FORCED OSCILLATIONS OF ISOTHERMAL AXISYMMETRIC LIQUID BRIDGES

José M. Perales, José Meseguer, Victoria Luengo

IDR/UPM, Universidad Politécnica de Madrid
c/o E.T.S.I. Aeronáuticos, Ciudad Universitaria
28040 Madrid, Spain
Tel: +34-91-3366353, e-mail: jmperales@idr.upm.es

ABSTRACT

The forced oscillations of a liquid column held by surface tension forces between two solid supports are analysed by using an inviscid linear three-dimensional model. The liquid bridge frequency response (interface deformation and pressure and velocity fields) when one (or both) of the supporting disks are oscillated or when the gravity field changes with time is calculated. The influence of a surrounding media of non-negligible density (outer bath) is included in the model. The latter consideration allows the use of experiments done on ground using the neutral buoyancy technique to validate the model for later use in flight experiments (where air is used as surrounding media and there is no need for considering its effect). In normal gravity conditions, the most appropriate way to obtain near cylindrical configurations is to surround the bridge with a matching density media and thus, the inertial effects of that fluid modifies significantly the dynamic behaviour.

INTRODUCTION

A liquid bridge is an idealisation of the fluid configuration appearing in the crystal growth process known as floating zone. When only mechanical aspects of the floating zone problem are considered, the simplest modelisation of the molten zone consists of an isothermal mass of liquid, of constant and uniform properties, filling the gap between two parallel solid disks placed a distance L apart, as sketched in Fig. 1.

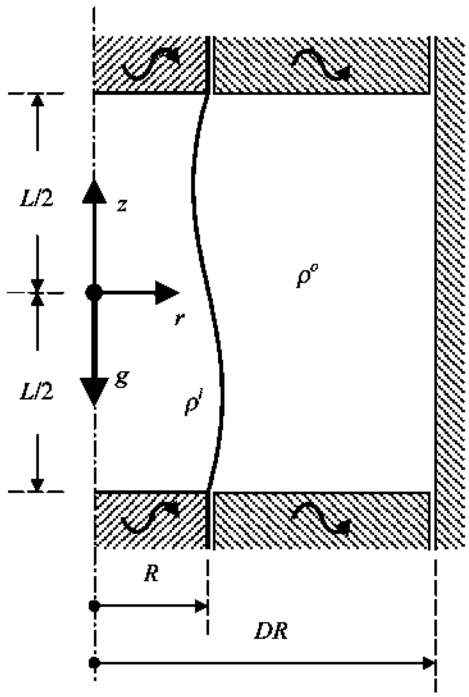

Figure 1. Geometry of the configuration.

Because of its interest not only as a mechanical model of the molten zone appearing in the floating zone process but its own scientific interest, liquid bridges have paid the attention of many scientists during the last two decades. Leaving apart the aspects of the liquid bridge problem related to the static (a review of the literature concerned with this topic can be found in Meseguer et al.[1]), a number of papers devoted to the dynamics of liquid columns have been published, most of them dealing with the oscillation of liquid bridges[2]. Because of the complexity of the problem formulation, most of the available results related to the liquid bridge dynamics are based on simplified formulations, which somehow reduce the range of validity of them. In some papers the analysis is performed by using one-dimensional models, which are of application only when it is assumed that the liquid bridge slenderness is large enough or, when three-dimensional models are used, other restricting hypothesis (dealing with the fluid properties -inviscid liquid- with the geometry of the fluid configuration -cylindrical liquid bridges- or both) are introduced in order to get treatable formulations.

The resonant frequencies have been calculated in the past by using three-dimensional models (which allow a precise consideration of inertial terms) in the inviscid case and considering a small viscosity for cylindrical liquid bridges[3]. If the volume is not cylindrical or the disks are not of equal diameter and concentrical, the analytical solutions based on three dimensional models are no longer possible and numerical schemes based in one-dimensional models are to be used[4,5].

Concerning the forced response of a liquid bridge when one (or both) of the supporting disks are oscillated or when the gravity field changes with time, one-dimensional models have been used mostly. These models, although allow an easy consideration of the viscous effects, introduce uncertainties in the consideration of inertial effects. In effect, they are based in averaging the velocity field and assuming a given radial variation of it. This hypothesis can be justified as far as the bridge is slender enough, but precludes the application of the model for short liquid bridges of large excitation frequencies.

