Dominance Measuring Approach using Stochastic Weights

A. Mateos, A. Jiménez, A. Moreno, E.A. Aguayo

Departamento de Inteligencia Artificial, Universidad Politécnica de Madrid (UPM), 28660 Boadilla del Monte, Madrid, Spain
{amateos, ajimenez, amoreno}@fi.upm.es

ABSTRACT. In this paper we propose an approach to obtain a ranking of alternatives in multicriteria decision-making problems when there is imprecision concerning the alternative performances, component utility functions and weights. We assume decision maker's preferences are represented by an additive multi-attribute utility function, in which weights are modeled by independent normal variables, the performance in each attribute for each alternative is an interval value and classes of utility functions are available for each attribute. The approach we propose is based on dominance measures, which are computed in a similar way that when the imprecision concerning weights is modeled by uniform distributions or by an ordinal relation. In this paper we will show how the approach can be applied when the imprecision concerning weights are represented by normal distributions. Extensions to other distributions, such as truncated normal or beta, can be feasible using Monte Carlo simulation techniques.

KEYWORDS. Additive multi-attribute utility function, imprecise weights, Monte Carlo simulation, dominance measures.

1. Introduction

In multicriteria decision-making, the classical additive multi-attribute utility model is considered to be a valid approach in most practical situations (Keeney and Raiffa, 1976). Considering a set of alternatives $A_1, \ldots, A_m$ and attributes $X_1, \ldots, X_n$ to evaluate them, a utility function $u_j(x_{ij})$ on the performance x_{ij} of alternative A_i under attribute X_j and a set of weights w_j representing the relative importance of each attribute, we have the well-known functional form

$$u(A_i) = \sum_{j=1}^{n} w_j u_j(x_{ij}), \qquad i = 1, \ldots, m. \tag{1}$$

where $u_j(\cdot)$ is the component utility function representing the decision maker's (DM) preferences on the values of attribute X_j. This represents the utility of each alternative A_i, $i = 1, \cdots, m$.

Incorporating imprecision concerning weights and/or component utilities is one way of extending the model to closely describe real situations: the information available is usually less than the information needed to determine the best alternative or strategy (Weber, 1987). This situation where it is not possible to indicate precise values for the parameters and quantities involved is often referred to as *decision-making with imprecise information*, *incomplete information* and *partial information*, together with *incomplete knowledge* or *linear partial information* (Kmietowicz and Pearman, 1984; Kirkwood and Sarin, 1985; Hazen, 1986). This concept is also widespread in the literature concerning multi-attribute utility theory. Sarabando and Dias (Sarabando and Dias, 2009) give a brief overview of approaches proposed by different authors within the multi-attribute utility theory framework to deal with incomplete information.

A more recent approach is to use information about each alternative's intensity dominance, known as *dominance measuring methods*. Ahn and Park (2008) first proposed a dominance measuring method, which computes both dominating and dominated measures from a dominance matrix and then derives a *net dominance*. Net dominance is used as a measure of the strength of preference in the sense that a greater net value is better. In Mateos et al. (2009) and Mateos et al. (2010) two alternative approaches aimed at improving Ahn and Park's methods are proposed and compared. First one considers uniformly distributed intervals to model imprecision concerning weights, whereas the second considers ordinal relations among attribute weights.

In this paper, we extend the methods proposed in Mateos et al. (2009) and Mateos et al. (2010). Instead of using uniform distributions on weight intervals for each attribute (i.e., $w_j \in [w_j^L, w_j^U],\ j = 1, \dots, n$) or ordinal relations among attribute weights (i.e., $w_1 \geq w_2 \geq \cdots \geq w_n$), we assume weights follow independent normal distributions (i.e., $w_j \sim N(\mu_j, \sigma_j^2),\ j = 1, \dots, n$, where μ_j and σ_j^2 are the mean and variance, respectively). Future research lines will consist on carrying out simulations to compare the performance of these models with models described in Mateos et al. (2009).

2. Model specification

First, we explain how partial information is represented in the decision-making problems we want to solve. Second, we show an approach to rank the alternatives based on dominance measures.

