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Modelling of the advection-diffusion equation with a meshless method without numerical diffusion

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Abstract

A comprehensive study is presented regarding the stability of the forward explicit integration technique with generalized finite difference spatial discretizations, free of numerical diffusion, applied to the advection-diffusion equation. The modified equivalent partial differential equation approach is used to demonstrate that the approximation is free of numerical diffusion. Two-dimensional results are obtained using the von Neumann method of stability analysis. Numerical results are presented showing the accuracy obtained.

Key words: advection-diffusion, generalized finite difference
MSC2000: AMS Codes (optional)

1. Introduction

With the development of modern industry, various pollulants discharge in the air rivers, lakes and oceans. The changes of pollulants in the air or in the water consist of the physical, chemical and biochemical process and so on. The physical changes of pollution involve two main important processes, that is, advection and diffusion. The mathematical model describing these two processes is the well known advection-diffusion equation. In two dimensions this equation is as follows

$$\frac{\partial U}{\partial t}+\beta_x\frac{\partial U}{\partial x}+\beta_y\frac{\partial U}{\partial y} = \alpha_x\frac{\partial^2 U}{\partial x^2}+\alpha_y\frac{\partial^2 U}{\partial y^2} \quad ;t>0, \quad \mathbf{x}=(\mathrm{x},\mathrm{y})^T\in\Omega$$

with the initial condition:

$$U(\mathbf{x},0)=f(\mathbf{x}) \tag{1}$$

and the boundary conditions:

$$aU(\mathbf{x}_0,\mathrm{t})+b\frac{\partial U(\mathbf{x}_0,\mathrm{t})}{\partial n}=g(t) \quad in\ \Gamma$$

being $f(\mathbf{x})$ and $g(t)$ two known functions, *a, b* are constants , Γ is the boundary of Ω, $U(x,y,t)$ is a transported (advected and diffused) scalar variable, $\beta_x>0$, $\beta_y>0$ are constant speeds of advection and $\alpha_x>0$, $\alpha_y>0$ are constant diffusivities in the x-, and y- direction respectively.

Various numerical techniques can be used to solve this partial differential equation with the associated initial and boundary conditions [1]. An active field of research is the use of meshless methods. An evolution of the method of finite differences has been the development of generalized finite difference method (GFDM) that can be applied as a meshless or meshfreee method to irregular grids or clouds of points. Benito, Ureña and Gavete have made interesting contributions to the development of this method [2-7]. This paper shows the application of the generalized finite difference method to solve the advection-diffusion equation by an explicit method.

The paper is structured in six sections. In section 2, we describe briefly the GFDM. In section 3 we describe the explicit scheme used to approximate the advection-diffusion equation. In section 4 we study the truncation error and the stability. In Section 5 an error analysis is done comparing with a test case. Section 6 contains concluding remarks.

2. The Generalized finite difference method

In the GFDM the intention is to obtain explicit linear expressions for the approximation of partial derivatives in the points of a domain. First of all, an irregular grid or cloud of points is generated in the domain $\Omega\cup\Gamma$. On defining the central node with a set of nodes surrounding that node, the star of nodes then refers to a group of established nodes in relation to a central node. Each node in the domain has an associated star assigned to it.

We define the following function based in the approximation of second order in Taylor series

$$B(u)=\sum_{j=1}^{N}\left[\left(u_0-u_j+h_j\frac{\partial U_0}{\partial x}+k_j\frac{\partial U_0}{\partial y}+\frac{1}{2}\left(h_j\frac{\partial U_0}{\partial x}+k_j\frac{\partial U_0}{\partial y}\right)^2\right)w(h_j,k_j)\right]^2$$

(2)

where u_0 is the approximated value of the function at the central node of the star, (x_0, y_0), u_j are the function values of the rest of the nodes, $h_j = x_j - x_0$, $k_j = y_j - y_0$ and $w(h_j, k_j)$ is the denominated weight function.

