

UNIVERSIDAD POLITÉCNICA DE MADRID
Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio



Impact of Aquifers on the Performance of Geothermal Boreholes for all Peclet Numbers

DOCTORAL THESIS

Submitted for the degree of Doctor by:

Javier Rico Cabrera
Master in Industrial Mathematics

Madrid, 2023



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Doctoral Degree in Aerospace Engineering

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A mis padres y María²

Without passion life is nothing

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Abstract

Energy plays an indispensable role in addressing nearly every challenge facing humanity today, with fighting climate change standing out as one of the most crucial ones. Hence, governmental and non-governmental institutions are promoting an energy transition to curtail greenhouse gas emissions, thereby combating the prevailing climate change crisis. This transition must be achieved without compromising the current level of human well-being, ensuring a continuous and cost-effective energy supply. In this scenario, the development of technologically and economically feasible energy storage solutions emerges as a crucial step toward successfully accomplishing the aforementioned energy transition.

Given that approximately 23% of mankind's final energy consumption is dedicated to heating and cooling buildings, the optimal and most efficient method for storing such a substantial amount of thermal energy is through direct thermal energy storage. Among existing technologies, borehole thermal energy storage (BTES) systems not only offer the highest potential energy savings but also stand out due to its versatility. For instance, surplus heat generated from an industrial process can be storage at temperatures ranging from 70°C to 80°C for direct use in heating the building in winter. Furthermore, there are feasible applications wherein the storage temperature is much closer to the unperturbed ground temperature. In these scenarios, the borehole field acts as a seasonal energy storage that along with a heat pump form a highly efficient heating, ventilation and air conditioning (HVAC) system.

The optimal design of these systems is critical. An excessively large size results in high initial investment costs and impractical payback periods. In contrast, an undersized system implies that, to meet the heating and cooling demands of the building, the heat-carrying liquid requires more extreme temperatures. This invariably results in a decrease in the overall efficiency of the HVAC system. The optimal design of the borehole field requires an accurate forecast of its thermal response over the whole lifespan of the building, of typically 100 years. Unfortunately, these forecasts cannot be obtained directly from detailed numerical simulations of the whole heat transfer problem as nowadays their computational cost makes them unfeasible for engineering purposes. Instead, simplified theoretical models are employed. Most of these theoretical models only take into account heat conduction in ground. However, in many real-world situations the presence of aquifers can highly affect the heat exchange between the boreholes and the ground.

This thesis exploits the presence of large disparities in time and length scales within the heat transfer problem to build simplified, albeit accurate, theoretical models that account for the presence of groundwater flows in the thermal response of geothermal boreholes. These models are rigorously derived mathematically using techniques commonly employed in aerospace engineering and fluid mechanics such as matched asymptotic expansion techniques. The main achievement of this thesis extends beyond enhancing the accuracy of existing models for creeping groundwater flows found in the literature, or the study of new scenarios involving strong groundwater flows. The presented theoretical models also serve to judge and clarify for the first time physical inconsistencies and mathematical-unjustified simplifications found in the state of the art.

Resumen

La energía juega un papel indispensable en abordar casi todos los desafíos que enfrenta la humanidad hoy en día, destacando la lucha contra el cambio climático como uno de los más cruciales. Por ende, instituciones gubernamentales y no gubernamentales están promoviendo una transición energética para reducir las emisiones de gases de efecto invernadero, combatiendo así la crisis climática actual. Esta transición debe lograrse sin comprometer el nivel actual de bienestar humano, asegurando un suministro continuo y rentable de energía. En este escenario, el desarrollo de soluciones de almacenamiento de energía tecnológica y económicamente factibles emerge como un paso crucial hacia el logro de la mencionada transición energética.

Dado que aproximadamente el 23% del consumo final de energía de la humanidad se dedica a calefacción y refrigeración de edificios, el método óptimo y más eficiente para almacenar tal cantidad de energía térmica es a través del almacenamiento directo de energía térmica. Entre las tecnologías existentes, los sistemas de almacenamiento de energía térmica en pozos (BTES, por sus siglas en inglés) no solo ofrecen el mayor potencial de ahorro de energía, sino que también se destacan por su versatilidad. Por ejemplo, el exceso de calor generado por un proceso industrial se puede almacenar a temperaturas que van desde 70°C hasta 80°C para su uso directo en la calefacción del edificio en invierno. Además, existen aplicaciones factibles en las que la temperatura de almacenamiento está mucho más cerca de la temperatura no alterada del suelo. En estos escenarios, el campo de pozos actúa como un almacenamiento de energía estacional que, junto con una bomba de calor, forma un sistema de calefacción, ventilación y aire acondicionado (HVAC) altamente eficiente.

El diseño óptimo de estos sistemas es fundamental. Un tamaño excesivamente grande resulta en costos iniciales de inversión elevados y períodos de retorno impracticables. Por el contrario, un sistema infradimensionado implica que, para satisfacer las demandas de calefacción y refrigeración del edificio, el líquido portador de calor requiere temperaturas más extremas. Esto inevitablemente conlleva una disminución en la eficiencia general del sistema HVAC. El diseño óptimo del campo de pozos requiere un pronóstico preciso de su respuesta térmica a lo largo de toda la vida útil del edificio, que suele ser de unos 100 años. Desafortunadamente, estos pronósticos no se pueden obtener directamente a partir de simulaciones numéricas detalladas de todo el problema de transferencia de calor, ya que en la actualidad su coste computacional los hace inadecuados para fines ingenieriles. En su lugar, se emplean modelos teóricos simplificados. La mayoría de estos modelos teóricos solo tienen en cuenta la conducción de calor en el suelo. Sin embargo, en muchas situaciones del mundo real, la presencia de acuíferos puede afectar considerablemente el intercambio de calor entre los pozos y el suelo.

Esta tesis explota la presencia de grandes disparidades en las escalas de tiempo y longitud dentro del problema de transferencia de calor para construir modelos teóricos simplificados, aunque precisos, que tengan en cuenta la presencia de flujos de aguas subterráneas en la respuesta térmica de los pozos geotérmicos. Estos modelos se derivan rigurosamente matemáticamente utilizando técnicas comúnmente empleadas en ingeniería aeroespacial y mecánica de fluidos, como el método de los desarrollos asintóticos acoplados. El logro principal de esta tesis se extiende más allá de mejorar la precisión de los modelos existentes para flujos

de aguas subterráneas reptantes encontrados en la literatura, o el estudio de nuevos escenarios que involucren flujos de aguas subterráneas intensos. Los modelos teóricos presentados también sirven para juzgar y aclarar por primera vez inconsistencias físicas y simplificaciones matemáticas injustificadas encontradas en el estado del arte.

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Nomenclature

$A, \tilde{A}$	Uniform non-dimensional temperature level imposed by the outer region onto the inner region $[-]$ and its Laplace-transformed counterpart $[-]$		
$A_{0m}, \tilde{A}_{0m}$	m th harmonic of the Fourier series expansion of the zeroth order solution to the inner region around the borehole $[-]$ and its Laplace-transformed counterpart $[-]$	C_{in}	Weight of F_{in} in the multipole expansion of Θ $[-]$
$A_{1m}, \tilde{A}_{1m}$	m th harmonic of the Fourier series expansion of the first order correction to the inner region around the borehole $[-]$ and its Laplace-transformed counterpart $[-]$	C_{ina}	Coefficient of $q_a/(2\pi\kappa)$ in C_{in} $[-]$
A_{0mj}	Coefficient of $q_j/(2\pi\kappa)$ in A_{0m} $[-]$	$C_{in,c}, C_{in,\bar{c}}$	Coefficient of C and of its complex conjugate $\bar{C}$ in C_{in} , respectively $[-]$
α_b	Thermal diffusivity of grout $[m^2/s]$	COP	Coefficient of Performance $[-]$
α_g	Effective thermal diffusivity of ground $[m^2/s]$	d_j	Thickness of the wall of pipe j $[m]$
$\tilde{\alpha}_n, \tilde{\beta}_n$	Laplace transformed integration constants to the inner region in the ground for $Pe \sim 1$ $[-]$	δ_{ij}	Kronecker's delta $[-]$
BTES	Borehole thermal energy storage	$\Delta\theta_0$	Angular deviation of the downstream direction from the borehole $[rad]$
$B, \tilde{B}$	Uniform non-dimensional temperature level imposed by the outer region onto the inner region $[-]$ and its Laplace-transformed counterpart $[-]$	E_1	Exponential integral $[-]$
$\mathcal{B}_{0\ell}$	Integration constants in Θ_p $[-]$	$f, \tilde{f}$	Non-dimensional fictitious heat injection rate per unit borehole length $[-]$ and its Laplace-transformed counterpart $[-]$
$\tilde{B}_{0n}, \tilde{B}_{1n}$	Integration constants of the zeroth order solution and first order correction to the inner region, respectively $[-]$	F_{in}	Multipole n centered at pipe i $[-]$
$C, \tilde{C}$	Coefficient specifying the non-dimensional temperature gradient imposed by the outer region onto the inner region $[-]$ and its Laplace-transformed counterpart $[-]$	$\mathcal{F}_{injm}, \mathcal{F}'_{injm}$	m th harmonic of the Fourier series expansions of F_{in} and of its radial derivative, respectively, around pipe j or around the borehole ($j = 0$) $[-]$
ce_n, se_n	Angular Mathieu function of order n $[-]$	$\mathcal{G}_{0m}, \mathcal{G}'_{0m}$	m th harmonic of the Fourier series expansions of Θ_g and of its radial derivative, respectively, around the borehole $[-]$
$\tilde{C}_\ell$	Laplace transformed ℓ th coefficient to the outer region for $Pe \sim 1$ $[-]$	γ	Euler's constant $[-]$
$\tilde{C}_{0n}, \tilde{C}_{1n}$	Laplace transformed integration constants of the zeroth order solution	h_j	Convective heat transfer coefficient of pipe j $[W/(m^2K)]$
		H	Depth of borehole $[m]$
		HVAC	Heating, Ventilation, and Air Conditioning
		ι	Imaginary unit $[-]$
		I_n	Modified Bessel function of the first kind of order n $[-]$
		K	Permeability of the ground $[m^2]$
		K_n	Modified Bessel function of the second kind of order n $[-]$
		$\underline{K}_n$	Mathematical function that tends to K_n for large times $[-]$

k_b, k_j	Thermal conductivity of grout and of the wall of pipe j , respectively [W/(m K)]	q_c	Characteristic value for q and q_j [W/m]
k_g	Effective thermal conductivity of ground [W/(m K)]	r_b	Borehole radius [m]
κ	Ratio between the thermal conductivity of grout and the effective thermal conductivity of ground [–]	r_j, ρ_j	Radial coordinate of the polar coordinate system centered at pipe j or at the borehole ($j = 0$) [m] and its non-dimensional counterpart [–]
λ	Dummy non-dimensional integration variable [–]	r_{0j}, ρ_{0j}	Radial position of pipe j in the polar coordinate system centered at the borehole [m] and its nondimensional counterpart [–]
$\mathcal{L}$	Non-dimensional Laplace operator [–]	r_{pj}, ρ_{pj}	Radius of pipe j or of the borehole ($j = 0$) [m] and its non-dimensional counterpart [–]
μ	Dynamic viscosity of groundwater [Pa s]	$R_{a,ij}$	Non-dimensional borehole's inner thermal resistances [–]
N_p	Number of pipes in the borehole [–]	R_b	Non-dimensional borehole's outer thermal resistance [–]
ω	Non-dimensional angular frequency [–]	R_{ij}	Non-dimensional thermal resistances of the borehole [–]
$\tilde{\Omega}_m$	m th coefficient of the Fourier series expansion to the inner region for $Pe \sim 1$ at the borehole wall	$\hat{R}_{ij}$	Non-dimensional thermal resistances of the borehole [–]
$\tilde{\Omega}_m^{(0)}, \tilde{\Omega}_m^{(1)}$	First two terms of the Taylor series expansion of $\tilde{\Omega}_m$ in the limit of small Peclet numbers [–]	$\hat{R}_{ij}^{(0)}, \hat{R}_{ij}^{(1)}$	Zeroth order solution and first order correction of the thermal resistances of the borehole, respectively [–]
P	Effective driving pressure [Pa]	R_{pj}, R_{pj}	Inner thermal resistance of pipe j [(m K)/W] and its non-dimensional counterpart [–]
Pe	Peclet number of the groundwater flow [–]	s	Non-dimensional position in the complex-valued Laplace plane [–]
$\mathcal{P}_{jm}, \mathcal{P}'_{jm}$	m th harmonic of the Fourier series expansions of Θ_p and of its radial derivative, respectively, around pipe j or around the borehole ($j = 0$) [–]	t, τ	Time [s] and its non-dimensional counterpart [–]
$\mathcal{P}_{jma}, \mathcal{P}'_{jma}$	Weights of $q_a/(2\pi\kappa)$ in $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$, respectively [–]	t_b	Characteristic transversal diffusion time [s]
$q, \tilde{q}, \hat{q}$	Heat injection rate per unit borehole length [W/m], its non-dimensional counterpart [–], its non-dimensional Laplace-transformed counterpart [–], and its non-dimensional harmonic counterpart [–]	t_c	Characteristic residence time of the groundwater stream in the vicinity of the borehole [s]
$q_j, \tilde{q}_j, \hat{q}_j$	Heat injection rate per unit pipe length of pipe j [W/m], its non-dimensional counterpart [–], its non-dimensional Laplace-transformed counterpart [–], and its non-dimensional harmonic counterpart [–]	t_H	Characteristic longitudinal diffusion time [s]
		t_q	Characteristic timescale of the variations in the heating and cooling needs of the building [s]
		t_r	Characteristic residence time of the heat carrying liquid in the pipes [s]

T_∞	Unperturbed ground temperature [K]		its Laplace-transformed counterpart [–]
$T, \Theta, \tilde{\Theta}, \hat{\Theta}$	Perturbed grout/ground temperature [K], its non-dimensional counterpart [–], its non-dimensional Laplace-transformed counterpart [–], and its non-dimensional harmonic counterpart [–]	$\Theta_{\text{out}}^{(0)}, \Theta_{\text{out}}^{(1)}, \tilde{\Theta}_{\text{out}}^{(0)}, \tilde{\Theta}_{\text{out}}^{(1)}$	Zeroth order solution and first order correction to the outer region, respectively, and their Laplace-transform counterparts [–]
Θ_a	Non-dimensional apparent ground temperature or weighted apparent ground temperature [–]	$\Theta_{\text{ex}}^{(1)}, \Theta_{\text{en}}^{(1)}$	Exogenous and endogenous first order corrections to the inner region, respectively [–]
Θ_{aj}	Non-dimensional apparent ground temperature for pipe j [–]	$\Theta_{j,\text{ex}}^{(1)}, \Theta_{j,\text{en}}^{(1)}$	Exogenous and endogenous first order corrections of the bulk fluid temperature in pipe j , respectively [–]
$\Theta_{aj}^{(0)}, \Theta_{aj}^{(1)}$	Zeroth order solution and first order correction of the apparent ground temperature for pipe j , respectively [–]	Θ_g	Particular solution of the governing equation in the ground [–]
$T_j, \Theta_j, \tilde{\Theta}_j, \hat{\Theta}_j$	Bulk temperature of the fluid in pipe j [K], its non-dimensional counterpart [–], its non-dimensional Laplace-transformed counterpart [–], and its non-dimensional harmonic counterpart [–]	Θ_m	Non-dimensional weighted mean fluid temperature of the borehole [–]
$\Theta_j^{(0)}, \Theta_j^{(1)}, \tilde{\Theta}_j^{(0)}, \tilde{\Theta}_j^{(1)}$	Zeroth order solution and first order correction of the bulk fluid temperature in pipe j [–], respectively, and their Laplace-transformed counterparts [–]	Θ_p	Particular solution [–]
θ_{0j}	Angular position of pipe j in the polar coordinate system centered at the borehole [rad]	$\Theta_b, \hat{\Theta}_b$	Non-dimensional mean azimuthal borehole wall temperature [–] and its harmonic counterpart [–]
θ_j	Angular coordinate of the polar coordinate system centered at pipe j or at the borehole ($j = 0$) [rad]	U_∞	Effective seepage velocity of the groundwater flow [m/s]
$\Theta_{\text{in}}, \tilde{\Theta}_{\text{in}}$	Solution to the inner region and its Laplace-transformed counterpart [–]	v_j	Secondary weighting coefficient of pipe j [–]
$\Theta_{\text{in}}^{(0)}, \Theta_{\text{in}}^{(1)}, \tilde{\Theta}_{\text{in}}^{(0)}, \tilde{\Theta}_{\text{in}}^{(1)}$	Zeroth order solution and first order correction to the inner region, respectively, and their Laplace-transform counterparts [–]	v_r, v_r	Radial component of the effective velocity of the groundwater flow [m/s] and its non-dimensional counterpart [–]
$\Theta_{\text{out}}, \tilde{\Theta}_{\text{out}}$	Solution to the outer region and	v_θ, v_θ	Azimuthal component of the effective velocity of the groundwater flow [m/s] and its non-dimensional counterpart [–]
		V	Bulk velocity of the heat carrying liquid [m/s]
		w_j	Primary weighting coefficient of pipe j [–]
		w	Second argument of the function $\underline{K}_n$ [–]
		x, y	Cartesian coordinates centered at the borehole [m]
		z	First argument of the function $\underline{K}_n$ [–]

Chapter 1

Introduction

1.1 Motivation

Throughout the past two centuries, societal and technological development along with the enhancement of human well-being, have been significantly dependent on the utilization of fossil fuels. Nonetheless, contemporary society is increasingly aware of the adverse environmental repercussions associated with the extensive burning of fossil fuels. For instance, one of the most important concerns is the ongoing climate change whose consequences could strongly impact the Earth if no action is taken.

In an effort to address the current situation, numerous governments are actively promoting the widespread adoption of renewable energy sources, which have minimal impact on climate change, along with enhancement of the energy efficiency in industrial and household processes. As a result, governmental and non-governmental organizations are advocating for the utilization of renewable energy sources like solar, wind, and geothermal for clean electricity and thermal energy generation along with an improvement of energy efficiency in transport, industry, and buildings.

For instance, the European Union has set ambitious goals for 2030, aiming to achieve a 55% reduction in greenhouse gas emissions compared to 1990 levels to ensure that 32% of the final energy consumption comes from renewable sources, and to improve energy efficiency by 32.5% [33–35].

This energetic transition must be accomplished without compromising the attained level of human well-being, ensuring an uninterrupted and affordable energy supply. While this energy supply is achievable for the baseload, addressing peak or seasonal energy demands presents a significant challenge. The direct approach to face this problem is to harvest the required energy from renewable sources at all times. However, the associated investment cost makes it economically unfeasible. A more worthwhile option involves storing large amounts of energy to meet those aforementioned peak or seasonal energy demands. Therefore, developing technologically and financially viable energy storage solutions is crucial for achieving the energetic transition goals.

Engineers are aware that energy storage stands among the most pivotal challenges to address. Therefore, extensive research has been conducted on this issue across various fields, resulting in diverse technologies such as thermal, chemical, electrical, gravitational, and more. Two well-known examples include lithium-ion batteries and redox flow batteries, both designed to store electrical energy in the form of chemical energy. Their largest installations report capacities of 1200 MWh and 800 MWh, respectively. These projects are critical in ensuring grid stability, providing short-term peak power supply, and facilitating frequency control.

According to the International Energy Agency [56], approximately 50% of global final energy consumption in 2018 was attributed to heat, with half of it being utilized for industrial processes, 46% for heating and cooling of buildings, and the remaining 4% for agricultural processes, mainly in greenhouse heating. When such amount of thermal energy is required, direct storage as thermal energy is more efficient than frequent conversions into electrical or chemical forms. This approach also allows for the utilization of other energy sources, such as waste heat, and enables the realization of larger storage capacities, up to 100 GWh, at a reasonable investment cost. Hence, large-scale, efficient, and cost-effective thermal energy storage stands as a vital asset in the battle against climate change.

The storage temperature assumes a decisive role as it governs the selection of viable technologies and appropriate materials, ranging from below 0°C, as in the cooling system implemented in the new terminal building at the airport in Oslo, Norway, till 800°C in concentrated solar power plants and adiabatic compressed air energy storage systems. Therefore, it becomes imperative to concentrate on specific temperature spectrums and certain technologies while analyzing and modeling thermal energy storage systems.

Among the existing technologies, borehole thermal energy storage (BTES) offers the highest storage capacity and energy potential savings [55]. It directly stores thermal energy into the ground. To this end, multiple boreholes are drilled vertically, as shown in Figure 1.1. These boreholes are equipped with one or more coaxial or U-shaped pipes, as the one illustrated in Figure 1.2. These pipes facilitate the flow of a heat-carrying liquid, enabling the exchange of heat with the surrounding ground. Thanks to the ground's low thermal diffusivity, the energy loss due to heat conduction is minimal, allowing for the potential recovery of between 75% to 85% of the injected thermal energy.

This technology mainly serves for the seasonal storage of thermal energy, enabling the recovery of excess of heat, either accumulated during summertime or generated during the whole year, for its utilization in wintertime. One of the first pilot installations employing this technology was the Luleåvärme Project in Luleå, Sweden, which was operational from 1983 to 1989. In this project, surplus industrial heat generated from a steel plant, provided at temperatures ranging from 70 – 80°C, was stored in granitic ground by means of 120 boreholes, each drilled to a depth of 64 meters. The overall storage efficiency reached was 43%, allowing the recovery of 1 GWh each year for the heating of buildings in wintertime [96]. Since then, numerous large-scale projects have been successfully constructed and operated [66, 71, 96]. One of the latest initiatives, currently in its initial design stage, forecasts storing 93 GWh of thermal energy, equivalent to roughly 10 million euros at current energy prices. This will be achieved through the utilization of 1500 boreholes, each with a depth of 300 meters. The expected overall storage efficiency for this project is 87% [71].

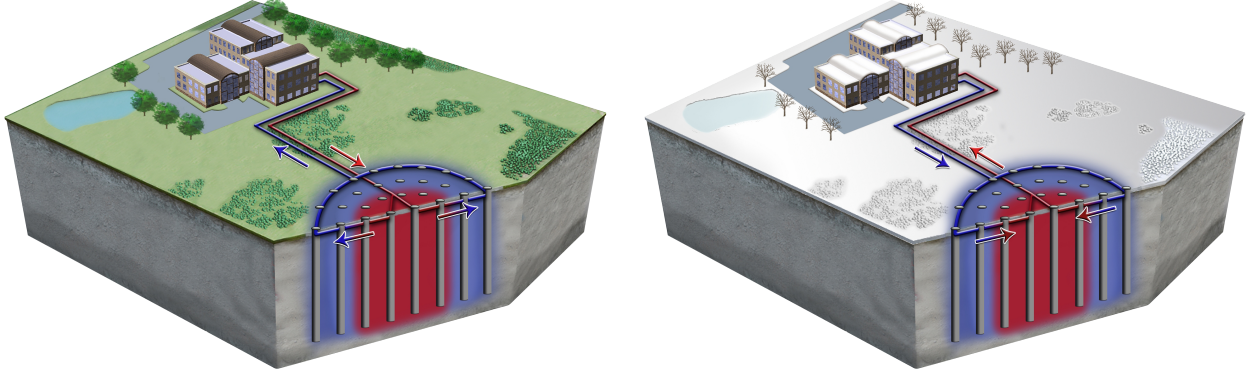


Figure 1.1: Sketches depicting a typical borehole thermal energy storage system consisting of multiple vertical boreholes drilled into the ground [69]. (Left) in summer, the surplus heat extracted from the buildings is stored in the ground. (Right) in winter, the stored thermal energy is extracted and utilized for heating the buildings.

Thanks to its adaptability, BTES technology is also viable for medium and small-scale installations, where the provided temperature is significantly closer to the unperturbed ground temperature. These smaller borehole fields are coupled with a heat pump to assemble energy-efficient heating, ventilation, and air conditioning (HVAC) systems in which the excess heat of the building in summertime is stored for wintertime. The development of this kind of installation is considered a crucial measure in fighting against climate change as, in the current scenario, replacing coal, fuel, and gas boilers with highly efficient electricity-driven systems arises as a top priority. More details about geothermal heat pumps and the role of geothermal boreholes are given in Section 1.2.

One relevant problem when storing thermal energy directly into the ground is the presence of aquifers as groundwater movements wash away the stored energy. To overcome this difficulty, large-scale BTES are placed in suitable geographical locations known for unfractured igneous and metamorphic rocks or clayey soils, which experience minimal groundwater movements. However, for small to medium-sized BTES systems, the locations are often predetermined, making it difficult to avoid the presence of groundwater movements. Hence, in locations characterized by fractured igneous and metamorphic rocks or gravelly soils, it becomes imperative to account for the presence of aquifers in the design stage of the installation as it can highly affect the thermal interaction between the boreholes and the ground.

Predicting the thermal response of the ground and the boreholes over the whole lifespan of a building, of typically 100 years, is crucial for correctly sizing the installation. An excessively large size results in high initial investment costs and impractical payback periods. In contrast, an undersized system implies that, to meet the heating and cooling demands of the building, the heat-carrying liquid requires more extreme temperatures, which invariably results in a decrease in the overall efficiency of the whole installation. Unfortunately, detailed numerical simulations of the unsteady three-dimensional heat transfer problem in ground and boreholes combined with the convective transport of heat along the pipes of a real-world installation cannot be employed to compute these forecasts as their computational cost make them unfeasible for engineering purposes [14, 18, 23, 36, 41, 67, 70, 77, 79, 83]. Therefore, simplified theoretical

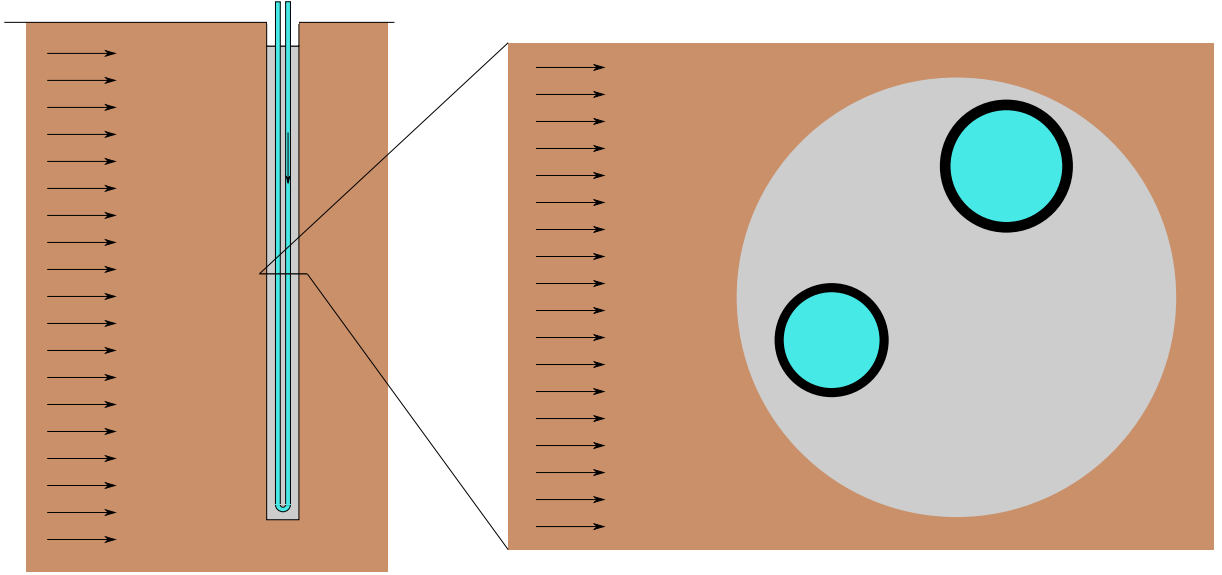


Figure 1.2: Sketch of a typical geothermal borehole.

models are employed to design the installations as they provide accurate, flexible, and fast solutions to the aforementioned problem. Ignoring the presence of groundwater streams during the installation design phase results in an overestimation of the BTES performance, thus accounting for them when forecasting the thermal response of the ground is critical. Hence, this thesis develops simplified, albeit accurate, analytical models for the thermal response of geothermal boreholes, incorporating the presence of aquifers, by means of asymptotic expansion techniques [64].

1.2 Geothermal heat pumps

In small to medium-sized BTES, in which the temperature of the heat carrying liquid is closer to the unperturbed temperature of the ground, a heat pump is required to adapt the available temperatures to the required temperatures by the HVAC system of the building. Briefly, a heat pump is a thermal engine that transfers thermal energy from a cold reservoir to a warmer reservoir by employing external work, usually sourced from electricity.

Figure 1.3 shows the temperature-entropy diagram of an ideal heat pump that follows the Carnot cycle. In the current section the discussion assumes that the ideal heat pump works for cooling the building which does not lead to a loss of generality since the discussion remains analogous to the heating case. Then, for the cooling scenario, process 1-2 extracts heat Q_c from the building while process 3-4 transfers energy Q_h to the ground through the borehole field with the aid of external work W_e applied in the 2-3 process.

The coefficient of performance (COP) is a widespread parameter that assesses the behavior of a heat pump, refrigerator, or air conditioning system. In the present case, it represents the ratio of useful heat extracted from the building to the external work required. For an ideal

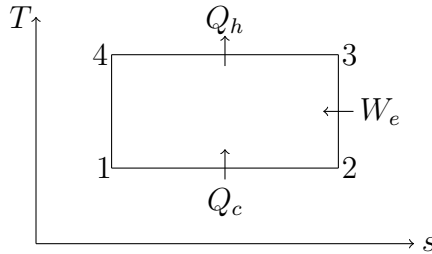


Figure 1.3: Temperature-entropy diagram of an ideal heat pump following a Carnot cycle.

refrigerator, the COP is defined as follows [75]:

$$\text{COP} = \frac{Q_c}{W_e} = \frac{T_c}{T_h - T_c}.$$

The temperature of the cold reservoir T_c is dictated by installation's secondary system which depends on the particular design of the building. The temperature of the warm reservoir T_h , however, is highly affected by the performance of the BTES as it is closely related to the temperature of the heat carrying liquid. Previous expression shows that minor temperature differences of reservoirs lead to higher COPs and, consequently, higher efficiencies of HVAC systems. Hence, the efficiency of the HVAC system not only depends on the storage efficiency of the BTES system but also on the temperature of the heat carrying liquid, impacting on the electricity consumption through the COP.

1.3 Thesis goals and organizational structure

The primary objective of this thesis is to develop theoretical models to accurately and efficiently forecast the thermal response of a geothermal borehole in presence of groundwater flows. In achieving this goal, the large disparities in time and length scales of the problem are exploited by means of matched asymptotic expansion techniques. The use of these techniques allows the derivation of simplified, albeit accurate, models that are suited for the design of optimally sized borehole fields.

This dissertation is structured in 5 chapters, including this introduction. The upcoming chapters describe and formulate the heat transfer problem to study, followed by the development of theoretical models that address different scenarios.

Chapter 2 describes the geometrical and physical attributes of a typical geothermal borehole, along with the relevant characteristics of the ground. Thereafter, a scale analysis is carried out to identify the disparities in time and length scales, which serves to justify and motivate the simplifications performed to address the problem. This analysis is successfully exploited to formulate and solve the fluid-mechanical problem that computes the velocity field of the groundwater flow in the ground. The appropriate simplifications along with the previous velocity field are used thereafter to formulate the thermal problem in the grout filling up the borehole and in the ground surrounding the borehole. Finally, a general discussion,

which applies to all chapters, of the asymptotic structure of the thermal problem is presented together with the asymptotic techniques to be employed in the next chapters.

Chapter 3 establishes a semi-analytical approach to address the problem in presence of creeping groundwater flows, characterized by small Peclet numbers compared to unity. Particularly, the zeroth order as well as its first order correction are derived. The developed asymptotic model not only exceeds by an order of magnitude the accuracy of models found in literature but also enables a critical review of the state of the art that allows to discern what is mathematically correct from what is not. All work contained in this chapter has already been published and can be found in Rico and Hermanns (2024) [87].

Chapter 4 presents then a pure analytical treatment of the inner region for small values of the Peclet number. In particular, a modification on the enhanced multipole method is performed so that this chapter presents, for the first time, the modified enhanced multipole method that accounts for the presence of creeping groundwater flows.

The application of this novel approach together with the proper manipulation and combination of its results enables to set up a network of thermal resistances that represents an exact relationship between the fluid temperatures, heat injection rates per unit pipe length, and apparent ground temperatures for the pipes. Comparison of this network of thermal resistances, that accounts for the presence of creeping groundwater flows, with existing models reveals not only qualitative differences between both models but also highlights the mathematical and conceptual inconsistencies of using, as done by the state of the art, a pure-conducting network of thermal resistances in presence of creeping groundwater flows. All the work presented in this chapter is currently undergoing preparation for publication, scheduled immediately following the submission of this thesis [86].

Finally, Chapter 5 addresses the heat transfer problem in presence of strong groundwater flows, characterized by Peclet numbers of order unity. The fact that the Peclet number attains values of order unity leads to qualitative changes in the asymptotic characteristics of the problem. This chapter discusses the new asymptotic features of the problem and develops a semi-analytical model following the same approach as in Chapter 3. Contrasting this model to detailed numerical simulations shows excellent results when strong groundwater flows are present. Furthermore, to assess its performance when creeping groundwater flows are present, the derived model is analyzed in the limit of small Peclet numbers, recovering the most relevant findings from previous chapters and from the state of the art. All the work presented in this chapter is currently undergoing preparation for publication, scheduled immediately following the submission of this thesis [89].

The work performed throughout this thesis has lead to two additional contributions, they are not included in the present document due to lack of time. The first one is another asymptotic model that works nicely for all possible Peclet numbers of the groundwater flow. The second one is an efficient algorithm for computing some special functions, closely related to the incomplete gamma functions [13, 80], that arise in Chapter 3. Both will be prepared and submitted for publication right after the submission of this thesis [85, 88].

Chapter 2

Description of the problem

2.1 Interaction of slender boreholes with aquifers

A geothermal borehole, like the one shown in Figure 2.1, consists in a vertical borehole of depth H and radius r_b into which several pipes are placed forming one or more coaxial or U-shaped probes. The heat carrying liquid flows with a bulk velocity V along these pipes to exchange heat with the surrounding ground. The void between these pipes and the ground is filled up with grout whose thermal conductivity k_b and thermal diffusivity α_b differ from the corresponding effective values in the ground, k_g and α_g .

Three characteristic times can be constructed out of the previous parameters, namely, the characteristic residence time $t_r \sim H/V$ of the heat carrying liquid in the pipes, the characteristic transversal diffusion time $t_b \sim r_b^2/\alpha_g$, and the characteristic longitudinal diffusion time $t_H \sim H^2/\alpha_g$. Computing these characteristic times using real-world values for H , r_b , V , and α_g reveals that t_r is of order minutes, t_b is of order hours, and t_H is of order centuries [49].

The heat injection/extraction imposed by the HVAC system onto the geothermal heat exchanger introduces a fourth characteristic time, namely, t_q . Since the heating and cooling needs of a building vary on an hourly, daily, weekly, monthly, and yearly basis, the characteristic time t_q presents a large spectrum of values that go from minutes up to decades [49]. The present work focuses on the most relevant operating conditions for which t_q is much larger than t_b but much smaller than t_H . This leads to the sequence of characteristic times

$$t_r \ll t_b \ll t_q \ll t_H.$$

This disparity in time scales has extensively been exploited in the literature for the development of theoretical models for the thermal response of geothermal heat exchangers in purely-conducting grounds [47–51, 53, 93].

The aim of this thesis is to incorporate the effect of groundwater flows in the thermal response of geothermal boreholes. Hence, a fifth characteristic time emerges, namely, the characteristic residence time $t_c \sim r_b/U_\infty$ of the groundwater stream in the vicinity of the borehole, where U_∞ is the effective seepage velocity of the groundwater flow [27, 74, 97]. The ratio of t_c to

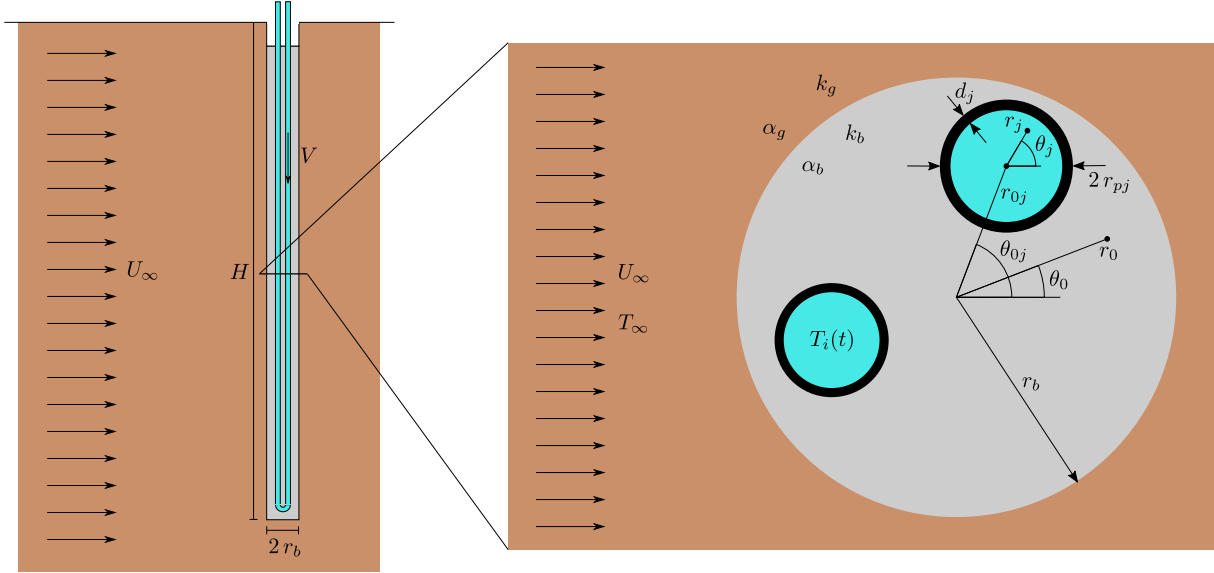


Figure 2.1: Sketch of a typical geothermal borehole.

the characteristic transversal diffusion time t_b is of the order of the Peclet number of the groundwater flow:

$$\text{Pe} = \frac{r_b U_\infty}{\alpha_g} \sim \frac{t_b}{t_c}. \quad (2.1)$$

The Peclet number measures the relative importance of convective terms over diffusive terms in the energy conservation equation in the ground [4]. This thesis explores a broad spectrum of Peclet numbers, ranging from creeping groundwater flows, in which Peclet numbers are small compared to unity, to strong groundwater flows, in which Peclet numbers attain values of order unity or above.

The presence of these large disparities in time and space scales is exploited to derive approximate, albeit accurate, solutions to the thermal interaction of geothermal boreholes with groundwater flows.

2.1.1 Negligible axial heat transfer

The extreme slenderness of geothermal boreholes enables the simplification of the three-dimensional heat transfer problem in the ground. On the one hand, heat conduction along the borehole is negligible compared to heat conduction in the radial direction as a consequence of t_b and t_q being small compared to t_H [49]. On the other hand, aquifers mostly flow perpendicular to the boreholes due to the small slopes of piezometric pressure levels in the ground [25, 99].

The aforementioned simplifications allow the three-dimensional heat transfer problem to be reinterpreted as two-dimensional heat transfer problems formulated in planes perpendicular to the borehole, only coupled to each other through the convective transport of heat along the pipes [49, 87]. This reinterpretation does not affect the possible dependence with depth of the thermal characteristics and state of the ground, aquifer, grout, etc. as each of these

two-dimensional heat transfer problems are solved using the values at the corresponding depth.

2.2 Fluid-mechanical problem

The thermal response of a geothermal borehole depends on the groundwater's velocity field as it flows around the borehole. Hence, its governing equations, i.e. mass and momentum conservation, need to be solved. To do so, the porous medium approach [4] is used to avoid formulating and solving the governing equations in the small voids of the soil.

The mass conservation equation for a constant-density flow in a porous medium is, for one of those two-dimensional planes perpendicular to the borehole shown in Figure 2.1,

$$\frac{\partial}{\partial r_0}(r_0 v_r) + \frac{\partial v_\theta}{\partial \theta_0} = 0,$$

where (r_0, θ_0) are the coordinates in the polar coordinate system centered at the borehole, with v_r and v_θ being the effective velocity components in the radial and azimuthal directions, respectively.

In the porous medium approach, the momentum conservation equation is always replaced by an empirical law. Since groundwater flows of interest are governed by viscous forces, Darcy's law is employed in the present work [4],

$$v_r = -\frac{K}{\mu} \frac{\partial P}{\partial r_0} \quad \text{and} \quad v_\theta = -\frac{K}{\mu} \frac{1}{r_0} \frac{\partial P}{\partial \theta_0},$$

being P the effective driving pressure. The permeability K of the ground and the dynamic viscosity μ of the groundwater are both functions of depth so that the values to use here correspond to the ones at the depth of the considered plane.

Once the governing equations for the groundwater motion have been established, conditions at each boundary of the problem must be formulated. So, a slip condition is enforced at the borehole wall to set there an impermeable behavior:

$$r_0 = r_b : v_r = 0.$$

And far from the borehole the velocity field must match the unperturbed effective seepage velocity U_∞ of the aquifer at the depth of the considered plane [27, 74, 97]. This uniform stream condition translates into the following boundary conditions for v_r and v_θ :

$$r_0 \rightarrow \infty : v_r \rightarrow U_\infty \cos(\theta_0), \quad v_\theta \rightarrow -U_\infty \sin(\theta_0).$$

For constant values of permeability K and dynamic viscosity μ , the just-formulated fluid-mechanical problem is analogous to the problem of potential flow past a circular cylinder. This fact is relevant as the latter problem can be solved exactly [2] leading to the following expressions for the effective velocity components of the groundwater flow:

$$v_r = U_\infty \left(1 - \frac{r_b^2}{r_0^2}\right) \cos(\theta_0) \quad \text{and} \quad v_\theta = -U_\infty \left(1 + \frac{r_b^2}{r_0^2}\right) \sin(\theta_0).$$

2.3 Thermal problem

As with the fluid-mechanical problem presented before, the porous medium approach is used to avoid formulating and solving the heat transfer problem in the intricate voids of the ground. Since heat conduction dominates over heat convection at the pore scale, thermal dispersion phenomenon is considered negligible [3]. For the same reason, local thermal equilibrium is assumed between the groundwater and the solid matrix [3]. As a result, a single unsteady convection-diffusion energy conservation equation with an effective thermal diffusivity α_g and with the effective groundwater velocity (v_r, v_θ) computed before governs the thermal response of the ground [4]:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r_0} + \frac{v_\theta}{r_0} \frac{\partial T}{\partial \theta_0} = \alpha_g \left[\frac{1}{r_0} \frac{\partial}{\partial r_0} \left(r_0 \frac{\partial T}{\partial r_0} \right) + \frac{1}{r_0^2} \frac{\partial^2 T}{\partial \theta_0^2} \right].$$

When writing this equation, constant thermal properties have been assumed for the ground. This simplification is justified by the small temperature variations expected in the problem, at most of 10°C-20°C, which lead to negligible variations in the involved thermal characteristics. The same assumption has also been used in the previous section for the permeability K and for the dynamic viscosity μ .

For the heat conduction taking place in the grout filling up the borehole, the previously-described simplifications apply as well so that the two-dimensional unsteady heat conduction equation to solve is

$$\frac{\partial T}{\partial t} = \alpha_b \left[\frac{1}{r_0} \frac{\partial}{\partial r_0} \left(r_0 \frac{\partial T}{\partial r_0} \right) + \frac{1}{r_0^2} \frac{\partial^2 T}{\partial \theta_0^2} \right],$$

with α_b being the thermal diffusivity of grout.

The solution to the formulated governing equations must also satisfy the following continuity conditions in temperature and normal heat flux at the borehole wall:

$$T \Big|_{r_0=r_b^-} = T \Big|_{r_0=r_b^+} \quad \text{and} \quad -k_b \frac{\partial T}{\partial r_0} \Big|_{r_0=r_b^-} = -k_g \frac{\partial T}{\partial r_0} \Big|_{r_0=r_b^+},$$

where k_g is the effective thermal conductivity of ground and k_b is the thermal conductivity of grout.

The solution must also obey boundary conditions at the N_p pipes forming the coaxial or U-shaped probes located inside the borehole. A prescribed heat injection rate per unit pipe length $q_j(t)$ is then enforced at each pipe j through the following expression:

$$q_j(t) = \int_{-\pi}^{\pi} -k_b \frac{\partial T}{\partial r_j} \Big|_{r_j=r_{pj}} r_{pj} d\theta_j,$$

where r_{pj} is the outer pipe radius and (r_j, θ_j) are the coordinates of a polar coordinate system centered at pipe j .

This condition, however, is not mathematically enough so that an additional condition is enforced at each point on the outer surface of pipe j [49]:

$$-k_b r_{pj} \frac{\partial T}{\partial r_j} \Big|_{r_j=r_{pj}} = \frac{T_j(t) - T \Big|_{r_j=r_{pj}}}{R_{pj}},$$

with the bulk temperature $T_j(t)$ of the fluid in pipe j conveniently set to ensure the prescribed heat injection rate per unit pipe length $q_j(t)$ is satisfied at all times [87]:

$$T_j(t) = \frac{R_{pj}}{2\pi} q_j(t) + \frac{1}{2\pi} \int_{-\pi}^{\pi} T \Big|_{r_j=r_{pj}} d\theta.$$

The inner thermal resistance of pipe j , R_{pj} , encompasses all heat transfer phenomena taking place inside the pipe. That is, from the heat-carrying liquid at $T_j(t)$ up to the outer pipe's wall at $T \Big|_{r_j=r_{pj}}$. In the present problem, two different phenomena are modeled by R_{pj} [49]. First, the turbulent transport of heat inside the fluid, conveniently modeled with a convective heat transfer coefficient h_j [49]. Second, the quasi-steady heat conduction occurring through the pipe's thin wall, of thickness d_j and thermal conductivity k_j :

$$R_{pj} = \frac{1}{(r_{pj} - d_j)h_j} + \frac{1}{k_j} \ln \left(\frac{r_{pj}}{r_{pj} - d_j} \right).$$

Additionally, ground temperature must tend to the unperturbed ground temperature T_∞ far away from the borehole and at the beginning of the problem:

$$r_0 \rightarrow \infty : T \rightarrow T_\infty \quad \text{and} \quad t = 0 : T = T_\infty.$$

Finally, there is a key quantity that awaits definition, namely, the heat injection rate per unit borehole length $\tilde{q}$. It represents the total amount of heat exchanged between borehole and ground per unit time and unit borehole length:

$$q(t) = \sum_{j=1}^{N_p} q_j(t).$$

This magnitude is conveniently defined here as it is widely employed in geothermal literature [26, 31, 32, 43, 49, 74, 87, 97] as well as in the following chapters of this thesis.

2.3.1 Non-dimensionalization

The formulated heat transfer problem in grout and ground is non-dimensionalized for convenience. For it, the borehole radius r_b is used to define the non-dimensional radial coordinates $\rho_0 = r_0/r_b$ and $\rho_j = r_j/r_b$, while $r_b^2/\alpha_g \sim t_b$ is used to define the non-dimensional time $\tau = \alpha_g t/r_b^2$.

