Viscosity Effects on the Dynamics of Long Axisymmetric Liquid Bridges

J. Meseguer and J.M. Perales
Lamf-µg, Laboratorio de Aerodinámica, E.T.S.I. Aeronáuticos,
Universidad Politécnica, 28040 Madrid, Spain

Abstract
In this paper the dynamics of axisymmetric liquid columns held by capillary forces between two circular, concentric, solid disks is considered. The problem has been solved by using an one-dimensional model known in the literature as the Cosserat model, which includes viscosity effects, where the axial velocity is considered constant in each section of the liquid bridge. The dynamic response of the bridge to an excitation consisting of a small amplitude vibration of the supporting disks has been solved by linearising the Cosserat model. It has been assumed that such excitation is harmonic so that the analysis has been performed in the frequency domain and the dependence of the frequency of resonance corresponding to the first oscillation mode on the parameters defining the liquid bridge configuration as well as the axial microgravity level has been calculated for several liquid bridge configurations.

I. Introduction

This paper deals with the dynamics of axisymmetric viscous liquid bridges. It is known as a liquid bridge the fluid configuration consisting of a mass of liquid held by surface tension forces between two parallel, coaxial, solid disks, as sketched in Fig. 1. Such a fluid configuration can be identified by the following dimensionless parameters: the slenderness, $\Lambda = L/2R_o$, where L stands for the distance between the disks and $R_o = (R_1+R_2)/2$ is a mean radius; the ratio of the radius of the smaller disk, R_1, to the radius of the larger one, R_2, $K = R_1/R_2$; the dimensionless volume of liquid, $V = \mathcal{V}/R_o^3$ $\mathcal{V}$ being the physical volume; the Bond number, $B = \rho g R_o^2/\sigma$, where ρ is the liquid density, g the axial acceleration and σ the surface tension; and the viscous to capillary forces ratio, $C = (Oh)^{1/2} = \nu(\rho/\sigma R_o)^{1/2}$, ν being the kinematic viscosity of the liquid, which is the square root of the Ohnesorge number, Oh.

Liquid bridges have focused the attention of numerous scientists during the last decades, and a large number of papers dealing with different aspects of the liquid bridge problem have been published. One of these aspects is that concerned with the frequencies of

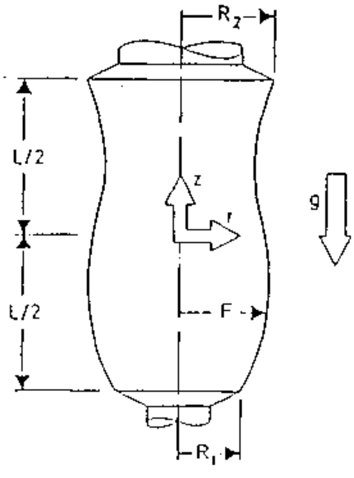

Fig. 1. Geometry and coordinate system for the liquid bridge problem.

resonance of liquid bridges, which have been extensively studied in the case of cylindrical liquid bridges ($K = 1$, $V = 2\pi\Lambda$) either in gravitationless conditions or taking into account the effect of the gravity acting parallel to the liquid bridge axis [1-5]. Some attempts have been made also to take into account volumes of liquid different from the cylindrical one and unequal disks [6-8]. However, most of the published papers deal with the dynamics of inviscid liquid bridges and, although some attempts to include viscosity effects in the analysis of liquid bridge dynamics have been performed [3,4], these studies are mainly devoted to the analysis of the free oscillations of cylindrical or almost cylindrical volume liquid bridges ($V \approx 2\pi\Lambda$). This paper is an extension of previous studies related to the forced oscillations of viscous liquid bridges [8]. The frequencies of resonance of liquid bridges between unequal disks, volume different from the cylindrical one, and in a small axial gravity field have been analyzed by using a one-dimensional Cosserat model and mappings of frequencies of resonance on Λ-V stability diagrams have been calculated.

2. Analytical Background

In the following, all physical quantities have been made dimensionless by using the characteristic length R_o and the characteristic time $(\rho R_o^3/\sigma)^{1/2}$. The mathematical model presented in this section is a simplified version of the one presented in [8]. If the slenderness of the liquid bridge is large enough, say $\Lambda > 1$, the dynamics of the liquid column can be described accurately enough by using one-dimensional theories such as the Cosserat model, which has been used to some extent either in capillary jets [9,10] or in liquid bridge problems [1,3].

