High-frequency propagation for the Schrödinger equation on the torus

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Abstract

The main objective of this paper is understanding the propagation laws obeyed by high-frequency limits of Wigner distributions associated to solutions to the Schrödinger equation on the standard d-dimensional torus $\mathbb{T}^d$. From the point of view of semiclassical analysis, our setting corresponds to performing the semiclassical limit at times of order $1/h$, as the characteristic wave-length h of the initial data tends to zero. It turns out that, in spite that for fixed h every Wigner distribution satisfies a Liouville equation, their limits are no longer uniquely determined by those of the Wigner distributions of the initial data. We characterize them in terms of a new object, the *resonant Wigner distribution*, which describes high-frequency effects associated to the fraction of the energy of the sequence of initial data that concentrates around the set of resonant frequencies in phase-space $T^*\mathbb{T}^d$. This construction is related to that of the so-called two-microlocal semiclassical measures. We prove that any limit μ of the Wigner distributions corresponding to solutions to the Schrödinger equation on the torus is completely determined by the limits of both the Wigner distribution and the resonant Wigner distribution of the initial data; moreover, μ follows a propagation law described by a family of density-matrix Schrödinger equations on the periodic geodesics of $\mathbb{T}^d$. Finally, we present some connections with the study of the dispersive behavior of the Schrödinger flow (in particular, with Strichartz estimates). Among these, we show that the limits of sequences of position densities of solutions to the Schrödinger equation on $\mathbb{T}^2$ are absolutely continuous with respect to the Lebesgue measure.

Keywords: Semiclassical (Wigner) measures; Schrödinger equation on the torus; Quantum limits; Two-microlocal Wigner measures; Resonances; Strichartz estimates

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1. Introduction

In this article, we shall consider solutions to Schrödinger's equation on the standard flat torus $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z}^d)$,

$$i\partial_t u(t,x) + \frac{1}{2}\Delta_x u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d. \tag{1}$$

We are interested in understanding the propagation of high-frequency effects associated to solutions to (1). More precisely, given a sequence (u_h) of initial data which oscillates at frequencies of the order of $1/h$ (see condition (5) below) we would like to describe in a quantitative manner the propagation of these oscillation effects under the action of the Schrödinger group $e^{it\Delta_x/2}$. Some understanding in this direction can be obtained by analysing the structure of weak $*$ limits of sequences of measures of the form:

$$\left|e^{it\Delta_x/2}u_h(x)\right|^2 dx, \tag{2}$$

where (u_h) is a bounded sequence in $L^2(\mathbb{T}^d)$ and dx denotes the Lebesgue measure on $\mathbb{T}^d$. These limiting measures give information about the regions on which the energy of $(e^{it\Delta_x/2}u_h)$ concentrates; a natural question in this context is to understand their dependence on t; and in particular, their dependence on the initial data (u_h). However, it is usually difficult to deal directly with (2). This is due to the presence of the modulus in (2) which prevents us from keeping track of the characteristic directions of oscillation of the functions u_h. It is preferable instead to consider their Wigner distributions, which are phase-space densities that take into account simultaneously concentration on physical and Fourier space, and which project onto (2). The main issue addressed in this article is to study the propagation laws obeyed by Wigner distributions of solutions to the Schrödinger equation (1). As a consequence of our results, we shall prove that for $d = 2$ a limit of a sequence of densities (2) (corresponding to sequence (u_h) satisfying a standard oscillation condition) is absolutely continuous with respect to the Lebesgue measure. At the end of this introduction we discuss the connections of this result with Strichartz estimates.

Let $\psi_k(x) := (2\pi)^{-d/2}e^{ik\cdot x}$, $k \in \mathbb{Z}^d$, denote the vectors of the standard orthonormal basis of $L^2(\mathbb{T}^d)$. The *Wigner distribution* of a function $u = \sum_{k\in\mathbb{Z}^d}\widehat{u}(k)\psi_k$ in $L^2(\mathbb{T}^d)$ is defined for $h > 0$ as:

$$w_u^h(x,\xi) := \sum_{k,j\in\mathbb{Z}^d} \widehat{u}(k)\overline{\widehat{u}(j)}\psi_k(x)\overline{\psi_j(x)}\delta_{\frac{h}{2}(k+j)}(\xi), \tag{3}$$

where δ_p stands for the Dirac delta centered at the point p. The distribution w_u^h is in fact a measure on $T^*\mathbb{T}^d \cong \mathbb{T}^d \times \mathbb{R}^d$; one easily checks that:

$$\int_{\mathbb{R}^d} w_u^h(x,d\xi) = \left|u(x)\right|^2, \qquad \int_{\mathbb{T}^d} w_u^h(dx,\xi) = \sum_{k\in\mathbb{Z}^d}\left|\widehat{u}(k)\right|^2\delta_{hk}(\xi).$$

Therefore, w_u^h may be viewed as a microlocal lift of the density $|u(x)|^2$ to phase space $T^*\mathbb{T}^d$; it allows to describe simultaneously the distribution of energy of u on physical and Fourier space. The distributions w_u^h are not positive, although their limits as $h \to 0^+$ are indeed finite positive

Radon measures.[2] Let us note that the definition of the Wigner distribution in a general compact Riemannian manifold usually depends on various choices (coordinate charts, partitions of unity) that however have no effect on their asymptotic behavior as $h \to 0^+$ (see, for instance, [13,17], and the references therein). Our formula (3) corresponds to identifying elements in $L^2(\mathbb{T}^d)$ to those functions in $L^2_{\text{loc}}(\mathbb{R}^d)$ that are $2\pi\mathbb{Z}^d$-periodic and then consider their canonical Euclidean Wigner distributions. One easily checks that this definition is consistent with the generally accepted one. The reader can refer to the book [10] for a comprehensive discussion on Wigner distributions.

We shall assume in the following that (u_h) is bounded in $L^2(\mathbb{T}^d)$ and write

$$w^h_{u_h}(t,\cdot) := w^h_{e^{it\Delta_x/2}u_h}.$$

Then, after possibly extracting a subsequence, the associated Wigner distributions at $t=0$ converge (see, for instance, [11–13,16]):

$$w^h_{u_h}(0,\cdot) \rightharpoonup \mu_0 \in \mathcal{M}_+\big(T^*\mathbb{T}^d\big), \quad \text{as } h \to 0^+ \text{ in } \mathcal{D}'\big(T^*\mathbb{T}^d\big). \tag{4}$$

The measure μ_0 is usually called the *semiclassical or Wigner measure* of (u_h). In addition, one can recover the asymptotic behavior of $|u_h|^2$ from μ_0. More precisely:

$$|u_h|^2\,dx \rightharpoonup \int_{\mathbb{R}^d} \mu_0(\cdot, d\xi), \quad \text{vaguely as } h \to 0^+,$$

provided the densities $|u_h|^2$ converge and the sequence (u_h) verifies the *h-oscillation property*[3]:

$$\limsup_{h\to 0^+} \sum_{|k|>R/h} \big|\widehat{u_h}(k)\big|^2 \to 0, \quad \text{as } R \to \infty. \tag{5}$$

Wigner distributions $w^h_{u_h}(t,\cdot)$ associated to solutions to the Schrödinger equation are completely determined by those of their initial data $w^h_{u_h}(0,\cdot)$ as they solve the classical Liouville equation:

$$\partial_t w^h_{u_h}(t,x,\xi) + \frac{\xi}{h}\cdot\nabla_x w^h_{u_h}(t,x,\xi) = 0. \tag{6}$$

As a consequence of this, it is possible to show that the rescaled Wigner distributions $w^h_{u_h}(ht,\cdot)$ converge (after possibly extracting a subsequence) locally uniformly in t to the measure $(\phi_{-t})_*\mu_0$, where $(\phi_t)_*$ denotes the push-forward operator on measures induced by the geodesic flow $\phi_t(x,\xi) := (x+t\xi,\xi)$ on $T^*\mathbb{T}^d$. This result, sometimes known as the *classical limit*, holds in a much more general setting, see, for instance, [12,14,16].

Eq. (6) provides interesting consequences when (u_n) is a sequence of (normalised) eigenfunctions $-\Delta_x$. Suppose that $-\Delta_x u_n = \lambda_n u_n$ and that $\lim_{n\to\infty}\lambda_n = \infty$. Since $w^h_{u_n}$ is quadratic in u_n,

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[2] From now on, we denote the set of such measures by $\mathcal{M}_+(T^*\mathbb{T}^d)$.

[3] This condition expresses that no positive fraction of the energy of the sequence (u_h) concentrates at frequencies asymptotically larger than $1/h$.

we have $w^h_{u_n}(t,\cdot) = w^h_{u_n}(0,\cdot)$, for every $t \in \mathbb{R}$. Setting $h := 1/\sqrt{\lambda_n}$ (so that (u_n) is h-oscillating) Eq. (6) implies that any semiclassical measure μ_0 of this sequence is invariant by the geodesic flow, *i.e.* $(\phi_s)_*\mu_0 = \mu_0$. Moreover, one can show that μ_0 is a probability measure concentrated on the unit cosphere bundle $S^*\mathbb{T}^d$; in this context, semiclassical measures are usually called *quantum limits*. The problem of classifying all possible quantum limits in $\mathbb{T}^d$ is very hard. Their structure has been clarified by Jakobson [15] for $d = 2$. For arbitrary $d \geqslant 1$, Bourgain has proved (see again [15]) that the projection of a quantum limit onto $\mathbb{T}^d$ (which is an accumulation point of the measures $(|u_n|^2)$) is absolutely continuous with respect to Lebesgue measure. In particular, the sequence (u_n) cannot concentrate on sets of dimension lower than d. The article [15] also provides partial results when $d \geqslant 3$ (see also [2]).

When (u_n) is not formed by eigenfunctions, it is not anymore clear that $w^h_{u_n}(t,\cdot)$ converges pointwise in t. However, we have the following result, proved in [17] for a general compact manifold:

Existence of limits. Given a bounded h-oscillating sequence (u_h) in $L^2(\mathbb{T}^d)$, there exists a subsequence such that, for every $a \in C_c^\infty(T^*\mathbb{T}^d)$ and every $\varphi \in L^1(\mathbb{R})$,

$$\lim_{h'\to 0^+} \int_{\mathbb{R}\times T^*\mathbb{T}^d} \varphi(t)a(x,\xi)w^{h'}_{u_{h'}}(t,dx,d\xi)\,dt = \int_{\mathbb{R}\times T^*\mathbb{T}^d} \varphi(t)a(x,\xi)\mu(t,dx,d\xi)\,dt, \tag{7}$$

where the limit μ is in $L^\infty(\mathbb{R};\mathcal{M}_+(T^*\mathbb{T}^d))$.[4] Moreover,

$$\lim_{h'\to 0^+} \int_{\mathbb{R}}\int_{\mathbb{T}^d} \varphi(t)m(x)\left|e^{it\Delta_x/2}u_{h'}(x)\right|^2 dx\,dt = \int_{\mathbb{R}}\int_{T^*\mathbb{T}^d} \varphi(t)m(x)\mu(t,dx,d\xi)\,dt, \tag{8}$$

for every $\varphi \in L^1(\mathbb{R})$ and $m \in C(\mathbb{T}^d)$.