The radial and azimuthal components of the velocity field are non-dimensionalized using the effective seepage velocity of the groundwater flow U_∞ , so that

$$v_\rho = \left(1 - \frac{1}{\rho_0^2}\right) \cos(\theta_0) \quad \text{and} \quad v_\theta = -\left(1 + \frac{1}{\rho_0^2}\right) \sin(\theta_0). \quad (2.2)$$

Next, all heat injection rate are non-dimensionalized using a characteristic value q_c that, for instance, is the mean or the maximum value attained by $q_j(t)$ or $q(t)$:

$$q_j = \frac{q_j}{q_c} \quad \text{and} \quad q = \frac{q}{q_c}.$$

Finally, temperatures are non-dimensionalized with the characteristic temperature difference q_c/k_g so that

$$\Theta = \frac{T - T_\infty}{q_c/k_g} \quad \text{and} \quad \Theta_j = \frac{T_j - T_\infty}{q_c/k_g}.$$

With the introduced definitions for the non-dimensional variables, the energy conservation equations in grout and grout become, respectively,

$$\frac{\alpha_g}{\alpha_b} \frac{\partial \Theta}{\partial \tau} = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \Theta}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \Theta}{\partial \theta_0^2} \quad (2.3)$$

and

$$\frac{\partial \Theta}{\partial \tau} + \text{Pe} \left[v_\rho \frac{\partial \Theta}{\partial \rho_0} + \frac{v_\theta}{\rho_0} \frac{\partial \Theta}{\partial \theta_0} \right] = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \Theta}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \Theta}{\partial \theta_0^2}, \quad (2.4)$$

while the non-dimensional continuity conditions at the borehole wall introduce the parameter $\kappa = k_b/k_g$:

$$\Theta \Big|_{\rho_0=1^-} = \Theta \Big|_{\rho_0=1^+} \quad \text{and} \quad -\kappa \frac{\partial \Theta}{\partial \rho_0} \Big|_{\rho_0=1^-} = -\frac{\partial \Theta}{\partial \rho_0} \Big|_{\rho_0=1^+}. \quad (2.5)$$

The heat injection rate per unit pipe length specified at pipe j non-dimensionalizes as follows

$$q_j(\tau) = \int_{-\pi}^{\pi} -\kappa \frac{\partial \Theta}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} \rho_{pj} d\theta_j, \quad (2.6)$$

with the non-dimensional pipe radius being $\rho_{pj} = r_{pj}/r_b$. The boundary condition enforced then at the outer surface of pipe j introduces the non-dimensional pipe's inner thermal resistance $R_{pj} = k_g R_{pj}$,

$$-\rho_{pj} \kappa \frac{\partial \Theta}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} = \frac{\Theta_j(\tau) - \Theta \Big|_{\rho_j=\rho_{pj}}}{R_{pj}}, \quad (2.7)$$

a quantity also present in the expression giving the non-dimensional fluid temperature in pipe j in terms of the specified heat injection rate per unit borehole length:

$$\Theta_j(\tau) = \frac{R_{pj}}{2\pi} q_j(\tau) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta \Big|_{\rho_j=\rho_{pj}} d\theta_j. \quad (2.8)$$

And the boundary condition far from the borehole as well as the initial condition for the problem become homogeneous thanks to the employed non-dimensionalization for the temperatures:

$$\rho_0 \rightarrow \infty : \Theta \rightarrow 0 \quad \text{and} \quad \tau = 0 : \Theta = 0. \quad (2.9)$$

2.3.2 Laplace transformation

Time dependence of the formulated heat transfer problem is addressed using the Laplace transform [7]. Most of the problem formulated and non-dimensionalized in previous sections remains thereby unaltered, with just temperatures and heat injection rates per unit pipe or borehole length bearing now a tilde ($\sim$) to denote their Laplace-transformed character.

Only the two energy conservation equations change with the application of the Laplace transform. So, the energy conservation equation in the ground gets transformed into the following expression in which s stands for the non-dimensional position in the complex-valued Laplace plane:

$$s \tilde{\Theta} + \text{Pe} \left[v_\rho \frac{\partial \tilde{\Theta}}{\partial \rho_0} + \frac{v_\theta}{\rho_0} \frac{\partial \tilde{\Theta}}{\partial \theta_0} \right] = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}}{\partial \theta_0^2}. \quad (2.10)$$

And the energy conservation equation in the grout becomes

$$s \frac{\alpha_g}{\alpha_b} \tilde{\Theta} = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}}{\partial \theta_0^2}. \quad (2.11)$$

2.4 Two-region structure

The heat transfer problem to solve, summarized in Table 2.1, is comprised of two governing equations, Eqs. (2.10) and (2.11), continuity conditions at the borehole wall, Eq. (2.5), boundary conditions at the pipe walls, Eqs. (2.7) and (2.8), and a boundary condition far from the borehole, Eq. (2.9).

In the absence of an exact solution, this thesis derives an approximate, albeit accurate, one using matched asymptotic expansion techniques [64]. These techniques exploit the presence of large disparities in time and length scales to decompose mathematically complex problems into a set of simpler ones. To effectively simplify the problem, an order of magnitude estimation is performed on the terms within the energy conservation equation in the ground:

$$\underbrace{s \tilde{\Theta}}_{\sim |s|} + \text{Pe} \underbrace{\left[v_\rho \frac{\partial \tilde{\Theta}}{\partial \rho_0} + \frac{v_\theta}{\rho_0} \frac{\partial \tilde{\Theta}}{\partial \theta_0} \right]}_{\sim \text{Pe}/\rho_0} = \underbrace{\frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}}{\partial \rho_0} \right)}_{\sim 1/\rho_0^2} + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}}{\partial \theta_0^2}. \quad (2.12)$$

Three factors influence the relative importance of these groups. First, the modulus of the position s in the complex-valued Laplace plane dictates the order of magnitude of the leftmost

Energy conservation in ground
$s \tilde{\Theta} + \text{Pe} \left[v_\rho \frac{\partial \tilde{\Theta}}{\partial \rho_0} + \frac{v_\theta}{\rho_0} \frac{\partial \tilde{\Theta}}{\partial \theta_0} \right] = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}}{\partial \theta_0^2}$ with $v_\rho = \left(1 - \frac{1}{\rho_0^2} \right) \cos(\theta_0) \quad \text{and} \quad v_\theta = - \left(1 + \frac{1}{\rho_0^2} \right) \sin(\theta_0)$
Energy conservation in grout
$s \frac{\alpha_g}{\alpha_b} \tilde{\Theta} = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}}{\partial \theta_0^2}$
Continuity conditions at borehole wall
$\tilde{\Theta} \Big _{\rho_0=1^-} = \tilde{\Theta} \Big _{\rho_0=1^+} \quad \text{and} \quad -\kappa \frac{\partial \tilde{\Theta}}{\partial \rho_0} \Big _{\rho_0=1^-} = -\frac{\partial \tilde{\Theta}}{\partial \rho_0} \Big _{\rho_0=1^+}$
Boundary condition far from the borehole
$\rho_0 \rightarrow \infty : \tilde{\Theta} \rightarrow 0$
Boundary condition at the wall of pipe j
$-\rho_{pj} \kappa \frac{\partial \tilde{\Theta}}{\partial \rho_j} \Big _{\rho_j=\rho_{pj}} = \frac{\tilde{\Theta}_j - \tilde{\Theta} \Big _{\rho_j=\rho_{pj}}}{R_{pj}}$ with $\tilde{\Theta}_j = \frac{R_{pj}}{2\pi} \tilde{q}_j + \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Theta} \Big _{\rho_j=\rho_{pj}} d\theta_j$

Table 2.1: Formulation of the heat transfer problem to solve.

group, linked to the thermal inertia of the ground. As explained in Rivero and Hermanns [93], $|s| \sim t_b/t_q$ so that $|s| \ll 1$ in the present problem. Second, the Peclet number of the flow influences the order of magnitude of the central group of terms, linked to the convective transport of heat in the ground. Since this thesis addresses all possible values for the Peclet number, distinct scenarios are treated in the following chapters. And third, the distance ρ_0 to the borehole affects the order of magnitude of the second and third groups of terms, with the third one representing heat conduction in the ground. All this allows the identification of two distinct regions in the problem.

In the vicinity of the borehole, i.e. at radial distances comparable to the borehole radius, $\rho_0 \sim 1$, thermal inertia is negligible compared to heat conduction as a consequence of $|s| \ll 1$, while the balance between heat convection and conduction is governed by the Peclet number

of the flow. Hence, this *inner region* is, in first approximation, quasi-steady and governed by heat conduction only or by heat conduction and heat convection depending on the Peclet number of the groundwater flow.

Further away from the borehole, however, the described balance of terms changes giving rise to a second region. So, at distances from the borehole of order $\rho_0 \sim 1/\sqrt{|s|} \gg 1$, thermal inertia and heat conduction become equally important in the governing equation. The relevance of the convective transport of heat in this *outer region* depends again on the order of magnitude of the Peclet number Pe . Particularly, it depends on how $Pe/\rho_0 \sim Pe\sqrt{|s|}$ compares to the order of magnitude of the dominant term. That is, to $|s|$.

2.4.1 Matched asymptotic expansion techniques

The two-region structure described before is exploited using matched asymptotic expansion techniques [64]. These mathematical tools address the inner and outer regions separately to afterwards match their solutions at an intermediate region in which both solutions are valid. To tackle each of these two regions, asymptotic expansion techniques exploit the presence of small parameters in the problem and express the solutions to the inner and outer regions, $\tilde{\Theta}_{\text{in}}$ and $\tilde{\Theta}_{\text{out}}$ respectively, as polynomial expansions of that small parameter.

As the selection of the small parameter relies on the specific problem under consideration, whether it involves creeping groundwater flows or strong groundwater flows, the expansions for the asymptotic solution are expressed in the present chapter using a general small parameter denoted as ϵ . This parameter is replaced by the appropriate one in Chapters 3 and 5. Thus, the expansions to use for the inner and outer solutions are

$$\tilde{\Theta}_{\text{in}} = \tilde{\Theta}_{\text{in}}^{(0)} + \epsilon \tilde{\Theta}_{\text{in}}^{(1)} + \mathcal{O}(Pe^2) \quad \text{and} \quad \tilde{\Theta}_{\text{out}} = \tilde{\Theta}_{\text{out}}^{(0)} + \epsilon \tilde{\Theta}_{\text{out}}^{(1)} + \mathcal{O}(Pe^2). \quad (2.13)$$

Since the formulated heat transfer problem also supplies the fluid temperatures in the pipes, $\tilde{\Theta}_j$, these must be expanded as well in terms of the small parameter so that

$$\tilde{\Theta}_j = \tilde{\Theta}_j^{(0)} + \epsilon \tilde{\Theta}_j^{(1)} + \mathcal{O}(Pe^2). \quad (2.14)$$

Substitution of all these expansions into the governing equations, continuity conditions, and boundary conditions of the inner and outer regions supplies a set of mathematical problems whose sequential solution delivers the different terms of the expansions.

Chapter 3

Creeping groundwater flows

For predicting the thermal response of a geothermal borehole all theoretical models take into account the conduction of heat in the ground [24, 68]. For unfractured igneous and metamorphic rocks and for clayey soils it is indeed the dominant heat transfer mechanism so the accuracy of predictions in such grounds is high. However, the convective transport of heat due to groundwater flows is often equally relevant, especially in fractured igneous and metamorphic rocks and in gravelly soils [9, 14, 18, 57, 84]. For those grounds, only models that incorporate this second heat transfer mechanism are able to correctly forecast the thermal response of the geothermal borehole.

The present chapter focuses on the presence of creeping groundwater flows, for which theoretical models already exist [17, 109]. They are mostly based on the moving point-source of heat theory extensively studied by Carslaw and Jaeger [10]. From it, several models have been derived like the moving infinite line source [9, 27, 42, 73, 97, 103, 104], the moving finite line source [20, 29–31, 39, 52, 74, 90, 91, 103, 104], the moving infinite line source [23, 60–63], or combinations of these three [8, 28, 101, 106]. These models work reasonably well for creeping groundwater flows but exploit assumptions and simplifications that lack proper mathematical justification or are inconsistent with the physics of the problem. This affects confidence in their use for real-world engineering tasks. Therefore, the mathematically rigorous derivation of a physically-sound theoretical model would help to judge and clarify these issues and explain why, after all, the existing models work well.

The aim of the present chapter is precisely the derivation of such a theoretical model. Three objectives are sought thereby. First and foremost, to judge and clarify the aforementioned issues in the state of the art. Second, improve the accuracy over the state of the art by an order of magnitude. And third, develop a model that seamlessly integrates into the coherent theoretical framework being pursued in the literature by means of asymptotic techniques since 2011 [45, 46].

The present chapter is organized as follows. Section 3.1 analyses the asymptotic structure of the heat transfer problem, while Sections 3.2 and 3.3 derive the asymptotic solution using matched asymptotic expansion techniques. Thereafter, Section 3.4 discusses the merits and limits of the proposed model while Section 3.5 does the same with the state of the art.

Sections 3.6 and 3.7 focus then on the integration of the proposed model into more complex models. Finally, some concluding remarks are presented in Section 3.8.

3.1 Asymptotic structure

While the existence of two asymptotic regions in the problem is not affected by the Peclet number, as discussed in Section 2.4, the distinctive features of the inner and outer regions vary depending on whether creeping or strong groundwater flows are present. Hence, the particular characteristics of both asymptotic regions in presence of creeping groundwater flows, in which the Peclet number is small compared to unity, are discussed in this section. The estimation of the order of magnitude for each term comprising the governing equation in the ground, Eq. (2.12), is written here again to enhance clarity and to facilitate the reading of this section:

$$\underbrace{\tilde{s}\tilde{\Theta}}_{\sim |s|} + \underbrace{\text{Pe} \left[v_\rho \frac{\partial \tilde{\Theta}}{\partial \rho_0} + \frac{v_\theta}{\rho_0} \frac{\partial \tilde{\Theta}}{\partial \theta_0} \right]}_{\sim \text{Pe}/\rho_0} = \underbrace{\frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}}{\partial \rho_0} \right)}_{\sim 1/\rho_0^2} + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}}{\partial \theta_0^2}. \quad (3.1)$$

In the inner region, at radial distances comparable to the borehole radius $\rho_0 \sim 1$, thermal inertia and convective transport of heat are negligible compared to heat conduction as a consequence of $|s| \ll 1$ and $\text{Pe} \ll 1$, respectively. Therefore, this region is, in first approximation, quasi-steady and solely governed by heat conduction.

However, in the outer region, at distances from the borehole of order $\rho_0 \sim 1/\sqrt{|s|} \gg 1$, thermal inertia and heat conduction become equally important in the governing equation. The relevance of the convective heat transport is dictated by the comparison between $\text{Pe}/\rho_0 \sim \text{Pe}\sqrt{|s|}$ and the order of magnitude of the dominant term. That is, to $|s|$.

This chapter focuses on the richest and most general case possible for which all three phenomena (thermal inertia, heat convection, and heat conduction) are important in the outer region. This occurs for the distinguished limit of $\text{Pe}\sqrt{|s|} \sim |s|$. This condition combined with Eq. (2.1) and with the fact that $|s| \sim t_b/t_q$ allows to properly incorporate the characteristic residence time t_c of the groundwater flow into the sequence of characteristic times presented in Chapter 1:

$$t_r \ll t_b \ll t_c \sim \text{Pe} t_q \ll t_q \ll t_H. \quad (3.2)$$

3.1.1 Matched asymptotic expansion techniques

The asymptotic structure previously described is exploited by means of matching asymptotic expansion techniques [64]. To do so, inner and outer solutions, $\tilde{\Theta}_{\text{in}}$ and $\tilde{\Theta}_{\text{out}}$ respectively, are expressed as polynomial expansions of the small parameter. In this chapter, $\text{Pe} \sim \sqrt{|s|} \ll 1$ acts as the small parameter and the expansions to use are

$$\tilde{\Theta}_{\text{in}} = \tilde{\Theta}_{\text{in}}^{(0)} + \text{Pe} \tilde{\Theta}_{\text{in}}^{(1)} + \mathcal{O}(\text{Pe}^2) \quad \text{and} \quad \tilde{\Theta}_{\text{out}} = \tilde{\Theta}_{\text{out}}^{(0)} + \text{Pe} \tilde{\Theta}_{\text{out}}^{(1)} + \mathcal{O}(\text{Pe}^2). \quad (3.3)$$

Although logarithmic dependencies on the small parameter, of the form $1/\ln(\text{Pe})$, do appear as well in the asymptotic solution of the problem, these are treated as simple numbers instead and are incorporated as such into the zeroth order solution and first order correction to obtain.

Since the fluid temperatures in the pipes, $\tilde{\Theta}_j$, result from the heat transfer problem formulated in Chapter 2, they must be expanded as well in terms of the small parameter Pe :

$$\tilde{\Theta}_j = \tilde{\Theta}_j^{(0)} + \text{Pe} \tilde{\Theta}_j^{(1)} + \mathcal{O}(\text{Pe}^2). \quad (3.4)$$

Replacing all these expansions into the governing equations, continuity conditions, and boundary conditions of the inner and outer regions results in a series of mathematical problems, the sequential solution of which leads to the different terms of the expansions.

3.2 Zeroth order solution

The zeroth order solution to the problem comprises $\tilde{\Theta}_{\text{in}}^{(0)}$, $\tilde{\Theta}_{\text{out}}^{(0)}$, and $\tilde{\Theta}_j^{(0)}$. The corresponding mathematical problems result from substituting Eqs. (3.3) and (3.4) into the heat transfer problem formulated in Chapter 2 and neglecting all terms that are small compared to unity. When doing so, special attention must be devoted to the order of magnitude of ρ_0 as it changes from the inner region to the outer region.

3.2.1 Inner region

The borehole itself and the ground located at non-dimensional distances of order unity, $\rho_0 \sim 1$, define the inner region. Since $|s| \ll 1$ and $\text{Pe} \ll 1$, the left hand sides of Eqs. (2.10) and (2.11) are negligible so that the zeroth order solution to the inner region satisfies the following equation in grout and ground:

$$0 = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \theta_0^2}. \quad (3.5)$$

This equation must be solved together with the continuity conditions at the borehole wall,

$$\tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0=1-} = \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0=1+} \quad \text{and} \quad -\kappa \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} \Big|_{\rho_0=1-} = -\frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} \Big|_{\rho_0=1+}, \quad (3.6)$$

the prescribed heat injection rates per unit pipe length,

$$\tilde{q}_j = \int_{-\pi}^{\pi} -\kappa \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} \rho_{pj} d\theta_j,$$

and the boundary conditions at the pipe walls:

$$-\rho_{pj} \kappa \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} = \frac{\tilde{\Theta}_j^{(0)} - \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_j=\rho_{pj}}}{R_{pj}}. \quad (3.7)$$

What cannot be imposed in the inner region is the boundary condition at infinity due to the existence of the outer region. Consequently, Eq. (2.9) is replaced by an asymptotic matching condition with the zeroth order solution to the outer region.

An exact solution to the formulated problem is unknown. Fortunately, it is not required for the theoretical work to perform as it is enough to know the general behavior of the zeroth order solution in the ground. This behavior is inferred from the general solution to the partial differential equation valid in the ground. That is, Eq. (3.5):

$$\tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0 \geq 1} = \tilde{A}_{00} + \tilde{B}_{00} \ln(\rho_0) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\frac{\tilde{A}_{0n}}{\rho_0^{|n|}} + \rho_0^{|n|} \tilde{B}_{0n} \right) e^{\iota n \theta_0}. \quad (3.8)$$

In this expression $\iota = \sqrt{-1}$ and the values of the integration constants $\tilde{A}_{0n}$ and $\tilde{B}_{0n}$ result either from the continuity conditions at the borehole wall, assuming the solution in the grout is known, or from the asymptotic matching with the zeroth order solution to the outer region.

For the time being, only $\tilde{B}_{00}$ is obtained. In the absence of thermal inertia, the heat injection rate per unit borehole length $\tilde{q}$ coincides with the heat flux at the borehole wall integrated over the whole borehole wall. The value of $\tilde{B}_{00}$ results from enforcing this equality:

$$\tilde{q} = \int_{-\pi}^{\pi} - \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} \Big|_{\rho_0=1} d\theta_0 \quad \rightarrow \quad \tilde{B}_{00} = - \frac{\tilde{q}}{2\pi}.$$

Finally, substitution of Eqs. (3.3) and (3.4) into Eq. (2.8) delivers the zeroth order approximation to the fluid temperatures in the pipes:

$$\tilde{\Theta}_j^{(0)} = \frac{R_{pj}}{2\pi} \tilde{q}_j + \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_j=\rho_{pj}} d\theta_j. \quad (3.9)$$

3.2.2 Outer region

Far from the borehole, at non-dimensional distances of order $\rho_0 \sim 1/\sqrt{|s|} \sim 1/\text{Pe} \gg 1$, all terms in Eq. (2.10) are important. Nonetheless, the governing equation becomes solvable in the outer region thanks to the simplifications experienced by the velocity field, Eq. (2.2), far from the borehole:

$$v_\rho \Big|_{\rho_0 \sim 1/\text{Pe}} = \cos(\theta_0) + \mathcal{O}(\text{Pe}^2) \quad \text{and} \quad v_\theta \Big|_{\rho_0 \sim 1/\text{Pe}} = -\sin(\theta_0) + \mathcal{O}(\text{Pe}^2). \quad (3.10)$$

Hence, the zeroth order solution to the outer region must satisfy the boundary condition far from the borehole, Eq (2.9), and the partial differential equation

$$s \tilde{\Theta}_{\text{out}}^{(0)} + \text{Pe} \left[\cos(\theta_0) \frac{\partial \tilde{\Theta}_{\text{out}}^{(0)}}{\partial \rho_0} - \frac{\sin(\theta_0)}{\rho_0} \frac{\partial \tilde{\Theta}_{\text{out}}^{(0)}}{\partial \theta_0} \right] = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}_{\text{out}}^{(0)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}_{\text{out}}^{(0)}}{\partial \theta_0^2}.$$

The solution to the formulated problem is

$$\tilde{\Theta}_{\text{out}}^{(0)} = e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \sum_{n=-\infty}^{\infty} \tilde{C}_{0n} K_n \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{in\theta_0}, \quad (3.11)$$

where $K_n(z)$ stands for the modified Bessel function of the second kind of order n [80]. The yet-unspecified integration constants $\tilde{C}_{0n}$ result from the asymptotic matching of this expression with the zeroth order solution to the inner region. This matching replaces all boundary and continuity conditions specified at the borehole as these cannot be enforced onto the outer region due to the existence of the inner region.

3.2.3 Matching of both regions

At the intermediate region, located beyond the inner region ($\rho_0 \gg 1$) but closer to the borehole than the outer region ($\rho_0 \ll 1/\text{Pe}$, or alternatively $\rho_0 \text{Pe} \ll 1$), both inner and outer solutions must be equivalent. Hence, their difference must be negligible in this intermediate region, a condition that is expressed mathematically as

$$\tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0 \gg 1} - \tilde{\Theta}_{\text{out}}^{(0)} \Big|_{\rho_0 \text{Pe} \ll 1} \ll 1.$$

To enforce it, the behavior of the inner solution far from the borehole, given directly by Eq. (3.8), has to be compared against Eq. (3.11) expanded for small values of $\rho_0 \text{Pe}$ but large values of ρ_0 :

$$\begin{aligned} \tilde{\Theta}_{\text{out}}^{(0)} \Big|_{\substack{\rho_0 \text{Pe} \ll 1 \\ \rho_0 \gg 1}} &= \tilde{C}_{00} \left[-\ln \left(\frac{\rho_0 \text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) - \gamma + \mathcal{O}(\text{Pe}) \right] \\ &+ \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \tilde{C}_{0n} \left[\frac{2^{|n|-1} (|n| - 1)!}{\left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right)^{|n|}} + \mathcal{O}(\text{Pe}^{1-|n|}) \right] e^{in\theta_0}. \end{aligned}$$

The comparison reveals that the integration constants $\tilde{C}_{00}$ and $\tilde{A}_{00}$ must be as follows in order for the first two terms in Eq. (3.8) match the corresponding terms in the zeroth order outer solution:

$$\tilde{C}_{00} = -\tilde{B}_{00} = \frac{\tilde{q}}{2\pi} \quad \text{and} \quad \tilde{A}_{00} = \frac{\tilde{q}}{2\pi} \left[-\ln \left(\frac{\text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) - \gamma \right]. \quad (3.12)$$

The comparison also reveals that all $\tilde{B}_{0n}$ must be zero as no positive powers of ρ_0 exist among the leading terms of the expansion close to the borehole of the zeroth order outer solution. Finally, also the values of $\tilde{C}_{0n}$ with $n \neq 0$ must be zero despite the existence of terms with negative powers of ρ_0 in the zeroth order inner solution. The reason is that the matching of both expressions requires the integration constants $\tilde{C}_{0n}$ to be of order $\text{Pe}^{|n|}$. Hence, they do

not belong to the zeroth order level of the asymptotic expansion of the problem but to higher order corrections.

In summary, the asymptotic matching of the zeroth order inner and outer solutions leads to the following final expressions for the zeroth order inner and outer solutions:

$$\tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0 \geq 1} = \tilde{A}_{00} - \frac{\tilde{q}}{2\pi} \ln(\rho_0) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{A}_{0n}}{\rho_0^{|n|}} e^{in\theta_0}, \quad (3.13)$$

$$\tilde{\Theta}_{\text{out}}^{(0)} = \frac{\tilde{q}}{2\pi} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_0 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right), \quad (3.14)$$

where for the sake of compactness the expression for $\tilde{A}_{00}$ has not been substituted.

3.3 First order correction

The first order correction to the problem consists in $\tilde{\Theta}_{\text{in}}^{(1)}$, $\tilde{\Theta}_{\text{out}}^{(1)}$, and $\tilde{\Theta}_j^{(1)}$. As before, the corresponding mathematical problems result from substituting Eqs. (3.3) and (3.4) into the heat transfer problem formulated in Chapter 2. Now, the terms of order Pe are retained so that the next level of the asymptotic expansion of the problem is analyzed. Thus, only the terms smaller than Pe are discarded.

3.3.1 Inner region

In the inner region, where $\rho_0 \sim 1$, the partial differential equation to solve in the ground includes now an additional term involving the zeroth order solution obtained before:

$$v_\rho \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} + \frac{v_\theta}{\rho_0} \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \theta_0} = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}_{\text{in}}^{(1)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}_{\text{in}}^{(1)}}{\partial \theta_0^2}. \quad (3.15)$$

In the grout, however, the equation to solve is the same as before as no convective heat transfer exists and thermal inertia is still negligible at this level of the asymptotic expansion of the problem:

$$0 = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}_{\text{in}}^{(1)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}_{\text{in}}^{(1)}}{\partial \theta_0^2}. \quad (3.16)$$

The two differential equations must be solved together with the continuity conditions at the borehole wall,

$$\tilde{\Theta}_{\text{in}}^{(1)} \Big|_{\rho_0=1^-} = \tilde{\Theta}_{\text{in}}^{(1)} \Big|_{\rho_0=1^+} \quad \text{and} \quad -\kappa \frac{\partial \tilde{\Theta}_{\text{in}}^{(1)}}{\partial \rho_0} \Big|_{\rho_0=1^-} = -\frac{\partial \tilde{\Theta}_{\text{in}}^{(1)}}{\partial \rho_0} \Big|_{\rho_0=1^+}, \quad (3.17)$$

and with the boundary conditions at the pipe walls

$$-\rho_{pj}\kappa\frac{\partial\tilde{\Theta}_{\text{in}}^{(1)}}{\partial\rho_j}\bigg|_{\rho_j=\rho_{pj}} = \frac{\tilde{\Theta}_{\text{in}}^{(1)} - \tilde{\Theta}_{\text{in}}^{(1)}\big|_{\rho_j=\rho_{pj}}}{R_{pj}}. \quad (3.18)$$

Also the prescribed heat injection rates per unit pipe length, $\tilde{q}_j$, must be enforced onto the first order correction to the inner region. But their values have already been imposed onto the zeroth order solution obtained before. Consequently, the first order correction must have zero heat injection rates per unit pipe length so that its combination with the zeroth order solution still yields the prescribed values $\tilde{q}_j$:

$$0 = \int_{-\pi}^{\pi} -\kappa\frac{\partial\tilde{\Theta}_{\text{in}}^{(1)}}{\partial\rho_j}\bigg|_{\rho_j=\rho_{pj}} \rho_{pj}d\theta_j.$$

Again, no exact solution to the formulated problem is known which does not pose a problem as only the general behavior of the first order correction in the ground is required. In this case, the general solution to the governing equation in the ground includes additional terms that represent the particular solution to the partial differential equation:

$$\begin{aligned} \tilde{\Theta}_{\text{in}}^{(1)}\big|_{\rho_0\geq 1} &= \tilde{A}_{10} + \tilde{B}_{10}\ln(\rho_0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{\tilde{A}_{1n}}{\rho_0^{|n|}} + \rho_0^{|n|}\tilde{B}_{1n} \right) e^{in\theta_0} \\ &\quad - \frac{\tilde{q}}{4\pi} \left(\rho_0 + \frac{1}{\rho_0} \right) \ln(\rho_0) \cos(\theta_0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\tilde{A}_{0n}}{4} \left[\frac{e^{i\frac{n}{|n|}(|n|+1)\theta_0}}{\rho_0^{|n|-1}} + \frac{e^{i\frac{n}{|n|}(|n|-1)\theta_0}}{\rho_0^{|n|+1}} \right]. \end{aligned}$$

In the absence of thermal inertia, the heat injection rate per unit borehole length $\tilde{q}$ coincides with the heat flux at the borehole wall integrated over the whole borehole wall. Substituting Eq. (3.3) into this condition leads to

$$\tilde{q} = \int_{-\pi}^{\pi} -\frac{\partial\tilde{\Theta}_{\text{in}}^{(0)}}{\partial\rho_0}\bigg|_{\rho_0=1} d\theta_0 + \text{Pe} \int_{-\pi}^{\pi} -\frac{\partial\tilde{\Theta}_{\text{in}}^{(1)}}{\partial\rho_0}\bigg|_{\rho_0=1} d\theta_0 + \mathcal{O}(\text{Pe}^2).$$

Since the zeroth order solution already matches the value of $\tilde{q}$, the first order correction must not contribute to this condition, a requirement that leads to the value of $\tilde{B}_{10}$:

$$0 = \int_{-\pi}^{\pi} -\frac{\partial\tilde{\Theta}_{\text{in}}^{(1)}}{\partial\rho_0}\bigg|_{\rho_0=1} d\theta_0 \quad \rightarrow \quad \tilde{B}_{10} = \frac{\tilde{A}_{0(+1)} + \tilde{A}_{0(-1)}}{2}.$$

Finally, the first order correction to the fluid temperatures results from substituting Eqs. (3.3) and (3.4) into Eq. (2.8) and grouping the terms of order Pe :

$$\tilde{\Theta}_j^{(1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Theta}_{\text{in}}^{(1)}\big|_{\rho_j=\rho_{pj}} d\theta_j. \quad (3.19)$$

3.3.2 Outer region

At distances from the borheole of order $\rho_0 \sim 1/\text{Pe}$, the velocity field is given by Eq. (3.10) which represents the only simplification made to the governing equation in the outer region. Since the terms being neglected in the velocity field are of order Pe^2 , exactly the same partial differential equation results for the first order correction as for the zeroth order solution to the outer region:

$$s\tilde{\Theta}_{\text{out}}^{(1)} + \text{Pe} \left[\cos(\theta_0) \frac{\partial \tilde{\Theta}_{\text{out}}^{(1)}}{\partial \rho_0} - \frac{\sin(\theta_0)}{\rho_0} \frac{\partial \tilde{\Theta}_{\text{out}}^{(1)}}{\partial \theta_0} \right] = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}_{\text{out}}^{(1)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}_{\text{out}}^{(1)}}{\partial \theta_0^2}.$$

Combined with the same homogeneous boundary condition at infinity, Eq. (2.9), the first order correction presents the same general expression as the zeroth order solution:

$$\tilde{\Theta}_{\text{out}}^{(1)} = e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \sum_{n=-\infty}^{\infty} \tilde{C}_{1n} K_n \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{in\theta_0}.$$

3.3.3 Matching of both regions

The asymptotic matching between the inner and outer solutions is carried out again at an intermediate region located beyond the inner region but before the outer region. The differences between both solutions must be negligible there, a condition mathematically expressed as

$$\left(\tilde{\Theta}_{\text{in}}^{(0)} + \text{Pe} \tilde{\Theta}_{\text{in}}^{(1)} \right) \Big|_{\rho_0 \sqrt{\text{Pe}} \gg 1} - \left(\tilde{\Theta}_{\text{out}}^{(0)} + \text{Pe} \tilde{\Theta}_{\text{out}}^{(1)} \right) \Big|_{\substack{\rho_0 \text{Pe} \ll 1 \\ \rho_0 \gg 1}} \ll 1. \quad (3.20)$$

The zeroth order outer and inner solutions appear in the formulated condition as they contribute terms that need to be picked up by the first order corrections. So, the zeroth order outer solution, expanded for small values of $\rho_0 \text{Pe}$ but large values of ρ_0 introduces terms of order Pe :

$$\tilde{\Theta}_{\text{out}}^{(0)} \Big|_{\substack{\rho_0 \text{Pe} \ll 1 \\ \rho_0 \gg 1}} = \frac{\tilde{q}}{2\pi} \left[1 + \frac{\rho_0 \text{Pe}}{2} \cos(\theta_0) \right] \left[-\ln \left(\frac{\rho_0 \text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) - \gamma \right] + \mathcal{O}(\text{Pe}^2).$$

These are matched to terms of the first order correction to the inner region. Some match without further intervention but others require certain integration constants to be properly fixed:

$$\tilde{B}_{1(\pm 1)} = \frac{\tilde{q}}{8\pi} \left[-\ln \left(\frac{\text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) - \gamma \right].$$

All other integration constants $\tilde{B}_{1n}$ are zero as no further positive powers of ρ_0 exist among the leading terms of the expansion of the outer solution for small value of $\rho_0 \text{Pe}$.

Also the zeroth order inner solution introduces terms, that go as $1/\rho_0^{|n|}$, which need to be taken into account. In this case, by the first order correction to the outer region whose expansion

for small values of $\rho_0 \text{Pe}$ is

$$\begin{aligned} \tilde{\Theta}_{\text{out}}^{(1)} \Big|_{\substack{\rho_0 \text{Pe} \ll 1 \\ \rho_0 \gg 1}} = & \tilde{C}_{10} \left[-\ln \left(\frac{\rho_0 \text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) - \gamma + \mathcal{O}(\text{Pe}) \right] \\ & + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \tilde{C}_{1n} \left[\left[1 + \frac{\rho_0 \text{Pe}}{2} \cos(\theta_0) \right] \frac{2^{|n|-1} (|n|-1)!}{\left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right)^{|n|}} + \mathcal{O}(\text{Pe}^2) \right] e^{\iota n \theta_0}. \end{aligned} \quad (3.21)$$

The outcome of the matching is

$$\tilde{C}_{1(\pm 1)} = \frac{\tilde{A}_{0(\pm 1)}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}}.$$

All other integration constants $\tilde{C}_{1n}$ are zero as the matching requires them to be of order $\text{Pe}^{|n|}$. Consequently, they do not belong to the level of approximation considered here. Nonetheless, setting $\tilde{C}_{1(\pm 2)} = 0$ has some consequences for the performed matching process as it narrows the intermediate region. Since the corresponding unmatched terms in the zeroth order inner solution must be negligible in the intermediate region, it is necessary to ensure they are small compared to Pe . This leads to the following condition on ρ_0 :

$$\frac{\tilde{A}_{0(\pm 2)}}{\rho_0^2} e^{\pm \iota 2 \theta_0} \ll \text{Pe} \quad \rightarrow \quad \rho_0 \gg \frac{1}{\sqrt{\text{Pe}}} \implies \rho_0 \sqrt{\text{Pe}} \gg 1.$$

Finally, expressions for two additional integration constants are obtained from matching terms between the outer and inner solutions:

$$\begin{aligned} \tilde{C}_{10} &= -\frac{\tilde{A}_{0(+1)} + \tilde{A}_{0(-1)}}{2}, \\ \tilde{A}_{10} &= \frac{\tilde{A}_{0(+1)} + \tilde{A}_{0(-1)}}{2} \left[\ln \left(\frac{\text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) + \gamma + \frac{1}{2} \right]. \end{aligned}$$

In summary, the asymptotic matching between first order corrections to the inner and outer regions leads to the following final expressions for the first order corrections:

$$\begin{aligned} \tilde{\Theta}_{\text{in}}^{(1)} \Big|_{\rho_0 \geq 1} = & \tilde{A}_{10} + \tilde{B}_{10} \ln(\rho_0) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{A}_{1n}}{\rho_0^{|n|}} e^{\iota n \theta_0} + \tilde{B}_{1(+1)} \rho_0 e^{\iota \theta_0} + \tilde{B}_{1(-1)} \rho_0 e^{-\iota \theta_0} \\ & - \frac{\tilde{q}}{4\pi} \left(\rho_0 + \frac{1}{\rho_0} \right) \ln(\rho_0) \cos(\theta_0) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{A}_{0n}}{4} \left[\frac{e^{\iota \frac{n}{|n|} (|n|+1) \theta_0}}{\rho_0^{|n|-1}} + \frac{e^{\iota \frac{n}{|n|} (|n|-1) \theta_0}}{\rho_0^{|n|+1}} \right], \end{aligned} \quad (3.22)$$

$$\begin{aligned} \tilde{\Theta}_{\text{out}}^{(1)} = & \tilde{C}_{10} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_0 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) + \tilde{C}_{1(+1)} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_1 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{+\iota \theta_0} \\ & + \tilde{C}_{1(-1)} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_1 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{-\iota \theta_0}, \end{aligned} \quad (3.23)$$

where for the sake of compactness the expressions for the integration constants have not been substituted.

3.4 Numerical examples

The capabilities and limitations of the developed model are demonstrated next by comparing its results against detailed numerical simulations of the unsteady heat transfer problem formulated in Chapter 2. Since no exact expressions are available for the zeroth order inner solution and its first order correction, both are obtained numerically as well. On the contrary, the asymptotic solution to the outer region is known analytically, only requiring the values of $\tilde{A}_{0(\pm 1)}$ to be obtained numerically.

3.4.1 Borehole description

All presented comparisons use the borehole configuration depicted in Figure 3.1. Although additional borehole configurations, with different geometrical and thermal characteristics as well as with coaxial and double U-shaped probes, have been analyzed, they are not discussed here as their results and conclusions are in line with the ones presented next.

The chosen configuration, whose geometrical and thermal characteristics are summarized in Table 3.1, consists in a borehole of diameter $2r_b = 152$ mm with two heterogeneous pipes inside. Pipe 1 has an outer diameter of $2r_{p1} = 40$ mm and a wall thickness of $d_1 = 3.7$ mm. Pipe 2 presents an outer diameter of $2r_{p2} = 50$ mm and a wall thickness of $d_2 = 4.6$ mm. Both pipes are placed at 5 mm from the borehole wall and at $\pm 45^\circ$ from the rear stagnation point. Hence, the polar coordinates of their centers are (51 mm, $+45^\circ$) and (46 mm, -45°), respectively.

Both pipes are surrounded by a conventional grout of thermal conductivity $k_b = 1.50$ W/(m K) [1] and thermal diffusivity $\alpha_b = 1.00 \cdot 10^{-7}$ m²/s. Its thermal properties differ from the effective ones for the ground that are $k_g = 3.00$ W/(m K) and $\alpha_g = 2.24 \cdot 10^{-7}$ m²/s [18].

Pure water is used as heat carrying liquid. Its density, specific heat capacity, thermal conductivity, and dynamic viscosity are 999 kg/m³, 4184 J/(kg K), 0.577 W/(m K), and $1.138 \cdot 10^{-3}$ kg/(m s), respectively. A turbulent flow regime is ensured in both pipes by imposing a mass flow rate of 0.25 kg/s. The corresponding convective heat transfer coefficients h_1 and h_2 are obtained using Gnielinski's correlations [38] which combined with the thermal conductivities of the pipes $k_1 = k_2 = 0.42$ W/(m K) lead to the following values for the inner thermal resistances of the pipes:

$$R_{p1} = 0.535 \frac{\text{m K}}{\text{W}} \quad \text{and} \quad R_{p2} = 0.544 \frac{\text{m K}}{\text{W}}.$$

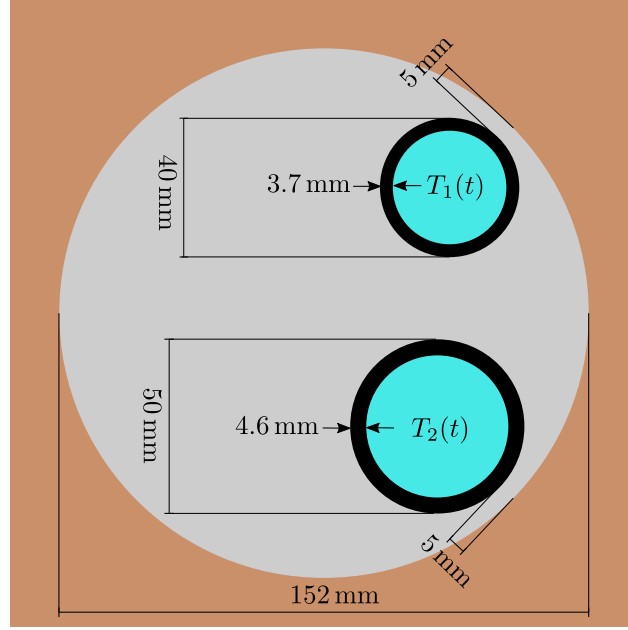


Figure 3.1: Borehole configuration for the numerical examples.

3.4.2 Simulation procedure

All required numerical simulations are performed using the commercial software package COMSOL [22]. The convenience of COMSOL is that mathematical problems can be specified in a very precise way and without restrictions, even allowing the use of complex variables and arithmetic in the formulation and solution of problems.

Reference solution

The benchmark solution is obtained by directly solving the unsteady heat transfer problem formulated in Section 2.3. That is, Eqs. (2.10) and (2.11) for the unsteady thermal response of ground and grout, respectively, together with Eq. (2.9) for the boundary condition far from the borehole and Eq. (2.7) for the boundary condition at the pipe walls. Since heat injection rates per unit pipe length are enforced at each pipe, Eq. (2.8) is used to compute the corresponding fluid temperatures that appear in the boundary conditions at the pipe walls. Finally, the continuity conditions at the borehole wall, Eq. (2.5), are enforced automatically by COMSOL.

A sufficiently large computational domain, that spans till $\rho_0 = 1000$, is defined in COMSOL so that truncation of the infinite ground has no negative consequences for the accuracy of the reference solution. And a sufficiently fine mesh, with the number of triangular elements ranging from 164,508 till 760,290 depending on the testcase, is used to ensure error levels below the threshold required by the present work.

Variable	Value
Borehole diameter $2r_b$	152 mm
Grout thermal conductivity k_b	1.50 W/(m K)
Ground thermal conductivity k_g	3.00 W/(m K)
Grout thermal diffusivity α_b	$1.00 \cdot 10^{-7} \text{ m}^2/\text{s}$
Ground thermal diffusivity α_g	$2.24 \cdot 10^{-7} \text{ m}^2/\text{s}$
Water density	999 kg/m ³
Water specific heat capacity	4184 J/(kg K)
Water thermal conductivity	0.577 W/(m K)
Water dynamic viscosity	$1.138 \cdot 10^{-3} \text{ kg}/(\text{m s})$
Pipe 1 outer diameter $2r_{p1}$	40 mm
Pipe 1 wall thickness d_1	3.7 mm
Pipe 1 wall thermal conductivity k_1	0.42 W/(m K)
Pipe 1 center radial coordinate	51 mm
Pipe 1 center angular coordinate	+45°
Pipe 1 mass flow rate	0.25 kg/s
Pipe 1 inner thermal resistance R_{p1}	0.535 m K/W
Pipe 2 outer diameter $2r_{p2}$	50 mm
Pipe 2 wall thickness d_2	4.6 mm
Pipe 2 wall thermal conductivity k_2	0.42 W/(m K)
Pipe 2 center radial coordinate	46 mm
Pipe 2 center angular coordinate	−45°
Pipe 2 mass flow rate	0.25 kg/s
Pipe 2 inner thermal resistance R_{p2}	0.544 m K/W

Table 3.1: Geometrical and thermal characteristics of the borehole.

Zeroth order solution to inner region

The zeroth order solution to the inner region is obtained by directly solving the mathematical problem formulated in Section 3.2.1. That is, Eq. (3.5) for the quasi-steady thermal response of grout and ground, the boundary condition at the pipe walls specified in Eq. (3.7), and the expression given in Eq. (3.9) for the fluid temperatures. As before, the continuity conditions in temperature and normal heat flux, Eq. (3.6), are enforced automatically by COMSOL.

Additionally, the asymptotic matching with the zeroth order solution to the outer region must be enforced as well in COMSOL. To do it, the behavior of the zeroth order inner solution far from the borehole, directly inferred from Eq. (3.13), is imposed as Dirichlet condition at a

sufficiently large distance from the borehole:

$$\tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0 \gg 1} = \frac{\tilde{q}}{2\pi} \left[-\ln \left(\frac{\rho_0 \text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) - \gamma \right] + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{A}_{0n}}{\rho_0^{|n|}} e^{in\theta_0}.$$

The values of $\tilde{A}_{0n}$ with $n \neq 0$ need to be obtained from the numerical simulation itself. This is achieved by computing the Fourier series expansion of the numerical simulation at the borehole wall and then realizing that the resulting Fourier coefficients coincide with $\tilde{A}_{0n}$:

$$\tilde{A}_{0n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0=1} d\theta_0.$$

Only the lowest harmonics are actually required for the computations as their relevance decays rapidly with the size of the computational domain. Thus, for an outer radius $\rho_0 = 1000$, as used here, the coefficients $\tilde{A}_{0(\pm 1)}$ and $\tilde{A}_{0(\pm 2)}$ are enough to exceed the desired accuracy level. Additionally, a fine mesh with a large number of triangular elements, between 46,208 and 1,068,172 depending on the testcase, is used as well to keep the discretization error below a certain threshold.

First order correction to inner region

The first order correction to the inner region is obtained directly solving the mathematical problem formulated in Section 3.3.1. That is, Eq. (3.15) for the quasi-steady thermal response of the ground, for which the numerically-obtained zeroth order solution is required for the left hand side, Eq. (3.16) for the quasi-steady thermal response of the grout, Eq. (3.18) for the boundary condition at the pipe walls, and Eq. (3.19) for the fluid temperature in each pipe. Again, the enforcement of the continuity conditions at the borehole wall, Eq. (3.17), is done automatically by COMSOL.

As before, the asymptotic matching far from the borehole must be enforced as well in COMSOL. To do it, the behavior of the first order correction far from the borehole, directly inferred from Eq. (3.22), is imposed as Dirichlet condition at a sufficiently large distance from the borehole:

$$\begin{aligned} \tilde{\Theta}_{\text{in}}^{(1)} \Big|_{\rho_0 \gg 1} = & \frac{\tilde{A}_{0(+1)} + \tilde{A}_{0(-1)}}{2} \left[\ln \left(\frac{\rho_0 \text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) + \gamma + \frac{1}{2} \right] \\ & - \frac{\tilde{q}}{4\pi} \rho_0 \cos(\theta_0) \left[\ln \left(\frac{\rho_0 \text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) + \gamma \right] - \frac{\tilde{q}}{4\pi} \frac{\ln(\rho_0)}{\rho_0} \cos(\theta_0) \\ & + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{A}_{1n}}{\rho_0^{|n|}} e^{in\theta_0} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{A}_{0n}}{4} \left[\frac{e^{i \frac{n}{|n|} (|n|+1)\theta_0}}{\rho_0^{|n|-1}} + \frac{e^{i \frac{n}{|n|} (|n|-1)\theta_0}}{\rho_0^{|n|+1}} \right]. \end{aligned} \quad (3.24)$$

Finally, the values of $\tilde{A}_{1n}$ are obtained in the same way as in the zeroth order solution to the inner region. The resulting expressions, however, are not that simple due to the particular

solution to the governing equation in the ground:

$$\tilde{A}_{1(\pm 1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Theta}_{\text{in}}^{(1)} \Big|_{\rho_0=1} e^{\mp \iota \theta_0} - \frac{\tilde{A}_{0(\pm 2)}}{4} + \frac{\tilde{q}}{8\pi} \left[\ln \left(\frac{\text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) + \gamma \right]$$

and, for $|n| \neq 1$,

$$\tilde{A}_{1n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Theta}_{\text{in}}^{(1)} e^{-\iota n \theta_0} d\theta_0 - \frac{\tilde{A}_{0(n+1)} + \tilde{A}_{0(n-1)}}{4}.$$

Again, inclusion of the coefficients $\tilde{A}_{1(\pm 1)}$ and $\tilde{A}_{1(\pm 2)}$ is enough to exceed the desired accuracy level, and a large number of triangular elements, between 46,208 and 1,068,172 depending on the testcase, is used to keep the discretization error below a certain threshold.

Outer region

No numerical simulations are required to obtain the solution to the outer region as analytical expressions have been derived in the present work. Nonetheless, these expressions make use of the coefficients $\tilde{A}_{0(\pm 1)}$ which are not known analytically. Hence, their values still need to be obtained from the numerical simulation performed in COMSOL for the zeroth order solution to the inner region.

To numerically evaluate the zeroth order solution and the first order correction to the outer region, Eqs. (3.14) and (3.23) are implemented in Fortran95 using the algorithms developed by Zhang and Jin [108] for the computation of the modified Bessel functions of the second kind.

3.4.3 Solutions in the Laplace plane

The thermal interaction of the considered borehole with a groundwater stream still has two unspecified parameters, namely, the Peclet number of the groundwater flow and the non-dimensional position s in the complex-valued Laplace plane. In the present section two different sets of values for Pe and s are considered. The first set serves to showcase the capabilities of the proposed asymptotic model while the second set highlights its limitations.

