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Spectral methods to approximate control and inverse problems for the wave equation and elasticity system

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Abstract

In this Thesis, we explore the use of spectral methods to approximate numerically some control and inverse problems for the wave equation and the elasticity system.

Roughly speaking, in a control problem one is interested in acting on the system (though a control) in order to drive an initial state into a desired final state after some prescribed time interval. Of course the main interest (and difficulty) arises when the control acts in a small part of the domain or the boundary. Here we focus on boundary controllability, i.e. the particular case where the control acts in one part of the boundary, since it is the case with more practical applications.

After the seminal work by Lions, 1988a it is well-known that boundary controllability of the wave equation can be reduced to a suitable observability inequality for the solutions of the adjoint uncontrolled wave equation. Roughly speaking, such observability inequalities give an estimate of global information, typically the energy of the solution, from local one obtained from the solution on the part of the boundary where the control acts, and for sufficiently large time. They are obtained by specific techniques such as the multipliers method, Carleman inequalities, moment problems, etc.

To obtain a numerical approximation of controls a natural procedure is as follows: first, replace the control problem by a discrete one where the controlled system is approximated by a suitable discretization of the continous one. This discrete approximation will depend on a small discretization parameter h. Then, solve the discrete control problem and, finally, prove the convergence of the discrete control to a control of the continuous system, as the discretization parameter h goes to zero.

It is important to note that a convergent numerical scheme for the continuous system does not guarantee the convergence of the discrete controls. In fact, the existence of discrete controls requires a discrete observability inequality for the discrete adjoint system, while the convergence will require the uniformity of this constant with respect to the discretization parameter h.

It turns out that this uniformity is not true for the classical discretization of the wave equation (finite difference or finite elements) and in fact, the discrete controls obtained by this

natural procedure constitute an unbounded sequence, as h goes to zero. This difficulty has given rise to an active research topic in the last 30 years and a number of different techniques have been studied to overcome this difficulty: Tychonoff regularization, bigrid techniques, space-time finite elements, perturbation, etc.

Here we focus on the spectral collocation method to discretize the wave equation. In contrast with other methods, the main advantage here is the high order accuracy that gives good approximation with a not-too-fine discretization, when the data are smooth. This is important in higher dimensions or vector problems like the elasticity system. On the other hand, the main drawback is that the method is mainly restricted to the rectangle domain due to the difficulty of imposing boundary conditions when considering polynomial approximation in separated variables.

In this Thesis, we find a method to obtain a discrete version of the boundary observability inequality which is uniform with respect to the discretization parameter. The method is applied to the wave model and the elasticity system. This allows us to find convergent numerical approximations of the associated control problems and some related inverse source problems. This is particularly significant as it overcomes the limitations encountered in classical discretization techniques which fail to achieve such uniformity. By harnessing the high-order accuracy inherent in spectral collocation methods, our approach enables efficient approximation of controls and solutions to inverse source problems. Furthermore, we demonstrate the applicability of our method across various dimensions and vectorial systems, exemplifying its versatility and effectiveness in practical settings.

Resumen

En esta tesis, exploramos el uso de métodos espectrales para aproximar numéricamente algunos problemas de control y problemas inversos para la ecuación de ondas y el sistema de elasticidad.

En términos generales, un problema de control consiste en actuar sobre el sistema a través de una fuerza externa (conocida como control) para llevar un estado inicial a un estado final deseado en un tiempo fijado de antemano. Por supuesto, el principal interés, y dificultad, surge cuando el control actúa en una pequeña parte del dominio o de la frontera. En este trabajo nos centramos en el control en la frontera ya que es el caso con más aplicaciones prácticas.

A partir del trabajo seminal de Lions, 1988a, es bien conocido que la controlabilidad desde la frontera de la ecuación de ondas puede reducirse a una desigualdad de observabilidad adecuada para la ecuación de ondas adjunta, sin control. Tales desigualdades de observabilidad estiman cantidades globales de las soluciones, típicamente la energía, a partir de información local obtenida en la parte de la frontera donde actúa el control durante un tiempo suficientemente largo. Se obtienen mediante técnicas específicas como el método de los multiplicadores, desigualdades de Carleman, problemas de momentos, etc.

Para obtener una aproximación numérica de los controles un procedimiento natural es el siguiente: en primer lugar, se sustituye el problema de control por uno discreto en el que la ecuación de ondas se reemplaza por una discretización convergente adecuada dependiendo de un parámetro h pequeño. A continuación, se resuelve el problema de control discreto y finalmente, se prueba la convergencia del control discreto a uno de los controles del problema continuo cuando h tiende a cero.

