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Sampling Associated with a Unitary Representation of a Semi-Direct Product of Groups: A Filter Bank Approach

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Received: 18 March 2019; Accepted: 9 April 2019; Published: 12 April 2019



Abstract: An abstract sampling theory associated with a unitary representation of a countable discrete non abelian group G , which is a semi-direct product of groups, on a separable Hilbert space is studied. A suitable expression of the data samples, the use of a filter bank formalism and the corresponding frame analysis allow for fixing the mathematical problem to be solved: the search of appropriate dual frames for $\ell^2(G)$. An example involving crystallographic groups illustrates the obtained results by using either average or pointwise samples.

Keywords: semi-direct product of groups; unitary representation of a group; LCA groups; dual frames; sampling expansions

1. Statement of the Problem

In this paper, an abstract sampling theory associated with non abelian groups is derived for the specific case of a unitary representation of a semi-direct product of groups on a separable Hilbert space. Semi-direct product of groups provide important examples of non abelian groups such as dihedral groups, infinite dihedral group, Euclidean motion groups or crystallographic groups. Concretely, let $(n, h) \mapsto U(n, h)$ be a unitary representation on a separable Hilbert space \mathcal{H} of a semi-direct product $G = N \rtimes_{\phi} H$, where N is a countable discrete LCA (locally compact abelian) group, H is a finite group, and ϕ denotes the action of the group H on the group N (see Section 2 infra for the details); for a fixed $a \in \mathcal{H}$ we consider the U -invariant subspace in \mathcal{H}

$$\mathcal{A}_a = \left\{ \sum_{(n,h) \in G} \alpha(n,h) U(n,h)a : \{\alpha(n,h)\}_{(n,h) \in G} \in \ell^2(G) \right\},$$

where we assume that $\{U(n, h)a\}$ is a Riesz sequence for \mathcal{H} , i.e., a Riesz basis for \mathcal{A}_a (see Ref. [1] for a necessary and sufficient condition). Given K elements b_k in \mathcal{H} , which do not belong necessarily to \mathcal{A}_a , the main goal in this paper is the stable recovery of any $x \in \mathcal{A}_a$ from the given data (generalized samples)

$$\mathcal{L}_k x(n) := \langle x, U(n, 1_H)b_k \rangle_{\mathcal{H}}, \quad n \in N \text{ and } k = 1, 2, \dots, K,$$

where 1_H denotes the identity element in H . These samples are nothing but a generalization of average sampling in shift-invariant subspaces of $L^2(\mathbb{R}^d)$; see, among others, Refs. [2–9]. The case where G is a discrete LCA group and the samples are taken at a uniform lattice of G has been solved in Ref. [10]; this work relies on the use of the Fourier analysis in the LCA group G (see also Ref. [11]). In the case involved here, a classical Fourier analysis is not available and, consequently, we need to overcome this drawback.

Having in mind the filter bank formalism in discrete LCA groups (see, for instance, Refs. [12–14]), the given data $\{\mathcal{L}_k x(n)\}_{n \in N; k=1,2,\dots,K}$ can be expressed as the output of a suitable K -channel analysis filter bank corresponding to the input $\alpha = \{\alpha(n, h)\}_{(n,h) \in G}$ in $\ell^2(G)$. As a consequence, the problem consists of finding a synthesis part of the former filter bank allowing perfect reconstruction; in addition, only Fourier analysis on the LCA group N is needed. Then, roughly speaking, substituting the output of the synthesis part in $x = \sum_{(n,h) \in G} \alpha(n, h) U(n, h)a$, we will obtain the corresponding sampling formula in \mathcal{A}_a .

This said, as it could be expected, the problem can be mathematically formulated as the search of dual frames for $\ell^2(G)$ having the form

$$\{T_n h_k\}_{n \in N; k=1,2,\dots,K} \quad \text{and} \quad \{T_n g_k\}_{n \in N; k=1,2,\dots,K}.$$

Here, $h_k, g_k \in \ell^2(G)$, $T_n h_k(m, h) = h_k(m - n, h)$ and $T_n g_k(m, h) = g_k(m - n, h)$, $(m, h) \in G$, where $n \in N$ and $k = 1, 2, \dots, K$. In addition, for any $x \in \mathcal{A}_a$, we have the expression for its samples

$$\mathcal{L}_k x(n) = \langle \alpha, T_n h_k \rangle_{\ell^2(G)}, \quad n \in N \text{ and } k = 1, 2, \dots, K.$$

Needless to say, frame theory plays a central role in what follows; the necessary background on Riesz bases or frame theory in a separable Hilbert space can be found, for instance, in Ref. [15]. Finally, sampling formulas in \mathcal{A}_a having the form

$$x = \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) U(n, 1_H) c_k \quad \text{in } \mathcal{H},$$

for some $c_k \in \mathcal{A}_a$, $k = 1, 2, \dots, K$, will come out by using, for $g \in \ell^2(G)$ and $n \in N$, the shifting property $\mathcal{T}_{U,a}(T_n g) = U(n, 1_H)(\mathcal{T}_{U,a} g)$ that satisfies the natural isomorphism $\mathcal{T}_{U,a} : \ell^2(G) \rightarrow \mathcal{A}_a$ which maps the usual orthonormal basis $\{\delta_{(n,h)}\}_{(n,h) \in G}$ for $\ell^2(G)$ onto the Riesz basis $\{U(n, h)a\}_{(n,h) \in G}$ for \mathcal{A}_a . All these steps will be carried out throughout the remaining sections. For the sake of completeness, Section 2 includes some basic preliminaries on semi-direct product of groups and Fourier analysis on LCA groups. The paper ends with an illustrative example involving the quasi regular representation of a crystallographic group on $L^2(\mathbb{R}^d)$; sampling formulas involving average or pointwise samples are obtained for the corresponding U -invariant subspaces in $L^2(\mathbb{R}^d)$.

2. Some Mathematical Preliminaries

In this section, we introduce the basic tools in semi-direct product of groups and in harmonic analysis in a discrete LCA group that will be used in the sequel.

2.1. Preliminaries on Semi-Direct Product of Groups

Given groups (N, \cdot) and (H, \cdot) , and a homomorphism $\phi : H \rightarrow \text{Aut}(N)$, their semi-direct product $G := N \rtimes_{\phi} H$ is defined as follows: The underlying set of G is the set of pairs (n, h) with $n \in N$ and $h \in H$, along with the multiplication rule

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1 \phi_{h_1}(n_2), h_1 h_2), \quad (n_1, h_1), (n_2, h_2) \in G,$$

where we denote $\phi(h) := \phi_h$; usually, the homomorphism ϕ is referred to as the action of the group H on the group N . Thus, we obtain a new group with identity element $(1_N, 1_H)$, and inverse $(n, h)^{-1} = (\phi_{h^{-1}}(n^{-1}), h^{-1})$.

In addition, we have the isomorphisms $N \simeq N \times \{1_H\}$ and $H \simeq \{1_N\} \times H$. Unless ϕ_h equals the identity for all $h \in H$, the group $G = N \rtimes_{\phi} H$ is not abelian, even for abelian N and H groups. The subgroup N is a normal subgroup in G . Some examples of semi-direct product of groups:

1. The dihedral group D_{2N} is the group of symmetries of a regular N -sided polygon; it is the semi-direct product $D_{2N} = \mathbb{Z}_N \rtimes_{\phi} \mathbb{Z}_2$ where $\phi_0 \equiv Id_{\mathbb{Z}_N}$ and $\phi_1(\bar{n}) = -\bar{n}$ for each $\bar{n} \in \mathbb{Z}_N$. The infinite dihedral group D_{∞} defined as $\mathbb{Z} \rtimes_{\phi} \mathbb{Z}_2$ for the similar homomorphism ϕ is the group of isometries of \mathbb{Z} .
2. The Euclidean motion group $E(d)$ is the semi-direct product $\mathbb{R}^d \rtimes_{\phi} O(d)$, where $O(d)$ is the orthogonal group of order d and $\phi_A(x) = Ax$ for $A \in O(d)$ and $x \in \mathbb{R}^d$. It contains as a subgroup any crystallographic group $M\mathbb{Z}^d \rtimes_{\phi} \Gamma$, where $M\mathbb{Z}^d$ denotes a full rank lattice of \mathbb{R}^d and Γ is any finite subgroup of $O(d)$ such that $\phi_{\gamma}(M\mathbb{Z}^d) = M\mathbb{Z}^d$ for each $\gamma \in \Gamma$.
3. The orthogonal group $O(d)$ of all orthogonal real $d \times d$ matrices is isomorphic to the semi-direct product $SO(d) \rtimes_{\phi} C_2$, where $SO(d)$ consists of all orthogonal matrices with determinant 1 and $C_2 = \{I, R\}$ a cyclic group of order 2; ϕ is the homomorphism given by $\phi_I(A) = A$ and $\phi_R(A) = RAR^{-1}$ for $A \in SO(d)$.