Additionally to Pe and s , time laws for the heat injection rates per unit pipe length are required as well. For the comparisons presented next, time-constant values are considered, namely, $q_1(\tau) = 1$ and $q_2(\tau) = 3$. Their Laplace transforms are $\tilde{q}_1(s) = 1/s$ and $\tilde{q}_2(s) = 3/s$, respectively.

$\text{Pe} = 0.03$ and $s = (1 + \iota)/2000$

Figure 3.2 shows the grout/ground temperature for the favorable set of values for Pe and s . Modulus and argument of $\tilde{\Theta}$ are represented on the left and right plots, respectively, and three different solutions are compared in each plot. The reference solution is depicted using

a solid color map, with the borehole wall and the two pipes marked by white circles. The zeroth order solution is represented using dashed red isolines and the complete asymptotic solution is drawn using solid black isolines.

The inner region, located at non-dimensional radial distances from the borehole of order unity, is shown at the top of Figure 3.2. Since the zeroth order solution to the inner region completely neglects the convective transport of heat, it is concentric with respect to the pipes that act as heat sources in the problem. The first order correction, on the contrary, accounts for the convective transport of heat and, therefore, shifts the solution towards the right. Since the considered Peclet number is small, the observed shift is small as well.

The performance of the proposed model is very good, closely following the isolines for the modulus of the reference solution. In this sense, addition of the first order correction to the zeroth order solution clearly improves the accuracy of the model, which is best close to the borehole and inside the borehole.

The performance of the proposed model, however, is less good for the argument of the grout/ground temperature. There, the lack of thermal inertia in the mathematical problems defining the zeroth order solution and first order correction leads to a faster-than-necessary thermal response. Close to the borehole, as well as inside the borehole, the discrepancies are minimal and the proposed model performs well. But predicted values deviate significantly from the expected ones for increasing radial distances to the borehole.

The outer region, located at non-dimensional radial distances from the borehole of order $\rho_0 \sim 1/\sqrt{|s|} \sim 40$, is shown at the middle of Figure 3.2. In this case, the zeroth order solution and first order correction to the outer region must be used in the comparison with the reference solution.

While the zeroth order solution closely follows the modulus and argument isolines of the reference solution, the complete asymptotic solution directly overlaps them. The good performance of both approximation levels is explained by their mathematical problems. In both cases the simplified governing equation differs from the full equation only by terms in the velocity field that decay quadratically with ρ_0 . Hence, both approximation levels are good representations of the full mathematical problem far from the borehole. Combined with the condensed, albeit valuable, information provided by the asymptotic matching with the inner region leads to the excellent results presented in Figure 3.2.

Although not shown here, the accuracy of the proposed model far from the borehole is also outstanding for combinations of Pe and s that do not follow the distinguished limit $Pe \sim \sqrt{|s|}$. This is so because the simplified governing equations defining the zeroth order solution and first order correction in the outer region include all three phenomena. That is, thermal inertia, heat convection, and heat conduction. Thus, when one phenomenon becomes dominant, the proposed model still represents it correctly.

The solution to the outer region is meant to perform well at large distances from the borehole, as demonstrated in Figure 3.2. Hence, good accuracy is not expected close to the borehole. Nonetheless, the solution to the outer region also performs well there, as shown at the bottom of Figure 3.2. While the zeroth order solution to the outer region follows more or less the

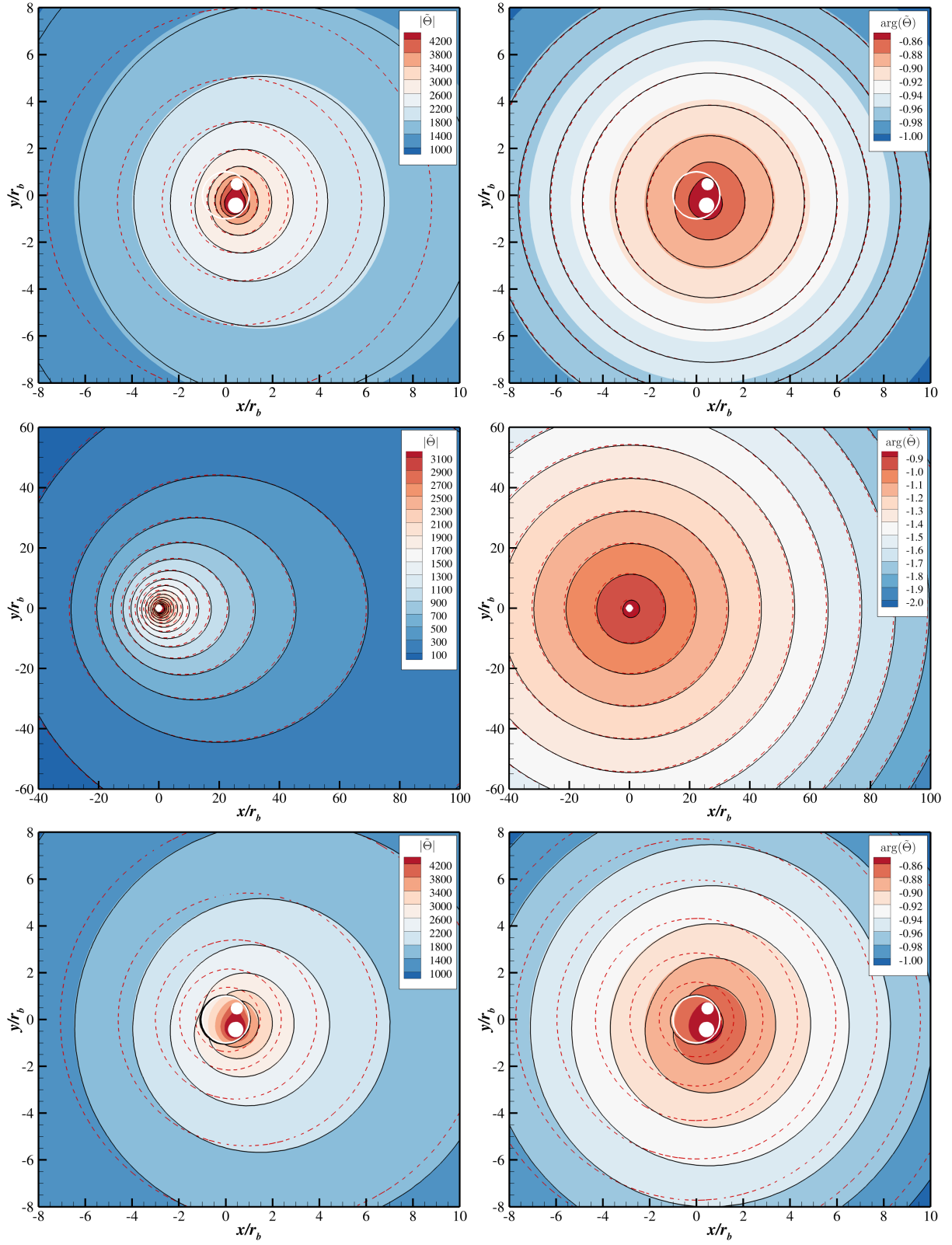


Figure 3.2: (Top) inner solution, (middle) outer solution, and (bottom) outer solution close to the borehole for $Pe = 0.03$ and $s = (1 + \iota)/2000$.

modulus and argument isolines of the reference solution, the complete asymptotic model follows them surprisingly good. In fact, better than the solution to the inner region presented before.

This *a-priori* unexpected accuracy is a consequence again of the presence of all three phenomena (thermal inertia, heat convection, and heat conduction) in the governing equations for the outer region. Then, the inner region can be seen as a simplified version of the outer region in which heat conduction dominates.

Of course, the outer solution is unable to represent the heat transfer taking place inside the borehole, reason why in the middle and bottom plots in Figure 3.2 the dashed red isolines and black isolines end at the borehole wall. Additionally, the outer solution requires information from the inner solution to correctly set the values of its integration constants $\tilde{C}_{00}$, $\tilde{C}_{10}$, and $\tilde{C}_{1(\pm 1)}$. Hence, an inner region with its asymptotic solution is always required regardless of the performance of the outer solution close to the borehole.

$Pe = 1.00$ and $s = (1 + \iota)/2000$

When the Peclet number of the groundwater flow or the non-dimensional position s in the complex-valued Laplace plane are not small compared to unity, the premise for the performed asymptotic analysis no longer exists. Then, the proposed model loses its validity and a sharp drop in its accuracy is expected.

Figure 3.3 shows the grout/ground temperature for the unfavorable set of values of $Pe = 1.00$ and $s = (1 + \iota)/2000$. Again, modulus and argument of $\tilde{\Theta}$ are represented on the left and right plots, respectively, and three different solutions are compared in each plot using the same representation convention as in the previous subsection.

As anticipated, the proposed model fails to correctly reproduce the temperature distribution in grout and ground. In fact, only the modulus of the grout temperature is predicted reasonably well. Everything else presents unacceptable error levels. Also the addition of the first order correction to the zeroth order solution, which in the previous example improved the results, does not help in this case.

Despite the bad results close to the borehole, the proposed model performs surprisingly well further away from the borehole, at the outer region. As seen in Figure 3.3, it correctly reproduces there the modulus and argument of the ground temperature. Again, this is a consequence of two factors. First, that all three phenomena (thermal inertia, heat convection, and heat conduction) are present in the governing equations for the outer region. Second, that the solution to the inner region is accurate enough at the borehole wall, where the values of $\tilde{A}_{0n}$ are defined, so that the integration constants $\tilde{C}_{00}$, $\tilde{C}_{10}$, and $\tilde{C}_{1(\pm 1)}$ are accurate as well.

The solution to the outer region also performs well close to the borehole, as observed at the bottom of Figure 3.3. In fact, it clearly outperforms the solution to the inner region presented before. This casts again doubts on the need and usefulness of the inner region. But as already pointed out in the previous example, the inner region is integral part of the asymptotic analysis leading to the proposed model so that it cannot be obviated or circumvented.

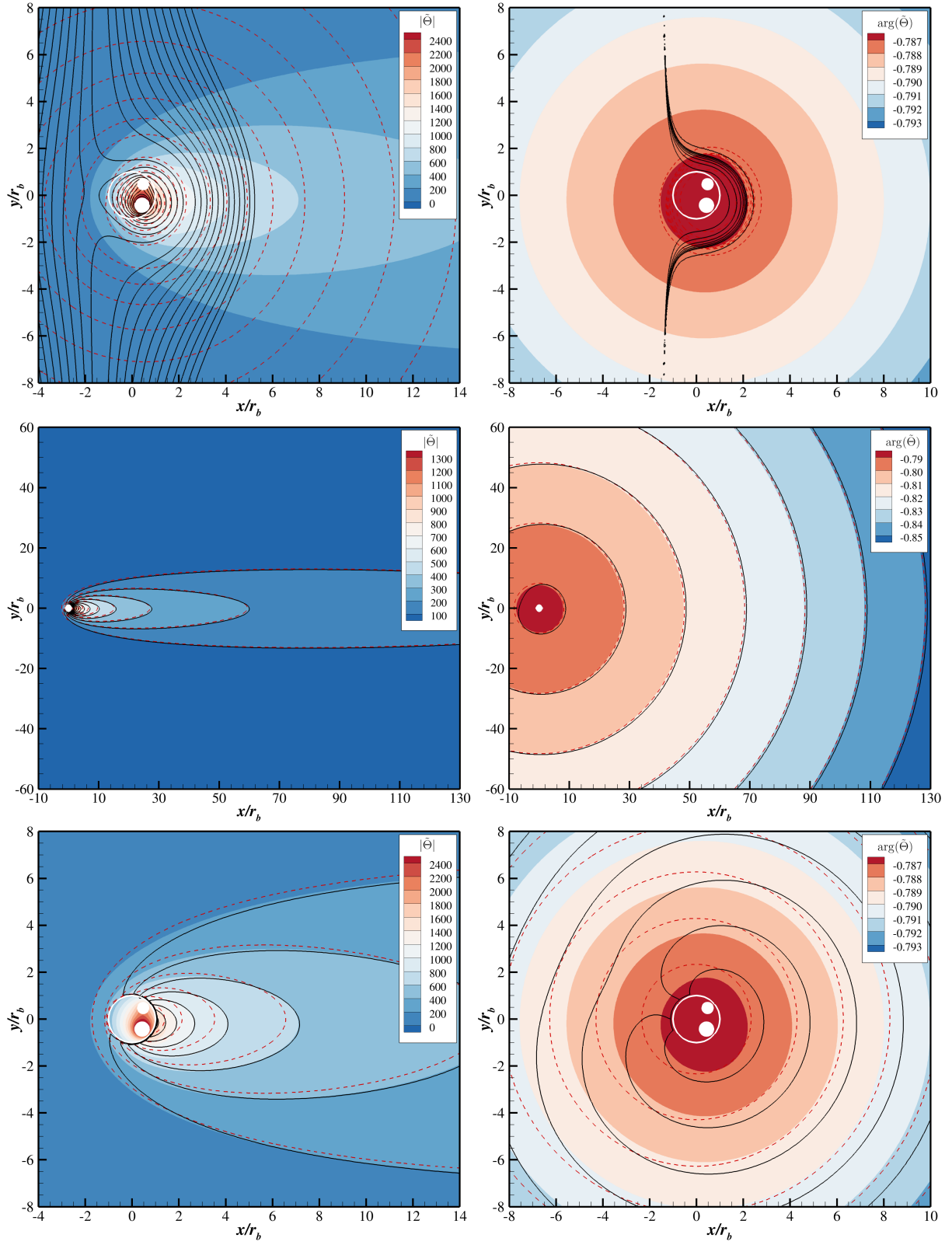


Figure 3.3: (Top) inner solution, (middle) outer solution, and (bottom) outer solution close to the borehole for $Pe = 1.00$ and $s = (1 + \iota)/2000$.

Although not shown here, the same conclusions regarding the performance of the proposed model apply to other combinations of unfavorable values for Pe and s . For instance, small Peclet numbers combined with order unity values of s . Or Peclet numbers and values of s that are both of order unity. In all these cases the proposed model performs well far from the borehole but fails to correctly reproduce the temperature field inside the borehole and close to it.

3.4.4 Solutions in the time domain

Known the grout/ground temperature in the complex-valued Laplace plane, the time evolution of the problem is recovered by applying the inverse Laplace transform. Due to the mathematical complexity of the developed model, this step is done numerically [44]. The algorithm to use requires the evaluation of the solution at a reduced set of positions s in the Laplace plane. The computation of these solutions as well as the subsequent application of the inversion algorithm is performed entirely within COMSOL.

The computational cost of the followed approach is lower than the one of time-marching the governing equations till the time instants of interest. For the two examples shown next, for which the complete grout/ground temperatures are required, a threefold speedup is achieved. Also overall grid independence is accomplished much easier for this unsteady problem as temporal discretization errors are nonexistent and the accuracy of the numerical inversion of the Laplace transform is on par with machine precision [44].

Pe = 0.03

Figure 3.4 shows the time evolution of the grout/ground temperature for the favorable Peclet number of 0.03. Three different instants in time are shown, namely, $\tau = 20$ at the top, $\tau = 200$ at the middle, and $\tau = 2000$ at the bottom. The left and right plots represent, respectively, the inner and outer solutions using the same representation convention as in previous figures, with the reference solution included as well in the plots.

One of the hypotheses underlying the asymptotic analysis of the present work is that the characteristic heat injection time t_q is large compared to the characteristic transversal diffusion time t_b . This requires the non-dimensional time τ to be large compared to unity for the developed model to perform well. The results shown in Figure 3.4 confirm this. The shown results also reveal that the model performs well for the not-so-large value of $\tau = 20$, especially the asymptotic solution to the outer region.

To quantitatively assess the accuracy of the proposed model, Figure 3.5 represents, for $\tau = 2000$, the different solutions along the green line depicted in Figure 3.4. This line starts at the rear stagnation point of the borehole, located at $(r_0, \theta_0) = (r_b, 0^\circ)$, and follows the groundwater flow downstream of the borehole. The left plot in Figure 3.5 represents the temperatures attained by the reference solution and by the different asymptotic solutions for the inner and outer regions. The right plot represents the absolute errors of the different asymptotic solutions when compared to the reference solution.

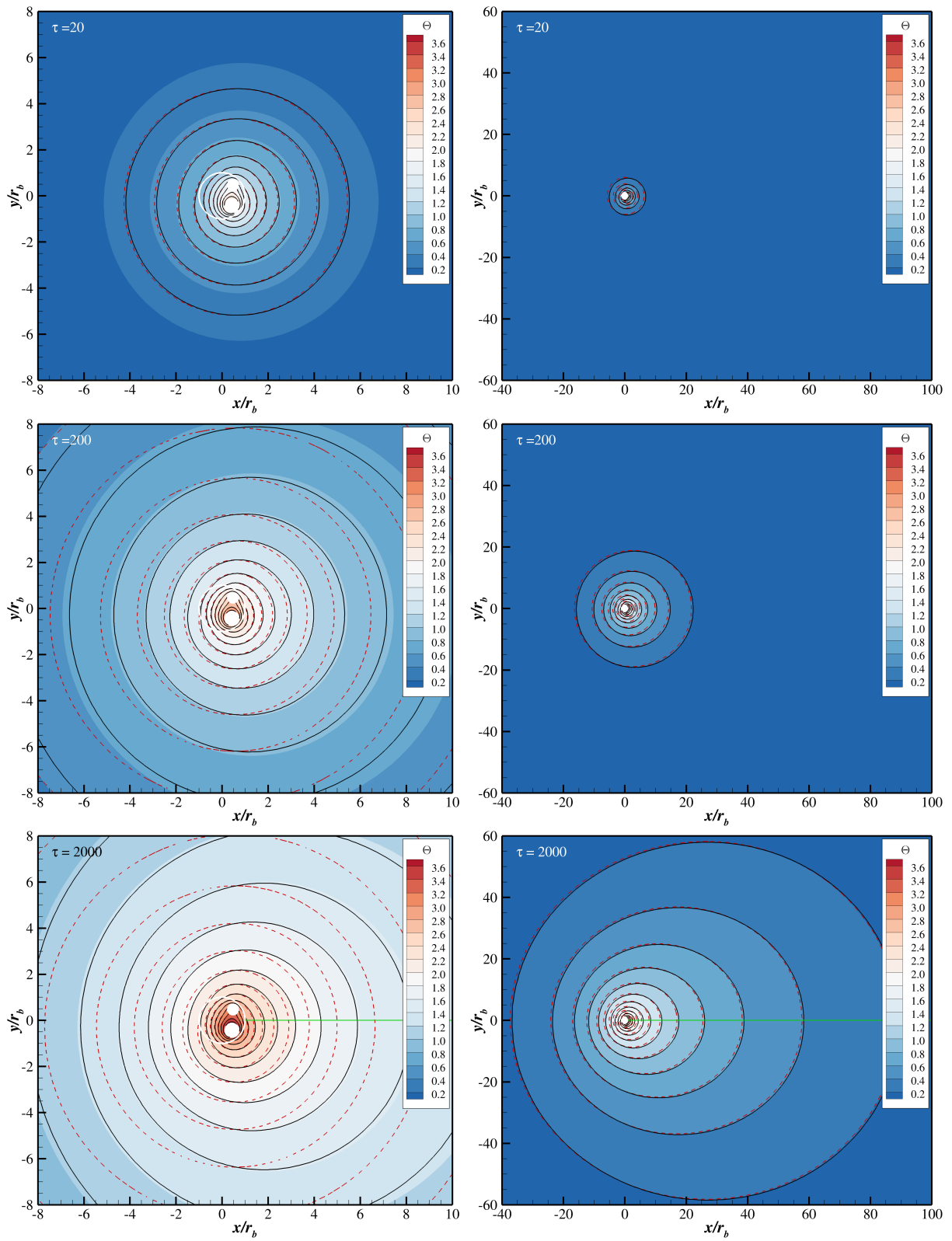


Figure 3.4: (Left) inner solution and (right) outer solution for $Pe = 0.03$.

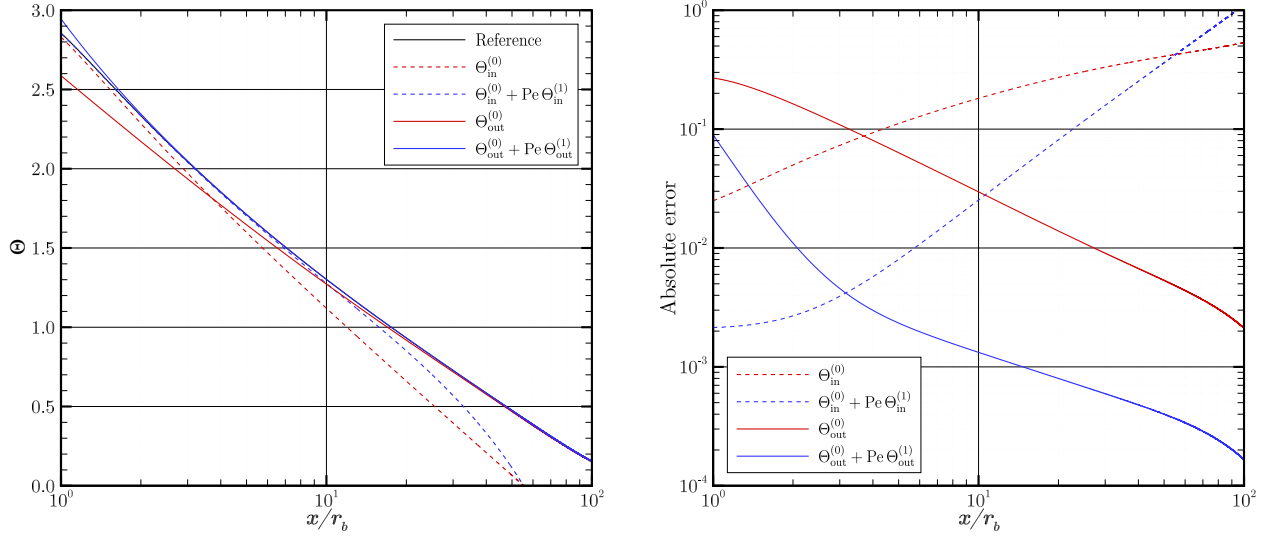


Figure 3.5: (Left) Temperature and (right) absolute errors for $Pe = 0.03$ and $\tau = 2000$.

Close to the borehole, the zeroth order solution to the inner region follows the reference solution with absolute (relative) errors in the range of $0.03 - 0.20$ ($1.1\% - 15.4\%$). The accuracy improves an order of magnitude by adding the first order correction to the inner region, leading to absolute (relative) errors in the range of $0.002 - 0.03$ ($0.1\% - 2.3\%$). As expected, the obtained solutions for the inner region perform worse with increasing distances from the borehole, where the solutions to the outer region must be used instead.

Far from the borehole, the zeroth order solution to the outer region already performs good, with absolute (relative) errors that start at 0.03 (2.3%) and decay with increasing distances to the borehole. Inclusion of the first order correction to the outer region improves again the accuracy by an order of magnitude, leading to absolute (relative) errors below 0.002 (0.2%). The complete asymptotic solution to the outer region also performs good at distances to the borehole belonging to the inner region of the problem: The crossover between the outer and inner solutions takes place at non-dimensional distances between 3 and 4.

To ensure the correctness of the absolute errors shown in Figure 3.5, successive mesh refinements were performed in COMSOL till the plotted curves stopped changing. Figure 3.6 exemplifies this by showing the evolution of the reference temperature at the rear stagnation point of the borehole, located at $(r_0, \theta_0) = (r_b, 0^\circ)$, with the number of mesh nodes.

Pe = 1.00

Figure 3.7 shows the time evolution of the grout/ground temperature for the unfavorable Peclet number of 1.00 using the same representation convention as before. Three different instants in time are shown, namely, $\tau = 20$ at the top, $\tau = 50$ at the middle, and $\tau = 200$ at the bottom.

Since the convective transport of heat is now stronger, the thermal wake behind the borehole builds up faster so that the final steady-state is already attained for $\tau = 200$. The steady-state

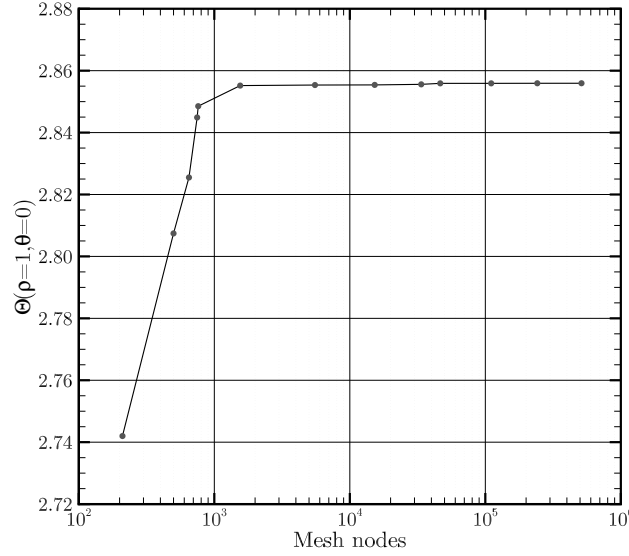


Figure 3.6: Rear stagnation point temperature versus number of mesh nodes.

is reached even sooner close to the borehole, where no relevant differences can be observed between the grout/ground temperatures at $\tau = 50$ and $\tau = 200$.

Regarding the performance of the proposed model, all remarks made so far also apply to this case so that its inclusion is more for the sake of completeness than for the discussion of anything new.

3.4.5 Time-harmonic responses

So far, the problem has been addressed both in the complex-valued Laplace plane and the time domain. This is of great practical interest as it allows arbitrarily-varying heat injection rates per unit pipe length to be specified at the pipes. Very enlightening is also the analysis of the thermal response to time-harmonic variations as then a sweep in angular frequencies can be performed. This sweep allows to show and discuss the rise and fall of thermal inertia with the aid of one single plot. As shown next, the proposed model is also applicable, with a minor tweak, to this case [93].

Consider the following time-harmonic variations for the grout/ground temperature, for the fluid temperatures in the pipes, for the heat injection rates per unit pipe length, and for the heat injection rate per unit borehole length:

$$\begin{aligned}\Theta &= \hat{\Theta} e^{i\omega\tau} + \hat{\Theta}^* e^{-i\omega\tau}, & \Theta_j &= \hat{\Theta}_j e^{i\omega\tau} + \hat{\Theta}_j^* e^{-i\omega\tau}, \\ q_j &= \hat{q}_j e^{i\omega\tau} + \hat{q}_j^* e^{-i\omega\tau}, & q &= \hat{q} e^{i\omega\tau} + \hat{q}^* e^{-i\omega\tau}.\end{aligned}$$

In these expressions ω is the non-dimensional angular frequency of the time-harmonic variations, $\hat{\Theta}$, $\hat{\Theta}_j$, $\hat{q}_j$, and $\hat{q}$ are the corresponding harmonics, and the superscript $*$ denotes the complex conjugate of a variable.

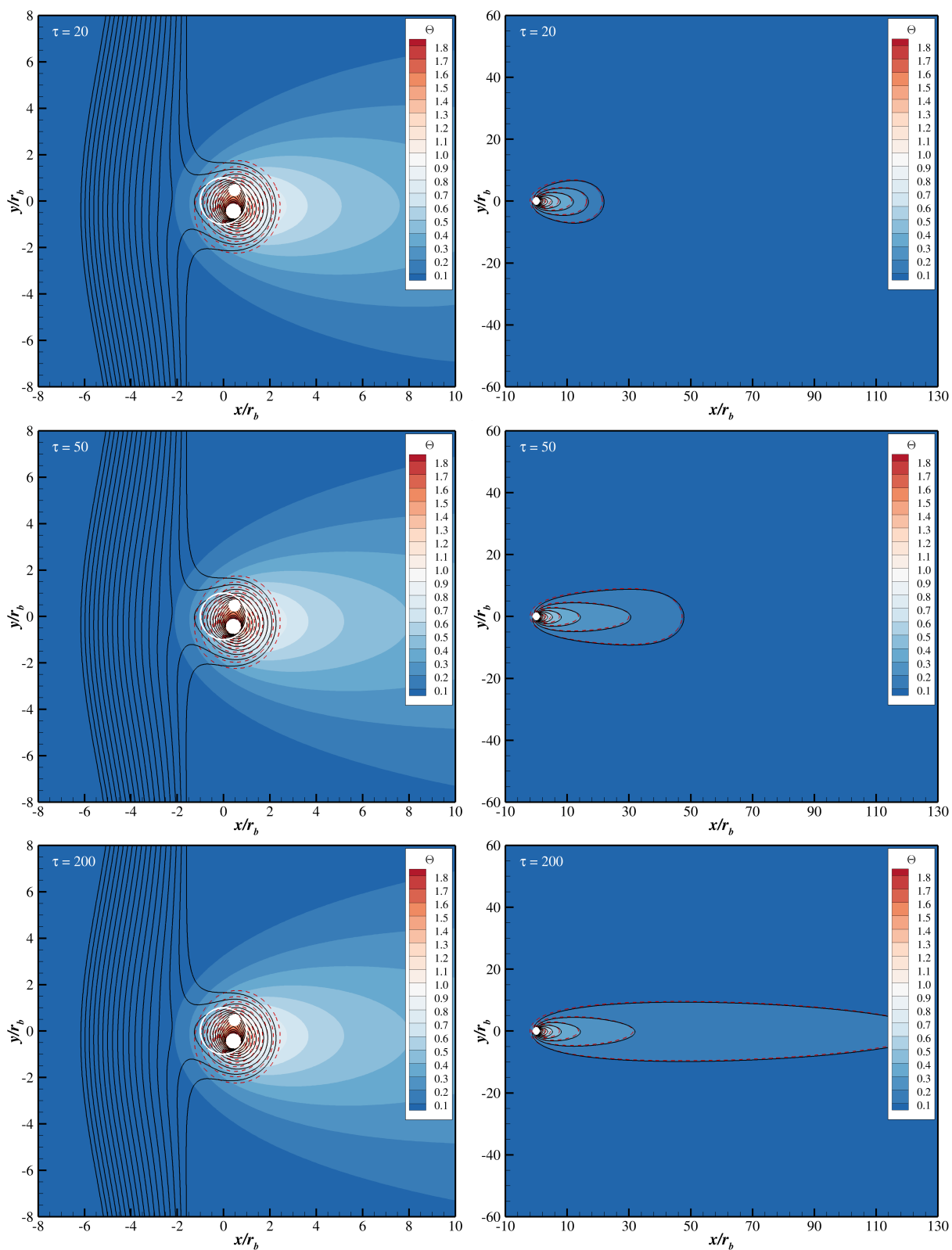


Figure 3.7: (Left) inner solution and (right) outer solution for $Pe = 1.00$.

Substitution of these time-harmonic variations into the heat transfer problem formulated in Section 2.3 leads to a mathematical problem for the harmonics $\hat{\Theta}$ and $\hat{\Theta}_j$ that is almost identical to the one for the Laplace-transformed functions $\tilde{\Theta}$ and $\tilde{\Theta}_j$. Only the governing equations in ground and grout change slightly, being now

$$\iota\omega \hat{\Theta} + \text{Pe} \left[v_\rho \frac{\partial \hat{\Theta}}{\partial \rho_0} + \frac{v_\theta}{\rho_0} \frac{\partial \hat{\Theta}}{\partial \theta_0} \right] = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \hat{\Theta}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \hat{\Theta}}{\partial \theta_0^2}$$

and

$$\iota\omega \frac{\alpha_g}{\alpha_b} \hat{\Theta} = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \hat{\Theta}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \hat{\Theta}}{\partial \theta_0^2}.$$

Thus, all formulae derived in Sections 3.2 and 3.3.1 is also applicable to time-harmonic problems once position s is replaced by $\iota\omega$. The same applies to the thermal influence of the borehole discussed later in Section 3.7.

The frequency response of the borehole and its surrounding ground is presented in Figure 3.8. The left and right plots show the modulus and argument of the mean azimuthal borehole wall temperature $\hat{\Theta}_b$, respectively, as returned by the reference solution, by the zeroth order solution, and by the complete asymptotic solution.

The represented temperature $\hat{\Theta}_b$, defined as

$$\hat{\Theta}_b = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\Theta} \Big|_{\rho_0=1} d\theta_0,$$

is considered by most researchers as the temperature at which the borehole perceives the surrounding ground [109]. This interpretation, however, is known to be flawed in the absence of groundwater flows [51, 93]. But the momentary lack of a better alternative, as explained later in Section 3.6, renders it as the best available quantity to represent here.

For values of the Peclet number of order 0.01, the proposed model accurately reproduces the mean azimuthal borehole wall temperature $\hat{\Theta}_b$, with the three curves for the reference solution, the zeroth order solution, and the complete asymptotic solution being almost indistinguishable in the plots.

The three curves become distinguishable for Peclet numbers an order of magnitude larger. That is, for $\text{Pe} \sim 0.1$. But the accuracy of the proposed model remains high, with relative errors below 1% for the modulus of $\hat{\Theta}_b$. It is not until Peclet numbers of order unity that the relative errors reach unacceptable levels of almost 23%. This makes sense as the performed asymptotic analysis is not valid for such large values of the Peclet number and, consequently, the accuracy of the proposed model cannot be assured.

The previous accuracy assessments and discussions hold as long as the non-dimensional angular frequency ω is small compared to unity. Then, the characteristic time t_q of the harmonic oscillation, closely related to its period, is large compared to the characteristic transversal diffusion time t_b and the performed asymptotic analysis is valid. Therefore, it comes as no surprise that the proposed model fails to correctly predict the mean azimuthal borehole wall temperature $\hat{\Theta}_b$ for non-dimensional angular frequencies of order unity.

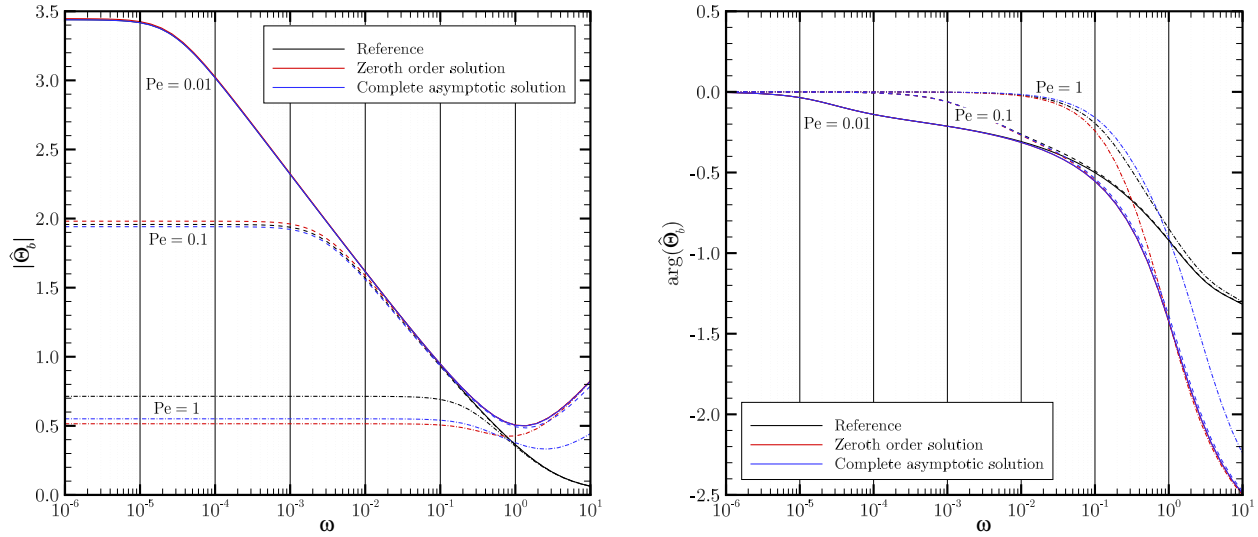


Figure 3.8: Mean azimuthal borehole wall temperature $\hat{\Theta}_b$.

In summary, the numerical examples presented and discussed show that the proposed model performs well as long as the underlying asymptotic analysis is valid. That is, as long as the Peclet number of the groundwater flow is small compared to unity and the characteristic heat injection time t_q is large compared to the characteristic transversal diffusion time t_b . If these conditions are not met, the proposed model fails to correctly represent the grout/ground temperature in the vicinity of the borehole, although it still succeeds in reproducing the ground temperature further away from the borehole.

3.5 Existing models for the problem

Most models for the thermal response of geothermal boreholes do not consider the presence of aquifers. For them, conduction is the solely heat transfer mechanism in the ground. Although apparently unrelated to the present work, these models are responsible for most of the characteristics of the models to review. Therefore, the discussion that follows next inevitably starts with them.

A common denominator to most models involving purely-conducting grounds is that the characteristic heat injection time t_q is considered large compared to the characteristic transversal diffusion time t_b [8, 31, 49, 93]. This assumption introduces a two-region structure into the problem that enables the development of efficient, albeit accurate, theoretical models for the thermal response of geothermal boreholes [68].

The inner region, located close to the borehole, is quasi-steady. In the case of purely-conducting grounds, its solution leads to a highly convenient network of thermal resistances that links the heat injection rates per unit pipe length of the pipes with certain temperature differences [19, 21, 49, 93].

One of the temperatures involved represents the thermal behavior of the ground located

further away from the borehole. To obtain its value, the solution to the unsteady outer region, located beyond the inner region, is required. That solution is usually obtained by superposition of point sources of heat that represent the thermal influence of the borehole onto the purely-conducting ground [10, 31, 54].

The average temperature of the outer solution at the borehole wall, known as the mean azimuthal borehole wall temperature [49], is considered by most researchers as the temperature representing the thermal behavior of the ground located further away from the borehole [21, 31, 65, 68]. In the limit of slowly-varying heat injection rates, for which $t_q \gg t_b$, this assumption is correct [49, 51, 93]. The same does not hold, though, for other relationships between t_q and t_b [51, 93].

Adding an aquifer to the ground significantly changes the heat transfer problem to solve. Hence, ideas, concepts, and modeling approaches meant for purely-conducting grounds do not necessarily extrapolate to the thermal interaction of geothermal boreholes with groundwater flows. Nonetheless, researchers have assumed they are applicable and have developed successful models that present the very same features [17, 109]. A formal proof that this can be done, at least for creeping groundwater flows, was not available until the work presented in this chapter and the follow-up.

So, models involving groundwater flows present two distinct regions. The inner region, located close to the borehole, is quasi-steady and heat conduction is the only heat transfer mechanism in it. The outer region, located further away from the borehole, is unsteady and considers both heat transfer mechanisms in the ground, namely, conduction and convection, with the latter one slightly simplified due to the assumption of a uniform velocity for the groundwater stream. Finally, both regions are interweaved through the mean azimuthal borehole wall temperature.

By comparing the described model structure with the present work it becomes evident that the state of the art corresponds to the zeroth order solution of the proposed model, with the integration constant $\tilde{A}_{00}$ embodying the mean azimuthal borehole wall temperature.

Most reviewed models obtain the solution to the outer region by superposing moving point-sources of heat that represent the thermal influence of the borehole onto a ground moving at the effective seepage velocity U_∞ [10]. If the borehole is assumed infinitely long, the so-called *moving infinite line source models* result [9, 27, 42, 73, 97, 103, 104]. On the contrary, the *moving finite line source models* take into account the finite length of the borehole [20, 29–31, 39, 52, 74, 90–92, 103, 104, 107].

Moving infinite line source models are meant, just as the model proposed in the present work, for operating conditions of the borehole in which the characteristic heat injection time t_q is small compared to the characteristic longitudinal diffusion time t_H . Then, three-dimensional effects at the borehole endings are not important and the finite length of the borehole plays a secondary role in the thermal response of the ground [45, 49].

Moving finite line source models, on the contrary, take into account the finite length of the borehole and the aforementioned three-dimensional effects at the borehole endings. Consequently, they are also valid for operating conditions in which t_q and t_H are comparable

in order. A downside of these models, however, is the significant increase in mathematical complexity and computational cost. To partially mitigate this, all moving finite line source models assume a uniform heat injection rate per unit borehole length q along the borehole. For purely-conducting grounds, however, it is known that this simplification is not correct [47, 48, 53]. Nonetheless, good results for the thermal response of geothermal boreholes are obtained with it [46].

Another simplification made by most moving finite line source models is to assume groundwater flows with the same effective seepage velocity U_∞ at all depths. This hypothesis oversimplifies the strata structure of the ground in which aquifers are confined to certain depth ranges and are surrounded by purely-conducting layers of rock or soil. Only the models developed by Hu [52] and by Erol and François [29] take into account the layered structure of aquifers, although they do it a clumsy way.

A different approach to the thermal response of the ground is offered by *infinite cylindrical source models*. These take into account the finite radius of the borehole when solving the heat transfer problem in the ground, a task accomplished either numerically [23, 60, 61] or analytically [62, 63]. Inclusion of the borehole radius is considered an improvement over models based on point-sources of heat. However, the present work shows that the improvement must be rather small as neither the zeroth order solution nor the first order correction to the outer region depend explicitly on the borehole radius ($\rho_0 \text{Pe}$ and s/Pe^2 do not depend on r_b). Consequently, its effect is relegated to the integration constants $\tilde{A}_{0(\pm 1)}$, present in $\tilde{C}_{10}$ and $\tilde{C}_{1(\pm 1)}$, and to higher order terms of the asymptotic expansion of the problem. Additionally, infinite cylindrical source models introduce important simplifications in order to account for the borehole radius. So, all reviewed models replace the interior of the borehole by a uniform Dirichlet or Neumann boundary condition at the borehole wall [23, 60–63]. And some assume as well the groundwater flow to be uniform everywhere, a simplification incompatible with the very presence of the borehole in the ground [62, 63].

Finally, hybrid models do exist as well that use all described modeling approaches [8, 28, 101, 106]. They choose one approach or the other depending on the value attained by the characteristic heat injection time t_q . This renders them more complex but also more efficient from a computational point of view.

3.6 Apparent ground temperature

The grout/ground temperature derived in Sections 3.2 and 3.3, although essential, is not the quantity of interest for the analysis of the thermal response of geothermal boreholes. What is relevant instead is how the perturbed grout/ground temperature appears in the governing equations for the convective transport of heat along the pipes, and whose solution represents the sought thermal response of the borehole.

In the absence of groundwater flows everything is well understood. All effects of the perturbed grout/ground temperature condense into one single quantity called the ground apparent temperature [49, 93]. This temperature differs from the unperturbed one, T_∞ , in the thermal

influence the borehole exerts onto itself as a consequence of the continuous exchange of heat with the ground. Therefore, it represents the temperature at which the borehole perceives the surrounding ground and is, as such, the same for all pipes. The derivation of mathematical expressions for it represents an essential aspect of the modeling of the thermal response of geothermal boreholes [47–49, 53].

In the presence of groundwater flows, though, the applicability of the aforementioned concept is not assured. First and foremost, the enhanced multipole method used for the rigorous solution of the heat conduction problem in grout and ground does not account for groundwater flows [93]. Hence, the resulting mathematical proof and physical interpretation of the apparent ground temperature do not necessarily extend to boreholes interacting with groundwater flows. Second, the moving groundwater introduces an asymmetry into the problem. This casts doubts on the spatial uniformity of the apparent ground temperature meaning that different pipes in a borehole could perceive the surrounding ground in different ways.

The answer to all these doubts is beyond the scope of this chapter as the exact solution to the formulated heat transfer problem is required for that. Only so is it possible to correctly identify what precisely is the apparent ground temperature. This challenge has been addressed in Chapter 4. However, not the full heat transfer problem formulated in Section 2.3 has been considered for that but the asymptotic approach presented in Sections 3.2 and 3.3. The outcome of that work is that an apparent ground temperature can still be defined in the presence of creeping groundwater flows, but different values result for each pipe in the borehole.

3.7 Thermal influence of a borehole

The thermal interaction between adjacent boreholes has negative consequences for the performance of geothermal HVAC systems. Hence, the thermal influence exerted by a geothermal borehole onto its surrounding ground is of utmost importance for the accurate forecasting of the thermal response of geothermal heat exchangers composed of multiple boreholes.

The typical distance between adjacent boreholes, of a few meters, is significantly larger than the borehole radius so that $\rho_0 \gg 1$. Consequently, the solution to the outer region is employed to describe the thermal influence a certain borehole exerts onto adjacent boreholes. By truncating the asymptotic expansion for the outer solution, Eq. (3.3), and substituting thereafter the derived expressions for the zeroth order solution and first order correction, the

following expression for the solution in the outer region results:

$$\begin{aligned}\tilde{\Theta}_{\text{out}} = & \left[\frac{\tilde{q}}{2\pi} + \text{Pe} \tilde{C}_{10} \right] e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_0 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) \\ & + \text{Pe} \tilde{C}_{1(+1)} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_1 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{+\iota\theta_0} \\ & + \text{Pe} \tilde{C}_{1(-1)} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_1 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{-\iota\theta_0}.\end{aligned}$$

Replacing the integration constants $\tilde{C}_{10}$ and $\tilde{C}_{1(\pm 1)}$ by their expressions leads, after some rearrangement, to

$$\begin{aligned}\tilde{\Theta}_{\text{out}} = & \frac{\tilde{f}}{2\pi} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_0 \left(\rho_0 \sqrt{s + \frac{\text{Pe}^2}{4}} \right) \\ & + \tilde{A}_{0(+1)} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \sqrt{s + \frac{\text{Pe}^2}{4}} K_1 \left(\rho_0 \sqrt{s + \frac{\text{Pe}^2}{4}} \right) e^{+\iota\theta_0} \\ & + \tilde{A}_{0(-1)} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \sqrt{s + \frac{\text{Pe}^2}{4}} K_1 \left(\rho_0 \sqrt{s + \frac{\text{Pe}^2}{4}} \right) e^{-\iota\theta_0}.\end{aligned}\tag{3.25}$$

A non-dimensional fictitious heat injection rate per unit borehole length $\tilde{f}$ is identified in the previous expression. It subtracts from the true value $\tilde{q}$ the contributions of the azimuthal corrections:

$$\tilde{f}(s) = \tilde{q}(s) - \text{Pe} \pi \left(\tilde{A}_{0(+1)}(s) + \tilde{A}_{0(-1)}(s) \right).$$

To complete the formulae for the thermal influence of a borehole, expressions for $\tilde{A}_{0(\pm 1)}$ are required. As shown in Hermanns and Rivero [51] and Chapter 4, they are linear combinations of the heat injection rates per unit pipe length $\tilde{q}_j$, with the weights not depending on the position s in the complex-valued Laplace plane:

$$\tilde{A}_{0n}(s) = \sum_{j=1}^{N_p} A_{0nj} \frac{\tilde{q}_j(s)}{2\pi\kappa}.$$

In the absence of an exact expression for the zeroth order solution to the inner region, as happens in the present chapter, the coefficients $\tilde{A}_{0(\pm 1)}$ are obtained numerically following the same procedure as in Section 3.4:

$$\tilde{A}_{0(\pm 1)}(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0=1} e^{\mp \iota\theta_0} d\theta_0.$$

3.7.1 Time-domain recovery

The formulae presented so far is best suited for theoretical models that describe and solve the thermal response of geothermal heat exchangers in the complex-valued Laplace plane. To obtain the corresponding time evolution, the inverse Laplace transform is applied numerically to their outcomes [44].

Models that obtain the thermal response directly in the time domain do also exist. For them it is necessary to first obtain the inverse Laplace transform of Eq. (3.25) so that the outcome is incorporated into them. Fortunately, this can be accomplished analytically by combining three properties of the Laplace transform theory, namely, a shift in s of magnitude $\text{Pe}^2/4$, the Laplace transform

$$\mathcal{L} \left[\left(\frac{\rho_0}{2} \right)^{|n|} \frac{e^{-\frac{\rho_0^2}{4\tau}}}{2\tau^{|n|+1}} \right] = (\sqrt{s})^{|n|} K_n(\rho_0 \sqrt{s}),$$

and the convolution theorem [7]. The result is the thermal influence exerted by a geothermal borehole onto its surrounding ground expressed directly in the time domain:

$$\begin{aligned} \Theta_{\text{out}} = & e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \int_0^\tau \frac{f(\lambda)}{4\pi} \frac{e^{-\frac{\rho_0^2}{4(\tau-\lambda)} - \frac{\text{Pe}^2}{4}(\tau-\lambda)}}{(\tau-\lambda)} d\lambda \\ & + e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \rho_0 e^{+\iota\theta_0} \int_0^\tau A_{0(+1)}(\lambda) \frac{e^{-\frac{\rho_0^2}{4(\tau-\lambda)} - \frac{\text{Pe}^2}{4}(\tau-\lambda)}}{4(\tau-\lambda)^2} d\lambda \\ & + e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \rho_0 e^{-\iota\theta_0} \int_0^\tau A_{0(-1)}(\lambda) \frac{e^{-\frac{\rho_0^2}{4(\tau-\lambda)} - \frac{\text{Pe}^2}{4}(\tau-\lambda)}}{4(\tau-\lambda)^2} d\lambda, \end{aligned} \quad (3.26)$$

where

$$f(\tau) = \sum_{j=1}^{N_p} \left[1 - \text{Pe} \pi (A_{0(+1)j} + A_{0(-1)j}) \right] q_j(\tau)$$

and

$$A_{0(\pm 1)}(\tau) = \sum_{j=1}^{N_p} A_{0(\pm 1)j} \frac{q_j(\tau)}{2\pi\kappa}.$$

The first term in Eq. (3.26) is well known from the state of art [10]. It represents the thermal influence exerted by a moving infinite line source of heat. The novelty of the present work resides in its mathematically rigorous derivation, using matched asymptotic expansion techniques, and in the presence of a fictitious heat injection rate per unit borehole length $f(\tau)$ replacing the real one $q(\tau)$.