La existencia de controles para el problema discreto requiere una desigualdad de observabilidad discreta para el sistema adjunto discreto, mientras que la convergencia va a requerir la uniformidad de esta constante con respecto al parámetro de discretización h. Curiosamente esta uniformidad no es cierta para la discretización clásica de la ecuación de ondas (diferencias finitas o elementos finitos). Este ha sido un tema de investigación activo en los últimos 30 años y se han analizado varias técnicas diferentes para superar esta dificultad: regularización de Tychonoff, elementos finitos mixtos, técnicas multimalla, elementos finitos espacio-temporales,

etc.

En esta Tesis nos centramos en el método de colocación espectral para discretizar el problema de control las ecuaciones de ondas y el sistema de la elasticidad lineal. La principal ventaja de este método, en contraste con otros, es la alta precisión. Esto proporciona una buena aproximación con una discretización no demasiado fina, cuando los datos son regulares. Esto es especialmente relevante en dimensiones 2 y 3 o problemas vectoriales, como el sistema de elasticidad. Por otro lado, la principal desventaja es que el método está principalmente restringido a dominios rectangulares debido a la dificultad de imponer condiciones de contorno en dominios generales cuando se considera una aproximación polinomial obtenida por separación de variables.

Como principal contribución, encontramos un método para obtener una versión discreta de la desigualdad de observabilidad que es uniforme con respecto al parámetro de discretización. El método se aplica tanto a la ecuación de ondas, como al sistema de elasticidad. Esto nos permite encontrar aproximaciones numéricas convergentes de problemas de control asociados a estas ecuaciones. Finalmente, aplicamos el resultado a la resolución de algunos problemas inversos cuya estabilidad se deduce también de desigualdades de observabilidad.

Résumé

Dans cette thèse, nous explorons l'utilisation de méthodes spectrales pour approximer numériquement certains problèmes de contrôle et problèmes inverses pour l'équation des ondes et le système d'élasticité.

En général, un problème de contrôle consiste à agir sur le système à travers une force externe (appelée contrôle) pour amener un état initial à un état final souhaité dans un temps prédéterminé. Bien sûr, l'intérêt principal, et la difficulté, survient lorsque le contrôle agit sur une petite partie du domaine ou de la frontière. Dans ce travail, nous nous concentrons sur le contrôle à la frontière car c'est le cas le plus courant dans les applications pratiques.

À partir du travail séminaire de Lions, 1988a, il est bien connu que la contrôlabilité frontière de l'équation des ondes peut être réduite à une inégalité d'observabilité appropriée pour l'équation des ondes adjointe, sans contrôle. De telles inégalités d'observabilité estiment des quantités globales des solutions, typiquement l'énergie, à partir d'informations locales obtenues sur la partie de la frontière où agit le contrôle pendant un temps suffisamment long. Elles sont obtenues grâce à des techniques spécifiques telles que la méthode des multiplicateurs, les inégalités de Carleman, les problèmes de moments, etc.

Pour obtenir une approximation numérique des contrôles, une procédure naturelle est la suivante: d'abord, on remplace le problème de contrôle par un problème discret où l'équation des ondes est remplacée par une discrétisation convergente appropriée dépendant d'un paramètre h petit. Ensuite, on résout le problème de contrôle discret et enfin, on teste la convergence du contrôle discret vers un contrôle du problème continu lorsque h tend vers zéro.

L'existence de contrôles pour le problème discret nécessite une inégalité d'observabilité discrète pour le système adjoint discret, tandis que la convergence nécessitera l'uniformité de cette constante par rapport au paramètre de discrétisation h. Curieusement, cette uniformité n'est pas vraie pour la discrétisation classique de l'équation des ondes (différences finies ou éléments finis). Cela a été un sujet de recherche actif au cours des 30 dernières années et plusieurs techniques différentes ont été analysées pour surmonter cette difficulté : régularisation de Tychonoff, éléments finis mixtes, techniques multi-maillages, éléments finis espace-temps, etc.

Dans cette thèse, nous nous concentrons sur la méthode de placement spectral pour discrétiser le problème de contrôle des équations des ondes et du système d'élasticité linéaire. L'avantage principal de cette méthode, contrairement aux autres, est sa haute précision. Cela fournit une bonne approximation avec une discrétisation pas trop fine, lorsque les données sont régulières. Cela est particulièrement pertinent en dimensions 2 et 3 ou pour les problèmes vectoriels, comme le système d'élasticité. D'autre part, le principal inconvénient est que la méthode est principalement restreinte aux domaines rectangulaires en raison de la difficulté d'imposer des conditions aux limites dans des domaines généraux lorsqu'on considère une approximation polynomiale obtenue par séparation de variables.