Concerning experiments, data exist for slender configurations as well as for short configurations. These data have been obtained in orbital laboratories[6] (Spacelab D-2), in sounding rockets[7] and on ground by simulating microgravity conditions (either with liquid bridges of millimetric dimensions or by using the neutral buoyancy technique). While for slender bridges the agreement between experimental results and one-dimensional ones is good, for shorter bridges some discrepancies appear. Three-dimensional models have also been used (Meseguer et al.[8]), but the influence of the surrounding media has not been considered.

In this paper the forced oscillation of a cylindrical liquid column, held by surface tension forces between two equal-in-diameter disks inside a surrounding bath of non-negligible density and considering oscillating disks and variable acceleration (gravity), is analysed by using an inviscid linear three-dimensional model. This axisymmetric fluid configuration is uniquely identified by two dimensionless parameters which are related to the geometry of the liquid column, namely: the slenderness $\Lambda = L/(2R)$, where R is the radius of the disks, and the dimensionless volume of liquid, $V = \boldsymbol{V}/(\pi L R^2)$, where $\boldsymbol{V}$ stands for the physical volume of liquid (although in the following only the case $V = 1$ –liquid bridges of cylindrical volume- is considered), the Bond number $\mathrm{Bo} = \rho_i \tilde{g} R^2/\sigma$, where $\tilde{g}$ is the acceleration acting on the liquid bridge and s the surface tension, the ratio between the surrounding media density ρ_o and that of the liquid bridge, ρ_i, $\rho=\rho_o/\rho_i$ and the dimensionless distance to the tank wall, DR.

PROBLEM FORMULATION

Let us assume an axisymmetric, inviscid liquid bridge having cylindrical volume ($V = 1$) and slenderness Λ, whose supporting disks can be axially vibrated in an arbitrary form. Under the simplifying assumptions already introduced and considering the influence of a surrounding fluid and an acceleration field varying with time, the set of differential equations for the three-dimensional axisymmetric, inviscid flow in dimensionless form are the following:

Continuity equation:

$$u_r^i + u^i/r + w_z^i = 0 \quad , \quad u_r^o + u^o/r + w_z^o = 0 . \tag{1}$$

Momentum equations:

$$u_t^i + \frac{1}{2}\left(u^{i2} + w^{i2}\right)_r + w^i\left(u_z^i - w_r^i\right) = -p_r^i \quad , \quad u_t^o + \frac{1}{2}\left(u^{o2} + w^{o2}\right)_r + w^o\left(u_z^o - w_r^o\right) = -\frac{1}{\rho} p_r^o , \tag{2}$$

$$w_t^i + \frac{1}{2}\left(u^{i2} + w^{i2}\right)_z + u^i\left(w_r^i - u_z^i\right) = -p_z^i - \mathrm{Bo} \quad , \quad w_t^o + \frac{1}{2}\left(u^{o2} + w^{o2}\right)_z + u^o\left(w_r^o - u_z^o\right) = -\frac{1}{\rho} p_z^o - \mathrm{Bo} . \tag{3}$$

The boundary conditions for the above equations are: the regularity of the speed at the liquid bridge axis and zero radial velocity at the tank wall:

$$u^i(0,z,t) = w_r^i(0,z,t) = u^o(D,z,t) = 0 , \tag{4}$$

the anchorage of the liquid bridge interface to the edges of the disks:

$$f\left(\Lambda_{\pm}(t),t\right)=1 \ , \tag{5}$$

zero normal velocity at the disks and at the tank wall (where the boundary conditions at the upper and lower wall have been chosen to fulfil mass conservation independently at both ends):

$$w^{i}\left(r,\Lambda_{\pm}(t),t\right)=\frac{d}{dt}\Lambda_{\pm}(t) \quad , \quad \left(D^{2}-1\right)w^{o}\left(r,\Lambda_{\pm}(t),t\right)=-w^{i}\left(r,\Lambda_{\pm}(t),t\right) \ , \tag{6}$$

and the kinematic and dynamic boundary conditions at the interface:

$$f_{t}(z,t)=u^{i}\left(f(z,t),z,t\right)-f_{z}(z,t)w^{i}\left(f(z,t),z,t\right)=u^{o}\left(f(z,t),z,t\right)-f_{z}(z,t)w^{o}\left(f(z,t),z,t\right) \ , \tag{7}$$