Completely nonexistent in the prevailing literature are the second and third terms in Eq. (3.26). They represent higher order corrections that can only be envisioned and derived through mathematically rigorous approaches. These novel terms account for the spatial position of the pipes inside the borehole and their relevance seems *a-priori* high, based on the absence of a multiplying Peclet number in Eq. (3.26). But it was shown in Section 3.4 that this is not

the case, with these terms indeed representing small corrections to the thermal influence of a borehole.

3.7.2 Constant heat injection rates

When heat injection rates per unit pipe length $q_j(\tau)$ are constant in time, f and $A_{0(\pm 1)}$ are constants as well. Then, Eq. (3.26) can be written in a more compact and enlightening way now that f and $A_{0(\pm 1)}$ can be taken out of their respective integrals:

$$\begin{aligned} \Theta_{\text{out}} = & \frac{f}{2\pi} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \underline{K}_0 \left(\frac{\rho_0 \text{Pe}}{2}, \frac{\text{Pe}^2}{4} \tau \right) \\ & + A_{0(+1)} e^{+\iota \theta_0} \frac{\text{Pe}}{2} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \underline{K}_1 \left(\frac{\rho_0 \text{Pe}}{2}, \frac{\text{Pe}^2}{4} \tau \right) \\ & + A_{0(-1)} e^{-\iota \theta_0} \frac{\text{Pe}}{2} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} \underline{K}_1 \left(\frac{\rho_0 \text{Pe}}{2}, \frac{\text{Pe}^2}{4} \tau \right). \end{aligned} \quad (3.27)$$

The functions $\underline{K}_n(z, w)$, defined as

$$\underline{K}_n(z, w) = \frac{1}{2} \left(\frac{z}{2} \right)^n \int_0^w \frac{e^{-\frac{z^2}{4\lambda} - \lambda}}{\lambda^{n+1}} d\lambda,$$

hide now all the mathematical complexity of Eq. (3.27). Their nomenclature is chosen on purpose as they are closely related to the modified Bessel functions of the second kind $K_n(z)$ to which they converge when their second argument tends to infinity [80]:

$$\lim_{w \rightarrow \infty} \underline{K}_n(z, w) = K_n(z).$$

However, they should not be called incomplete modified Bessel functions of the second kind as better candidates for that name already exist in the literature [40].

Regarding the computation of $\underline{K}_n(z, w)$, these functions can handily be expressed in terms of generalized incomplete gamma functions whose evaluation has thoroughly been studied in the literature, both analytically [6, 11–13] and numerically [98, 105]. Since the present application leads to very demanding values for the arguments of $\underline{K}_n(z, w)$, the authors have developed their own expansions for the generalized incomplete gamma functions which outperform the existing ones [89].

Figure 3.9 shows, for the borehole configuration considered in Section 3.4, the time evolution of the ground temperature on a circle located at a distance $\rho_0 = 50$ from the borehole. Left and right plots represent the evolution for two different values of the Peclet number, namely, 0.01 and 0.1 respectively, obtained from implementing in Fortran95 Eq. (3.27) and the expansions for $\underline{K}_n(z, w)$ [89].

First and foremost, the complete asymptotic solution as well as the zeroth order solution perform very good, with deviations from the reference solution that are negligible from an

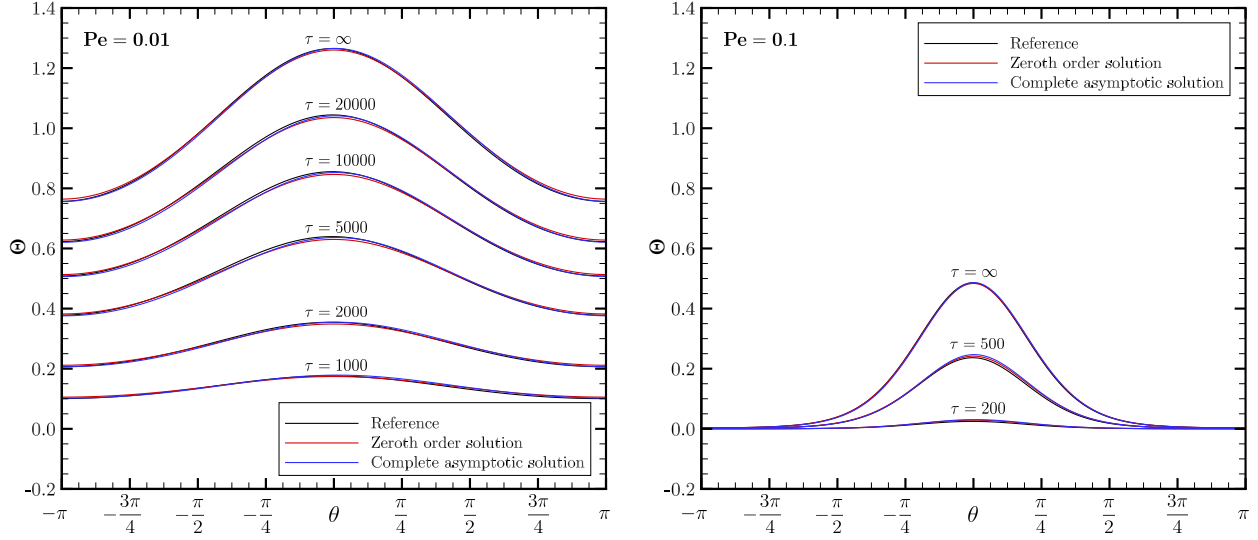


Figure 3.9: Unsteady thermal influence of the borehole.

engineering point of view. The shown results also evidence the little impact the Peclet number of the groundwater flow has on the accuracy of the model, a conclusion in line with the observations made in Section 3.4.

The Peclet number of the groundwater flow has, however, a strong impact on the thermal influence of the borehole, as seen in Figure 3.9. On one hand, the intensity of the thermal influence grows with decreasing Peclet numbers. This is a consequence of the inherent inefficiency of heat conduction that leads to a build up of heat around the borehole. This build up takes place in all spatial directions, not only downwards of the borehole, leading to the shift in temperatures observed on the left plot. On the other hand, the final steady-state of the thermal influence is reached sooner for increasing Peclet numbers of the groundwater flow. This makes sense as the convective transport of heat is then more intense, washing away faster the intermediate temperature states.

The previous two statements regarding the influence of the Peclet number can also be inferred from Eq. (3.27). The aforementioned build up in temperatures is the result of $\rho_0 Pe$ becoming smaller in the argument of the exponential function. And the product $Pe^2 \tau$ becoming larger in the second argument of the functions $\underline{K}_n(z, w)$ is the reason for the faster transition towards the final steady-state.

3.7.3 Steady-state solution

The final steady state reached by the thermal influence of a geothermal borehole, with constant heat injection rates per unit pipe length, can be obtained in two different ways. The first one consists in letting τ tend to infinity in Eq. (3.27). This approach exploits that $\underline{K}_n(z, w)$ tends to the modified Bessel function of the second kind $K_n(z)$ under that limit. The second one consists in using the Final Value Theorem applied to Eq. (3.25) [7]. Both approaches

obviously lead to the same result:

$$\begin{aligned}\Theta_{\text{out}}\Big|_{\tau \rightarrow \infty} &= \frac{f}{2\pi} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_0\left(\frac{\rho_0 \text{Pe}}{2}\right) \\ &+ A_{0(+1)} e^{+\iota\theta_0} \frac{\text{Pe}}{2} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_1\left(\frac{\rho_0 \text{Pe}}{2}\right) \\ &+ A_{0(-1)} e^{-\iota\theta_0} \frac{\text{Pe}}{2} e^{\frac{\rho_0 \text{Pe}}{2} \cos(\theta_0)} K_1\left(\frac{\rho_0 \text{Pe}}{2}\right).\end{aligned}$$

Figure 3.10 shows, for the borehole configuration considered in Section 3.4, the steady-state ground temperature on a circle located at the distance $\rho_0 = 50$ from the borehole center. The plotted results, obtained from implementing in Fortran95 the previous equation, evidence once again the capability of the proposed model to correctly predict the thermal influence of a borehole, regardless of the Peclet number of the groundwater flow. Also the build up of heat around the borehole, pointed out in the previous section, and its dependence on the Peclet number are perfectly identifiable in Figure 3.10 in which results for Peclet numbers up to 10 are represented.

3.7.4 Purely-conducting limit

Whenever an existing model is improved by the addition of a new physical phenomenon, it is desirable that the resulting model reverts to the existing one whenever the new phenomenon becomes negligible. For the present work it means the developed formulae should revert to the purely-conducting case for vanishing groundwater velocities.

The purely-conducting limit of the thermal influence exerted by a borehole onto its surrounding ground is obtained by letting the Peclet number tend to zero in any of the expressions presented in Section 3.7. The best suited expression for that is Eq. (3.26) which simplifies to

$$\begin{aligned}\Theta_{\text{out}} &= \int_0^\tau \frac{f(\lambda)}{4\pi} \frac{e^{-\frac{\rho_0^2}{4(\tau-\lambda)}}}{(\tau-\lambda)} d\lambda \\ &+ \rho_0 e^{+\iota\theta_0} \int_0^\tau A_{0(+1)}(\lambda) \frac{e^{-\frac{\rho_0^2}{4(\tau-\lambda)}}}{4(\tau-\lambda)^2} d\lambda \\ &+ \rho_0 e^{-\iota\theta_0} \int_0^\tau A_{0(-1)}(\lambda) \frac{e^{-\frac{\rho_0^2}{4(\tau-\lambda)}}}{4(\tau-\lambda)^2} d\lambda\end{aligned}\tag{3.28}$$

with

$$f(\tau) = \sum_{j=1}^{N_p} q_j(\tau) = q(\tau).$$

The first term in Eq. (3.28) is again well known from the state of the art [10]. It represents the thermal influence exerted by an infinite line source of heat. Again, the novelty of the

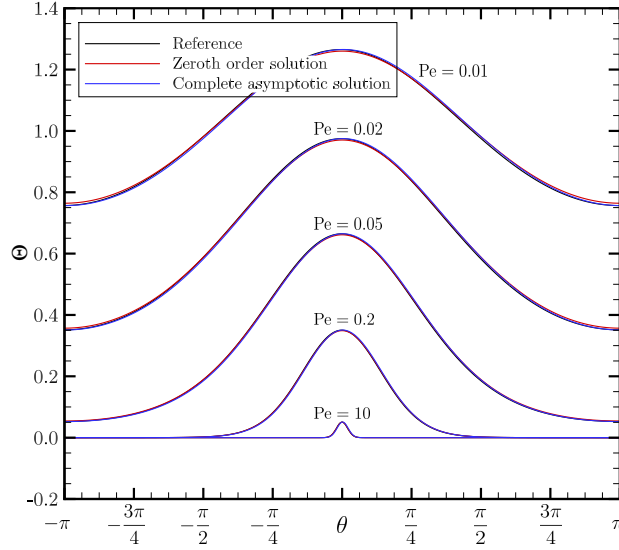


Figure 3.10: Steady-state thermal influence of the borehole.

present work resides in the followed derivation procedure and not in the term itself. The remaining two terms are again the higher order corrections introduced here for the first time.

Finally, and for the sake of completeness, for constant heat injection rates per unit pipe length a simpler and more convenient expression for the thermal influence of a borehole is obtained, namely,

$$\Theta_{\text{out}} = \frac{q}{4\pi} E_1\left(\frac{\rho_0^2}{4\tau}\right) + \left(A_{0(+1)}e^{+\iota\theta_0} + A_{0(-1)}e^{-\iota\theta_0}\right) \frac{e^{-\frac{\rho_0^2}{4\tau}}}{\rho_0},$$

in which $E_1(z)$ represents the exponential integral [80]. As well known for two-dimensional heat conduction problems, no steady state solution exists in this case [10].

3.8 Conclusions

The present chapter develops a theoretical model that accounts for the presence of creeping groundwater flows. In it, the convective transport of heat is considered to be weak enough that the Peclet number of the flow is small compared to unity. Additionally, heating and cooling needs of the building are assumed to vary slowly, a simplification that matches most operating conditions of real-world geothermal HVAC systems.

Scale analysis of the considered heat transfer problem has revealed the existence of two distinct regions. The inner region encompasses the borehole and the ground located close to the borehole. In it, heat conduction dominates, with heat convection and thermal inertia being second and third ranked, respectively. The relevance of the different phenomena changes, though, further away from the borehole giving rise to the outer region. In the distinguished limit considered in the present work, all three physical phenomena are equally important in the outer region.

The developed model for the interaction of geothermal boreholes with groundwater flows is the result of exploiting the aforementioned two-region structure using matched asymptotic expansion techniques. For the solution to the outer region, closed-form expressions have been obtained. For the solution to the inner region, though, only semi-analytical results have been presented, with the derivation of closed-form expressions postponed to the next chapter.

Comparisons with detailed numerical simulations of the complete heat transfer problem have shown that the proposed model exceeds by an order of magnitude the accuracy of the state of the art, leading to absolute (relative) errors below 0.03 (2.3%). This is accomplished by incorporating, for the first time, azimuthal corrections which can only be envisioned and derived following rigorous mathematical approaches, something the state of the art was lacking. The performed comparisons have also highlighted the limitations of the proposed model as its accuracy worsens when the assumptions underlying the performed asymptotic analysis are not met. That is, when the Peclet number becomes of order unity or the heating and cooling needs change too fast.

It has also been shown that the state of the art coincides with the zeroth order solution derived in the present work and that some of the improvements proposed in the literature, like the infinite cylindrical source models, have in reality a limited impact onto the accuracy of the results. The underlying asymptotic analysis of the problem conducted in this chapter serves as an initial step to critically assess, for the first time, the theoretical and conceptual merits and limits of the state of the art. The next chapter completes this discussion and determines whether the ideas, concepts, and modeling approaches meant for purely-conducting grounds are also applicable to grounds with creeping groundwater flows or not.

Chapter 4

Network of thermal resistances for creeping groundwater flows

While Chapter 3 analyzes the asymptotic structure of the problem and derives a semi-analytical approach for the grout/ground temperature in presence of creeping groundwater flows, the current chapter develops a pure analytical treatment of the grout filling up the boreholes and of the adjacent ground. That is, the inner region.

Pure analytical models for the thermal response of geothermal boreholes already exist in the literature. The first ones were introduced in the 1980s by researchers in Sweden [5, 19, 31, 32]. By exploiting the large disparity in time and length scales of the problem they succeeded in deriving models for geothermal boreholes constructed in purely-conducting grounds. These models proposed the use of networks of thermal resistances to embody the quasi-steady thermal response of the grout filling up the boreholes and of the ground located close to the boreholes [32]. They introduced as well the notion of borehole wall temperature, also known as apparent ground temperature nowadays [49], as a representation of the thermal state of the surrounding ground.

The validity and correctness of these concepts was given for granted at that time, and comparisons with experiments and numerical simulations reinforced that [24, 68]. The lack of a formal mathematical proof, though, spawned the exploration of alternative definitions for the network of thermal resistances and for the borehole wall temperature in the quest for better results [65, 76, 95]. With the advent of a formal mathematical proof in 2014 [49], almost 30 years after the seminal work by Per Eskilson [31], light was finally shed onto the topic allowing to discern what was mathematically correct from what was not [49].

Theoretical models that account for the presence of aquifers, with the resulting convective transport of heat in the ground, do also exist [17, 109]. They replicate the methodology and ideas of the models meant for purely-conductive grounds. So, they use a network of thermal resistances to represent the quasi-steady thermal response of the grout filling up the boreholes and of the ground located close to the boreholes [20, 97] and employ a borehole wall temperature, or apparent ground temperature, to describe the thermal state of the surrounding ground.

The validity and correctness of these two concepts, now in the context of groundwater flows, is again given for granted in the literature and comparisons with experiments numerical simulations support that [109]. But without a formal mathematical proof of their validity and correctness it is impossible to truly discuss the merits and limits of the state of the art, and to envision mathematically correct ways of improving upon the state of the art.

The current chapter addresses the aforementioned lack of mathematical rigorousness by delivering a formal proof for the case of creeping groundwater flows. In this flow regime, which is the heart of most theoretical models available nowadays, heat convection in the vicinity of the borehole is weak compared to heat conduction.

Three main conclusions are drawn from the presented proof. First, the quasi-steady thermal response of the grout filling up the borehole and of the ground located close to the borehole can also be represented by a network of thermal resistances when creeping groundwater flows are present. Second, the structure of that network differs from the one employed in the literature. And third, the identified inconsistencies in the state of the art do not affect the performance of existing models as long as creeping groundwater flows are considered.

The present chapter is organized as follows. Section 4.1 analyzes the impact of creeping groundwater flows in the region close to the borehole, by addressing both the state of the art and the outcome of the present chapter. Then, Section 4.2 discusses at length the matching condition with the outer region, formulating both zeroth and first order inner problems using a generalized matching condition with the outer solution. Thereafter, Section 4.3 introduces a modified version of the enhanced multipole method to cope with the presence of creeping groundwater flows. Section 4.4 solves the formulated problem using the modified enhanced multipole method. The obtained solutions are then exploited in Sections 4.5 and 4.6 to explain how the network of thermal resistances must be and which temperature, or temperatures, must be used to describe the thermal state of the ground surrounding the borehole. Thereafter, some numerical examples are given in Section 4.7. Finally, Section 4.8 is devoted to the conclusions of the chapter.

4.1 Impact of aquifers in the inner region

The role played by groundwater flows in the inner region is governed by the Peclet number of the groundwater flow as discussed in Chapter 2. Although already presented in previous chapters it is defined here again to enhance the readability of the present chapter:

$$\text{Pe} = \frac{r_b U_\infty}{\alpha_g}.$$

This non-dimensional number measures the relative importance of heat convection versus heat conduction in the energy conservation equation in the ground [4]. Two different cases are discussed next to explain the goal of the present chapter and its outcome.

4.1.1 Purely-conducting limit

In the absence of groundwater flows, heat transfer in ground is solely due to conduction and the Peclet number of the problem is zero. The exact solution to the quasi-steady inner region can then be obtained using the enhanced multipole method [51, 93]. The main outcome is the following exact relationship between the heat injection rates per unit pipe length $q_j(\tau)$, specified at the N_p pipes of the borehole, and the resulting temperatures $\Theta_i(\tau)$ of the liquid in the pipes:

$$\sum_{j=1}^{N_p} \hat{R}_{ij} q_j(\tau) = \Theta_i(\tau) - \Theta_a(\tau). \quad (4.1)$$

Non-dimensional thermal resistances $\hat{R}_{ij}$, which relate the two quantities of interest in the previous expression, depend on the geometry and thermal characteristics of the borehole and on the thermal conductivity of the ground. Their values, given by the enhanced multipole method, do not depend on time in the limit of slowly-varying heat injection rates [51].

The non-dimensional apparent ground temperature $\Theta_a(\tau)$ represents the temperature at which the borehole perceives the surrounding ground [93]. It is a quantity imposed by the outer region onto the inner region that includes two distinct phenomena: the thermal influence exerted by neighboring boreholes, and the thermal influence exerted by the borehole onto itself [48, 49].

4.1.2 Creeping groundwater flows

The heat transfer problem to solve in the inner region changes with the addition of a groundwater stream to the ground. So the following question arises: Does a network of thermal resistances, equal or similar to Eq. (4.1), still link heat injection rates per unit pipe length $q_j(\tau)$ with fluid temperatures $\Theta_i(\tau)$ in the pipes? The current chapter answers this precise question by obtaining the exact solution to the inner region. For it, a modified version of the enhanced multipole method is developed and used. The delivered proof, however, is limited to creeping groundwater flows for which the Peclet number is small compared to unity.

Anticipating the outcome of this chapter, the exact solution to the inner region leads to the following relationship between heat injection rates per unit pipe length $q_j(\tau)$ and fluid temperatures $\Theta_i(\tau)$ in the pipes:

$$\sum_{j=1}^{N_p} \hat{R}_{ij} q_j(\tau) = \Theta_i(\tau) - \Theta_{ai}(\tau). \quad (4.2)$$

Two differences exist compared to the purely-conducting case of Eq. (4.1). First, each pipe perceives the surrounding ground at a different temperature $\Theta_{ai}(\tau)$. That is the result of the convective transport of heat in the outer region introducing an asymmetry into the problem. This asymmetry is perceived by the inner region as a temperature gradient being imposed by the outer region. The thermal response of the inner region to that temperature gradient is what leads to a different apparent ground temperature for each pipe.

The second difference with the purely-conducting case is that thermal resistances $\hat{R}_{ij}$ are now functions of the Peclet number of the groundwater flow. This is a direct consequence of the convective transport of heat acting in the vicinity of the borehole, even in the here-considered case of creeping groundwater flows.

4.1.3 State of the art

As highlighted in the prevailing subsection, the addition of a groundwater stream changes the mathematical structure and the values of the network of thermal resistances linking heat injection rates per unit pipe length $q_j(\tau)$ with fluid temperatures $\Theta_i(\tau)$ in the pipes. As explained next, this has important implications for the state of the art.

Many theoretical models exist in the literature for the thermal interaction of boreholes with aquifers. Most of them incorporate a network of thermal resistances to represent the quasi-steady thermal response of the grout filling up the borehole and of the ground located close to the borehole [8, 9, 20, 27–31, 39, 42, 52, 73, 74, 90–92, 97, 101, 103, 104, 106, 107]. The adoption of this modeling approach has its roots in the successful use of such networks by theoretical models for the thermal response of geothermal boreholes constructed in purely-conducting grounds.

All reviewed models adopt without changes the structure and values of the network of thermal resistances, Eq. (4.1), valid for purely-conducting ground. In the absence of mathematical proofs telling researchers how to proceed in a mathematically correct way, the taken decision is the most logical one. But the availability now of Eq. (4.2), result of a mathematical proof valid for creeping groundwater flows, allows for the first time to assess the merits and limits of the state of the art.

So two conceptual errors are identified in the state of the art as a result of the differences discussed in the previous subsection. First, existing models for the thermal interaction of boreholes with aquifers neglect the impact of the aquifers on the thermal resistances. Second, only one apparent ground temperature is considered for all pipes. The analysis of the practical implications of these two discrepancies is the subject of Section 4.7.

4.2 Matching with outer problem

The inner problem formulated in Sections 3.2.1 and 3.3.1 replaces the boundary condition at infinity, Eq. (2.9), with a matching condition with the outer region, Eq. (3.20). In the present section, however, a broader and more general discussion of the matching condition is performed.

In absence of aquifers, which corresponds to zero Peclet number, the matching condition to enforce at the outer boundary of the inner region is well known and was mathematically proven in 2014 [49]:

$$\rho_0 \rightarrow \infty : \Theta_{\text{in}} \rightarrow \Theta_a(\tau) - \frac{q(\tau)}{2\pi} \ln(\rho_0). \quad (4.3)$$

The first term in Eq. (4.3) is the apparent ground temperature, the same one that appears in the network of thermal resistances given in Eq. (4.1). It represents a uniform temperature level imposed by the outer region onto the inner region [49]. It differs from the non-dimensional unperturbed ground temperature, Eq. (2.9), in the thermal influence exerted by neighboring boreholes and in the thermal influence exerted by the borehole onto itself [48, 49], with its value resulting from the solution to the outer region. The second term in Eq. (4.3) is imposed by the inner region onto the outer region. It is an inherent part of the solution to the inner region and ultimately the cause for the thermal reaction of the ground located in the outer region.

The matching condition to enforce at the outer boundary of the inner region changes with the presence of aquifers. For creeping groundwater flows, whose Peclet numbers are small compared to unity, the matching condition was mathematically proven in Chapters 3 and 5. The resulting expression, in which $\iota = \sqrt{-1}$, can be generalized as follows to also include the thermal influence exerted by neighboring boreholes:

$$\begin{aligned} \rho_0 \rightarrow \infty: \Theta_{\text{in}} \rightarrow & A(\tau) + \frac{q(\tau)}{2\pi} \ln(\rho_0) \\ & + \text{Pe} \left[B(\tau) + C(\tau)\rho_0 e^{+\iota\theta_0} + \overline{C}(\tau)\rho_0 e^{-\iota\theta_0} \right] \\ & + \text{Pe} \left[\frac{A_{0(+1)}(\tau) + A_{0(-1)}(\tau)}{2} - \frac{q(\tau)}{4\pi} \rho_0 \cos(\theta_0) \right] \ln(\rho_0). \end{aligned} \quad (4.4)$$

The uniform temperature level imposed by the outer region onto the inner region is now represented by $A(\tau)$ and $B(\tau)$. Its value includes the thermal self-influence of the borehole, the effect of the groundwater flow and the influence exerted by neighboring boreholes.

The pair $C(\tau)$ and $\overline{C}(\tau)$ represent a temperature gradient that is also imposed by the outer region onto the inner region. The two quantities are complex conjugates and together specify the intensity and spatial direction of the temperature gradient. The specified direction does not necessary coincide with the groundwater flow direction when the borehole is affected by the thermal wakes of neighboring boreholes located upstream of the borehole of interest. In the absence of those boreholes, however, $C(\tau)$ and $\overline{C}(\tau)$ become real numbers and the imposed temperature gradient is perfectly aligned with the flow direction of the aquifer.

The remaining terms in Eq. (4.4) are all imposed by the inner region onto the outer region and are inherent part of the solution to the inner region. Apart from the heat injection rate per unit borehole length $q(\tau)$, these terms also involve the first harmonics $A_{0(+1)}(\tau)$ and $A_{0(-1)}(\tau)$ of the Fourier series expansion of the grout/ground temperature around the borehole wall. These harmonics, however, correspond to the solution to the inner region without any convection at all. That is, the zeroth order solution:

$$A_{0(\pm 1)}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta_{\text{in}}^{(0)} \Big|_{\rho_0=1} e^{\mp \iota \theta_0} d\theta_0. \quad (4.5)$$

The meaning and reason for this was clear when the asymptotic matching for the first order correction was developed in Section 3.3.

4.2.1 Zeroth order inner problem

The formulation of the zeroth order inner problem addressed in this chapter mirrors what was presented in Section 3.2.1, differing only in the generalized matching condition enforced far from the borehole

$$\rho_0 \rightarrow \infty : \Theta_{\text{in}}^{(0)} \rightarrow A(\tau) - \frac{q(\tau)}{2\pi} \ln(\rho_0).$$

For convenience, the formulated zeroth order problem is summarized in Table 4.1. Its solution are $\Theta_{\text{in}}^{(0)}$ and $\Theta_j^{(0)}$, the zeroth order terms of the expansion of the unknowns. From the obtained grout/ground temperature it is necessary to compute the first harmonics $A_{0(\pm 1)}(\tau)$ of its Fourier series expansion around the borehole wall, Eq. (4.5), as these are required for the matching condition in the first order problem.

4.2.2 First order problem

The formulation of the first order correction to the inner solution is the same as the one presented in Section 3.3.1, differing again only in the matching condition enforced far from the borehole:

$$\begin{aligned} \rho_0 \rightarrow \infty : \Theta_{\text{in}}^{(1)} \rightarrow & B(\tau) + C(\tau)\rho_0 e^{+i\theta_0} + \bar{C}(\tau)\rho_0 e^{-i\theta_0} \\ & + \ln(\rho_0) \left[\frac{A_{0(+1)}(\tau) + A_{0(-1)}(\tau)}{2} - \frac{q(\tau)}{4\pi} \rho_0 \cos(\theta_0) \right]. \end{aligned}$$

To tackle the formulated first order problem, its solution is split into two parts by taking advantage of the linearity of governing equations and boundary equations. Thus, the first order corrections to the grout/ground temperature and to the fluid temperatures in the pipes are written as

$$\Theta_{\text{in}}^{(1)} = \Theta_{\text{ex}}^{(1)} + \Theta_{\text{en}}^{(1)} \quad \text{and} \quad \Theta_j^{(1)} = \Theta_{j,\text{ex}}^{(1)} + \Theta_{j,\text{en}}^{(1)}.$$

The exogenous first order problem, comprised of $\Theta_{\text{ex}}^{(1)}$ and $\Theta_{j,\text{ex}}^{(1)}$, accounts for everything the outer region imposes onto the inner region. That is, the temperature level $B(\tau)$ and the temperature gradient specified by the pair $C(\tau)$ and $\bar{C}(\tau)$. Its governing equations and boundary conditions are summarized in Table 4.2.

The endogenous first order problem, comprised of $\Theta_{\text{en}}^{(1)}$ and $\Theta_{j,\text{en}}^{(1)}$, accounts for everything that is purely related to the inner region. That is, the forcing term in the energy conservation equation in the ground and all terms the inner region imposes onto the outer region in the matching condition far from the borehole. Its governing equations and boundary conditions are summarized in Table 4.3.

Energy conservation in ground
$0 = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \Theta_{\text{in}}^{(0)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \Theta_{\text{in}}^{(0)}}{\partial \theta_0^2}$
Energy conservation in grout
$0 = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \Theta_{\text{in}}^{(0)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \Theta_{\text{in}}^{(0)}}{\partial \theta_0^2}$
Continuity conditions at borehole wall
$\Theta_{\text{in}}^{(0)} \Big _{\rho_0=1^-} = \Theta_{\text{in}}^{(0)} \Big _{\rho_0=1^+} \quad \text{and} \quad -\kappa \frac{\partial \Theta_{\text{in}}^{(0)}}{\partial \rho_0} \Big _{\rho_0=1^-} = -\frac{\partial \Theta_{\text{in}}^{(0)}}{\partial \rho_0} \Big _{\rho_0=1^+}$
Boundary condition at wall of pipe j
$-\rho_{pj} \kappa \frac{\partial \Theta_{\text{in}}^{(0)}}{\partial \rho_j} \Big _{\rho_j=\rho_{pj}} = \frac{\Theta_j^{(0)}(\tau) - \Theta_{\text{in}}^{(0)} \Big _{\rho_j=\rho_{pj}}}{R_{pj}}$ with $\Theta_j^{(0)}(\tau) = \frac{R_{pj}}{2\pi} q_j(\tau) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta_{\text{in}}^{(0)} \Big _{\rho_j=\rho_{pj}} d\theta_j$
Matching condition far from borehole
$\rho_0 \rightarrow \infty: \Theta_{\text{in}}^{(0)} \rightarrow A(\tau) - \frac{q(\tau)}{2\pi} \ln(\rho_0)$ with $q(\tau) = \sum_{j=1}^{N_p} q_j(\tau)$

Table 4.1: Formulation of zeroth order problem.

Energy conservation in ground
$0 = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \Theta_{\text{ex}}^{(1)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \Theta_{\text{ex}}^{(1)}}{\partial \theta_0^2}$
Energy conservation in grout
$0 = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \Theta_{\text{ex}}^{(0)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \Theta_{\text{ex}}^{(0)}}{\partial \theta_0^2}$
Continuity conditions at borehole wall
$\Theta_{\text{ex}}^{(1)} \Big _{\rho_0=1^-} = \Theta_{\text{ex}}^{(1)} \Big _{\rho_0=1^+} \quad \text{and} \quad -\kappa \frac{\partial \Theta_{\text{ex}}^{(1)}}{\partial \rho_0} \Big _{\rho_0=1^-} = -\frac{\partial \Theta_{\text{ex}}^{(1)}}{\partial \rho_0} \Big _{\rho_0=1^+}$
Boundary condition at wall of pipe j
$-\rho_{pj} \kappa \frac{\partial \Theta_{\text{ex}}^{(1)}}{\partial \rho_j} \Big _{\rho_j=\rho_{pj}} = \frac{\Theta_{j,\text{ex}}^{(1)}(\tau) - \Theta_{\text{ex}}^{(1)} \Big _{\rho_j=\rho_{pj}}}{R_{pj}}$ with $\Theta_{j,\text{ex}}^{(1)}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta_{\text{ex}}^{(1)} \Big _{\rho_j=\rho_{pj}} d\theta_j$
Matching condition far from borehole
$\rho_0 \rightarrow \infty: \Theta_{\text{ex}}^{(1)} \rightarrow B(\tau) + C(\tau) \rho_0 e^{+\iota \theta_0} + \bar{C}(\tau) \rho_0 e^{-\iota \theta_0}$

Table 4.2: Formulation of exogeneous first order problem.

4.3 Modified Enhanced Multipole Method

The three problems formulated in the previous section need to be solved exactly for which the enhanced multipole method is used [51, 93]. This method gives the exact solution to the quasi-steady heat conduction problem in grout and ground. Hence, it is able to deliver the exact solution to the zeroth order problem summarized in Table 4.1.

To address the exogenous and endogenous first order problems, summarized in Tables 4.2 and 4.3, the enhanced multipole method needs to be modified to incorporate the presence of forcing terms in the governing equation in the ground and of a temperature gradient in the matching condition far from the borehole. The resulting method, called modified enhanced multipole method, is the subject of the present section.

Energy conservation in ground
$v_\rho \frac{\partial \Theta^{(0)}}{\partial \rho_0} + \frac{v_\theta}{\rho_0} \frac{\partial \Theta^{(0)}}{\partial \theta_0} = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \Theta_{\text{en}}^{(1)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \Theta_{\text{en}}^{(1)}}{\partial \theta_0^2}$ with $v_\rho = \left(1 - \frac{1}{\rho_0^2} \right) \cos(\theta_0) \quad \text{and} \quad v_\theta = - \left(1 + \frac{1}{\rho_0^2} \right) \sin(\theta_0)$
Energy conservation in grout
$0 = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \Theta_{\text{en}}^{(0)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \Theta_{\text{en}}^{(0)}}{\partial \theta_0^2}$
Continuity conditions at borehole wall
$\Theta_{\text{en}}^{(1)} \Big _{\rho_0=1^-} = \Theta_{\text{en}}^{(1)} \Big _{\rho_0=1^+} \quad \text{and} \quad -\kappa \frac{\partial \Theta_{\text{en}}^{(1)}}{\partial \rho_0} \Big _{\rho_0=1^-} = -\frac{\partial \Theta_{\text{en}}^{(1)}}{\partial \rho_0} \Big _{\rho_0=1^+}$
Boundary condition at wall of pipe j
$-\rho_{pj} \kappa \frac{\partial \Theta_{\text{en}}^{(1)}}{\partial \rho_j} \Big _{\rho_j=\rho_{pj}} = \frac{\Theta_{j,\text{en}}^{(1)}(\tau) - \Theta_{\text{en}}^{(1)} \Big _{\rho_j=\rho_{pj}}}{R_{pj}}$ with $\Theta_{j,\text{en}}^{(1)}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta_{\text{en}}^{(1)} \Big _{\rho_j=\rho_{pj}} d\theta_j$
Matching condition far from borehole
$\rho_0 \rightarrow \infty: \quad \Theta_{\text{en}}^{(1)} \rightarrow \ln(\rho_0) \left[\frac{A_{0(+1)}(\tau) + A_{0(-1)}(\tau)}{2} - \frac{q(\tau)}{4\pi} \rho_0 \cos(\theta_0) \right]$ with $A_{0(\pm 1)}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta_{\text{in}}^{(0)} \Big _{\rho_0=1} e^{\mp i\theta_0} d\theta_0$

Table 4.3: Formulation of endogeneous first order problem.

4.3.1 Multipole expansion

The idea behind the enhanced multipole method is to expand the grout/ground temperature in terms of conveniently chosen eigenfunctions F_{in} which in the literature are called multipoles [5, 19, 21, 51, 93]. Each pipe i contributes with an infinite number of them, indicating subindex n the azimuthal order of each multipole. The expansion of the grout/ground temperature is then written as follows:

$$\Theta(\rho_0, \theta_0, \tau) = \Theta_p(\rho_0, \theta_0, \tau) + \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in}(\tau) F_{in}(\rho_0, \theta_0). \quad (4.6)$$

Each multipole F_{in} is obtained from solving a quasi-steady heat conduction problem in grout and ground [93],

$$0 = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial F_{in}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 F_{in}}{\partial \theta_0^2},$$

with its continuity conditions at the borehole wall:

$$F_{in} \Big|_{\rho_0=1^-} = F_{in} \Big|_{\rho_0=1^+} \quad \text{and} \quad -\kappa \frac{\partial F_{in}}{\partial \rho_0} \Big|_{\rho_0=1^-} = -\frac{\partial F_{in}}{\partial \rho_0} \Big|_{\rho_0=1^+}.$$

Finally, different behaviors far from the borehole are enforced depending on n . While for $n \neq 0$ the corresponding multipole decays to zero,

$$\rho_0 \rightarrow \infty : F_{in} \rightarrow 0,$$

for $n = 0$ the multipole F_{i0} grows logarithmically as

$$\rho_0 \rightarrow \infty : F_{i0} \rightarrow -\kappa \ln(\rho_0).$$

Explicit expressions for the multipoles F_{in} are given in the literature [51]. These are not required, though, for the theoretical consideration of this thesis or for the numerical examples presented in Section 4.7.

The proposed expansion for the grout/ground temperature also includes a particular solution to the problem, namely, Θ_p . This particular solution incorporates everything not covered by the multipoles F_{in} . That is, the forcing terms in the governing equation in the ground and everything the outer region imposes onto the inner region.

Last, the proposed expansion also includes a-priori unknown weights C_{in} that multiply the corresponding multipoles F_{in} in the expansion. Their values result from enforcing the expansion of the grout/ground temperature to comply with the boundary conditions at the N_p pipes. That is, with the specified heat injection rates per unit pipe length,

$$\int_{-\pi}^{\pi} \rho_{pj} \frac{\partial \Theta}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} d\theta_j = -\frac{q_j(\tau)}{\kappa}, \quad (4.7)$$

and with the additional boundary conditions enforced at the outer surface of the pipes:

$$-\kappa R_{pj} \rho_{pj} \frac{\partial \Theta}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} = \Theta_j(\tau) - \Theta \Big|_{\rho_j=\rho_{pj}}. \quad (4.8)$$

4.3.2 Fourier expansion of F_{in} and Θ_p

To enforce the previous two conditions at the pipes, the Fourier series expansion of the grout/ground temperature around each pipe j is obtained. For it, Fourier series expansions of multipole F_{in} and its derivative with respect to ρ_j are defined as follows [93]:

$$F_{in} \Big|_{\rho_j=\rho_{pj}} = \sum_{m=-\infty}^{\infty} \mathcal{F}_{injm} e^{\iota m \theta_j} \quad \text{and} \quad \rho_{pj} \frac{\partial F_{in}}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} = \sum_{m=-\infty}^{\infty} \mathcal{F}'_{injm} e^{\iota m \theta_j},$$

with the Fourier series coefficients being given by

$$\mathcal{F}_{injm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{in} \Big|_{\rho_j=\rho_{pj}} e^{-\iota m \theta_j} d\theta_j \quad \text{and} \quad \mathcal{F}'_{injm} = \rho_{pj} \frac{\partial \mathcal{F}_{injm}}{\partial \rho_{pj}}.$$

Explicit expressions for $\mathcal{F}_{injm}$ and $\mathcal{F}'_{injm}$ are readily available in the literature [51]. In the current chapter these are required for some of the theoretical considerations made and for the numerical examples presented in Section 4.7.

The Fourier series expansions of the particular solution Θ_p and its derivative with respect to ρ_j are defined following the same idea as for the multipoles:

$$\Theta_p \Big|_{\rho_j=\rho_{pj}} = \sum_{m=-\infty}^{\infty} \mathcal{P}_{jm} e^{\iota m \theta_j} \quad \text{and} \quad \rho_{pj} \frac{\partial \Theta_p}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} = \sum_{m=-\infty}^{\infty} \mathcal{P}'_{jm} e^{\iota m \theta_j},$$

with the Fourier series coefficients being given by

$$\mathcal{P}_{jm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta_p \Big|_{\rho_j=\rho_{pj}} e^{-\iota m \theta_j} d\theta_j \quad \text{and} \quad \mathcal{P}'_{jm} = \rho_{pj} \frac{\partial \mathcal{P}_{jm}}{\partial \rho_{pj}}.$$

Since the particular solution Θ_p is a function of time, its coefficients $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$ will also be functions of τ . This contrasts with the Fourier series coefficients $\mathcal{F}_{injm}$ and $\mathcal{F}'_{injm}$ which, in the considered limit of slowly-varying heat injection rates, do not depend on time [51].

4.3.3 Heat injection rates enforcement

To enforce the prescribed heat injection rates per unit pipe length $q_j(\tau)$, the multipole expansion for the grout/ground temperature, Eq. (4.6), is substituted into Eq. (4.7):

$$\int_{-\pi}^{\pi} \left[\rho_{pj} \frac{\partial \Theta_p}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} + \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} \rho_{pj} \frac{\partial F_{in}}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} \right] d\theta_j = -\frac{q_j}{\kappa}.$$

Replacing the radial derivatives of F_{in} and Θ_p by their Fourier series expansions around pipe j leads to

$$\int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} \left[\mathcal{P}'_{jm} + \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} \mathcal{F}'_{injm} \right] e^{\iota m \theta_j} d\theta_j = -\frac{q_j}{\kappa}.$$

The azimuthal integration of all terms in the Fourier series expansions around pipe j reveals that the only non-zero contribution to the resulting expression comes from $m = 0$:

$$\sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} \mathcal{F}'_{inj0} = -\frac{q_j}{2\pi\kappa} - \mathcal{P}'_{j0}. \quad (4.9)$$

Compared to the original enhanced multipole method [51, 93], a new term arises on the right hand side of the equation. It represents the heat injection rate per unit pipe length at pipe j due to the particular solution Θ_p .

4.3.4 Boundary condition at pipe walls enforcement

Substitution of the multipole expansion for the grout/ground temperature, Eq. (4.6), into the additional boundary condition at the outer surface of the pipes, Eq. (4.8), leads to

$$\begin{aligned} -\kappa R_{pj} \left[\rho_{pj} \frac{\partial \Theta_p}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} + \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} \rho_{pj} \frac{\partial F_{in}}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} \right] = \\ \Theta_j - \left[\Theta_p \Big|_{\rho_j=\rho_{pj}} + \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} F_{in} \Big|_{\rho_j=\rho_{pj}} \right]. \end{aligned}$$

Making again use of the Fourier series expansions around pipe j of Θ_p and F_{in} and of their derivatives with respect to ρ_j allows the previous expression to be rewritten as

$$\sum_{m=-\infty}^{\infty} \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} [\mathcal{F}_{injm} - \kappa R_{pj} \mathcal{F}'_{injm}] e^{im\theta_0} = \Theta_j - \sum_{m=-\infty}^{\infty} [\mathcal{P}_{jm} - \kappa R_{pj} \mathcal{P}'_{jm}] e^{im\theta_0}.$$

This condition can now be enforced independently onto each harmonic of the Fourier series expansion around pipe j so that an infinite set of linear algebraic equations, with m ranging from $-\infty$ till $+\infty$, is obtained in which δ_{mn} stands for Kronecker's delta:

$$\sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} [\mathcal{F}_{injm} - \kappa R_{pj} \mathcal{F}'_{injm}] = \Theta_j \delta_{m0} - [\mathcal{P}_{jm} - \kappa R_{pj} \mathcal{P}'_{jm}]. \quad (4.10)$$

Compared with the original enhanced multipole method [51, 93], these equations present additional terms on their right hand sides. While for the enhanced multipole method only the equations with $m = 0$ had additional terms to Θ_j , which were all equal to the apparent ground temperature, now contributions from the particular solution Θ_p are present in all equations, regardless of the value of m .

4.3.5 Fluid temperatures

To obtain the weights C_{in} that make the multipole expansion of the grout/ground temperature, Eq. (4.6), satisfy the boundary conditions at the N_p pipes, an infinite system of linear algebraic

equations is solved. It is comprised of Eq. (4.9) and of all equations in Eq. (4.10) with $m \neq 0$. Thus,

$$\sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} \mathcal{F}'_{inj0} = -\frac{q_j}{2\pi\kappa} - \mathcal{P}'_{j0} \quad (4.11)$$

$$\sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} [\mathcal{F}_{injm} - \kappa R_{pj} \mathcal{F}'_{injm}] = -[\mathcal{P}_{jm} - \kappa R_{pj} \mathcal{P}'_{jm}],$$

with index j going from 1 till N_p in both sets of equations and index m going from $-\infty$ till $+\infty$ excluding $m = 0$.

Once weights C_{in} are known, the N_p fluid temperatures Θ_j ensuring the prescribed heat injection rates per unit pipe length are met result from the N_p equations in Eq. (4.10) with $m = 0$:

$$\sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} [\mathcal{F}_{inj0} - \kappa R_{pj} \mathcal{F}'_{inj0}] = \Theta_j - [\mathcal{P}_{j0} - \kappa R_{pj} \mathcal{P}'_{j0}]. \quad (4.12)$$

4.3.6 Temperature around borehole wall

Apart from the fluid temperatures Θ_j , another useful result is the temperature distribution around the borehole wall. Particularizing Eq. (4.6) at the borehole wall, where $\rho_0 = 1$, leads to

$$\Theta|_{\rho_0=1} = \Theta_p|_{\rho_0=1} + \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} F_{in}|_{\rho_0=1}.$$

If the borehole is seen as just another pipe, namely, pipe 0 with $\rho_{p0} = 1$ [51, 93], this expression for the temperature distribution along the borehole wall can also be written as

$$\Theta|_{\rho_0=1} = \sum_{m=-\infty}^{\infty} \left[\mathcal{P}_{0m} + \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} \mathcal{F}_{in0m} \right] e^{\iota m \theta_0}, \quad (4.13)$$

with Fourier series coefficients $\mathcal{P}_{0m}$ and $\mathcal{F}_{in0m}$ being fully equivalent to the ones defined in Section 4.3.2. Additionally, explicit expressions for $\mathcal{F}_{in0m}$ are readily available in the literature [51].

4.4 Solution to inner problem

The modified enhanced multipole method is used next to tackle the three problems presented in Sections 4.2.1 and 4.2.2. The combination of their solutions will then deliver in Sections 4.5 and 4.6 the sought conclusions regarding the impact of creeping groundwater flows onto the network of thermal resistances and onto the apparent ground temperature.

4.4.1 Zeroth order solution

The first problem to address is the zeroth order one summarized in Table 4.1. No forcing terms are present in the governing equations so that the only phenomenon to incorporate into the particular solution Θ_p is the uniform temperature level $A(\tau)$ imposed by the outer region onto the inner region. Consequently, $\Theta_p = A(\tau)$.

Next, coefficients $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$ of the Fourier series expansions around pipe j of Θ_p and its derivative with respect to ρ_j are obtained. Since the particular solution is uniform in space, they are straightforward to get:

$$\mathcal{P}_{jm} = A(\tau) \delta_{m0} \quad \text{and} \quad \mathcal{P}'_{jm} = 0. \quad (4.14)$$

The Fourier series coefficients are then substituted into Eq. (4.11) to derive the infinite system of linear algebraic equations whose solution are the weights C_{in} :

$$\begin{aligned} \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} \mathcal{F}'_{inj0} &= -\frac{q_j}{2\pi\kappa} \\ \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in} [\mathcal{F}_{injm} - \kappa R_{pj} \mathcal{F}'_{injm}] &= 0, \end{aligned}$$

where index j goes from 1 till N_p in both sets of equations and m goes from $-\infty$ till $+\infty$ excluding $m = 0$.

This system of equations coincides with the one solved by the original enhanced multipole method [51, 93]. This comes as no surprise since the zeroth order problem is a purely-conducting one with no additional imposition by the outer region than a spatially-uniform temperature level.

