

Hermitian positive definite Toeplitz matrices and Hessenberg matrices

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In the context of orthogonal polynomials, an interesting class of Hermitian positive definite (HPD) matrices are those that are moment matrices with respect to a measure μ with support on the complex plane. In a more general framework, we establish a one-to-one correspondence between infinite upper Hessenberg matrices with positive subdiagonal and HPD matrices. In the particular case of an HPD Toeplitz matrix \mathbf{T} , the properties and the description of its associated Hessenberg matrix in terms of the well-known recursion coefficients, and in the context of orthogonal polynomials in the unit circle, can be obtained using only an algebraical approach. We give some definition of Hessenberg matrices $\mathbf{D}(\alpha)$ associated to a certain sequence $(\alpha_n)_{n=0}^{\infty}$, and we characterize when such matrices are asymptotically Toeplitz.

KEYWORDS

Hermitian positive definite matrices, infinite Hessenberg matrices, orthogonal polynomials, Toeplitz matrices

1 | INTRODUCTION

Let $\mathbf{M} = (c_{i,j})_{i,j=0}^{\infty}$ be an infinite Hermitian matrix, ie, $c_{i,j} = \overline{c_{j,i}}$ for all i, j nonnegative integers. Following our previous work,¹ we say that \mathbf{M} is positive definite if $|\mathbf{M}_n| > 0$ for all $n \geq 0$, where \mathbf{M}_n is the truncated matrix of size $(n+1) \times (n+1)$ of \mathbf{M} . An infinite Hermitian positive definite matrix (in short, an HPD matrix) defines an inner product $\langle \cdot, \cdot \rangle$ in the space $\mathbb{P}[\mathbb{Z}]$ of all polynomials with complex coefficients in the following way: If $p(z) = \sum_{k=0}^n v_k z^k$ and $q(z) = \sum_{k=0}^m w_k z^k$, then

$$\langle p(z), q(z) \rangle = v \mathbf{M} w^*,$$

where $v = (v_0, \dots, v_n, 0, 0, \dots)$ and $w = (w_0, \dots, w_m, 0, 0, \dots) \in c_{00}$, with c_{00} being the space of all complex sequences with only finitely many nonzero entries. The associated norm is $\|p(z)\|^2 = \langle p(z), p(z) \rangle$ for every $p(z) \in \mathbb{P}[\mathbb{Z}]$.

An interesting class of HPD matrices is the one of moment matrices with respect to a measure μ , ie, HPD matrices $\mathbf{M} = (c_{i,j})_{i,j=0}^{\infty}$ such that there exists a representing measure μ with infinite support on \mathbb{C} and finite moments for all $i, j \geq 0$,

$$c_{i,j} = \int z^i \overline{z}^j d\mu.$$

In this case, we denote by $\mathbf{M} = \mathbf{M}(\mu)$. For more information concerning the characterization of HPD matrices that are moment matrices with respect to a certain measure μ with support on \mathbb{C} , see the works of Atzmon,² Berg and Maserik,³ and Szafraniec,⁴ among others.

An HPD matrix $\mathbf{M} = (c_{i,j})_{i,j=0}^\infty$ is the Gram matrix of the inner product in the vector space $\mathbb{P}[z]$ with respect to $\{z^n\}_{n=0}^\infty$. Let $\{\varphi_n(z)\}_{n=0}^\infty$ be the sequence of orthonormal polynomials uniquely determined by the requirements that

$$\varphi_n(z) = \sum_{k=0}^n \kappa_{k,n} z^k$$

with positive leading coefficient $\kappa_n = \kappa_{n,n} > 0$ and the orthonormality condition

$$\langle \varphi_n(z), \varphi_m(z) \rangle = 0 \quad \text{if } n \neq m, \quad \text{and} \quad \langle \varphi_n(z), \varphi_n(z) \rangle = 1,$$

where the inner product is the one defined above.

We denote by $\Phi_n(z) = \kappa_n^{-1} \varphi_n(z)$ the monic orthogonal polynomial of degree n .

