$$

Proof: $\Phi(-1) = \mathrm{diag}(c_{-1}, c_0, c_1, \ldots, c_{k-2})$ is taken as initial fundamental matrix without loss of generality. The coefficients *c* are any kth-set of non null complex numbers. The transition matrix is the companion matrix:

$$\mathbf{A}(k-2) = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{I}_{k-1} \\ \hline p_k(k-2) & \mathbf{p}(k-2) \end{array}\right), \ldots\ldots, \mathbf{A}(n) = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{I}_{k-1} \\ \hline p_k(n) & \mathbf{p}(n) \end{array}\right).$$

$$\mathbf{\Phi}(n) = \left(\prod_{i=k-2}^{n-1} \mathbf{A}(i) \right) \mathbf{\Phi}(-1) =$$

$$\left(\begin{array}{c|c|c|c|c}
c_{-1}\alpha_k (k-3+l)\Phi^{(k,1)}_{(n-k+1)-1} & \cdots & c_{k-2-j} \sum_{l=1}^{k-j} \alpha_{j+l} (k-3+l)\Phi^{(k,l)}_{n-k+1-l} & \cdots & c_{k-2}\Phi^{(k,0)}_{n-k+1} \\
c_{-1}\alpha_k (k-3+l)\Phi^{(k,1)}_{(n-k+2)-1} & \cdots & c_{k-2-j} \sum_{l=1}^{k-j} \alpha_{j+l} (k-3+l)\Phi^{(k,l)}_{n-k+2-l} & \cdots & c_{k-2}\Phi^{(k,0)}_{n-k+2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c_{-1}\alpha_k (k-3+l)\Phi^{(k,1)}_{(n-1)-1} & \cdots & c_{k-2-j} \sum_{l=1}^{k-j} \alpha_{j+l} (k-3+l)\Phi^{(k,l)}_{n-1-l} & \cdots & c_{k-2}\Phi^{(k,0)}_{n-1} \\
c_{-1}\alpha_k (k-3+l)\Phi^{(k,1)}_{n-1} & \cdots & c_{k-2-j} \sum_{l=1}^{k-j} \alpha_{j+l} (k-3+l)\Phi^{(k,l)}_{n-l} & \cdots & c_{k-2}\Phi^{(k,0)}_{n}
\end{array} \right)$$

By induction on n.
The last row is the representation of $x^{(k)}(n)$ as linear combination of
k l.i. particular solutions.
Here, the multiplicative factor of p_1 terms of $x^{(k)}(i)$ has been omitted for clarity.
Index *j* increases in inverse order in relation to columns order.

All results are justified in the proof of following proposition

Theorem: If $p_k(n) \neq 0$, a expression of general solution for *n > 2k-3* of equation

$$x^{(k)}(n+1) = \sum_{i=0}^{k-1} p_{i+1}(n) x^{(k)}(n-i) \qquad (1)$$

is through a linear combination of k nested fundamental solutions:

$$x^{(k)}(n) = \sum_{l=0}^{k-1} t_{k-2+l} \Psi_{n-l}^{(k,l)}, \text{ with } \Psi_{n-l}^{(k,l)} = \left(\prod_{i=k-2+l}^{n-1} p_1(i) \right) \Phi_{n-l}^{(k,l)}. \qquad (2)$$

Proof: With the same conditions that previous proposition, $\Phi(-1) = \mathrm{diag}(c_{-1}, c_0, c_1, \ldots, c_{k-2})$.

After a transition stage, for *n > 2k-3*, a representation of general solution is of the form (2)

The complex coefficients *t* are related with the previous *c* and *p* coefficients by the relations:

$$t_{k-2+l} = \begin{cases} c_{k-2} & if \quad l = 0. \\ \sum_{j=1}^{k-l} c_{k-2-j} p_{l+j}(k-3+l) & if \quad 0 < l < k. \end{cases}$$

In matrix form: $\vec{t} = \mathbf{T}\vec{c}$

With the k *x* k block matrix T : $\mathbf{T} = \left(\begin{array}{c|c} \mathbf{0} & 1 \\ \hline \mathbf{U} & \mathbf{0} \end{array}\right)$