Suppose that N is an LCA group with Haar measure μ_N and H is a locally compact group with Haar measure μ_H . Then, the semi-direct product $G = N \rtimes_{\phi} H$ endowed with the product topology becomes also a topological group. For the left Haar measure on G , see Ref. [1].

2.2. Some Preliminaries on Harmonic Analysis on Discrete LCA Groups

The results about harmonic analysis on locally compact abelian (LCA) groups are borrowed from Ref. [16]. Notice that, in particular, a countable discrete abelian group is a second countable Hausdorff LCA group.

For a countable discrete group (N, \cdot) , not necessarily abelian, the convolution of $x, y : N \rightarrow \mathbb{C}$ is formally defined as $(x * y)(m) := \sum_{n \in N} x(n)y(n^{-1}m)$, $m \in N$. If, in addition, the group is abelian, therefore denoted by $(N, +)$, the convolution reads as

$$(x * y)(m) := \sum_{n \in N} x(n)y(m - n), \quad m \in N.$$

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unidimensional torus. We said that $\zeta : N \mapsto \mathbb{T}$ is a character of N if $\zeta(n + m) = \zeta(n)\zeta(m)$ for all $n, m \in N$. We denote $\zeta(n) = \langle n, \zeta \rangle$. Defining $(\zeta + \gamma)(n) = \zeta(n)\gamma(n)$, the set of characters \hat{N} with the operation $+$ is a group, called the dual group of N ; since N is discrete \hat{N} is compact ([16], Prop. 4.4). For $x \in \ell^1(N)$, we define its Fourier transform as

$$X(\zeta) = \hat{x}(\zeta) := \sum_{n \in N} x(n)\overline{\langle n, \zeta \rangle} = \sum_{n \in N} x(n)\langle -n, \zeta \rangle, \quad \zeta \in \hat{N}.$$

It is known ([16], Theorem 4.5) that $\hat{\hat{N}} \cong \mathbb{T}$, with $\langle n, z \rangle = z^n$, and $\hat{\mathbb{Z}_s} \cong \mathbb{Z}_s := \mathbb{Z}/s\mathbb{Z}$, with $\langle n, m \rangle = W_s^{nm}$, where $W_s = e^{2\pi i/s}$.

There exists a unique measure, the Haar measure μ on \hat{N} satisfying $\mu(\zeta + E) = \mu(E)$, for every Borel set $E \subset \hat{N}$ ([16], Section 2.2), and $\mu(\hat{N}) = 1$. We denote $\int_{\hat{N}} X(\zeta)d\zeta = \int_{\hat{N}} X(\zeta)d\mu(\zeta)$. If $N = \mathbb{Z}$,

$$\int_{\hat{N}} X(\zeta)d\zeta = \int_{\mathbb{T}} X(z)dz = \frac{1}{2\pi} \int_0^{2\pi} X(e^{iw})dw,$$

and, if $N = \mathbb{Z}_s$,

$$\int_{\hat{N}} X(\zeta)d\zeta = \int_{\mathbb{Z}_s} X(n)dn = \frac{1}{s} \sum_{n \in \mathbb{Z}_s} X(n).$$

If N_1, N_2, \dots, N_d are abelian discrete groups, then the dual group of the product group is $(N_1 \times N_2 \times \dots \times N_d)^\wedge \cong \widehat{N}_1 \times \widehat{N}_2 \times \dots \times \widehat{N}_d$ (see ([16], Prop. 4.6)) with

$$\langle (n_1, n_2, \dots, n_d), (\xi_1, \xi_2, \dots, \xi_d) \rangle = \langle n_1, \xi_1 \rangle \langle n_2, \xi_2 \rangle \cdots \langle n_d, \xi_d \rangle.$$

The Fourier transform on $\ell^1(N) \cap \ell^2(N)$ is an isometry on a dense subspace of $L^2(\widehat{N})$; Plancherel theorem extends it in a unique manner to a unitary operator of $\ell^2(N)$ onto $L^2(\widehat{N})$ ([16], p. 99). The following lemma, giving a relationship between Fourier transform and convolution, will be used later (see Ref. [17]):

Lemma 1. Assume that $a, b \in \ell^2(N)$ and $\widehat{a}(\xi) \widehat{b}(\xi) \in L^2(\widehat{N})$. Then, the convolution $a * b$ belongs to $\ell^2(N)$ and $\widehat{a * b}(\xi) = \widehat{a}(\xi) \widehat{b}(\xi)$, a.e. $\xi \in \widehat{N}$.

3. Filter Bank Formalism on Semi-Direct Product of Groups

In what follows, we will assume that $G = N \rtimes_\phi H$ where $(N, +)$ is a countable discrete abelian group and (H, \cdot) is a finite group. Having in mind the operational calculus $(n, h) \cdot (m, l) = (n + \phi_h(m), hl)$, $(n, h)^{-1} = (\phi_{h^{-1}}(-n), h^{-1})$ and $(n, h)^{-1} \cdot (m, l) = (\phi_{h^{-1}}(m - n), h^{-1}l)$, the convolution $\alpha * h \in \ell^2(G)$ can be expressed as

$$\begin{aligned} (\alpha * h)(m, l) &= \sum_{(n, h) \in G} \alpha(n, h) h[(n, h)^{-1} \cdot (m, l)] \\ &= \sum_{(n, h) \in G} \alpha(n, h) h(\phi_{h^{-1}}(m - n), h^{-1}l), \quad (m, l) \in G. \end{aligned} \tag{1}$$

For a function $\alpha : G \rightarrow \mathbb{C}$, its H -decimation $\downarrow_H \alpha : N \rightarrow \mathbb{C}$ is defined as $(\downarrow_H \alpha)(n) := \alpha(n, 1_H)$ for any $n \in N$. Thus, we have

$$\begin{aligned} \downarrow_H (\alpha * h)(m) &= (\alpha * h)(m, 1_H) = \sum_{(n, h) \in G} \alpha(n, h) h(\phi_{h^{-1}}(m - n), h^{-1}) \\ &= \sum_{(n, h) \in G} \alpha(n, h) h[(n - m, h)^{-1}], \quad m \in N. \end{aligned} \tag{2}$$

Defining the polyphase components of α and h as $\alpha_h(n) := \alpha(n, h)$ and $h_h(n) := h[(-n, h)^{-1}]$ respectively, we write

$$\downarrow_H (\alpha * h)(m) = \sum_{h \in H} \sum_{n \in N} \alpha_h(n) h_h(m - n) = \sum_{h \in H} (\alpha_h *_{N} h_h)(m), \quad m \in N.$$

For a function $c : N \rightarrow \mathbb{C}$, its H -expander $\uparrow_H c : G \rightarrow \mathbb{C}$ is defined as

$$(\uparrow_H c)(n, h) = \begin{cases} c(n) & \text{if } h = 1_H, \\ 0 & \text{if } h \neq 1_H. \end{cases}$$

In case $\uparrow_H c$ and g belong to $\ell^2(G)$, we have

$$\begin{aligned} (\uparrow_H c * g)(m, l) &= \sum_{(n, h) \in G} (\uparrow_H c)(n, h) g[(n, h)^{-1} \cdot (m, l)] \\ &= \sum_{(n, h) \in G} (\uparrow_H c)(n, h) g(\phi_{h^{-1}}(m - n), h^{-1}l) \\ &= \sum_{n \in N} c(n) g(m - n, l) = (c *_{N} g_l)(m), \quad m \in N, l \in H, \end{aligned}$$

where $g_l(n) := g(n, l)$ is the polyphase component of g .