Let $\Lambda_z : \mathbb{P}[z] \rightarrow \mathbb{P}[z]$ be the linear mapping (not necessarily continuous) in $\mathbb{P}[z]$ defined by $\Lambda_z(p(z)) := zp(z)$ and denote by $\mathbf{D} = (d_{i,j})_{i,j=0}^\infty$ the matrix representation of Λ_z with respect to the orthonormal basis $\{\varphi_n(z)\}_{n=0}^\infty$; clearly, this matrix is a Hessenberg matrix, ie, $d_{i,j} = 0$, if $i > j$, with a positive subdiagonal, ie, $d_{i+1,i} > 0$ for all $i \in \mathbb{N}_0$. In the case that the orthonormal basis is $\{z^n\}_{n=0}^\infty$, the associated matrix \mathbf{D} is the shift-right matrix \mathbf{S}_R , which is a Toeplitz matrix with all the entries 0 except the subdiagonal whose entries are one.

In this research, we study the relationship between the infinite Hessenberg matrices with positive subdiagonal and the *normalized* HPD matrices, that is, with the first entry equals to 1. In this sense, for finite-dimensional matrices, Parlett⁵ studies the connections of normal Hessenberg matrices with positive subdiagonal to moment matrices associated to measures with finite support and conclude that there are few normal finite-dimensional Hessenberg matrices. We always consider infinite-dimensional matrices. We prove that there is a one-to-one correspondence between the set of all Hessenberg matrices with a positive subdiagonal and the normalized HPD matrices. In this context, normal Hessenberg matrices are those verifying that the corresponding HPD matrix is indeed a moment matrix with respect to a compactly supported measure μ in the complex plane verifying that the polynomials are dense in $L^2(\mu)$.

In this correspondence, HPD Toeplitz matrices are associated to a special class of Hessenberg matrices that we can describe using only algebraical techniques in terms of the coefficients of a certain recurrence relation that, in the context of orthogonal polynomials in the unit circle \mathbb{T} , is the well-known Szegő recurrence. Indeed, in the theory of the orthogonal polynomials in the unit circle, the associated Hessenberg matrix is known as the *GGT representation* after Geronimus,⁶ and it is a key point that the matrix elements can be expressed in terms of the Verblunsky coefficients (see the work of Simon⁷ for more information). It is interesting to point out that our approach only uses algebraical techniques in order to obtain this matrix representation and does not need to use the fact of the existence of a measure. Moreover, it can be obtained in a more general framework, that is, in the vector space c_{00} with an inner product induced by an HPD Toeplitz matrix as we see in Proposition 5. These results agree with those in the work of Gragg.⁸

We use this matrix representation in order to study the asymptotic properties of the entries of the Hessenberg matrix \mathbf{D} associated to an HPD Toeplitz matrix \mathbf{T} in terms of the asymptotics of the associated coefficients. We⁹ introduced the notion of an *asymptotic Toeplitz* matrix, very closely related to the notion of asymptotic Toeplitz operator that appears in the work of Barriá and Halmos.¹⁰ In this sense, the property $\lim_{n \rightarrow \infty} \Phi_{n+1}(0) = 0$ can be characterized with \mathbf{D} being asymptotic Toeplitz to the shift-right matrix \mathbf{S}_R . In this situation, one can wonder if the operator \mathcal{D} induced by \mathbf{D} in ℓ_2 could be indeed a compact perturbation of the forward shift operator S .

First, we introduce some notation. We are mainly concerned with infinite matrices and, in order to avoid confusion, we distinguish infinite matrices and operators using different notation: An infinite matrix \mathbf{A} defines a linear operator from the sequence space c_{00} to c_{00} if for every $v = (v_i)_{i=0}^\infty \in c_{00}$ the formal matrix product $\mathbf{A}v$ is defined and belongs to c_{00} . We denote by $\mathbf{A}[i,j]$ the entries of \mathbf{A} . Every Hessenberg matrix defines a linear operator. If a matrix \mathbf{A} induces a bounded operator from c_{00} to c_{00} with the ℓ_2 norm, it can be extended to a bounded operator on ℓ_2 that will be denoted by \mathcal{A} . On the other hand, if $\mathcal{A} : \ell_2 \rightarrow \ell_2$ is an operator on a pre-Hilbert space ℓ_2 , the representation of \mathcal{A} using an orthonormal basis $\mathfrak{B} = \{v_i\}_{i=0}^\infty$ in ℓ_2 is by constructing the infinite matrix with entries $a_{i,j} = \langle \mathcal{A}v_j, v_i \rangle$, ie, $\mathbf{A} = (a_{i,j})_{i,j=0}^\infty$. On the Hilbert space ℓ_2 , we always use the standard basis $\{e_n\}_{n=0}^\infty$ and denote by S the forward shift operator such that $S(e_n) = e_{n+1}$ whose representing matrix is \mathbf{S}_R .