The original enhanced multipole method expresses the solution to the formulated system of equations as a linear combination of heat injection rates per unit pipe length $q_j(\tau)$ [51, 93]. The same approach is followed here so that weights C_{in} are written as follows:

$$C_{in}(\tau) = \sum_{a=1}^{N_p} C_{ina} \frac{q_a(\tau)}{2\pi\kappa}. \quad (4.15)$$

Substitution of this relationship into the formulated system of linear algebraic equations leads to an alternative system of equations directly for the coefficients C_{ina} in which indexes j and a go from 1 till N_p in both sets of equations and index m goes from $-\infty$ till $+\infty$ excluding $m = 0$:

$$\begin{aligned} \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{ina} \mathcal{F}'_{inj0} &= -\delta_{aj} \\ \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{ina} [\mathcal{F}_{injm} - \kappa R_{pj} \mathcal{F}'_{injm}] &= 0. \end{aligned} \quad (4.16)$$

Once coefficients C_{ina} are known, which by the way do not depend on time, zeroth order fluid temperatures $\Theta_j^{(0)}(\tau)$ result from combining Eq. (4.15) with Eq. (4.12) and with the values for $\mathcal{P}_{j0}$ and $\mathcal{P}'_{j0}$:

$$\sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} \sum_{a=1}^{N_p} C_{ina} [\mathcal{F}_{inj0} - \kappa R_{pj} \mathcal{F}'_{inj0}] \frac{q_a(\tau)}{2\pi\kappa} = \Theta_j^{(0)}(\tau) - A(\tau).$$

Network of thermal resistances

The last expression is conveniently reinterpreted as a network of thermal resistances leading to the mathematically equivalent expression

$$\sum_{j=1}^{N_p} \hat{R}_{ij}^{(0)} q_j(\tau) = \Theta_i^{(0)}(\tau) - \Theta_{ai}^{(0)}(\tau).$$

The zeroth order thermal resistances $\hat{R}_{ij}^{(0)}$, which do not depend on time, are given by

$$2\pi\kappa\hat{R}_{ij}^{(0)} = \sum_{a=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{anj} [\mathcal{F}_{ani0} - \kappa R_{pi} \mathcal{F}'_{ani0}] = \kappa R_{pi} \delta_{ij} + \sum_{a=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{anj} \mathcal{F}_{ani0},$$

where the latter expression results from simplifying the former one using the first set of equations in Eq. (4.16).

The zeroth order apparent ground temperature for pipe i , $\Theta_{ai}^{(0)}(\tau)$, is equal to the temperature level $A(\tau)$ imposed by the outer region onto the inner region. Consequently, for the zeroth order approximation to the problem, all apparent ground temperatures for the pipes are equal:

$$\Theta_{ai}^{(0)}(\tau) = A(\tau).$$

Borehole wall harmonics

Next, grout/ground temperature distribution around the borehole wall is obtained as its Fourier series expansion is required for the endogenous first order problem and its solution. The coefficients of that Fourier series expansion are

$$A_{0m}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta_{in}^{(0)} \Big|_{\rho_0=1} e^{-im\theta_0} d\theta_0.$$

Using Eq. (4.13) for the temperature around the borehole wall, the following expression for the Fourier series coefficients is directly obtained:

$$A_{0m}(\tau) = \mathcal{P}_{0m}(\tau) + \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in}(\tau) \mathcal{F}_{in0m}.$$

Taking into account expressions for $\mathcal{P}_{0m}$, Eq. (4.14), and the linear dependence of weights C_{in} with heat injection rates per unit pipe length $q_j(\tau)$, Eq. (4.15), the following results:

$$A_{0m}(\tau) = A(\tau) \delta_{m0} + \sum_{i=1}^{N_p} A_{0mj} \frac{q_j(\tau)}{2\pi\kappa},$$

with coefficients A_{0mj} not depending on time:

$$A_{0mj} = \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{inj} \mathcal{F}_{in0m}.$$

4.4.2 Exogenous first order correction

The second problem to address is the exogenous first order one summarized in Table 4.2. The particular solution Θ_p must contain in this case the imposed temperature level $B(\tau)$ as well as the thermal response to the imposed temperature gradient specified by $C(\tau)$ and $\bar{C}(\tau)$.

To obtain the thermal response to the imposed temperature gradient, the particular solution must satisfy the governing equations in grout and ground as well as the continuity conditions at the borehole wall. This is achieved with the aid of Appendix 4.A, leading to the following expressions for the particular solution:

$$\Theta_p = \begin{cases} B(\tau) + \frac{2\rho_0}{1+\kappa} \left(C(\tau) e^{+\iota\theta_0} + \bar{C}(\tau) e^{-\iota\theta_0} \right) & \text{if } \rho_0 \leq 1 \\ B(\tau) + \left[\rho_0 + \frac{1-\kappa}{1+\kappa} \frac{1}{\rho_0} \right] \left(C(\tau) e^{+\iota\theta_0} + \bar{C}(\tau) e^{-\iota\theta_0} \right) & \text{if } \rho_0 \geq 1. \end{cases}$$

Next, coefficients of the Fourier series expansions around pipe j of Θ_p and its derivative with respect to ρ_j are obtained. For it, complex arithmetic is used to express position (ρ_0, θ_0) in the polar coordinate system centered at the borehole in terms of position (ρ_j, θ_j) in the polar coordinate system centered at pipe j :

$$\rho_0 e^{\pm\iota\theta_0} = \rho_{0j} e^{\pm\iota\theta_{0j}} + \rho_{pj} e^{\pm\iota\theta_j}.$$

Combining this relationship with the expression for Θ_p valid inside the borehole, where $\rho_0 \leq 1$, it can be shown that all Fourier series coefficients $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$ with $|m| > 1$ are zero, that for $m = 0$ the coefficients are given by

$$\mathcal{P}_{j0}(\tau) = B(\tau) + \frac{2\rho_{0j}}{1+\kappa} \left(C(\tau) e^{+\iota\theta_{0j}} + \bar{C}(\tau) e^{-\iota\theta_{0j}} \right)$$

and $\mathcal{P}'_{j0} = 0$, and that for $m = \pm 1$ they are

$$\mathcal{P}_{j(+1)}(\tau) = \mathcal{P}'_{j(+1)}(\tau) = \frac{2\rho_{pj}}{1+\kappa} C(\tau),$$

$$\mathcal{P}_{j(-1)}(\tau) = \mathcal{P}'_{j(-1)}(\tau) = \frac{2\rho_{pj}}{1+\kappa} \bar{C}(\tau).$$

Thereafter, weights C_{in} are determined for which the previous Fourier series coefficients are substituted into Eq. (4.11). When writing the resulting system of linear algebraic equations, two additional remarks need to be made. First, in the exogenous first order problem no heat injection rates per unit pipe length are specified at the pipes so that the first set of equations in Eq. (4.11) will have zero right hand sides. Second, only $C(\tau)$ and $\bar{C}(\tau)$ trigger the thermal response of the inner region. Hence, weights C_{in} can be expressed as linear combinations of these two quantities:

$$C_{in} = C_{in,c}C(\tau) + C_{in,\bar{c}}\bar{C}(\tau).$$

Substitution of this relationship into Eq. (4.11) leads to independent systems of linear algebraic equations for each set of coefficients $C_{in,c}$ and $C_{in,\bar{c}}$. So, for the former set of coefficients the equations to solve are

$$\begin{aligned} \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in,c} \mathcal{F}'_{inj0} &= 0 \\ \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in,c} [\mathcal{F}_{injm} - \kappa R_{pj} \mathcal{F}'_{injm}] &= 0 \quad \forall m \neq 0, +1 \\ \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in,c} [\mathcal{F}_{inj(+1)} - \kappa R_{pj} \mathcal{F}'_{inj(+1)}] &= -[1 - \kappa R_{pj}] \frac{2\rho_{pj}}{1 + \kappa}, \end{aligned} \quad (4.17)$$

while for the latter set of coefficients the system of linear algebraic equations to solve is

$$\begin{aligned} \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in,\bar{c}} \mathcal{F}'_{inj0} &= 0 \\ \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in,\bar{c}} [\mathcal{F}_{injm} - \kappa R_{pj} \mathcal{F}'_{injm}] &= 0 \quad \forall m \neq 0, -1 \\ \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{in,\bar{c}} [\mathcal{F}_{inj(-1)} - \kappa R_{pj} \mathcal{F}'_{inj(-1)}] &= -[1 - \kappa R_{pj}] \frac{2\rho_{pj}}{1 + \kappa}. \end{aligned} \quad (4.18)$$

In both cases index j goes from 1 till N_p and the middle set of equations applies to all values of m ranging from $-\infty$ till $+\infty$ excluding the two values specified on the right.

Known the coefficients $C_{in,c}$ and $C_{in,\bar{c}}$, whose values result from solving the previous two systems of linear algebraic equations, exogenous first order corrections $\Theta_{j,\text{ex}}^{(1)}(\tau)$ of the fluid temperatures in the pipes result from Eq. (4.12):

$$\sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} [C_{in,c}C(\tau) + C_{in,\bar{c}}\bar{C}(\tau)] [\mathcal{F}_{inj0} - \kappa R_{pj} \mathcal{F}'_{inj0}] = \Theta_{j,\text{ex}}^{(1)}(\tau) - \mathcal{P}_{j0}(\tau),$$

where $\mathcal{P}_{j0}(\tau)$ has not been substituted for the sake of clarity and compactness.

Network of thermal resistances

The last expression is conveniently reinterpreted as a network of thermal resistances leading to the mathematically equivalent expression

$$0 = \Theta_{i,\text{ex}}^{(1)}(\tau) - \Theta_{ai}^{(1)}(\tau).$$

Since the prescribed heat injection rates per unit pipe length $q_j(\tau)$ do not participate in the exogenous first order problem and its solution, no left hand side appears in the expression.

The first order correction to the apparent ground temperature for pipe i is defined as

$$\Theta_{ai}^{(1)}(\tau) = \mathcal{P}_{i0}(\tau) + \sum_{a=1}^{N_p} \sum_{n=-\infty}^{\infty} \left[C_{an,c} C(\tau) + C_{an,\bar{c}} \bar{C}(\tau) \right] \mathcal{F}_{ani0},$$

where advantage has been taken, when writing this expression, of the first set of equations in Eqs. (4.17) and (4.18). Substituting now $\mathcal{P}_{i0}$ and rearranging the output leads to

$$\begin{aligned} \Theta_{ai}^{(1)}(\tau) = B(\tau) &+ \left[\frac{2\rho_{0i}}{1+\kappa} e^{+\iota\theta_{0i}} + \sum_{a=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{an,c} \mathcal{F}_{ani0} \right] C(\tau) \\ &+ \left[\frac{2\rho_{0i}}{1+\kappa} e^{-\iota\theta_{0i}} + \sum_{a=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{an,\bar{c}} \mathcal{F}_{ani0} \right] \bar{C}(\tau). \end{aligned}$$

The first term, $B(\tau)$, is common to all pipes and represents a correction to the uniform temperature level $A(\tau)$ defining the zeroth order apparent ground temperatures for the pipes. The remaining terms are due to the temperature gradient imposed by the outer region onto the inner region. The first part of the coefficients multiplying $C(\tau)$ and $\bar{C}(\tau)$ is the particular solution evaluated at the position of pipe i while the second part is the thermal response of the inner region to that particular solution.

The dependence of the apparent ground temperatures for the pipes on the thermal response of the inner region complicates their computation as both regions are now interwoven and can no longer be treated separately, as done in purely-conducting grounds.

4.4.3 Endogenous first order correction

The last problem to address is the endogenous first order one summarized in Table 4.3. The particular solution Θ_p includes in this case all forcing terms present in the governing equation for the ground and all terms imposed by the inner region onto the outer region. This renders Θ_p much more complex so that its derivation is relegated to Appendix 4.A. The same applies to the corresponding Fourier series coefficients $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$.

Next, weights C_{in} are determined for which the Fourier series coefficients given in Appendix 4.A are substituted into Eq. (4.11). Of special interest is $\mathcal{P}'_{j0} = 0$ which combined with the fact that no heat injection rates per unit pipe length are specified at the pipes leads to zero right hand sides in the first set of equations in Eq. (4.11).

Also shown in Appendix 4.A is that $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$ are linear combinations of the heat injection rates per unit pipe length specified at the pipes. This dependency has its origin in the presence of the zeroth order solution in the forcing terms of the governing equation in the ground. Hence, Fourier series coefficients $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$ can be written as

$$\mathcal{P}_{jm}(\tau) = \sum_{a=1}^{N_p} \mathcal{P}_{jma} \frac{q_a(\tau)}{2\pi\kappa} \quad \text{and} \quad \mathcal{P}'_{jm}(\tau) = \sum_{a=1}^{N_p} \mathcal{P}'_{jma} \frac{q_a(\tau)}{2\pi\kappa},$$

which implies that weights C_{in} can also be expressed as linear combinations of the heat injection rates per unit pipe length specified at the pipes:

$$C_{in}(\tau) = \sum_{a=1}^{N_p} C_{ina} \frac{q_a(\tau)}{2\pi\kappa}.$$

Substitution of everything into Eq. (4.11) supplies the system of linear algebraic equations for the coefficients C_{ina} in which indexes j and a go from 1 till N_p and index m goes from $-\infty$ till $+\infty$ excluding $m = 0$:

$$\begin{aligned} \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{ina} \mathcal{F}'_{inj0} &= 0 \\ \sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{ina} [\mathcal{F}_{injm} - \kappa R_{pj} \mathcal{F}'_{injm}] &= -[\mathcal{P}_{jma} - \kappa R_{pj} \mathcal{P}'_{jma}]. \end{aligned} \tag{4.19}$$

Known the coefficients C_{ina} , which do not depend on time, endogenous first order corrections $\Theta_{j,\text{en}}^{(1)}(\tau)$ of the fluid temperatures in the pipes result from Eq. (4.12):

$$\sum_{i=1}^{N_p} \sum_{n=-\infty}^{\infty} \sum_{a=1}^{N_p} C_{ina} [\mathcal{F}_{inj0} - \kappa R_{pj} \mathcal{F}'_{inj0}] \frac{q_a(\tau)}{2\pi\kappa} = \Theta_{j,\text{en}}^{(1)}(\tau) - \sum_{a=1}^{N_p} [\mathcal{P}_{j0a} - \kappa R_{pj} \mathcal{P}'_{j0a}] \frac{q_a(\tau)}{2\pi\kappa}.$$

Network of thermal resistances

The last expression is conveniently reinterpreted as a network of thermal resistances leading to the mathematically equivalent expression

$$\sum_{j=1}^{N_p} \hat{R}_{ij}^{(1)} q_j(\tau) = \Theta_{i,\text{en}}^{(1)}(\tau).$$

Since the endogenous first order problem is characterized by the absence of impositions by the outer region onto the inner region, no apparent ground temperature appears on the right hand side of the expression.

The first order corrections $\hat{R}_{ij}^{(1)}$ to the thermal resistances, which do not depend on time, are

$$2\pi\kappa \hat{R}_{ij}^{(1)} = \sum_{a=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{anj} [\mathcal{F}_{ani0} - \kappa R_{pi} \mathcal{F}'_{ani0}] + [\mathcal{P}_{i0j} - \kappa R_{pi} \mathcal{P}'_{i0j}].$$

This expression can be simplified twice. First, it is shown in Appendix 4.A that $\mathcal{P}'_{i0} = 0$. Consequently, $\mathcal{P}'_{i0j} = 0$. Second, the first set of equations in Eq. (4.19) holds. Hence, the final expression for the first order corrections to the thermal resistances becomes

$$2\pi\kappa\hat{R}_{ij}^{(1)} = \sum_{a=1}^{N_p} \sum_{n=-\infty}^{\infty} C_{anj} \mathcal{F}_{ani0} + \mathcal{P}_{i0j}.$$

4.5 Apparent ground temperature

Each of the three problems solved in the previous section introduces a network of thermal resistances that relates specified heat injection rates per unit pipe length with resulting fluid temperatures in the pipes and with apparent ground temperatures for the pipes. In the present section these results are combined to obtain one single network of thermal resistances for the thermal interaction of geothermal boreholes with creeping groundwater flows.

The followed combination procedure seeks to reconstruct the fluid temperatures in the pipes out of their asymptotic constituents:

$$\Theta_i(\tau) = \Theta_i^{(0)}(\tau) + \text{Pe} \left[\Theta_{i,\text{ex}}^{(1)}(\tau) + \Theta_{i,\text{en}}^{(1)}(\tau) \right].$$

Obeying to this combination procedure, the following network of thermal resistances is obtained,

$$\sum_{j=1}^{N_p} \hat{R}_{ij} q_j(\tau) = \Theta_i(\tau) - \Theta_{ai}(\tau), \quad (4.20)$$

in which thermal resistances $\hat{R}_{ij}$ and apparent ground temperatures for the pipes $\Theta_{ai}(\tau)$ are given by

$$\begin{aligned} \hat{R}_{ij} &= \hat{R}_{ij}^{(0)} + \text{Pe} \hat{R}_{ij}^{(1)} \\ \Theta_{ai}(\tau) &= \Theta_{ai}^{(0)}(\tau) + \text{Pe} \Theta_{ai}^{(1)}(\tau). \end{aligned} \quad (4.21)$$

These expressions make sense as they are equivalent to Taylor series expansions for small values of the Peclet number.

4.5.1 Single borehole

The case of a single borehole is considered next in order to obtain more specific expressions for the apparent ground temperatures for the pipes. The aim is to use them in a forthcoming section to assess the merits and limits of the state of the art.

To obtain the sought expressions, values for $A(\tau)$, $B(\tau)$, and $C(\tau)$ are required. These are derived from results for the outer region of the thermal interaction of single geothermal boreholes with creeping groundwater flows shown in Chapter 3. In that work, the Laplace

transform is used to deal with time dependence of the problem [7] so that expressions for the Laplace transforms of $A(\tau)$, $B(\tau)$, and $C(\tau)$ are given. In what follows a tilde ($\sim$) denotes the Laplace transformed character of a variable and s represents the position in the complex-valued Laplace plane.

By comparing the matching condition, Eq. (4.4), with the work done in Chapter 3, the following expressions for the Laplace transforms of $A(\tau)$, $B(\tau)$, and $C(\tau)$ are identified:

$$\begin{aligned}\tilde{A}(s) &= -\frac{\tilde{q}(s)}{2\pi} \left[\ln \left(\frac{\text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) + \gamma \right] \\ \tilde{B}(s) &= \frac{\tilde{A}_{0(+1)}(s) + \tilde{A}_{0(-1)}(s)}{2} \left[\ln \left(\frac{\text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) + \gamma + \frac{1}{2} \right] \\ \tilde{C}(s) &= -\frac{\tilde{q}(s)}{8\pi} \left[\ln \left(\frac{\text{Pe}}{4} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) + \gamma \right].\end{aligned}$$

In these expressions γ represents Euler's constant, $\tilde{q}(s)$ is the Laplace transform of the heat injection rate per unit borehole length,

$$\tilde{q}(s) = \sum_{j=1}^{N_p} \tilde{q}_j(s),$$

and $\tilde{A}_{0(\pm 1)}(s)$ are the Laplace transforms of the first harmonics of the Fourier series expansion of the grout/ground temperature around the borehole wall:

$$\tilde{A}_{0(\pm 1)}(s) = \sum_{j=1}^{N_p} A_{0(\pm 1)j} \frac{\tilde{q}_j(s)}{2\pi\kappa}.$$

Time-domain recovery

To recover the time evolution of the problem, the inverse Laplace transform has to be applied to the previous expressions for $\tilde{A}(s)$, $\tilde{B}(s)$, and $\tilde{C}(s)$. In general, applying the inverse Laplace transform cannot be done analytically and needs to be done numerically instead [44]. Fortunately, this is not the case here and an exact inversion is possible for which the known Laplace transform of the exponential integral $E_1(z)$ is exploited [7]:

$$\mathcal{L} \left[E_1 \left(\frac{\text{Pe}^2}{4} \tau \right) \right] = \frac{\ln \left(1 + \frac{4s}{\text{Pe}^2} \right)}{s}.$$

The Laplace inversion will be described in detail for $\tilde{A}(s)$, being the inversion of $\tilde{B}(s)$ and $\tilde{C}(s)$ identical. First, $\tilde{A}(s)$ needs to be rewritten to resemble the above Laplace transform of the exponential integral. Then, the initial value $q(0)$ for the heat injection rate per unit

borehole length is added and subtracted conveniently to construct the Laplace transform of the time derivative of $q(\tau)$. The result is

$$\tilde{A}(s) = -\frac{\tilde{q}(s)s - q(0)}{4\pi} \frac{\ln\left(1 + \frac{4s}{Pe^2}\right) + \ln\left(\frac{Pe^2}{16}\right) + 2\gamma}{s} - \frac{q(0)}{4\pi} \frac{\ln\left(1 + \frac{4s}{Pe^2}\right) + \ln\left(\frac{Pe^2}{16}\right) + 2\gamma}{s}.$$

Thanks to the convolution theorem [7], the inverse Laplace transform of the previous expression is straightforward to obtain:

$$\begin{aligned} A(\tau) = & -\frac{1}{4\pi} \int_0^\tau \frac{dq}{d\lambda} \left[E_1\left(\frac{Pe^2}{4}(\tau - \lambda)\right) + \ln\left(\frac{Pe^2}{16}\right) + 2\gamma \right] d\lambda \\ & - \frac{q(0)}{4\pi} \left[E_1\left(\frac{Pe^2}{4}\tau\right) + \ln\left(\frac{Pe^2}{16}\right) + 2\gamma \right], \end{aligned}$$

which upon integration by parts finally leads to the sought result for $A(\tau)$:

$$A(\tau) = -\frac{q(\tau)}{4\pi} \left[E_1\left(\frac{Pe^2}{4}\tau\right) + \ln\left(\frac{Pe^2}{16}\right) + 2\gamma \right] - \frac{1}{4\pi} \int_0^\tau \frac{q(\tau) - q(\lambda)}{\tau - \lambda} e^{-\frac{Pe^2}{4}(\tau - \lambda)} d\lambda.$$

For the outcome of the Laplace inversion of $\tilde{B}(s)$, it is more convenient to express it in combination with $A(\tau)$ as

$$\begin{aligned} A(\tau) + Pe B(\tau) = & -\frac{f(\tau)}{4\pi} \left[E_1\left(\frac{Pe^2}{4}\tau\right) + \ln\left(\frac{Pe^2}{16}\right) + 2\gamma \right] - \frac{1}{4\pi} \int_0^\tau \frac{f(\tau) - f(\lambda)}{\tau - \lambda} e^{-\frac{Pe^2}{4}(\tau - \lambda)} d\lambda \\ & + Pe \frac{A_{0(+1)}(\tau) + A_{0(-1)}(\tau)}{4} \end{aligned} \quad (4.22)$$

since then the fictitious heat injection rate per unit borehole length $f(\tau)$ emerges in the expression:

$$f(\tau) = q(\tau) - Pe \pi (A_{0(+1)}(\tau) + A_{0(-1)}(\tau)).$$

This is the same quantity present in the solution to the outer region developed in Chapter 3. This makes sense as the outer solution has to be expanded close to the borehole in order to perform its asymptotic matching with the inner solution.

Last, but not least, the Laplace inversion of $\tilde{C}(s)$ leads to the following expression for $C(\tau)$:

$$C(\tau) = -\frac{q(\tau)}{16\pi} \left[E_1\left(\frac{Pe^2}{4}\tau\right) + \ln\left(\frac{Pe^2}{16}\right) + 2\gamma \right] - \frac{1}{16\pi} \int_0^\tau \frac{q(\tau) - q(\lambda)}{\tau - \lambda} e^{-\frac{Pe^2}{4}(\tau - \lambda)} d\lambda.$$

As anticipated in Section 4.2, for a single borehole interacting with a groundwater stream the temperature gradient imposed by the outer region onto the inner region is perfectly aligned with the flow direction and $C(\tau)$ becomes real valued, which implies that $\bar{C}(\tau) = C(\tau)$.

Constant heat injection rates

When heat injection rates per unit pipe length are constant in time, $f(\tau)$ and $A_{0(\pm 1)}(\tau)$ become constants as well and the previous expressions for $A(\tau)$, $B(\tau)$, and $C(\tau)$ simplify considerably. In the case of $A(\tau)$ and $B(\tau)$,

$$A(\tau) + \text{Pe} B(\tau) = -\frac{f}{4\pi} \left[E_1 \left(\frac{\text{Pe}^2}{4} \tau \right) + \ln \left(\frac{\text{Pe}^2}{16} \right) + 2\gamma \right] + \text{Pe} \frac{A_{0(+1)} + A_{0(-1)}}{4}, \quad (4.23)$$

while for $C(\tau)$ the simplified expression is

$$C(\tau) = -\frac{q}{16\pi} \left[E_1 \left(\frac{\text{Pe}^2}{4} \tau \right) + \ln \left(\frac{\text{Pe}^2}{16} \right) + 2\gamma \right]. \quad (4.24)$$

As time passes, $A(\tau)$, $B(\tau)$, and $C(\tau)$ evolve towards steady-state values that can be computed by letting τ tend to infinity in the previous expressions:

$$\begin{aligned} [A + \text{Pe} B]_{\tau \rightarrow \infty} &= -\frac{f}{4\pi} \left[\ln \left(\frac{\text{Pe}^2}{16} \right) + 2\gamma \right] + \text{Pe} \frac{A_{0(+1)} + A_{0(-1)}}{4} \\ C|_{\tau \rightarrow \infty} &= -\frac{q}{16\pi} \left[\ln \left(\frac{\text{Pe}^2}{16} \right) + 2\gamma \right]. \end{aligned}$$

For non-zero Peclet numbers, a steady-state value is always reached as evidenced by the just-presented expressions. When the Peclet number is zero, though, no steady-state is reached due to the divergent behavior of these expressions when $\text{Pe} = 0$. This absence of a steady-state solution comes as no surprise and was already pointed out in Chapter 3. It is also well documented and explained in the literature [10].

Purely-conducting limit

For the sake of completeness, the purely-conducting limit is recovered from the results presented so far. By letting the Peclet number tend to zero, the first order correction to the apparent ground temperatures for the pipes vanishes and only the zeroth order values $\Theta_{ai}^{(0)}(\tau)$ prevail. Thus, only $A(\tau)$ is relevant in the purely-conducting limit. Computing its behavior as the Peclet number tends to zero leads to the following result for the apparent ground temperatures for the pipes:

$$\Theta_{ai}(\tau) \Big|_{\text{Pe} \rightarrow 0} = \frac{q(\tau)}{4\pi} [\ln(4\tau) - \gamma] - \frac{1}{4\pi} \int_0^\tau \frac{q(\tau) - q(\lambda)}{\tau - \lambda} d\lambda.$$

This expression, that specifies the same apparent ground temperature for all pipes, coincides with the expression found in the literature for the apparent temperature of a purely-conducting ground [49, 93].

4.6 Borehole thermal resistances

The impact of a groundwater stream on the thermal resistances of the borehole is analyzed next. It is common practice in the literature devoted to purely-conducting grounds to invert the relationship of the network of thermal resistances so that heat injection rates per unit pipe length are given explicitly in terms of fluid temperatures and the apparent ground temperature [32, 43, 49, 93].

Proceeding in the same way with Eq. (4.20) leads to the following result

$$q_i(\tau) = \sum_{j=1}^{N_p} \frac{\Theta_j(\tau) - \Theta_{aj}(\tau)}{R_{ij}} \quad (4.25)$$

in which thermal resistances R_{ij} are defined as shown in Rivero and Hermanns (2021) [93]:

$$\begin{bmatrix} \frac{1}{R_{11}} & \frac{1}{R_{12}} & \cdots \\ \frac{1}{R_{21}} & \frac{1}{R_{22}} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} & \cdots \\ \hat{R}_{21} & \hat{R}_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}^{-1}$$

Eq. (4.25) is normally recast in a more enlightening way with the steps to follow described in detail in the literature [93]. Applying the same steps and ideas to the present problem, in which different apparent ground temperatures are defined for each pipe, the following recast expression results:

$$q_i(\tau) = w_i \frac{\Theta_m(\tau) - \Theta_a(\tau)}{R_b} + \sum_{j=1}^{N_p} \frac{\Theta_i(\tau) - \Theta_j(\tau)}{R_{a,ij}} - \sum_{j=1}^{N_p} \frac{\Theta_{ai}(\tau) - \Theta_{aj}(\tau)}{R_{a,ij}}. \quad (4.26)$$

Most quantities are defined in exactly the same way as in the existing literature for purely-conducting grounds [93]. So, the borehole's outer thermal resistance R_b , which governs the overall heat exchange between borehole and ground, is defined as

$$\frac{1}{R_b} = \sum_{i=1}^{N_p} \sum_{j=1}^{N_p} \frac{1}{R_{ij}}.$$

The primary weighting coefficients w_i , which give the share of each pipe in the overall heat exchange between borehole and ground, are defined as

$$\frac{w_i}{R_b} = \sum_{j=1}^{N_p} \frac{1}{R_{ij}} \quad \text{with} \quad \sum_{i=1}^{N_p} w_i = 1.$$

The weighted mean fluid temperature $\Theta_m(\tau)$, which is the temperature with which the borehole exchanges heat with the ground, is defined as

$$\Theta_m = \sum_{j=1}^{N_p} v_j \Theta_j,$$

with the contribution of each pipe specified by the secondary weighting coefficients v_j defined as

$$\frac{v_j}{R_b} = \sum_{i=1}^{N_p} \frac{1}{R_{ij}} \quad \text{with} \quad \sum_{j=1}^{N_p} v_j = 1.$$

Also the borehole's inner thermal resistances $R_{a,ij}$, which govern the heat exchange between pairs of pipes inside the borehole, are defined in the same way as in the existing literature for purely-conducting grounds,

$$\frac{1}{R_{a,ij}} = \frac{w_i v_j}{R_b} - \frac{1}{R_{ij}},$$

with the complete set of inner thermal resistances satisfying the following relationship required, among other things, for the recast of Eq. (4.25) into Eq. (4.26) [50]:

$$\sum_{i=1}^{N_p} \frac{1}{R_{a,ij}} = \sum_{j=1}^{N_p} \frac{1}{R_{a,ij}} = 0. \quad (4.27)$$

A completely new concept, introduced here for the first time, is the weighted apparent ground temperature $\Theta_a(\tau)$ which represents the temperature at which the borehole as a whole perceives the surrounding ground:

$$\Theta_a = \sum_{j=1}^{N_p} v_j \Theta_{aj}.$$

Its physical meaning is understood by summing the equations for all pipes in Eq. (4.26) so that on the left hand side the heat injection rate per unit borehole length $q(\tau)$ emerges and on the right hand side multiple terms vanish as a consequence of the property given in Eq. (4.27):

$$q = \frac{\Theta_m - \Theta_a}{R_b}. \quad (4.28)$$

Clearly, Θ_m is the temperature with which the borehole exchanges heat with the ground, Θ_a is the temperature at which the borehole as a whole perceives the ground, and R_b is the thermal resistance governing the overall heat exchange between borehole and ground.

4.7 Numerical examples

To assess the impact of creeping groundwater flows on the apparent ground temperature for the pipes and on the borehole's thermal resistances and weighting coefficients, the present section solves numerically the systems of linear algebraic equations formulated in Eqs. (4.16) to (4.19). Since those systems contain an infinite number of equations, indexes n and m need to be truncated which inevitably introduces truncation errors into the results. Fortunately, spectral convergence of the enhanced multipole method [93] ensures that these truncation errors decay rapidly with the number of multipoles retained in the expansion of the grout/ground temperature. So, limiting n and m to values between -20 and $+20$ already ensures an accuracy level that exceeds the requirements of the present work.

4.7.1 Borehole description

All comparisons and discussions presented in the forthcoming subsections make use of the borehole configuration shown in Figure 4.1. Although other borehole configurations with different geometrical and thermal characteristics as well as with coaxial and double U-shaped probes have been analyzed as well, their results are not reported here as they are in line with what is presented.

The chosen configuration, borrowed from Chapter 3, consists in a borehole of diameter $2r_b = 152$ mm with an heterogeneous single U-shaped probe inside. Pipe 1 of the probe has an outer diameter of $2r_{p1} = 40$ mm and a wall thickness of $d_1 = 3.7$ mm while pipe 2 of the probe has an outer diameter of $2r_{p2} = 50$ mm and a wall thickness of $d_2 = 4.6$ mm. The two pipes are placed at 5 mm from the borehole wall and at $\pm 45^\circ$ from the rear stagnation point, respectively. Consequently, the polar coordinates of their centers are (51 mm, $+45^\circ$) and (46 mm, -45°), respectively.

The void between the two pipes and the ground, of effective thermal conductivity $k_g = 3.00$ W/(m K) and effective thermal diffusivity $\alpha_g = 2.24 \cdot 10^{-7}$ m²/s [18], is filled up with conventional grout of thermal conductivity $k_b = 1.50$ W/(m K) [1].

Pure water acts as heat carrying liquid. Its density, specific heat capacity, thermal conductivity, and dynamic viscosity are 999 kg/m³, 4184 J/(kg K), 0.577 W/(m K), and $1.138 \cdot 10^{-3}$ kg/(m s), respectively. Thanks to the pumped mass flow rate of 0.25 kg/s, a turbulent flow regime is sustained in both pipes. The resulting turbulent transport of heat inside the liquid is modeled using convective heat transfer coefficients h_1 and h_2 whose values are set using Gnielinski's correlations [38]. Combined with the thermal conductivities of the pipes $k_1 = k_2 = 0.42$ W/(m K), the following inner thermal resistances of the pipes are obtained:

$$R_{p1} = 0.535 \frac{\text{m K}}{\text{W}} \quad \text{and} \quad R_{p2} = 0.544 \frac{\text{m K}}{\text{W}}.$$

For all results presented in the following subsections, time-constant heat injection rates per unit pipe length are specified at the pipes so that $q_1(\tau) = 5$ and $q_2(\tau) = 1$. These values differ from the ones used in Chapters 3 and 5.

4.7.2 Apparent ground temperature

The first quantities to analyze are the apparent ground temperatures for the two pipes, $\Theta_{a1}(\tau)$ and $\Theta_{a2}(\tau)$. Figure 4.2 represents their time evolution together with the time evolution of the apparent ground temperature used by the state of the art, namely, $\Theta_{ai}^{(0)}(\tau) = A(\tau)$. Initially, an identical logarithmic growth with time is observed for all represented temperatures and for all considered Peclet numbers of the groundwater flow. This common behavior finishes once the steady-state value given in Section 4.5.1 is attained. The transition between the two behaviors takes place for non-dimensional times $\tau \sim 4\text{Pe}^{-2}$ for which the argument of the exponential integral E_1 in Eqs. (4.23) and (4.24) becomes of order unity.

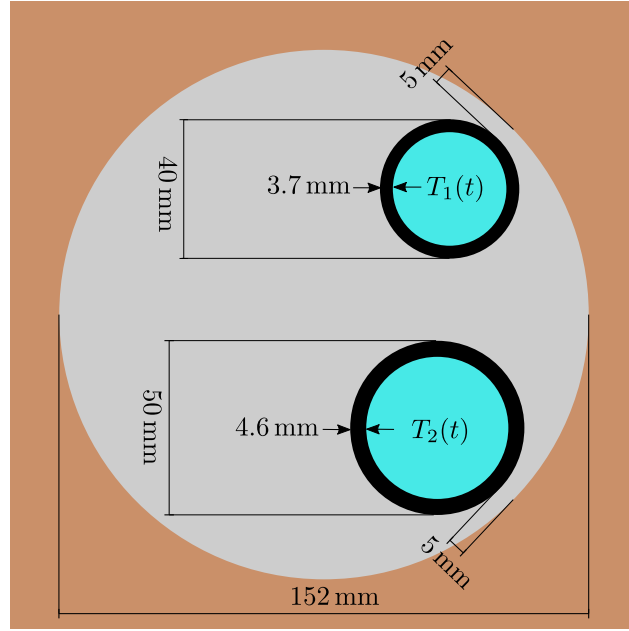


Figure 4.1: Borehole configuration for the numerical examples.

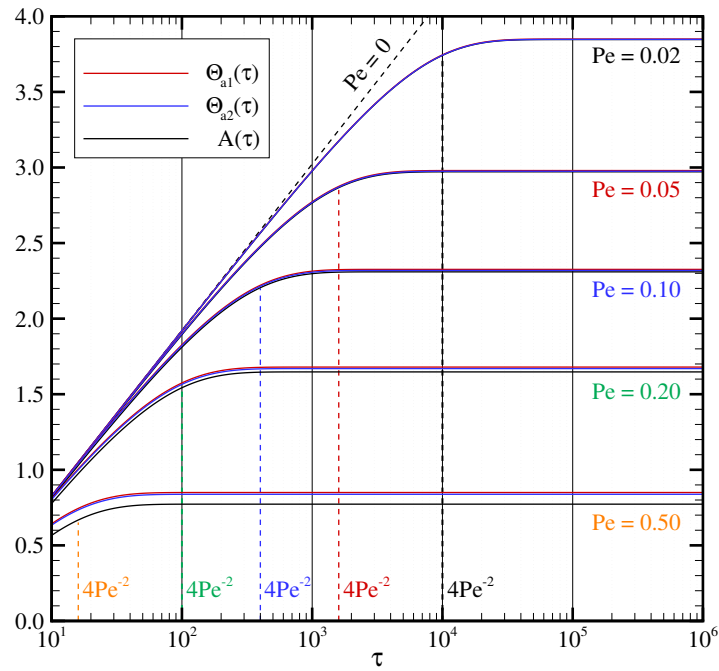


Figure 4.2: Time evolution of the apparent ground temperatures for the pipes $\Theta_{ai}(\tau)$ for different Peclet numbers of the groundwater flow.

The attained steady-state value also depends on the Peclet number of the groundwater flow, being its value higher as the Peclet number gets smaller. In fact, in the absence of groundwater flows no steady-state is reached at all, as already mentioned in Section 4.5.1. This limiting case of zero Peclet number is represented in Figure 4.2 with an oblique dashed line.

Looking at the individual pipes, differences in their apparent ground temperatures are solely due to the first order corrections $\Theta_{ai}^{(1)}(\tau)$. In general, these differences are small which is something to expect as the two pipes are very similar. Larger differences can be triggered, though, by considering very different pipes or by assigning heat injection rates per unit pipe length of opposite signs to the pipes. Both options, however, are too far from reality to be meaningful.

More relevant is the difference of the two pipes with respect to the apparent ground temperature used by the state of the art. That is, to $A(\tau)$. For small Peclet numbers, that difference is minimal so that the apparent ground temperature $A(\tau)$ is accurate enough. The difference grows, however, with increasing values of the Peclet number so that for $Pe = 0.5$, for instance, the non-negligible quantity of 10% is reached. Such large Peclet numbers do actually occur in real world installations once unconsolidated grounds are present or energy piles are used instead of boreholes [84].

The first order correction, responsible for the aforementioned differences, also reveals a new phenomenon. In the zeroth order solution, equivalent to the state of the art, the orientation of the borehole is irrelevant as the outer region enforces a uniform temperature level $A(\tau)$ onto the inner region. Hence, all pipes perceive the same apparent ground temperature $\Theta_{ai}^{(0)}(\tau) = A(\tau)$ regardless of their position inside the borehole. This changes, however, with the first order correction as then the outer region enforces also a temperature gradient onto the inner region. This spatial gradient combined with the pipe positions inside the borehole leads to apparent ground temperatures that depend on the borehole's orientation with respect to the groundwater flow.

To showcase this phenomenon, four different orientations of the considered borehole are analyzed next. Apart from the original orientation depicted in Figure 4.1, three additional counterclockwise rotations of 90° , 180° , and 270° are considered. Figure 4.3 represents the time evolution of the corresponding weighted apparent ground temperatures $\Theta_a(\tau)$ and of the apparent ground temperature $A(\tau)$ used by the state of the art. Clearly, different borehole orientations lead to different weighted apparent ground temperatures. For $Pe = 0.5$, for instance, a variation of 20% exists between the original orientation of the borehole, for which $\Theta_a(\tau)$ attains the highest values, and the 180° rotation, for which $\Theta_a(\tau)$ attains the lowest values.

Since the weighted apparent ground temperature represents the temperature at which the borehole as a whole perceives the surrounding ground, the observed variations in $\Theta_a(\tau)$ lead to different heat exchanges between borehole and ground. Unfortunately, it is difficult to exploit this effect in real world installations as no control on the location of pipes inside the borehole is exerted during the construction of boreholes, leading to pipe positions that change arbitrarily with depth.

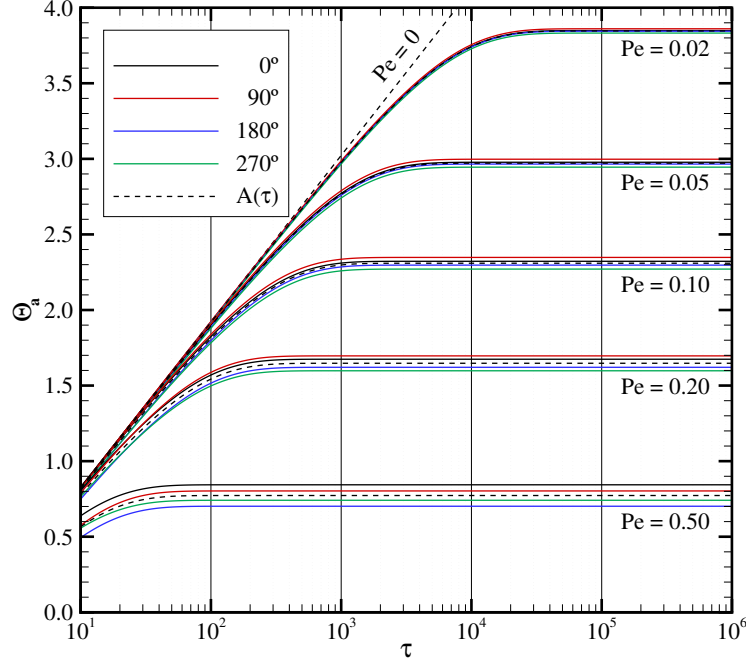


Figure 4.3: Time evolution of the weighted apparent ground temperature $\Theta_a(\tau)$ for different Peclet numbers of the groundwater flow and for four different orientations of the borehole.

4.7.3 Borehole thermal resistances and weights

The first thermal resistance to discuss is the borehole's inner thermal resistance $R_{a,12}$ that governs, as explained in Section 4.6, the heat exchange between the two pipes. Its expected dependence with the Peclet number of the groundwater flow can be inferred from its expression in terms of the thermal resistances $\hat{R}_{ij}$ that results from applying the formulae presented in Section 4.6:

$$R_a = \hat{R}_{11} + \hat{R}_{22} - \hat{R}_{12} - \hat{R}_{21}.$$

Since thermal resistances $\hat{R}_{ij}$ vary linearly with the Peclet number of the groundwater flow, as shown in Eq. (4.21), this expression reveals that the inner thermal resistance must behave in the same way. However, the obtained numerical results indicate that its value remains unaltered, locked at $R_{a,12} = 1.0996$, regardless of the Peclet number of the groundwater flow. Although most of the heat exchange between the two pipes takes place through the grout filling up the borehole, this complete lack of dependence on the groundwater motion around the borehole is surprising. Further research is required to fully understand this.

Next thermal resistance to discuss is the borehole's outer thermal resistance R_b that governs the overall heat exchange between borehole and ground [93]. Figure 4.4 represents its value in terms of the Peclet number of the groundwater flow and for the same four borehole orientations as in the previous subsection. Now, a clear dependence with the Peclet number exists that leads to variations close to 8% when the Peclet number reaches one. These large variations occur for the borehole orientations with the pipes located at the back and at the front of the borehole. That is, for the original borehole configuration and for its rotation by 180° .

Since R_b has a direct impact onto the overall heat exchange between borehole and ground, the best location for the pipes to promote that heat exchange is at the back of the borehole, as in Figure 4.1. The advantage of this pipe location over the other three is that there the advected heat is pulled away sooner from the borehole. In the other three locations the supplied heat must first be advected around the borehole which interferes with the heat exchange at other positions around the borehole wall. Unfortunately, it is very difficult to exploit this effect in real-world installations due to the uncontrolled meandering of pipes along boreholes.

The observed dependency of R_b with the Peclet number of the groundwater flow can be explained with the expression for R_b in terms of the thermal resistances $\hat{R}_{ij}$ that results from applying the formulae presented in Section 4.6:

$$R_b = \frac{\hat{R}_{11}\hat{R}_{22} - \hat{R}_{12}\hat{R}_{21}}{\hat{R}_{11} + \hat{R}_{22} - \hat{R}_{12} - \hat{R}_{21}}.$$

It represents a rational dependence, with the denominator and numerator being linear and quadratic polynomials in the Peclet number, respectively. The denominator, however, coincides with the borehole's inner thermal resistance so that its value remains unchanged. Consequently, the borehole's inner thermal resistance presents a quadratic dependence with the Peclet number that can best be seen for the borehole orientations corresponding to rotations of 90° and 270° .

Finally, primary and secondary weighting coefficients, w_i and v_j , are discussed. They represent the contribution of each pipe to the overall heat exchange between borehole and ground and to the weighted mean fluid temperature, respectively [93]. In the absence of thermal inertia in grout and ground, both weighting coefficients coincide and no distinction is made among them [50]. This, however, only applies to purely-conducting grounds so that in the present case they attain different values. Their discrepancies, however, are small, in the range of 0.1% for the considered borehole configuration. Therefore, only primary weighting coefficients w_i are presented and discussed next.

Figure 4.5 represents the primary weighting coefficients of the two pipes, w_1 and w_2 , as functions of the Peclet number of the groundwater flow and for the four borehole orientations of interest. Since the two pipes are different, their contributions to the overall heat exchange are also different, with $w_1 = 0.477$ and $w_2 = 0.523$ in the absence of groundwater flows. With moving groundwater, primary weighting coefficients vary linearly with Peclet number. This dependence can be explained with the expressions for w_i derived from the formulae presented in Section 4.6:

$$w_1 = \frac{\hat{R}_{22} - \hat{R}_{12}}{\hat{R}_{11} + \hat{R}_{22} - \hat{R}_{12} - \hat{R}_{21}} \quad \text{and} \quad w_2 = \frac{\hat{R}_{11} - \hat{R}_{21}}{\hat{R}_{11} + \hat{R}_{22} - \hat{R}_{12} - \hat{R}_{21}}.$$

Again, denominators coincide with the immutable borehole's inner thermal resistance R_a so that the only dependency left is the linear one from the numerators.

The most interesting borehole orientations to discuss are the ones corresponding to rotations of 90° and 270° . For both, influence of the Peclet number is the highest, always increasing the heat exchange of the pipe located downstream and decreasing the heat exchange of the

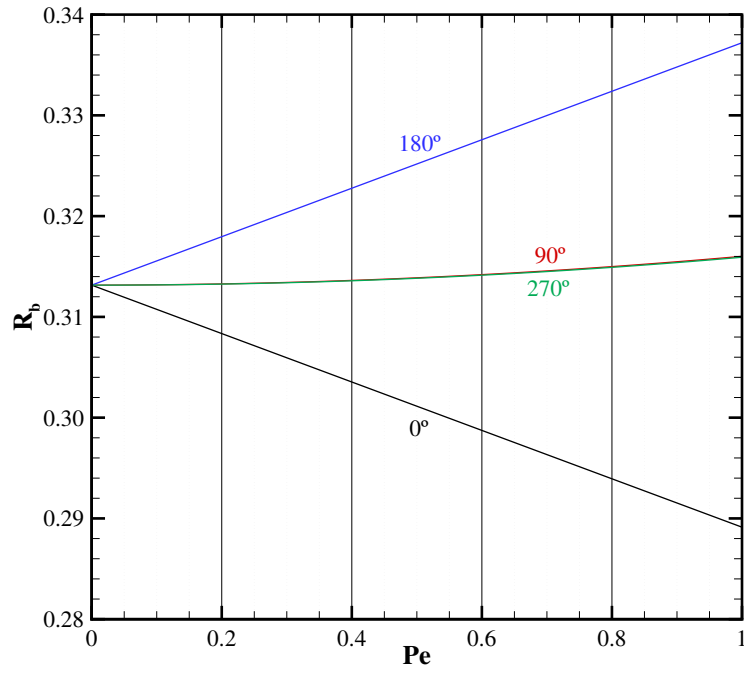


Figure 4.4: Dependence of outer thermal resistance R_b on the Peclet number of the groundwater flow for four different orientations of the borehole.