In carrying out the analysis the following assumptions are introduced: it is assumed that the properties of both the liquid (density and viscosity) and the interface (surface tension) are uniform and constant, and the effects of the gas surrounding the liquid bridge are negligible. In addition, since only axisymmetric configurations are considered, the problem is assumed to be independent of the azimuthal coordinate. Under such assumptions the set of nondimensional differential equations and boundary conditions for the axisymmetric, non-rotating viscous flow, according to the Cosserat model, are the following:

$$S_t + Q_z = 0 \ , \tag{2.1}$$

$$Q_t + (Q^2/S)_z - \frac{1}{8}\left\{S^2\left[\left(\frac{Q_t + (Q^2/S)_z}{S}\right)_z - \frac{3}{2}(Q/S)_z^2\right]\right\}_z = \tag{2.2}$$

$$= -S\left(4(2S + S_z^2 - SS_{zz})(4S + S_z^2)^{-3/2} + Bz\right)_z - \frac{1}{8}C\left[S^2(Q/S)_{zz}\right]_{zz} + 3C\left[S(Q/S)_z\right]_z$$

In these expressions $S = F^2$ and $Q = F^2W$, where $F(z,t)$ is the dimensionless equation of the liquid-gas interface and $W(z,t)$ the axial velocity at each plane parallel to the disks; $P(z,t)$ accounts for capillary pressure jump across the interface. The subscripts t and z indicate derivatives with respect to the time and the axial coordinate, respectively. Boundary conditions are: (1) the interface must remain anchored to the disk edges and (2) the axial velocity at each one of the disks must be equal to that of the corresponding supporting disks (which are assumed to be in a known position as a function of the time given by $z_1 = -\Lambda + \Lambda\lambda_1(t)$ and $z_2 = \Lambda + \Lambda\lambda_2(t)$), its velocity being $z'_1(t) = \Lambda\lambda'_1(t)$ and $z'_2(t) = \Lambda\lambda'_2(t)$

$$S(\Lambda + \Lambda\lambda_2(t),t) = \left(\frac{2}{1+K}\right)^2 , \quad S(-\Lambda + \Lambda\lambda_1(t),t) = \left(\frac{2K}{1+K}\right)^2 , \tag{2.3}$$

$$Q(\Lambda + \Lambda\lambda_2(t),t) = \Lambda\lambda_2'(t)\left(\frac{2}{1+K}\right)^2 , \quad Q(-\Lambda + \Lambda\lambda_1(t),t) = \Lambda\lambda_1'(t)\left(\frac{2K}{1+K}\right)^2 \tag{2.4}$$

where prime means time derivative of the function considered. Initial conditions are $S(z,0) = S_i(z)$ and $Q(z,0) = Q_i(z)$; in addition, one more condition could be introduced imposing the overall mass conservation during the evolution. Concerning the above formulation it should be pointed out that boundary conditions must be fulfilled in two points whose position, although known, varies with time. To avoid the difficulties of these moving boundary conditions a contraction of the axial coordinate is made and a new variable x is defined so that the interval of variation of the coordinate z (function of time) is mapped into a fixed interval. Amongst the different possibilities, a simple linear mapping has been chosen:

$$x = x(z,t) = \Lambda\frac{z - g(t)}{\Lambda + h(t)} = \frac{z - g(t)}{1 + \frac{1}{\Lambda}h(t)} \tag{2.5}$$

where $g(t) = (\Lambda/2)(\lambda_2(t)+\lambda_1(t))$ and $h(t) = (\Lambda/2)(\lambda_2(t)-\lambda_1(t))$. The function $g(t)$ gives the variation with time of the position of the center of the liquid bridge (that point of the axis placed at every moment in the middle of the segment defined by the centres of the disks) and $h(t)$ is the variation with time of the distance between the disks. In the coordinates x,t the disk positions are fixed and given by $x(-\Lambda+\Lambda\lambda_1(t),t) = -\Lambda$ and $x(\Lambda+\Lambda\lambda_2(t),t) = \Lambda$.