Following the work of Barriá and Halmos,¹⁰ a bounded operator $\mathcal{A} : \ell_2 \rightarrow \ell_2$ in a Hilbert space ℓ_2 is a *weakly asymptotic Toeplitz operator* if the sequence $S^{*n} \mathcal{A} S^n$ is strongly convergent. In this direction, we⁹ introduced the following notion: $\mathbf{A} = (a_{i,j})_{i,j=0}^\infty$ is an *asymptotic Toeplitz matrix* if for every $k \in \mathbb{Z}$, there exists $\lim_{n \rightarrow \infty} a_{n,n+k} = \alpha_k$. In such a case, it is denoted by $\text{Lim}(\mathbf{A}) = (\alpha_{i-j})_{i,j=0}^\infty$, which obviously is a Toeplitz matrix. It is clear that the matrix representation of any weakly asymptotic Toeplitz operator on ℓ_2 is an asymptotic Toeplitz matrix.

2 | HESSENBERG MATRICES AND HPD MATRICES: THE TOEPLITZ CASE.

As we saw in the introduction, from an HPD matrix \mathbf{M} with $\mathbf{M}[0, 0] = 1$, there is a unique Hessenberg matrix with positive subdiagonal \mathbf{D} , which is the matrix representation of the multiplication by z linear mapping with respect to the orthonormal basis induced in $\mathbb{P}[z]$ by \mathbf{M} . There is a reciprocal procedure to obtain from an infinite Hessenberg matrix with a subdiagonal positive being the unique suitable HPD matrix; we show it for the sake of completeness.

Proposition 1. *Let $\mathbf{D} = (d_{i,j})_{i,j=0}^\infty$ be an upper Hessenberg matrix with a positive subdiagonal, ie, $d_{i+1,i} > 0$ for every $i \in \mathbb{N}_0$. Then, there exists a unique HPD matrix \mathbf{M} with $\mathbf{M}[0, 0] = 1$ such that \mathbf{D} is the matrix representation of the multiplication by z operator Λ_z defined in $\mathbb{P}[z]$ with respect to the orthonormal basis induced by the inner product given by \mathbf{M} .*

Proof. Consider the following recurrence sequence of polynomials $\{\varphi_n(z)\}_{n=0}^\infty$ with each $\varphi_n(z)$ with degree n associated to matrix \mathbf{D} given by

$$z\varphi_n(z) = \sum_{k=0}^{n+1} d_{k,n} \varphi_k(z), \quad n \geq 0.$$

By taking $\varphi_0(z) = 1$, because $d_{n+1,n} > 0$,

$$\varphi_{n+1}(z) = \frac{1}{d_{n+1,n}} \left((d_{n,n} - z)\varphi_n(z) + \sum_{k=0}^{n-1} d_{k,n} \varphi_k(z) \right).$$

It is easy to check that $\kappa_{i+1} = \frac{1}{d_{i+1,i}} \kappa_i$ and $\kappa_0 = 1$. Moreover, κ_n are positive because $d_{i+1,i} > 0$ for every $i \in \mathbb{N}_0$. We define an inner product $\langle \cdot, \cdot \rangle$ in the space $\mathbb{P}[z]$ of polynomials in the following way for $p(z) = \sum_{k=0}^r a_k \varphi_k(z)$ and $q(z) = \sum_{k=0}^s b_k \varphi_k(z)$:

$$\langle p(z), q(z) \rangle = \sum_{k=0}^{\min r,s} a_k \overline{b_k}.$$

The matrix \mathbf{M} is the Gram matrix of the inner product in the vector space of polynomials $\mathbb{P}[z]$, ie, $\langle z^i, z^j \rangle = c_{i,j}$. Note that $\mathbf{M}[0, 0] = 1$. On the other way, let $\Lambda_z : \mathbb{P}[z] \rightarrow \mathbb{P}[z]$ be the linear mapping (not necessarily continuous) in $\mathbb{P}[z]$ defined by $\Lambda_z(p(z)) := zp(z)$; then, $\mathbf{D} = (d_{i,j})_{i,j=0}^\infty$ is the Hessenberg matrix representation of Λ_z with respect to $\{\varphi_n(z)\}_{n=0}^\infty$. \square

Using the above result, we may establish a correspondence between the subset of normalized HPD matrices, say, $\mathcal{M}_0 = \{\mathbf{M} : \mathbf{M} \text{ is HPD and } \mathbf{M}[0, 0] = 1\}$, and the subset of Hessenberg matrices with a positive subdiagonal, say, $\mathcal{H}^+ = \{\mathbf{D} = (d_{i,j})_{i,j=0}^\infty : \mathbf{D} \text{ is Hessenberg with } d_{i+1,i} > 0, i \in \mathbb{N}_0\}$.