And the *k-1* x k-1 upper triangular matrix $\mathbf{U} = \begin{pmatrix} p_k(k-2) & p_{k-1}(k-2) & \ddots & p_2(k-2) \\ 0 & p_k(k-1) & \ddots & \ddots \\ \cdots & \vdots & \ddots & p_{k-1}(2k-5) \\ 0 & \cdots & 0 & p_k(2k-4) \end{pmatrix}$

$$x^{(k)}(n) = \left\langle \vec{t}^{\,*}_{k-2} \middle| \overline{\Psi}^{(k)}_{n} \right\rangle = \sum_{l=0}^{k-1} t_{k-2+l} \Psi^{(k,l)}_{n-l}.$$

Standard complex inner vector product:

The representation (2) is a linear combination of the $\Psi^{(k,l)}_{n-l}$, *l = 0, … k-1*.
These functions, fundamental solutions, are independent solutions, with similar nested structure.
Each $\Psi^{(k,l)}_{n-l}$ accomplishes the “shifted” initial condition, with *i,l = 0*, …, *k-1:*

$$x_{k-2+i} = \begin{cases} 0 \text{ if } i < l \\ \Psi^{(k,l)}_{k-2+i-l} \text{ if } l \le i < k \end{cases}$$

Thus, for linearity, it is sufficient to show that any $\Psi^{(k,l)}_{n-l}$ accomplishes the difference equation.

$$\Psi^{(k,l)}_{n+1-l} = \sum_{i=0}^{k-1} p_{i+1}(n)\Psi^{(k,l)}_{n-i-l}.$$

$$\Psi^{(k,l)}_{n+1-l} = \left(\prod_{i=k-2+l}^{n} p_1(i)\right)\sum_{m=0}^{\left\lfloor\frac{n+1-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+(k-3)+(i_1-2)+l}^{n}\alpha_{i_1}(k_1)\left(\sum_{i_2=2}^{k}\left(\sum_{k_2=2(m-1)+(k-3)+(i_2-2)+l}^{k_2-i_1}\alpha_{i_2}(k_2)\left(\cdots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)\right)$$

Coefficients $\alpha_{i1}(n)$ are separated

$$\Psi^{(k,l)}_{n+1-l} = p_1(n)\left(\prod_{i=k-2+l}^{n-1} p_1(i)\right)\sum_{m=0}^{\left\lfloor\frac{n-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+(k-3)+(i_1-2)+l}^{n-1}\alpha_{i_1}(k_1)\left(\sum_{i_2=2}^{k}\left(\sum_{k_2=2(m-1)+(k-3)+(i_2-2)+l}^{k_1-i_1}\alpha_{i_2}(k_2)\left(\cdots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)\right)$$

$$+\left(\prod_{i=k-2+l}^{n} p_1(i)\right)\sum_{m=1}^{\left\lfloor\frac{n+1-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_1=2}^{k}\alpha_{i_1}(n)\left(\sum_{i_2=2}^{k}\left(\sum_{k_2=2(m-1)+(k-3)+(i_2-2)+l}^{k_1-i_1}\alpha_{i_2}(k_2)\left(\cdots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right).$$

This is equal to

$$\Psi_{n+1-l}^{(k,l)} = p_1(n)\Psi_{n-l}^{(k,l)} +$$

$$+p_2(n)\left(\prod_{i=k-2+l}^{n-2} p_1(i)\right)\left(\sum_{m=1}^{\left\lfloor\frac{n+1-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_2=2}^{k}\left(\sum_{k_2=2(m-1)+(k-3)+(i_2-2)+l}^{k_1-2}\alpha_{i_2}(k_2)\left(\ldots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)$$

$$+p_3(n)\left(\prod_{i=k-2+l}^{n-3} p_1(i)\right)\left(\sum_{m=1}^{\left\lfloor\frac{n+1-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_2=2}^{k}\left(\sum_{k_2=2(m-1)+(k-3)+(i_2-2)+l}^{k_1-3}\alpha_{i_2}(k_2)\left(\ldots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)$$