In addition, as one can check by taking limits in Eq. (6), the invariance property satisfied by semiclassical measures corresponding to sequences of eigenfunctions also holds in this more general setting.

Invariance. For a.e. $t \in \mathbb{R}$, the measure $\mu(t,\cdot)$ is invariant by the geodesic flow on $T^*\mathbb{T}^d$:

$$(\phi_s)_*\mu(t,\cdot) = \mu(t,\cdot), \quad \text{for every } s \in \mathbb{R}. \tag{9}$$

See also [17] for a proof in a more general context. In that reference, a characterization of the propagation law for these measures in the class of compact manifolds with periodic geodesic flow (the so-called *Zoll manifolds*) was given. In fact, a formula relating μ and μ_0 exists: $\mu(t,\cdot)$ equals the average of μ_0 along the geodesic flow for a.e. $t \in \mathbb{R}$ (which is well-defined due to the periodicity of the geodesic flow); note, in particular, that μ is constant in time. This fits our setting when $d = 1$; but when $d \geqslant 2$, the dynamics of the geodesic flow in the torus are more complex than those in Zoll manifolds. In both cases, the geodesic flow is completely integrable. However, the torus possesses geodesics of arbitrary large minimal periods, as well as non-periodic, dense, geodesics.

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[4] Note that, in contrast with the semiclassical limit, it is not true that $(w^h_{u_h}(t,\cdot))$ (or any subsequence) converges to $\mu(t,\cdot)$, even almost everywhere.

It will turn out that this added complexity will have an effect on our problem. However, there is still a class of sequences of initial data for which the measures μ and μ_0 are related by an averaging process. More precisely, μ is obtained by averaging μ_0 when the initial data do not see the set of *resonant frequencies*:

$$\Omega := \{\xi \in \mathbb{R}^d\colon\ k \cdot \xi = 0 \text{ for some } k \in \mathbb{Z}^d \setminus \{0\}\}.$$

Proposition 1 *(Non-resonant case). (See [17, Proposition 10].) Suppose μ and μ_0 are given respectively by* (7) *and* (4) *for some sequence* (u_h) *bounded in* $L^2(\mathbb{T}^d)$ *and verifying* (5). *If* $\mu_0(\mathbb{T}^d \times \Omega) = 0$ *then, for a.e.* $t \in \mathbb{R}$,

$$\mu(t, x, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mu_0(dy, \xi).$$

Note that in this case, any limit (8) of the densities $|e^{it\Delta_x/2}u_h(x)|^2$ is a constant function in t and x.[5]

The role of resonances. Therefore, all the difficulties in analysing the structure of μ rely on understanding its behavior when μ_0 actually sees the set of resonant frequencies Ω. Given $\xi_0 \in \Omega$, in [17], Proposition 11, sequences of initial data (u_h) and (v_n) are constructed such that both have $|\rho(x)|^2\,dx\,\delta_{\xi_0}(\xi)$ as a semiclassical measure. However, when $\rho \in L^2(\mathbb{T}^d)$ is invariant in the ξ_0-direction, the corresponding limits (7) of the evolved Wigner distributions are, respectively,

$$\left|e^{it\Delta_x/2}\rho(x)\right|^2 dx\,\delta_{\xi_0}(\xi) \quad \text{and} \quad \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \left|\rho(y)\right|^2 dy\delta_{\xi_0}(\xi).$$

Two conclusions can be extracted from this fact: (i) The measures $\mu(t, \cdot)$ may have a non-trivial dependence on t, which is not directly related to the dynamics of the geodesic flow and, most importantly, (ii) the measure μ_0 corresponding to the initial data *no longer determines uniquely* the measures $\mu(t, \cdot)$ corresponding to the evolution.

The structure of μ is therefore much more complex when μ_0 sees the set of resonant frequencies; before stating our main result, let us introduce some notation.

We define set $\mathbb{W}$ of *simple resonant directions* in $\mathbb{R}^d$ as follows. Consider the subset $\Omega_1 \subset \Omega$ consisting of simple resonances (that is, Ω_1 is formed by the $\xi \in \Omega$ such that $\lambda\xi \in \mathbb{Z}^d$ for some real $\lambda \neq 0$). Consider the equivalence relation $\sim$ on $\Omega_1 \setminus \{0\}$ defined by $x \sim y$ if and only if $x, y \in \Omega_1 \setminus \{0\}$ lie on a line through the origin. Define $\mathbb{W}$ as the set of equivalence classes of $\sim$. In other words, $\mathbb{W}$ is the subset of the real projective space $\mathbb{RP}^{d-1}$ obtained by projecting $\Omega_1 \setminus \{0\} \subset \mathbb{R}^d$ using the canonical covering projection.

For each $\omega \in \mathbb{W}$ define,

$$\gamma_\omega := \{t v_\omega\colon\ t \in \mathbb{R}\}/2\pi\mathbb{Z}^d \subset \mathbb{T}^d, \quad \text{where } v_\omega \in \omega.$$

[5] Note also that for $d = 1$, the condition $\mu_0(\mathbb{T}^d \times \Omega) = 0$ reduces to $\mu_0(\{\xi = 0\}) = 0$. This has to be interpreted as the requirement that no positive fraction of the energy of the sequence of initial data concentrates at frequencies asymptotically smaller than $1/h$.

This is the (closed) geodesic of $\mathbb{T}^d$ issued from the point 0 in the direction ν_ω. There is a bijection between $\mathbb{W}$ and the set of closed geodesics in $\mathbb{T}^d$ that pass through 0. In what follows, $L^2(\gamma_\omega)$ will denote the space of (equivalence classes of) square integrable functions on γ_ω with respect to arc-length measure.

Given $\omega \in \mathbb{W}$, we shall denote by $I_\omega \subset \mathbb{R}^d$ the hyperplane through the origin orthogonal to ω. The structure of the measures $\mu(t, \cdot)$ is given by the next result (in Theorem 9 in Section 3, we precise the nature of the propagation law for $\mu(t, \cdot)$).

Theorem 2. *Let (u_h) be a bounded, h-oscillating sequence in $L^2(\mathbb{T}^d)$ with semiclassical measure μ_0. Suppose that (7) holds for some measure μ. Then, for a.e. $t \in \mathbb{R}$,*

$$\mu(t, x, \xi) = \sum_{\omega \in \mathbb{W}} \rho_\omega^t(x, \xi) + \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mu_0(dy, \xi), \tag{10}$$

where ρ_ω^t, for $t \in \mathbb{R}$ and $\omega \in \mathbb{W}$, is a signed measure concentrated on $\mathbb{T}^d \times I_\omega$ whose projection on the first component is absolutely continuous with respect to the Lebesgue measure.
Moreover, the measure ρ_ω^t follows a propagation law related to the Schrödinger flow on $L^2(\gamma_\omega)$, and can be computed solely in terms of ρ_ω^0, which is in turn completely determined by the initial data (u_h).

Note that the sum $\sum_{\omega \in \mathbb{W}} \rho_\omega^t$ is concentrated on $\mathbb{T}^d \times \Omega$, and, as we shall prove in Theorem 9, ρ_ω^t is related to the trace of a density matrix in $L^2(\gamma_\omega)$ evolving according to the Schrödinger equation on γ_ω. The measure ρ_ω^0 will be obtained as the limit of a new object, the *resonant Wigner distribution* of the initial data u_h, which describes the concentration of energy of the sequence (u_h) over $\mathbb{T}^d \times \Omega$ at scales of order one. We introduce its definition, along with a description of the properties that are relevant to our analysis in the next section. Let us just mention that the resonant Wigner distribution may be viewed as a two-microlocal object, in the spirit of the 2-microlocal semiclassical measures introduced by Fermanian-Kammerer [7,8], Gérard and Fermanian-Kammerer [9], Miller [18], and Nier [19].

In particular, ρ_ω^0 might vanish even if $\mu_0(\mathbb{T}^d \times I_\omega) > 0$. The condition for ρ_ω^0 to be zero is the following (see Proposition 12 in Section 4).

Suppose that ρ_ω^t are given by formula (10)*. If (u_h) satisfies:*

$$\lim_{h \to 0^+} \sum_{|k \cdot \nu_\omega| < N} \left|\widehat{u_h}(k)\right|^2 = 0, \quad \text{for every } N > 0,$$

where $\nu_\omega \in \omega$ is a unit vector, then $\rho_\omega^t = 0$ for every $t \in \mathbb{R}$.

Using this characterization, we are able to describe the propagation of wave-packet type solutions, see Proposition 13 in Section 4. We also give there an example of sequence (u_h) for which some of the ρ_ω^t are non-zero.

As a consequence of formula (10) we prove in Section 3 the following result for the position densities (2).

Corollary 3. *Let $d = 2$ and (u_h) be a bounded, h-oscillating sequence in $L^2(\mathbb{T}^2)$ with a semiclassical measure μ_0. If $\mu_0(\{\xi = 0\}) = 0$ then, up to some subsequence, for every $\varphi \in L^1(\mathbb{R})$ and $m \in C(\mathbb{T}^2)$,*

$$\lim_{h'\to 0^+}\int_{\mathbb{R}}\int_{\mathbb{T}^2}\varphi(t)m(x)\left|e^{it\Delta_x/2}u_{h'}(x)\right|^2 dx\,dt=\int_{\mathbb{R}}\int_{\mathbb{T}^2}\varphi(t)m(x)\nu(t,dx)\,dt,$$

and the measure $\nu\in L^\infty(\mathbb{R};\mathcal{M}_+(\mathbb{T}^2))$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{T}^2$.

