An improvement of the lower bound on the maximum number of halving lines in planar sets with 32 points

Javier Rodrigo
Departamento de Matemática Aplicada. E.T.S. de Ingeniería.
Universidad Pontificia Comillas de Madrid
Madrid, Spain

Mª Dolores López
Departamento de Matemática e Informática Aplicadas a las Ingenierías Civil y Naval de la E.T.S.I. Caminos, Canales y Puertos
Universidad Politécnica de Madrid
Madrid, Spain

Abstract
In this paper we give a recursive lower bound on the maximum number of halving lines for sets in the plane and as a consequence we improve the current best lower bound on the maximum number of halving lines for sets in the plane with 32 points.

Keywords: Combinatorial Geometry, Halving lines, Rectilinear Crossing number, Optimization

1 Introduction
The search for upper and lower bounds for the maximum number of halving lines in sets of \( n \) points of the plane, is a challenging task in the existing Combinatorial Geometry literature due to its relation with the rectilinear crossing number problem[1], [4].
An example of this relation is a conjecture by Aichholzer et al. [3] that says that every set attaining the rectilinear crossing number maximizes the number of halving lines.

Roughly speaking, a halving line of a finite planar set $P$ is a line passing by two points of $P$ that splits $P$ in two equally sized subsets (see definitions 1.1, 1.2 below for a more formal definition). With respect to the lower bounds for the maximum number of these halving lines in sets of $n$ points for even $n$ ($h_n$), we have that $h_n \geq \frac{n}{4} \log_2 \left(\frac{n}{4}\right)$, given by Erdős, Lovász, Simmons and Strauss in 1973 [6]. Later, this bound was improved to $h_n \geq n \log_4 \left(\frac{2n}{3}\right)$ by Eppstein [5]. More recently, Tóth [8] found a better asymptotic lower bound: $h_n \geq n^2 e^{0.744 \sqrt{\log(4) \log(n)}} - 2.7$. Nivasch [7] improved the constant in the exponent and found the up-to-date best asymptotic lower bound: $h_n \geq Cn^2 e^{0.744 \sqrt{\log(4) \log(n)}}$ for some fixed constant $C$.

For small $n$, the exact value of $h_n$ is known for $n \leq 27$. For $28 \leq n \leq 32$, there are small gaps between the best lower bound and the best upper bound of $h_n$.

In this paper, we improve by one unit the current best lower bound of $h_n$ for $n = 32$ reported in [2],[1]. Concretely, we prove that $h_{32} \geq 74$. This reduces the gap with the up-to-date best upper bound of $h_{32}$, namely $h_{32} \leq 79$ [1]. This also yields a reduction by one unit of the current best upper bound of the rectilinear crossing number for sets of 32 points ($cr(32)$, see definition 1.3 below), assuming that the aforementioned conjecture is true.

The formal definition of a halving line is as follows:

**Definition 1.1** Given a set of points in the plane $A = \{p_1, \ldots, p_n\}$, a $k$-edge of $A$ is a line $R$ that joins two points of $A$ and leaves $k$ points of $A$ in one of the open half planes (that is to say, in one connected component of $\mathbb{R}^2 - R$).

**Definition 1.2** Given a set of points in the plane $A = \{p_1, \ldots, p_n\}$, a halving line of $A$ is an $\left\lfloor \frac{n-2}{2} \right\rfloor$-edge of $A$.

**Remark:** For even $n$, a halving line of $A$ leaves the same number of points of $A$ in each half plane: $\frac{n-2}{2}$.

We also give the following definitions:

**Definition 1.3** Given a finite set of points in the plane $P$, assume that we join each pair of points of $P$ with a straight line segment. The rectilinear crossing number of $P$ ($cr(P)$) is the number of segment-crossings. The rectilinear
crossing number of \( n \) \((cr(n))\) is the minimum of \( cr(P) \) over all the sets \( P \) with \( n \) points.