From now on, we will refer to a K -channel filter bank with analysis filters h_k and synthesis filters g_k , $k = 1, 2, \dots, K$ as the one given by (see Figure 1)

$$c_k := \downarrow_H(\alpha * h_k), \quad k = 1, 2, \dots, K, \quad \text{and} \quad \beta = \sum_{k=1}^K (\uparrow_H c_k) * g_k, \quad (3)$$

where α and β denote, respectively, the input and the output of the filter bank. In polyphase notation,

$$\begin{aligned} c_k(m) &= \sum_{h \in H} (\alpha_h *_{N} h_{k,h})(m), \quad m \in N, \quad k = 1, 2, \dots, K, \\ \beta_l(m) &= \sum_{k=1}^K (c_k *_{N} g_{l,k})(m), \quad m \in N, \quad l \in H, \end{aligned} \quad (4)$$

where $\alpha_h(n) := \alpha(n, h)$, $\beta_l(n) := \beta(n, l)$, $h_{k,h}(n) := h_k[(-n, h)^{-1}]$ and $g_{l,k}(n) := g_k(n, l)$ are the polyphase components of α , β , h_k and g_k , $k = 1, 2, \dots, K$, respectively. We also assume that $h_k, g_k \in \ell^2(G)$ with $\widehat{h}_{k,h}, \widehat{g}_{h,k} \in L^\infty(\widehat{N})$ for $k = 1, 2, \dots, K$ and $h \in H$; from Lemma 1, the filter bank (3) is well defined in $\ell^2(G)$.

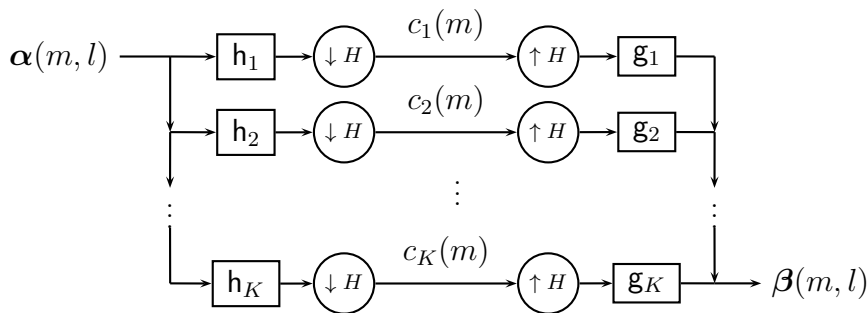


Figure 1. The K -channel filter bank scheme.

The above K -channel filter bank (3) is said to be a *perfect reconstruction* filter bank if and only if it satisfies $\alpha = \sum_{k=1}^K (\uparrow_H c_k) * g_k$ for each $\alpha \in \ell^2(G)$, or equivalently, $\alpha_h = \sum_{k=1}^K (c_k *_{N} g_{h,k})$ for each $h \in H$.

Since N is an LCA group where a Fourier transform is available, the polyphase expression (4) of the filter bank (3) allows us to carry out its polyphase analysis.

Polyphase Analysis: Perfect Reconstruction Condition

For notational ease, we denote $L := |H|$, the order of the group H , and its elements as $H = \{h_1, h_2, \dots, h_L\}$. Having in mind Lemma 1, the N -Fourier transform in $c_k(m) = \sum_{h \in H} (\alpha_h *_{N} h_{k,h})(m)$ gives $\widehat{c}_k(\gamma) = \sum_{h \in H} \widehat{h}_{k,h}(\gamma) \widehat{\alpha}_h(\gamma)$ a.e. $\gamma \in \widehat{N}$ for each $k = 1, 2, \dots, K$. In matrix notation,

$$\mathbf{C}(\gamma) = \mathbf{H}(\gamma) \mathbf{A}(\gamma) \quad \text{a.e. } \gamma \in \widehat{N},$$

where $\mathbf{C}(\gamma) = (\widehat{c}_1(\gamma), \widehat{c}_2(\gamma), \dots, \widehat{c}_K(\gamma))^T$, $\mathbf{A}(\gamma) = (\widehat{\alpha}_{h_1}(\gamma), \widehat{\alpha}_{h_2}(\gamma), \dots, \widehat{\alpha}_{h_L}(\gamma))^T$, and $\mathbf{H}(\gamma)$ is the $K \times L$ matrix

$$\mathbf{H}(\gamma) = \begin{pmatrix} \widehat{h}_{1,h_1}(\gamma) & \widehat{h}_{1,h_2}(\gamma) & \cdots & \widehat{h}_{1,h_L}(\gamma) \\ \widehat{h}_{2,h_1}(\gamma) & \widehat{h}_{2,h_2}(\gamma) & \cdots & \widehat{h}_{2,h_L}(\gamma) \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{h}_{K,h_1}(\gamma) & \widehat{h}_{K,h_2}(\gamma) & \cdots & \widehat{h}_{K,h_L}(\gamma) \end{pmatrix}, \quad (5)$$

where $\widehat{h}_{k,h_i} \in L^2(\widehat{N})$ is the Fourier transform of $h_{k,h_i}(n) := h_k[(-n, h_i)^{-1}] \in \ell^2(N)$.

The same procedure for $\beta_l(m) = \sum_{k=1}^K (\mathbf{c}_k * \mathbf{g}_{l,k})(m)$ gives $\widehat{\beta}_l(\gamma) = \sum_{k=1}^K \widehat{\mathbf{g}}_{l,k}(\gamma) \widehat{\mathbf{c}}_k(\gamma)$ a.e. $\gamma \in \widehat{N}$. In matrix notation,

$$\mathbf{B}(\gamma) = \mathbf{G}(\gamma) \mathbf{C}(\gamma) \quad \text{a.e. } \gamma \in \widehat{N},$$

where $\mathbf{B}(\gamma) = (\widehat{\beta}_{h_1}(\gamma), \widehat{\beta}_{h_2}(\gamma), \dots, \widehat{\beta}_{h_L}(\gamma))^T$, $\mathbf{C}(\gamma) = (\widehat{\mathbf{c}}_1(\gamma), \widehat{\mathbf{c}}_2(\gamma), \dots, \widehat{\mathbf{c}}_K(\gamma))^T$ and $\mathbf{G}(\gamma)$ is the $L \times K$ matrix

$$\mathbf{G}(\gamma) = \begin{pmatrix} \widehat{\mathbf{g}}_{h_1,1}(\gamma) & \widehat{\mathbf{g}}_{h_1,2}(\gamma) & \cdots & \widehat{\mathbf{g}}_{h_1,K}(\gamma) \\ \widehat{\mathbf{g}}_{h_2,1}(\gamma) & \widehat{\mathbf{g}}_{h_2,2}(\gamma) & \cdots & \widehat{\mathbf{g}}_{h_2,K}(\gamma) \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{\mathbf{g}}_{h_L,1}(\gamma) & \widehat{\mathbf{g}}_{h_L,2}(\gamma) & \cdots & \widehat{\mathbf{g}}_{h_L,K}(\gamma) \end{pmatrix}, \tag{6}$$

where $\widehat{\mathbf{g}}_{h_i,k} \in L^2(\widehat{N})$ is the Fourier transform of $\mathbf{g}_{h_i,k}(n) := \mathbf{g}_k(n, h_i) \in \ell^2(N)$.