Comme principale contribution, nous avons trouvé une méthode pour obtenir une version discrète de l'inégalité d'observabilité qui est uniforme par rapport au paramètre de discrétisation. La méthode est appliquée à la fois à l'équation des ondes et au système d'élasticité. Cela nous permet de trouver des approximations numériques convergentes des problèmes de contrôle associés à ces équations. Enfin, nous appliquons le résultat à la résolution de certains problèmes inverses dont la stabilité est également déduite des inégalités d'observabilité.

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Abbreviations and acronyms

HUM	Hilbert Uniquesse Method
ISP	Inverse Source Problem
ODE	Ordinary differential equation
PDE	Partial differential equations

Chapter 1

Introduction

An Overview of the Introduction

In this introduction, we begin by delving into the historical aspects surrounding the concepts of controllability in ordinary differential equations (ODE) and partial differential equations (PDE). These important concepts lay the groundwork for understanding the subsequent sections of the thesis. Following the historical context, we provide an exploration of controllability and observability, elucidating their significance in analyzing and controlling dynamical systems. Then we transition to the thesis's core focus, where we indicate the numerical approximation of control for PDEs using various numerical techniques and the natural strategy followed for finding a discrete control. Then, we introduce the concept of spectral collocation methods as a powerful tool for approximating control, particularly within the context of the wave equation. Finally, we provide an overview of the thesis structure and outline the key themes and methodologies explored in each chapter.

Historical aspects

Throughout history, humans have always been in constant search for laws and principles that govern natural forms or phenomena, with the purpose of explaining or understanding the behavior of nature. Thus, the Frenchman Pierre Louis Moreau de Maupertuis (1698-1759) formulated in 1744 the Principle of Least Action, by which it is established that:

"In any change that occurs in Nature, the amount of necessary action must be minimized."

This is a physical principle that mathematics has rigorously substantiated. In particular, the necessary techniques have been developed to address a broad class of optimization problems.

Later, in the late 17th century, works on the pendulum by Ch. Huygens and R. Hooke emerged, in which there was an attempt to measure time accurately. In these works, we can clearly observe elements that are now part of what we call Control Theory.

J.C. Maxwell's 1868 analysis marked the first rigorous examination of mechanisms associated with system control, using tools from the qualitative theory of ordinary differential equations (ODEs). The period from 1960 onward is identified as Classical Control Theory, characterized by efforts to model real-world complexity, incorporating nonlinear and nondeterministic behavior. Key contributors such as R. Bellman and R. Kalman played pivotal roles in shaping the field, with D.E. Russel and J.L. Lions influenced the development of Control Theory through their work in partial differential equations (PDEs), numerical analysis, and industrial applications.

Recent attention has increasingly focused on controlling differential equations and systems, particularly nonlinear ODEs and PDEs. This research has provided invaluable insights across diverse scientific fields, including Physics, Biology, Economics, Medicine, and Engineering (Intriligator, 1980, Levine, 1999, Boyle and Caviness, 1988, Palm, 1975). Despite significant progress, challenges and unanswered questions persist, fueling ongoing research efforts in Control Theory see for example (Fabre, 1996, Fernández-Cara and Zuazua Iriondo, 2003, Fernández-Cara and Zuazua, 2000, A. V. Fursikov, 1999, A. Fursikov, 1996, Emanuilov, 1995, Lions and Zuazua, 2017, Murray, 2003, Zuazua, 1991).

In tackling controllability problems, a critical distinction arises between finite-dimensional systems modelled by ODEs and infinite-dimensional distributed systems described by PDEs. This modeling issue is pivotal in practice, given the divergent properties of finite-dimensional and infinite-dimensional systems from a control theoretical standpoint see Castro, 1999. Resolving these challenges hinges on understanding system characteristics such as temporal reversibility, state regularity, and the structure of admissible control (Fernández-Cara and Guerrero, 2006, Komornik, 1994, Lasiecka and Triggiani, 2000, Zuazua, 1995, Zuazua, 1998).

The controllability of PDEs has been the subject of intense research for more than 40 years.