Corollary 1. *There is a one-to-one correspondence $\mathcal{B} : \mathcal{M}_0 \rightarrow \mathcal{H}^+$ in such a way that, for every $\mathbf{M} \in \mathcal{M}_0$, the Hessenberg matrix $\mathcal{B}(\mathbf{M})$ is the matrix representation of the multiplication operator $\Lambda_z : \mathbb{P}[z] \rightarrow \mathbb{P}[z]$ with respect to the orthonormal basis induced by the inner product given by \mathbf{M} .*

Recall that a bounded operator on ℓ_2 is a *normal* operator if $D^*D = DD^*$, and *unitary* if $D^*D = DD^* = I$, where D^* is the adjoint operator. It is a *subnormal* operator if it is the restriction of a normal operator on an invariant subspace. From the results in other works,^{2,11,12} we have the following proposition.

Proposition 2. *Let $\mathbf{M} \in \mathcal{M}_0$,*

1. *There is a representing probability measure μ with bounded support on \mathbb{C} such that $\mathbf{M} = \mathbf{M}(\mu)$ if and only if $\mathcal{B}(\mathbf{M}) = \mathbf{D}$ induces a subnormal bounded operator on ℓ_2 .*
2. *There is a representing probability measure μ with bounded support on \mathbb{C} such that $\mathbf{M} = \mathbf{M}(\mu)$ and $P^2(\mu) = L^2(\mu)$ if and only if $\mathcal{B}(\mathbf{M}) = \mathbf{D}$ induces a bounded normal operator on ℓ_2 .*

Remark 1. From the above result, it follows that, for compactly supported measures μ , the completeness of the polynomials in the space $L^2(\mu)$ can be characterized in terms of the *normality* of the associated Hessenberg operator. Nevertheless, we do not know a characterization of such completeness in terms of the moment matrix $\mathbf{M}(\mu)$. In this direction, we point out that, in our previous work,¹ a necessary and not sufficient condition is given in terms of the

behavior of the smallest eigenvalues λ_n of the finite sections \mathbf{M}_n of the matrix $\mathbf{M}(\mu)$; in particular, for compactly supported measures, it is proved that if $P^2(\mu) = L^2(\mu)$, then $\lim_{n \rightarrow \infty} \lambda_n = 0$.

From now on, we consider an HPD Toeplitz \mathbf{T} . We prove, only requiring algebraical techniques, the recurrence relation of the associated monic orthogonal polynomials. In order to do it, we need some preliminary lemmas and definitions. Recall that, for $p(z) \in \mathbb{P}[z]$, the *reversed polynomial* is defined by $p^*(z) = z^n p\left(\frac{1}{z}\right)$; in particular, if $p(z) = \sum_{k=0}^n a_k z^k$, $p^*(z) = \sum_{k=0}^n \overline{a_{n-k}} z^k$. In an analogous way, for $\mathbf{v} = (v_0, \dots, v_n) \in \mathbb{C}^{n+1}$, we define the *reversed vector* $\mathbf{v}^r = (\overline{v_n}, \dots, \overline{v_0})$.

Lemma 1. *Let $n \in \mathbb{N}_0$ and \mathbf{T}_n be a $(n+1) \times (n+1)$ Hermitian Toeplitz matrix; for every $\mathbf{v} = (v_0, \dots, v_n) \in \mathbb{C}^{n+1}$, it follows*

$$\mathbf{v} \mathbf{T}_n \mathbf{v}^* = \mathbf{v}^r \overline{\mathbf{T}_n(\mathbf{v}^r)^*}.$$

Proof. First of all, denoting by

$$\mathbf{F}_n = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix},$$

it is easy to check that, for every $n \in \mathbb{N}$, $\mathbf{F}_n \mathbf{T}_n \mathbf{F}_n = \mathbf{T}_n$, and $\mathbf{v}^r = \mathbf{v} \mathbf{F}_n$, consequently

$$\mathbf{v} \mathbf{T}_n \mathbf{v}^* = \mathbf{v} \mathbf{F}_n \mathbf{T}_n \mathbf{F}_n \mathbf{v}^* = \mathbf{v}^r \overline{\mathbf{T}_n(\mathbf{v}^r)^*} \quad \square$$

Lemma 2. *Let \mathbf{T} be an infinite HPD Toeplitz matrix and $\langle \cdot, \cdot \rangle$ be the inner product induced by it in $\mathbb{P}[z]$. For every $p(z), q(z) \in \mathbb{P}[z]$, it holds the following:*

1. $\langle p(z), q(z) \rangle = \langle q^*(z), p^*(z) \rangle$. In particular, $\|p(z)\|^2 = \|p^*(z)\|^2$.
2. $\langle zp(z), zp(z) \rangle = \langle p(z), p(z) \rangle$.