This result is somehow related to the analysis of dispersion (Strichartz) estimates for the Schrödinger equation. For instance, when $d=1$ we have the following inequality due to Zygmund [20]:

$$\left\|e^{it\Delta/2}u\right\|_{L^4(\mathbb{T}_t\times\mathbb{T}_x)}\leqslant C\|u\|_{L^2(\mathbb{T})},\tag{11}$$

for some constant $C>0$ (note that $e^{it\Delta/2}$ is $2\pi\mathbb{Z}$-periodic in t). This estimate implies that for $d=1$ any limit of averages in time of the densities (2) is absolutely continuous with respect to Lebesgue measure (and is in fact an $L^2(\mathbb{T})$-function). However, as shown by Bourgain [1], estimate (11) fails when $d=2$; although a version of (11) with a loss of derivatives does hold.[6] Therefore, the result given in Corollary 3 supports in some sense the possibility that an inequality such as (11) holds on $\mathbb{T}_t\times\mathbb{T}_x^2$ in some L^p-space with $2<p<4$.

The present analysis can be extended to more general tori and Schrödinger-type equations arising as the quantization of completely integrable Hamiltonian systems. These issues will be addressed elsewhere.

2. The resonant Wigner distribution

2.1. Preliminaries and definition

Let $\omega\in\mathbb{W}$ be a simple resonant direction; as before, denote by $I_\omega\subset\mathbb{R}^d$ the linear hyperplane orthogonal to ω. Then there exists a unique $p_\omega\in\omega\cap\mathbb{Z}^d$ such that:

(i) the (non-zero) components of p_ω are coprime;
(ii) the first non-zero component of p_ω is positive.

Clearly, $\omega\cap\mathbb{Z}^d=\{np_\omega\colon n\in\mathbb{Z}\}$; therefore, the component in the direction p_ω of any $k\in\mathbb{Z}^d$ is of the form

$$\frac{n}{|p_\omega|}\nu_\omega,\quad \text{where } n=k\cdot p_\omega\in\mathbb{Z}\text{ and }\nu_\omega:=\frac{p_\omega}{|p_\omega|}.$$

Moreover, since Bezout's theorem ensures the existence of $c\in\mathbb{Z}^d$ satisfying $p_\omega\cdot c=1$, we have that for any given $n\in\mathbb{Z}$ there exists at least one $k\in\mathbb{Z}^d$ such that $k\cdot p_\omega=n$. In other words, the set of orthogonal projections onto ω of points in $\mathbb{Z}^d$ consists of the vectors $n/|p_\omega|\nu_\omega$ for $n\in\mathbb{Z}$. Note that, in particular, the sets

[6] References [1,3] describes also positive results. In [4–6] Strichartz estimates in general compact manifolds are established, together with a detailed analysis of the loss of derivatives phenomenon in specific geometries.

$$\omega_n^\perp := \left\{ r \in I_\omega : \frac{n}{|p_\omega|} v_\omega + r \in \mathbb{Z}^d \right\} \subset I_\omega,$$

are non-empty.

It is not difficult to see that $\omega_n^\perp \cap \omega_m^\perp \neq \emptyset$ if and only if $m \equiv n \pmod{|p_\omega|^2}$, in which case $\omega_n^\perp = \omega_m^\perp$. This implies that the set $\omega^\perp := \bigcup_{n \in \mathbb{Z}} \omega_n^\perp$ consisting of the orthogonal projections on I_ω of vectors in $\mathbb{Z}^d$ is a subgroup of $\mathbb{R}^d$.

The results discussed so far imply the following.

Proposition 4. *For $p \in \mathbb{Z}$, denote by $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ the group of congruence classes modulo p. The map:*

$$\bigcup_{[c] \in \mathbb{Z}_{|p_\omega|^2}} \left([c] \times \omega_c^\perp \right) \to \mathbb{Z}^d : (n, r) \mapsto \frac{n}{|p_\omega|} v_\omega + r$$

is well-defined and bijective. Moreover, the map $h : \omega^\perp \to \mathbb{Z}_{|p_\omega|^2}$ defined by $h(r) := [n]$ if $r \in \omega_n^\perp$ is a well-defined group homomorphism whose kernel is $\omega_0^\perp \subset \mathbb{Z}^d$. Therefore, the quotient group $\omega^\perp/\omega_0^\perp$ is isomorphic to $\mathbb{Z}_{|p_\omega|^2}$ and consists of the cosets $\omega_n^\perp = r + \omega_0^\perp$ where r is any element of $\omega_n^\perp$.

The geodesic γ_ω, passing through 0 and pointing in the direction ω, has length $2\pi|p_\omega|$. Therefore, it can be identified to $\mathbb{T}_\omega := \mathbb{R}/(2\pi|p_\omega|\mathbb{Z})$ in such a way that arc-length measure on γ_ω corresponds to a (suitably normalized) Haar measure on $\mathbb{T}_\omega$. The functions:

$$\phi_m^\omega(s) := \frac{e^{i \frac{m}{|p_\omega|} s}}{\sqrt{2\pi|p_\omega|}}, \quad m \in \mathbb{Z},$$

for an orthonormal basis of $L^2(\gamma_\omega)$. For $n, m \in \mathbb{Z}$, we shall denote by $\phi_m^\omega \otimes \overline{\phi_n^\omega}$ the operator on $L^2(\gamma_\omega)$ given by:

$$\phi_m^\omega \otimes \overline{\phi_n^\omega}\left(\phi_k^\omega\right) = \begin{cases} \phi_m^\omega & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

We shall denote by $\mathcal{L}(L^2(\gamma_\omega))$, $\mathcal{K}(L^2(\gamma_\omega))$ and $\mathcal{L}^1(L^2(\gamma_\omega))$, the space of linear bounded, compact and trace-class operators on $L^2(\gamma_\omega)$, respectively.

We write:

$$\mathcal{J} := \bigcup_{\omega \in \mathbb{W}} \{\omega\} \times I_\omega;$$

consider on each $\{\omega\} \times I_\omega$ the topology induced by $\mathbb{R}^d$ and endow $\mathcal{J}$ with the disjoint union topology. To every $(\omega, \xi) \in \mathcal{J}$ we associate the vector spaces $\mathcal{L}(L^2(\gamma_\omega))$ and $\mathcal{K}(L^2(\gamma_\omega))$; this defines vector bundles over $\mathcal{J}$:

$$\pi_{\mathcal{L}} : \bigcup_{(\omega,\xi) \in \mathcal{J}} \mathcal{L}\left(L^2(\gamma_\omega)\right) \to \mathcal{J}, \qquad \pi_{\mathcal{K}} : \bigcup_{(\omega,\xi) \in \mathcal{J}} \mathcal{K}\left(L^2(\gamma_\omega)\right) \to \mathcal{J}.$$

Let us denote respectively by $\mathcal{X}(\mathcal{J})$ and $\mathcal{X}_0(\mathcal{J})$ the spaces of continuous, compactly supported, sections of the bundles $\pi_{\mathcal{L}}$ and $\pi_{\mathcal{K}}$. That is, $k \in \mathcal{X}(\mathcal{J})$ whenever $k(\omega, \xi) \in \mathcal{L}(L^2(\gamma_\omega))$ for every $(\omega, \xi) \in \mathcal{J}$ and $k(\omega, \cdot)$ is continuous, compactly supported, and non-zero for at most a finite number of ω. Similar considerations hold for the elements of $\mathcal{X}_0(\mathcal{J})$. The dual of $\mathcal{X}(\mathcal{J})$ (resp. $\mathcal{X}_0(\mathcal{J})$) will be denoted by $\mathcal{X}'(\mathcal{J})$ (resp. $\mathcal{X}'_0(\mathcal{J})$). Given $\mu \in \mathcal{X}'(\mathcal{J})$ (resp. $\mathcal{X}'_0(\mathcal{J})$), $\mu(\omega, \cdot)$ can be identified to a measure on I_ω taking values on $\mathcal{L}(L^2(\gamma_\omega))$ (resp. $\mathcal{L}^1(L^2(\gamma_\omega))$). Finally, $\mathcal{X}'_{0,+}(\mathcal{J})$ will stand for the cone of positive elements of $\mathcal{X}'_0(\mathcal{J})$; hence, $\mu \in \mathcal{X}'_0(\mathcal{J})$ if $\int_{I_\omega} b(\xi)\mu(\omega, d\xi) \in \mathcal{L}^1(L^2(\gamma_\omega))$ is positive and Hermitian whenever $b \in C_c(I_\omega)$ is positive. Appendix A provides background material, additional details and references on operator-valued measures.

The *resonant Wigner distribution* of $u \in L^2(\mathbb{T}^d)$, is defined as:

$$\mathcal{R}_u^h(\omega, \xi) := \sum_{[c] \in \mathbb{Z}_{|p_\omega|^2}} \sum_{\substack{m,n \in [c] \\ r \in \omega_c^\perp}} \widehat{u}\left(\frac{m}{|p_\omega|} v_\omega + r\right) \overline{\widehat{u}\left(\frac{n}{|p_\omega|} v_\omega + r\right)} \delta_{hr}(\xi) \phi_m^\omega \otimes \overline{\phi_n^\omega}. \tag{12}$$

Clearly, $\mathcal{R}_u^h \in \mathcal{X}'(\mathcal{J})$.

2.2. *Boundedness and convergence*

Our next result shows, in particular, that $\mathcal{R}_u^h \in \mathcal{X}'_{0,+}(\mathcal{J})$.

Proposition 5. *Let $u \in L^2(\mathbb{T}^d)$. Then for every $\omega \in \mathbb{W}$ and $b \in C_c(I_\omega)$,*

$$\int_{I_\omega} b(\xi)\mathcal{R}_u^h(\omega, d\xi) \quad \text{is a Hermitian, trace-class operator of } L^2(\gamma_\omega),$$

which is positive if b is non-negative. In addition, we have the following bound:

$$\operatorname{tr}\left|\int_{I_\omega} b(\xi)\mathcal{R}_u^h(\omega, d\xi)\right| \leqslant \|u\|^2_{L^2(\mathbb{T}^d)} \sup_{r \in I_\omega} |b(r)|. \tag{13}$$

In particular, $\mathcal{R}_u^h \in \mathcal{X}'_{0,+}(\mathcal{J})$.