Russell, 1978 conducted a fairly comprehensive study of the most relevant results available in the literature at that time where described a series of different tools that were developed to address controllability problems, often inspired and related to other topics close to PDEs: the method of multipliers, problems and methods of moments, non-harmonic Fourier series, etc. Shortly after, J.L. Lions introduced the so-called Hilbert Uniqueness Method (H.U.M) see Lions, 1988a, which allowed connecting the concepts of controllability, observability, unique continuation, etc.

Controllability and observability for PDE

In this section, we delve into the fundamental concepts of controllability and observability for PDEs. In order to describe the standard controllability problem, let us consider X be a Hilbert space endowed with the norm $\|\cdot\|_X$. Let $A : \mathcal{D}(A) \subset X \to X$ be an unbounded skew-adjoint operator, i.e. $A^* = -A$, with domain $\mathcal{D}(A) \subset X$ that is the generator of a strongly continuous group in X. Consider the system,

$$\begin{cases} z_t - Az = Bv, \quad t \in (0,T), \\ z(0) = z_0, \end{cases} \tag{1.1}$$

where $B \in \mathcal{L}(Y, X_{-1})$, Y is a Hilbert space, X_{-1} is the completion of X with respect to the norm $\|\cdot\|_{-1} = \|A^{-1}\cdot\|_X$, $v(t) \in Y$ and $z_0 \in \mathcal{D}(A)$ is the initial data. Here z is usually referred as the state (identifying the physical properties of the system) and v is the control, determining the action we take on the system, on a time interval $(0,T)$.

Roughly speaking, in a control problem one is interested in acting on the system (though a control) in order to drive an initial state to a desired final state after some time (see Figure 1.1). Of course, the main interest (and difficulty) arises when the control acts in a small part of the domain or the boundary (see Figure 1.2).

We can formulate various controllability problems:

- **Exact controllability:** For each z_0, find v such that $z(T)$ is equal to a given data $z_{obj}(T)$.
- **Null controllability:** For each z_0, find v such that the corresponding function z verifies

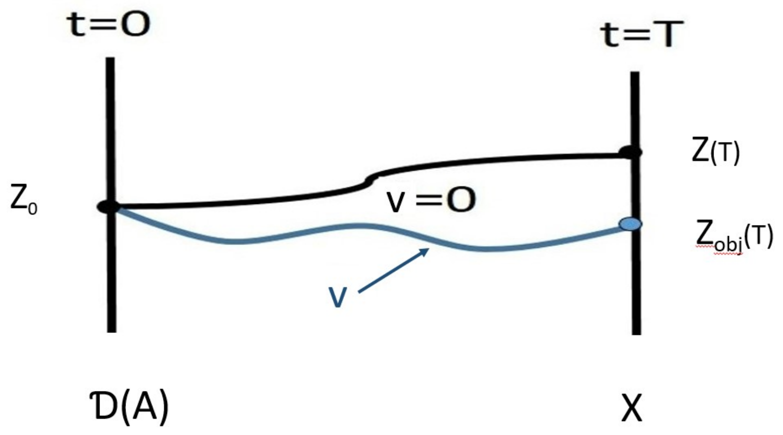
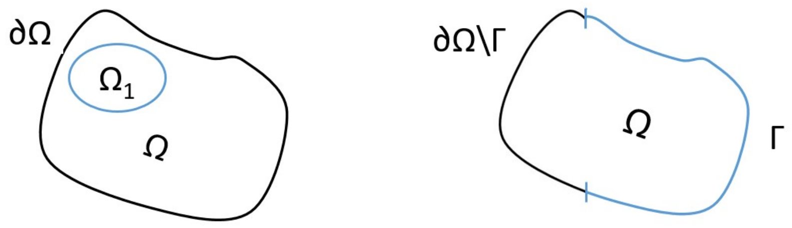


Figure 1.1: The effect of control on the behaviour of final data to reach the objective z_{obj}.



Internal control act on Ω_1. Boundary control act on Γ.

Figure 1.2: The types of controls.

$z(T) = 0$. In this linear reversible setting this is equivalent to the exact controllability.

- **Approximate controllability:** For each z_0, each z_d, and each $\delta > 0$, find v such that $|z(T) - z_d| \leq \delta$.
- **Exact controllability to trajectories:** For each z_0 and any free trajectory z^* of the system, i.e., a solution of the equation

$$z_t^* - Az^* = 0, \quad z^*(0) = z_0^*,$$

for some z_0^*, find v such that $z(T) = z^*(T)$.

Note that exact controllability to trajectories is a very useful property from the applications point of view: if we can find a control such that $z(T) = z^*(T)$, after some time, we can remove the control and let the system "follow" the trajectory autonomously.