Proof. The first part is a consequence of Lemma 1. To prove the second part, we only have to take in mind that being a Toeplitz matrix is equivalent to verify $\mathbf{S}_L \mathbf{T} \mathbf{S}_R = \mathbf{T}$. Thus, for every $p(z) = \sum_{k=0}^n v_k z^k$ if $\mathbf{v} = (v_0, v_1, \dots, v_n, 0, \dots) \in c_{00}$,

$$\langle zp(z), zp(z) \rangle = \mathbf{v} \mathbf{S}_L \mathbf{T} \mathbf{S}_R \mathbf{v}^* = \mathbf{v} \mathbf{T} \mathbf{v}^* = \langle p(z), p(z) \rangle. \quad \square$$

Proposition 3. *Let \mathbf{T} be an infinite HPD Toeplitz matrix and let $\{\Phi_n(z)\}_{n=0}^\infty$ be the associated sequence of monic orthogonal polynomials. For every $n \in \mathbb{N}_0$,*

$$\begin{aligned} \Phi_0(z) &= 1 \\ \Phi_{n+1}(z) &= z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z). \end{aligned}$$

Proof. First of all, by Lemma 2, it follows that, for every $k = 1, \dots, n$,

$$\langle z\Phi_n(z), z^k \rangle = \langle \Phi_n(z), z^{k-1} \rangle.$$

By construction, $\Phi_n(z) \perp [1, z, \dots, z^{n-1}]$, and therefore, $z\Phi_n(z) \perp [z, z^2, \dots, z^n]$. Because $\Phi_{n+1}(z) \perp [z, \dots, z^n]$, we have that $\Phi_{n+1} - z\Phi_n(z) \perp [z, \dots, z^n]$.

On the other hand, by Lemma 2 since $\Phi_n \perp [1, z, \dots, z^{n-1}]$, it follows that $\Phi_n^*(z) \perp [1^*, z^*, \dots, (z^{n-1})^*] = [z, \dots, z^n]$. By orthogonality, there is only a polynomial of degree n , which is orthogonal to $[z, z^2, \dots, z^n]$; therefore, there exists a constant $c \neq 0$ such that

$$\Phi_{n+1}(z) - z\Phi_n(z) = c \Phi_n^*(z).$$

Note that $\Phi_n^*(0) = 1$; thus,

$$\Phi_{n+1}(0) = c \Phi_n^*(0) = c,$$

as we required. □

From the above results, we may associate to every infinite HPD Toeplitz matrix a sequence of coefficients $(\Phi_{n+1}(0))_{n=0}^\infty$; if we consider $\alpha_n = -\overline{\Phi_{n+1}(0)}$, these are the so-called Verblunsky or Geronimus coefficients associated to the corresponding measure ν whose associated moment matrix is \mathbf{T} . As it is well known, $|\alpha_n| < 1$ for all $n \in \mathbb{N}$. We show it as an algebraical consequence of the orthogonality properties in the general framework.

Corollary 2. Let \mathbf{T} be an infinite HPD Toeplitz matrix and let \mathbf{D} be the associated Hessenberg matrix. For every $n \in \mathbb{N}_0$, $d_{n+1,n} = \frac{\kappa_n}{\kappa_{n+1}} < 1$ and

$$\frac{\kappa_n}{\kappa_{n+1}} = \sqrt{1 - |\Phi_{n+1}(0)|^2}.$$

Proof. From the construction of \mathbf{D} , it easily follows that $d_{n+1,n} = \frac{\kappa_n}{\kappa_{n+1}}$. Now, because $\Phi_{n+1}(z) \perp [1, z, \dots, z^n]$ by using Lemma 2, it follows

$$\begin{aligned} \|\Phi_{n+1}(z) - \Phi_{n+1}(0)\Phi_n^*(z)\|^2 &= \|\Phi_{n+1}(z)\|^2 + |\Phi_{n+1}(0)|^2\|\Phi_n^*(z)\|^2 = \\ &= \|\Phi_{n+1}(z)\|^2 + |\Phi_{n+1}(0)|^2\|\Phi_n(z)\|^2 = \|z\Phi_n(z)\|^2 = \|\Phi_n(z)\|^2. \end{aligned}$$