Proof. Define

$$\|u\|^2_{m,\omega} := \sum_{r \in \omega_m^\perp} \left|\widehat{u}\left(\frac{m}{|p_\omega|} v_\omega + r\right)\right|^2;$$

because of Proposition 4, one has $\|u\|^2_{L^2(\mathbb{T}^d)} = \sum_{m \in \mathbb{Z}} \|u\|^2_{m,\omega}$. Take $b \in C_c(I_\omega)$ and set

$$K_b := \int_{I_\omega} b(\xi)\mathcal{R}_u^h(\omega, d\xi);$$

then $K_b = \sum_{m\equiv n \pmod{|p_\omega|^2}} k_b(m,n)\phi_m^\omega \otimes \overline{\phi_n^\omega}$ with:

$$k_b(m,n) = \sum_{r\in\omega_c^\perp} b(hr)\widehat{u}\left(\frac{m}{|p_\omega|}v_\omega + r\right)\overline{\widehat{u}\left(\frac{n}{|p_\omega|}v_\omega + r\right)}.$$

The operator K_b is bounded, since:

$$\sum_{m\equiv n \pmod{|p_\omega|^2}} \left|k_b(m,n)\right|^2 \leqslant \|b\|^2_{L^\infty(I_\omega)} \sum_{m\equiv n \pmod{|p_\omega|^2}} \|u\|^2_{m,\omega}\|u\|^2_{n,\omega}$$

$$\leqslant \|b\|^2_{L^\infty(I_\omega)}\left(\sum_{n\in\mathbb{Z}} \|u\|^2_{n,\omega}\right)^2 = \|b\|^2_{L^\infty(I_\omega)}\|u\|^4_{L^2(\mathbb{T}^d)}.$$

Moreover, K_b is Hermitian as soon as b is real valued, since $k_b(m,n) = \overline{k_b(n,m)}$; therefore, K_b is a Hilbert–Schmidt operator on $L^2(\gamma_\omega)$.

Now, given $v \in L^2(\gamma_\omega)$ write $v = \sum_{m\in\mathbb{Z}} v_m\phi_m^\omega$. Then

$$(K_b v|v)_{L^2(\gamma_\omega)} = \sum_{[c]\in\mathbb{Z}_{|p_\omega|^2}} \sum_{r\in\omega_c^\perp} b(hr)\left|\sum_{n\in[c]} v_n\widehat{u}\left(\frac{n}{|p_\omega|}v_\omega + r\right)\right|^2.$$

This quantity is positive whenever $b \geqslant 0$ and $b \not\equiv 0$. Therefore, for such b the operator K_b is Hilbert–Schmidt (and hence compact), Hermitian and positive. Thus, it will be trace-class as soon as its trace is finite. This is clearly the case, since

$$\operatorname{tr} K_b = \sum_{n\in\mathbb{Z}} \sum_{r\in\omega_n^\perp} b(hr)\left|\widehat{u}\left(\frac{n}{|p_\omega|}v_\omega + r\right)\right|^2 \leqslant \sup_{r\in I_\omega} b(r)\|u\|^2_{L^2(\mathbb{T}^d)}.$$

For a general b non-necessarily positive, the result follows by expressing $b = b_+ - b_-$ and applying the above estimate to each term separately. □

If (u_h) is a bounded family in $L^2(\mathbb{T}^d)$, estimate (13) then shows that $\mathcal{R}^h_{u_h}(\omega,\cdot)$ is a uniformly bounded family in $\mathcal{X}'_{0,+}(\mathcal{J})$.

Proposition 6. *Let (u_h) be a bounded sequence in $L^2(\mathbb{T}^d)$. Then, there exist a subsequence $(u_{h'})$ and a finite measure $\mu_{\mathcal{R}} \in \mathcal{X}'_{0,+}(\mathcal{J})$, such that, for every $\omega \in \mathbb{W}$ and $b \in \mathcal{X}_0(\mathcal{J})$:*

$$\lim_{h'\to 0^+} \operatorname{tr} \int_{I_\omega} b(\omega,\xi)\mathcal{R}^{h'}_{u_{h'}}(\omega,d\xi) = \operatorname{tr}\int_{\mathbb{R}^d} b(\omega,\xi)\mu_{\mathcal{R}}(\omega,d\xi). \tag{14}$$

Moreover, the total mass of $\mu_{\mathcal{R}}(\omega,\cdot)$ satisfies:

$$\operatorname{tr}\int_{I_\omega} \mu_{\mathcal{R}}(\omega,d\xi) \leqslant \liminf_{h'\to 0^+} \|u_{h'}\|_{L^2(\mathbb{T}^d)};$$

and, due to the structure of $\mathcal{R}^h_{u_h}$,

$$\left(\int_{I_\omega} b(\xi)\mu_{\mathcal{R}}(\omega, d\xi)\phi_n^\omega \middle| \phi_m^\omega\right)_{L^2(\gamma_\omega)} = 0, \quad \textit{if } m \not\equiv n \ (\mathrm{mod}\ |p_\omega|^2).$$

Proof. Estimate (13) implies that each $\mathcal{R}^h_{u_h}(\omega, \cdot)$ is uniformly bounded in the space[7] $\mathcal{M}_+(I_\omega; \mathcal{L}^1(L^2(\gamma_\omega)))$ of positive measures on I_ω with values in $\mathcal{L}^1(L^2(\gamma_\omega))$ by a constant $C > 0$ independent of $\omega \in \mathbb{W}$. Since $\mathcal{M}_+(I_\omega; \mathcal{L}^1(L^2(\gamma_\omega)))$ may be identified to the cone of positive elements of the dual of $C_c(I_\omega; \mathcal{K}(L^2(\gamma_\omega)))$, statement (14) follows from the Banach–Alaoglu theorem and a standard diagonal argument. Finally, the bound on the total mass of $\mu_{\mathcal{R}}(\omega, \cdot)$ is a consequence of estimate (13). □

In what follows, we shall refer to a measure $\mu_{\mathcal{R}} \in \mathcal{X}'_{0,+}(\mathcal{J})$ obtained as a limit (14) as a *resonant Wigner measure* of the sequence (u_h).

As we mentioned in the introduction, resonant Wigner measures are closely related to the two-microlocal semiclassical measures introduced in [7–9,18,19]. However, our definition of the resonant Wigner distribution gives rise to a global object (see the discussion in [7, p. 518]); moreover, resonant Wigner measures describe the energy concentration (at scales of order one) of the sequence (u_h) on the non-smooth set $\mathbb{T}^d \times \Omega$ in phase space.

2.3. Additional properties

Our next result is a manifestation of the two-microlocal character of resonant Wigner measures. It characterizes the sequences (u_h) for which $\mu_{\mathcal{R}}$ is identically zero.

Proposition 7. *Let* (u_h) *be h-oscillatory and suppose that* (14) *holds for the sequence* (u_h)*. Given any* $\omega \in \mathbb{W}$*, one has* $\mu_{\mathcal{R}}(\omega, \cdot) = 0$ *if and only if*:

$$\lim_{h \to 0^+} \sum_{|k \cdot p_\omega| < N} \left|\widehat{u_h}(k)\right|^2 = 0, \quad \textit{for every } N > 0. \tag{15}$$

Proof. Let $N \in \mathbb{N}$ and denote by π_N the projection in $L^2(\gamma_\omega)$ onto the subspace spanned by $(\phi_j^\omega)_{0 \leqslant |j| \leqslant N}$. Then π_N is compact and

$$\operatorname{tr}\left(\pi_N \mathcal{R}^h_{u_h}(\omega, \xi)\right) = \sum_{|n| \leqslant N} \sum_{r \in \omega_n^\perp} \left|\widehat{u_h}\left(\frac{n}{|p_\omega|}\nu_\omega + r\right)\right|^2 \delta_{hr}(\xi).$$

Therefore, for every $\varphi \in C_c(I_\omega)$,

$$\lim_{h \to 0^+} \sum_{|n| \leqslant N} \sum_{r \in \omega_n^\perp} \varphi(hr) \left|\widehat{u_h}\left(\frac{n}{|p_\omega|}\nu_\omega + r\right)\right|^2 = \operatorname{tr}\left(\pi_N \int_{I_\omega} \varphi(\xi)\mu_{\mathcal{R}}(\omega, d\xi)\right). \tag{16}$$

[7] We refer the reader to Appendix A for precise definitions of spaces of operator-valued measures.

Since (u_h) is h-oscillatory, we can suppose without loss of generality that there exists $R > 0$ such that $\widehat{u_h}(k) = 0$ if $|hk| > R$. Now suppose $\mu_{\mathcal{R}}(\omega, \cdot) = 0$; by taking $\varphi(\xi) = 1$ for $|\xi| \leqslant R$ in (16) we conclude (15). Now suppose that (15) holds. Then $\operatorname{tr}(\pi_N \int_{I_\omega} \varphi(\xi) \mathcal{R}^h_{u_h}(\omega, d\xi)) = 0$ for every $\varphi \in C_c(I_\omega)$ and every $N > 0$. By letting N tend to infinity we conclude that $\operatorname{tr} \int_{I_\omega} \varphi(\xi) \mu_{\mathcal{R}}(\omega, d\xi) = 0$, which implies, since φ is arbitrary, $\mu_{\mathcal{R}}(\omega, \cdot) = 0$. $\square$

Since the energy of sequence (u_h) may concentrate on I_ω at scales larger than one, typically the restriction to I_ω of the semiclassical measure of (u_h) is larger than $\operatorname{tr} \mu_{\mathcal{R}}(\omega, \xi)$. This is the content of our next result.