For each system of the form (1.1), these problems lead to several interesting questions, such as those related to the existence of v in different forms (continuous, bang-bang, sparse, positive, etc.) or with different properties related with the optimization of its norm or its support.

In the following, we shall focus on the null controllability and we shall refer to it simply as controllability. The existence of an exact control v can be obtained from a suitable variational characterization, which is the heart of the duality methods we shall use in this work, the so-called H.U.M introduced by Lions, 1988a. To characterize this control first, let us consider the adjoint system:

$$\begin{cases} \varphi_t - A\varphi = 0, \qquad t \in (0,T), \\ \varphi(T) = \varphi_T. \end{cases} \tag{1.2}$$

Note that system (1.2) is backwards in time. Now, multiplying (1.1) by φ and integrating it on $(0,T)$, one immediately gets that for all $\varphi_T \in X$,

$$0 = \int_0^T \langle v, B^*\varphi \rangle_Y \, dt + \langle z_0, \varphi(0) \rangle_X \, ,$$

where B^* is the adjoint operator of B. Hence v is a control function for (1.1) if and only if for all φ_T

$$0 = \int_0^T \langle v, B^*\varphi \rangle_Y \, dt + \langle z_0, \varphi(0) \rangle_X \, . \tag{1.3}$$

This is known as a variational characterization of controls.

In instances where a system is controllable and according to HUM, the control strategy can be constructed by minimizing a relevant quadratic coercive function defined within the class of solutions of the adjoint system $J : X \to \mathbb{R}$ defined by,

$$J(\varphi_T) = \frac{1}{2} \int_0^T |B^*\varphi(t)|_Y^2 dt + \langle z_0, \varphi(0) \rangle_X \, , \tag{1.4}$$

where φ is the solution of (1.2) with final data φ_T at time $t = T$. Note that, if $\hat{\varphi}_T \in X$ is a minimizer for J, then the optimality condition $DJ(\hat{\varphi}_T) = 0$ provides the control

$$v = B^* \hat{\varphi}. \tag{1.5}$$

It is important to note that the control, when it exists, is not unique in general. Among the set of controls, a natural choice is to consider the minimal L^2-norm control which is usually unique. This control is the one given by (1.5).

The existence of a minimizer for the quadratic function J (and therefore control for system (1.1)) is guaranteed as soon as we prove its coercivity. This is a consequence of the following observability inequality:

$$|\varphi(0)|_X^2 \leq C_{obs} \int_0^T |B^* \varphi|_Y^2 dt, \tag{1.6}$$

where $C_{obs} = C_{obs}(T)$ is a positive constant and φ is a solution of (1.2) with final data φ_T.

The proof of observability inequality can be approached through various methods, including Fourier series (with extensions to non-harmonic Fourier series, as discussed in Young, 1981), multipliers (explored in works by Komornik, 1989, Lions, 1988a, 1988b, Ho, 1986, and Osses, 2001) which we shall use in this work, also sidewise energy estimates Zuazua, 1993, Carleman inequalities (as proved by Zhang, 2000), and microlocal tools (employed in studies by Bardos et al., 1992 and Burq and Gérard, 1997).

Numerical approximation of controls

Significant advancements have been made recently in the numerical approximation of controllability issues for PDEs using various numerical techniques like (Finite Difference Methods, Finite Element Methods, Model Predictive Control, Mesh-Free Methods, Mixed formulation, etc,...) see for example (Burman et al., 2021, Burman et al., 2023, Ervedoza and Zuazua, 2012, Font and Periago, 2010, Glowinski et al., 1989 and Munch and Montaner, 2018). The choice of method depends on the specific requirements and characteristics of the control problem at hand. Researchers and engineers often select the most suitable numerical approach based on the nature of the system, available computational resources, and desired accuracy.

To approximate the control v in (1.1) with a finite difference or finite element method the natural strategy consists in:

I. Introduce a finite dimensional problem approximating of (1.1), depending on a discretization parameter $h \to 0$, typically related with the mesh size. This is usually obtained by a natural discretization of the continuous problem.

II. Prove the controllability of the discrete system for each $h > 0$.

III. Construct a uniform bounded sequence of discrete controls v^h that converges to a continuous control v.

In this way, one introduces finite dimensional sub-spaces $X^h \subset X, Y^h \subset Y$ and we approximate the continuous model (1.1) by a sequence of finite-dimensional system:

$$\begin{cases} z_t^h - A^h z^h = B^h v^h, \quad t \in (0,T), \\ z^h(0) = z_0^h, \end{cases} \tag{1.7}$$

where $z_0^h \in X^h$, $v^h \in Y^h$ and (A^h, B^h) is a sequence of finite-dimensional approximations of the operators (A, B), respectively, with $A^h \in \mathcal{L}(X^h, X^h)$ and $B^h \in \mathcal{L}(Y^h, X^h)$.