Moreover, because $\varphi_n(z) = \kappa_n\Phi_n(z)$,

$$\begin{aligned} 1 &= \kappa_{n+1}^2\|\Phi_{n+1}(z)\|^2 = \kappa_{n+1}^2(1 - |\Phi_{n+1}(0)|^2)\|\Phi_n(z)\|^2 = \\ &= \frac{\kappa_{n+1}^2}{\kappa_n^2}(1 - |\Phi_{n+1}(0)|^2)\|\Phi_n(z)\|^2\kappa_n^2 = \frac{\kappa_{n+1}^2}{\kappa_n^2}(1 - |\Phi_{n+1}(0)|^2). \end{aligned}$$

Then, it follows that, for every $n \in \mathbb{N}_0$, $|\Phi_{n+1}(0)| < 1$ and

$$\frac{\kappa_n}{\kappa_{n+1}} = \sqrt{1 - |\Phi_{n+1}(0)|^2}. \quad \square$$

Recall that if $\{\varphi_n(z)\}_{n=0}^\infty$ is the sequence of orthonormal polynomials associated to a measure and, in a more general context, orthonormal polynomials with respect to an infinite HPD matrix, the n -reproducing kernels are defined as

$$K_n(z, w) = \sum_{k=0}^n \overline{\varphi_k(z)}\varphi_k(w).$$

In the context of the theory of orthogonal polynomials, the extremal properties of the monic orthogonal polynomials and the n -reproducing kernels at 0 play an important role (see, eg, the work of Van Assche¹³). As it was proved in our previous work,¹ these extremal properties remain true when we consider orthogonal polynomials associated to an inner product induced by a general HPD matrix \mathbf{M} , not necessarily a moment matrix. We rewrite these minimization results (see Lemma 1 in our previous work¹) with our terminology.

Proposition 4 (See our previous work¹). Let \mathbf{M} be an infinite HPD matrix,

1. Extremal property of n -reproducing kernels at 0:

$$\frac{1}{\sum_{n=0}^\infty |\varphi_n(0)|^2} = \inf \{ \|p(z)\|^2 : p(z) \in \mathbb{P}[z], p(0) = 1 \},$$

where the left side is zero if $\sum_{k=0}^\infty |\varphi_k(0)|^2 = \infty$.

2. Extremal property of monic polynomials: For every $n \in \mathbb{N}$,

$$\|\Phi_n(z)\| = \inf \{ \|p(z)\|^2, p(z) \in \mathbb{P}_n[z] \text{ monic polynomial} \}.$$

3. In the particular case of an HPD Toeplitz matrix \mathbf{T} , for every $n \in \mathbb{N}$,

$$\|\Phi_n(z)\|^2 = \frac{1}{\sum_{k=0}^n |\varphi_k(0)|^2} \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\Phi_n(z)\| = \frac{1}{\sum_{k=0}^\infty |\varphi_k(0)|^2}.$$

Although all of the results are obtained in the context of the space of the polynomials, the same arguments provide the same kind of results in the space \mathbb{C}^{n+1} and c_{00} . We establish the recurrence relation that, in the context of linear algebra, the approach can be of independent interest and a very useful computational tool. The proof is directly from Proposition 3.

Proposition 5. Let \mathbf{T}_n be an $(n + 1) \times (n + 1)$ HPD Toeplitz matrix and let $\{\mathbf{v}_0^n, \dots, \mathbf{v}_n^n\}$ be the orthonormal basis in \mathbb{C}^{n+1} with respect the inner product induced by \mathbf{T}_n with $\mathbf{v}_k^n = (v_{0,k}, \dots, v_{k,k}, 0, \dots, 0)$ for $i = 0 \dots n$ and $v_{k,k} = 1$. Then, for every $0 \leq k \leq n - 1$,

$$\begin{aligned} \mathbf{v}_0^n &= (1, 0, \dots, 0) \\ \mathbf{v}_{k+1}^n &= \mathbf{S}_{R,n} \mathbf{v}_k^n + v_{k,0} (\mathbf{v}_k^n)^r. \end{aligned}$$