$$p^{o}\left(f(z,t),z,t\right)-p^{i}\left(f(z,t),z,t\right)+\frac{f_{zz}(z,t)}{\left(1+f_{z}^{2}(z,t)\right)^{3/2}}-\frac{1}{f(z,t)\left(1+f_{z}^{2}(z,t)\right)^{1/2}}=0 \ , \tag{8}$$

plus the condition of volume preservation:

$$\int_{\Lambda_{-}(t)}^{\Lambda_{+}(t)} f^{2}(z,t)\,dz=2\Lambda \ . \tag{9}$$

To write down these dimensionless expressions R and $(\rho_l R^3/\sigma)^{1/2}$. The subscripts t, r and z mean partial derivatives with respect to time, r-coordinate and z-coordinate of the respective functions, and the subscript $\pm$ refers to conditions either in the upper disk (+) or in the lower one (−).

LINEAR ANALYSIS

The change in the position of the disks and the Bond number are assumed to be small and of the form

$$\Lambda_{\pm}(t)=\pm\Lambda+\varepsilon\lambda_{\pm}e^{i\Omega t} \quad , \quad \mathrm{Bo}=\varepsilon Be^{i\Omega t} \ , \tag{10}$$

where Λ is the zero amplitude position, $\lambda_{\pm}$ stands for the amplitude of the oscillation of the disks, ε is a small parameter that gives the order of magnitude of the oscillations and Ω is the frequency of the forced oscillations. If only small amplitudes for these perturbations are considered, ε<<1, the solution for the problem can be expressed as:

$$f(z,t)=1+\varepsilon F(z)e^{i\Omega t}, \quad p^{i}(r,z,t)=1+\varepsilon P^{i}(r,z)e^{i\Omega t}, \quad p^{o}(r,z,t)=1+\varepsilon P^{o}(r,z)e^{i\Omega t}$$
$$u^{i}(r,z,t)=\varepsilon U^{i}(r,z)e^{i\Omega t}\ , u^{o}(r,z,t)=\varepsilon U^{o}(r,z)e^{i\Omega t}\ , w^{i}(r,z,t)=\varepsilon W^{i}(r,z)e^{i\Omega t}\ , w^{o}(r,z,t)=\varepsilon W^{o}(r,z)e^{i\Omega t} \ ,$$

and the introduction of these expressions in (1)-(9), neglecting higher order terms, gives the following irrotational linearised problem:

$$U_{r}^{i}+U^{i}/r+W_{z}^{i}=0 \quad , \quad U_{r}^{o}+U^{o}/r+W_{z}^{o}=0 \ , \tag{11}$$

$$i\Omega U^{i}=-P_{r}^{i} \quad , \quad i\Omega U^{o}=-P_{r}^{o}/\rho \ , \tag{12}$$

$$i\Omega W^{i}=-P_{z}^{i}-B \quad , \quad i\Omega W^{o}=-P_{z}^{o}/\rho-B \ , \tag{13}$$

$$U^{i}(0,z)=W_{r}^{i}(0,z)=U^{o}(D,z)=0 \ , \tag{14}$$

$$F(\pm\Lambda)=0 \ , \tag{15}$$

$$W^{i}(r,\pm\Lambda)=i\lambda_{\pm}\Omega \quad , \quad W^{o}(r,\pm\Lambda)=-\frac{i\lambda_{\pm}\Omega}{D^{2}-1} \ , \tag{16}$$

$$i\Omega F(z)=U^{i}(1,z)=U^{o}(1,z) \ , \tag{17}$$

$$F_{zz}(z)+F(z)=-P^{i}(1,z)+P^{o}(1,z) \ , \tag{18}$$

$$\int_{-\Lambda}^{\Lambda} F(z)dz = -\frac{1}{2}(\lambda_{+} - \lambda_{-}) \; . \tag{19}$$

Momentum equations (12) and (13) give the velocities U^i, U^o, W^i and W^o in terms of the pressures P^i and P^o. The introduction of these expressions for U^i, U^o, W^i and W^o in the continuity equations (11) and in the boundary conditions at the liquid bridge axis, expressions (14), and at the disks, (16), gives the following formulation for the pressure problems

$$P_{rr}^i + P_r^i / r + P_{zz}^i = 0 \quad , \quad P_{rr}^o + P_r^o / r + P_{zz}^o = 0 \; , \tag{20}$$

$$P_r^i(0,z) = P_{rz}^i(0,z) = P_r^o(D,z) = 0 \; , \tag{21}$$

$$P_z^i(r,\pm\Lambda) = \lambda_{\pm}\Omega^2 - B \quad , \quad P_z^o(r,\pm\Lambda) = -\lambda_{\mp}\Omega^2 \frac{\rho}{D^2-1} - \rho B \; . \tag{22}$$