As in the infinite dimensional model, the existence of controls can be derived from the following discrete observability inequality,

$$|\varphi^h(0)|_{X^h}^2 \leq M_{obs} \int_0^T |B^{h^*} \varphi^h|_{Y^h}^2 dt, \tag{1.8}$$

where $M_{obs} = M_{obs}(T) > 0$ is a constant and φ^h is the solution of

$$\begin{cases} \varphi_t^h - A^h \varphi^h = 0, \quad t \in (0,T), \\ \varphi^h(T) = \varphi_T^h, \end{cases} \tag{1.9}$$

with final data ${\varphi^h}_T \in X^h$.

An important issue here is to obtain a constant M_{obs} in (1.8) uniform in $h \to 0$, since this guarantees the existence of a bounded sequence of controls v^h. But this is not enough to prove the convergence of discrete controls to a control of the infinite dimensional system. To derive this convergence, the following hypotheses are required (see Ervedoza and Zuazua, 2012):

- **H1 : Uniform observability**. The observability inequality (1.8) holds with a constant M_{obs} that does not depend on h.

- **H2 : Consistency.** For $\varphi_T \in \mathcal{D}(A)$, there exists a sequence of $\varphi_T^h \in X^h$ such that,

  $$\varphi_T^h \to \varphi_T \text{ in } X \quad \text{as } h \to 0.$$

  Furthermore, if φ is the solution of (1.2) with final data φ_T and φ_T^h is the solution of (1.9) with final data φ_T^h the following holds:

  $$\begin{aligned} \varphi^h &\to \varphi \text{ in } C^0([0,T];\mathcal{D}(A)) \cap C^1([0,T];X) \quad \text{as } h \to 0, \\ B^{h^*}\varphi^h &\to B^*\varphi \text{ in } L^2((0,T);Y) \quad \text{as } h \to 0. \end{aligned}$$

Under these hypotheses, a convergence result is obtained in Ervedoza and Zuazua, 2012.

It turns out that these hypotheses are not satisfied for most of the numerical schemes used to approximate the wave equation. For instance, when considering finite elements or finite differences approaches the discrete observability inequality above is not uniform with respect to h. In this context, the usual convergence hypotheses (consistency + stability) for the discrete control problem are not sufficient to ensure the convergence of discrete controls to the limit one. In the context of the wave equation, this has been extensively analyzed in terms of the artificial dispersion phenomenon that arises in the discrete models. Somehow, high-frequency oscillations interact with the numerical mesh and alter the infinite dimension dynamics (Trefethen, 1982, Vichnevetsky and Bowles, 1982). Specifically, when the wavelength of solutions is of the order of the size of the mesh and the latter tends to zero, the numerical wave propagation velocity, or more precisely, the so-called group velocity, may converge to zero because of this nonphysical interaction of waves with the discrete medium. This fact leads to the possibility that when the mesh size is reduced, the time required to uniformly (with respect to the mesh size) watch (or control) the numerical waves from the observation zone may tend to infinity. As the mesh size goes to zero, this is what causes the control and observation properties of the majority of numerical approximation methods to behave in an unstable manner. This was observed forthe first time by Glowinski, 1992, Glowinski et al., 1990, and analyzed in detail later on by other authors (see Infante and Zuazua, 1999, Zuazua, 1999). Since then, several techniques have been proposed to recover convergence approximations of the controls, such as bigrid algorithms, Tychonoff regularization, filtering, mixed finite elements, etc. (see, for example, Glowinski et al., 1989, Glowinski, 1992, Glowinski et al., 1990, Glowinski and LI, 1990, Glowinski and Lions, 1995). We also highlight more contemporary methods, such as those based on a space-time or mixed formulations that do not require regularizations (Burman et al., 2021, Munch and Montaner, 2018) or those where controls are created by minimizing a cost function that penalizes both the control and the state Cîndea et al., 2013.

Spectral collocation method

In this thesis we focus on a group of numerical approaches known as spectral methods. The main idea consists in representing the solution as a weighted sum of basis functions, which are normally selected to be orthogonal or nearly orthogonal. The reader is referred to Gottlieb and Lustman, 1983, Canuto et al., 1988, Ciarlet and Lions, 1997 for background information and details on this method. Spectral methods are distinguished by,

1. The type of method (Galerkin, collocation, or tau). In collocation, the basis functions are assessed at predetermined locations (referred to as collocation points) within the domain to approximate the solution. In the Galerkin approach, a weighted residual equation is built using the basis functions.
2. The specific selection of the basis functions. Trigonometric polynomials, Chebyshev polynomials, and Legendre polynomials are the most often used trial functions.