Moreover, when \mathbf{M} is an infinite HPD matrix and $\{\mathbf{v}_0, \dots, \mathbf{v}_n, \dots\}$ is the orthonormal basis in c_{00} with respect to the inner product induced by \mathbf{M} with $\mathbf{v}_k = (v_{0,k}, \dots, v_{k,k}, 0, \dots)$ and $v_{k,k} = 1$ for every $k \in \mathbb{N}$, it holds

$$\begin{aligned} \mathbf{v}_0 &= (1, 0, \dots) \\ \mathbf{v}_{k+1} &= \mathbf{S}_R \mathbf{v}_k + v_{k,0} \mathbf{v}_k^r \end{aligned}$$

Remark 2. In the same way as in Corollary 2, we may obtain that $|v_{0,k}| < 1$ for every $k \in \mathbb{N}$, and therefore, from an infinite HPD Toeplitz matrix, we obtain a sequence of coefficients $(v_{k,0})_{k=1}^\infty \in \mathbb{D}^\mathbb{N}$, which determines the recurrence relation associated to the Toeplitz matrix. Note that this recurrence relation provides all the coordinates of the orthonormal basis only from the knowledge of the first coordinate. This is analogous of the Verblunsky coefficients in the general framework of infinite HPD Toeplitz matrices.

We may now prove that the matrix elements of the Hessenberg matrix \mathbf{D} associated to an infinite HPD matrix are expressible in terms of the associated coefficients $(\Phi_n(0))_{n=0}^\infty$.

Proposition 6. Let \mathbf{T} be an infinite HPD Toeplitz matrix. For every $n \in \mathbb{N}$, it follows that

$$\mathbf{D} = \begin{pmatrix} -\Phi_1 \bar{\Phi}_0 & -\frac{\kappa_0}{\kappa_1} \Phi_2 \bar{\Phi}_0 & -\frac{\kappa_0}{\kappa_2} \Phi_3 \bar{\Phi}_0 & \dots & -\frac{\kappa_0}{\kappa_{n-2}} \Phi_{n-1} \bar{\Phi}_0 & -\frac{\kappa_0}{\kappa_{n-1}} \Phi_n \bar{\Phi}_0 & \dots \\ \frac{\kappa_0}{\kappa_1} & -\Phi_2 \bar{\Phi}_1 & -\frac{\kappa_1}{\kappa_2} \Phi_3 \bar{\Phi}_1 & \dots & -\frac{\kappa_{n-2}}{\kappa_{n-1}} \Phi_{n-1} \bar{\Phi}_1 & -\frac{\kappa_{n-1}}{\kappa_{n-1}} \Phi_n \bar{\Phi}_1 & \dots \\ 0 & \frac{\kappa_1}{\kappa_2} & -\Phi_3 \bar{\Phi}_2 & \dots & -\frac{\kappa_{n-2}}{\kappa_{n-1}} \Phi_{n-1} \bar{\Phi}_2 & -\frac{\kappa_{n-1}}{\kappa_{n-1}} \Phi_n \bar{\Phi}_2 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\Phi_{n-1} \bar{\Phi}_{n-2} & -\frac{\kappa_{n-2}}{\kappa_{n-1}} \Phi_n \bar{\Phi}_{n-2} & \dots \\ 0 & 0 & 0 & \dots & \frac{\kappa_{n-2}}{\kappa_{n-1}} & -\Phi_n \bar{\Phi}_{n-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\Phi_i := \Phi_i(0)$ and $\frac{\kappa_i}{\kappa_{i+1}} = \sqrt{1 - |\Phi_i(0)|^2}$ for all $i \geq 0$.

Proof. By Proposition 4, it holds

$$\frac{1}{\sum_{k=0}^n |\varphi_n(0)|^2} = \min \{ \|p(z)\|^2 : p(z) \text{ is monic} \} = \|\Phi_n(z)\|^2.$$

Moreover, this minimum is attained only in a point. Consequently, because the polynomials $K_n(z; 0) = \frac{1}{\kappa_n} \sum_{k=0}^n \varphi_n(0) \varphi_n(z)$ and $\varphi_n^*(z)$ are polynomials of degree n , which attained the minimum in the extremal problem, both polynomials coincide, and therefore,

$$K_n(z, 0) = \kappa_n \varphi_n^*(z).$$

Finally, using the recurrence relation of Proposition 3, it follows that, for every $n \in \mathbb{N}$,