Expressions (20) and (21) are equal to the corresponding ones appearing in the free oscillation problem solved by Sanz[9], but the boundary conditions are now rather different (the quoted problem has an homogeneous formulation and thus only the eigenproblem was solved there). If we now define two new functions $Q^i(r,z)$ and $Q^o(r,z)$ by

$$P^i(r,z) = \lambda_{+}\frac{\Omega^2}{2\Lambda}\left[\frac{z^2}{2} + \Lambda z - \frac{r^2}{4}\right] + \lambda_{-}\frac{\Omega^2}{2\Lambda}\left[-\frac{z^2}{2} + \Lambda z + \frac{r^2}{4}\right] - Bz + Q^i(r,z) \; , \tag{23a}$$

$$P^o(r,z) = -\lambda_{-}\frac{\Omega^2}{2\Lambda}\frac{\rho}{D^2-1}\left[\frac{z^2}{2} + \Lambda z + \frac{D^2}{2}\ln r - \frac{r^2}{4}\right] - \lambda_{-}\frac{\Omega^2}{2\Lambda}\frac{\rho}{D^2-1}\left[-\frac{z^2}{2} + \Lambda z - \frac{D^2}{2}\ln r + \frac{r^2}{4}\right] - \rho Bz + Q^o(r,z) \; , \tag{23b}$$

the following formulation for $Q^i(r,z)$ and $Q^o(r,z)$ results

$$Q_{rr}^i + Q_r^i / r + Q_{zz}^i = 0 \quad , \quad Q_{rr}^o + Q_r^o / r + Q_{zz}^o = 0 \; , \tag{24}$$

$$Q_r^i(0,z) = Q_{rz}^i(0,z) = Q_r^o(D,z) = 0 \; , \tag{25}$$

$$Q_z^i(r,\pm\Lambda) = Q_z^o(r,\pm\Lambda) = 0 \; . \tag{26}$$

Therefore, the problem for $Q^i(r,z)$ and $Q^o(r,z)$ is now the same problem solved by Sanz[9] and a similar solution technique will be used. A general solution of the differential equations (24) fulfilling the boundary conditions (25) and (26) can be found by looking a solution in terms of a function only of r times a function only of z, the solution being:

$$Q^i(r,z) = g^i + \sum_{n=1}^{\infty} a_n \Omega^2 \, \mathrm{I}_0(\mu_n r)\cos(\mu_n(z+\Lambda)) \; , \tag{27}$$

$$Q^o(r,z) = g^o + \sum_{n=1}^{\infty} b_n \Omega^2 \left[\mathrm{K}_0(\mu_n r) + \mathrm{I}_0(\mu_n r)\frac{\mathrm{K}_1(\mu_n D)}{\mathrm{I}_1(\mu_n D)}\right]\cos(\mu_n(z+\Lambda)) \; , \tag{28}$$

where $\mu_n = n\pi/(2\Lambda)$ and g^i, g^o and a_n are unknown constants; I_0 and I_1 are the modified Bessel function of first kind of zero and first order and K_0 and K_1 are the modified Bessel function of second kind of zero and first order.

Taking into account the equations (12) from the boundary conditions (17) it is deduced that $F(z) = P_r^i(1,z)/\Omega^2 = P_r^o(1,z)/\Omega^2\rho$. From these equalities, it is deduced that

$$F(z) = -\frac{1}{4\Lambda}(\lambda_{+} - \lambda_{-}) + \sum_{n=1}^{\infty} a_n \mu_n \mathrm{I}_1(\mu_n)\cos(\mu_n(z+\Lambda)) \; . \tag{29}$$

and

$$b_n = -\rho a_n \left[\frac{\mathrm{K}_1(\mu_n)}{\mathrm{I}_1(\mu_n)} - \frac{\mathrm{K}_1(\mu_n D)}{\mathrm{I}_1(\mu_n D)}\right]^{-1} .$$