If the function (2) is minimized with respect to the partial derivatives, the following linear equation system is obtained

$$\mathbf{A}\,\mathbf{D}_{\mathbf{u}} = \mathbf{b} = \left\{ \sum_{j=1}^{N} \left(-u_0 + u_j\right) h_j w^2 \; \ldots \; \sum_{j=1}^{N} \left(-u_0 + u_j\right) \frac{h_j^2}{2} w^2 \; \ldots \; \sum_{j=1}^{N} \left(-u_0 + u_j\right) h_j k_j w^2 \; \ldots \right\}^T \quad (3)$$

$$\mathbf{D}_u = \left\{ \frac{\partial U_0}{\partial x}, \frac{\partial U_0}{\partial y}, \frac{\partial^2 U_0}{\partial x^2}, \frac{\partial^2 U_0}{\partial y^2}, \frac{\partial^2 U_0}{\partial x \partial y} \right\}^T$$

on solving the system (3) the explicit finite difference formulae are obtained.

$$\begin{cases} \mathbf{Y}(k) = -u_0 \sum_{i=1}^{p} M_{ki} c_i + \sum_{j=1}^{N} u_j \left(\sum_{i=1}^{p} M_{ki} d_{ji} \right), & k = 1,..,5 \\ \mathbf{D}_u(k) = \frac{1}{q_{kk}} \left(\mathbf{Y}(k) - \sum_{i=1}^{p-k} q(k+1, k)\, \mathbf{D}_u(k+i) \right), & k = 1,..,5 \end{cases} \quad (4)$$

where

$$M_{ij} = \begin{cases} (-1)^{1-\delta_{ij}} \sum_{k=j}^{i-1} q(i,k)\, M(k,j) & \textit{for}: j < i, \quad i = 1,..,5 \quad j = 1,..,5 \\ \frac{1}{q_{ij}} & \textit{for}: j = i, \quad i = 1,..,5 \quad j = 1,..,5 \\ 0 & \textit{for}: j > i, \quad i = 1,..,5 \quad j = 1,..,5 \end{cases}$$

with δ_{ij} the Kronecker delta function, and:

$$c_i = \sum_{j=1}^{N} d_{ji}$$

$$d_{j1} = h_j w^2;\; d_{j2} = k_j w^2;\; d_{j4} = \frac{h_j^2}{2} w^2;\; d_{j5} = \frac{k_j^2}{2} w^2;\; d_{j6} = h_j k_j w^2$$

3. The advection-diffusion GFDM explicit scheme

On including the explicit expressions for the values of partial derivatives (5) in the differential equation of problem, we obtain the star equation (explicit difference scheme)(6):

$$\frac{\partial U_0}{\partial \mathrm{t}} = \frac{u_0^{n+1} - u_0^n}{\Delta t}$$
$$\frac{\partial U_0}{\partial \mathrm{x}} = -\lambda_0 u_0^n + \sum_{j=1}^{N} \lambda_j u_j^n \; ; \quad \frac{\partial U_0}{\partial \mathrm{y}} = -\mu_0 u_0^n + \sum_{j=1}^{N} \mu_j u_j^n \; ; \tag{5}$$
$$\frac{\partial^2 U_0}{\partial \mathrm{x}^2} = -m_0 u_0^n + \sum_{j=1}^{N} m_j u_j^n \; ; \quad \frac{\partial^2 U_0}{\partial \mathrm{y}^2} = -\eta_0 u_0^n + \sum_{j=1}^{N} \eta_j u_j^n$$

$$u_0^{n+1} = u_0^n - \Delta t \left[\beta_x \left(-\lambda_0 u_0^n + \sum_{j=1}^{N} \lambda_j u_j^n \right) + \beta_y \left(-\mu_0 u_0^n + \sum_{j=1}^{N} \mu_j u_j^n \right) \right] +$$
$$\Delta t \left[\alpha_x \left(-m_0 u_0^n + \sum_{j=1}^{N} m_j u_j^n \right) + \alpha_y \left(-\eta_0 u_0^n + \sum_{j=1}^{N} \eta_j u_j^n \right) \right] \tag{6}$$
$$with: \; \lambda_0 = \sum_{j=1}^{N} \lambda_j \; ; \quad \mu_0 = \sum_{j=1}^{N} \mu_j \; ; \quad m_0 = \sum_{j=1}^{N} m_j ; \quad \eta_0 = \sum_{j=1}^{N} \eta_j$$