$$z\Phi_n(z) = \Phi_{n+1}(z) - \Phi_{n+1}(0)\Phi_n^*(z).$$

Therefore,

$$z\kappa_n \Phi_n(z) = \kappa_n \Phi_{n+1}(z) - \Phi_{n+1}(0)\kappa_n \Phi_n^*(z)$$

$$z\varphi_n(z) = \frac{\kappa_n}{\kappa_{n+1}} \varphi_{n+1}(z) - \Phi_{n+1}(0)\varphi_n^*(z),$$

and using that

$$K_n(z, 0) = \kappa_n \varphi_n^*(z),$$

we obtain

$$z\varphi_n(z) = \frac{\kappa_n}{\kappa_{n+1}} \varphi_{n+1}(z) - \sum_{i=0}^n \frac{\kappa_i}{\kappa_n} \Phi_{n+1}(0)\Phi_i(0)\varphi_n(z),$$

as we required. □

The matrix representation of Hessenberg matrices associated to HPD Toeplitz matrices in Proposition 6 has the advantage that it can be parameterized by a sequence. We give some definition of Hessenberg matrices $\mathbf{D}(\alpha)$ associated to a certain sequence $(\alpha_n)_{n=0}^\infty$ and we characterize when such matrices are asymptotically Toeplitz. In the sequel, in order to simplify the notation, we give the following definition.

Definition 1. Let $(\alpha_n)_{n=0}^\infty$ with $\alpha_0 = 1$ and $(\alpha_n)_{n=1}^\infty \in \mathbb{D}^\mathbb{N}$; we denote by $\mathbf{D}(\alpha_n)$ the Hessenberg matrix $\mathbf{D}(\alpha_n)$ with $d_{i+1,i} = \frac{\kappa_i}{\kappa_{i+1}}$, $d_{i,i} = -\alpha_{i+1}\overline{\alpha_i}$ and $d_{i,n} = -\frac{\kappa_i}{\kappa_n}\alpha_{n+1}\overline{\alpha_i}$ for $0 \leq i < n$, where $\frac{\kappa_{n-1}}{\kappa_n} = \sqrt{1 - |\alpha_n|^2}$, for $n \in \mathbb{N}$ and $\kappa_0 = 1$.

Lemma 3. Let $(\alpha_n)_{n=0}^\infty$ with $\alpha_0 = 1$ and $(\alpha_n)_{n=1}^\infty \in \mathbb{D}^\mathbb{N}$ and the Hessenberg matrix $\mathbf{D}(\alpha_n)$. The following statements are equivalent:

1. $\lim_{n \rightarrow \infty} \alpha_n = 0$.
2. The matrix $\mathbf{D}(\alpha_n)$ is asymptotic Toeplitz and $\text{Lim}(\mathbf{D}(\alpha_n)) = \mathbf{S}_R$.

Proof. Assume that $\lim_{n \rightarrow \infty} \alpha_n = 0$; because $\frac{\kappa_{n-1}}{\kappa_n} = \sqrt{1 - |\alpha_n|^2}$, it follows that the subdiagonal of the matrix $\mathbf{D}(\alpha_n)$ converges to 1. On the other hand, for every $k \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \alpha_n \overline{\alpha_{n+k}} = 0,$$

and because $\frac{\kappa_{n-1}}{\kappa_n} \leq 1$, all the diagonals converge to zero. The reciprocal result is obvious. □

3 | CONCLUSIONS

In the polynomials orthogonal framework, one type of HPD matrices that are moment matrices with respect to a measure μ with support on the complex plane is handled. In a more general framework, we establish in this paper a one-to-one correspondence between the set of all upper Hessenberg matrices with positive subdiagonal \mathcal{H}^+ and the infinite normalized HPD matrices \mathcal{M}_0 . In the particular case of Hermitian Toeplitz matrices, an algebraical approach has been developed to obtain properties and recurrence relations in accordance with the well-known recursion coefficients in the context of orthogonal polynomials in the unit circle. The fact that the matrix representation of Hessenberg matrices associated \mathbf{D} to HPD Toeplitz matrices can be parameterized by a sequence allows us to ask ourselves about the behavior of $\mathbf{D}(\alpha)$ associated to a $(\alpha_n)_{n=0}^\infty$ and to characterize when such matrices are asymptotically Toeplitz. From this result, one can wonder if in the case that $\lim_{n \rightarrow \infty} \alpha_n = 0$ the bounded operator $D(\alpha_n)$ on ℓ_2 induced by the Hessenberg matrix $\mathbf{D}(\alpha_n)$ is a compact perturbation of the forward shift operator.

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