Proposition 8. *Let (u_h) have a semiclassical measure μ_0 and satisfy* (14). *Then, for every non-negative $\varphi \in C_c(\mathbb{R}^d)$ and every $\omega \in \mathbb{W}$,*

$$\operatorname{tr} \int_{I_\omega} \varphi(\xi) \mu_{\mathcal{R}}(\omega, d\xi) \leqslant \int_{T^*\mathbb{T}^d} \varphi(\xi) \mu_0(dx, d\xi). \tag{17}$$

Proof. Consider the projector π_N defined in the proof of Proposition 7. For every $\varphi \in C_c^1(\mathbb{R}^d)$ and every $N > 0$ one has:

$$\left| \operatorname{tr}\left(\pi_N \int_{I_\omega} \varphi(\xi) \mathcal{R}^h_{u_h}(\omega, d\xi) \right) - \sum_{|p_\omega \cdot k| \leqslant N} \varphi(hk) \left| \widehat{u_h}(k) \right|^2 \right| \\ \leqslant hN \|\nabla_\xi \varphi\|_{L^\infty(I_\omega)} \|u_h\|^2_{L^2(\mathbb{T}^d)}.$$

Therefore, if in addition φ is non-negative,

$$\operatorname{tr}\left(\pi_N \int_{I_\omega} \varphi(\xi) \mathcal{R}^h_{u_h}(\omega, d\xi) \right) \leqslant \sum_{k \in \mathbb{Z}^d} \varphi(hk) \left| \widehat{u_h}(k) \right|^2 + \mathcal{O}(h);$$

taking limits as $h \to 0^+$ we get, for every $N > 0$:

$$\operatorname{tr}\left(\pi_N \int_{I_\omega} \varphi(\xi) \mu_{\mathcal{R}}(\omega, d\xi) \right) \leqslant \int_{T^*\mathbb{T}^d} \varphi(\xi) \mu_0(dx, d\xi).$$

Letting $N \to \infty$ and using the density of $C_c^1(\mathbb{R}^d)$ in $C_c(\mathbb{R}^d)$ we conclude the proof of the proposition. $\square$

3. Resonant Wigner measures and the Schrödinger flow

Before stating our main result, we need some more notation. Given $\omega \in \mathbb{W}$, denote by L_ω the length of γ_ω, equal to $2\pi |p_\omega|$. There is a well-defined extension operator $\mathcal{E}_\omega$ from the space of $L_\omega / |p_\omega|^2 \mathbb{Z}$-periodic functions in $L^1(\gamma_\omega)$ to the space $L^1(\mathbb{T}^d)$. If $f \in L^1(\gamma_\omega)$ is $L_\omega / |p_\omega|^2 \mathbb{Z}$-

periodic then put

$$\mathcal{E}_\omega f(x) := \frac{|p_\omega|}{(2\pi)^{d-1}} f(x \cdot v_\omega), \quad x \in \mathbb{T}^d.$$

With that normalization, it is not difficult to check that $\|\mathcal{E}_\omega f\|_{L^1(\mathbb{T}^d)} = \|f\|_{L^1(\gamma_\omega)}$. Moreover, $\mathcal{E}_\omega f$ is invariant by translations along vectors orthogonal to ω.

The following is a reformulation of Proposition 15 from Appendix A in our setting. Let $\mu_{\mathcal{R}} \in \mathcal{X}_{0,+}(\mathcal{J})$; for $f_\omega \in L^\infty(\gamma_\omega)$ denote by m_{f_ω} the operator in $L^2(\gamma_\omega)$ defined by multiplication by f_ω. The measures $\tilde{\rho}_\omega \in \mathcal{M}(\gamma_\omega \times I_\omega)$, $\omega \in \mathbb{W}$, defined by:

$$\int_{\gamma_\omega \times I_\omega} f_\omega(s)\varphi(\xi)\tilde{\rho}_\omega(ds, d\xi) := \operatorname{tr}\left[m_{f_\omega} \int_{I_\omega} \varphi(\xi)\mu_{\mathcal{R}}(\omega, d\xi)\right],$$

for $f_\omega \in C(\gamma_\omega)$ and $\varphi \in C_c(I_\omega)$ are positive and $\tilde{\rho}_\omega(\cdot, \xi)$ is absolutely continuous with respect to arc-length measure ds in γ_ω. Clearly, $\int_{\gamma_\omega} \tilde{\rho}_\omega(ds, \cdot) = \operatorname{tr} \mu_{\mathcal{R}}(\omega, \cdot)$.

We shall say that $\tilde{\rho}_\omega$ is the *trace density* of $\mu_{\mathcal{R}}(\omega, \cdot)$. Let ∂_ω^2 denote the Laplacian in γ_ω (with respect to the arc-length metric, that is, when γ_ω is identified to $\mathbb{R}/(2\pi|p_\omega|\mathbb{Z})$).

The next result complements Theorem 2.

Theorem 9. *Let (u_h) satisfy the hypotheses of Theorem 2. Then the measures $\rho_\omega^t \in \mathcal{M}(\mathbb{T}^d \times I_\omega)$, $\omega \in \mathbb{W}$, $t \in \mathbb{R}$, appearing in formula (10) are uniquely determined by the initial data (u_h) as follows. Let $\mu_{\mathcal{R}}^0 \in \mathcal{X}_{0,+}(\mathcal{J})$ be a resonant Wigner measure corresponding to (u_h). Let $\mu_{\mathcal{R}}^t \in \mathcal{X}_{0,+}(\mathcal{J})$, $t \in \mathbb{R}$, solve the density matrix Schrödinger equation:*

$$\begin{cases} i\partial_t \mu_{\mathcal{R}}^t(\omega, \xi) = \left[-\frac{1}{2}\partial_\omega^2, \mu_{\mathcal{R}}^t(\omega, \xi)\right], \\ \mu_{\mathcal{R}}^t\big|_{t=0}(\omega, \xi) = \mu_{\mathcal{R}}^0(\omega, \xi). \end{cases} \tag{18}$$

Let $\tilde{\rho}_\omega^t$ be the trace density of $\mu_{\mathcal{R}}^t(\omega, \cdot)$. Then

$$\rho_\omega^t(\cdot, \xi) = \mathcal{E}_\omega\big(\tilde{\rho}_\omega^t(\cdot, \xi) - \operatorname{tr} \mu_{\mathcal{R}}^t(\omega, \xi)\big).$$

In particular, each $\rho_\omega^t(\cdot, \xi)$ is absolutely continuous with respect to Lebesgue measure in $\mathbb{T}^d$, has zero total mass and is invariant under the geodesic flow.

Before giving the proof of Theorems 2 and 9, we shall first need some preparatory lemmas.

Let $a \in C_c^\infty(T^*\mathbb{T}^d)$ and write it as $a(x, \xi) = \sum_{k \in \mathbb{Z}^d} a_k(\xi)\psi_k(x)$. Given $\varphi \in \mathcal{S}(\mathbb{R})$ define for every $\omega \in \mathbb{W}$ the operator-valued function:

$$k_{a,\varphi}(\omega, \xi) := \frac{1}{(2\pi)^{d/2}} \sum_{[c] \in \mathbb{Z}_{|p_\omega|^2}} \sum_{\substack{n,m \in [c] \\ m \neq n}} \widehat{\varphi}\left(\frac{n^2 - m^2}{2|p_\omega|^2}\right) a_{\frac{m-n}{|p_\omega|} v_\omega}(\xi) \phi_m^\omega \otimes \overline{\phi_n^\omega}. \tag{19}$$

Lemma 10. *For every $(\omega,\xi)\in\mathcal{J}$, the operator $k_{a,\varphi}(\omega,\xi)$ is a Hilbert–Schmidt operator on $L^2(\gamma_\omega)$. Moreover,*

$$\sum_{\omega\in\mathbb{W}}\left\|k_{a,\varphi}(\omega,\xi)\right\|^2_{\mathcal{L}^2(L^2(\gamma_\omega))}\leqslant\frac{1}{(2\pi)^d}\sum_{n\in\mathbb{Z}^d}\left|\widehat{\varphi}\left(\frac{n}{2}\right)\right|^2\left\|a(\cdot,\xi)-a_0(\xi)\psi_0\right\|^2_{L^2(\mathbb{T}^d)}. \tag{20}$$

In particular, $k_{a,\varphi}\in\mathcal{X}_0(\mathcal{J})$.

Proof. A direct computation of the Hilbert–Schmidt norm gives:

$$\begin{aligned}\left\|k_{a,\varphi}(\omega,\xi)\right\|^2_{\mathcal{L}^2(L^2(\gamma_\omega))}&=\frac{1}{(2\pi)^d}\sum_{l\in\mathbb{Z}\setminus\{0\}}\left|a_{lp_\omega}(\xi)\right|^2\sum_{n\in\mathbb{Z}}\left|\widehat{\varphi}\left(l\left(\frac{l|p_\omega|^2}{2}+n\right)\right)\right|^2\\&\leqslant\frac{1}{(2\pi)^d}\sum_{l\in\mathbb{Z}\setminus\{0\}}\left|a_{lp_\omega}(\xi)\right|^2\sum_{n\in\mathbb{Z}}\left|\widehat{\varphi}\left(\frac{n}{2}\right)\right|^2,\end{aligned}$$

since, for every fixed $l\neq0$ the map that associates $n\in\mathbb{Z}$ to $l(l|p_\omega|^2+2n)\in\mathbb{Z}$ is injective. Therefore, summing in ω gives the estimate (20). □

Lemma 11. *Let (u_h) be a bounded sequence in $L^2(\mathbb{T}^d)$ such that (7) holds. Then, given $a\in C_c^\infty(T^*\mathbb{T}^d)$ such that $\int_{\mathbb{T}^d}a(x,\cdot)\,dx=0$ and $\varphi\in\mathcal{S}(\mathbb{R})$, we have:*

$$\left|\int_{\mathbb{R}}\varphi(t)\left\langle w^h_{u_h}(t,\cdot),a\right\rangle dt-\sum_{\omega\in\mathbb{W}}\operatorname{tr}\left(\int_{I_\omega}k_{a,\varphi}(\omega,\xi)\mathcal{R}^h_{u_h}(\omega,d\xi)\right)\right|\leqslant C_{a,\varphi}h,$$

for some constant $C_{a,\varphi}>0$.

Proof. Let $\varphi\in\mathcal{S}(\mathbb{R})$ and take $a\in C_c^\infty(T^*\mathbb{T}^d)$ such that $a_0\equiv0$; from formula (3) we deduce:

$$\int_{\mathbb{R}}\varphi(t)\left\langle w^h_{u_h}(t,\cdot),a\right\rangle dt=\frac{1}{(2\pi)^{d/2}}\sum_{k,j\in\mathbb{Z}^d}\widehat{\varphi}\left(\frac{|k|^2-|j|^2}{2}\right)a_{j-k}\left(h\frac{k+j}{2}\right)\widehat{u_h}(k)\overline{\widehat{u_h}(j)}.$$

This expression can be written as:

$$\frac{1}{(2\pi)^{d/2}}\sum_{\omega\in\mathbb{W}}\sum_{\substack{k,j\in\mathbb{Z}^d\\k-j\in\omega}}\widehat{\varphi}\left(\frac{|k|^2-|j|^2}{2}\right)a_{j-k}\left(h\frac{k+j}{2}\right)\widehat{u_h}(k)\overline{\widehat{u_h}(j)}, \tag{21}$$

since, recall $a_0\equiv0$, and, by definition, the lines $\omega\in\mathbb{W}$ do not contain the origin. Now, for $\omega\in\mathbb{W}$ fixed, the sum in $k-j\in\omega$ in (21) may be rewritten in terms of the parametrization introduced in Proposition 4 to give:

$$\sum_{\substack{[c]\in\mathbb{Z}_{|p_\omega|^2}\\(m,n,r)\in\mathcal{C}_{\omega,[c]}}} \widehat{\varphi}\left(\frac{m^2-n^2}{2|p_\omega|^2}\right) a_{\frac{(n-m)}{|p_\omega|}\nu_\omega}\left(h\frac{m+n}{2|p_\omega|}\nu_\omega + hr\right)\widehat{u_h}\left(\frac{m}{|p_\omega|}\nu_\omega + r\right)\overline{\widehat{u_h}\left(\frac{n}{|p_\omega|}\nu_\omega + r\right)}, \tag{22}$$

where, for $\omega \in \mathbb{W}$ and $[c] \in \mathbb{Z}_{|p_\omega|^2}$ we have set:

$$\mathcal{C}_{\omega,[c]} := \left\{(m,n,r)\colon\ m,n\in[c],\ m\neq n,\ r\in\omega_c^\perp\right\}.$$