2.1. Imprecise Decision-Making Problems

In this paper we consider a decision-making problem with m alternatives, A_i, $i = 1,\dots,m$ and n attributes, X_j, $j = 1,\dots,n$, where incomplete information about input parameters has been incorporated into the decision-making process:

- alternative performances under uncertainty $\left(x_{ij} \in \left[x_{ij}^L, x_{ij}^U\right], i = 1,\dots,m, j = 1,\dots,n\right)$, where x_{ij}^L and x_{ij}^U are the lower and the upper end-points of the uniformly distributed performance interval of the attribute X_j for the alternative A_i, respectively,
- imprecision concerning utility function assessment $\left(u_j(\cdot) \in \left[u_j^L(\cdot), u_j^U(\cdot)\right], j = 1,\dots,n\right)$, where $u_j^L(\cdot)$ and $u_j^U(\cdot)$ are the lower and the upper utility function of the attribute X_j, and
- uncertainty about weights, which is represented by independent normal distributions with means $(\mu_1, \dots, \mu_n)$ and variances $(\sigma_1^2, \cdots, \sigma_n^2)$, i.e., $\left(w_j \sim N(\mu_j, \sigma_j^2), j = 1, \dots, n\right)$.

One possibility described in the literature to deal with imprecision attempts to eliminate inferior alternatives based on the concept of dominance. Given two alternatives A_k and A_l, alternative A_k dominates A_l if $D_{kl} \geq 0$, being D_{kl} the optimum value of the optimization problem:

$$
\begin{aligned}
D_{kl} \quad &= \min \left\{u(A_k) - u(A_l) = \textstyle\sum_j w_j\, u_j(x_{kj}) - \sum_j w_j\, u_j(x_{lj})\right\} \\
\text{s.t.} \quad & w_j \sim N(\mu_j, \sigma_j^2), j = 1, \dots, n \\
& x_{kj}^L \leq x_{kj} \leq x_{kj}^U, j=1,\dots,n \\
& x_{lj}^L \leq x_{lj} \leq x_{lj}^U, j=1,\dots,n \\
& u_j^L(x_{kj}) \leq u_j(x_{kj}) \leq u_j^U(x_{kj}), j=1,\dots,n \\
& u_j^L(x_{lj}) \leq u_j(x_{lj}) \leq u_j^U(x_{lj}), j=1,\dots,n
\end{aligned}
\qquad (2)
$$

Examining the objective function, we find that it can be rewritten as $\sum_j w_j\, [u_j(x_{kj}) - u_j(x_{lj})]$, where $u_j(x_{kj})$ -

$u_j(x_{lj})$ does not depend on weights w_j. Moreover, if we carefully observe the constraints, we discover that variables w_j are independent of the other variables. So, taking into account that weights w_j are nonnegative, solving problem (2) is equivalent to solve the optimization problem

$$\begin{aligned} D_{kl} &= \textstyle\sum_j w_j z_{klj} \qquad (3) \\ \text{s.t.} \quad & w_j \sim N(\mu_j, \sigma_j^2), j = 1, \dots, n \end{aligned}$$

where z_{klj} are the optimal values of the optimization problem

$$\begin{aligned} z_{klj} &= \min \; u_j(x_{kj}) - u_j(x_{lj}) \\ \text{s.t.} \quad & x_{kj}^L \le x_{kj} \le x_{kj}^U, j=1,\dots,n \\ & x_{lj}^L \le x_{lj} \le x_{lj}^U, j=1,\dots,n \qquad (4) \\ & u_j^L(x_{kj}) \le u_j(x_{kj}) \le u_j^U(x_{kj}), j=1,\dots,n \\ & u_j^L(x_{lj}) \le u_j(x_{lj}) \le u_j^U(x_{lj}), j=1,\dots,n \end{aligned}$$

Solution to problem (4) can be determined depending on what the characteristics of the utility function for attribute X_j are (Mateos et al., 2003, Mateos et al., 2007):

- If the utility function is increasing monotone, then $z_{klj} = u_j^L(x_{kj}^L) - u_j^U(x_{lj}^U)$.
- If the utility function is decreasing monotone, then $z_{klj} = u_j^L(x_{kj}^U) - u_j^U(x_{lj}^L)$.

Our aim is to take advantage of normal distribution properties, such as closure under linear combinations. For example, any linear combination of a number of independent normal distributions also follows a normal distribution. Therefore, it is well-know that if $w_j \sim N(\mu_j, \sigma_j^2), j = 1, \dots, n$, then D_{kl} has a normal distribution with mean $\sum_{j=1}^{n} z_{klj}\mu_j$ and variance $\sum_{j=1}^{n} z_{klj}^2 \sigma_j^2$, because D_{kl} is a linear combination of the w_j.

A way to overcome the infinity range of normal distributions is to consider truncated normal distributions. However, distributions of functions of truncated normal variables cannot usually be expressed in mathematically elegant forms. Weinstein (1964) made an early attempt to derive the distribution of the sum of two independent random variables, one normal and the other truncated normal.