Thus, in terms of the polyphase matrices $\mathbf{G}(\gamma)$ and $\mathbf{H}(\gamma)$, the filter bank (3) can be expressed as

$$\mathbf{B}(\gamma) = \mathbf{G}(\gamma) \mathbf{H}(\gamma) \mathbf{A}(\gamma) \quad \text{a.e. } \gamma \in \widehat{N}. \tag{7}$$

As a consequence of Equation (7), we have:

Theorem 1. *The K -channel filter bank given in Equation (3), where $\mathbf{h}_k, \mathbf{g}_k$ belong to $\ell^2(G)$ and $\widehat{\mathbf{h}}_{k,h_i}, \widehat{\mathbf{g}}_{h_i,k}$ belong to $L^\infty(\widehat{N})$ for $k = 1, 2, \dots, K$ and $i = 1, 2, \dots, L$, satisfies the perfect reconstruction property if and only if $\mathbf{G}(\gamma) \mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$, where \mathbf{I}_L denotes the identity matrix of order L .*

Proof. First of all, note that the mapping $\alpha \in \ell^2(G) \mapsto \mathbf{A} \in L_L^2(\widehat{N})$ is a unitary operator. Indeed, for each $\alpha, \beta \in \ell^2(G)$, we have the isometry property

$$\begin{aligned} \langle \alpha, \beta \rangle_{\ell^2(G)} &= \sum_{(m,h) \in G} \alpha(m,h) \overline{\beta(m,h)} = \sum_{h \in H} \langle \alpha_h, \beta_h \rangle_{\ell^2(N)} \\ &= \sum_{h \in H} \langle \widehat{\alpha}_h, \widehat{\beta}_h \rangle_{L^2(\widehat{N})} = \langle \mathbf{A}, \mathbf{B} \rangle_{L_L^2(\widehat{N})}. \end{aligned}$$

It is also surjective since the N -Fourier transform is a surjective isometry between $\ell^2(N)$ and $L^2(\widehat{N})$. Having in mind this property, Equation (7) tells us that the filter bank satisfies the perfect reconstruction property if and only if $\mathbf{G}(\gamma) \mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$. \square

Notice that, in the perfect reconstruction setting, the number of channels K must be necessarily bigger or equal that the order L of the group H , i.e., $K \geq L$.

4. Frame Analysis

For $m \in N$, the translation operator $T_m : \ell^2(G) \rightarrow \ell^2(G)$ is defined as

$$T_m \alpha(n, h) := \alpha((m, 1_H)^{-1} \cdot (n, h)) = \alpha(n - m, h), \quad (n, h) \in G. \tag{8}$$

The involution operator $\alpha \in \ell^2(G) \mapsto \widetilde{\alpha} \in \ell^2(G)$ is defined as $\widetilde{\alpha}(n, h) := \overline{\alpha((n, h)^{-1})}$, $(n, h) \in G$. As expected, the classical relationship between convolution and translation operators holds. Thus, for the K -channel filter bank (3), we have (see (2)):

$$\mathbf{c}_k(m) = \downarrow_H (\alpha * \mathbf{h}_k)(m) = \langle \alpha, T_m \widetilde{\mathbf{h}}_k \rangle_{\ell^2(G)}, \quad m \in N, \quad k = 1, 2, \dots, K.$$

In addition,

$$(\uparrow_H \mathbf{c}_k * \mathbf{g}_k)(m, h) = \sum_{n \in N} \mathbf{c}_k(n) \mathbf{g}_k(m - n, h) = \sum_{n \in N} \langle \alpha, T_n \widetilde{\mathbf{h}}_k \rangle_{\ell^2(G)} T_n \mathbf{g}_k(m, h).$$

In the perfect reconstruction setting, for any $\alpha \in \ell^2(G)$, we have

$$\alpha = \sum_{k=1}^K \sum_{n \in N} \langle \alpha, T_n \tilde{h}_k \rangle_{\ell^2(G)} T_n g_k \quad \text{in } \ell^2(G). \tag{9}$$

Given K sequences $f_k \in \ell^2(G)$, $k = 1, 2, \dots, K$, our main tasks now are: (i) to characterize the sequence $\{T_n f_k\}_{n \in N; k=1,2,\dots,K}$ as a frame for $\ell^2(G)$, and (ii) to find its dual frames having the form $\{T_n g_k\}_{n \in N; k=1,2,\dots,K}$.

To the first end, we consider a K -channel analysis filter bank with analysis filters $h_k := \tilde{f}_k$, i.e., the involution of f_k , $k = 1, 2, \dots, K$; let $\mathbf{H}(\gamma)$ be its associated $K \times L$ polyphase matrix (5). First, we check that Equation (5) is:

$$\mathbf{H}(\gamma) = \left(\overline{\widehat{f}_{k,h_i}(\gamma)} \right)_{\substack{k=1,2,\dots,K \\ i=1,2,\dots,L}} \tag{10}$$

where $\widehat{f}_{k,h_i}(\gamma)$ denotes the Fourier transform in $L^2(\widehat{N})$ of $f_{k,h_i}(n) = f_k(n, h_i)$ in $\ell^2(N)$. Indeed, for $k = 1, 2, \dots, K$ and $i = 1, 2, \dots, L$, having in mind that $h_{k,h_i}(n) = h_k[(-n, h_i)^{-1}]$ for analysis filters, we have

$$\begin{aligned} \widehat{h}_{k,h_i}(\gamma) &= \sum_{n \in N} h_{k,h_i}(n) \langle -n, \gamma \rangle = \sum_{n \in N} h_k[(-n, h_i)^{-1}] \langle -n, \gamma \rangle = \sum_{n \in N} \tilde{f}_k[(-n, h_i)^{-1}] \langle -n, \gamma \rangle \\ &= \sum_{n \in N} \overline{\widehat{f}_k(-n, h_i)} \langle -n, \gamma \rangle = \overline{\sum_{n \in N} f_k(n, h_i) \langle -n, \gamma \rangle} = \overline{\widehat{f}_{k,h_i}(\gamma)}, \quad \gamma \in \widehat{N}. \end{aligned}$$

Next, we consider its associated constants

$$A_{\mathbf{H}} := \operatorname{ess\,inf}_{\gamma \in \widehat{N}} \lambda_{\min} [\mathbf{H}^*(\gamma) \mathbf{H}(\gamma)] \quad \text{and} \quad B_{\mathbf{H}} := \operatorname{ess\,sup}_{\gamma \in \widehat{N}} \lambda_{\max} [\mathbf{H}^*(\gamma) \mathbf{H}(\gamma)].$$

Theorem 2. For f_k in $\ell^2(G)$, $k = 1, 2, \dots, K$, consider the associated matrix $\mathbf{H}(\gamma)$ given in Equation (10). Then,

1. The sequence $\{T_n f_k\}_{n \in N; k=1,2,\dots,K}$ is a Bessel sequence for $\ell^2(G)$ if and only if $B_{\mathbf{H}} < \infty$.
2. The sequence $\{T_n f_k\}_{n \in N; k=1,2,\dots,K}$ is a frame for $\ell^2(G)$ if and only if the inequalities $0 < A_{\mathbf{H}} \leq B_{\mathbf{H}} < \infty$ hold.

Proof. Using Plancherel theorem ([16], Theorem 4.25), for each $\alpha \in \ell^2(G)$, we get

$$\begin{aligned} \langle \alpha, T_n f_k \rangle_{\ell^2(G)} &= \sum_{h \in H} \langle \alpha_h, f_{k,h}(\cdot - n) \rangle_{\ell^2(N)} = \sum_{h \in H} \int_{\widehat{N}} \widehat{\alpha}_h(\gamma) \overline{\widehat{f}_{k,h}(\gamma)} \langle -n, \gamma \rangle d\gamma \\ &= \int_{\widehat{N}} \sum_{h \in H} \widehat{\alpha}_h(\gamma) \overline{\widehat{f}_{k,h}(\gamma)} \langle -n, \gamma \rangle d\gamma = \int_{\widehat{N}} \mathbf{H}_k(\gamma) \mathbf{A}(\gamma) \overline{\langle -n, \gamma \rangle} d\gamma, \end{aligned}$$

where $\mathbf{A}(\gamma) = (\widehat{\alpha}_{h_1}(\gamma), \widehat{\alpha}_{h_2}(\gamma), \dots, \widehat{\alpha}_{h_L}(\gamma))^{\top}$ and $\mathbf{H}_k(\gamma)$ denotes the k -th row of $\mathbf{H}(\gamma)$.