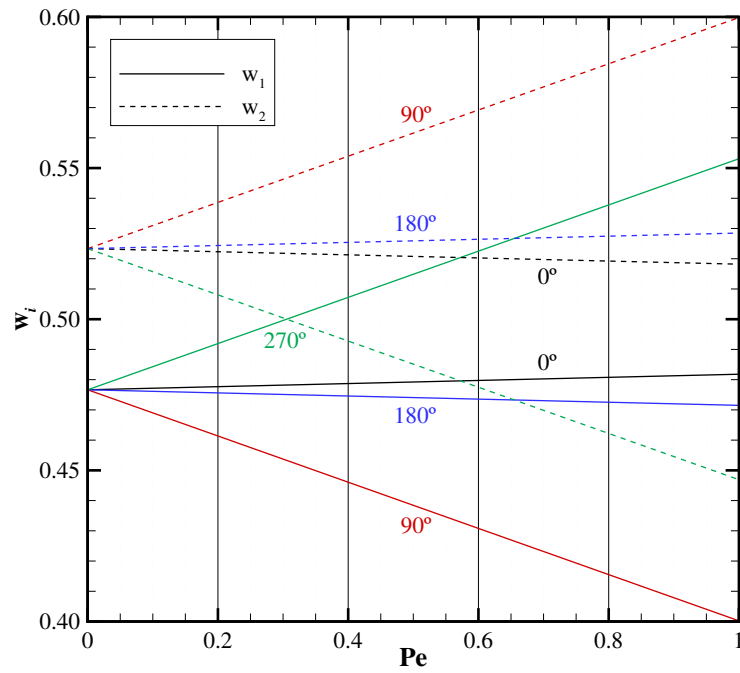


Figure 4.5: Dependence of primary weighting coefficients w_i on the Peclet number of the groundwater flow for four different orientations of the borehole.

pipe located upstream. So, for $Pe = 1$ and a 90° rotation of the original borehole, pipe 2 sees its contribution increased by 14.6% while pipe 1 decreases its contribution accordingly. When rotated 270° , the trend reverts and pipe 1 is now the one that sees its contribution increased by 16% while pipe 2 sees its contribution decreased accordingly. As seen in Figure 4.5, these large changes in the primary weighting coefficients lead even to an inversion in the relevance of the pipes for Peclet numbers beyond 0.30.

For the sake of completeness, expressions for the secondary weighting coefficients v_i are given as well. These result from the formulae presented in Section 4.6:

$$v_1 = \frac{\hat{R}_{22} - \hat{R}_{21}}{\hat{R}_{11} + \hat{R}_{22} - \hat{R}_{12} - \hat{R}_{21}} \quad \text{and} \quad v_2 = \frac{\hat{R}_{11} - \hat{R}_{12}}{\hat{R}_{11} + \hat{R}_{22} - \hat{R}_{12} - \hat{R}_{21}}.$$

4.7.4 Weighted mean fluid temperature

The thermal response of the borehole is given by the time evolution of the weighted mean fluid temperature Θ_m . This temperature has a direct impact onto the efficiency of the heat pump, driving the geothermal installation, as its coefficient of performance (COP) depends on the temperature of the heat carrying liquid, as explained in Chapter 1.

Figure 4.6 shows the time evolution of the weighted mean fluid temperature for different Peclet numbers of the groundwater flow and for the four borehole orientations considered so far. Overall, the behavior of Θ_m is very similar to the one of the weighted apparent ground temperature Θ_a , a similarity that comes as no surprise since both are tightly linked to each other through Eq. (4.28).

Differences exist, though, between Θ_m and Θ_a as the best and worst performing borehole orientations change, corresponding now to rotations of 90° and 270° instead of 0° and 180° . This change is caused by the counteracting effects of the Peclet number in the weighted apparent ground temperature and in the borehole's outer thermal resistance.

The difference in Θ_m between the best and worst performing borehole orientations is almost 3% for $Pe = 0.2$. This is less than the previously reported differences in weighted apparent ground temperature and borehole's outer thermal resistance. The explanation is the larger value of Θ_m that reduces the relative impact of the differences in Θ_a and R_b . Another interesting aspect is that the best and worst performing orientations depend on the Peclet number of the groundwater flow. Although all represented cases show the same ranking, a change of order takes place for larger values of the Peclet number. In fact, for $Pe = 0.5$ the best and worst performing orientations are already in a converging trend instead of a diverging one.

4.8 Conclusions

The pure analytical treatment of the inner region has already been addressed in the literature for purely-conducting grounds. Most existing models that account for the presence of creeping

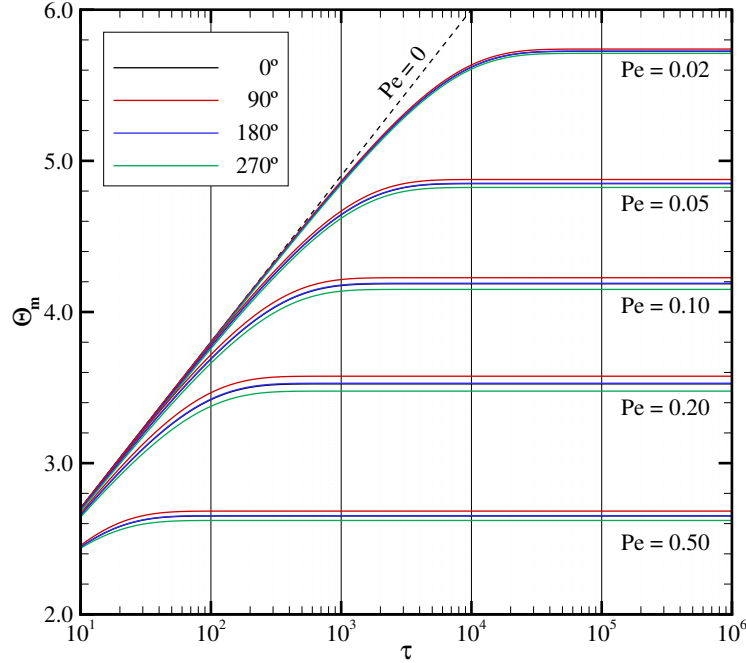


Figure 4.6: Time evolution of the weighted mean fluid temperature $\Theta_m(\tau)$ for different Peclet numbers of the groundwater flow and for four different orientations of the borehole.

groundwater flows endorse the well-established concepts, ideas, and procedures meant for purely-conducting grounds.

Although the state of the art performs reasonably well, the followed procedure of "*adopting what is done for purely-conducting grounds*" leads to theoretical inconsistencies in the models and to overlook interesting and/or relevant phenomena introduced by the presence of aquifers. To overcome these limitations, a mathematically rigorous derivation of a physically sound model is required. This chapter, together with the previous one, have precisely addressed this need in the case of creeping groundwater flows for which the Peclet number of the flow is small compared to unity.

The previous chapter dissected the overall heat transfer problem with the aid of asymptotic expansion techniques, identifying the existence of two distinct regions and formulating the mathematical problems to solve in each region. The present chapter, on the contrary, is entirely devoted to the inner region encompassing the borehole, with its pipes and grout, and the ground located close to the borehole.

To obtain exact solutions to the formulated problems in the inner region, the present chapter has modified the enhanced multipole method to cope with new phenomena introduced by the aquifer, namely, the spatial temperature gradient enforced by the outer region onto the inner region and the weak convective transport of heat in the vicinity of the borehole.

By proper manipulation and combination of the resulting solutions, a network of thermal resistances has been set up. It represents an exact relationship between heat injection rates per unit pipe length of the pipes, fluid temperatures in the pipes, and apparent ground

temperatures for the pipes, with proportionality constants being the thermal resistances.

Compared to the state of the art, in which a similar network of thermal resistances is embraced, two important differences are revealed. First, each pipe perceives the surrounding ground at a different apparent ground temperature, while in the state of the art the same apparent ground temperature is used for all pipes. This change is caused by the temperature gradient imposed by the outer region onto the inner region and whose origin is the flowing aquifer. Second, thermal resistances depend on the Peclet number of the groundwater flow, while in the state of the art their values are constant. This change is caused by the weak convective transport of heat in the vicinity of the borehole that is obviated by the state of the art.

Apart from uncovering the aforementioned theoretical inconsistencies, the present work has also assessed the accuracy of the state of the art using the here-developed model as benchmark. For creeping groundwater flows, accuracy is not affected by the identified inconsistencies which explains why the state of the art performs reasonably well. For larger Peclet numbers, however, stronger discrepancies are observed. So, variations of up to 20% exist in the apparent ground temperatures for the pipes and changes of up to 8% are reported for the thermal resistances.

Although the present chapter shows that identified discrepancies worsen with increasing values of the Peclet number, no confident conclusion regarding their real magnitude and relevance can be drawn as validity and accuracy of the developed asymptotic model are not assured for such large values of the Peclet number. Therefore, rigorous development of a theoretical model for the thermal interaction of geothermal boreholes with strong groundwater flows, as initiated in Chapter 5, is of interest.

4.A Endogenous particular solution

The solution to the endogenous first order problem requires, as an intermediate step, the particular solution Θ_p to the problem summarized in Table 4.3. This particular solution has to incorporate all forcing terms of the governing equation in the ground and all terms imposed by the inner region onto the outer region. In the following subsections different steps are taken to obtain this particular solution. Also the coefficients $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$ of the Fourier series expansions around pipe j of Θ_p and its derivative with respect to ρ_j are obtained.

4.A.1 Particular solution to ground equation

First, a particular solution Θ_g to the governing equation in the ground is proposed. It must include as well all terms imposed by the inner region onto the outer region which is not a

difficult task as they are closely related to the forcing terms of the equation in the ground:

$$\begin{aligned} \Theta_g(\rho_0, \theta_0, \tau) = & \frac{A_{0(+1)}(\tau) + A_{0(-1)}(\tau)}{2} \ln \rho_0 - \frac{q(\tau)}{4\pi} \left(\rho_0 + \frac{1}{\rho_0} \right) \ln \rho_0 \cos \theta_0 \\ & + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{A_{0m}(\tau)}{4} \left[\frac{e^{\iota \frac{m}{|m|} (|m|+1) \theta_0}}{\rho_0^{|m|-1}} + \frac{e^{\iota \frac{m}{|m|} (|m|-1) \theta_0}}{\rho_0^{|m|+1}} \right]. \end{aligned} \quad (4.29)$$

This particular solution Θ_g is only valid in the ground. Consequently, it does not obey the governing equation in the grout and the continuity conditions at the borehole wall. Thus, further work is required to obtain Θ_p .

4.A.2 Fourier expansion around the borehole wall

Next, Fourier series expansions around the borehole wall of Θ_g and its derivative with respect to ρ_0 are obtained. These are required for the enforcement of the continuity conditions at the borehole wall and the extension of Θ_g towards the interior of the borehole.

Following the definitions introduced in Section 4.3.2 and assimilating the borehole to pipe $j = 0$, as done in Section 4.3.6, the Fourier series expansions for Θ_g and its radial derivative with respect to ρ_0 are defined as

$$\Theta_g \Big|_{\rho_0=\rho_{p0}} = \sum_{\ell=-\infty}^{\infty} \mathcal{G}_{0\ell} e^{\iota \ell \theta_0} \quad \text{and} \quad \rho_{p0} \frac{\partial \Theta_g}{\partial \rho_0} \Big|_{\rho_0=\rho_{p0}} = \sum_{\ell=-\infty}^{\infty} \mathcal{G}'_{0\ell} e^{\iota \ell \theta_0}.$$

Expressions for $\mathcal{G}_{0\ell}$ and $\mathcal{G}'_{0\ell}$ result from comparing these Fourier series expansions with Eq. (4.29), and its derivative with respect to ρ_0 , particularized at the borehole wall. So, for $|\ell| > 1$ the corresponding Fourier series coefficients are

$$\mathcal{G}_{0\ell} = \frac{A_{0(\ell+1)}(\tau) + A_{0(\ell-1)}(\tau)}{4}$$

and

$$\mathcal{G}'_{0\ell} = -\frac{|\ell|}{\ell} (\ell + 2) \frac{A_{0(\ell+1)}(\tau)}{4} - \frac{|\ell|}{\ell} (\ell - 2) \frac{A_{0(\ell-1)}(\tau)}{4},$$

while for $\ell = \pm 1$ the expressions to use are

$$\mathcal{G}_{0(\pm 1)} = \frac{A_{0(\pm 2)}(\tau)}{4} \quad \text{and} \quad \mathcal{G}'_{0(\pm 1)} = -\frac{q(\tau)}{4\pi} - \frac{3}{4} A_{0(\pm 2)}(\tau).$$

Finally, for $\ell = 0$,

$$\mathcal{G}_{00} = \frac{A_{0(+1)}(\tau) + A_{0(-1)}(\tau)}{4} \quad \text{and} \quad \mathcal{G}'_{00} = 0.$$

Apart from the expressions for the Fourier series coefficients, an important outcome of the present subsection is that $\mathcal{G}_{0\ell}$ and $\mathcal{G}'_{0\ell}$ are linear combinations of the specified heat injection rates per unit pipe length $q_j(\tau)$ as a result of all involved quantities having that property.

4.A.3 Continuity at borehole wall

To enforce the continuity conditions at the borehole wall, and extend the particular solution Θ_g towards the interior of the borehole, homogeneous solutions to the governing equations in grout and ground are added to Θ_g . The result is the following expression for the sought particular solution Θ_p :

$$\Theta_p = \begin{cases} \sum_{\ell=-\infty}^{\infty} \mathcal{P}_{0\ell} \rho_0^{|\ell|} e^{\iota\ell\theta_0} & \text{if } \rho_0 \leq 1 \\ \sum_{\ell=-\infty}^{\infty} \mathcal{B}_{0\ell} \rho_0^{-|\ell|} e^{\iota\ell\theta_0} + \Theta_g & \text{if } \rho_0 \geq 1. \end{cases} \quad (4.30)$$

The values for $\mathcal{P}_{0\ell}$ and $\mathcal{B}_{0\ell}$ result from enforcing continuity conditions at the borehole wall,

$$\mathcal{P}_{0\ell} = \mathcal{B}_{0\ell} + \mathcal{G}_{0\ell} \quad \text{and} \quad \mathcal{B}_{0\ell} = \frac{\mathcal{G}'_{0\ell} - \kappa|\ell|\mathcal{G}_{0\ell}}{(1 + \kappa)|\ell|},$$

while for $\ell = 0$ the expression for the latter integration constant is $\mathcal{B}_{00} = 0$. Again, $\mathcal{P}_{0\ell}$ and $\mathcal{B}_{0\ell}$ are linear combinations of the specified heat injection rates per unit pipe length $q_j(\tau)$ as all involved quantities share that same property.

4.A.4 Computation of $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$

Finally, coefficients of the Fourier series expansions around pipe j of Θ_p and its derivative with respect to ρ_j are obtained. For it, the series defining Θ_p inside the borehole, where $\rho_0 \leq 1$, is particularized at the wall of pipe j and split into two. The result, after convenient addition and subtraction of $\mathcal{P}_{00}$, is

$$\Theta_p \Big|_{\rho_j = \rho_{pj}} = \sum_{\ell=0}^{\infty} \left[\mathcal{P}_{0\ell} [\rho_0 e^{\iota\theta_0}]^{\ell} + \mathcal{P}_{0(-\ell)} [\rho_0 e^{-\iota\theta_0}]^{\ell} \right]_{\rho_j = \rho_{pj}} - \mathcal{P}_{00}. \quad (4.31)$$

Next, complex arithmetic is used to express position (ρ_0, θ_0) in the polar coordinate system centered at the borehole in terms of position (ρ_j, θ_j) in the polar coordinate system centered at pipe j :

$$\rho_0 e^{\pm\iota\theta_0} = \rho_{0j} e^{\pm\iota\theta_{0j}} + \rho_{pj} e^{\pm\iota\theta_j}.$$

How to combine these two expressions is explained next using the first of the series in Eq. (4.31) as example. After substituting one into the other and applying the binomial theorem to the outcome,

$$\sum_{\ell=0}^{\infty} \mathcal{P}_{0\ell} [\rho_0 e^{\iota\theta_0}]^{\ell} = \sum_{\ell=0}^{\infty} \mathcal{P}_{0\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} [\rho_{0j} e^{\iota\theta_{0j}}]^{\ell-m} [\rho_{pj} e^{\iota\theta_j}]^m.$$

After careful inspection of the involved pairs of ℓ and m it becomes clear that the summation order can be reversed, leading to

$$\sum_{\ell=0}^{\infty} \mathcal{P}_{0\ell} [\rho_0 e^{\iota\theta_0}]^{\ell} = \sum_{m=0}^{\infty} \sum_{\ell=m}^{\infty} \mathcal{P}_{0\ell} \binom{\ell}{m} [\rho_{0j} e^{\iota\theta_{0j}}]^{\ell-m} [\rho_{pj} e^{\iota\theta_j}]^m.$$

Proceeding in the same way with the second series in Eq. (4.31) allows to rewrite Θ_p particularized at pipe j as

$$\begin{aligned} \Theta_p|_{\rho_j=\rho_{pj}} &= \sum_{m=0}^{\infty} \left[\sum_{\ell=m}^{\infty} \mathcal{P}_{0\ell} \binom{\ell}{m} [\rho_{0j} e^{\iota\theta_{0j}}]^{\ell-m} \rho_{pj}^m \right] e^{+\iota m\theta_j} \\ &+ \sum_{m=0}^{\infty} \left[\sum_{\ell=m}^{\infty} \mathcal{P}_{0(-\ell)} \binom{\ell}{m} [\rho_{0j} e^{-\iota\theta_{0j}}]^{\ell-m} \rho_{pj}^m \right] e^{-\iota m\theta_j} - \mathcal{P}_{00}. \end{aligned}$$

Now it is straightforward to identify the coefficients $\mathcal{P}_{jm}$ of the Fourier series expansion of Θ_p around pipe j . For $m > 0$ it can be concluded that

$$\mathcal{P}_{jm} = \sum_{\ell=m}^{\infty} \mathcal{P}_{0\ell} \binom{\ell}{m} [\rho_{0j} e^{\iota\theta_{0j}}]^{\ell-m} \rho_{pj}^m,$$

while for $m < 0$ it can be shown that

$$\mathcal{P}_{jm} = \overline{\mathcal{P}_{j(-m)}}.$$

Finally, for $m = 0$ the corresponding coefficient is

$$\mathcal{P}_{j0} = \sum_{\ell=-\infty}^{\infty} \mathcal{P}_{0\ell} \rho_{0j}^{|\ell|} e^{\iota\ell\theta_{0j}}.$$

Regarding the coefficients $\mathcal{P}'_{jm}$ of the Fourier series expansion around pipe j of the derivative of Θ_p with respect to ρ_j , their values are straightforward to obtain:

$$\mathcal{P}'_{jm} = \rho_{pj} \frac{\partial \mathcal{P}_{jm}}{\partial \rho_{pj}} = |m| \mathcal{P}_{jm}.$$

In particular, the case $m = 0$ for which $\mathcal{P}'_{j0} = 0$ is of special interest as it is exploited in Sections 4.4.2 and 4.4.3 to simplify the formulated systems of linear algebraic equations.

Just as in all intermediate steps presented before, Fourier series coefficients $\mathcal{P}_{jm}$ and $\mathcal{P}'_{jm}$ are linear combinations of the specified heat injection rates per unit pipe length $q_j(\tau)$, a property of great importance for the solution of the endogenous first order problem in Section 4.4.3.

Chapter 5

Strong groundwater flows

The previous chapters assume heat convection to be negligible near the borehole, providing good results in presence of creeping groundwater flows. However, these models are not suitable when the Peclet number attains values of order unity, notably in scenarios involving energy piles or fractured igneous and metamorphic rocks and in gravelly soils.

The present chapter tackles the problem of the thermal interaction between geothermal boreholes and strong groundwater flows, characterized by Peclet numbers of order unity, following the asymptotic framework established in previous chapters. That is, the analysis of the two-region structure in which heat convection will now be relevant in both regions, the inner one located close to the borehole and the outer one located further away.

The need to perform an ad-hoc analysis for the case of strong groundwater flows, as done in this chapter, is motivated by the unsuitability of existing models to correctly address this case.

Most models in the literature focus solely on the outer problem. They obtain the perturbed ground temperature in the outer region in terms of the heat injection rate per unit borehole length by means of superposition of moving point sources of heat [9, 20, 27, 29–31, 39, 42, 52, 73, 74, 90–92, 97, 103, 104, 107]. Although valid for all Peclet numbers, none of these works address the groundwater flow in the inner region. In essence, these models, albeit implicitly, neglect heat convection near the borehole when they combine their outer solutions with a classical network of thermal resistances [5, 19, 21].

Other approaches solve the unsteady convection-conduction problem in the ground up to the borehole wall, reporting a model valid for all Peclet numbers. Nonetheless, they simplify the velocity field to be an uniform stream even near the borehole in order to analytically address the problem [62, 63]. This assumption not only contradicts the very presence of the borehole in the ground but also fails when the Peclet number attains values of order unity.

Further models for the interaction of boreholes with aquifers can be found in the field of fluid mechanics, namely, the high Reynolds-number flow past a cylinder at small Prandtl numbers and the Darcy flow past a cylinder in a porous media [4, 15, 16, 58, 59, 78, 82, 94, 100, 102]. Unfortunately, they are also not suitable for the case of strong groundwater flows.

This chapter is organized as follows. Section 5.1 analyzes the asymptotic structure of the problem, considering the presence of strong groundwater flows, and derives an asymptotic solution by means of matched asymptotic expansion techniques. Thereafter, Section 5.2 examines the merits and limits of the proposed model while Section 5.3 performs a similar analysis concerning the state of the art. Section 5.4 analyzes the behavior of the presented asymptotic model under creeping groundwater flow conditions to facilitate a comprehensive comparison with the state-of-the-art approaches. Finally, some concluding remarks are presented in Section 5.5.

5.1 Asymptotic structure

The presence of two distinct regions in the heat transfer problem to address does not rely on the value of the Peclet number, as discussed in Section 2.4. However, the particular characteristics of each region vary based on the presence of either creeping or strong groundwater flows. Hence, these particular characteristics of both asymptotic regions in presence of strong groundwater flows, in which the Peclet number is of order unity, are discussed in the current section. To facilitate this discussion, an order of magnitude estimation for each term comprising the governing equation in the ground is carried out:

$$\underbrace{s\tilde{\Theta}}_{\sim |s|} + \text{Pe} \left[\underbrace{v_\rho \frac{\partial \tilde{\Theta}}{\partial \rho_0}}_{\sim \frac{\text{Pe } v_\rho}{\rho_0}} + \underbrace{\frac{v_\theta}{\rho_0} \frac{\partial \tilde{\Theta}}{\partial \theta_0}}_{\sim \frac{\text{Pe } v_\theta}{\rho_0 \Delta \theta_0}} \right] = \underbrace{\frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}}{\partial \rho_0} \right)}_{\sim \frac{1}{\rho_0^2}} + \underbrace{\frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}}{\partial \theta_0^2}}_{\sim \frac{1}{\rho_0^2 \Delta^2 \theta_0}}. \quad (5.1)$$

Regarding the left hand side of the equation, the modulus of position s in the complex-valued Laplace plane governs the order of magnitude of its first term, which is related to thermal inertia of the ground. In the current chapter, this term is small compared to unity as $|s| \sim t_b/t_q \ll 1$ [93]. The remaining two terms of the left hand side are linked to heat convection and their order of magnitude depends on four factors, namely, the velocity field, the Peclet number, the distance from the borehole ρ_0 , and the deviation from the downstream direction $\Delta \theta_0$. Regarding the velocity field, the order of magnitudes of its components are related through the mass conservation equation which imposes $v_\rho \sim v_\theta/\Delta \theta_0$ [4]. Furthermore, the order of magnitude of its radial component v_ρ is of order unity as shown in Eq. (2.2). Thus, the order of magnitude of both convective terms in the energy conservation equation in the ground is of order Pe/ρ_0 . Finally, the right hand side of the equation represents heat conduction in the ground. It diminishes quadratically as the distance from the borehole ρ_0 increases, resulting in a faster decay compared to the convective terms. This discussion, despite differences in the specific characteristics of each region, is analogous to the ones performed in Chapters 2 and 3. Those analyses identified two distinct regions within the problem, which depend on the distance from the borehole.

In the vicinity of the borehole, where $\rho_0 \sim 1$, thermal inertia is negligible compared to heat convection and heat conduction due to $|s| \ll 1$ and $\text{Pe} \sim 1$. Hence, this *inner region* is, in first approximation, quasi-steady and governed by conduction and convection.

Far from the borehole, particularly at distances $\rho_0 \sim \text{Pe}/|s| \gg 1$, thermal inertia and heat convection become equally relevant, while heat conduction is negligible in all directions except downstream of the borehole. This is explained by the presence of a thermal wake in which the aforementioned balance of terms slightly changes as transversal heat conduction plays also a relevant role there [4]. By enforcing this condition, namely, transversal heat conduction to be equally relevant to thermal inertia and convection, the wake thickness, $\rho_0 \Delta\theta_0$, is found: $\rho_0 \Delta\theta_0 \sim \sqrt{\rho_0/\text{Pe}} \ll \rho_0$.

The existence of a slender thermal wake has been successfully exploited in the literature on porous media and on fluid mechanics to solve the problem in the outer region [4]. However, this approach is based on simplifications that, albeit accurate for $\text{Pe} \sim 1$, do not allow to recover valid results for small values of the Peclet number. Since the ultimate goal of this thesis is the development of a single model that works well for any Peclet number, this approach is not pursued in the present work. Instead, all conductive terms, transversal and radial, are retained throughout the entire outer region following the asymptotic framework outlined in Chapter 3.

5.1.1 Matched asymptotic expansion techniques

The aforementioned two-regions structure is exploited by means of matched asymptotic expansion techniques [64]. These mathematical methods are employed to address the inner and outer problems independently and then match their solutions at an intermediate distance in which both solutions are valid.

To deal with inner and outer problems, the unknown variables are asymptotically expanded in terms of the small parameter of the problem which in the current chapter is the position s in the complex-valued Laplace plane:

$$\tilde{\Theta}_{\text{in}} = \tilde{\Theta}_{\text{in}}^{(0)} + s \tilde{\Theta}_{\text{in}}^{(1)} + \mathcal{O}(s^2) \quad \text{and} \quad \tilde{\Theta}_{\text{out}} = \tilde{\Theta}_{\text{out}}^{(0)} + s \tilde{\Theta}_{\text{out}}^{(1)} + \mathcal{O}(s^2). \quad (5.2)$$

Furthermore, since fluid temperatures are also part of the solution to the heat transfer problem, they are expanded in terms of the small parameter as well:

$$\tilde{\Theta}_j = \tilde{\Theta}_j^{(0)} + s \tilde{\Theta}_j^{(1)} + \mathcal{O}(s^2). \quad (5.3)$$

Through the substitution of these expansions into the governing equations, continuity conditions, and boundary conditions for both the inner and outer regions, a series of mathematical problems arises. Their sequential resolution provides the various terms of the asymptotic expansion of the solution.

5.1.2 Solution to inner region

The inner region includes the borehole and the adjacent ground located at distances of order unity, $\rho_0 \sim 1$. At such distances thermal inertia is negligible and, consequently, the thermal

response of the grout filling up the borehole is governed solely by heat conduction:

$$0 = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \theta_0^2}. \quad (5.4)$$

While thermal inertia remains negligible in the ground near the borehole, both heat convection and heat conduction are relevant as the Peclet number is of order unity. Therefore, the governing equation in the surrounding ground is

$$\text{Pe} \left[v_\rho \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} + \frac{v_\theta}{\rho_0} \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \theta_0} \right] = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \theta_0^2}. \quad (5.5)$$

The previous two equations must be solved to obtain the grout/ground temperature along with the continuity conditions at the borehole wall,

$$\tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0=1^-} = \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0=1^+} \quad \text{and} \quad -\kappa \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} \Big|_{\rho_0=1^-} = -\frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} \Big|_{\rho_0=1^+}, \quad (5.6)$$

the specified heat injection rates per unit pipe length,

$$\tilde{q}_j = \int_{-\pi}^{\pi} -\kappa \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} \rho_{pj} d\theta_j,$$

and the additional boundary conditions enforced at the outer surface of each pipe:

$$-\rho_{pj} \kappa \frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_j} \Big|_{\rho_j=\rho_{pj}} = \frac{\tilde{\Theta}_j^{(0)} - \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_j=\rho_{pj}}}{R_{pj}}, \quad (5.7)$$

where the zeroth order fluid temperature $\tilde{\Theta}_j^{(0)}$ is given by

$$\tilde{\Theta}_j^{(0)} = \frac{R_{pj}}{2\pi} \tilde{q}_j + \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_j=\rho_{pj}} d\theta_j. \quad (5.8)$$

The enforcement of the boundary condition at infinity is not feasible due to the presence of the outer region. Consequently, this condition is replaced by an asymptotic matching with the solution to that outer region.

An exact solution to the formulated inner problem is unknown but, fortunately, such a solution is not required for the theoretical approach pursued in this work. What is crucial instead is to know the general behavior of the zeroth order inner solution in the ground, which is inferred from the general solution to the governing equation there, Eq. (5.5):

$$\begin{aligned} \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0 \geq 1} = e^{\frac{\text{Pe}}{2} \cos(\theta_0) \left(\rho_0 + \frac{1}{\rho_0} \right)} \sum_{n=0}^{\infty} \left[\tilde{\alpha}_n \frac{\text{Ke}_n \left(\ln \rho_0, \frac{\text{Pe}}{2} \right)}{\text{Ke}_n \left(0, \frac{\text{Pe}}{2} \right)} \text{ce}_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) \right. \\ \left. + \tilde{\beta}_n \frac{\text{Ko}_n \left(\ln \rho_0, \frac{\text{Pe}}{2} \right)}{\text{Ko}_n \left(0, \frac{\text{Pe}}{2} \right)} \text{se}_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) \right]. \end{aligned} \quad (5.9)$$

The separation of variables method [81] allows to express the sought solution in terms of angular Mathieu functions, ce_n and se_n , and modified Mathieu functions, Ke_n and Ko_n , [72, 80], with $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ being integration constants. More details about these special functions can be found in Appendices 5.A and 5.B.

It is important to highlight that the general solution presented for the ground, Eq. (5.9), already fulfills the boundary condition at infinity, so there are no remaining constants to be determined through matching with the outer solution. Therefore, while the matching condition is crucial to find the zeroth order approximation to the outer solution, it is not required to obtain the zeroth order solution to the inner problem.

Due to the absence of an exact solution, the integration constants $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ remain undetermined. To facilitate the theoretical work carried out here, these constants are expressed in terms of more convenient unknowns, which are also part of the solution to the zeroth inner problem, namely, the coefficients $\tilde{\Omega}_m$ of the Fourier series expansion of the grout/ground temperature at the borehole wall:

$$\tilde{\Omega}_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0=1} e^{-\iota m \theta_0} d\theta_0, \quad (5.10)$$

where $\iota = \sqrt{-1}$. The sought relationship is found through the substitution of the general solution in the ground, particularized at the borehole wall, into the definition of the Fourier series coefficients $\tilde{\Omega}_m$, Eq. (5.10):

$$e^{\text{Pe} \cos(\theta_0)} \sum_{n=0}^{\infty} \left[\tilde{\alpha}_n ce_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) + \tilde{\beta}_n se_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) \right] = \sum_{m=-\infty}^{\infty} \tilde{\Omega}_m e^{\iota m \theta_0}.$$

The real-valued exponential function is shifted to the right hand side of the equation to exploit the orthogonality of angular Mathieu functions shown in Appendix 5.A:

$$\begin{aligned} \tilde{\alpha}_n &= \frac{1}{\varepsilon_n \pi} \int_{-\pi}^{\pi} e^{-\text{Pe} \cos(\theta_0)} \sum_{m=-\infty}^{\infty} \tilde{\Omega}_m e^{\iota m \theta_0} ce_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) d\theta_0 \\ \tilde{\beta}_n &= \frac{1}{\varepsilon_n \pi} \int_{-\pi}^{\pi} e^{-\text{Pe} \cos(\theta_0)} \sum_{m=-\infty}^{\infty} \tilde{\Omega}_m e^{\iota m \theta_0} se_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) d\theta_0, \end{aligned} \quad (5.11)$$

where $\varepsilon_0 = 2$ and $\varepsilon_n = 1$ in any other case.

Equation (5.11) successfully relates the integration constants $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ with the Fourier series coefficients $\tilde{\Omega}_m$ of the grout/ground temperature at the borehole wall. However, this relationship can be further simplified by also expressing the angular Mathieu functions as Fourier series expansions:

$$ce_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) = \sum_{\ell=-\infty}^{\infty} c_{n\ell} e^{\iota \ell \theta_0} \quad \text{and} \quad se_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) = \sum_{\ell=-\infty}^{\infty} s_{n\ell} e^{\iota \ell \theta_0},$$

where the coefficients $c_{n\ell}$ and $s_{n\ell}$ depend on the Peclet number as shown in Appendix 5.A.

Substitution of these expansions into Eq. (5.11) leads to

$$\begin{aligned}\tilde{\alpha}_n &= \frac{1}{\varepsilon_n \pi} \sum_{m=-\infty}^{\infty} \tilde{\Omega}_m \sum_{\ell=-\infty}^{\infty} c_{n\ell} \int_{-\pi}^{\pi} e^{-\text{Pe} \cos(\theta_0)} e^{\iota m \theta_0} e^{\iota \ell \theta_0} d\theta_0 \\ \tilde{\beta}_n &= \frac{1}{\varepsilon_n \pi} \sum_{m=-\infty}^{\infty} \tilde{\Omega}_m \sum_{\ell=-\infty}^{\infty} s_{n\ell} \int_{-\pi}^{\pi} e^{-\text{Pe} \cos(\theta_0)} e^{\iota m \theta_0} e^{\iota \ell \theta_0} d\theta_0.\end{aligned}$$

The simplification arises from the fact that the imaginary part of the integrands is odd, causing no effect on the values of $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ when integrated over a 2π domain. Additionally, the remaining real part of the integrands can be expressed using modified Bessel functions of the first kind $I_n(z)$ [80], leading to

$$\begin{aligned}\tilde{\alpha}_n &= \frac{2}{\varepsilon_n} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \tilde{\Omega}_m c_{np} I_{m+p}(-\text{Pe}) \\ \tilde{\beta}_n &= \frac{2}{\varepsilon_n} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \tilde{\Omega}_m s_{np} I_{m+p}(-\text{Pe}).\end{aligned}\tag{5.12}$$

Heat injection rate per unit borehole length

In the field of geothermal energy, a frequently used parameter, already introduced in previous chapters, is the heat injection rate per unit borehole length $\tilde{q}$. This parameter represents the total amount of heat exchanged between the borehole and ground per unit time and unit borehole length.

The interest of reintroducing this quantity here is that most models in the literature report their results in terms of the heat injection rate per unit borehole length, $\tilde{q}$, [19, 27, 31, 32, 49, 74, 97], including the model developed in Chapter 3. Therefore, it is particularly useful, especially when comparing results with the state of the art in Section 5.4, to establish a relationship between $\tilde{q}$ and the integration constants $\tilde{\alpha}_n$ and $\tilde{\beta}_n$.

This link is achieved, analogously to Chapter 3, thanks to the absence of thermal inertia, so that the heat injection rate per unit borehole length, $\tilde{q}$, coincides with the heat flux at the borehole integrated over the whole borehole wall:

$$\frac{\tilde{q}}{2\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} -\frac{\partial \tilde{\Theta}_{\text{in}}^{(0)}}{\partial \rho_0} \Big|_{\rho_0=1} d\theta_0.$$

Thereafter, the zeroth order inner solution in the ground, Eq. (5.9), is differentiated, evaluated at the borehole wall, and substituted into previous equation to obtain

$$\frac{\tilde{q}}{2\pi} = - \sum_{n=0}^{\infty} \tilde{\alpha}_n \frac{\text{Ke}'_n\left(0, \frac{\text{Pe}}{2}\right)}{\text{Ke}_n\left(0, \frac{\text{Pe}}{2}\right)} \sum_{\ell=-\infty}^{\infty} c_{n\ell} I_{\ell}(\text{Pe}).\tag{5.13}$$

5.1.3 Solution to outer region

Far from the borehole, at non-dimensional distances of order $\rho_0 \sim \text{Pe}/|s| \gg 1$, thermal inertia and heat convection are equally relevant, being heat conduction negligible in all directions except downstream of the borehole. Nonetheless, all three phenomena are retained throughout the entire outer region following the asymptotic framework presented in Chapter 3. Thankfully, the governing equation that holds all three phenomena becomes solvable in the outer region since the velocity field far from the borehole is, in first approximation, an uniform stream:

$$v_\rho \Big|_{\rho_0 \gg 1} = \cos(\theta_0) + \mathcal{O}(s^2) \quad \text{and} \quad v_\theta \Big|_{\rho_0 \gg 1} = -\sin(\theta_0) + \mathcal{O}(s^2).$$

Hence, the zeroth order solution to the outer region must satisfy the boundary condition at infinity, Eq. (2.9), and the partial differential equation

$$s\tilde{\Theta}_{\text{out}}^{(0)} + \text{Pe} \left[\cos(\theta_0) \frac{\partial \tilde{\Theta}_{\text{out}}^{(0)}}{\partial \rho_0} - \frac{\sin(\theta_0)}{\rho_0} \frac{\partial \tilde{\Theta}_{\text{out}}^{(0)}}{\partial \theta_0} \right] = \frac{1}{\rho_0} \frac{\partial}{\partial \rho_0} \left(\rho_0 \frac{\partial \tilde{\Theta}_{\text{out}}^{(0)}}{\partial \rho_0} \right) + \frac{1}{\rho_0^2} \frac{\partial^2 \tilde{\Theta}_{\text{out}}^{(0)}}{\partial \theta_0^2}.$$

The solution to the zeroth order outer problem is

$$\tilde{\Theta}_{\text{out}}^{(0)} = e^{\frac{\text{Pe}}{2} \rho_0 \cos(\theta_0)} \sum_{\ell=-\infty}^{\infty} \tilde{C}_\ell K_\ell \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{\ell \theta_0}, \quad (5.14)$$

with $K_n(z)$ being the modified Bessel functions of the second kind [80]. The unspecified integration constants $\tilde{C}_\ell$ are determined by imposing the matching with the inner solution.

5.1.4 Matching of both regions

The final step to construct an asymptotic model by means of matched asymptotic expansion techniques consists in the matching of both solutions [64]. This process imposes that at an intermediate region located far from the borehole, $\rho_0 \gg 1$, but closer to the borehole than the outer region, $\rho_0 |s| \ll 1$, both solutions must be asymptotically equivalent. That is, their difference must be negligible:

$$\tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0 \gg 1} - \tilde{\Theta}_{\text{out}}^{(0)} \Big|_{\rho_0 |s| \ll 1} \ll 1.$$

The inner solution in the ground, Eq. (5.9), is also expressed as Fourier series in order to facilitate the term-by-term comparison with the outer solution, Eq. (5.14):

$$\tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0 \geq 1} = e^{\frac{\text{Pe}}{2} \left(\rho_0 + \frac{1}{\rho_0} \right) \cos(\theta_0)} \sum_{\ell=-\infty}^{\infty} \sum_{n=0}^{\infty} \left[\tilde{\alpha}_n \frac{\text{Ke}_n \left(\ln \rho_0, \frac{\text{Pe}}{2} \right)}{\text{Ke}_n \left(0, \frac{\text{Pe}}{2} \right)} c_{n\ell} + \tilde{\beta}_n \frac{\text{Ko}_n \left(\ln \rho_0, \frac{\text{Pe}}{2} \right)}{\text{Ko}_n \left(0, \frac{\text{Pe}}{2} \right)} s_{n\ell} \right] e^{\ell \theta_0}. \quad (5.15)$$

Its behavior at the intermediate region is obtained by expanding the real-valued exponential function and the modified Mathieu functions for large values of ρ_0 , following Appendix 5.B:

$$\tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0 \gg 1} = e^{\frac{\text{Pe}}{2} \rho_0 (\cos(\theta_0) - 1)} \sqrt{\frac{\pi}{\rho_0 \text{Pe}}} \left[1 + \mathcal{O}\left(\frac{1}{\rho_0}\right) \right] \sum_{\ell=-\infty}^{\infty} \sum_{n=0}^{\infty} \left(\frac{\tilde{\alpha}_n c_{n\ell}}{\text{Ke}_n\left(0, \frac{\text{Pe}}{2}\right)} + \frac{\tilde{\beta}_n s_{n\ell}}{\text{Ko}_n\left(0, \frac{\text{Pe}}{2}\right)} \right) e^{\ell \theta_0}.$$

This simplified version of the inner solution at the intermediate region has to be compared with the outer solution, Eq. (5.14), expanded for small values of $\rho_0|s|$ but large values of ρ_0 :

$$\tilde{\Theta}_{\text{out}}^{(0)} \Big|_{\substack{\rho_0|s| \ll 1 \\ \rho_0 \gg 1}} = e^{\frac{\text{Pe}}{2} \rho_0 (\cos(\theta_0) - 1)} \sqrt{\frac{\pi}{\rho_0 \text{Pe}}} \left[1 + \mathcal{O}\left(\frac{1}{\rho_0}, \rho_0|s|\right) \right] \sum_{\ell=-\infty}^{\infty} \tilde{C}_{\ell} e^{\ell \theta_0}.$$

Finally, a term-by-term comparison between the inner and outer solutions, both expanded at the intermediate region, shows that the integration constants $\tilde{C}_{\ell}$ of the outer solution must be

$$\tilde{C}_{\ell} = \sum_{n=0}^{\infty} \left(\frac{\tilde{\alpha}_n c_{n\ell}}{\text{Ke}_n\left(0, \frac{\text{Pe}}{2}\right)} + \frac{\tilde{\beta}_n s_{n\ell}}{\text{Ko}_n\left(0, \frac{\text{Pe}}{2}\right)} \right). \quad (5.16)$$

5.2 Numerical examples

The capabilities and limitations of the developed model are shown next through a comparison with detailed numerical simulations of the unsteady heat transfer problem formulated in Chapter 2. All required numerical simulations are performed using the commercial software package COMSOL [22]. The advantage of using COMSOL is that it allows for the precise specification of mathematical problems without limitations, including the use of complex variables and arithmetic in problem formulation and solution.

5.2.1 Borehole description

The borehole configuration shown in Figure 5.1 is employed for all comparisons presented in the following subsections. Although various other borehole configurations with different geometrical and thermal characteristics as well as with coaxial and double U-shaped probes have been studied, they are not discussed here since their results and conclusions align with those presented next.

The geometrical and thermal characteristics of the chosen configuration, already used in Chapters 3 and 4, are summarized in Table 3.1. It includes a borehole of diameter $2r_b = 152$ mm along with two heterogeneous pipes inside. Pipe 1 has an outer diameter of $2r_{p1} = 40$ mm and a wall thickness of $d_1 = 3.7$ mm, while pipe 2 presents an outer diameter of $2r_{p2} = 50$ mm and a wall thickness of $d_2 = 4.6$ mm. Both pipes are placed at 5 mm from the borehole wall and at $\pm 45^\circ$ from the rear stagnation point. Thus, the polar coordinates of their centers are (51 mm, $+45^\circ$) and (46 mm, -45°), respectively.

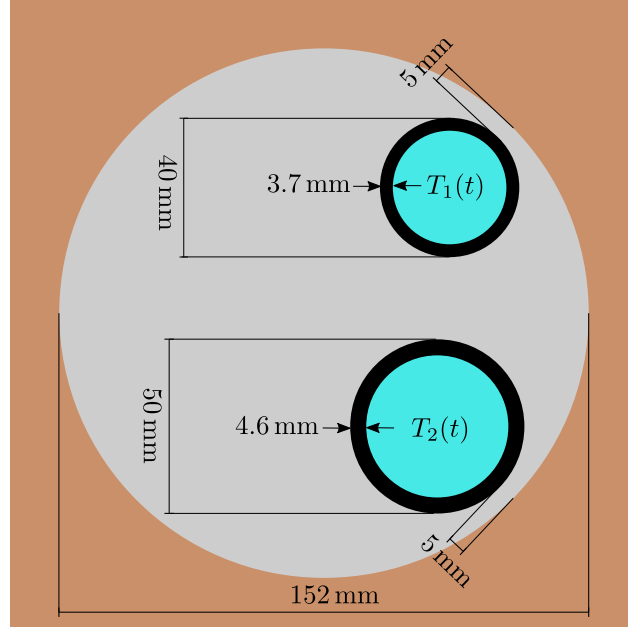


Figure 5.1: Borehole configuration for the numerical examples.

The void between the two pipes and the ground, of effective thermal conductivity $k_g = 3.00$ W/(m K) and effective thermal diffusivity $\alpha_g = 2.24 \cdot 10^{-7}$ m²/s [18], is filled up with conventional grout of thermal conductivity $k_b = 1.5$ W/(m K) [1] and thermal diffusivity $1.00 \cdot 10^{-7}$ m²/s.

The heat carrying liquid inside the pipes is pure water, characterized by a density of 999 kg/m³, a specific heat capacity of 4184 J/(kg K), a thermal conductivity of 0.577 W/(m K), and a dynamic viscosity of $1.138 \cdot 10^{-3}$ kg/(m s). A turbulent flow regime is ensured in both pipes by enforcing a mass flow rate of 0.25 kg/s. The turbulent transport of heat within the liquid is modeled using convective heat transfer coefficients, h_1 and h_2 , whose values are set using Gnielinski's correlations [38]. Their combination with the thermal conductivities of the pipes $k_1 = k_2 = 0.42$ W/(m K) leads to the following inner thermal resistances of the pipes:

$$R_{p1} = 0.535 \frac{\text{m K}}{\text{W}} \quad \text{and} \quad R_{p2} = 0.544 \frac{\text{m K}}{\text{W}}. \quad (5.17)$$

5.2.2 Simulation procedure

The interaction between the just-described borehole and the groundwater flow involves an additional parameter, namely, the Peclet number of the groundwater flow. Two different values for the Peclet number are considered next. The first one, $\text{Pe} = 1.0$, demonstrates the capabilities of the here-developed asymptotic model in presence of strong groundwater flows. The second value, $\text{Pe} = 0.03$, addresses situations in which the state of the art performs well and the present model is expected to fail.

In order to report results in the time domain, the inverse Laplace transform is applied to the grout/ground temperature computed in the complex-valued Laplace plane. This temperature is derived from solving the reference heat transfer problem formulated in Chapter 2 as well as from the inner and outer solutions developed in Section 5.1. The employed numerical Laplace inversion algorithm requires the evaluation of the obtained temperatures at a reduced set of positions s in the Laplace plane [44]. All required numerical simulations as well as their numerical inverse Laplace transforms are entirely performed within COMSOL.

Finally, apart from the Peclet number, temporal laws governing the heat injection rates per unit pipe length are necessary as well. For the comparisons conducted in this section, time-constant values are selected, $q_1(\tau) = 1$ and $q_2(\tau) = 3$, whose Laplace transforms are $\tilde{q}_1(\tau) = 1/s$ and $\tilde{q}_2(\tau) = 3/s$, respectively.

Reference solution

The grout/ground temperature used as benchmark is obtained from numerically solving the formulated problem presented in Section 2.3. It is comprised of the governing equations in grout and ground, Eqs. (2.10) and (2.11), along with the boundary conditions at the pipe walls, Eqs. (2.7) and (2.8). Finally, continuity conditions at the borehole wall, Eq. (2.5), are enforced automatically by COMSOL.

This reference problem is formulated in an unbounded domain wherein, infinitely far from the borehole, the temperature and its Laplace-transformed counterpart are zero, Eq. (2.9). To ensure accuracy of the numerical solution on a finite computational domain, a convenient domain size and shape is chosen and suitable boundary conditions are enforced at its outer rim. Since the optimal numerical characteristics of each simulation depend on the Peclet number of the groundwater flow, they are different for each numerical example shown in this section.

In presence of strong groundwater flows, characterized by a Peclet number of order unity, ground temperature exhibits a rapid decay in all directions except downstream of the borehole, where a thermal wake is present. Consequently, the computational domain consists in a semicircle of radius $\rho_0 = 30$ centered at the borehole and a rectangle extended up to $\rho_0 = 1000$ in the downstream direction, to correctly account for the thermal wake. To best mimic the behavior of the problem far from the borehole, an outflow condition is specified downstream of the borehole, while in the remaining directions a zero-temperature boundary condition is enforced.

When creeping groundwater flows are present, for which the Peclet number is small compared to unity, the temperature decreases at the same rate in all spatial directions. Hence, the employed computational domain is a circle of radius $\rho_0 = 1000$ centered at the borehole, with a zero-temperature boundary condition enforced at its outer boundary.

Solution to inner region

The zeroth order solution to the inner problem is obtained by numerically solving the problem described in Section 5.1.2. It is comprised of the quasi-steady thermal response of grout and ground, Eqs. (5.4) and (5.5), the boundary condition at the pipe walls, Eqs. (5.7) and (5.8), and a zero temperature condition far from the borehole Eq. (2.9). Finally, the continuity conditions at the borehole wall, Eq. (5.6), are enforced automatically by COMSOL.

The numerical solution in the inner region represents the final steady-state of the reference problem. Hence, the numerical characteristics chosen in the previous subsection for the reference problem are adopted as well for the numerical simulation of the inner problem, including computational domain, mesh, and boundary conditions.

Solution to outer region

No numerical simulation is required to obtain the zeroth order solution to the outer region as its analytical expression, Eq. (5.14), is known. Nonetheless, this expression is written in terms of the integration constants $\tilde{C}_\ell$ which ultimately depend on the coefficients $\tilde{\Omega}_m$ of the Fourier series expansion of the zeroth order inner solution around the borehole wall. Hence, these values must be computed using the numerical solution to the inner problem.