Notice that the reason for (22) to hold is that the condition $k - j \in \omega$ results in the fact that k, j can be written as $k = \frac{m}{|p_\omega|}\nu_\omega + r$ and $j = \frac{n}{|p_\omega|}\nu_\omega + r$ for a unique $(m,n,r) \in \bigcup_{[c]\in\mathbb{Z}_{|p_\omega|^2}} \mathcal{C}_{\omega,[c]}$. Comparing (22) with the expression (12) defining the resonant Wigner distribution of u_h we find that:

$$\int_{\mathbb{R}} \varphi(t)\langle w_{u_h}^h(t,\cdot), a\rangle dt = \sum_{\omega\in\mathbb{W}} \mathrm{tr}\left(\int_{I_\omega} k_{a,\varphi}(\omega,\xi)\mathcal{R}_{u_h}^h(\omega,d\xi)\right) + \sum_{\omega\in\mathbb{W}} \mathrm{tr}\left(\int_{I_\omega} r_{a,\varphi}(\omega,\xi)\mathcal{R}_{u_h}^h(\omega,d\xi)\right),$$

where:

$$r_{a,\varphi}(\omega,\xi) := \frac{1}{(2\pi)^{d/2}} \sum_{\substack{[c]\in\mathbb{Z}_{|p_\omega|^2}\\ m,n\in[c]}} \widehat{\varphi}\left(\frac{n^2-m^2}{2|p_\omega|^2}\right) l(m,n,\omega,\xi)\phi_m^\omega \otimes \overline{\phi_n^\omega},$$

with

$$l(m,n,\omega,\xi) := a_{\frac{m-n}{|p_\omega|}\nu_\omega}\left(h\frac{m+n}{2|p_\omega|}\nu_\omega + \xi\right) - a_{\frac{m-n}{|p_\omega|}\nu_\omega}(\xi).$$

Let us estimate the remainder term. First note that:

$$\sup_{\xi\in I_\omega}\left|l(m,n,\omega,\xi)\right| \leqslant \frac{h}{|p_\omega|}\left|\frac{m+n}{2}\right| \sup_{\xi\in I_\omega}\left|\nabla_\xi a_{\frac{n-m}{|p_\omega|}\nu_\omega}(\xi)\right|. \tag{23}$$

Proceeding as in the proof of Lemma 10, we use (23) to estimate:

$$\left\| r_{a,\varphi}(\omega,\xi)\right\|^2_{\mathcal{L}^2(L^2(\gamma_\omega))} \leqslant \frac{h^2}{(2\pi)^d}\sum_{n\in\mathbb{Z}}\left|\frac{n}{2}\widehat{\varphi}\left(\frac{n}{2}\right)\right|^2 \sum_{l\in\mathbb{Z}\setminus\{0\}} \frac{\sup_{\xi\in I_\omega}|\nabla_\xi a_{lp_\omega}(\xi)|^2}{l^2|p_\omega|^2}.$$

Since $a \in C_c^\infty(T^*\mathbb{T}^d)$ and $a_0 \equiv 0$,

$$\sum_{\omega \in \mathbb{W}} \Bigg(\sum_{l \in \mathbb{Z}\setminus\{0\}} l^{-2} |p_\omega|^{-2} \sup_{\xi \in I_\omega} \big| \nabla_\xi a_{l p_\omega}(\xi) \big|^2 \Bigg)^{1/2} \quad \text{is finite.}$$

Therefore:

$$\Bigg| \sum_{\omega \in \mathbb{W}} \operatorname{tr}\Bigg(\int_{I_\omega} r_{a,\varphi}(\omega,\xi) \mathcal{R}^h_{u_h}(\omega, d\xi) \Bigg) \Bigg| \leqslant C_{a,\varphi} \|u_h\|^2_{L^2(\mathbb{T}^d)} h,$$

and the result follows. $\square$

Proof of Theorems 2 and 9. Suppose that $(\mathcal{R}^{h'}_{u_{h'}})$ converges along some subsequence $(u_{h'})$ to the resonant Wigner measure $\mu^0_{\mathcal{R}}$ (this is the case, by Proposition 6). Let $a \in C_c^\infty(T^*\mathbb{T}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R})$; in view of estimate (20) we have that:

$$\lim_{h' \to 0^+} \sum_{\omega \in \mathbb{W}} \operatorname{tr}\Bigg(\int_{I_\omega} k_{a,\varphi}(\omega,\xi) \mathcal{R}^{h'}_{u_{h'}}(\omega, d\xi) \Bigg) = \sum_{\omega \in \mathbb{W}} \operatorname{tr}\Bigg(\int_{I_\omega} k_{a,\varphi}(\omega,\xi) \mu^0_{\mathcal{R}}(\omega, d\xi) \Bigg).$$

Applying Lemma 11 we obtain:

$$\begin{aligned} &\int_{\mathbb{R}} \int_{T^*\mathbb{T}^d} \varphi(t) a(x,\xi) \mu(t, dx, d\xi) \\ &\quad = \sum_{\omega \in \mathbb{W}} \operatorname{tr} \int_{I_\omega} k_{a,\varphi}(\omega,\xi) \mu^0_{\mathcal{R}}(\omega, d\xi) + \int_{T^*\mathbb{T}^d} \bar{a}(\xi) \mu_0(dx, d\xi), \end{aligned} \tag{24}$$

where $\bar{a}(\xi) := (2\pi)^{-d} \int_{\mathbb{T}^d} a(x,\xi)\, dx$. Therefore, it only remains to identify the term involving the resonant Wigner measure $\mu^0_{\mathcal{R}}$.

Simple inspection gives:

$$\operatorname{tr} \int_{I_\omega} k_{a,\varphi}(\omega,\xi) \mu^0_{\mathcal{R}}(\omega, d\xi) = \int_{\mathbb{R}} \varphi(t) \operatorname{tr} \int_{I_\omega} p_a(\omega,\xi) e^{it\partial^2_\omega/2} \mu^0_{\mathcal{R}}(\omega, d\xi) e^{-it\partial^2_\omega/2}, \tag{25}$$

where ∂^2_ω denotes the Laplacian on $L^2(\gamma_\omega)$ and

$$p_a(\omega,\xi) := \frac{1}{(2\pi)^{d/2}} \sum_{[c] \in \mathbb{Z}_{|p_\omega|}} \sum_{\substack{n,m \in [c] \\ m \neq n}} a_{\frac{m-n}{|p_\omega|}\nu_\omega}(\xi) \phi^\omega_m \otimes \overline{\phi^\omega_n}. \tag{26}$$

Note that

$$\mu^t_{\mathcal{R}}(\omega,\xi) := e^{it\partial^2_\omega/2} \mu^0_{\mathcal{R}}(\omega,\xi) e^{-it\partial^2_\omega/2}$$

solves (18). For $m \equiv n \pmod{|p_\omega|^2}$, define the measures $\mu^t_{\mathcal{R}}(\omega,\xi)(m,n)$:

$$\int_{I_\omega} \varphi(\xi)\mu^t_{\mathcal{R}}(\omega,d\xi)(m,n) := \left(\int_{I_\omega} \varphi(\xi)\mu^t_{\mathcal{R}}(\omega,d\xi)\phi^\omega_n \middle| \phi^\omega_m\right)_{L^2(\gamma_\omega)}, \quad \varphi \in C_c(I_\omega), \tag{27}$$

so

$$\mu^t_{\mathcal{R}}(\omega,\xi) = \sum_{m\equiv n \pmod{|p_\omega|^2}} \mu^t_{\mathcal{R}}(\omega,\xi)(m,n)\phi^\omega_m \otimes \overline{\phi^\omega_n}.$$

Given $k \in \mathbb{Z}^d$, let $\omega \in \mathbb{W}$ be the unique resonant direction such that $k \in \omega$. In view of (24)–(26), we have that, for a.e. $t \in \mathbb{R}$ the kth Fourier coefficient of $\mu(t,\cdot)$ is given by:

$$\int_{\mathbb{T}^d} \overline{\psi_k(x)}\mu(t,dx,\xi) = \frac{1}{(2\pi)^{d/2}} \sum_{m-n=k\cdot p_\omega} \mu^t_{\mathcal{R}}(\omega,\xi)(m,n).$$

The trace density of $\mu^t_{\mathcal{R}}(\omega,\xi)$ is precisely the measure

$$\tilde{\rho}^t_\omega(\cdot,\xi) := \frac{1}{\sqrt{2\pi|p_\omega|}} \sum_{k\in\omega\cap\mathbb{Z}^d} \sum_{m-n=k\cdot p_\omega} \mu^t_{\mathcal{R}}(\omega,\xi)(m,n)\phi^\omega_{k\cdot p_\omega} + \frac{1}{2\pi|p_\omega|}\operatorname{tr}\mu^t_{\mathcal{R}}(\omega,\xi).$$

As, for $k \in \omega \cap \mathbb{Z}^d$ one has that $\phi^\omega_{k\cdot p_\omega}$ is $L_\omega/|p_\omega|^2$-periodic and

$$\mathcal{E}_\omega \phi^\omega_{k\cdot p_\omega} = \sqrt{\frac{|p_\omega|}{(2\pi)^{d-1}}}\psi_k,$$

we find that

$$\frac{1}{(2\pi)^{d/2}} \sum_{k\in\omega\cap\mathbb{Z}^d} \sum_{m-n=k\cdot p_\omega} \mu^t_{\mathcal{R}}(\omega,\xi)(m,n)\psi_k = \mathcal{E}_\omega\big(\tilde{\rho}^t_\omega(\cdot,\xi) - \operatorname{tr}\mu^t_{\mathcal{R}}(\omega,\xi)\big) = \rho^t_\omega(\cdot,\xi),$$

and identity (10) follows. □

The proof of Corollary 3 is an easy consequence of formula (10) and the properties of resonant Wigner distributions.