If only small perturbations are considered ($g(t) << 1$ and $h(t) << 1$) the solution of the problem can be written as a static solution plus a small perturbation i.e.

$$S(x,t) = S_o(x) + s(x,t) \quad , \quad Q(x,t) = q(x,t) \quad . \tag{2.6}$$

After introduction of eq. (2.6) in the above formulation, the resulting zeroth order problem consists of the determination of the equilibrium shape of a liquid bridge at rest, $S_o(x)$, and it can be solved with a method similar to the one used in [11]. Concerning the first order problem, $s(x,t)$ can be eliminated from the formulation and the whole problem formulated in terms of $q(x,t)$, the resulting equation being

$$C_{41}q_{xxxxt} + C_{40}q_{xxxx} + C_{30}q_{xxx} + C_{22}q_{xxtt} + C_{21}q_{xxt} +$$

$$C_{20}q_{xx} + C_{11}q_{xt} + C_{10}q_x + C_{02}q_{tt} + C_{01}q_t = C_{g1}g' + C_{h1}h' \tag{2.7}$$

where C_{ij} are functions of $S_o(x)$ and its derivatives (additional details can be obtained upon request from the authors). Equation (2.7) is fourth order in the variable x and, therefore, needs four boundary conditions to be solved. Two of them are derived from eq. (2.4) and the two remaining can be deduced from the boundary condition (2.3) which implies $s_t(\pm\Lambda,t) = 0$ and, using the continuity equation (2.1), the boundary conditions become

$$q(\pm\Lambda,t) = [g' \pm h']S_o(\pm\Lambda) \quad , \quad q_x(\pm\Lambda,t) = [g' \pm h']S_{ox}(\pm\Lambda) \tag{2.8}$$

3. Harmonic Oscillations

Since in the modelling of most of the technological applications (e.g. floating zone technique) the distance between the disks should be considered constant and the perturbation is assumed to be due to g-jitter, only in-phase vibration of disks ($h(t) = 0$, $g(t) \neq 0$) will be considered from now on. In the following it is assumed that both the liquid bridge perturbation, $g(t)$, and the liquid bridge response, $q(x,t)$ and $s(x,t)$, are harmonic functions of time, i.e.

$$g(t) = \mathrm{Re}\left(Ge^{i\omega t}\right) \quad , \quad q(x,t) = \mathrm{Re}\left(\mathrm{Q}(x)e^{i\omega t}\right) \quad , \quad s(x,t) = \mathrm{Re}\left(\mathrm{S}(x)e^{i\omega t}\right) \quad , \tag{3.1}$$

where G is a real constant and $\mathrm{Q}(x)$ and $\mathrm{S}(x)$ are complex functions of the real variable x. Introduction of these expressions in the first order problem yields:

$$\mathrm{S}(x) = i\omega \mathrm{Q}_x + GS_{ox} \tag{3.2}$$

$$C_4(x)\mathrm{Q}_{xxxx} + C_3(x)\mathrm{Q}_{xxx} + C_2(x)\mathrm{Q}_{xx} + C_1(x)\mathrm{Q}_x + C_0(x)\mathrm{Q} = C_g(x)G \tag{3.3}$$

$$Q(\pm\Lambda) = i\omega G S_o(\pm\Lambda) \quad , \quad Q_x(\pm\Lambda) = i\omega G S_{o_x}(\pm\Lambda) \tag{3.4}$$

where C_i are complex functions of the real variable x derived from the functions C_{ij} appearing in eq. (2.7). To solve the above formulation an implicit finite-difference method is used, with a centered five-point scheme for the evaluation of the spatial derivatives. This method is similar to the one used by Meseguer [6] to solve the slice model in the case of an inviscid liquid bridge in an oscillatory axial microgravity field. Once the value of Q is known (note that these values must be computed using complex algebra) the value of S can be obtained through continuity equation (3.2). Additional details on the numerical scheme can be obtained upon request to the authors.