This scheme uses the forward-difference form for the time derivative and generalized finite difference forms for all spatial derivatives.
By using the modified equivalent partial differential equation approach of Warming and Hyett [8], we obtain the following expansion equation

$$\frac{\partial U}{\partial t} + \frac{\Delta t}{2} \frac{\partial^2 U}{\partial t^2} + \frac{(\Delta t)^2}{6} \frac{\partial^3 U}{\partial t^3} + \frac{(\Delta t)^3}{24} \frac{\partial^4 U}{\partial t^4} + \beta_x \left(\frac{\partial U}{\partial x} + \sum_{j=1}^{N} \gamma_{1,j} \;\; h_j, k_j \;\; \frac{\partial^3 U}{\partial x^3} + \ldots \right)$$
$$+\beta_y \left(\frac{\partial U}{\partial y} + \sum_{j=1}^{N} \gamma_{2,j} \;\; h_j, k_j \;\; \frac{\partial^3 U}{\partial y^3} + \ldots \right) - \alpha_x \left(\frac{\partial^2 U}{\partial x^2} + \sum_{j=1}^{N} \gamma_{3,j} \;\; h_j, k_j \;\; \frac{\partial^3 U}{\partial x^3} + \ldots \right) \tag{7}$$
$$-\alpha_y \left(\frac{\partial^2 U}{\partial y^2} + \sum_{j=1}^{N} \gamma_{4,j} \;\; h_j, k_j \;\; \frac{\partial^3 U}{\partial x^3} + \ldots \right) = 0$$

and the modified equation

$$\frac{\partial U}{\partial t} + \beta_x \frac{\partial U}{\partial x} + \beta_y \frac{\partial U}{\partial y} - \left(\alpha_x - \frac{\beta_x^2}{2} \Delta t \right) \frac{\partial^2 U}{\partial x^2} - \left(\alpha_y - \frac{\beta_y^2}{2} \Delta t \right) \frac{\partial^2 U}{\partial y^2} + \ldots\ldots = 0 \tag{8}$$

This method incorporates numerical diffusion.
A new GFD scheme free of numerical diffusion can be created as follows

$$u_0^{n+1} = u_0^n - \Delta t\left[\beta_x\left(-\lambda_0 u_0^n + \sum_{j=1}^{N}\lambda_j u_j^n\right) + \beta_y\left(-\mu_0 u_0^n + \sum_{j=1}^{N}\mu_j u_j^n\right)\right] +$$

$$\Delta t\left[\left(\alpha_x + \frac{\beta_x^2}{2}\Delta t\right)\left(-m_0 u_0^n + \sum_{j=1}^{N} m_j u_j^n\right) + \left(\alpha_y + \frac{\beta_y^2}{2}\Delta t\right)\left(-\eta_0 u_0^n + \sum_{j=1}^{N}\eta_j u_j^n\right)\right] \qquad (9)$$

$$with:\ \lambda_0 = \sum_{j=1}^{N}\lambda_j\ ;\quad \mu_0 = \sum_{j=1}^{N}\mu_j\ ;\quad m_0 = \sum_{j=1}^{N} m_j;\quad \eta_0 = \sum_{j=1}^{N}\eta_j$$

Then by using the modified equivalent partial differential equation approach of Warming and Hyett [8] we obtain the following expansion equation

$$\frac{\partial U}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 U}{\partial t^2} + \frac{(\Delta t)^2}{6}\frac{\partial^3 U}{\partial t^3} + \frac{(\Delta t)^3}{24}\frac{\partial^4 U}{\partial t^4} + \beta_x\left(\frac{\partial U}{\partial x} + \sum_{j=1}^{N}\gamma_{1,j}\;\; h_j, k_j\;\; \frac{\partial^3 U}{\partial x^3} + \ldots\right) +$$

$$\beta_y\left(\frac{\partial U}{\partial y} + \sum_{j=1}^{N}\gamma_{2,j}\;\; h_j, k_j\;\; \frac{\partial^3 U}{\partial y^3} + \ldots\right) - (\alpha_x + \frac{\beta_x^2\Delta t}{2})\left(\frac{\partial^2 U}{\partial x^2} + \sum_{j=1}^{N}\gamma_{3,j}\;\; h_j, k_j\;\; \frac{\partial^3 U}{\partial x^3} + \ldots\right) - \qquad (10)$$

$$(\alpha_y + \frac{\beta_y^2\Delta t}{2})\left(\frac{\partial^2 U}{\partial y^2} + \sum_{j=1}^{N}\gamma_{4,j}\;\; h_j, k_j\;\; \frac{\partial^3 U}{\partial x^3} + \ldots\right) = 0$$

and the modified equation

$$\frac{\partial U}{\partial t} + \beta_x\frac{\partial U}{\partial x} + \beta_y\frac{\partial U}{\partial y} - \alpha_x\frac{\partial^2 U}{\partial x^2} - \alpha_y\frac{\partial^2 U}{\partial y^2} + \ldots\ldots = 0 \qquad (11)$$

The modified equivalent partial differential equation of this method shows that this GFD formula (9) is free of numerical diffusion.