To evaluate the analytical solution shown in Eq. (5.14), its infinite series needs to be truncated. The number of terms required to ensure error levels below the desired threshold depends on the Peclet number of the groundwater flow. For small values, five terms are enough. When the Peclet number attains values of order unity, fifteen terms are needed instead.

Furthermore, integration constants $\tilde{C}_\ell$ are also expressed in terms of infinite series, Eq. (5.16), so truncation of those series is again necessary. Fortunately, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ decrease rapidly as n increases so that fifteen terms are enough to accurately compute $\tilde{C}_\ell$ for any value of the Peclet number.

Finally, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ themselves are also expressed as infinite series so truncation of these series is again mandatory. Thankfully, Eq. (5.12) involves two contributions that rapidly decay in importance, the Fourier series coefficients $\tilde{\Omega}_m$ and the modified Bessel functions of the first kind $I_{m+p}(-\text{Pe})$, and quantities that are at most of order unity, namely, coefficients $c_{n\ell}$ and $s_{n\ell}$. Consequently, thirteen terms in m , ranging from -6 till 6, and thirty-one values of p , ranging from -15 till 15, are retained to accurately compute the values of $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ for any possible Peclet number.

Once integration constants $\tilde{C}_\ell$ are obtained, the implementation of Eq. (5.14), using the numerical algorithms developed by Zhang and Jin [108] for the modified Bessel functions of the second kind, as well as its numerical inverse Laplace transform is performed in FORTRAN95.

5.2.3 $Pe = 1$

Temporal evolution of the grout/ground temperature for the favorable Peclet number of 1.0 is depicted in Figure 5.2 through three snapshots at different times, namely, $\tau = 20$ at the top, $\tau = 50$ at the middle and $\tau = 200$ at the bottom. Inner and outer solutions are presented using solid black lines on the left and right sides of the figure, respectively, while the reference solution is plotted using a solid color map.

This figure displays the time evolution up to non-dimensional times of $\tau = 200$ as by that time the final steady-state is essentially reached. However, even at a tenfold lower value of $\tau = 20$, the reference solution exhibits minimal variations close to the borehole when compared to the plot corresponding to $\tau = 200$. This phenomenon is explained by the presence of the strong groundwater flow that swiftly eliminates intermediate temperature states, leading to a rapid approach to the steady-state.

Hence, owing to this behavior, the inner solution reports excellent performance within the inner region even for the not-so-large value of $\tau = 20$. As time goes by and τ increases, the accuracy of the inner solution enhances, eventually overlapping the reference solution from $\tau = 50$ onwards.

The outer solution exhibits great performance in general in all snapshots shown in Figure 5.2, with the exception in the vicinity of the borehole. This good performance is explained by its mathematical formulation. In contrast to the inner region, in which the governing equations are heavily simplified, the governing equation in the outer region maintains all three phenomena, namely, thermal inertia, heat convection and heat conduction. It only differs from the full governing equation in terms related to the velocity field that decay quadratically with ρ_0 . Hence, far from the borehole the differences are minimal leading to the excellent results shown in Figure 5.2.

To quantitatively assess the accuracy of the presented asymptotic model, Figure 5.3 shows all three solutions along the green line depicted in the second plot of Figure 5.2 that corresponds to $\tau = 50$. This green line starts at the rear stagnation point of the borehole, at $(r_0, \theta_0) = (r_b, 0^\circ)$, and extends in the downstream direction from the borehole.

The left part of Figure 5.3 represents the temperatures provided by the reference solution, by the inner solution, and by the outer solution along the aforementioned green line. This plot perfectly illustrates the asymptotic structure of the problem where three distinct regions, namely, the inner region, the intermediate region, and the outer region, are easily identified. The right part of Figure 5.3 shows the absolute errors of both inner and outer solutions along the green line, compared to the reference solution.

Close to the borehole, where the inner region is located at distances ρ_0 of order unity, the inner solution matches the reference solution with absolute/relative errors below 10^{-5} . On the other hand, the outer solution clearly diverges from the reference solution in the inner region as it is outside of its region of validity.

Further away from the borehole, at distances $5 < \rho_0 < 30$, both asymptotic solutions coincide, leading to absolute (relative) errors below 0.01 (4%). This is precisely the intermediate region

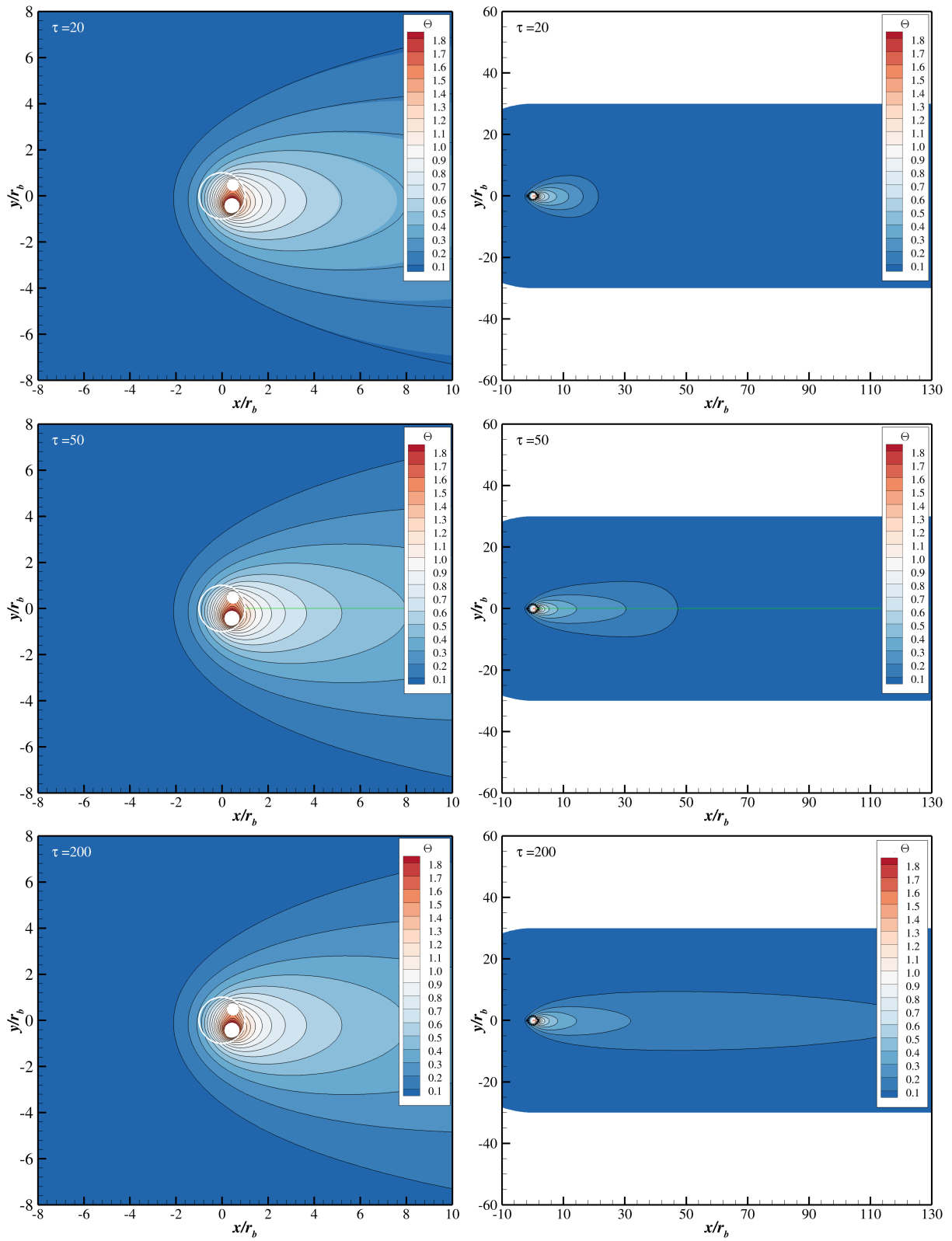


Figure 5.2: (Left) inner solution and (right) outer solution for $Pe = 1$.

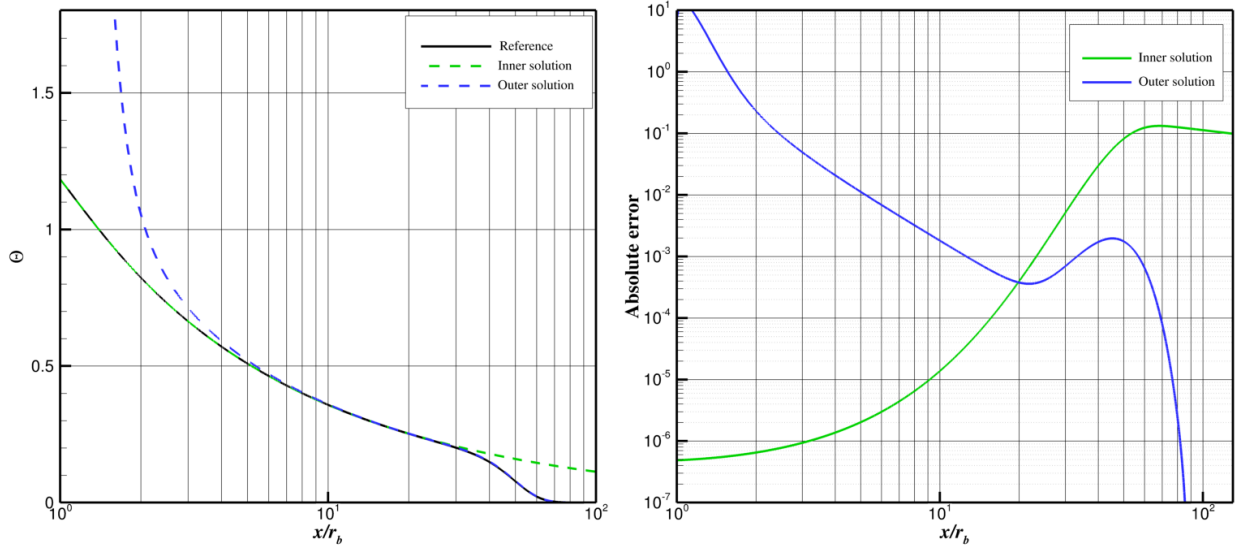


Figure 5.3: (Left) Temperature and (right) absolute errors for $Pe = 1$ and $\tau = 50$.

at which the asymptotic matching between the inner and outer solutions was performed. Thus, it was expected that both solutions would coincide.

Finally, far from the borehole, where $\rho_0 > 50$, the outer solution perfectly follows the reference solution to the unperturbed value $\Theta = 0$ since, at $\tau = 50$, the thermal influence of the borehole has not yet extended to those distances. The outer solution presents absolute (relative) errors below $2 \cdot 10^{-3}$ (2%). On the other hand, Figure 5.3 shows how the inner solution deviates from the reference solution at such large distances from the borehole, as it is there beyond its region of validity.

5.2.4 $Pe = 0.03$

Figure 5.4 shows the time evolution of the grout/ground temperature for the unfavorable case of $Pe = 0.03$. Three snapshots for different times display this temporal evolution, namely, $\tau = 20$ at the top, $\tau = 200$ at the middle, and $\tau = 2000$ at the bottom. The left side of the figure shows the inner solution with solid black isolines while the right side depicts the outer solution using the same black isolines. The reference solution is represented in both plots using a solid color map.

Solutions are displayed up to $\tau = 2000$, when the final steady-state is essentially achieved. Compared to Figure 5.2, the time needed to reach the steady-state is tenfold higher for $Pe = 0.03$. This fact reinforces the idea that strong groundwater flows promote a faster transition to the final steady-state than creeping groundwater flows as the intermediate states are flushed away faster.

Another significant difference is the double-fold increase in non-dimensional temperature when comparing Figure 5.4 with Figure 5.2. This is explained by the inefficiency in transferring heat to the ground when creeping instead of strong groundwater flows are present in the

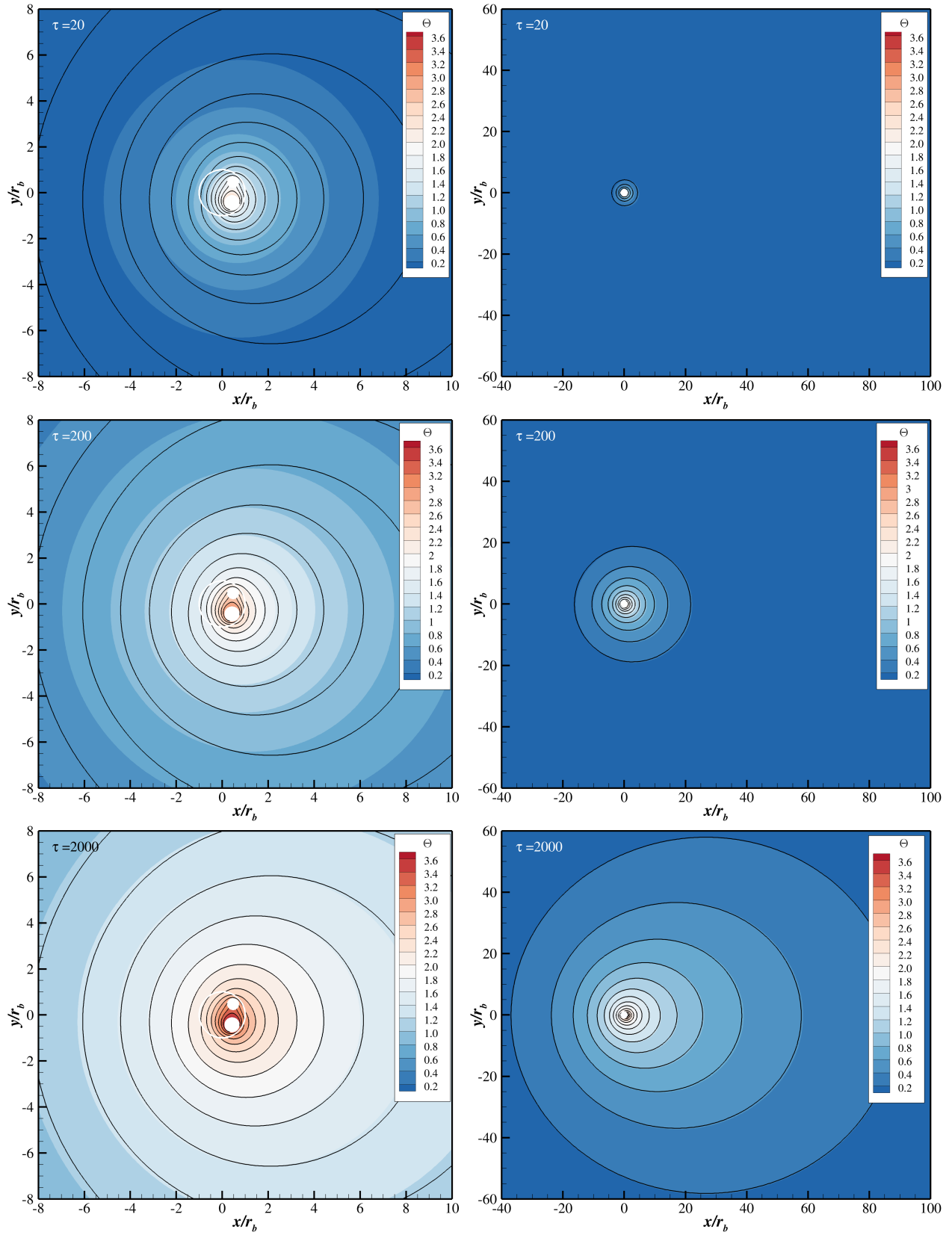


Figure 5.4: (Left) inner solution and (right) outer solution for $Pe = 0.03$.

ground, with heat conduction arising then as the dominant heat transfer mechanism near the borehole.

The most unfavorable snapshot corresponding to $\tau = 20$ shows relevant differences between the inner solution and the reference solution near the borehole. In absence of strong groundwater flows the quasi-steady region grows slower, thus for such not-so-large values of τ the inner solution fails even close to the borehole. As time evolves, though, the quasi-steady region expands and the accuracy of the inner solution enhances, practically overlapping the reference solution for $\tau = 2000$.

The outer solution, however, demonstrates excellent performance at all times, despite the unfavorable value of $Pe = 0.03$. This exceptional result is explained similarly to the previous case. Since the outer solution incorporates all three phenomena in the governing equation, only differing from the complete governing equation in terms that decay quadratically with the distance to the borehole, at the outer region this difference is negligible leading to the outstanding performance shown in Figure 5.4.

5.3 Existing models for the problem

The interaction of geothermal boreholes with groundwater flows has extensively been studied in the literature. Existing models for that interaction assume, either explicitly or implicitly, that the characteristic heat injection time t_q is large compared to the characteristic transversal diffusion time t_b so that the same two-region structure as in the present work emerges [87, 109].

Most work in the literature is focused on the outer region to obtain there the perturbed ground temperature in terms of the heat injection rate per unit borehole length. For it, a superposition of moving point sources of heat, all of equal intensity, is used to represent the borehole. If the finite length of the borehole is thereby taken into account, so-called *moving finite line source models* result [20, 29–31, 39, 52, 74, 90–92, 103, 104, 107]. If its finite length is neglected instead, then so-called *moving infinite line source models* are obtained [9, 27, 42, 73, 97, 103, 104].

The aforementioned outer solutions are valid for all Peclet numbers of the groundwater flow as the equation solved in the outer region coincides with the one in Chapter 3 and in the present chapter. None of these models, though, addresses the groundwater flow in the inner region. In fact, they implicitly assume that inner region to be purely-conducting when they propose to pair their outer solutions with a classical network of thermal resistances [5, 19, 21]. Hence, these models are only correct for small values of the Peclet number, leading to wrong results in the vicinity of the borehole for Peclet numbers of order unity [86, 87].

A different approach is followed by the models recently proposed by Al-Khoury and coworkers [62, 63]. They solve the heat convection-conduction problem in the ground up to the borehole wall, effectively addressing the outer and inner regions simultaneously. To obtain a solution, though, they simplify the velocity field in a way that clashes with the very existence of the borehole as they consider it uniform everywhere in the ground. Also the heat exchange inside

the borehole is completely skipped as a uniform temperature is enforced by them at the borehole wall.

The interaction of geothermal boreholes with groundwater flows is mathematically equivalent to two other problems extensively studied in the field of fluid mechanics, namely, the high Reynolds-number flow past a cylinder at small Prandtl numbers and the Darcy flow past a cylinder in a porous media. In both cases the energy conservation equation to solve and the velocity field to consider are the same, playing the effective driving pressure in the latter one the role of the velocity potential in the former one.

Most work in the literature considers steady-state conditions and a uniform temperature at the cylinder wall. Then, the resulting heat transfer problem for large values of the Peclet number is tackled using the boundary layer approach that supplies closed-form expressions for the heat flux at the cylinder [4, 15, 78, 82, 94]. These solutions, however, are not valid for Peclet numbers of order unity due to the simplifications introduced by the boundary layer approach.

The boundary layer approach, or wake approach, is also used by some authors to study the thermal wake that forms behind the cylinder [78]. This wake is of relevance for the thermal interaction of neighboring boreholes as it negatively affects the performance of the geothermal HVAC system. Unfortunately, the presented solutions for the thermal wake are not suited for small values of the Peclet number so that the approach followed in the present work is preferred.

Also the case of small Peclet numbers has been analyzed in the literature, although in an indirect way. The classical solution for the viscous flow past a cylinder also presents a two-region structure, with its Stokes region being equivalent to the inner region considered here and its Oseen region matching the outer region [102]. Then, the laplacian of the stream function plays the same role as the ground temperature in the present work. Despite these similarities, though, the resulting solutions are not good for Peclet numbers of order unity due to their poor accuracy in the vicinity of the borehole [87].

Exact solutions valid for all Peclet numbers can also be found in the literature [16, 58, 59, 100]. They involve angular and modified Mathieu functions just as the zeroth order solution to the inner region does. In fact, they are fully equivalent as both are steady-state solutions to the convection-conduction problem around a cylinder/borehole.

Finally, the unsteady case with a uniform temperature at the cylinder wall has also been addressed by the prevailing literature [94]. By means of asymptotic expansion techniques, Sano derived approximate solutions for small and large values of the Peclet number, but not for the Peclet numbers of order unity that are of interest for the present chapter.

5.4 Limit of creeping groundwater flows

The state of the art provides reliable results concerning the thermal interaction of geothermal boreholes with creeping groundwater flows [27, 74, 87, 97]. Hence, this section conducts an

analysis of the developed asymptotic model in the limit of small Peclet numbers to facilitate a rigorous comparison with existing well-established models.

5.4.1 Inner region

An exhaustive term-by-term analysis is only accomplished for the zeroth order inner solution in the ground as no analytical expression is available for the grout. The zeroth order inner solution given by Eq. (5.9) is written here again to facilitate the reading of this section:

$$\begin{aligned} \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\rho_0 \geq 1} = e^{\frac{\text{Pe}}{2} \cos(\theta_0) \left(\rho_0 + \frac{1}{\rho_0} \right)} \sum_{n=0}^{\infty} \left[\tilde{\alpha}_n \frac{\text{Ke}_n \left(\ln \rho_0, \frac{\text{Pe}}{2} \right)}{\text{Ke}_n \left(0, \frac{\text{Pe}}{2} \right)} \text{ce}_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) \right. \\ \left. + \tilde{\beta}_n \frac{\text{Ko}_n \left(\ln \rho_0, \frac{\text{Pe}}{2} \right)}{\text{Ko}_n \left(0, \frac{\text{Pe}}{2} \right)} \text{se}_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) \right]. \end{aligned} \quad (5.18)$$

Exponential function

Since distance to the borehole, ρ_0 , is of order unity in the inner region and the Peclet number is considered small compared to unity, the argument of the real-valued exponential function in Eq. (5.18) is small as well. Consequently, it can be approximated by the first terms of its Taylor series expansion:

$$e^{\frac{\text{Pe}}{2} \cos(\theta_0) \left(\rho_0 + \frac{1}{\rho_0} \right)} \Big|_{\text{Pe} \ll 1} = 1 + \frac{\text{Pe}}{2} \cos(\theta_0) \left(\rho_0 + \frac{1}{\rho_0} \right) + \mathcal{O}(\text{Pe}^2). \quad (5.19)$$

Mathieu functions

The zeroth order inner solution in the ground, Eq. (5.18), relies on the angular Mathieu functions ce_n and se_n . It is well-known in the literature [37, 72, 80], and also shown in Appendix 5.A, that they tend to cosine and sine functions, respectively, as their second argument, $-\text{Pe}^2/4$ in this case, approaches zero:

$$\begin{aligned} \text{ce}_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) \Big|_{\text{Pe} \ll 1} &= \cos(\theta_0) + \mathcal{O}(\text{Pe}^2), \\ \text{se}_n \left(\theta_0, -\frac{\text{Pe}^2}{4} \right) \Big|_{\text{Pe} \ll 1} &= \sin(\theta_0) + \mathcal{O}(\text{Pe}^2). \end{aligned} \quad (5.20)$$

The zeroth order inner solution also depends on the modified Mathieu functions Ke_n and Ko_n . Therefore, these functions must also be expanded in the limit of creeping groundwater flows. These expansions are rigorously addressed in Appendix 5.B, as they involve some lengthy

mathematical intricacies, leading to the following results for the ratios of modified Mathieu functions present in Eq. (5.18):

$$\begin{aligned}
 \left. \frac{\text{Ke}_0\left(\ln \rho_0, \frac{\text{Pe}}{2}\right)}{\text{Ke}_0\left(0, \frac{\text{Pe}}{2}\right)} \right|_{\text{Pe} \ll 1} &= 1 + \frac{\ln \rho_0}{\ln\left(\frac{\text{Pe}}{4}\right) + \gamma} + \mathcal{O}(\text{Pe}^2), \\
 \left. \frac{\text{Ke}_n\left(\ln \rho_0, \frac{\text{Pe}}{2}\right)}{\text{Ke}_n\left(0, \frac{\text{Pe}}{2}\right)} \right|_{\text{Pe} \ll 1} &= \frac{1}{\rho_0^{|n|}} + \mathcal{O}(\text{Pe}^2) \quad \forall n > 0, \\
 \left. \frac{\text{Ko}_n\left(\ln \rho_0, \frac{\text{Pe}}{2}\right)}{\text{Ko}_n\left(0, \frac{\text{Pe}}{2}\right)} \right|_{\text{Pe} \ll 1} &= \frac{1}{\rho_0^{|n|}} + \mathcal{O}(\text{Pe}^2) \quad \forall n > 0.
 \end{aligned} \tag{5.21}$$

Integration constants

Integration constants $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ are given in Eq. (5.12) in terms of the coefficients $\tilde{\Omega}_m$ of the Fourier series expansion of the ground temperature around the borehole wall. Consequently, these coefficients need to be expanded as well as they are functions of the Peclet number of the flow:

$$\tilde{\Omega}_m \Big|_{\text{Pe} \ll 1} = \tilde{\Omega}_m^{(0)} + \text{Pe} \tilde{\Omega}_m^{(1)} + \mathcal{O}(\text{Pe}^2).$$

Furthermore, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ also depend on $c_{n\ell}$ and $s_{n\ell}$, the coefficients of the Fourier series expansions of the angular Mathieu functions, and on the modified Bessel functions of the first kind I_n . Therefore, an analysis of their behavior as the Peclet number tends to zero is required as well. The former coefficients are expanded in Appendix 5.A while the behavior of the latter functions in the limit of interest can be found in the literature [80]. By combining all aforementioned results, the expansion of the integration constants $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ as the Peclet number approaches zero is obtained:

$$\begin{aligned}
 \tilde{\alpha}_0 \Big|_{\text{Pe} \ll 1} &= \tilde{\Omega}_0^{(0)} + \text{Pe} \left(\tilde{\Omega}_0^{(1)} - \frac{\tilde{\Omega}_{-1}^{(0)} + \tilde{\Omega}_{+1}^{(0)}}{4} \right) + \mathcal{O}(\text{Pe}^2), \\
 \tilde{\alpha}_n \Big|_{\text{Pe} \ll 1} &= \tilde{\Omega}_n^{(0)} + \tilde{\Omega}_{-n}^{(0)} - \frac{\text{Pe}}{2} \left(\tilde{\Omega}_{n+1}^{(0)} + \tilde{\Omega}_{n-1}^{(0)} + \tilde{\Omega}_{1-n}^{(0)} \right. \\
 &\quad \left. + \tilde{\Omega}_{-1-n}^{(0)} - 2\tilde{\Omega}_{-n}^{(1)} - 2\tilde{\Omega}_n^{(1)} \right) + \mathcal{O}(\text{Pe}^2) \quad \forall n > 0, \\
 \tilde{\beta}_n \Big|_{\text{Pe} \ll 1} &= \iota(\tilde{\Omega}_n^{(0)} - \tilde{\Omega}_{-n}^{(0)}) - \iota \frac{\text{Pe}}{2} \left(\tilde{\Omega}_{n+1}^{(0)} + \tilde{\Omega}_{n-1}^{(0)} - \tilde{\Omega}_{1-n}^{(0)} \right. \\
 &\quad \left. - \tilde{\Omega}_{-1-n}^{(0)} - 2\tilde{\Omega}_{-n}^{(1)} - 2\tilde{\Omega}_n^{(1)} \right) + \mathcal{O}(\text{Pe}^2) \quad \forall n > 0.
 \end{aligned} \tag{5.22}$$

Once each element comprising the inner solution has been analyzed separately, the outcomes are substituted into Eq. (5.18) to obtain the sought expansion of the inner solution in the

limit of creeping groundwater flows. However, its comparison with the state of the art is still challenging as models in the literature are commonly expressed using not only coefficients $\tilde{\Omega}_m^{(0)}$ and $\tilde{\Omega}_m^{(1)}$ but also the heat injection rate per unit borehole length $\tilde{q}$.

To facilitate a meaningful comparison between the present model and the state of the art, an alternative expansion of the integration constant $\tilde{\alpha}_0$, that incorporates $\tilde{q}$, is developed. First, the heat injection rate per unit borehole length is written in the limit of small Peclet numbers by substituting the obtained expansions of Mathieu and Bessel functions into Eq. (5.13):

$$\left. \frac{\tilde{q}}{2\pi} \right|_{\text{Pe} \ll 1} = -\frac{\tilde{\alpha}_0}{\ln\left(\frac{\text{Pe}}{4}\right) + \gamma} + \tilde{\alpha}_1 \frac{\text{Pe}}{2} + \mathcal{O}(\text{Pe}^2).$$

Then, expansions of the integration constants $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ given in Eq. (5.22) are used to obtain

$$\left. \frac{\tilde{q}}{2\pi} \right|_{\text{Pe} \ll 1} = -\frac{\tilde{\Omega}_0^{(0)} + \text{Pe}\tilde{\Omega}_0^{(1)}}{\ln\left(\frac{\text{Pe}}{4}\right) + \gamma} + \frac{\text{Pe}}{2} \left(\tilde{\Omega}_1^{(0)} + \tilde{\Omega}_{-1}^{(0)} \right) \left(1 + \frac{1}{2} \frac{1}{\ln\left(\frac{\text{Pe}}{4}\right) + \gamma} \right) + \mathcal{O}(\text{Pe}^2). \quad (5.23)$$

Treating $\ln(\text{Pe})$ as a simple number, as already done in Chapter 3, the previous expression presents $\tilde{q}$ as a polynomial expansion of the small parameter Pe . By neglecting terms of order Peclet, this expression enables to relate the first approximation of the mean azimuthal borehole wall temperature $\tilde{\Omega}_0^{(0)}$ with $\tilde{q}$:

$$\tilde{\Omega}_0^{(0)} = -\frac{\tilde{q}}{2\pi} \left[\ln\left(\frac{\text{Pe}}{4}\right) + \gamma \right].$$

The neglected terms of order Peclet can now be used to derive an expression for the first order correction to the mean azimuthal borehole wall temperature following Eq. (5.23):

$$\tilde{\Omega}_0^{(1)} = \frac{\tilde{\Omega}_1^{(0)} + \tilde{\Omega}_{-1}^{(0)}}{2} \left[\ln\left(\frac{\text{Pe}}{4}\right) + \gamma + \frac{1}{2} \right].$$

Finally, these results are substituted into Eq. (5.22) to obtain the sought alternative expansion of $\tilde{\alpha}_0$ that also involves the heat injection rate per unit borehole length $\tilde{q}$:

$$\tilde{\alpha}_0 \Big|_{\text{Pe} \ll 1} = \left[\ln\left(\frac{\text{Pe}}{4}\right) + \gamma \right] \left[-\frac{\tilde{q}}{2\pi} + \frac{\text{Pe}}{2} \left(\tilde{\Omega}_1^{(0)} + \tilde{\Omega}_{-1}^{(0)} \right) \right] + \mathcal{O}(\text{Pe}^2). \quad (5.24)$$

Comparison with state of the art

All previous results allow to write the expansion of the inner solution for small values of the Peclet number in terms of the mean azimuthal borehole wall temperature, $\tilde{\Omega}_0^{(0)}$ and $\tilde{\Omega}_0^{(1)}$, and the heat injection rate per unit borehole length $\tilde{q}$ in such a way that a direct comparison with

the state of the art is straightforward:

$$\begin{aligned}
 \tilde{\Theta}_{\text{in}}^{(0)} \Big|_{\text{Pe} \ll 1} &= \tilde{\Omega}_0^{(0)} - \frac{\tilde{q}}{2\pi} \ln(\rho_0) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{\Omega}_n^{(0)}}{\rho_0^{|n|}} e^{\iota n \theta_0} \\
 &+ \text{Pe} \left[\tilde{\Omega}_0^{(1)} + \frac{\tilde{\Omega}_1^{(0)} + \tilde{\Omega}_{-1}^{(0)}}{2} \ln(\rho_0) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{A}_{1n}}{\rho_0^{|n|}} e^{\iota n \theta_0} \right] \\
 &+ \text{Pe} \left[\frac{\tilde{\Omega}_0^{(0)}}{2} \rho_0 \cos(\theta_0) - \frac{\tilde{q}}{4\pi} \left(\rho_0 + \frac{1}{\rho_0} \right) \ln(\rho_0) \cos(\theta_0) \right] \\
 &+ \text{Pe} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{\Omega}_n^{(0)}}{4} \left[\frac{e^{\iota \frac{n}{|n|} (|n|+1) \theta_0}}{\rho_0^{|n|-1}} + \frac{e^{\iota \frac{n}{|n|} (|n|-1) \theta_0}}{\rho_0^{|n|+1}} \right] + \mathcal{O}(\text{Pe}^2).
 \end{aligned} \tag{5.25}$$

The first row corresponds to the zeroth order approximation given in Chapter 3, with coefficients $\tilde{A}_{1n}$ with $n \neq 0$ defined as follows:

$$\tilde{A}_{1n} = \tilde{\Omega}_n^{(1)} - \frac{\tilde{\Omega}_{n+1}^{(0)} + \tilde{\Omega}_{n-1}^{(0)}}{4}.$$

The second row represents the first order corrections to the three terms from the first row. The last two rows arise from combinations of the first order correction to the exponential function, Eq. (5.19), and the three terms of order unity from the first row in Eq. (5.25).

A comparison between the inner solution presented here and its counterpart given in Chapter 3 reveals one single difference, namely, the absence of position s in the complex-valued Laplace plane. Unfortunately, fully restoring the inner solution from the literature, including s , is not possible as the matching process conducted in the present chapter is qualitatively distinct.

The discrepancy arises from the different behaviors exhibited by the inner and outer solutions within the intermediate region in presence of creeping and strong groundwater flows. This behavior is dictated by $\rho_0 \text{Pe}$ that governs the expansions of the inner and outer solutions within the intermediate region, as shown in Eq. (5.14) and in Appendix 5.B. The state of the art deals with the presence of creeping groundwater flows for which $\rho_0 \text{Pe} \ll 1$ in the intermediate region. The present work, on the contrary, focuses on strong groundwater flows for which $\rho_0 \text{Pe} \gg 1$ in the intermediate region. This qualitative change affects the expansions of inner and outer solutions in an irreconcilable way, leading to different matching processes and the impossibility of fully recovering the state of the art.

5.4.2 Outer solution

An exhaustive term-by-term analysis of the zeroth order outer solution is accomplished next. To enhance the readability of this section, the zeroth order solution to the outer region

obtained in Section 5.1.3 is shown here again:

$$\tilde{\Theta}_{\text{out}}^{(0)} = e^{\frac{\text{Pe}}{2} \rho_0 \cos(\theta_0)} \sum_{\ell=-\infty}^{\infty} \tilde{C}_{\ell} K_{\ell} \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{i\ell\theta_0}. \quad (5.26)$$

Exponential function

Although in the limit of creeping groundwater flows the Peclet number is small compared to unity, $\text{Pe} \ll 1$, the outer region is located at distances $\rho_0 \text{Pe} \gg 1$. Consequently, the argument of the exponential function in Eq. (5.26) is large so that no simplification is possible.

Integration constants

Integration constants $\tilde{C}_{\ell}$ of the outer solution are dependent on three distinct sets of quantities. As shown in Eq. (5.16), these are the integration constants $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ of the inner solution, the coefficients $c_{n\ell}$ and $s_{n\ell}$ of the Fourier series expansions of the angular Mathieu functions, and the modified Mathieu functions evaluated at the borehole wall. That is, at $\rho_0 = 1$. Consequently, the behavior of these three sets in the limit of creeping groundwater flows, which ultimately defines the behavior of $\tilde{C}_{\ell}$, is discussed next.

The behavior of the integration constants of the inner solution, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$, in the limit of creeping groundwater flows has already been obtained in Section 5.4.1. Hence, their resulting expansions given by Eqs. (5.22) and (5.24) are employed in the present discussion.

On the other hand, the analysis of the coefficients $c_{n\ell}$ and $s_{n\ell}$ as the Peclet number approaches zero is performed in Appendix 5.A:

$$\begin{aligned} c_{n\ell} \Big|_{\text{Pe} \ll 1} &= \frac{\varepsilon_n \delta_{n|\ell|}}{2} + \mathcal{O}(\text{Pe}^2), \\ s_{n\ell} \Big|_{\text{Pe} \ll 1} &= \frac{\ell}{|\ell|} \frac{\delta_{n|\ell|}}{2i} (1 - \delta_{|\ell|0}) + \mathcal{O}(\text{Pe}^2). \end{aligned}$$

Finally, expansions for the modified Mathieu functions evaluated at $\rho_0 = 1$ are also required. The mathematical derivation, with its intricate details, is reported in Appendix 5.B leading to

$$\begin{aligned} \text{Ke}_0 \left(0, \frac{\text{Pe}}{2} \right) \Big|_{\text{Pe} \ll 1} &= - \left[\ln \left(\frac{\text{Pe}}{4} \right) + \gamma \right] + \mathcal{O}(\text{Pe}^2), \\ \text{Ke}_n \left(0, \frac{\text{Pe}}{2} \right) \Big|_{\text{Pe} \ll 1} &= \left(\frac{\text{Pe}}{2} \right)^{-n} + \mathcal{O}(\text{Pe}^{-n+2}) \quad \forall n > 0, \\ \text{Ko}_n \left(0, \frac{\text{Pe}}{2} \right) \Big|_{\text{Pe} \ll 1} &= \left(\frac{\text{Pe}}{2} \right)^{-n} + \mathcal{O}(\text{Pe}^{-n+2}) \quad \forall n > 0. \end{aligned}$$

All discussed expansions are substituted into Eq.(5.16) to obtain the behavior of the integration constants $\tilde{C}_\ell$ in the limit of creeping groundwater flows:

$$\begin{aligned}\tilde{C}_0|_{\text{Pe} \ll 1} &= \frac{\tilde{q}}{2\pi} - \frac{\text{Pe}}{2} (\tilde{\Omega}_1^{(0)} + \tilde{\Omega}_{-1}^{(0)}) + \mathcal{O}(\text{Pe}^2), \\ \tilde{C}_{\pm 1}|_{\text{Pe} \ll 1} &= \text{Pe} \frac{\tilde{\Omega}_{\pm 1}^{(0)}}{2} + \mathcal{O}(\text{Pe}^2), \\ \tilde{C}_{\pm n}|_{\text{Pe} \ll 1} &= \mathcal{O}(\text{Pe}^{n+1}) \quad |n| > 1.\end{aligned}\tag{5.27}$$

Substitution of Eq. (5.27) into Eq. (5.26) provides the following expansion for the zeroth order outer solution in the limit of creeping groundwater flows in which only terms up to order Pe are retained:

$$\begin{aligned}\tilde{\Theta}_{\text{out}}^{(0)}|_{\text{Pe} \ll 1} &= \frac{\tilde{f}}{2\pi} e^{\frac{\text{Pe}}{2} \rho_0 \cos(\theta_0)} K_0 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) \\ &+ \text{Pe} \frac{\tilde{\Omega}_{+1}^{(0)}}{2} e^{\frac{\text{Pe}}{2} \rho_0 \cos(\theta_0)} K_1 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{+\iota \theta_0} \\ &+ \text{Pe} \frac{\tilde{\Omega}_{-1}^{(0)}}{2} e^{\frac{\text{Pe}}{2} \rho_0 \cos(\theta_0)} K_1 \left(\frac{\rho_0 \text{Pe}}{2} \sqrt{1 + \frac{4s}{\text{Pe}^2}} \right) e^{-\iota \theta_0} + \mathcal{O}(\text{Pe}^2).\end{aligned}\tag{5.28}$$

The quantity $\tilde{f}$ is the non-dimensional fictitious heat injection rate per unit borehole length previously introduced in Chapter 3. It subtracts from the true value $\tilde{q}$ the contributions of the azimuthal corrections:

$$\tilde{f}(s) = \tilde{q}(s) - \text{Pe} \pi (\tilde{\Omega}_{+1}^{(0)} + \tilde{\Omega}_{-1}^{(0)}).$$

As opposed to the inner solution discussed before, position s in the complex-valued Laplace plane does appear in the just-developed outer solution. Particularly, s appears within the argument of the modified Bessel functions of the second kind K_n , perfectly matching the literature [62, 63, 87]. This was to expect as the presence of the modified Bessel functions in the outer solution is due to the governing equation being solve and not from the matching process with the inner solution.

A comparison between the integration constants, Eq.(5.27), and results from Chapter 3 reveals, though, that the presence of s in $\tilde{C}_{\pm 1}$ is missing. This absence is due to the fact that $\tilde{C}_\ell$ result from the asymptotic matching with the inner region. And as discussed in Section 5.4.1, the irreconcilable differences in the matching processes performed in this chapter and in Chapter 3 make it impossible to fully recover the pre-existing models for creeping groundwater flows.

5.5 Conclusions

Most models in the literature as well as the model developed in the previous two chapters neglect heat convection near the borehole. That is, in the inner region. This approach provides good results for creeping groundwater flows but fails in many real-world situations in which heat convection is as relevant as heat conduction in the vicinity of the borehole. Hence, simplified, albeit accurate, theoretical models that account for the convection of heat even near the borehole are crucial to tackle those scenarios.

This chapter derives an analytical model that retains the convective phenomenon in both asymptotic regions. In particular, a closed-form expression is derived for the outer region, while a semi-analytical solution is developed for the inner region. Comparisons with detailed numerical simulations for a Peclet number equal to one show excellent performance of both inner and outer solutions, keeping absolute (relative) errors below 0.01 (4%) even downstream of the borehole.

Furthermore, the derived model demonstrates good performance, even for the unfavorable value of $Pe = 0.03$, within both the inner and outer regions. This is attributed to the mathematical formulation of the outer problem and is further reinforced when comparing its behavior for small Peclet numbers with models specifically designed for creeping groundwater flows.

This chapter promotes further research aimed at developing the pure analytical treatment of the inner region so that the questions left open at the end of Chapter 4 can be answered. Additionally, all models derived in this document serve to encourage the development of a unique asymptotic model valid for all Peclet numbers, encompassing even the extreme case of $Pe = 0.0$ that represents the purely-conduction case.

5.A Angular Mathieu functions

In the literature, the preferred form of expressing the angular Mathieu functions ce_n and se_n is as real Fourier series expansions [72, 80]:

$$\begin{aligned}
 ce_{2n}\left(\theta_0, -\frac{Pe^2}{4}\right) &= (-1)^n \sum_{\ell=0}^{\infty} (-1)^\ell A_{2\ell}^{(2n)} \cos(2\ell\theta_0), \\
 ce_{2n+1}\left(\theta_0, -\frac{Pe^2}{4}\right) &= (-1)^n \sum_{\ell=0}^{\infty} (-1)^\ell B_{2\ell+1}^{(2n+1)} \cos[(2\ell+1)\theta_0], \\
 se_{2n+1}\left(\theta_0, -\frac{Pe^2}{4}\right) &= (-1)^n \sum_{\ell=0}^{\infty} (-1)^\ell A_{2\ell+1}^{(2n+1)} \sin[(2\ell+1)\theta_0], \\
 se_{2n+2}\left(\theta_0, -\frac{Pe^2}{4}\right) &= (-1)^n \sum_{\ell=0}^{\infty} (-1)^\ell B_{2\ell+2}^{(2n+2)} \sin[(2\ell+2)\theta_0].
 \end{aligned} \tag{5.29}$$

The relationship of these functions with the Peclet number is established via the Fourier coefficients $A_\ell^{(n)}$ and $B_\ell^{(n)}$ whose values result from recurrence relations and a normalization criterion for the eigenfunctions. In the present work the normalization proposed by Frenkel [37] is adopted.

The chosen normalization along with the orthogonality property of the angular Mathieu functions leads to

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} \text{ce}_n\left(\theta_0, -\frac{\text{Pe}^2}{4}\right) \text{se}_m\left(\theta_0, -\frac{\text{Pe}^2}{4}\right) d\theta_0 &= 0 & \forall n, m \geq 0, \\ \frac{1}{\pi} \int_{-\pi}^{\pi} \text{ce}_n\left(\theta_0, -\frac{\text{Pe}^2}{4}\right) \text{ce}_m\left(\theta_0, -\frac{\text{Pe}^2}{4}\right) d\theta_0 &= \varepsilon_n \delta_{nm} & \forall n, m \geq 0, \\ \frac{1}{\pi} \int_{-\pi}^{\pi} \text{se}_n\left(\theta_0, -\frac{\text{Pe}^2}{4}\right) \text{se}_m\left(\theta_0, -\frac{\text{Pe}^2}{4}\right) d\theta_0 &= \delta_{nm} & \forall n, m > 0, \end{aligned}$$

where $\varepsilon_0 = 2$ and $\varepsilon_n = 1$ for all other values of n . Additionally, to streamline the expressions given in Eqs. (5.9) and (5.15) for the inner solution in the ground, the dummy angular Mathieu function se_0 is introduced and defined as

$$\text{se}_0\left(\theta_0, -\frac{\text{Pe}^2}{4}\right) = 0.$$

Despite the utility and widespread use of the previous expressions for the angular Mathieu functions, the present work introduces alternative expressions that resemble the structure of complex Fourier series expansions:

$$\text{ce}_n\left(\theta_0, -\frac{\text{Pe}^2}{4}\right) = \sum_{\ell=-\infty}^{\infty} c_{n\ell} e^{i\ell\theta_0} \quad \text{and} \quad \text{se}_n\left(\theta_0, -\frac{\text{Pe}^2}{4}\right) = \sum_{\ell=-\infty}^{\infty} s_{n\ell} e^{i\ell\theta_0}. \quad (5.30)$$

Comparing these expansions with the ones given before allows to relate the complex Fourier coefficients $c_{n\ell}$ and $s_{n\ell}$ with the real ones:

$$c_{n\ell} = \begin{cases} (-1)^{\frac{n+|\ell|}{2}} \frac{\varepsilon_n A_{|\ell|}^{(n)}}{2} & \text{for } n \text{ and } \ell \text{ even or } 0, \\ (-1)^{\frac{n+|\ell|-2}{2}} \frac{B_{|\ell|}^{(n)}}{2} & \text{for } n \text{ and } \ell \text{ odd,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.31)$$

$$s_{n\ell} = \begin{cases} \frac{\ell}{|\ell|} (-1)^{\frac{n+|\ell|}{2}} \frac{B_{|\ell|}^{(n)}}{2i} & \text{for } n \text{ and } \ell \text{ even } \neq 0, \\ \frac{\ell}{|\ell|} (-1)^{\frac{n+|\ell|-2}{2}} \frac{A_{|\ell|}^{(n)}}{2i} & \text{for } n \text{ and } \ell \text{ odd,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.32)$$

5.A.1 Limit of creeping groundwater flows

Angular Mathieu functions depend on the Peclet number of the groundwater flow through the Fourier coefficients $A_\ell^{(n)}$ and $B_\ell^{(n)}$, or alternatively, through $c_{n\ell}$ and $s_{n\ell}$. This subsection examines the behavior of the aforementioned Fourier coefficients to thereafter present the expansion of angular Mathieu functions in the limit of creeping groundwater flows.

Literature shows that angular Mathieu functions can be expanded into power series of their second parameter, $-\text{Pe}^2/4$ in the present case [37, 72, 80]. Therefore, Fourier coefficients $A_\ell^{(n)}$ and $B_\ell^{(n)}$ can be written, in first approximation, as

$$\begin{cases} A_\ell^{(n)}|_{\text{Pe} \ll 1} = B_\ell^{(n)}|_{\text{Pe} \ll 1} = 1 + \mathcal{O}(\text{Pe}^2) & \text{if } \ell = n, \\ A_\ell^{(n)}|_{\text{Pe} \ll 1} = B_\ell^{(n)}|_{\text{Pe} \ll 1} = \mathcal{O}(\text{Pe}^{2|n-\ell|}) & \text{if } \ell \neq n. \end{cases} \quad (5.33)$$

These expansions are substituted into Eqs. (5.31) and (5.32) to obtain the behavior of $c_{n\ell}$ and $s_{n\ell}$ in the limit of creeping groundwater flows:

$$c_{n\ell}|_{\text{Pe} \ll 1} = \frac{\varepsilon_n \delta_{n|\ell|}}{2} + \mathcal{O}(\text{Pe}^2) \quad \text{and} \quad s_{n\ell}|_{\text{Pe} \ll 1} = \frac{\ell}{|\ell|} \frac{\delta_{n|\ell|}}{2\ell} (1 - \delta_{|\ell|0}) + \mathcal{O}(\text{Pe}^2). \quad (5.34)$$

Finally, the first approximation of the angular Mathieu functions in the limit of small Peclet numbers is obtained by substituting the above results for $c_{n\ell}$ and $s_{n\ell}$ into Eq. (5.30), leading to

$$\begin{aligned} \text{ce}_n\left(\theta_0, -\frac{\text{Pe}^2}{4}\right)\bigg|_{\text{Pe} \ll 1} &= \frac{e^{i n \theta_0} + e^{-i n \theta_0}}{2} + \mathcal{O}(\text{Pe}^2) \\ \text{se}_n\left(\theta_0, -\frac{\text{Pe}^2}{4}\right)\bigg|_{\text{Pe} \ll 1} &= \frac{e^{i n \theta_0} - e^{-i n \theta_0}}{2i} + \mathcal{O}(\text{Pe}^2). \end{aligned} \quad (5.35)$$

It becomes evident that angular Mathieu functions ce_n and se_n tend to $\cos(n\theta_0)$ and $\sin(n\theta_0)$, respectively, for small values of their second argument, $-\text{Pe}^2/4$ in this case. This behavior is the reason why angular Mathieu functions are also known in the literature as elliptical cosines and sines.