4. Liquid Bridge Resonances

Before present theoretical results concerning the mappings of frequencies of resonance it would be convenient to introduce some previous comments on the influence of viscosity on the dynamic response of liquid bridges. To evaluate such influence, attention has been focused mainly on the resonances (formally, on the resonance pulsations, $\omega = 2\pi f$) corresponding to the first oscillation modes: the first mode and the third one (note that according to the kind of perturbation considered -in phase vibration of both disks- only odd oscillation modes are excited). In Fig. 2, the variation with the viscosity parameter, C, of the response of two liquid bridges between equal disks and in gravitationless conditions

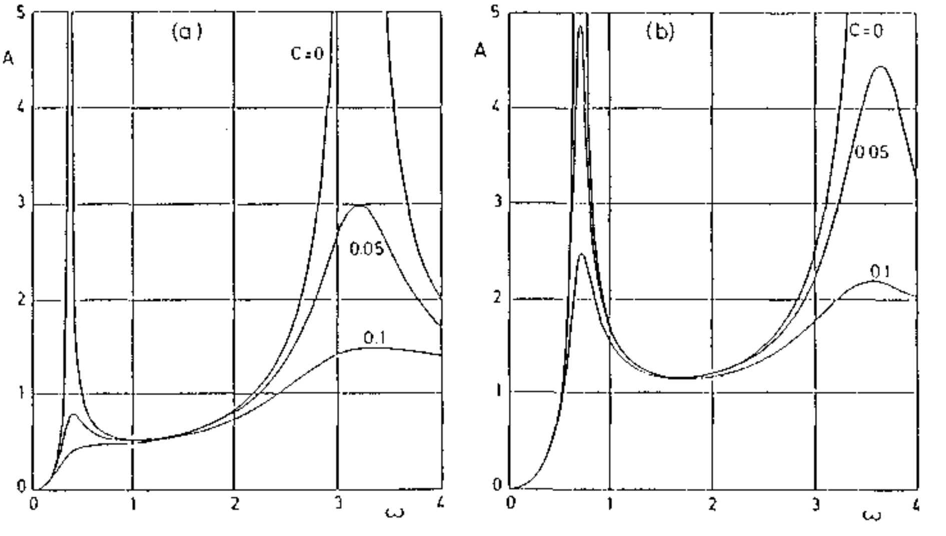

Fig. 2. Variation with the pulsation, ω, of the ratio of the maximum interface deformation to the amplitude of the oscillation of the disks, A, of liquid bridges with slenderness $\Lambda = 2$, between equal disks, $K = 1$, in gravitationless conditions, $B = 0$, and dimensionless volume $V = 8$ (a) or $V = 12$ (b). Numbers on the curves indicate the value of the parameter of viscosity, C.

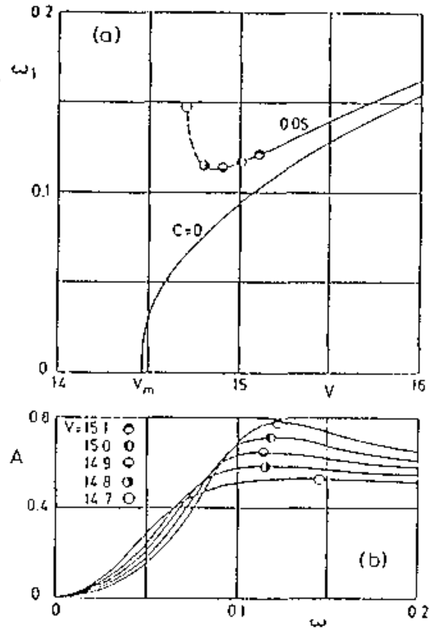

Fig. 3. (a) Variation with the volume of the liquid bridge, V, of the pulsation of resonance corresponding to the first oscillation mode, ω_1, of liquid bridges with Λ = 2.8, K = 1 and B = 0. Numbers on the curves indicate the value of the parameter of viscosity, C. The dependence on the liquid bridge volume, V, of the response of the liquid bridge (A vs. ω) for a particular value of the viscosity parameter, C = 0.05, is shown in (b).