Figure 1 shows a simulation approach to the distribution of linear combinations of truncated normal variables (TN). Blue line (c371155) represents the distribution of the linear combination $0.5 * TN(0.3, 0.1) + 0.5 * TN(0.7, 0.1)$. Truncation is on interval $[0,1]$. Green line (c373355) represents the distribution of the linear combination $0.5 * TN(0.3, 0.3) + 0.5 * TN(0.7, 0.3)$. Red line (c374482) represents the distribution of the linear combination $0.8 * TN(0.3, 0.4) + 0.2 * TN(0.7, 0.4)$. Finally, brown line (c374428) represents the distribution of $0.2 * TN(0.3, 0.4) + 0.8 * TN(0.7, 0.4)$.

As it is well known, approximately 99% of the mass of normal distributions is concentrated in $(\mu \pm 3\sigma)$. If the truncation is two-sided, μ-symmetric and within this range, it hardly changes its form (blue line). The narrower the truncation interval, the more the resulting distribution departs from normality.

To begin with, 99% of the mass of the normal distributions considered will be concentrated in intervals $w_j \in [w_j^L, w_j^U]$, for each j. Then, we will consider the use of truncated normal and beta distributions. Given their variety of shapes, beta distributions provide a flexible way to model different kinds of uncertainty on these weights. In fact, the uniform distributions considered previously (Mateos et al., 2009) are particular cases of beta distributions.

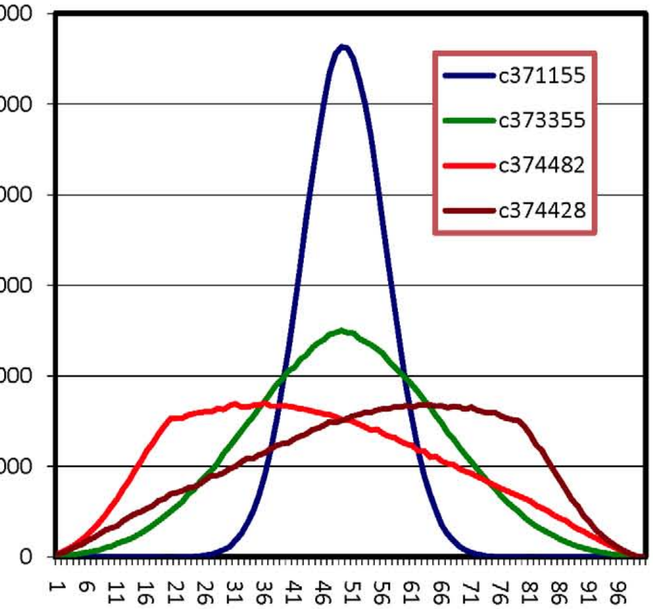

Figure 1. Linear combinations of truncated normal distributions.

2.2. Approach based on dominance measures

In this section, an approach is proposed when DM weights are modeled by independent normal distributions. The approach is based on the same idea as methods introduced in Mateos et al. (2009), where only uniform distributions on interval values are considered to represent weights.

The approach can be implemented in the following 4 steps. In first step, the method computes the optimal solution of the optimization problems (4) for each pair of alternatives A_k and A_l ($k, l=1,\dots,m$). In second step the probability for alternative A_k to dominate the others is computed. In third step a *net dominance measure* for each alternative A_k is computed, which represents a dominance intensity measure of A_k. The third step ranks the alternatives considering the net dominance measure. Alternative A_k is better than alternative A_l if the net dominance measure of alternative A_k is greater than the net dominance measure of alternative A_l.

1. Compute z_{klj} for alternatives A_k and A_l ($k, l=1,\dots,m$) and each attribute X_j ($j = 1,\dots,n$) following (4).

2. Compute

$$P_{kl} = \int_0^\infty f_{kl}(x)dx$$

where $f_{kl}(x)$ is the density function of the variable D_{kl}, i.e., $f_{kl}(x)$ is the density function of a normal distribution variable with mean $\sum_{j=1}^n z_{klj}\mu_j$ and variance $\sum_{j=1}^n z_{klj}^2\sigma_j^2$.

3. Compute a dominance probability measure (P) for each alternative A_k:

$$P_k = \sum_{\substack{l=1 \\ l \neq k}}^{m} P_{kl} \, .$$

4. Rank alternatives according to the P_k values, where the best (rank 1) is the alternative with greatest P_k and the worst is the alternative with the least P_k.

If $f_{kl}(x)$ doesn't have a 'treatable' analytical expression, for example when w_j follows truncated normal or beta distributions, the approach is still valid using Monte Carlo simulation in step two or analogous methods to those shown in (Mateos et al., 2009).

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