5.B Modified Mathieu functions

Modified Mathieu functions can be expressed in different ways [72]. For convenience, in the present work they are written as series of cross-products of modified Bessel functions of the

first and second kind, I_n and K_n respectively [80]:

$$\begin{aligned}
 \text{Ke}_{2n} \left(\ln \rho_0, \frac{\text{Pe}}{2} \right) &= \frac{1}{\varepsilon_n} \sum_{\ell=0}^{\infty} \frac{A_{2\ell}^{(2n)}}{A_{2n}^{(2n)}} \left[I_{\ell-n} \left(\frac{\text{Pe}}{2} \frac{1}{\rho_0} \right) K_{\ell+n} \left(\frac{\text{Pe}}{2} \rho_0 \right) + I_{\ell+n} \left(\frac{\text{Pe}}{2} \frac{1}{\rho_0} \right) K_{\ell-n} \left(\frac{\text{Pe}}{2} \rho_0 \right) \right], \\
 \text{Ke}_{2n+1} \left(\ln \rho_0, \frac{\text{Pe}}{2} \right) &= \sum_{\ell=0}^{\infty} \frac{B_{2\ell+1}^{(2n+1)}}{B_{2n+1}^{(2n+1)}} \left[I_{\ell-n} \left(\frac{\text{Pe}}{2} \frac{1}{\rho_0} \right) K_{\ell+n+1} \left(\frac{\text{Pe}}{2} \rho_0 \right) - I_{\ell+n+1} \left(\frac{\text{Pe}}{2} \frac{1}{\rho_0} \right) K_{\ell-n} \left(\frac{\text{Pe}}{2} \rho_0 \right) \right], \\
 \text{Ko}_{2n+2} \left(\ln \rho_0, \frac{\text{Pe}}{2} \right) &= \sum_{\ell=0}^{\infty} \frac{B_{2\ell+2}^{(2n+2)}}{B_{2n+2}^{(2n+2)}} \left[I_{\ell-n} \left(\frac{\text{Pe}}{2} \frac{1}{\rho_0} \right) K_{\ell+n+2} \left(\frac{\text{Pe}}{2} \rho_0 \right) - I_{\ell+n+2} \left(\frac{\text{Pe}}{2} \frac{1}{\rho_0} \right) K_{\ell-n} \left(\frac{\text{Pe}}{2} \rho_0 \right) \right], \\
 \text{Ko}_{2n+1} \left(\ln \rho_0, \frac{\text{Pe}}{2} \right) &= \sum_{\ell=0}^{\infty} \frac{A_{2\ell+1}^{(2n+1)}}{A_{2n+1}^{(2n+1)}} \left[I_{\ell-n} \left(\frac{\text{Pe}}{2} \frac{1}{\rho_0} \right) K_{\ell+n+1} \left(\frac{\text{Pe}}{2} \rho_0 \right) + I_{\ell+n+1} \left(\frac{\text{Pe}}{2} \frac{1}{\rho_0} \right) K_{\ell-n} \left(\frac{\text{Pe}}{2} \rho_0 \right) \right],
 \end{aligned} \tag{5.36}$$

where $\varepsilon_0 = 2$ and $\varepsilon_n = 1$ for all other values of n . Additionally, to streamline the expressions given in Eqs. (5.9) and (5.15) for the inner solution in the ground, the dummy angular Mathieu function Ko_0 is introduced and defined as

$$\text{Ko}_0 \left(\ln \rho_0, \frac{\text{Pe}}{2} \right) = 0.$$

The following two subsections analyze the behavior of the modified Mathieu functions in two distinguished limits. Firstly, in the intermediate region where $\rho_0 \gg 1$, to facilitate the asymptotic matching of inner and outer solutions presented in Section 5.1.4. Secondly, in the limit of creeping groundwater flows for which $\text{Pe} \ll 1$, to enable the comparison performed in Section 5.4 between the developed asymptotic model and the state of the art.

5.B.1 Behavior in the intermediate region

Modified Mathieu functions are expressed as infinite series, with each term comprising two distinct contributions. Therefore, the present subsection derives simpler asymptotic expansions to facilitate the matching of inner and outer solutions. To do so, the potential existence of a dominant term within the series is analyzed by estimating the order of magnitude of each contribution.

All contributions in the series involve modified Bessel functions. For the modified Bessel functions of the first kind I_n , their argument $\text{Pe}/(2\rho_0)$ is small compared to unity in the intermediate region where $\rho_0 \gg 1$. This enables to identify I_0 as the dominant function among the modified Bessel functions of the first kind [80]. In contrast, at distances far from the borehole, the argument $\text{Pe}\rho/2$ of the modified Bessel function of the second kind K_n is large compared to unity. Given the behavior of these functions for large values of their argument [80], all modified Bessel functions of the second kind are equally relevant in the intermediate region.

The previous analysis indicates that, in the intermediate region, a dominant term exists in the infinite series given in Eq. (5.36) which corresponds to the contribution that includes the modified Bessel function of the first kind of order zero I_0 . In the particular case of the modified Mathieu function Ke_0 , both contributions in the term $n = \ell = 0$ involve I_0 so both must be retained. For all other modified Mathieu functions, the first contribution of the term $n = \ell$ dominates. Hence, in the intermediate region, modified Mathieu functions can be simplified as follows:

$$\begin{aligned} \text{Ke}_0\left(\ln \rho_0, \frac{\text{Pe}}{2}\right) \Big|_{\rho_0 \gg 1} &= K_0\left(\frac{\text{Pe}}{2} \rho_0\right) + \mathcal{O}\left(\frac{1}{\rho_0}\right), \\ \text{Ke}_m\left(\ln \rho_0, \frac{\text{Pe}}{2}\right) \Big|_{\rho_0 \gg 1} &= K_m\left(\frac{\text{Pe}}{2} \rho_0\right) + \mathcal{O}\left(\frac{1}{\rho_0^m}\right) \quad \forall m > 0, \\ \text{Ko}_m\left(\ln \rho_0, \frac{\text{Pe}}{2}\right) \Big|_{\rho_0 \gg 1} &= K_m\left(\frac{\text{Pe}}{2} \rho_0\right) + \mathcal{O}\left(\frac{1}{\rho_0^m}\right) \quad \forall m > 0. \end{aligned} \quad (5.37)$$

Finally, to facilitate the matching with the outer solution, these expressions are further simplified by expanding the modified Bessel functions of the second kind for large values of their argument:

$$\begin{aligned} \text{Ke}_m\left(\ln \rho_0, \frac{\text{Pe}}{2}\right) \Big|_{\rho_0 \gg 1} &= \frac{\sqrt{\pi}}{\sqrt{\rho_0 \text{Pe}}} e^{-\frac{\rho_0 \text{Pe}}{2}} \left[1 + \mathcal{O}\left(\frac{1}{\rho_0}\right)\right] \quad \forall m \geq 0, \\ \text{Ko}_m\left(\ln \rho_0, \frac{\text{Pe}}{2}\right) \Big|_{\rho_0 \gg 1} &= \frac{\sqrt{\pi}}{\sqrt{\rho_0 \text{Pe}}} e^{-\frac{\rho_0 \text{Pe}}{2}} \left[1 + \mathcal{O}\left(\frac{1}{\rho_0}\right)\right] \quad \forall m > 0. \end{aligned} \quad (5.38)$$

5.B.2 Limit of creeping groundwater flows

As explained before, modified Mathieu functions are given by cumbersome expressions. Therefore, this subsection aims to derive simplified asymptotic expansions to facilitate the comparisons with the state of the art performed in Section 5.4. Again, these simpler expressions are obtained by identifying the most relevant contribution within the series defining the modified Mathieu functions.

Each contribution to each term of the series involves modified Bessel functions of the first and second kind. Thus, the primary focus lies in examining the arguments of these functions, considering $\text{Pe} \ll 1$, as they govern all expansions shown next. In contrast to the previous subsection, in the limit of small Peclet numbers the argument of both kinds of modified Bessel functions, $\text{Pe}/(2\rho_0)$ and $\text{Pe}\rho_0/2$, are small compared to unity.

Bearing this in mind, two comparisons are carried out to identify the most relevant contribution within the series. For simplicity, these comparisons are exclusively made for the modified Mathieu functions Ke_{2n} as the discussion is analogous for the other modified Mathieu functions Ke_{2n+1} , Ko_{2n+1} , and Ko_{2n+2} .

The first comparison is carried out between the two contributions within each term of the series. For the trivial case of the modified Mathieu function of order zero, Ke_0 , both contributions are equal so both must be retained. For the remaining modified Mathieu functions Ke_{2n} with $n \neq 0$, an order of magnitude estimation of each contribution in each term of the series is accomplished:

$$\begin{aligned} \text{Ke}_{2n}\left(\ln \rho_0, \frac{\text{Pe}}{2}\right) &= \overbrace{\text{I}_0\left(\frac{\text{Pe}}{2} \frac{1}{\rho_0}\right) \text{K}_{2n}\left(\frac{\text{Pe}}{2} \rho_0\right)}^{\sim \left(\frac{1}{\rho_0 \text{Pe}}\right)^{2n}} + \overbrace{\text{I}_{2n}\left(\frac{\text{Pe}}{2} \frac{1}{\rho_0}\right) \text{K}_0\left(\frac{\text{Pe}}{2} \rho_0\right)}^{\sim \left(\frac{\text{Pe}}{\rho_0}\right)^{2n} \ln(\rho_0 \text{Pe})} \\ &+ \sum_{\substack{\ell=0 \\ \ell \neq n}}^{\infty} \frac{A_{2\ell}^{(2n)}}{A_{2n}^{(2n)}} \left[\underbrace{\text{I}_{\ell-n}\left(\frac{\text{Pe}}{2} \frac{1}{\rho_0}\right) \text{K}_{\ell+n}\left(\frac{\text{Pe}}{2} \rho_0\right)}_{\sim \left(\frac{\text{Pe}}{\rho_0}\right)^{|\ell-n|} \left(\frac{1}{\rho_0 \text{Pe}}\right)^{\ell+n}} + \underbrace{\text{I}_{\ell+n}\left(\frac{\text{Pe}}{2} \frac{1}{\rho_0}\right) \text{K}_{\ell-n}\left(\frac{\text{Pe}}{2} \rho_0\right)}_{\sim \left(\frac{\text{Pe}}{\rho_0}\right)^{\ell+n} \left(\frac{1}{\rho_0 \text{Pe}}\right)^{|\ell-n|}} \right]. \end{aligned} \quad (5.39)$$

The first two contributions of the previous equation correspond to the term $\ell = n$ of the series which is analyzed separately due to the logarithmic instead of potential behavior of the modified Bessel function of the second kind of order zero K_0 [80]. Among them, the second contribution is negligible compared to the first one in the limit of small Peclet numbers. For the remaining terms in the series, with $\ell \neq n$, a comparison of the order of magnitudes of the two contributions shows that the first one always dominates over the second one since ℓ and n are always positive integers:

$$\frac{\left(\frac{\text{Pe}}{\rho_0}\right)^{|\ell-n|} \left(\frac{1}{\rho_0 \text{Pe}}\right)^{\ell+n}}{\left(\frac{\text{Pe}}{\rho_0}\right)^{\ell+n} \left(\frac{1}{\rho_0 \text{Pe}}\right)^{|\ell-n|}} \sim \frac{\text{Pe}^{2|\ell-n|}}{\text{Pe}^{2(\ell+n)}} \gg 1. \quad (5.40)$$

These outcomes allow to write, in first approximation, the modified Mathieu functions of even order Ke_{2n} for small Peclet numbers as

$$\begin{aligned} \text{Ke}_0\left(\ln \rho_0, \frac{\text{Pe}}{2}\right) \Big|_{\text{Pe} \ll 1} &= \sum_{\ell=0}^{\infty} \frac{A_{2\ell}^{(0)}}{A_0^{(0)}} \text{I}_{\ell}\left(\frac{\text{Pe}}{2} \frac{1}{\rho_0}\right) \text{K}_{\ell}\left(\frac{\text{Pe}}{2} \rho_0\right), \\ \text{Ke}_{2n}\left(\ln \rho_0, \frac{\text{Pe}}{2}\right) \Big|_{\text{Pe} \ll 1} &= \sum_{\ell=0}^{\infty} \varepsilon_{\ell} \frac{A_{2\ell}^{(2n)}}{A_{2n}^{(2n)}} \text{I}_{\ell-n}\left(\frac{\text{Pe}}{2} \frac{1}{\rho_0}\right) \text{K}_{\ell+n}\left(\frac{\text{Pe}}{2} \rho_0\right) + \mathcal{O}\left(\text{Pe}^{2n}\right), \end{aligned} \quad (5.41)$$

where $\varepsilon_0 = 2$ and $\varepsilon_{\ell} = 1$ for all other values of n .

These expressions can be further simplified as one of the terms of the series always dominates. For convenience, the modified Mathieu functions of non-zero order are discussed first. That is, Ke_{2n} with $n \neq 0$. Taking into account the behavior of the real Fourier coefficients $A_{\ell}^{(n)}$ given in Appendix 5.A.1, the following order of magnitude results for each term of the series:

$$\text{Ke}_{2n}\left(\ln \rho_0, \frac{\text{Pe}}{2}\right) \Big|_{\text{Pe} \ll 1} = \underbrace{\sum_{\ell=0}^{\infty} \varepsilon_{\ell} \frac{A_{2\ell}^{(2n)}}{A_{2n}^{(2n)}} \text{I}_{\ell-n}\left(\frac{\text{Pe}}{2} \frac{1}{\rho_0}\right) \text{K}_{\ell+n}\left(\frac{\text{Pe}}{2} \rho_0\right)}_{\sim \text{Pe}^{2|\ell-n|} \text{Pe}^{|\ell-n|} \text{Pe}^{-|\ell+n|}}, \quad (5.42)$$

This comparison reveals that, for $n \neq 0$, the term corresponding to $\ell = n$ dominates in the series.

Next, the modified Mathieu function of order zero, Ke_0 , is analyzed. Since the modified Bessel function of the second kind of order zero, K_0 , behaves different, the order of magnitude of the term $\ell = 0$ of the series is treated apart, revealing that for creeping groundwater flows the dominant terms of the series is the one corresponding to $\ell = 0$:

$$\text{Ke}_0 \left(\ln \rho_0, \frac{\text{Pe}}{2} \right) \Big|_{\text{Pe} \ll 1} = \underbrace{\text{I}_0 \left(\frac{\text{Pe}}{2} \frac{1}{\rho_0} \right) \text{K}_0 \left(\frac{\text{Pe}}{2} \rho_0 \right)}_{\sim \ln(\rho_0 \text{Pe})} + \underbrace{\sum_{\ell=1}^{\infty} \frac{A_{2\ell}^{(0)}}{A_0^{(0)}} \text{I}_\ell \left(\frac{\text{Pe}}{2} \rho_0 \right) \text{K}_\ell \left(\frac{\text{Pe}}{2} \rho_0 \right)}_{\sim \text{Pe}^{2\ell}}. \quad (5.43)$$

An equivalent analysis is carried out for the other modified Mathieu functions Ke_{2n+1} , Ko_{2n+1} , and Ko_{2n+2} . All results can be condensed into the following two expressions:

$$\begin{aligned} \text{Ke}_m \left(\ln \rho_0, \frac{\text{Pe}}{2} \right) \Big|_{\text{Pe} \ll 1} &= \text{K}_m \left(\frac{\text{Pe}}{2} \rho_0 \right) + \mathcal{O}(\text{Pe}^2) \quad \forall m \geq 0, \\ \text{Ko}_m \left(\ln \rho_0, \frac{\text{Pe}}{2} \right) \Big|_{\text{Pe} \ll 1} &= \text{K}_m \left(\frac{\text{Pe}}{2} \rho_0 \right) + \mathcal{O}(\text{Pe}^2) \quad \forall m > 0, \end{aligned} \quad (5.44)$$

The zeroth order solution to the inner region presents ratios of modified Mathieu functions evaluated at different radial positions. Using the obtained approximations for small Peclet numbers, these ratios can be simplified as follows:

$$\begin{aligned} \frac{\text{Ke}_0 \left(\ln \rho_0, \frac{\text{Pe}}{2} \right)}{\text{Ke}_0 \left(0, \frac{\text{Pe}}{2} \right)} \Big|_{\text{Pe} \ll 1} &= \frac{\text{K}_0 \left(\frac{\text{Pe}}{2} \rho_0 \right)}{\text{K}_0 \left(\frac{\text{Pe}}{2} \right)} + \mathcal{O}(\text{Pe}^2) = 1 + \frac{\ln \rho_0}{\ln \left(\frac{\text{Pe}}{4} \right) + \gamma} + \mathcal{O}(\text{Pe}^2), \\ \frac{\text{Ke}_m \left(\ln \rho_0, \frac{\text{Pe}}{2} \right)}{\text{Ke}_m \left(0, \frac{\text{Pe}}{2} \right)} \Big|_{\text{Pe} \ll 1} &= \frac{\text{K}_m \left(\frac{\text{Pe}}{2} \rho_0 \right)}{\text{K}_m \left(\frac{\text{Pe}}{2} \right)} + \mathcal{O}(\text{Pe}^2) = \frac{1}{\rho_0^{|m|}} + \mathcal{O}(\text{Pe}^2) \quad \forall n > 0, \\ \frac{\text{Ko}_m \left(\ln \rho_0, \frac{\text{Pe}}{2} \right)}{\text{Ko}_m \left(0, \frac{\text{Pe}}{2} \right)} \Big|_{\text{Pe} \ll 1} &= \frac{\text{K}_m \left(\frac{\text{Pe}}{2} \rho_0 \right)}{\text{K}_m \left(\frac{\text{Pe}}{2} \right)} + \mathcal{O}(\text{Pe}^2) = \frac{1}{\rho_0^{|m|}} + \mathcal{O}(\text{Pe}^2) \quad \forall n > 0. \end{aligned} \quad (5.45)$$

References

- [1] Borja Badenes et al. “Development of Advanced Materials guided by numerical simulations to improve performance and cost-efficiency of Borehole Heat Exchangers (BHEs)”. In: *Energy* 201 (June 2020), p. 117628. DOI: <https://doi.org/10.1016/j.energy.2020.117628>.
- [2] George K. Batchelor. *An introduction to fluid dynamics*. 2nd. Cambridge: Cambridge University Press, 2000.
- [3] Jacob Bear. *Dynamics of fluids in porous media*. Courier Corporation, 2013.
- [4] Adrian Bejan. *Convection Heat Transfer*. 3rd. Hoboken, NJ: Wiley, 2004.
- [5] Johan Bennet, Johan Claesson, and Göran Hellström. *Multipole method to compute the conductive heat flows to and between pipes in a composite cylinder*. Tech. rep. Lund, Sweden: Department of Building Technology and Mathematical Physics, Lund Institute of Technology, 1987.
- [6] M.T. Boudjelkha and Muhammad Aslam Chaudhry. “On the approximation of a generalized incomplete gamma function arising in heat conduction problems”. In: *Journal of Mathematical Analysis and Applications* 248.2 (Aug. 2000), pp. 509–519. DOI: [10.1006/jmaa.2000.6925](https://doi.org/10.1006/jmaa.2000.6925).
- [7] William E Boyce, Richard C. DiPrima, and Douglas B. Meade. *Elementary Differential Equations and Boundary Value Problems*. 11th. Hoboken, NJ: Wiley, 2017.
- [8] Shanshan Cai et al. “An analytical full-scale model to predict thermal response in boreholes with groundwater advection”. In: *Applied Thermal Engineering* 168 (Mar. 2020), p. 114828. DOI: [10.1016/j.applthermaleng.2019.114828](https://doi.org/10.1016/j.applthermaleng.2019.114828).
- [9] Antonio Capozza, Michele De Carli, and Angelo Zarrella. “Investigations on the influence of aquifers on the ground temperature in ground-source heat pump operation”. In: *Applied Energy* 107 (July 2013), pp. 350–363. DOI: [10.1016/j.apenergy.2013.02.043](https://doi.org/10.1016/j.apenergy.2013.02.043).
- [10] H.S. Carslaw and J.C. Jaeger. *Conduction of Heat in Solids*. 2nd. Oxford: Oxford University Press, 1959.
- [11] Muhammad Aslam Chaudhry, N.M. Temme, and E.J.M. Velting. “Asymptotics and closed form of a generalized incomplete gamma function”. In: *Journal of Computational and Applied Mathematics* 67.2 (Mar. 1996), pp. 371–379. DOI: [10.1016/0377-0427\(95\)00018-6](https://doi.org/10.1016/0377-0427(95)00018-6).
- [12] Muhammad Aslam Chaudhry and Syed M Zubair. “Generalized incomplete gamma functions with applications”. In: *Journal of Computational and Applied Mathematics* 55.1 (Oct. 1994), pp. 99–123. DOI: [10.1016/0377-0427\(94\)90187-2](https://doi.org/10.1016/0377-0427(94)90187-2).

- [13] Muhammad Aslam Chaudhry and Syed M. Zubair. *On a class of incomplete gamma functions with applications*. New York: Chapman and Hall/CRC, 2001.
- [14] Shangyuan Chen, Jinfeng Mao, and Xu Han. “Heat transfer analysis of a vertical ground heat exchanger using numerical simulation and multiple regression model”. In: *Energy and Buildings* 129 (Oct. 2016), pp. 81–91. DOI: [10.1016/j.enbuild.2016.07.010](https://doi.org/10.1016/j.enbuild.2016.07.010).
- [15] P. Cheng. “Mixed convection about a horizontal cylinder and sphere in a fluid-saturated porous medium”. In: *International Journal of Heat and Mass Transfer* 25.8 (Aug. 1982), pp. 1245–1246. DOI: [10.1016/0017-9310\(82\)90219-8](https://doi.org/10.1016/0017-9310(82)90219-8).
- [16] Genady P Cherepanov. “Two-dimensional convective heat/mass transfer for low Prandtl and any Peclet numbers”. In: *SIAM Journal on Applied Mathematics* 58.3 (1998), pp. 942–960. DOI: [10.1137/S0036139995282311](https://doi.org/10.1137/S0036139995282311).
- [17] Andrew Chiasson and Amanda O’Connell. “New analytical solution for sizing vertical borehole ground heat exchangers in environments with significant groundwater flow: Parameter estimation from thermal response test data”. In: *HVAC&R Research* 17.6 (Dec. 2011), pp. 1000–1011. DOI: [10.1080/10789669.2011.609926](https://doi.org/10.1080/10789669.2011.609926).
- [18] Andrew D. Chiasson, Simon J. Rees, and Jeffrey D. Spitler. “A Preliminary Assessment of the Effects of Groundwater Flow on Closed-Loop Ground-Source Heat Pump Systems”. In: *ASHRAE Transactions* 106.1 (Jan. 2000), pp. 380–393.
- [19] Johan Claesson and Johan Bennet. *Multipole method to compute the conductive heat flows to and between pipes in a cylinder*. Tech. rep. Lund, Sweden: Department of Building Technology and Mathematical Physics, Lund Institute of Technology, 1987.
- [20] Johan Claesson and Göran Hellström. “Analytical Studies of the Influence of Regional Groundwater Flow by on the Performance of Borehole Heat Exchangers”. In: *8th International Conference on Thermal Energy Storage*. 2000.
- [21] Johan Claesson and Göran Hellström. “Multipole method to calculate borehole thermal resistances in a borehole heat exchanger”. In: *HVAC&R Research* 17.6 (Dec. 2011), pp. 895–911. DOI: [10.1080/10789669.2011.609927](https://doi.org/10.1080/10789669.2011.609927).
- [22] Comsol Inc. *COMSOL Multiphysics Version 5.4. Reference Manual*. 2018. URL: <https://www.comsol.com/>.
- [23] Paolo Conti, Daniele Testi, and Walter Grassi. “Transient forced convection from an infinite cylindrical heat source in a saturated Darcian porous medium”. In: *International Journal of Heat and Mass Transfer* 117 (2018), pp. 154–166. DOI: [10.1016/j.ijheatmasstransfer.2017.10.012](https://doi.org/10.1016/j.ijheatmasstransfer.2017.10.012).
- [24] Ping Cui et al. “Advances in ground heat exchangers for space heating and cooling: Review and perspectives”. In: *Energy and Built Environment* 5.2 (Apr. 2024), pp. 255–269. DOI: [10.1016/j.enbenv.2022.10.002](https://doi.org/10.1016/j.enbenv.2022.10.002).
- [25] E Custodio and MR Llamas. “Hidrología subterránea. Barcelona, España”. In: *Omega* 1 (2001), p. 1157.
- [26] N.R. Diao, H.Y. Zeng, and Z.H. Fang. “Improvement in Modeling of Heat Transfer in Vertical Ground Heat Exchangers”. In: *HVAC&R Research* 10.4 (Oct. 2004), pp. 459–470. DOI: [10.1080/10789669.2004.10391114](https://doi.org/10.1080/10789669.2004.10391114).
- [27] Nairen Diao, Qinyun Li, and Zhaohong Fang. “Heat transfer in ground heat exchangers with groundwater advection”. In: *International Journal of Thermal Sciences* 43.12 (Dec. 2004), pp. 1203–1211.

-
- [28] Stephan Düber, Martin Ziegler, and Raul Fuentes. “Development and validation of a computationally efficient hybrid model for temporal high-resolution simulations of geothermal bore fields”. In: *International Journal for Numerical and Analytical Methods in Geomechanics* 46.14 (Oct. 2022), pp. 2792–2813. DOI: [10.1002/nag.3427](https://doi.org/10.1002/nag.3427).
 - [29] Selçuk Erol and Bertrand François. “Multilayer analytical model for vertical ground heat exchanger with groundwater flow”. In: *Geothermics* 71 (Jan. 2018), pp. 294–305. DOI: [10.1016/j.geothermics.2017.09.008](https://doi.org/10.1016/j.geothermics.2017.09.008).
 - [30] Selçuk Erol, Mir Amid Hashemi, and Bertrand François. “Analytical solution of discontinuous heat extraction for sustainability and recovery aspects of borehole heat exchangers”. In: *International Journal of Thermal Sciences* 88 (Feb. 2015), pp. 47–58. DOI: [10.1016/j.ijthermalsci.2014.09.007](https://doi.org/10.1016/j.ijthermalsci.2014.09.007).
 - [31] Per Eskilson. “Thermal Analysis of Heat Extraction Boreholes”. PhD thesis. Lund, Sweden: Department of Mathematical Physics, Lund Institute of Technology, 1987.
 - [32] Per Eskilson and Johan Claesson. “Simulation Model for Thermally Interacting Heat Extraction Boreholes”. In: *Numerical Heat Transfer* 13.2 (1988), pp. 149–165. DOI: [10.1080/10407788808913609](https://doi.org/10.1080/10407788808913609).
 - [33] European Union. *Directive (EU) 2018/2001 of the European Parliament and of the Council of 11 December 2018 on the promotion of the use of energy from renewable sources*. Brussels, Belgium, Dec. 2018.
 - [34] European Union. *Directive (EU) 2018/2002 of the European Parliament and of the Council of 11 December 2018 amending Directive 2012/27 on energy efficiency*. Brussels, Belgium, Dec. 2018.
 - [35] European Union. *Directive (EU) 2018/844 of the European Parliament and of the Council of 30 May 2018 amending Directive 2010/31/EU on the energy performance of buildings and Directive 2012/27/EU on energy efficiency*. Brussels, Belgium, May 2018.
 - [36] Rui Fan et al. “A study on the performance of a geothermal heat exchanger under coupled heat conduction and groundwater advection”. In: *Energy* 32 (2017), pp. 2199–2209. DOI: [10.1016/j.energy.2007.05.001](https://doi.org/10.1016/j.energy.2007.05.001).
 - [37] D Frenkel and R Portugal. “Algebraic methods to compute Mathieu functions”. In: *Journal of Physics A: Mathematical and General* 34.17 (2001), p. 3541. DOI: [10.1088/0305-4470/34/17/302](https://doi.org/10.1088/0305-4470/34/17/302).
 - [38] Volker Gnielinski. “Turbulent Heat Transfer in Annular Spaces – A New Comprehensive Correlation”. In: *Heat Transfer Engineering* 36.9 (June 2015), pp. 787–789. DOI: [10.1080/01457632.2015.962953](https://doi.org/10.1080/01457632.2015.962953).
 - [39] Y. Guo et al. “Considering buried depth in the moving finite line source model for vertical borehole heat exchangers — A new solution”. In: *Energy and Buildings* 214 (May 2020), p. 109859. DOI: [10.1016/j.enbuild.2020.109859](https://doi.org/10.1016/j.enbuild.2020.109859).
 - [40] Frank E. Harris. “Incomplete Bessel, generalized incomplete gamma, or leaky aquifer functions”. In: *Journal of Computational and Applied Mathematics* 215.1 (May 2008), pp. 260–269. DOI: [10.1016/j.cam.2007.04.008](https://doi.org/10.1016/j.cam.2007.04.008).
 - [41] Jozsef Hecht-Méndez et al. “Evaluating MT3DMS for Heat Transport Simulation of Closed Geothermal Systems”. In: *GROUND WATER* 48.5 (Sept. 2010), pp. 741–756. DOI: [10.1111/j.1745-6584.2010.00678.x](https://doi.org/10.1111/j.1745-6584.2010.00678.x).

- [42] Jozsef Hecht-Méndez et al. “Optimization of energy extraction for vertical closed-loop geothermal systems considering groundwater flow”. In: *Energy Conversion and Management* 66 (Feb. 2013), pp. 1–10. DOI: [10.1016/j.enconman.2012.09.019](https://doi.org/10.1016/j.enconman.2012.09.019).
- [43] Göran Hellström. “Ground Heat Storage: Thermal Analyses of Duct Storage Systems, I. Theory”. PhD thesis. Lund, Sweden: Department of Mathematical Physics, Lund Institute of Technology, 1991.
- [44] Miguel Hermanns. “Fast Inverse Laplace Transform for the Unsteady Thermal Response of Geothermal Heat Exchangers”. In: *AIP Conference Proceedings* 2186.1 (Dec. 2019), p. 170002. DOI: [10.1063/1.5138081](https://doi.org/10.1063/1.5138081).
- [45] Miguel Hermanns. “New generation of theoretical models for the thermal response of geothermal heat exchangers”. In: *IOP Conference Series: Earth and Environmental Science* 410 (Jan. 2020), p. 012042. DOI: [10.1088/1755-1315/410/1/012042](https://doi.org/10.1088/1755-1315/410/1/012042).
- [46] Miguel Hermanns. “Modelling the Thermal Response of Geothermal Heat Exchangers Using Asymptotic Expansion Techniques”. In: *Proceedings of the World Geothermal Congress 2020+1*. Reykjavik, Iceland, 2021.
- [47] Miguel Hermanns and Santiago Ibáñez. “Thermal response of slender geothermal boreholes to subannual harmonic excitations”. In: *SIAM Journal on Applied Mathematics* 79.1 (Jan. 2019), pp. 230–256. DOI: [10.1137/17M1161324](https://doi.org/10.1137/17M1161324).
- [48] Miguel Hermanns and Santiago Ibáñez. “Harmonic thermal response of thermally interacting geothermal boreholes”. In: *SIAM Journal on Applied Mathematics* 80.1 (Jan. 2020), pp. 262–288. DOI: [10.1137/18M119001X](https://doi.org/10.1137/18M119001X).
- [49] Miguel Hermanns and José Miguel Pérez. “Asymptotic Analysis of Vertical Geothermal Boreholes in the Limit of Slowly Varying Heat Injection Rates”. In: *SIAM Journal on Applied Mathematics* 74.1 (Jan. 2014), pp. 60–82. DOI: [10.1137/130930170](https://doi.org/10.1137/130930170).
- [50] Miguel Hermanns and Juan Manuel Rivero. “On the symmetry properties of the network of thermal resistances representing the thermal response of slender geothermal boreholes”. In: *Geothermics* 94 (July 2021), p. 102078. DOI: [10.1016/j.geothermics.2021.102078](https://doi.org/10.1016/j.geothermics.2021.102078).
- [51] Miguel Hermanns and Juan Manuel Rivero. “On the quasi-steady limit of the enhanced multipole method for the thermal response of geothermal boreholes”. In: *Applied Thermal Engineering* 225 (May 2023), p. 120121. DOI: [10.1016/j.applthermaleng.2023.120121](https://doi.org/10.1016/j.applthermaleng.2023.120121).
- [52] Jinzhong Hu. “An improved analytical model for vertical borehole ground heat exchanger with multiple-layer substrates and groundwater flow”. In: *Applied Energy* 202 (Sept. 2017), pp. 537–549. DOI: [10.1016/j.apenergy.2017.05.152](https://doi.org/10.1016/j.apenergy.2017.05.152).
- [53] Santiago Ibáñez and Miguel Hermanns. “On the steady-state thermal response of slender geothermal boreholes”. In: *SIAM Journal on Applied Mathematics* 78.3 (June 2018), pp. 1658–1681. DOI: [10.1137/17M1122566](https://doi.org/10.1137/17M1122566).
- [54] Leonard Rose Ingersoll, Otto Julius Zabel, and Alfred Cajori Ingersoll. *Heat Conduction - With Engineering, Geological, and other Applications*. Madison, WI: University of Wisconsin Press, 1954.
- [55] International Energy Agency. *Renewables for Heating and Cooling: Untapped Potential*. Paris, France, July 2007.
- [56] International Energy Agency. *Renewables 2019: market analysis and forecast from 2019 to 2024*. Paris, France, 2019.

-
- [57] G.S. Jia et al. “Influence of groundwater flow on the ground heat exchanger performance and ground temperature distributions: A comprehensive review of analytical, numerical and experimental studies”. In: *Geothermics* 100 (2022), p. 102342. DOI: [10.1016/j.geothermics.2021.102342](https://doi.org/10.1016/j.geothermics.2021.102342).
 - [58] A.V. Kashevarov. “Exact solution of the problem of convective heat exchange for a circular cylinder at small Prandtl numbers”. In: *Fluid dynamics* 29.1 (1994), pp. 33–37. DOI: [10.1007/BF02330618](https://doi.org/10.1007/BF02330618).
 - [59] A.V. Kashevarov. “Heat transfer under conditions of plane potential flow past obstacles”. In: *High Temperature* 38.5 (2000), pp. 748–752. DOI: [10.1007/BF02755928](https://doi.org/10.1007/BF02755928).
 - [60] Takao Katsura et al. “Erratum to "Method for calculation of ground temperature in scenarios involving multiple ground heat exchangers and groundwater advection" [Energy Build. 220 (2020) 110000]”. In: *Energy and Buildings* 227 (Nov. 2020), p. 110409. DOI: [10.1016/j.enbuild.2020.110409](https://doi.org/10.1016/j.enbuild.2020.110409).
 - [61] Takao Katsura et al. “Method for calculation of ground temperature in scenario involving multiple ground heat exchangers considering groundwater advection”. In: *Energy and Buildings* 220 (Aug. 2020), p. 110000. DOI: [10.1016/j.enbuild.2020.110000](https://doi.org/10.1016/j.enbuild.2020.110000).
 - [62] Rafid Al-Khoury et al. “A spectral model for a moving cylindrical heat source in a conductive-convective domain”. In: *International Journal of Heat and Mass Transfer* 163 (Dec. 2020), p. 120517. DOI: [10.1016/j.ijheatmasstransfer.2020.120517](https://doi.org/10.1016/j.ijheatmasstransfer.2020.120517).
 - [63] Rafid Al-Khoury et al. “Analytical model for arbitrarily configured neighboring shallow geothermal installations in the presence of groundwater flow”. In: *Geothermics* 93 (June 2021), p. 102063. DOI: [10.1016/j.geothermics.2021.102063](https://doi.org/10.1016/j.geothermics.2021.102063).
 - [64] Paco A. Lagerstrom. *Matched Asymptotic Expansions: Ideas and Techniques*. New York: Springer, 1988.
 - [65] Louis Lamarche, Stanislaw Kajl, and Benoit Beauchamp. “A review of methods to evaluate borehole thermal resistances in geothermal heat-pump systems”. In: *Geothermics* 39.2 (June 2010), pp. 187–200. DOI: [10.1016/j.geothermics.2010.03.003](https://doi.org/10.1016/j.geothermics.2010.03.003).
 - [66] Michael Lanahan and Paulo Cesar Tabares-Velasco. “Seasonal thermal-energy storage: A critical review on BTES systems, modeling, and system design for higher system efficiency”. In: *Energies* 10.6 (2017), p. 743. DOI: [10.3390/en10060743](https://doi.org/10.3390/en10060743).
 - [67] Biao Li et al. “Study on the influence of the design method of the ground source heat pump system with considering groundwater seepage”. In: *Applied Thermal Engineering* 160 (Sept. 2019), p. 114016. DOI: [10.1016/j.applthermaleng.2019.114016](https://doi.org/10.1016/j.applthermaleng.2019.114016).
 - [68] Min Li and Alvin C K Lai. “Review of analytical models for heat transfer by vertical ground heat exchangers (GHEs): A perspective of time and space scales”. In: *Applied Energy* 151 (Aug. 2015), pp. 178–191. DOI: [10.1016/j.apenergy.2015.04.070](https://doi.org/10.1016/j.apenergy.2015.04.070).
 - [69] Underground Energy LLC. *Underground Energy*. URL: <https://underground-energy.com/our-technology/btes/#what-is-btes>.
 - [70] Yang Lou et al. “Numerical research on thermal response for geothermal energy pile groups under groundwater flow”. In: *Geomechanics for Energy and the Environment* 28 (Dec. 2021), p. 100257. DOI: [10.1016/j.gete.2021.100257](https://doi.org/10.1016/j.gete.2021.100257).
 - [71] Malin Malmberg et al. “High temperature borehole thermal energy storage-A case study”. In: (2018). DOI: [10.22488/okstate.18.000036](https://doi.org/10.22488/okstate.18.000036).

- [72] Norman William McLachlan. “Theory and application of Mathieu functions”. In: *Publisher to the University Geoffrey Cumberlege, Oxford University Press* (1951).
- [73] T. Metzger, S. Didierjean, and D. Maillat. “Optimal experimental estimation of thermal dispersion coefficients in porous media”. In: *International Journal of Heat and Mass Transfer* 47 (2004), pp. 3341–3353. DOI: [10.1016/j.ijheatmasstransfer.2004.02.024](https://doi.org/10.1016/j.ijheatmasstransfer.2004.02.024).
- [74] Nelson Molina-Giraldo et al. “A moving finite line source model to simulate borehole heat exchangers with groundwater advection”. In: *International Journal of Thermal Sciences* 50.12 (Dec. 2011), pp. 2506–2513. DOI: [10.1016/j.ijthermalsci.2011.06.012](https://doi.org/10.1016/j.ijthermalsci.2011.06.012).
- [75] Michael J Moran et al. *Fundamentals of engineering thermodynamics*. John Wiley & Sons, 2010.
- [76] Katsuori Nagano, Takao Katsura, and Sayaka Takeda. “Development of a design and performance prediction tool for the ground source heat pump system”. In: *Applied Thermal Engineering* 26.14–15 (Oct. 2006), pp. 1578–1592. DOI: [10.1016/j.applthermaleng.2005.12.003](https://doi.org/10.1016/j.applthermaleng.2005.12.003).
- [77] Yujin Nam, Ryoza Ooka, and Suckho Hwang. “Development of a numerical model to predict heat exchange rates for a ground-source heat pump system”. In: *Energy and Buildings* 40.12 (Oct. 2008), pp. 2133–2140. DOI: [10.1016/j.enbuild.2008.06.004](https://doi.org/10.1016/j.enbuild.2008.06.004).
- [78] Donald A. Nield and Adrian Bejan. *Convection in porous media*. 5th. Cham: Springer, 2017.
- [79] Yuichi Niibori et al. “Design of the BHP System Considering the Heat Transport of Groundwater Flow”. In: *Proceedings of the World Geothermal Congress 2005*. Antalya, Turkey, 2005.
- [80] Frank W.J. Olver et al., eds. *NIST Handbook of Mathematical Functions*. New York: Cambridge University Press, 2010.
- [81] Peter J Olver et al. *Introduction to partial differential equations*. Vol. 1. Springer, 2014.
- [82] I Pop and B Yan. “Forced convection flow past a circular cylinder and a sphere in a darcian fluid at large peclet numbers”. In: *International communications in heat and mass transfer* 25.2 (1998), pp. 261–267. DOI: [10.1016/S0735-1933\(98\)00013-X](https://doi.org/10.1016/S0735-1933(98)00013-X).
- [83] J. Raymond et al. “Numerical analysis of thermal response tests with a groundwater flow and heat transfer model”. In: *Renewable Energy* 36.1 (Jan. 2011), pp. 315–324. DOI: [10.1016/j.renene.2010.06.044](https://doi.org/10.1016/j.renene.2010.06.044).
- [84] Javier Rico and Miguel Hermanns. “Modeling the thermal interaction of geothermal boreholes with aquifers using asymptotic expansion techniques”. In: *IOP Conference Series: Earth and Environmental Science* 588.5 (Nov. 2020), p. 052033. DOI: [10.1088/1755-1315/588/5/052033](https://doi.org/10.1088/1755-1315/588/5/052033).
- [85] Javier Rico and Miguel Hermanns. “Computation of generalized incomplete gamma functions for parameter values spanning several orders of magnitude”. In: *In preparation for Journal of Computational and Applied Mathematics* (2024).
- [86] Javier Rico and Miguel Hermanns. “On the network of thermal resistances embodying the response of geothermal boreholes to creeping groundwater flows”. In: *In preparation for International Journal of Thermal Sciences* (2024).

-
- [87] Javier Rico and Miguel Hermanns. “Thermal interaction of slender geothermal boreholes with creeping groundwater flows”. In: *Applied Thermal Engineering* 236.C (Jan. 2024), p. 121626. DOI: [10.1016/j.applthermaleng.2023.121626](https://doi.org/10.1016/j.applthermaleng.2023.121626).
 - [88] Javier Rico and Miguel Hermanns. “Thermal interaction of slender geothermal boreholes with groundwater flows”. In: *In preparation for Journal of Fluid Mechanics* (2024).
 - [89] Javier Rico and Miguel Hermanns. “Thermal interaction of slender geothermal boreholes with strong groundwater flows”. In: *In preparation for Applied Thermal Engineering* (2024).
 - [90] Jaime A. Rivera, Philipp Blum, and Peter Bayer. “Ground energy balance for borehole heat exchangers: Vertical fluxes, groundwater and storage”. In: *Renewable Energy* 83 (Nov. 2015), pp. 1341–1351. DOI: [10.1016/j.renene.2015.05.051](https://doi.org/10.1016/j.renene.2015.05.051).
 - [91] Jaime A. Rivera, Philipp Blum, and Peter Bayer. “A finite line source model with Cauchy-type top boundary conditions for simulating near surface effects on borehole heat exchangers”. In: *Energy* 98 (Mar. 2016), pp. 50–63. DOI: [10.1016/j.energy.2015.12.129](https://doi.org/10.1016/j.energy.2015.12.129).
 - [92] Jaime A. Rivera, Philipp Blum, and Peter Bayer. “Influence of spatially variable ground heat flux on closed-loop geothermal systems: Line source model with nonhomogeneous Cauchy-type top boundary conditions”. In: *Applied Energy* 180 (Oct. 2016), pp. 572–585. DOI: [10.1016/j.apenergy.2016.06.074](https://doi.org/10.1016/j.apenergy.2016.06.074).
 - [93] Juan Manuel Rivero and Miguel Hermanns. “Enhanced multipole method for the transient thermal response of slender geothermal boreholes”. In: *International Journal of Thermal Sciences* 164 (June 2021), p. 106531. DOI: [10.1016/j.ijthermalsci.2020.106531](https://doi.org/10.1016/j.ijthermalsci.2020.106531).
 - [94] Takao Sano. “Unsteady heat transfer from a circular cylinder immersed in a Darcy flow”. In: *Journal of Engineering Mathematics* 14.3 (1980), pp. 177–190. DOI: [10.1007/BF00039314](https://doi.org/10.1007/BF00039314).
 - [95] Mostafa H. Sharqawy, Esmail M. Mokheimer, and Hassan M. Badr. “Effective pipe-to-borehole thermal resistance for vertical ground heat exchangers”. In: *Geothermics* 38.2 (June 2009), pp. 271–277. DOI: [10.1016/j.geothermics.2009.02.001](https://doi.org/10.1016/j.geothermics.2009.02.001).
 - [96] Helge Skarphagen et al. “Design considerations for borehole thermal energy storage (BTES): A review with emphasis on convective heat transfer”. In: *Geofluids* 2019 (2019). DOI: [10.1155/2019/4961781](https://doi.org/10.1155/2019/4961781).
 - [97] Matthew G. Sutton, Darin W. Nutter, and Rick J. Couvillion. “A ground resistance for vertical bore heat exchangers with groundwater flow”. In: *Journal of Energy Resources Technology* 125.3 (Sept. 2003), pp. 183–189. DOI: [doi.org/10.1115/1.1591203](https://doi.org/doi.org/10.1115/1.1591203).
 - [98] Nico M. Temme. “The leaky aquifer function revisited”. In: *International Journal of Quantum Chemistry* 109.13 (Apr. 2009), pp. 2826–2830. DOI: [10.1002/qua.22130](https://doi.org/10.1002/qua.22130).
 - [99] David Keith Todd and Larry W Mays. *Groundwater hydrology*. John Wiley & Sons, 2004.
 - [100] S Tomotika and H Yosinobu. “On the Convection of Heat from Cylinders Immersed in a Low-Speed Stream of Incompressible Fluid”. In: *Journal of Mathematics and Physics* 36.1-4 (1957), pp. 112–120. DOI: [10.1002/sapm1957361112](https://doi.org/10.1002/sapm1957361112).
 - [101] Maxime Tye-Gingras and Louis Gosselin. “Generic ground response functions for ground exchangers in the presence of groundwater flow”. In: *Renewable Energy* 72 (Dec. 2014), pp. 354–366. DOI: [10.1016/j.renene.2014.07.026](https://doi.org/10.1016/j.renene.2014.07.026).

- [102] Milton Van Dyke. *Perturbation Methods in Fluid Mechanics*. Annotated. Stanford: The Parabolic Press, 1975.
- [103] Massimo Verdoya and Paolo Chiozzi. “Influence of groundwater flow on the estimation of subsurface thermal parameters”. In: *International Journal of Earth Sciences* 107 (Sept. 2018), pp. 137–144. DOI: [10.1007/s00531-016-1397-x](https://doi.org/10.1007/s00531-016-1397-x).
- [104] Massimo Verdoya et al. “Thermophysical parameters from laboratory measurements and in-situ tests in borehole heat exchangers”. In: *Applied Thermal Engineering* 144 (Nov. 2018), pp. 711–720. DOI: [10.1016/j.applthermaleng.2018.08.039](https://doi.org/10.1016/j.applthermaleng.2018.08.039).
- [105] Visual Numerics. *IMSL MATH/LIBRARY FORTRAN Subroutines for Mathematical Applications*. Houston, 1991.
- [106] Changlong Wang et al. “A semi-analytical heat transfer model for deep borehole heat exchanger considering groundwater seepage”. In: *International Journal of Thermal Sciences* 175 (May 2022), p. 107465. DOI: [10.1016/j.ijthermalsci.2022.107465](https://doi.org/10.1016/j.ijthermalsci.2022.107465).
- [107] Zhang Wenke et al. “The annual fluctuation of underground temperature response caused by ground heat exchanger in the condition of groundwater seepage”. In: *Energy and Buildings* 186 (Mar. 2019), pp. 37–45. DOI: [10.1016/j.enbuild.2019.01.004](https://doi.org/10.1016/j.enbuild.2019.01.004).
- [108] Shanjie Zhang and Jianming Jin. *Computation of Special Functions*. New York: John Wiley & Sons Inc., 1996.
- [109] Zilong Zhao et al. “Assessing impacts of groundwater on geothermal heat exchangers: A review of methodology and modeling”. In: *Renewable Energy* 190 (May 2022), pp. 121–147. DOI: [10.1016/j.renene.2022.03.089](https://doi.org/10.1016/j.renene.2022.03.089).