Since $\{\langle -n, \gamma \rangle\}_{n \in N}$ is an orthonormal basis for $L^2(\widehat{N})$, in case that $\mathbf{H}(\gamma) \mathbf{A}(\gamma) \in L^2_{\mathbb{K}}(\widehat{N})$, we have

$$\sum_{k=1}^K \sum_{n \in N} |\langle \alpha, T_n f_k \rangle|^2 = \sum_{k=1}^K \int_{\widehat{N}} |\mathbf{H}_k(\gamma) \mathbf{A}(\gamma)|^2 d\gamma = \int_{\widehat{N}} \|\mathbf{H}(\gamma) \mathbf{A}(\gamma)\|^2 d\gamma.$$

If $B_{\mathbf{H}} < \infty$, having in mind that $\|\alpha\|_{\ell^2(G)}^2 = \|\mathbf{A}\|_{L^2_{\mathbb{K}}(\widehat{N})}^2 = \int_{\widehat{N}} \|\mathbf{A}(\gamma)\|^2 d\gamma$, the above equality and the Rayleigh–Ritz theorem ([18], Theorem 4.2.2) prove that $\{T_n f_k\}_{n \in N; k=1,2,\dots,K}$ is a Bessel sequence for $\ell^2(G)$ with Bessel bound less or equal than $B_{\mathbf{H}}$.

On the other hand, if $K < B_H$, then there exists a set $\Omega \subset \widehat{N}$ having a strictly positive measure such that $\lambda_{\max}[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)] > K$ for $\gamma \in \Omega$. Consider α such that its associated $\mathbf{A}(\gamma)$ is 0 if $\gamma \notin \Omega$, and $\mathbf{A}(\gamma)$ is a unitary eigenvector corresponding to the largest eigenvalue of $\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)$ if $\gamma \in \Omega$. Thus, we have that

$$\sum_{k=1}^K \sum_{n \in \widehat{N}} |\langle \alpha, T_n f_k \rangle|^2 = \int_{\widehat{N}} \|\mathbf{H}(\gamma)\mathbf{A}(\gamma)\|^2 d\gamma > K \int_{\widehat{N}} \|\mathbf{A}(\gamma)\|^2 d\gamma = K \|\alpha\|_{\ell^2(G)}^2.$$

As a consequence, if $B_H = \infty$, the sequence is not Bessel, and, if $B_H < \infty$, the optimal bound is precisely B_H .

Similarly, by using inequality $\|\mathbf{H}(\gamma)\mathbf{A}(\gamma)\|^2 \geq \lambda_{\min}[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)]\|\mathbf{A}(\gamma)\|^2$, and that equality holds whenever $\mathbf{A}(\gamma)$ is a unitary eigenvector corresponding to the smallest eigenvalue of $\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)$; one proves the other inequality in part 2. \square

Corollary 1. *The sequence $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ is a Bessel sequence for $\ell^2(G)$ if and only if for each $k = 1, 2, \dots, K$ and $i = 1, 2, \dots, L$ the function \widehat{f}_{k,h_i} belongs to $L^\infty(\widehat{N})$.*

Proof. It is a direct consequence of the equivalence between the spectral and Frobenius norms for matrices [18]. \square

It is worth mentioning that f_k in $\ell^1(G)$, $k = 1, 2, \dots, K$, implies that the sequence $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ is always a Bessel sequence for $\ell^2(G)$ since each function \widehat{f}_{k,h_i} is continuous and \widehat{N} is compact. In this case, the frame condition for $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ reduces to $\text{rank } \mathbf{H}(\gamma) = L$ for all $\gamma \in \widehat{N}$ or, equivalently,

$$\min_{\gamma \in \widehat{N}} (\det[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)]) > 0.$$

To the second end, a K -channel filter bank formalism allows, in a similar manner, to obtain properties in $\ell^2(G)$ of the sequences $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ and $\{T_n g_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$. In case they are Bessel sequences for $\ell^2(G)$, the idea is to consider a K -channel filter bank (3) where the analysis filters are $h_k := \widehat{f}_k$ and the synthesis filters are g_k , $k = 1, 2, \dots, K$. As a consequence, the corresponding polyphase matrices $\mathbf{H}(\gamma)$ and $\mathbf{G}(\gamma)$, given in Equations (5) and (6), are

$$\mathbf{H}(\gamma) = \left(\overline{\widehat{f}_{k,h_i}(\gamma)} \right)_{\substack{k=1,2,\dots,K \\ i=1,2,\dots,L}} \quad \text{and} \quad \mathbf{G}(\gamma) = \left(\widehat{g}_{h_i,k}(\gamma) \right)_{\substack{i=1,2,\dots,L \\ k=1,2,\dots,K}}, \quad \gamma \in \widehat{N}. \tag{11}$$

Theorem 3. *Let $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ and $\{T_n g_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ be two Bessel sequences for $\ell^2(G)$, and $\mathbf{H}(\gamma)$ and $\mathbf{G}(\gamma)$ their associated matrices (11). Under the above circumstances, we have:*

- (a) *The sequences $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ and $\{T_n g_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ are dual frames for $\ell^2(G)$ if and only if condition $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$ holds.*
- (b) *The sequences $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ and $\{T_n g_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ are biorthogonal sequences in $\ell^2(G)$ if and only if condition $\mathbf{H}(\gamma)\mathbf{G}(\gamma) = \mathbf{I}_K$ a.e. $\gamma \in \widehat{N}$ holds.*
- (c) *The sequences $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ and $\{T_n g_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ are dual Riesz bases for $\ell^2(G)$ if and only if $K = L$ and $\mathbf{G}(\gamma) = \mathbf{H}(\gamma)^{-1}$ a.e. $\gamma \in \widehat{N}$.*
- (d) *The sequence $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ is an A -tight frame for $\ell^2(G)$ if and only if condition $\mathbf{H}^*(\gamma)\mathbf{H}(\gamma) = A\mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$ holds.*
- (e) *The sequence $\{T_n f_k\}_{n \in \widehat{N}; k=1,2,\dots,K}$ is an orthonormal basis for $\ell^2(G)$ if and only if $K = L$ and $\mathbf{H}^*(\gamma) = \mathbf{H}(\gamma)^{-1}$ a.e. $\gamma \in \widehat{N}$.*

Proof. Having in mind Equation (9) and Corollary 1, part (a) is nothing but Theorem 1.

The output of the analysis filter bank (3) corresponding to the input $g_{k'}$ is a K -vector whose k -entry is

$$c_{k,k'}(m) = \downarrow_H (g_{k'} * h_k)(m) = \langle g_{k'}, T_m \tilde{h}_k \rangle_{\ell^2(G)} = \langle g_{k'}, T_m f_k \rangle_{\ell^2(G)},$$

and whose N -Fourier transform is $C_{k'}(\gamma) = H(\gamma) G_{k'}(\gamma)$ a.e. $\gamma \in \hat{N}$, where $G_{k'}$ is the k' -column of the matrix $G(\gamma)$. Note that $\{T_n f_k\}_{n \in N; k=1,2,\dots,K}$ and $\{T_n g_k\}_{n \in N; k=1,2,\dots,K}$ are biorthogonal if and only if $\langle g_{k'}, T_m f_k \rangle_{\ell^2(G)} = \delta(k - k')\delta(m)$. Therefore, the sequences $\{T_n f_k\}_{n \in N; k=1,2,\dots,K}$ and $\{T_n g_k\}_{n \in N; k=1,2,\dots,K}$ are biorthogonal if and only if $H(\gamma)G(\gamma) = I_K$. Thus, we have proved (b).

Having in mind ([15], Theorem 7.1.1), from (a) and (b), we obtain (c).