Proof of Corollary 3. The weak form of Egorov's theorem proved in [17], Theorem 2(ii), gives in this case:

$$\int_{T^*\mathbb{T}^2} b(\xi)\mu(t,dx,d\xi) = \int_{T^*\mathbb{T}^2} b(\xi)\mu_0(dx,d\xi), \tag{28}$$

for every $b \in C_c(\mathbb{R}^2)$ and a.e. $t \in \mathbb{R}$ (this can also be directly deduced from Eq. (6)). Identity (28), together with our hypothesis $\mu_0(\{\xi = 0\}) = 0$ implies $\mu(t,\{\xi = 0\}) = 0$ for a.e. $t \in \mathbb{R}$. Notice that since $d=2$ the lines I_ω only intersect at the origin. As a consequence of this, we deduce that

$$\mu(t,\cdot)=\sum_{\omega\in\mathbb{W}}\mu(t,\cdot)\rceil_{\mathbb{T}^2\times I_\omega}+\mu(t,\cdot)\rceil_{\mathbb{T}^2\times(\mathbb{R}^2\setminus\Omega)}. \tag{29}$$

Let $\nu_0:=(2\pi)^{-2}\int_{\mathbb{T}^2}\mu_0(dy,\cdot)$, then we obtain from formula (10) the following expressions:

$$\mu(t,\cdot)\rceil_{\mathbb{T}^2\times I_\omega}=\rho_\omega^t+\nu_0\rceil_{I_\omega},\qquad \mu(t,\cdot)\rceil_{\mathbb{T}^2\times(\mathbb{R}^2\setminus\Omega)}=\nu_0\rceil_{\mathbb{R}^d\setminus\Omega}.$$

Recall that all the measures involved in the right-hand side of Eq. (29) are mutually disjoint. In particular, the fact that $\mu(t,\cdot)\geqslant 0$ for a.e. $t\in\mathbb{R}$ implies that $\rho_\omega^t+\nu_0\rceil_{I_\omega}\geqslant 0$ as well. Theorem 2 shows that the projection on $\mathbb{T}^2$ of every $\rho_\omega^t+\nu_0\rceil_{I_\omega}$ is absolutely continuous with respect to the Lebesgue measure. Therefore, the monotone convergence theorem ensures that $\int_{\mathbb{R}^2}\mu(t,\cdot,d\xi)$ is also absolutely continuous with respect to the Lebesgue measure for a.e. $t\in\mathbb{R}$. The result then follows applying identity (8). $\square$

4. Additional properties and examples

The following is a direct consequence of Proposition 7 and Theorem 9.

Proposition 12. *Let (u_h) be an h-oscillating sequence such that* (7) *holds. If in addition, one has*

$$\lim_{h\to 0^+}\sum_{|k\cdot p_\omega|<N}\left|\widehat{u_h}(k)\right|^2=0,\quad \textit{for every } N>0,$$

for some $\omega\in\mathbb{W}$ then the corresponding term ρ_ω^t in (10) *vanishes identically.*

As an example, we shall apply Proposition 12 to analyse the propagation of wave-packet type solutions to the Schrödinger equation in this context. Given $(x_0,\xi_0)\in T^*\mathbb{T}^d$ and $\rho\in C_c^\infty((-\pi,\pi)^d)$ define $c^h_{(x_0,\xi_0)}\in L^2(\mathbb{T}^d)$ to be the $2\pi\mathbb{Z}^d$-periodization of the function:

$$u_h(x):=\frac{1}{h^{d/4}}\rho\left(\frac{x-x_0}{\sqrt{h}}\right)e^{i\xi_0/h\cdot x}.$$

The Poisson summation formula ensures that the Fourier coefficients of $c^h_{(x_0,\xi_0)}$ are:

$$\widehat{c^h_{(x_0,\xi_0)}}(k)=(2\pi)^{d/2}h^{d/4}\widehat{\rho}\big(\sqrt{h}(k-\xi_0/h)\big)e^{-i(k-\xi_0/h)\cdot x_0}.$$

It is not difficult to prove that

$$w^h_{c^h_{(x_0,\xi_0)}}(0,\cdot)\rightharpoonup\|\rho\|^2_{L^2(\mathbb{R}^d)}\delta_{x_0}\otimes\delta_{\xi_0},\quad \text{as } h\to 0^+.$$

Proposition 13. *Let μ be the semiclassical measure given by the limit* (7) *corresponding to the initial data $c^h_{(x_0,\xi_0)}$. Then, for almost every $t\in\mathbb{R}$,*

$$\mu(t,x,\xi)=\|\rho\|^2_{L^2(\mathbb{R}^d)}\,dx\,\delta_{\xi_0}(\xi).$$

Proof. Let $\omega \in \mathbb{W}$; if $\xi_0 \notin I_\omega$ then $\rho^t_\omega = 0$; therefore we may suppose that $\xi_0 \in I_\omega$. The conclusion will follow as soon as we show that $\rho^t_\omega = 0$ also holds in this case. Take a function $\chi \in C^\infty_c((-2,2))$, identically equal to one in $[-1,1]$ and taking values between 0 and 1. Let $N > 0$ and write $\chi_N(\xi) := \chi(p_\omega \cdot \xi / N)$. Then

$$\sum_{|k\cdot p_\omega| < N} \left|\widehat{c^h_{(x_0,\xi_0)}}(k)\right|^2 \leqslant (2\pi)^d h^{d/2} \sum_{k\in\mathbb{Z}^d} \left|\chi_N(k)\widehat{\rho}(\sqrt{h}k - \xi_0/\sqrt{h})\right|^2.$$

Applying the Poisson summation formula, we find that the right-hand side of the above inequality equals $\|v_h\|^2_{L^2(\mathbb{T}^d)}$, where v_h stands for the $2\pi\mathbb{Z}^d$-periodization of $\chi_N(D_x)u_h$. Since this function is in $\mathcal{S}(\mathbb{R}^d)$, we have, for every $s > d$ an estimate:

$$\|v_h\|^2_{L^2(\mathbb{T}^d)} \leqslant C_s \int_{\mathbb{R}^d} \left|\chi_N(D_x)u_h(x)\right|^2 \left(1 + |x|^2\right)^{s/2} dx.$$

Applying Plancherel's identity, we get, after changing variables and taking into account that $\xi_0 \cdot p_\omega = 0$,

$$\int_{\mathbb{R}^d} \left|\chi_N(D_x)u_h(x)\right|^2 \left(1 + |x|^2\right)^{s/2} dx = \int_{\mathbb{R}^d} \left|(1 - h\Delta_\xi)^{s/4} w_h(\xi)\right|^2 \frac{d\xi}{(2\pi)^d},$$

where $w_h(\xi) := \chi_N(\xi/\sqrt{h})\widehat{\rho}(\xi)$. The functions $(1 - h\Delta_\xi)^{s/4} w_h$ are uniformly bounded in $\mathcal{S}(\mathbb{R}^d)$, and supported on the strips $S_h := \{\xi \colon |\xi \cdot p_\omega| \leqslant 2\sqrt{h}N\}$. Taking, for instance, $s/4 \in \mathbb{N}$, we find that, for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that:

$$\int_{\mathbb{R}^d} \left|(1 - h\Delta_\xi)^{s/4} w_h(\xi)\right|^2 \frac{d\xi}{(2\pi)^d} \leqslant C\left|S_h \cap B(0; R_\varepsilon)\right| + \varepsilon.$$

This expression tends to ε as $h \to 0^+$. Therefore, as ε is arbitrary, Proposition 12 ensures that $\rho^t_\omega = 0$, and the conclusion follows. □

If instead we consider the purely oscillating profiles $u_h(x)$ defined as the periodizations of:

$$\rho(x)e^{i\xi_0/h\cdot x},$$

with $\xi_0 \in \Omega$ a simple resonance (*i.e.* such that $\lambda\xi_0 \in \mathbb{Z}^d$ for some $\lambda \in \mathbb{R} \setminus \{0\}$) we find that some of the terms ρ^t_ω are non-zero. More precisely, write ρ_{per} to denote the periodization of ρ and set, for $f \in L^1(\mathbb{T}^d)$:

$$\langle f\rangle_{\xi_0}(x) := \frac{|\xi_0|}{L} \int_0^{L/|\xi_0|} f(x + s\xi_0)\, ds,$$

where L denotes the length of the periodic geodesic issued from (x, ξ_0). We have the following.

Proposition 14. *Let $\xi_0 \in \Omega \setminus \{0\}$ be a simple resonance. The semiclassical measure μ given by* (7) *corresponding to the initial data u_{h_n}, where for simplicity we have taken $\xi_0/h_n \in \mathbb{Z}^d$, is given by:*

$$\mu(t,x,\xi) = \langle |e^{it\Delta_x/2}\rho_{\mathrm{per}}|^2\rangle_{\xi_0}(x)\,dx\,\delta_{\xi_0}(\xi).$$

Proof. It is easy to check that the Wigner measure corresponding to the sequence of initial data (u_{h_n}) is precisely:

$$\mu_0(x,\xi) = |\rho_{\mathrm{per}}(x)|^2\,dx\,\delta_{\xi_0}(\xi).$$

Therefore, in view of (17), the resonant Wigner measure of (u_{h_n}) satisfies $\mu_{\mathcal{R}}(\omega,\cdot) \equiv 0$, whenever $\xi_0 \notin I_\omega$. Let us compute the measures $\mu_{\mathcal{R}}(\omega,\cdot)$ when $\xi_0 \in I_\omega$. Start noticing that the Poisson summation formula gives:

$$u_h = \sum_{k\in\mathbb{Z}^d} \widehat{\rho}(k - \xi_0/h_n)\psi_k;$$

hence, for $b \in C^\infty(I_\omega)$:

$$\int_{I_\omega} b(\xi)\mathcal{R}^{h_n}_{u_{h_n}}(\omega, d\xi)$$

$$= \sum_{[c]\in\mathbb{Z}_{|p_\omega|^2}} \sum_{\substack{m,n\in[c]\\ r\in\omega_c^\perp}} \widehat{\rho}\left(\frac{m}{|p_\omega|}v_\omega + r - \frac{\xi_0}{h_n}\right)\overline{\widehat{\rho}\left(\frac{n}{|p_\omega|}v_\omega + r - \frac{\xi_0}{h_n}\right)}b(hr)\phi^\omega_m \otimes \overline{\phi^\omega_n}.$$