$$+\ldots$$

$$+p_j(n)\left(\prod_{i=k-2+l}^{n-j} p_1(i)\right)\left(\sum_{m=1}^{\left\lfloor\frac{n+1-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_2=2}^{k}\left(\sum_{k_2=2(m-1)+(k-3)+(i_2-2)+l}^{k_1-j}\alpha_{i_2}(k_2)\left(\ldots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)$$

$$+\ldots$$

$$+p_k(n)\left(\prod_{i=k-2+l}^{n-k} p_1(i)\right)\left(\sum_{m=1}^{\left\lfloor\frac{n+1-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_2=2}^{k}\left(\sum_{k_2=2(m-1)+(k-3)+(i_2-2)+l}^{k_1-k}\alpha_{i_2}(k_2)\left(\ldots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right).$$

Change of indices $m_{new} = m_{old} -1$*, and then changing the mute indices* k_m by k_{m-1} and i_i by i_{i-1}*:*

$$\Psi_{n+1-l}^{(k,l)} = p_1(n)\Psi_{n-l}^{(k,l)} +$$

$$+p_2(n)\left(\prod_{i=k-2+l}^{n-2} p_1(i)\right)\left(\sum_{m=0}^{\left\lfloor \frac{n-1-(k-2)-l}{2} \right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+(k-3)+(i_1-2)+l}^{n-2}\alpha_{i_1}(k_1)\left(\cdots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)$$

$$+p_3(n)\left(\prod_{i=k-2+l}^{n-3} p_1(i)\right)\left(\sum_{m=0}^{\left\lfloor \frac{n-1-(k-2)-l}{2} \right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+(k-3)+(i_1-2)+l}^{n-3}\alpha_{i_1}(k_1)\left(\cdots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)$$

$$+\ldots$$

$$+p_j(n)\left(\prod_{i=k-2+l}^{n-j} p_1(i)\right)\left(\sum_{m=0}^{\left\lfloor \frac{n-1-(k-2)-l}{2} \right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+(k-3)+(i_1-2)+l}^{n-j}\alpha_{i_1}(k_1)\left(\cdots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)$$

$$+\ldots$$

$$+p_k(n)\left(\prod_{i=k-2+l}^{n-k} p_1(i)\right)\left(\sum_{m=0}^{\left\lfloor \frac{n-1-(k-2)-l}{2} \right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+(k-3)+(i_1-2)+l}^{n-k}\alpha_{i_1}(k_1)\left(\cdots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right).$$

Remark: When $m = 0$ the sum is 1 by convention.

In the *j*-th term of this sum:

$$p_j(n)\left(\prod_{i=k-2+l}^{n-j} p_1(i)\right)\left(\sum_{m=0}^{\left\lfloor \frac{n-1-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+(k-3)+(i_1-2)+l}^{n-j}\alpha_{i_1}(k_1)\left(\cdots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)$$

Only contribute to sum of index k_1 values that accomplish:

$$2m+(k-3)+(i_1-2)+l\le n-j\rightarrow$$

$$2m+(i_1-2)\le n-(j-1)-(k-2)-l\rightarrow m\le\frac{n-(j-1)-(k-2)-l}{2}$$

But: $$\frac{n-(j-1)-(k-2)-l}{2}-1<\left\lfloor\frac{n-(j-1)-(k-2)-l}{2}\right\rfloor\le\frac{n-(j-1)-(k-2)-l}{2}$$

Thus, the *j*-th term in the previous sum can be represented as :

$$p_j(n)\left(\prod_{i=k-2+l}^{n-(j-1)-1} p_1(i)\right)\left(\sum_{m=0}^{\left\lfloor \frac{n-(j-1)-(k-2)-l}{2}\right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+(k-3)+(i_1-2)+l}^{n-(j-1)-1}\alpha_{i_1}(k_1)\left(\cdots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+k-3+l}^{k_{m-1}-i_{m-1}}\alpha_{i_m}(k_m)\right)\right)\right)\right)\right)\right)$$

$$=p_j(n)\Psi^{(k,l)}_{n-(j-1)-l}$$

Then, the result holds

$$\Psi^{(k,l)}_{n+1-l} = \sum_{i=0}^{k-1} p_{i+1}(n)\Psi^{(k,l)}_{n-i-l}.$$

k-2+l ≤ n.
l = 0,1,…,k-1.