4. Convergence

According to Lax's equivalence theorem, if the consistency condition is satisfied, stability is the necessary and sufficient condition for convergence. In this section we study firstly the truncation error of the advection-diffusion equation, and secondly consistency and stability.

We split the truncation error (TTE) in time derivative error (TE_t) and space derivatives error ($TE_{\mathbf{x}}$). As the first order time derivative is given by

$$\frac{\partial U\;\; \mathbf{x_0}, t}{\partial t} = \frac{U\;\; \mathbf{x_0}, t + \Delta t\;\; - U\;\; \mathbf{x_0}, t}{\Delta t} - \frac{\Delta t}{2}\frac{\partial^2 U\;\; \mathbf{x_0}, t_1}{\partial t^2} + O\;\;\Delta t\;\;^2 \qquad \forall\; t < t_1 < t + \Delta t$$

then the truncation time error is given by

$$TE_t = -\frac{\Delta t}{2}\frac{\partial^2 u(x_0, y_0, t_1)}{\partial t^2} + \Theta((\Delta t)^2), \quad t < t_1 < t + \Delta t \tag{12}$$

In order to obtain the truncation error for space GFD derivatives, Taylor's series expansion including higher order derivatives is used and then higher order function $B^*(u)$ is obtained

$$B^*(u) = \sum_{j=1}^{N}\left[\left(\begin{array}{l} u_0 - u_i + h_j\frac{\partial U_0}{\partial x} + k_j\frac{\partial U_0}{\partial y} + \frac{1}{2}\left(h_j\frac{\partial U_0}{\partial x} + k_j\frac{\partial U_0}{\partial y}\right)^{2)} + \\ \frac{1}{6}\left(h_j\frac{\partial U_0}{\partial x} + k_j\frac{\partial U_0}{\partial y}\right)^{3)} + \frac{1}{24}\left(h_j\frac{\partial U_0}{\partial x} + k_j\frac{\partial U_0}{\partial y}\right)^{4)} + \ldots \end{array}\right) w(h_j, k_j)\right]^2$$

If $B^*(u)$ is minimized with respect to the partial derivatives up to second order, the following linear equation system is defined

$$\mathbf{A}\ \mathbf{D_u} = \left(\sum_{j=1}^{N}\Xi h_j \quad \sum_{j=1}^{N}\Xi k_j \quad \sum_{j=1}^{N}\Xi\frac{h_j^2}{2} \quad \sum_{j=1}^{N}\Xi\frac{k_j^2}{2} \quad \sum_{j=1}^{N}\Xi h_j k_j\right)^T \tag{13}$$

where

$$\Xi = \left(u_0 - u_i - \frac{1}{6}\left(h_j\frac{\partial U_0}{\partial x} + k_j\frac{\partial U_0}{\partial y}\right)^{3)} - \frac{1}{24}\left(h_j\frac{\partial U_0}{\partial x} + k_j\frac{\partial U_0}{\partial y}\right)^{4)} - \ldots\right) w(h_j, k_j)^2$$

with N=8, and then

$$TE_{(x,y)} = \mathbf{CA}^{-1}\left(\sum_{j=1}^{N}\Upsilon h_j \quad \sum_{j=1}^{N}\Upsilon h_j \quad \sum_{j=1}^{N}\Upsilon\frac{h_j^2}{2} \quad \sum_{j=1}^{N}\Upsilon\frac{k_j^2}{2} \quad \sum_{j=1}^{N}\Upsilon h_j k_j\right)^T \tag{14}$$

where

$$\Upsilon = \left(-\frac{1}{6}\left(h_j\frac{\partial U_0}{\partial x} + k_j\frac{\partial U_0}{\partial y}\right)^{3)} - \frac{1}{24}\left(h_j\frac{\partial U_0}{\partial x} + k_j\frac{\partial U_0}{\partial y}\right)^{4)} - \ldots\right) w(h_j, k_j)^2$$