On the other hand, $F(z)$ must fulfil the interface mechanical equilibrium, expression (18), as well as boundary conditions (15) and (19), that is

$$
\begin{aligned}
F+F_{zz} &= -P^{i}(1,z)+P^{o}(1,z) = -g-\sum_{n=1}^{\infty} a_n \Omega^2 \mathrm{I}_0(\mu_n) S_n \cos\left(\mu_n (z+\Lambda)\right) - \\
&\quad -\lambda_{+} \frac{\Omega^2}{2\Lambda}\left(1+\frac{\rho}{D^2-1}\right)\left[\frac{z^2}{2}+\Lambda z-\frac{1}{4}\right]-\lambda_{-} \frac{\Omega^2}{2\Lambda}\left(1+\frac{\rho}{D^2-1}\right)\left[-\frac{z^2}{2}+\Lambda z+\frac{1}{4}\right]+(1-\rho)Bz
\end{aligned}
\tag{30}
$$

$$
F(\pm\Lambda)=0 \ , \tag{31}
$$

$$
\int_{-\Lambda}^{\Lambda} F(z)\,dz = -\frac{1}{2}(\lambda_{-}-\lambda_{-}) \ . \tag{32}
$$

where $g = g^i - g^o$ and $S_n = 1+\rho\left[\frac{\mathrm{K}_0(\mu_n)}{\mathrm{I}_0(\mu_n)}+\frac{\mathrm{K}_1(\mu_n D)}{\mathrm{I}_1(\mu_n D)}\right]\left[\frac{\mathrm{K}_1(\mu_n)}{\mathrm{I}_1(\mu_n)}-\frac{\mathrm{K}_1(\mu_n D)}{\mathrm{I}_1(\mu_n D)}\right]^{-1}$.

Expressions (30)-(32) are the formulation to be solved to calculate $F(z)$, and particularly to determine the constants g and a_n. Once the constants are known, the functions $Q^i(r,z)$ and $Q^o(r,z)$ and the pressure fields can be calculated from equations (23a), (23b), (27) and (28). The velocity field can then be obtained by using equations (12) and (13). To calculate the constants g and a_n appearing in (30) the procedure is as follows: the general solution of the differential equation (30) can be written as

$$
\begin{aligned}
F(z) &= a\sin(z)+b\cos(z)-g-\sum_{n=1}^{\infty}\frac{a_n\Omega^2 \mathrm{I}_0(\mu_n)S_n}{1-\mu_n{}^2}\cos\left(\mu_n(z+\Lambda)\right)+ \\
&\quad +(\lambda_{+}-\lambda_{-})\Omega^2\frac{1}{4\Lambda}\left(1+\frac{\rho}{D^2-1}\right)\left[\frac{5}{2}-z^2\right]-(\lambda_{+}+\lambda_{-})\Omega^2\left(1+\frac{\rho}{D^2-1}\right)\frac{z}{2}+(1-\rho)Bz
\end{aligned}
\tag{33}
$$

and from boundary conditions (31) and (32) onc obtains:

$$
\begin{aligned}
&-a\sin(\Lambda)-b\cos(\Lambda)+g+\sum_{n=1}^{\infty}\frac{a_n\Omega^2 \mathrm{I}_0(\mu_n)S_n}{1-\mu_n{}^2}\cos(n\pi)= \\
&\quad =(\lambda_{+}-\lambda_{-})\Omega^2\frac{1}{4\Lambda}\left(1+\frac{\rho}{D^2-1}\right)\left[\frac{5}{2}-\Lambda^2\right]-(\lambda_{+}+\lambda_{-})\Omega^2\left(1+\frac{\rho}{D^2-1}\right)\frac{\Lambda}{2}+(1-\rho)B\Lambda
\end{aligned}
\ , \tag{34}
$$

$$
\begin{aligned}
&a\sin(\Lambda)-b\cos(\Lambda)+g+\sum_{n=1}^{\infty}\frac{a_n\Omega^2 \mathrm{I}_0(\mu_n)S_n}{1-\mu_n{}^2}= \\
&\quad =(\lambda_{+}-\lambda_{-})\Omega^2\frac{1}{4\Lambda}\left(1+\frac{\rho}{D^2-1}\right)\left[\frac{5}{2}-\Lambda^2\right]+(\lambda_{+}+\lambda_{-})\Omega^2\left(1+\frac{\rho}{D^2-1}\right)\frac{\Lambda}{2}-(1-\rho)B\Lambda
\end{aligned}
\ , \tag{35}
$$

$$
g\Lambda-b\sin(\Lambda)=(\lambda_{+}-\lambda_{-})\Omega^2\left(1+\frac{\rho}{D^2-1}\right)\left(\frac{5}{8}-\frac{\Lambda^2}{12}\right)+\frac{1}{4}(\lambda_{+}-\lambda_{-}) \ . \tag{36}
$$