Here we focus on the polynomial spectral collocation technique, which appears to have been initially applied by Slater, 1934 and Fihtengol'c and Kantorovic, 1934. The technique was later developed as a versatile method for solving ordinary differential equations by Jones et al., 1937, utilizing a random distribution of collocation points and various trial functions. Lanczos, 1938 contributed significantly by demonstrating the importance of an appropriate distribution of collocation points and trial functions for solution accuracy, laying the foundation for the orthogonal collocation method.

The main advantages of the spectral methods are:

- **High Accuracy:** Spectral methods provide high accuracy in approximating solutions to differential equations. They can achieve exponential convergence rates, meaning that the error decreases rapidly with an increasing number of basis functions.
- **Global Approximation:** Unlike finite difference or finite element methods that use local approximations, spectral methods use global basis functions (e.g., Chebyshev, Legendre polynomial) to represent the solution over the entire domain. This allows for an accurate representation of functions with sharp gradients or singularities. These methods are particularly effective when the solution can be well-represented by a low-degree polynomial.
- **Rate of Convergence:** It is typically exponential for smooth solutions. However, it is influenced by the regularity of the solution, the choice of basis functions, and the presence of singularities or discontinuities.

On the other hand, we can indicate some drawbacks of these methods:

- **Limited Applicability to Irregular Geometries:** Spectral methods are typically applied to regular domains with simple geometries. Extending them to irregular domains or domains with complex boundaries can be challenging due to the difficulty to impose boundary conditions with polynomial basis functions, for example.
- **Ill-Conditioning:** The matrices arising from spectral methods are often ill-conditioned, which may lead to numerical instabilities. They usually require preconditioning and a careful consideration in the implementation.

Spectral collocation to approximate control problems

In this section, we delineate the discrete control problem within the framework of the spectral collocation method.

The resulting discrete system can be written in the general form (1.7) but now the discretization parameter h is a natural number $N = 1/h$ which is the maximum degree of the polynomial approximation. On the other hand, the numerical solutions of (1.7) and (1.9) are regarded as a smooth global polynomial of degree N denote $\mathbb{P}_N$ (typically quite large).

The use of spectral methods to approximate the control of the wave equation has been previously investigated by Boulmezaoud and Urquiza, 2007 where, instead of collocation, a Galerkin spectral method is considered. The proposed approximation is not uniformly controllable. However, a bounded sequence of controls is obtained when trying to control the projection of the solution in a suitable low-frequency space, similar to the result obtained for finite differences in Infante and Zuazua, 1999. From a practical point of view, this is not satisfactory since it requires an accurate representation of the eigenfunctions associated with the discrete problem, something which is not available in general.

The novelties in this Thesis are the following:

- **Uniform observability:** The method we present here to obtain the uniform observability inequality is new and considers, instead of the discrete collocation system, the equivalent continuous error equation associated to the polynomial approximation (see Gottlieb and Lustman, 1983). This error equation is the same continuous equation but with a non-homogeneous second-hand term, known as the error term. Therefore, the observability inequality can be derived using the same techniques as in the continuous model and we only have to estimate this extra error term. This is an important advantage of the method since it can be easily extended to more general equations (elasticity, fluid dynamics, etc.) and higher dimensions, as long as we consider rectangular domains.
- **Numerical approximation of control:** We are able to prove the uniform controllability and find a numerical approximation of the boundary control for the wave equation and the elasticity system. The result relies on the uniform observability inequality for the associated discrete adjoint system which allows us to obtain the uniform boundedness of discrete controls.
- **Convergence of controls for wave equation:** for the wave equation, we are able to prove a convergent sequence of controls as $N \to \infty$, by adding an extra discrete boundary control that vanishes as $N \to \infty$. This provides an accurate approximation of the continuous control. The result relies on a detailed spectral analysis of the discrete low frequencies which is used to obtain the convergence of the discrete control to the continuous one, such convergence results are more difficult to prove for the elasticity system, we just have found numerical evidence.