(K = 1, B = 0), both with the same slenderness, Λ = 2.0, but with different volumes, V = 8.0 and V = 12.0, respectively, has been represented (the response has been defined as the ratio of the maximum interface deformation to the amplitude of the perturbation: $A(\omega) = (F_{max}-F_{min})/G$, where F_{max} and F_{min} are the maximum and minimum values of the dimensionless radius of the interface in each cycle). According to this plot, the amplitude of the liquid bridge response decreases as the viscosity of the liquid increases and that amplitude decreases faster in the case V=8.0 (this liquid bridge configuration is closer to the minimum volume stability limit than the second one, V = 12.0). Even more, as it can be observed, there is a critical value of the viscosity parameter, C_1^*, for which the resonance corresponding to the first oscillation mode disappears (there is a critical value of the viscosity parameter associated with each oscillation mode, C_n^*, with $C_n^* > C_{n+1}^*$). This value C_1^*, becomes smaller as the volume of liquid approaches the corresponding minimum volume stability limit, V_m. This behaviour is summarized in Fig. 3a, where the dependence of the resonance pulsation corresponding to the first oscillation mode, ω_1, with the volume of the liquid bridge has been represented in the case of liquid bridges with Λ = 2.8, K = 1 and B = 0. As it can be observed, ω_1 decreases as the volume of the liquid bridge V decreases and, in the case of inviscid liquid bridges (C = 0), ω_1 becomes zero when the corresponding stability limit of minimum volume, V_m, is reached. The same trends are shown when viscous liquid bridges are considered (C = 0.05): ω_1 decreases as V decreases, although in this case the resonance disappears at a value of ω_1 different from zero for a volume greater than V_m (that means that the considered value of C has become critical for such configuration). Such behaviour is illustrated in Fig. 3b, where the variation of the transfer function, $A(\omega)$, with the liquid bridge volume, close to the stability limit, is shown for a liquid bridge with $\Lambda = 2.8$, $K = 1$, $B = 0$ and $C = 0.05$. Note that if $V < 15.1$ the maximum in the curve disappears and that, close to this limiting value of V, the pulsation of resonance slightly increases. This rise in the value of ω_1 is due to the definition of ω_1 as the value of ω where $A(\omega)$ becomes maximum, no matter how relatively small it is. However, it is questionable that such increment in the value of ω_1 could be detected experimentally and therefore, that last part of the curve corresponding to $C = 0.05$ in Fig. 3a has been plotted by using a dashed line instead of a continuous one to indicate that this phenomenon could not be detectable under normal experimental conditions. Obviously this behaviour is qualitatively similar no matter what the values of Λ, K and B are.

These features of the liquid bridge response, mainly the dependence of C_n^* on the parameters defining the liquid bridge configuration, Λ, V, K and B, are of paramount importance when designing any experiment related to liquid bridge resonances; it would be even possible to select a fluid and a liquid column configuration for which it were impossible to observe any resonance.

The mappings of frequencies of resonance corresponding to the first oscillation mode (the curves of constant ω_1 on the Λ-V stability diagrams) have been plotted in Figs. 4, 5 and 6 for different values of the geometry parameter, K, and Bond number, B. The first of these

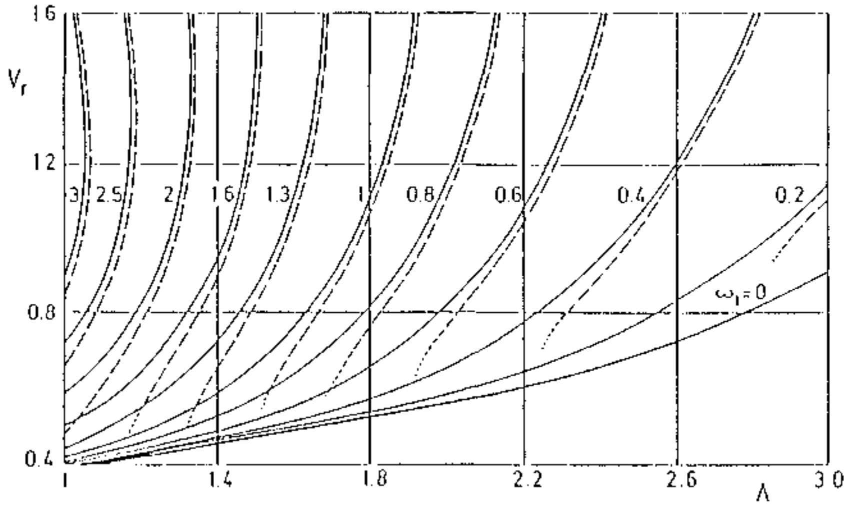

Fig. 4. Variation with the slenderness, Λ, and the reduced volume of the liquid bridge, $V_r = V/(2\pi\Lambda)$, of the resonance pulsation corresponding to the first oscillation mode, ω_1, of liquid bridges between equal disks, $K = 1$, Bond number $B = 0$ and viscosity parameter $C = 0$ (——) and $C = 0.1$ (-----). Numbers on the curves indicate the value of ω_1. The curve labelled as $\omega_1 = 0$ corresponds to the minimum volume stability limit.