Expressions (34) and (35) can be replaced by the following two equivalent conditions:

$$
a\sin(\Lambda)+\sum_{m=0}^{\infty}\frac{a_{2m-1}\Omega^2 \mathrm{I}_0(\mu_{2m+1})S_{2m+1}}{1-\mu_{2m+1}^2}=(\lambda_{+}+\lambda_{-})\Omega^2\left(1+\frac{\rho}{D^2-1}\right)\frac{\Lambda}{2}-(1-\rho)B\Lambda \ , \tag{37}
$$

$$
-b\cos(\Lambda)+g+\sum_{m=1}^{\infty}\frac{a_{2m}\Omega^2 \mathrm{I}_0(\mu_{2m})S_{2m}}{1-\mu_{2m}^2}=(\lambda_{-}-\lambda_{-})\Omega^2\left(1+\frac{\rho}{D^2-1}\right)\left(\frac{5}{8\Lambda}-\frac{\Lambda}{4}\right), \tag{38}
$$

and taking into account (36) the constant g can be replaced in (38), the resulting expression being

$$
b\cos(\Lambda)\left[1-\frac{\tan(\Lambda)}{\Lambda}\right]-\sum_{m=1}^{\infty}\frac{a_{2m}\Omega^2 \mathrm{I}_0(\mu_{2m})S_{2m}}{1-\mu_{2m}^2}=(\lambda_{+}-\lambda_{-})\Omega^2\left(1+\frac{\rho}{D^2-1}\right)\frac{\Lambda}{6}+(\lambda_{+}-\lambda_{-})\frac{1}{4\Lambda} \ . \tag{39}
$$

From expressions (37) and (39) the constants a and b are calculated in terms of the coefficients a_n. To calculate such coefficients, first of all the following expansions for cos(z), sin(z), z and z^2 are needed:

$$\sin(z)=2\frac{\cos(\Lambda)}{\Lambda}\sum_{m=0}^{\infty}\frac{1}{1-\mu_{2m-1}^2}\cos\left(\mu_{2m+1}(z+\Lambda)\right) \quad , \quad \cos(z)=\frac{\sin(\Lambda)}{\Lambda}\left[1+2\sum_{m=1}^{\infty}\frac{1}{1-\mu_{2m}^2}\cos\left(\mu_{2m}(z+\Lambda)\right)\right] ,$$

$$z=-\frac{2}{\Lambda}\sum_{m=0}^{\infty}\frac{1}{\mu_{2m+1}^2}\cos\left(\mu_{2m+1}(z+\Lambda)\right) \quad , \quad z^2=\frac{\Lambda^2}{3}+4\sum_{m=1}^{\infty}\frac{1}{\mu_{2m}^2}\cos\left(\mu_{2m}(z+\Lambda)\right) , \tag{40}$$

which are introduced in the expression of $F(z)$ given by (33). Then, since we have now two different expressions for $F(z)$, equations (29) and (33), by equating the coefficients of $\cos(\mu_n(z+\Lambda))$ one obtains

$$a_{2m+1}\left[\Omega^2 \mathrm{I}_0(\mu_{2m+1})S_{2m+1}+\mu_{2m+1}\left(1-\mu_{2m+1}^2\right)\mathrm{I}_1(\mu_{2m+1})\right]=2a\frac{\cos(\Lambda)}{\Lambda}+\left[(\lambda_+ +\lambda_-)\Omega^2\left(1+\frac{\rho}{D^2-1}\right)-2(1-\rho)B\right]\frac{1-\mu_{2m+1}^2}{\Lambda\mu_{2m+1}^2} \text{ ,} \tag{41}$$

$$a_{2m}\left[\Omega^2 \mathrm{I}_0(\mu_{2m})S_{2m}+\mu_{2m}\left(1-\mu_{2m}^2\right)\mathrm{I}_1(\mu_{2m})\right]=2b\frac{\sin(\Lambda)}{\Lambda}-(\lambda_+ -\lambda_-)\Omega^2\left(1+\frac{\rho}{D^2-1}\right)\frac{1-\mu_{2m}^2}{\Lambda\mu_{2m}^2} . \tag{42}$$