$$\mathbf{A} = \begin{pmatrix} \sum_{j=1}^{N} h_j^2 w^2 & \sum_{j=1}^{N} h_j k_j w^2 & \sum_{j=1}^{N} \frac{h_j^3}{2} w^2 & \sum_{j=1}^{N} \frac{h_j k_j^2}{2} w^2 & \sum_{j=1}^{N} h_j^2 k_j w^2 \\ & \sum_{j=1}^{N} k_j^2 w^2 & \sum_{j=1}^{N} \frac{h_j^2 k_j}{2} w^2 & \sum_{j=1}^{N} \frac{k_j^3}{2} w^2 & \sum_{j=1}^{N} h_j k_j^2 w^2 \\ & & \sum_{j=1}^{N} \frac{h_j^4}{4} w^2 & \sum_{j=1}^{N} \frac{h_j^2 k_j^2}{4} w^2 & \sum_{j=1}^{N} \frac{h_j^3 k_j}{2} w^2 \\ & & & \sum_{j=1}^{N} \frac{k_j^4}{4} w^2 & \sum_{j=1}^{N} \frac{h_j k_j^3}{2} w^2 \\ & & & & \sum_{j=1}^{N} h_j^2 k_j^2 w^2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -\beta_x & -\beta_y & -\left(\alpha_x + \frac{\beta_x^2 \Delta t}{2}\right) & -\left(\alpha_y + \frac{\beta_y^2 \Delta t}{2}\right) & 0 \end{pmatrix}$$

where

$$w^2 = w(h_j, k_j)^{2}$$

Then by solving (14), we obtain

$$\begin{aligned} TE_{(x,y)} = {} & -\beta_x \left[\sum_{j=1}^{N} \left(\psi_{1,j} \frac{\partial^3 U}{\partial x^3} + \psi_{2,j} \frac{\partial^3 U}{\partial x^2 \partial y} + \psi_{3,j} \frac{\partial^3 U}{\partial x \partial y^2} + \psi_{4,j} \frac{\partial^3 U}{\partial y^3} + \ldots \right) \right] - \\ & \beta_y \left[\sum_{j=1}^{N} \left(\psi_{5,j} \frac{\partial^3 U}{\partial x^3} + \psi_{6,j} \frac{\partial^3 U}{\partial x^2 \partial y} + \psi_{7,j} \frac{\partial^3 U}{\partial x \partial y^2} + \psi_{8,j} \frac{\partial^3 U}{\partial y^3} + \ldots \right) \right] - \\ & \left(\alpha_x + \frac{\beta_x^2 \Delta t}{2} \right) \left[\sum_{j=1}^{N} \left(\psi_{9,j} \frac{\partial^3 U}{\partial x^3} + \psi_{10,j} \frac{\partial^3 U}{\partial x^2 \partial y} + \psi_{11,j} \frac{\partial^3 U}{\partial x \partial y^2} + \psi_{12,j} \frac{\partial^3 U}{\partial y^3} + \ldots \right) \right] - \\ & \left(\alpha_y + \frac{\beta_y^2 \Delta t}{2} \right) \left[\sum_{j=1}^{N} \left(\psi_{13,j} \frac{\partial^3 U}{\partial x^3} + \psi_{14,j} \frac{\partial^3 U}{\partial x^2 \partial y} + \psi_{15,j} \frac{\partial^3 U}{\partial x \partial y^2} + \psi_{16,j} \frac{\partial^3 U}{\partial y^3} + \ldots \right) \right] + \Theta(h_j, k_j) \end{aligned} \tag{15}$$

where $\psi_{i,j}(h_j, k_j)$ are second-order rational functions and $\Theta(h_j, k_j)$ is a series of third- and higher-order functions.

The total truncation error for advection-diffusion equation is

$$TTE = TE_t + TE_{(x,y)} \tag{16}$$

By considering bounded derivatives in *TTE*, we have consistency

$$\lim_{\Delta t,\, h_j, k_j \to 0,\, 0,\, 0} TTE \to 0 \tag{17}$$

“Boundary conditions are neglected by the von Neumann method which applies in theory only to pure initial value problems with periodic initial data. It does however provide necessary conditions for stability of constant coefficient problems regardless of the type of boundary condition” [9].
From the previously obtained formula (9)

$$u_0^{n+1} = \left(1 + \Delta t\left[\beta_x\lambda_0 + \beta_y\mu_0 - (\alpha_x + \beta_x^2\frac{\Delta t}{2})m_0 - (\alpha_y + \beta_y^2\frac{\Delta t}{2})\eta_0\right]\right)u_0^n -$$

$$\Delta t\left[\beta_x\sum_{j=1}^{N}\lambda_j u_j^n + \beta_y\sum_{j=1}^{N}\mu_j u_j^n - (\alpha_x + \beta_x^2\frac{\Delta t}{2})\sum_{j=1}^{N}m_j u_j^n - (\alpha_y + \beta_y^2\frac{\Delta t}{2})\sum_{j=1}^{N}\eta_j u_j^n\right] \quad (18)$$