We can read the frame operator corresponding to the sequence $\{T_n f_k\}_{n \in N; k=1,2,\dots,K}$ i.e.,

$$S(\alpha) = \sum_{k=1}^K \sum_{n \in N} \langle \alpha, T_n f_k \rangle_{\ell^2(G)} T_n f_k, \quad \alpha \in \ell^2(G),$$

as the output of the filter bank (3), whenever $h_k = \tilde{f}_k$ and $g_k = f_{k'}$ for the input α . For this filter bank, the (k, h_l) -entry of the analysis polyphase matrix $H(\gamma)$ is $\tilde{f}_{k,h_l}(\gamma)$ and the (h_l, k) -entry of the synthesis polyphase matrix $G(\gamma)$ is $\tilde{f}_{k,h_l}(\gamma)$; in other words, $G(\gamma) = H^*(\gamma)$. Hence, the sequence $\{T_n f_k\}_{n \in N; k=1,2,\dots,K}$ is an A -tight frame for $\ell^2(G)$, i.e.,

$$\alpha = \frac{1}{A} \sum_{k=1}^K \sum_{n \in N} \langle \alpha, T_n f_k \rangle_{\ell^2(G)} T_n f_k, \quad \alpha \in \ell^2(G),$$

if and only if $H^*(\gamma)H(\gamma) = AI_L$ for all $\gamma \in \hat{N}$. Thus, we have proved (d).

Finally, from (c) and (d), the sequence $\{T_n f_k\}_{n \in N; k=1,2,\dots,K}$ is an orthonormal system if and only if $H^*(\gamma) = H(\gamma)^{-1}$ a.e. $\gamma \in \hat{N}$. \square

5. Getting on with Sampling

Suppose that $\{U(n, h)\}_{(n,h) \in G}$ is a unitary representation of the group $G = N \rtimes_{\phi} H$ on a separable Hilbert space \mathcal{H} , and assume that for a fixed $a \in \mathcal{H}$ the sequence $\{U(n, h)a\}_{(n,h) \in G}$ is a Riesz sequence for \mathcal{H} (see Ref. ([1], Theorem A)). Thus, we consider the U -invariant subspace in \mathcal{H}

$$\mathcal{A}_a = \left\{ \sum_{(n,h) \in G} \alpha(n, h) U(n, h)a : \{\alpha(n, h)\}_{(n,h) \in G} \in \ell^2(G) \right\}.$$

For K fixed elements $b_k \in \mathcal{H}, k = 1, 2, \dots, K$, not necessarily in \mathcal{A}_a , we consider for each $x \in \mathcal{A}_a$ its generalized samples defined as

$$\mathcal{L}_k x(m) := \langle x, U(m, 1_H) b_k \rangle_{\mathcal{H}}, \quad m \in N \text{ and } k = 1, 2, \dots, K. \tag{12}$$

The task is the stable recovery of any $x \in \mathcal{A}_a$ from the data $\{\mathcal{L}_k x(m)\}_{m \in N; k=1,2,\dots,K}$.

In what follows, we propose a solution involving a perfect reconstruction K -channel filter bank. First, we express the samples in a more suitable manner. Namely, for each $x = \sum_{(n,h) \in G} \alpha(n, h) U(n, h)a$ in \mathcal{A}_a , we have

$$\begin{aligned} \mathcal{L}_k x(m) &= \sum_{(n,h) \in G} \alpha(n, h) \langle U(n, h)a, U(m, 1_H) b_k \rangle \\ &= \sum_{(n,h) \in G} \alpha(n, h) \langle a, U[(n, h)^{-1} \cdot (m, 1_H)] b_k \rangle = \downarrow_H (\alpha * h_k)(m), \quad m \in N, \end{aligned}$$

where $\alpha = \{\alpha(n, h)\}_{(n,h) \in G} \in \ell^2(G)$, and $h_k(n, h) := \langle a, U(n, h) b_k \rangle_{\mathcal{H}}$ also belongs to $\ell^2(G)$ for each $k = 1, 2, \dots, K$.

Suppose also that there exists a perfect reconstruction K -channel filter-bank with analysis filters the above h_k and synthesis filters g_k , $k = 1, 2, \dots, K$, such that the sequences $\{T_n h_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$ and $\{T_n g_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$ are Bessel sequences for $\ell^2(G)$. Having in mind Equation (9), for each $\alpha = \{\alpha(n, h)\}_{(n,h) \in G}$ in $\ell^2(G)$, we have

$$\alpha = \sum_{k=1}^K \sum_{n \in \mathbb{N}} \downarrow_H(\alpha * h_k)(n) T_n g_k = \sum_{k=1}^K \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) T_n g_k \quad \text{in } \ell^2(G). \tag{13}$$

In order to derive a sampling formula in \mathcal{A}_a , we consider the natural isomorphism $\mathcal{T}_{U,a} : \ell^2(G) \rightarrow \mathcal{A}_a$ which maps the usual orthonormal basis $\{\delta_{(n,h)}\}_{(n,h) \in G}$ for $\ell^2(G)$ onto the Riesz basis $\{U(n, h) a\}_{(n,h) \in G}$ for \mathcal{A}_a , i.e.,

$$\mathcal{T}_{U,a} : \delta_{(n,h)} \mapsto U(n, h) a \quad \text{for each } (n, h) \in G.$$

This isomorphism $\mathcal{T}_{U,a}$ possesses the following shifting property:

Lemma 2. For each $m \in \mathbb{N}$, consider the translation operator T_m operator defined in Equation (8). For each $m \in \mathbb{N}$, the following shifting property holds

$$\mathcal{T}_{U,a}(T_m f) = U(m, 1_H)(\mathcal{T}_{U,a} f), \quad f \in \ell^2(G). \tag{14}$$

Proof. For each $\delta_{(n,h)}$, it is easy to check that $T_m \delta_{(n,h)} = \delta_{(m+n,h)}$. Hence,

$$\mathcal{T}_{U,a}(T_m \delta_{(n,h)}) = U(m + n, h) a = U(m, 1_H) U(n, h) a = U(m, 1_H)(\mathcal{T}_{U,a} \delta_{(n,h)}).$$

A continuity argument proves the result for all f in $\ell^2(G)$. \square

Now, for each $x = \mathcal{T}_{U,a} \alpha \in \mathcal{A}_a$, applying the isomorphism $\mathcal{T}_{U,a}$ and the shifting property (14) in Equation (13), we get for each $x \in \mathcal{A}_a$ the expansion

$$\begin{aligned} x &= \sum_{k=1}^K \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) \mathcal{T}_{U,a}(T_n g_k) = \sum_{k=1}^K \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n, 1_H)(\mathcal{T}_{U,a} g_k) \\ &= \sum_{k=1}^K \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n, 1_H) c_{k,g} \quad \text{in } \mathcal{H}, \end{aligned} \tag{15}$$

where $c_{k,g} = \mathcal{T}_{U,a} g_k$, $k = 1, 2, \dots, K$. In fact, the following sampling theorem in the subspace \mathcal{A}_a holds:

Theorem 4. For K fixed $b_k \in \mathcal{H}$, let $\mathcal{L}_k : \mathcal{A}_a \rightarrow \mathbb{C}^{\mathbb{N}}$ be its associated U -system defined in Equation (12) with corresponding $h_k \in \ell^2(G)$, $k = 1, 2, \dots, K$. Assume that its polyphase matrix $\mathbf{H}(\gamma)$ given in Equation (5) has all its entries in $L^\infty(\hat{N})$. The following statements are equivalent:

1. The constant $A_{\mathbf{H}} = \operatorname{ess\,inf}_{\gamma \in \hat{N}} \lambda_{\min}[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)] > 0$.
2. There exist g_k in $\ell^2(G)$, $k = 1, 2, \dots, K$, such that the associated polyphase matrix $\mathbf{G}(\gamma)$ given in (6) has all its entries in $L^\infty(\hat{N})$, and it satisfies $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \hat{N}$.
3. There exist K elements $c_k \in \mathcal{A}_a$ such that the sequence $\{U(n, 1_H)c_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$ is a frame for \mathcal{A}_a and, for each $x \in \mathcal{A}_a$, the sampling formula

$$x = \sum_{k=1}^K \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n, 1_H) c_k \quad \text{in } \mathcal{H} \tag{16}$$

holds.