Since $\xi_0/h_n \in I_\omega \cap \mathbb{Z}^d$, each of the summands in m and n can be rewritten as:

$$\sum_{\substack{k,j\in\mathbb{Z}^d, k-j=\lambda p_\omega\\ k\cdot p_\omega=m,\, j\cdot p_\omega=n}} \widehat{\rho}(k)\overline{\widehat{\rho}(j)}b(hk^\perp + \xi_0)\phi^\omega_m \otimes \overline{\phi^\omega_n},$$

for some $\lambda \in \mathbb{Z}$ and where $k^\perp$ denotes the projection of k onto I_ω. As $h_n \to 0$ this converges to:

$$\sum_{\substack{k,j\in\mathbb{Z}^d, k-j=\lambda p_\omega\\ k\cdot p_\omega=m,\, j\cdot p_\omega=n}} \widehat{\rho}(k)\overline{\widehat{\rho}(j)}b(\xi_0)\phi^\omega_m \otimes \overline{\phi^\omega_n}.$$

Therefore,

$$\mu_{\mathcal{R}}(\omega,\xi) = \sum_{\lambda\in\mathbb{Z}} \sum_{\substack{k,j\in\mathbb{Z}^d, k-j=\lambda p_\omega\\ k\cdot p_\omega=m,\, j\cdot p_\omega=n}} \widehat{\rho}(k)\overline{\widehat{\rho}(j)}\phi^\omega_m \otimes \overline{\phi^\omega_n}\delta_{\xi_0}(\xi).$$

The conclusion then follows using the formula defining ρ^t_ω. □

Appendix A. Operator-valued measures

Let H be a separable Hilbert space, we denote by $\mathcal{L}(H)$, $\mathcal{K}(H)$, and $\mathcal{L}^1(H)$ the spaces of bounded, compact and trace-class operators on H, respectively. If $A \in \mathcal{L}^1(H)$, $\operatorname{tr} A$ denotes the trace of A; $\|A\|_{\mathcal{L}^1(H)} := \operatorname{tr}|A|$ defines a norm on $\mathcal{L}^1(H)$. With this norm, $\mathcal{L}^1(H)$ is the dual of $\mathcal{K}(H)$, the duality being $\operatorname{tr}(AB)$.

When is X a locally compact, σ-compact, Hausdorff metric space, the space $\mathcal{M}(X; \mathcal{L}^1(H))$ of trace-operator-valued Radon measures on X consists of linear operators $\mu : C_c(X) \to \mathcal{L}^1(H)$ bounded in the following sense: given $K \subset X$ compact there exists $C_K > 0$ such that for every $\varphi \in C_c(K)$,

$$\big\|\langle \mu, \varphi\rangle\big\|_{\mathcal{L}^1(H)} \leqslant C_K \sup_{x \in K}\big|\varphi(x)\big|.$$

Note that $\mathcal{M}(X; \mathcal{L}^1(H))$ is the dual of $C_c(X; \mathcal{K}(H))$, the space of compactly supported functions from X into $\mathcal{K}(H)$.

An element $\mu \in \mathcal{M}(X; \mathcal{L}^1(H))$ is positive if for every non-negative $\varphi \in C_c(X)$ the operator $\langle \mu, \varphi\rangle$ is Hermitian and positive. The set of such positive elements is denoted by $\mathcal{M}_+(X; \mathcal{L}^1(H))$. Given a positive measure μ on defines the scalar valued positive measure $\operatorname{tr}\mu$ as

$$\langle \operatorname{tr}\mu, \varphi\rangle := \operatorname{tr}\langle \mu, \varphi\rangle, \quad \text{for } \varphi \in C_c\big(\mathbb{R}^d\big).$$

We refer the reader to [11] for a clear presentation of operator-valued measures, as well as a proof of a Radon–Nykodim theorem in this context.

When $H = L^2(T, \nu)$, where T is locally compact, σ-compact, Hausdorff metric space equipped with a Radon measure ν, then the operators $\langle \mu, b\rangle$ may be represented by their integral kernels $k_b \in L^2(T \times T)$. Thus, μ can be viewed as an $L^2(T \times T)$-valued measure.

Given $f \in C_c(T \times X)$, we denote by $m_f(x)$ the operator acting on $L^2(T, \nu)$ by multiplication by $f(\cdot, x)$. Clearly, $m_f \in C_c(X; \mathcal{L}(L^2(T)))$. The following construction is used in the proof of Theorems 2 and 9.

Proposition 15. *Let $H = L^2(T, \nu)$ and $\mu \in \mathcal{M}_+(X; \mathcal{L}^1(H))$. The linear operator*

$$\rho_\mu : C_c(T \times X) \to \mathbb{R} : g \mapsto \operatorname{tr}\langle \mu, m_g\rangle$$

extends to a positive Radon measure on $T \times X$, which is finite if μ is. Moreover, for every $\varphi \in C_c(X)$,

$$\int_X \varphi(x)\rho_\mu(\cdot, dx) \in L^1(T, \nu). \tag{30}$$

In addition

$$\rho_\mu = 0 \quad \textit{if and only if} \quad \mu = 0.$$

Proof. Given compact sets $S \subset T$ and $K \subset X$ and functions $f \in C_c(S)$ and $\varphi \in C_c(K)$ we have, as $\mathcal{L}^1(H)$ is an ideal in $\mathcal{L}(H)$:

$$\left|\operatorname{tr}\left(m_f \langle \mu, \varphi \rangle\right)\right| \leqslant \|m_f\|_{\mathcal{L}(H)} \operatorname{tr}\left|\langle \mu, \varphi \rangle\right| \leqslant C_K \|f\|_{L^\infty(T,\nu)} \sup_{x \in K} \left|\varphi(x)\right|$$

for some constant $C_K > 0$. Therefore, as $C_c(T) \otimes C_c(X)$ is dense in $C_c(T \times X)$, the functional ρ_μ extends to a measure on $T \times X$. The positivity of ρ_μ follows easily from the properties of the trace of linear operators. Let $\varphi \in C_c(X)$ and write

$$\langle \mu, \varphi \rangle = \sum_{j=1}^{\infty} \lambda_j \phi_j \otimes \overline{\phi_j}$$

where λ_j are the eigenvalues of $\langle \mu, \varphi \rangle$, the ϕ_j, $j \in \mathbb{N}$, form an orthonormal basis of H consisting of eigenvectors, and $\phi_j \otimes \overline{\phi_j}$ is the projection on the linear span of ϕ_j. Now,

$$\int_{T \times X} f(t)\varphi(x)\rho_\mu(dt, dx) = \operatorname{tr}\left(m_f \sum_{j=1}^{\infty} \lambda_j \phi_j \otimes \overline{\phi_j}\right) = \sum_{j=0}^{\infty} \lambda_j \int_T f|\phi_j|^2 \, d\nu. \tag{31}$$

Since $\langle \mu, \varphi \rangle$ is trace class, $\sum_{j=0}^{\infty} \lambda_j$ is absolutely convergent; therefore,

$$\int_X \varphi(x)\rho_\mu(\cdot, dx) = \sum_{j=0}^{\infty} \lambda_j |\phi_j|^2$$

is in $L^1(T)$ as we wanted to show. Finally, to see that $\rho_\mu = 0$ implies $\mu = 0$ simply take f and φ non-negative in formula (31). As the left-hand side is zero, all λ_j must vanish, that is $\langle \mu, \varphi \rangle = 0$. Since φ is arbitrary, we must have $\mu = 0$. □

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References

[1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, I, Schrödinger equations, Geom. Funct. Anal. 3 (2) (1993) 107–156.

[2] J. Bourgain, Eigenfunction bounds for the Laplacian on the n-torus, Int. Math. Res. Not. 3 (1993) 61–66.

[3] J. Bourgain, Global Solutions of Nonlinear Schrödinger Equations, Amer. Math. Soc. Colloq. Publ., vol. 46, American Mathematical Society, Providence, RI, 1999.

[4] N. Burq, P. Gérard, N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, Amer. J. Math. 126 (3) (2004) 569–605.

[5] N. Burq, P. Gérard, N. Tzvetkov, Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces, Invent. Math. 159 (1) (2005) 187–223.
[6] N. Burq, P. Gérard, N. Tzvetkov, Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations, Ann. Sci. École Norm. Sup. (4) 38 (2) (2005) 255–301.
[7] C. Fermanian-Kammerer, Mesures semi-classiques 2-microlocales, C. R. Math. Acad. Sci. Paris Sér. I 331 (7) (2000) 515–518.
[8] C. Fermanian-Kammerer, Propagation and absorption of concentration effects near shock hypersurfaces for the heat equation, Asymptot. Anal. 24 (2) (2000) 107–141.
[9] C. Fermanian-Kammerer, P. Gérard, Mesures semi-classiques et croisement de modes, Bull. Soc. Math. France 130 (1) (2002) 123–168.
[10] G.B. Folland, Harmonic Analysis in Phase Space, Ann. of Math. Stud., vol. 122, Princeton University Press, Princeton, NJ, 1989.
[11] P. Gérard, Microlocal defect measures, Comm. Partial Differential Equations 16 (11) (1991) 1761–1794.
[12] P. Gérard, Mesures semi-classiques et ondes de Bloch, Séminaire sur les Équations aux Dérivées Partielles, 1990–1991, Exp. No. XVI, Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 1991.
[13] P. Gérard, E. Leichtnam, Ergodic properties of eigenfunctions for the Dirichlet problem, Duke Math. J. 71 (2) (1993) 559–607.
[14] P. Gérard, P.A. Markowich, N.J. Mauser, F. Poupaud, Homogenization limits and Wigner transforms, Comm. Pure Appl. Math. 50 (4) (1997) 323–379.
[15] D. Jakobson, Quantum limits on flat tori, Ann. of Math. 145 (2) (1997) 235–266.
[16] P.-L. Lions, T. Paul, Sur les mesures de Wigner, Rev. Mat. Iberoamericana 9 (3) (1993) 553–618.
[17] F. Macià, Semiclassical measures and the Schrödinger flow on Riemannian manifolds, Nonlinearity 22 (2009) 1003–1020.
[18] L. Miller, Propagation d'ondes semi-classiques à travers une interface et mesures 2-microlocales, PhD thesis, École Polytechnique, 1996.
[19] F. Nier, A semi-classical picture of quantum scattering, Ann. Sci. École Norm. Sup. (4) 29 (2) (1996) 149–183.
[20] A. Zygmund, On Fourier coefficients and transforms of functions of two variables, Studia Math. 50 (1974) 189–201.