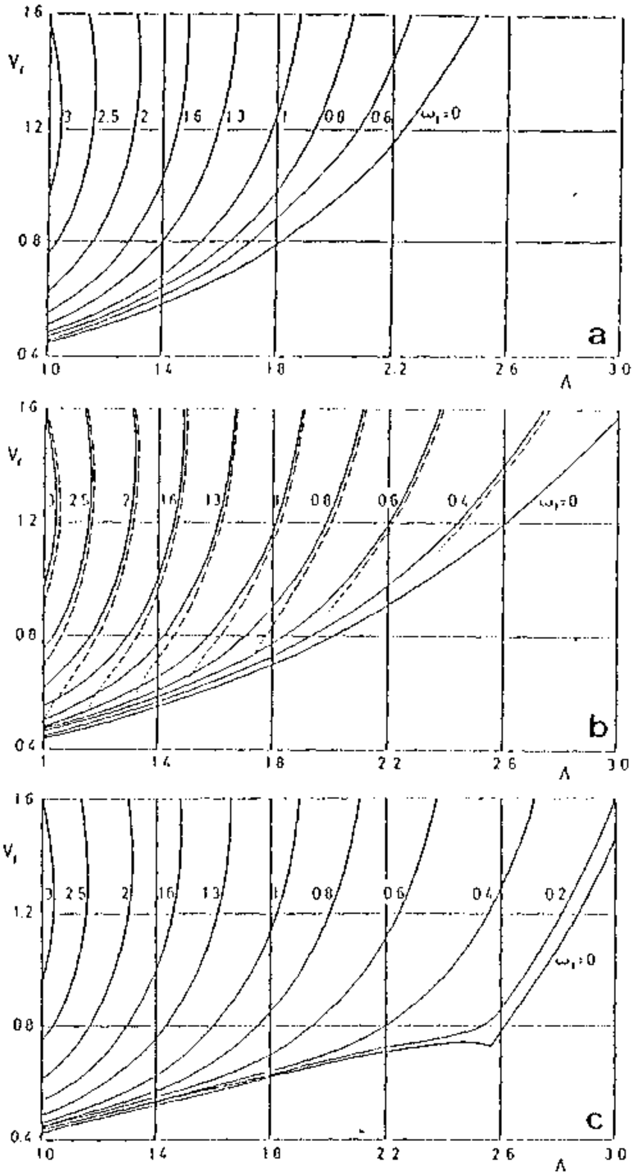

Fig. 5. Variation with the slenderness, Λ, and the reduced volume of the liquid bridge, $V_r = V/(2\pi\Lambda)$ of the resonance pulsation corresponding to the first oscillation mode, ω_1, of liquid bridges between unequal disks, $K = 0.7$. Numbers on the curves indicate the value of ω_1. The curve labelled as $\omega_1 = 0$ corresponds to the minimum volume stability limit. Bond numbers are (a) $B = -0.1$, (b) $B = 0$ and (c) $B = 0.1$. In (a) and (c) only zero viscosity ($C = 0$) has been considered whereas in (b) results corresponding to viscosity parameters $C = 0$ (——) and $C = 0.1$ (-----) has been plotted.

plots, Fig. 4, corresponds to the case of liquid bridges between equal disks, $K = 1$, and in gravitationless conditions, $B = 0$. As it can be observed, in the inviscid case the resonance frequency becomes zero at the minimum volume stability limit and, in a region close to this stability limit, the pulsation of resonance increases as the volume of liquid grows. Note that for each value of the slenderness there is a value of the volume of liquid for which ω_1 reaches a maximum, the values of the pulsation of resonance decreasing again if the liquid bridge volume exceeds this maximum frequency volume (this is clearly seen in the left most curves of the diagram). This phenomenon, namely, the existence for each value of the slenderness of a volume of liquid for which there is a maximum in the value of the pulsation of resonance corresponding to the first oscillation mode, has been already pointed out in [8]. The influence of viscosity is also shown in Fig. 4. Observe that the different curves of constant ω_1 disappear in a region close to the curve of minimum volume in which the damping becomes critical. In the case of liquid bridge configurations represented by points inside this region in the Λ–V stability diagram it is not possible to detect any resonance corresponding to the first oscillation mode. The behaviour for higher values of the liquid bridge volume is in this case similar to that of the inviscid case, at least within the range of values of volume analyzed.