4. There exists a frame $\{C_{k,n}\}_{n \in N; k=1,2,\dots,K}$ for \mathcal{A}_a such that for each $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) C_{k,n} \quad \text{in } \mathcal{H}$$

holds.

Proof. (1) implies (2). The $L \times K$ Moore–Penrose pseudo-inverse $\mathbf{H}^\dagger(\gamma)$ of $\mathbf{H}(\gamma)$ is given by $\mathbf{H}^\dagger(\gamma) = [\mathbf{H}^*(\gamma) \mathbf{H}(\gamma)]^{-1} \mathbf{H}^*(\gamma)$. Its entries are essentially bounded in \widehat{N} since the entries of $\mathbf{H}(\gamma)$ belong to $L^\infty(\widehat{N})$ and $\det^{-1} [\mathbf{H}^*(\gamma) \mathbf{H}(\gamma)]$ is essentially bounded \widehat{N} since $0 < A_H$. In addition, $\mathbf{H}^\dagger(\gamma) \mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$. The inverse N -Fourier transform in $L^2(\widehat{N})$ of the k -th column of $\mathbf{H}^\dagger(\gamma)$ gives $g_k, k = 1, 2, \dots, K$.

(2) implies (3). According to Theorems 2 and 3, the sequences $\{T_n \tilde{h}_k\}_{n \in N; k=1,2,\dots,K}$ and $\{T_n g_k\}_{n \in N; k=1,2,\dots,K}$ form a pair of dual frames for $\ell^2(G)$. We deduce the sampling expansion as in Formula (15). In addition, the sequence $\{U(n, 1_H) c_{k,g}\}_{n \in N; k=1,2,\dots,K}$ is a frame for \mathcal{A}_a .

Obviously, (3) implies (4). Finally, (4) implies (1). Applying $\mathcal{T}_{U,a}^{-1}$, we get that the sequences $\{T_n \tilde{h}_k\}_{n \in N; k=1,2,\dots,K}$ and $\{\mathcal{T}_{U,a}^{-1}(C_{k,n})\}_{n \in N; k=1,2,\dots,K}$ form a pair of dual frames for $\ell^2(G)$; in particular, by using Theorem 2, we obtain that $0 < A_H$. \square

All the possible solutions of $\mathbf{G}(\gamma) \mathbf{H}(\gamma) = \mathbf{I}_L$ a.e. $\gamma \in \widehat{N}$ with entries in $L^\infty(\widehat{N})$ are given in terms of the Moore–Penrose pseudo inverse by the $L \times K$ matrices $\mathbf{G}(\gamma) := \mathbf{H}^\dagger(\gamma) + \mathbf{U}(\gamma) [\mathbf{I}_K - \mathbf{H}(\gamma) \mathbf{H}^\dagger(\gamma)]$, where $\mathbf{U}(\gamma)$ denotes any $L \times K$ matrix with entries in $L^\infty(\widehat{N})$.

Notice that $K \geq L$ where L is the order of the group H . In case $K = L$, we obtain:

Corollary 2. In the case $K = L$, assume that its polyphase matrix $\mathbf{H}(\gamma)$ given in Equation (5) has all entries in $L^\infty(\widehat{N})$. The following statements are equivalent:

1. The constant $A_H = \text{ess inf}_{\gamma \in \widehat{N}} \lambda_{\min} [\mathbf{H}^*(\gamma) \mathbf{H}(\gamma)] > 0$.
2. There exist L unique elements $c_k, k = 1, 2, \dots, L$, in \mathcal{A}_a such that the associated sequence $\{U(n, 1_H) c_k\}_{n \in N; k=1,2,\dots,L}$ is a Riesz basis for \mathcal{A}_a and the sampling formula

$$x = \sum_{k=1}^L \sum_{n \in N} \mathcal{L}_k x(n) U(n, 1_H) c_k \quad \text{in } \mathcal{H}$$

holds for each $x \in \mathcal{A}_a$.

Moreover, the interpolation property $\mathcal{L}_k c_{k'}(n) = \delta_{k,k'} \delta_{n,0_N}$, where $n \in N$ and $k, k' = 1, 2, \dots, L$, holds.

Proof. In this case, the square matrix $\mathbf{H}(\gamma)$ is invertible and the result comes out from Theorem 3. From the uniqueness of the coefficients in a Riesz basis expansion, we get the interpolation property. \square

Denote $H = \{h_1, h_2, \dots, h_L\}$; for a fixed $b \in \mathcal{H}$, we consider the samples

$$\mathcal{L}_k x(m) := \langle x, U(m, h_k) b \rangle, \quad m \in N \text{ and } k = 1, 2, \dots, L,$$

of any $x \in \mathcal{A}_a$. Since $U(m, h_k) b = U(m, 1_H) U(0_N, h_k) b = U(m, 1_H) b_k$, where $b_k := U(0_N, h_k) b, k = 1, 2, \dots, L$, we are in a particular case of Equation (12) with $K = L$.

Notice also that the subspace \mathcal{A}_a can be viewed as the multiple generated U -invariant subspace of \mathcal{H}

$$\overline{\text{span}}\{U(n, 1_H) a_h : n \in N, h \in H\}$$

with L generators $a_h := U(0_N, h) a \in \mathcal{H}, h \in H$, obtained from $a \in \mathcal{H}$ by the action of the group H .

5.1. An Example Involving Crystallographic Groups

The Euclidean motion group $E(d)$ is the semi-direct product $\mathbb{R}^d \rtimes_{\phi} O(d)$ corresponding to the homomorphism $\phi : O(d) \rightarrow \text{Aut}(\mathbb{R}^d)$ given by $\phi_A(x) = Ax$, where $A \in O(d)$ and $x \in \mathbb{R}^d$. The composition law on $E(d) = \mathbb{R}^d \rtimes_{\phi} O(d)$ reads $(x, A) \cdot (x', A') = (x + Ax', AA')$.

Let M be a non-singular $d \times d$ matrix and Γ a finite subgroup of $O(d)$ of order L such that $A(M\mathbb{Z}^d) = M\mathbb{Z}^d$ for each $A \in \Gamma$. We consider the crystallographic group $C_{M,\Gamma} := M\mathbb{Z}^d \rtimes_{\phi} \Gamma$ and its quasi regular representation (see Ref. [1]) on $L^2(\mathbb{R}^d)$

$$U(n, A)f(t) = f[A^{\top}(t - n)], \quad n \in M\mathbb{Z}^d, A \in \Gamma \text{ and } f \in L^2(\mathbb{R}^d).$$

For a fixed $\varphi \in L^2(\mathbb{R}^d)$ such that the sequence $\{U(n, A)\varphi\}_{(n,A) \in C_{M,\Gamma}}$ is a Riesz sequence for $L^2(\mathbb{R}^d)$ (see, for instance, Refs. [19,20]) we consider the U -invariant subspace in $L^2(\mathbb{R}^d)$

$$\begin{aligned} \mathcal{A}_{\varphi} &= \left\{ \sum_{(n,A) \in C_{M,\Gamma}} \alpha(n, A) \varphi[A^{\top}(t - n)] : \{\alpha(n, A)\} \in \ell^2(C_{M,\Gamma}) \right\} \\ &= \left\{ \sum_{(n,A) \in C_{M,\Gamma}} \alpha(n, A) \varphi(At - n) : \{\alpha(n, A)\} \in \ell^2(C_{M,\Gamma}) \right\}. \end{aligned}$$

Choosing K functions $b_k \in L^2(\mathbb{R}^d)$, $k = 1, 2, \dots, K$, we consider the average samples of $f \in \mathcal{A}_{\varphi}$

$$\mathcal{L}_k f(n) = \langle f, U(n, I)b_k \rangle = \langle f, b_k(\cdot - n) \rangle, \quad n \in M\mathbb{Z}^d.$$

Under the hypotheses in Theorem 4, there exist $K \geq L$ sampling functions $\psi_k \in \mathcal{A}_{\varphi}$ for $k = 1, 2, \dots, K$, such that the sequence $\{\psi_k(\cdot - n)\}_{n \in M\mathbb{Z}^d, k=1,2,\dots,K}$ is a frame for \mathcal{A}_{φ} , and the sampling expansion

$$f(t) = \sum_{k=1}^K \sum_{n \in M\mathbb{Z}^d} \langle f, b_k(\cdot - n) \rangle_{L^2(\mathbb{R}^d)} \psi_k(t - n) \quad \text{in } L^2(\mathbb{R}^d) \tag{17}$$

holds.