These last two equations express the coefficients a_n in terms of the constants a and b. The substitution of a_n as given by (41) and (42) in (37) and (39) finally yields

$$a\left[\sin(\Lambda)+2\frac{\cos(\Lambda)}{\Lambda}\sum_{m=0}^{\infty}\frac{\Gamma_{2m-1}}{1-\mu_{2m+1}^2}\right]=\left[(\lambda_+ +\lambda_-)\Omega^2\left(1+\frac{\rho}{D^2-1}\right)-2(1-\rho)B\right]\left[\frac{\Lambda}{2}-\frac{1}{\Lambda}\sum_{m=0}^{\infty}\frac{\Gamma_{2m+1}}{\mu_{2m+1}^2}\right] , \tag{43}$$

$$b\left[\cos(\Lambda)\left[1-\frac{\tan(\Lambda)}{\Lambda}\right]-2\frac{\sin(\Lambda)}{\Lambda}\sum_{m=1}^{\infty}\frac{\Gamma_{2m}}{1-\mu_{2m}^2}\right]=(\lambda_+ -\lambda_-)\Omega^2\left(1+\frac{\rho}{D^2-1}\right)\left[\frac{\Lambda}{6}-\frac{1}{\Lambda}\sum_{m=1}^{\infty}\frac{\Gamma_{2m}}{\mu_{2m}^2}\right]+(\lambda_+ -\lambda_-)\frac{1}{4\Lambda} , \tag{44}$$

where $\Gamma_n=\dfrac{\Omega^2 \mathrm{I}_0(\mu_n)S_n}{\left[\Omega^2 \mathrm{I}_0(\mu_n)S_n+\mu_n\left(1-\mu_n^2\right)\mathrm{I}_1(\mu_n)\right]}$.

Therefore, once a and b are determined by using (43) and (44), the coefficients a_{2m} and a_{2m+1} are obtained from (41) and (42), the constant g results from expression (36) and $F(z)$ can be obtained either from (29) or (33).

It is remarkable that responses to symmetric and antisymmetric (with respect to z = 0) excitations are fully decoupled. In effect, the coefficient a depends only on $\lambda_+ + \lambda_-$ and B and the coefficients b and g on $\lambda_+ - \lambda_-$. Thus an antisymmetric excitations produces $a_{2m+1} \neq 0$, $a_{2m} = 0$ (an antisymmetric response) whereas a symmetric excitation produces a symmetric response ($a_{2m+1} = 0$, $a_{2m} \neq 0$).

Before pursuing further it is convenient to compare the results here obtained with other results already published. The forced oscillation of long axisymmetric liquid bridges was analysed by Perales and Meseguer[4] by using a linearised one-dimensional model based on the Cosserat formulation for the liquid bridge problem. According to this study, when the frequency of the forcing perturbation vanishes ($\Omega\to 0$), in the case of in-phase oscillation the liquid bridge interface can be expressed as

$$F(z)=\frac{1}{4}(\lambda_+ -\lambda_-)\frac{\cos(z)-\cos(\Lambda)}{\Lambda\cos(\Lambda)-\sin(\Lambda)} . \tag{45}$$

According to the results here obtained, when $\Omega\to 0$ from expression (43) one gets a=0, and from (44) it is obtained

$$b\left[\Lambda\cos(\Lambda)-\sin\Lambda\right]=\frac{1}{4}(\lambda_+ -\lambda_-) .$$

On the other hand, from (38) results $g = b\cos(\Lambda)$. Therefore, the substitution of these expressions in equation (33), which now is reduced to $F(z) = b\cos(z)-g$, yields to an expression for $F(z)$ identical to (45).

The transfer function of the liquid bridge interface, defined as the variation of the maximum interface deformation of the liquid bridge as a function of the frequency of the forcing perturbation, $\Delta F = (F_{max} - F_{min})/(\lambda_+ + \lambda_-)$ for the case of antisymmetric excitation has been represented in figure 2 for two liquid bridges of slendernesses Λ=1.5 and Λ=2.5. In these plots the transfer functions resulting from the three-dimensional (3-D) model considering the effect of a bath isodense with the liquid bridge, ρ=1, and walls at a dimensionless distance D=4 have been represented (solid lines), as well as the results obtained neglecting the bath effect (dashed lines) and the transfer function resulting from the Cosserat one-dimensional (1-D) model[4] (dotted lines).