Fort the stability analysis a harmonic decomposition is made of the approximate solution at grid points at a given time level *n*

$$u_0^n = \xi^n e^{i(\kappa_x x_0 + \kappa_y y_0)};\; u_j^n = \xi^n e^{i(\kappa_x(x_0+h_j)+\kappa_y\ y_0+k_j\ )}$$

where (x_0, y_0) are the coordinates in the central node of the star, and (h_j, k_j) are the coordinates of the other nodes of the star with respect to the central node.

$$\xi = \left(1 - \Delta t\sum_{j=1}^{N}\left[(\alpha_x + \beta_x^2\frac{\Delta t}{2})m_j + (\alpha_y + \beta_y^2\frac{\Delta t}{2})\eta_j - \beta_x\lambda_j - \beta_y\mu_j\right]\ 1 - \cos(\kappa_x h_j + \kappa_y k_j)\ \right)$$

$$+ i\Delta t\sum_{j=1}^{N}\left[(\alpha_x + \beta_x^2\frac{\Delta t}{2})m_j + (\alpha_y + \beta_y^2\frac{\Delta t}{2})\eta_j - \beta_x\lambda_j - \beta_y\mu_j\right] sen(\kappa_x h_j + \kappa_y k_j)$$

Taking into account that the stability condition is $\|\xi\| \leq 1$, then

a) $|\text{Real}(\xi)| < 1$

$$\left(1 - \Delta t\sum_{j=1}^{N}\left[(\alpha_x + \beta_x^2\frac{\Delta t}{2})m_j + (\alpha_y + \beta_y^2\frac{\Delta t}{2})\eta_j - \beta_x\lambda_j - \beta_y\mu_j\right]\ 1 - \cos(\kappa_x h_j + \kappa_y k_j)\ \right) < 1 \Rightarrow$$

$$-1 < \left(1 - \Delta t\sum_{j=1}^{N}\left[(\alpha_x + \beta_x^2\frac{\Delta t}{2})m_j + (\alpha_y + \beta_y^2\frac{\Delta t}{2})\eta_j - \beta_x\lambda_j - \beta_y\mu_j\right]\ 1 - \cos(\kappa_x h_j + \kappa_y k_j)\ \right) < 1 \Rightarrow$$

$$\Delta t\left[-\beta_x\lambda_0 - \beta_y\mu_0 + (\alpha_x + \beta_x^2\frac{\Delta t}{2})m_0 + (\alpha_y + \beta_y^2\frac{\Delta t}{2})\eta_0\right] \leq 1 \quad (19)$$

b) $\|\xi\| \leq 1$

$$\|\xi\|^2 = \left(1-\Delta t\sum_{j=1}^{N}\left[(\alpha_x+\beta_x^2\frac{\Delta t}{2})m_j+(\alpha_y+\beta_y^2\frac{\Delta t}{2})\eta_j-\beta_x\lambda_j-\beta_y\mu_j\right]\ 1-\cos(\kappa_x h_j+\kappa_y k_j)\ \right)^2$$
$$+\left(\Delta t\sum_{j=1}^{N}\left[(\alpha_x+\beta_x^2\frac{\Delta t}{2})m_j+(\alpha_y+\beta_y^2\frac{\Delta t}{2})\eta_j-\beta_x\lambda_j-\beta_y\mu_j\right]sen(\kappa_x h_j+\kappa_y k_j)\right)^2\le 1$$

$$\Delta t\left(\sum_{j=1}^{N}\left[(\alpha_x+\beta_x^2\frac{\Delta t}{2})m_j+(\alpha_y+\beta_y^2\frac{\Delta t}{2})\eta_j-\beta_x\lambda_j-\beta_y\mu_j\right]sen(\kappa_x h_j+\kappa_y k_j)\right)^2\le$$
$$\sum_{j=1}^{N}\left[(\alpha_x+\beta_x^2\frac{\Delta t}{2})m_j+(\alpha_y+\beta_y^2\frac{\Delta t}{2})\eta_j-\beta_x\lambda_j-\beta_y\mu_j\right]\ 1-\cos(\kappa_x h_j+\kappa_y k_j)\ \times$$
$$\left[2-\Delta t\sum_{j=1}^{N}\left[(\alpha_x+\beta_x^2\frac{\Delta t}{2})m_j+(\alpha_y+\beta_y^2\frac{\Delta t}{2})\eta_j-\beta_x\lambda_j-\beta_y\mu_j\right]\left(1-\cos(\kappa_x h_j+\kappa_y k_j)\right)\right]$$