If the generator $\varphi \in C(\mathbb{R}^d)$ and the function $t \mapsto \sum_n |\varphi(t - n)|^2$ is bounded on \mathbb{R}^d , a standard argument shows that \mathcal{A}_{φ} is a reproducing kernel Hilbert space (RKHS) of bounded continuous functions in $L^2(\mathbb{R}^d)$. As a consequence, convergence in $L^2(\mathbb{R}^d)$ -norm implies pointwise convergence which is absolute and uniform on \mathbb{R}^d .

Notice that the infinite dihedral group $D_{\infty} = \mathbb{Z} \rtimes_{\phi} \mathbb{Z}_2$ is a particular crystallographic group with lattice \mathbb{Z} and $\Gamma = \mathbb{Z}_2$. Its quasi regular representation on $L^2(\mathbb{R})$ reads

$$U(n, 0)f(t) = f(t - n) \quad \text{and} \quad U(n, 1)f(t) = f(-t + n), \quad n \in \mathbb{Z} \text{ and } f \in L^2(\mathbb{R}).$$

Thus, we could obtain sampling formulas as (17) for $K \geq 2$ average functions b_k .

The quasi regular unitary representation of a crystallographic group $C_{M,\Gamma}$ on $L^2(\mathbb{R}^d)$ motivates the next section:

5.2. The Case of Pointwise Samples

Let $\{U(n, h)\}_{(n,h) \in G}$ be a unitary representation of the group $G = N \rtimes_{\phi} H$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. If the generator $\varphi \in L^2(\mathbb{R}^d)$ satisfies that, for each $(n, h) \in G$, the function $U(n, h)\varphi$ is continuous on \mathbb{R}^d , and the condition

$$\sup_{t \in \mathbb{R}^d} \sum_{(n,h) \in G} |[U(n, h)\varphi](t)|^2 < \infty,$$

then the subspace \mathcal{A}_φ is an RKHS of bounded continuous functions in $L^2(\mathbb{R}^d)$; proceeding as in [21], one can prove that the above conditions are also necessary.

For K fixed points $t_k \in \mathbb{R}^d$, $k = 1, 2, \dots, K$, we consider for each $f \in \mathcal{A}_\varphi$ the new samples given by

$$\mathcal{L}_k f(n) := [U(-n, 1_H)f](t_k), \quad n \in N \text{ and } k = 1, 2, \dots, K. \quad (18)$$

For each $f = \sum_{(m,h) \in G} \alpha(m, h) U(m, h) \varphi$ in \mathcal{A}_φ and $k = 1, 2, \dots, K$, we have

$$\begin{aligned} \mathcal{L}_k f(n) &= \left[\sum_{(m,h) \in G} \alpha(m, h) U[(-n, 1_H) \cdot (m, h)] \varphi \right](t_k) \\ &= \sum_{(m,h) \in G} \alpha(m, h) [U(m - n, h) \varphi](t_k) = \langle \alpha, T_n f_k \rangle_{\ell^2(G)}, \quad n \in N, \end{aligned}$$

where $\alpha = \{\alpha(m, h)\}_{(m,h) \in G}$ and $f_k(m, h) := \overline{[U(m, h) \varphi](t_k)}$, $(m, h) \in G$. Notice that f_k belongs to $\ell^2(G)$, $k = 1, 2, \dots, K$. As a consequence, under the hypotheses in Theorem 4 (on these new $h_k := \tilde{f}_k$, $k = 1, 2, \dots, K$), a sampling formula such as (16) holds for the data sequence $\{\mathcal{L}_k f(n)\}_{n \in N; k=1,2,\dots,K}$ defined in Equation (18).

In the particular case of the quasi regular representation of a crystallographic group $\mathcal{C}_{M,\Gamma} = M\mathbb{Z}^d \rtimes_\varphi \Gamma$, for each $f \in \mathcal{A}_\varphi$, the samples (18) read

$$\mathcal{L}_k f(n) = [U(-n, I)f](t_k) = f(t_k + n), \quad n \in M\mathbb{Z}^d \text{ and } k = 1, 2, \dots, K.$$

Thus (under hypotheses in Theorem 4), there exist K functions $\psi_k \in \mathcal{A}_\varphi$, $k = 1, 2, \dots, K$, such that for each $f \in \mathcal{A}_\varphi$ the sampling formula

$$f(t) = \sum_{k=1}^K \sum_{n \in M\mathbb{Z}^d} f(t_k + n) \psi_k(t - n), \quad t \in \mathbb{R}^d$$

holds. The convergence of the series in the $L^2(\mathbb{R}^d)$ -norm sense implies pointwise convergence which is absolute and uniform on \mathbb{R}^d .

6. Conclusions

In this paper, we have derived an abstract regular sampling theory associated with a unitary representation $(n, h) \mapsto U(n, h)$ of a group G which is a semi-direct product of two groups, N countable discrete abelian group and H finite, on a separable Hilbert space \mathcal{H} ; here, regular sampling means that we are taken the samples at the group N . Concretely, the sampling theory is obtained in the U -invariant subspace of \mathcal{H} generated by $a \in \mathcal{H}$ that is

$$\mathcal{A}_a = \left\{ \sum_{(n,h) \in G} \alpha(n, h) U(n, h)a : \{\alpha(n, h)\}_{(n,h) \in G} \in \ell^2(G) \right\},$$

and the samples of $x \in \mathcal{A}_a$ are given by $\mathcal{L}_k x(n) := \langle x, U(n, 1_H)b_k \rangle_{\mathcal{H}}$, $n \in N$, where b_k , $k = 1, 2, \dots, K$, denote K fixed elements in \mathcal{H} which do not belong necessarily to \mathcal{A}_a . We look for K elements $c_k \in \mathcal{A}_a$ such that the sequence $\{U(n, 1_H)c_k\}_{n \in N; k=1,2,\dots,K}$ is a frame for \mathcal{A}_a and, for each $x \in \mathcal{A}_a$, the sampling formula $x = \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) U(n, 1_H)c_k$ holds.

A similar problem was solved when the group G is a discrete LCA group and the samples are taken at a uniform lattice of G (see Ref. [10]). In the case of an abelian group, we have the Fourier transform, a basic tool in this previous work. In the present work, a classical Fourier analysis on G is not available, but if G is a semi-direct product of the form $G = N \rtimes_\varphi H$, where N is a countable discrete abelian group and H is a finite group, the Fourier transform on the abelian group N allows us to solve the problem by means of a filter bank formalism. Recalling the filter bank formalism in discrete LCA

groups, the defined samples are expressed as the output of a suitable K -channel analysis filter bank corresponding to the input $x \in \mathcal{A}_a$. The frame analysis of this filter bank along with the synthesis one giving perfect reconstruction allows us to obtain a pair of suitable dual frames for obtaining the desired sampling result, which is written as a list of equivalent statements (see Theorem 4).

Although the semi-direct product of groups represents, so to speak, the simplest case of non-abelian groups, this paper can be a good starting point for finding sampling theorems associated with unitary representations of non-abelian groups that are not isomorphic to a semi-direct product of groups.

Author Contributions: The authors contributed equally in the aspects concerning this work: conceptualization, methodology, writing—original draft preparation, writing—review and editing and funding acquisition.

Funding: This research was funded by the grant MTM2017-84098-P from the Spanish Ministerio de Economía y Competitividad (MINECO).

Acknowledgments: The authors wish to thank the referees for their valuable and constructive comments.

Conflicts of Interest: The authors declare no conflict of interest.

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