As it can be observed there is a remarkable influence of the outer bath in the transfer function. Resonance frequencies decrease due to the added inertia of the outer bath and the curve changes so much that it would be very difficult to compare the results of experiments done on ground with the Plateau tank technique with the results of any model that do not take the bath effect into account (especially for high frequencies).

The Cosserat model yields results very similar to those of the 3-D model when bath effect is not considered as can be expected as this model does not take into account the bath. The agreement is worse with decreasing slenderness, as expected, since 1-D models show a decreasing accuracy as the slenderness decreases (in 1-D models it is assumed that the velocity does not depend on the radial coordinate r, and this hypothesis is only valid provided the slenderness is large enough[4], $\Lambda >> 1$).

Another aspect to be pointed out from figure. 2 is that the differences between 1-D and 3-D models increases as the frequency of the forcing perturbation increases. This fact can be easily explained as the characteristic length to be taken to compare with the radius is not really the liquid bridge length but the wavelength of the perturbation. Thus, for the one-dimensional models yielding accurate results it is needed not only to have $\Lambda >> 1$ but $k/R >> 1$, k being the wavelength. As the wavelength decreases as the excitation frequency increases, one-dimensional models yield worse results for higher frequencies.

CONCLUSIONS

A three-dimensional model to study forced oscillations of liquid bridges including surrounding media effect has been developed. This model (exact as long as neither viscosity effects nor non-linear effects are considered) compares well with previous results obtained using one-dimensional models (which, in turn, can easily include viscous effects and non-linear effects). Thus the range of validity of existing one-dimensional models has been determined developing a method to check its accuracy (limited to large slendernesses and small excitation frequencies). The model allows an easy checking with experiments done with the Plateau technique.

ACKNOWLEDGEMENT

This work has been supported by the Spanish Ministry for Science and Technology, MCYT.

REFERENCES

1. Meseguer, J., Perales, J.M., Martínez, I., Bezdenejnykh, N.A. and Sanz, A, Hydrostatic Instabilities in Floating Zone Crystal Growth Process, Current Topics in Crystal Growth Research, Vol. 5, pp. 27-42, 1999.
2. Sanz, A., Static and Dynamic Response of Liquid Bridges, in Microgravity Fluid Mechanics, H.J. Rath, Ed., Springer-Verlag, Berlin, pp. 3-17, 1992.
3. Higuera, M., Nicolas, J.A. and Vega, J.M., Linear Oscillations of Weakly Dissipative Axisymmetric Liquid Bridges, Physics of Fluids, Vol. 6, pp. 438-450, 1994.
4. Perales, J.M. and Meseguer, J., Theoretical and Experimental Study of the Vibration of Axisymmetric Viscous Liquid Bridges, Physics of Fluids A, Vol. 4, pp. 1110-1130, 1992.
5. Meseguer, J. and Perales, J.M., Viscosity Effects on the Dynamics of Long Axisymmetric Liquid Bridges, in Microgravity Fluid Mechanics, H.J. Rath, Ed., Springer-Verlag, Berlin, pp. 37-46, 1992.
6. Langbein, D., Falk, F., Grossbach, R., Heide, W., Bauer, H., Meseguer, J., Perales, J.M. and Sanz, A., LICOR-Liquid Columns' Resonances, in Scientific Results of the German Spacelab Mission D-2, Sahm, P.R., Keller, M.H. and Schieve, B., Eds., Wissenschaftliche Projektfuhrung Spacelabmission D-2, pp. 209-219, 1995.
7. Martínez, I., Perales, J.M. and Meseguer, J., Response of a Liquid Bridge to an Acceleration Varying Sinusoidally with Time, in Materials and Fluids under Low Gravity, L. Ratke, H. Walter and B. Feuerbacher, Eds., Springer, Berlin, pp. 271-279, 1996.
8. Meseguer, J., Perales, J.M. and Laverón-Simavilla, A., Forced Oscillations of Isothermal Liquid Bridges, Proceedings of the 20th International Symposium on Space Technology and Science, Institute of Space and Aeronautical Science of Japan, pp. 963-968, 1996.
9. Sanz, A., The influence of the outer bath on the dynamics of axisymmetric liquid bridges, J. Fluid Mech., Vol. 156, pp. 101-140, 1985.