For each *l*, "shifted" initial condition also it accomplishes

$$x_{k\text{-}2+i} = 0 = \Psi^{(k,l)}_{k\text{-}2+i-l} \quad \text{For } i < l.$$

$$x_{k\text{-}2+l} = 1 = \Psi^{(k,l)}_{k-2}.$$

$$x_{k\text{-}1+l} = p_1(k-2+l)\Psi^{(k,l)}_{k-1+l-l} = \Psi^{(k,l)}_{k-1}.$$

$$x_{k+l} = p_1(k-1+l)\Psi^{(k,l)}_{k-1+l-l} + p_2(k-1+l)\Psi^{(k,l)}_{k-2+l-l} = \Psi^{(k,l)}_{k}.$$

....

$$x_{2k-3} = \Psi^{(k,l)}_{2k-3-l}.$$

l = 0,1,…,k-1.

The set of solutions $\Psi^{(k,l)}{}_{n-l}$ is composed by *k* l.i. solutions, with initial fundamental matrix:

$$\Phi\left(k-2\right)=\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \Psi_{k-1}^{(k,0)} & 1 & 0 & \ddots & 0 \\ \vdots & \Psi_{k-1}^{(k,1)} & 1 & 0 & \vdots \\ \Psi_{2k-4}^{(k,0)} & \ddots & \ddots & 1 & 0 \\ \Psi_{2k-3}^{(k,0)} & \Psi_{2k-4}^{(k,1)} & \cdots & \Psi_{k-1}^{(k,k-2)} & 1 \end{pmatrix}$$

Also this proof validates previous results.

For any $z_0 \in Z$, results are equivalent and expressions for $\Phi^{(k,l)}{}_{z-l}$ are, with $z_0 + k \leq z$:

$$\Phi_{z-l}^{(k,l)} =$$

$$\sum_{m=0}^{\left\lfloor \frac{z-1-z_0-(k-2)-l}{2} \right\rfloor}\left(\sum_{i_1=2}^{k}\left(\sum_{k_1=2m+z_0+(k-2)+(i_1-2)+l}^{z-1}\alpha_{i_1}\left(k_1\right)\left(\sum_{i_2=2}^{k}\left(\sum_{k_2=2(m-1)+z_0+(k-2)+(i_2-2)+l}^{k_1-i_1}\alpha_{i_2}\left(k_2\right)\left(\ldots\left(\sum_{i_m=2}^{k}\left(\sum_{k_m=i_m+z_0+k+l-2}^{k_{m-1}-i_{m-1}}\alpha_{i_m}\left(k_m\right)\right)\right)\right)\right)\right)\right)\right)$$

Application*. A compact representation of Orthogonal Polynomials sequences with a quasy-definite moment functional.

Particular 2th order case, $p_1(n) = A_n x + B_n$, $p_2(n) = -C_n$, $x_{-1} = 0$, $x_0 = 1$.

Orthogonal sequence $\{P_n(x)\}^{\infty}_{n=0}$ with terms of order n in x, $P_n(x) = \Psi^{(2,0)}_n(x) =$

$$\left(\prod_{j=0}^{n-1}\left(A_j x + B_j\right)\right)\sum_{l=0}^{\left\lfloor \frac{n}{2} \right\rfloor}(-1)^l\left(\sum_{k_1=2l}^{n-1}\frac{C_{k_1}}{\left(A_{k_1-1}x + B_{k_1-1}\right)\left(A_{k_1}x + B_{k_1}\right)}\left(\sum_{k_2=2(l-1)}^{k_1-2}\frac{C_{k_2}}{\left(A_{k_2-1}x + B_{k_2-1}\right)\left(A_{k_2}x + B_{k_2}\right)}\left(\cdots\left(\sum_{k_l=2}^{k_{l-1}-2}\frac{C_{k_l}}{\left(A_{k_l-1}x + B_{k_l-1}\right)\left(A_{k_l}x + B_{k_l}\right)}\right)\right)\right)\right)$$

*Abderramán, Escribano, Giraldo and Sastre.
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