It is clear from Fig. 4 that the mapping of resonances strongly depends on the minimum volume stability limit, V_m. Then, since V_m varies as K and B change, one could expect that the frequencies of resonance vary with these parameters in a similar fashion. To visualize

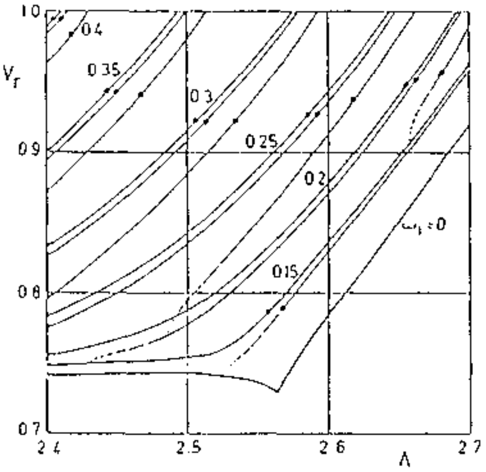

Fig. 6. Variation with the slenderness, Λ, and the reduced volume of the liquid bridge, $V_r = V/(2\pi\Lambda)$ of the resonance pulsation corresponding to the first oscillation mode, ω_1, of liquid bridges between unequal disks, $K = 0.7$, Bond number $B = 0.1$ and viscosity parameter $C = 0$ (●), $C = 0.04$ (◆) and $C = 0.08$ (■). Numbers on the curves indicate the value of ω_1.

such dependence, the mappings corresponding to liquid bridges between unequal disks, $K = 0.7$, subjected to different values of Bond number have been represented in Fig. 5a ($B = -0.1$), Fig. 5b ($B = 0$) and Fig. 5c ($B = 0.1$). Note that in each one of these plots the behaviour is similar to that shown in Fig. 4, the different mappings being only different in the low frequency range to become adapted to the corresponding minimum volume stability limit (corresponding to $\omega_1 = 0$). Additional details on the influence of viscosity on the frequencies of resonance corresponding to the first oscillation mode of liquid bridge configurations close to the minimum volume stability limit ($K = 0.7$, $B = 0.1$) are shown in Fig. 6, which provides a close look at the region where the stability limit curve presents a discontinuity in the slope.

Acknowledgements

This work has been supported by the Spanish Comisión Interministerial de Ciencia y Tecnología (CICYT), Project No. ESP88-0359.

References

1. Meseguer, J., The breaking of axisymmetric liquid bridges, J. Fluid Mech. 130 (1983), 123-151.

2. Sanz, A., The influence of the outer bath on the dynamics of axisymmetric liquid bridges, J. Fluid Mech. 156 (1985), 101-140.

3. Rivas, D. and Meseguer, J., One-dimensional self-similar solution of the dynamics of axisymmetric slender liquid bridges, J. Fluid Mech. 138 (1984), 417-429.

4. Zhang, Y. and Alexander, J.I.D., Sensitivity of liquid bridges subjected to axial residual acceleration, Phys. Fluids A 2 (1990), 1966-1974.

5. Nicolás, J.A., Frequency response of axisymmetric liquid bridges to an oscillatory microgravity field, Microgravity Sci. Technol., in press.

6. Meseguer, J., Axisymmetric long liquid bridges in a time-dependent microgravity field, Appl. Microgravity Tech. 1 (1988), 136-141.

7. Meseguer, J., Sanz, A. and Perales, J.M., Axisymmetric long liquid bridges stability and resonances, Appl. Microgravity Tech. 2 (1990), 186-192.

8. Perales, J.M., Dinámica de columnas líquidas, Tesis Doctoral, Universidad Politécnica de Madrid, 1990.

9. Green, A.E., On the non-linear behaviour of fluid jets, Int. J. Engng. Sci. 14 (1976), 49–63.

10. Bogy, D.B., Drop formation in a circular liquid jet, Ann. Rev. Fluid Mech. 11 (1979), 207-228.

11. Perales, J.M., Meseguer, J. and Martínez, I., Minimum volume of axisymmetric liquid bridges between unequal disks in an axial microgravity field, J. Crystal Growth 110 (1991), 855-861.