**Definition 1.4** Given a finite set of points in the plane \( P = \{p_1, \ldots, p_n\} \), the subjacent graph of \( P \) is the graph \( G = (V,E) \) such that \( V = P \) and \( \{p_i,p_j\} \in E \) if the line joining \( p_i,p_j \) is a halving line of \( P \) (sometimes we will identify the edge \( \{p_i,p_j\} \) with the halving line of \( P \) containing \( p_i,p_j \). We will call this line \( p_i - p_j \)). The valence of a vertex is the number of edges incident with the vertex in the subjacent graph.

The outline of the rest of the paper is as follows: In Section 2 we give a technical lemma, in Section 3 we apply said lemma to state the main result, the aforementioned improvement of the lower bound and in Section 4 we give some concluding remarks.

## 2 A technical Lemma

For the following lemma we need two previous definitions:

**Definition 2.1** Let \( A \) be a finite set with \(| A | = n \), \( n \) an even number, \( Hal = \{\text{halving lines of } A\} = \{R_1, \ldots, R_m\} \) and let \( R \) be a line such that \( R_i \cap R \neq \emptyset \) and \( R_i \cap R \neq R_j \cap R \) for \( i \neq j \), \( i,j \in \{1, \ldots, m\} \).

The arrangement of the elements of \( Hal \) with respect to \( R \) is the order according to the \( y \) coordinate of the intersection points of \( R \) and \( R_i \), \( i = 1, \ldots, m \). That is to say, \( R_i \leq R_j \) if \( y_i \leq y_j \), being \( R_i \cap R = \{(x_i,y_i)\} \), \( R_j \cap R = \{(x_j,y_j)\} \).

**Definition 2.2** Let \( R \) be a line and let \( p_i, p_j \) be two points that do not belong to \( R \), we say that \( R \) separates \( p_i, p_j \) if it does not leave \( p_i, p_j \) in the same open half plane.

**Lemma 2.3** Consider a set \( Q = \{p_1, \ldots, p_n\} \) where \( n \) is an even number, \( n > 2 \), let \( R' \) be an \( \frac{n-4}{2} \)-edge of \( Q \) and let \( R \) be a line satisfying the following conditions: (a) \( R \) meets all the halving lines of \( Q \) at different points; (b) \( R \) separates the points of \( Q \) in the first and last halving lines that it crosses with the order of Definition 2.1; (c) \( R \) is parallel to \( R' \) and it is contained in the half plane generated by \( R' \) with \( \frac{n-4}{2} \) points of \( Q \).

Then we can locate two points \( p_{n+1}, p_{n+2} \) in such a way that \( P := Q \cup \{p_{n+1}, p_{n+2}\} \) satisfies \( h(P) \geq h(Q) + 5 \).

**Proof.** Let \( R_1 = p_i - p_j, R_2 = p_k - p_l \) respectively be the maximum and minimum of \( Hal = \{\text{halving lines of } A\} = \{R_1, \ldots, R_m\} \) with the order of
Definition 2.1. Then $R$ separates $p_i, p_j$ and also separates $p_k, p_l$ as per condition (b) above.

If we consider two points $p_{n+1}, p_{n+2}$ in $R$ satisfying

(i) The $y$ coordinate of $p_{n+1}$ is lower than the one of $R_1 \cap R$ and higher than the one of $R_i \cap R$, $\forall i \neq 1$.

(ii) The $y$ coordinate of $p_{n+2}$ is higher than the one of $R_2 \cap R$ and lower than the one of $R_i \cap R$, $\forall i \neq 2$.

Then the number of halving lines of the set $P = Q \cup \{p_{n+1}, p_{n+2}\}$ is greater than or equal to the number of halving lines of $Q$ separating $p_{n+1}, p_{n+2}$ plus the number of halving lines of $Q$ containing to $p_{n+1}$ or $p_{n+2}$ plus one because $R'$ is a halving line of $P$ as per condition (c) above.

We have that every halving line of $Q$ except for $R_1, R_2$ separates $p_{n+1}, p_{n+2}$, so the number of halving lines of $Q$ separating $p_{n+1}, p_{n+2}$ is $h(Q) - 2$.