$$\Delta t\left[\frac{\beta_x^2\left(\sum_{j=1}^{N}|\lambda_j|\right)^2}{\alpha_x m_0+\ {}^5\!/_6\ \beta_x^2\left(\sum_{j=1}^{N}|\lambda_j|\right)^2\ \Delta t}+\frac{\beta_y^2\left(\sum_{j=1}^{N}|\mu_j|\right)^2}{\alpha_y \eta_0+\ {}^5\!/_6\ \beta_y^2\left(\sum_{j=1}^{N}|\mu_j|\right)^2\ \Delta t}\right]\le 1 \qquad (20)$$

Both formulae (19) and (20) give us the conditions for the stability.

5. Numerical results

In order to illustrate the application of the numerical explicit GFD scheme developed previously, a problem for which an exact solution is available is required so that approximate results obtained can be compared with an exact solution. The problem to be solved is

$$\frac{\partial U\ \ \mathbf{x},t}{\partial t}+\frac{\partial U\ \ \mathbf{x},t}{\partial x}+\frac{\partial U\ \ \mathbf{x},t}{\partial y}=0.1\left(\frac{\partial U^2\ \ \mathbf{x},t}{\partial x^2}+\frac{\partial U^2\ \ \mathbf{x},t}{\partial y^2}\right) \qquad (21)$$
$$t>0,\quad 9<x^2+\mathrm{y}^2<25$$

The exact solution is

$$U(x, y, t) = e^{-1.8t+x+y}$$

$$\begin{cases} \text{Initial conditions:} \quad U(\mathbf{x}, 0) \\ \textit{Dirichlet} \text{ boundary conditions at } 9 = x^2 + y^2 = 25 \end{cases}$$

The weight function is: $$w(x_j, y_j) = \frac{1}{\sqrt{\left(x_j^2 + y_j^2\right)^3}} \tag{22}$$

The global error is evaluated in the last step considered, using the following formula

$$\text{global error} = \frac{\sqrt{\dfrac{\sum_{1}^{M} \left(sol(i) - exac(i)\right)^2}{M}}}{|exac|_{max}} \tag{23}$$

where sol(i) is the GFDM solution at the node i, exac(i) is the exact value of solution at the node (i), $|exac_{max}|$ is the maximum value of the exact solution in the cloud of nodes considered and M is the total number of nodes of the domain.

In this problem we consider different irregular clouds of points as given in Fig. 1.