Now, if we have located $p_{n+1}, p_{n+2}$ close enough to $R_1, R_2$, then we can get that $p_i - p_{n+1}, p_j - p_{n+1}, p_k - p_{n+2}, p_l - p_{n+2}$ are halving lines of $P$.

Therefore, since $p_{n+1} - p_{n+2}$ is not a halving line of $P$ because it is parallel to $R'$ by hypothesis, we have two more halving lines of $P$ including $p_{n+1}$ or $p_{n+2}$ to ensure that the degrees of $p_{n+1}, p_{n+2}$ in the underlying graph are odd numbers, and then $h(P) \geq (h(Q) - 2) + 6 + 1 = h(Q) + 5$ as desired. \hfill $\square$

3 The improvement of the lower bound

In this section we show examples of sets that shift the lower bound of $h_{32}$. They are constructed by adding two points to a set $Q$ that gives the current best lower bound of $h_{30}$, in such a way that every halving line but two of $Q$ are preserved and seven new halving lines are created by following the procedure of the proof of Lemma 2.3. The proof of the following proposition gives a first example:

**Proposition 3.1** It is satisfied that $h_{32} \geq 74$.

**Proof.** We consider as basis set, the set $Q = \{p_1, \ldots, p_{30}\}$ with 30 points included in [2], and we find a line $R$ satisfying the conditions of Lemma 2.3 (for instance, a parallel line to $R' = p_2 - p_{18}$). Then we add two points $p_{31}, p_{32}$ according to the proof of Lemma 2.3 to get a set $P = \{p_1, \ldots, p_{32}\}$ such
that $h(P) \geq h(Q) + 5 = 74$. The coordinates of the points of $P$ are:

\[
\begin{align*}
  v_1 &= (9259, 16958), \quad v_2 = (9763, 16199), \quad v_3 = (9977, 16397), \\
  v_4 &= (10248, 16225), \quad v_5 = (10666, 16385), \quad v_6 = (12849, 16335), \\
  v_7 &= (18577, 16451), \quad v_8 = (10391, 16281), \quad v_9 = (28477, 16613), \\
  v_{10} &= (15909, 16415), \quad v_{11} = (9446, 15905), \quad v_{12} = (9540, 16541), \\
  v_{13} &= (9262, 16627), \quad v_{14} = (9282, 16947), \quad v_{15} = (8912, 17261), \\
  v_{16} &= (7842, 19232), \quad v_{17} = (5141, 23755), \quad v_{18} = (9154, 17055), \\
  v_{19} &= (0, 32394), \quad v_{20} = (6820, 20291), \quad v_{21} = (9949, 16415), \\
  v_{22} &= (9355, 16177), \quad v_{23} = (9419, 15893), \quad v_{24} = (9146, 15771), \\
  v_{25} &= (9075, 15420), \quad v_{26} = (7921, 13407), \quad v_{27} = (5206, 8451), \\
  v_{28} &= (9121, 15603), \quad v_{29} = (480, 0), \quad v_{30} = (6432, 10637), \\
  v_{31} &= (8525, 17938.4), \quad v_{32} = (9855, 16069)
\end{align*}
\]

This implies that $h_{32} \geq 74$ as desired.

**Remark:** Since the set included in [2] that gives $cr(32) \leq 12836$ has 73 halving lines, we have that if the conjecture of [3] was true, then Proposition 3.1 would imply that $cr(32) \leq 12835$.

### 4 Conclusions

We have considered the task of sharpening the best lower bound for the maximum number of halving lines in planar sets of 32 points. We have achieved it by means of an example that improve by one unit the current best lower bound for these sets. This carries with it either the improvement of the upper bound of the rectilinear crossing number for planar sets of 32 points or the refutation of a conjecture included in [3].

A future line of work could be the search of similar examples for $n = 28, 30$ by means of an improvement of Lemma 2.3.

**Funding:** This work was supported by FEDER and the State Research Agency (AEI) of the Spanish Ministry of Economy and Competition [Grant number TIN2016-76843-C4-2-R]


References