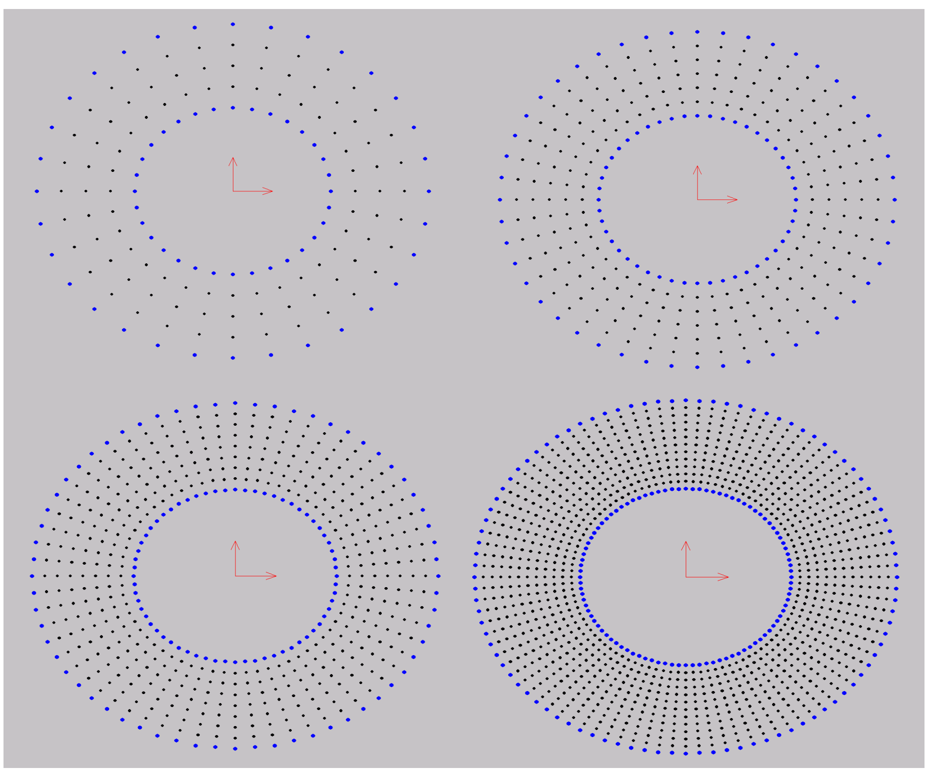

Fig. 1. Different irregular clouds of points.

The influence on global error of using different number of nodes is given in Fig. 2.

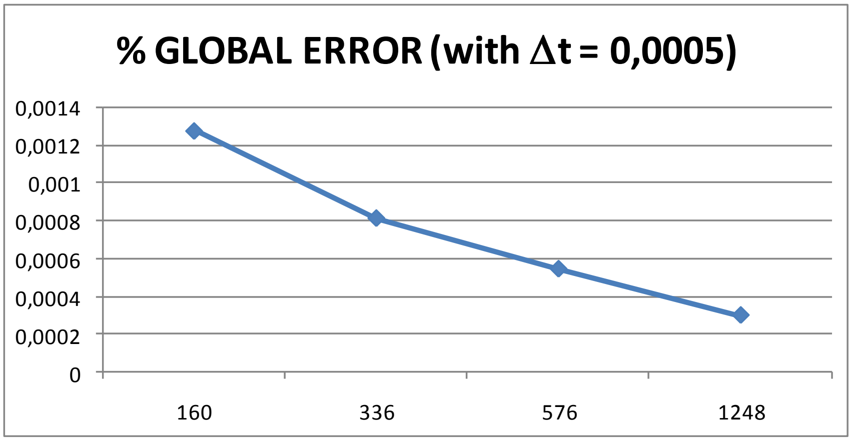

Fig. 2 Global error versus the number of nodes.

Also we consider for the irregular cloud of 1248 nodes the influence on global error versus different values of time increment in Fig.3.

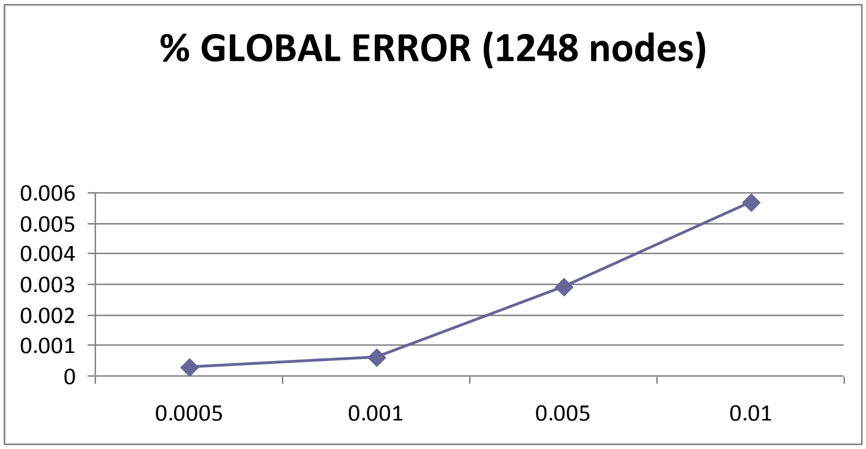

Fig. 3 Global error versus the time increment.

As it is shown in Fig.2 and 3, the global error decreases by increasing the number of nodes or decreasing the time increment.

6. Conclusions

An explicit solution of advection-diffusion equation has been presented for the case of using the GFDM over irregular grids. We have defined the truncation error of the scheme in the case of irregular grids of nodes. Then, we have established the consistency and stability (following the von Neumann stability analysis) criteria for this scheme. An academic test has been presented to illustrate the application of this method.

The fully explicit generalized finite difference schemes are simple to implement and economical to use. They are very efficient and very quick. They are conditionally stable.

The modified equivalent partial differential equation approach of Warming and Hyett[8] has been employed which permits to demonstrate that the GFD scheme is free of numerical diffusion.

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