- **Application to an inverse source problem for the wave equation:** As a further application we consider a related inverse source problem for the wave equation which consists in recovering the non-homogeneous term in the wave equation from boundary data of a single solution, usually referred as observation. For such inverse problems, it is well-known that the stability can be obtained from a suitable boundary observability inequality. To approximate numerically such inverse problems we introduce a spectral method. We give a numerical algorithm to approximate the source term from the observation, when the associated discrete observability inequality is uniform with respect to the discretization parameter. As we analyzed in the previous chapters this is the case for the spectral collocation method.

An Overview of the Thesis

The aim of this thesis is to implement the spectral collocation method to approximate the boundary controls for both the wave and the elasticity equations in a particular square domain with a Dirichlet control acting on two adjacent edges. We focus on the space discretization of the control problem. A complete numerical approximation would also require a suitable time discretization and this could obviously affect the convergence of the fully discrete controls (both in space in time). This is an interesting question that we do not address here. Let us simply mention that we did not observe any singular phenomenon associated to time discretization in the numerical experiments. In particular, our work is divided into four chapters.

First chapter: we indicate the main results of boundary controllability, uniform observability and convergence for the discrete 1-d wave equation given by the spectral collocation methods. According to HUM, we prove that a particular discrete control can be obtained as the minimizer of a convex quadratic function defined on a polynomial space. Here we only prove that convergence holds but the numerical experiments illustrate that one recovers the high accuracy expected by a spectral method, even when nonsmooth data are considered. For example, in experiment 2 we estimated rates of convergence for the L^2-norm of the order $N^{-1.5}$ and $N^{-0.4}$ for two chosen Lipschitz and discontinuous initial data respectively. Published in Boumimez and Castro, 2024.

Second chapter: We extend the results of the first chapter on the square domain with some numerical experiments to illustrate the convergence that we expected. The results are easily extended by the separation of variables. The fact that we only consider rectangular domains is due to the particular spectral method that we analyze. Published in Boumimez and Castro, 2024.

Third chapter: We extended the main theoretical results in our previous chapters for a more general problem which is the elasticity on higher dimensions on $d-$cube. Such convergence result is more difficult to prove, we just have found numerical evidence. As we mentioned above to recover the convergence of controls, the important hypotheses mentioned above related to the convergence of the solutions of discrete adjoint system must be checked. Such convergent results are not standard with the usual numerical analysis techniques and require further investigation which is our goal at the moment. Preprint in Boumimez and Castro, 2023.

Fourth chapter: we deal with a different problem which is the inverse source problem which involves determining the causes or parameters of a system based on observed or measured outcomes. Unlike traditional problems where the inputs lead to predictable outcomes, the inverse problem seeks to infer the hidden or internal dynamics of a system from measurable or observable data. It is well known that observability inequality plays a crucial role in the inverse problem. Here, we provide a general framework allowing to use of exact observability inequality for the approximation of the system on a finite-dimensional system to solve a class of discrete inverse source problems. More precisely, we show that if a discrete system is exactly observable, then a discrete source term in this system can be identified and converged to the source term of the continuous one by knowing its intensity and appropriate observations which often correspond to measurements of some boundary. As an application, we reconstruct a source of the discrete system resulting when approximating the wave equation with two different numerical approaches, the spectral collocation method (chapter 1) and the projection method onto the finite-dimensional subspace formed by the first eigenfunctions of the Laplacian.

For the numerical experiences that are included in the work, we have used MatLab software.

Chapter 2

Numerical approximation of the boundary control for the 1-d wave equation with a spectral collocation method

In this chapter, we propose a spectral collocation method to approximate the exact boundary control of the 1-d wave equation. First, we introduce a suitable approximate control problem that we solve in the finite-dimensional space of polynomials of degree $N \in \mathbb{N}$ in space. Then, we prove that we can choose a sequence of controls f^N associated to the approximate control problem in such a way that they converge, as $N \to \infty$, to a control of the continuous 1-d wave equation. Finally, we illustrate the method in several experiments which give numerical evidence of the highly accurate approximation inherent to spectral methods.

2.1 Problem statement

Consider the 1-d wave equation on a domain $\Omega = (-1, 1)$ and we assume that the control f acts at the right extreme $x = 1$ for some time $t \in (0, T)$:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (t, x) \in \ (0, T) \times \Omega, \\ u(t, 1) = f(t) & \text{in } t \in \ (0, T), \\ u(t, -1) = 0 & \text{in } t \in \ (0, T), \\ u(0, x) = u^0, \quad u_t(0, x) = u^1 & \text{in } x \in \ \Omega. \end{cases} \tag{2.1}$$

Given any $f \in L^2(0, T)$ and some initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, problem (2.1) has a unique solution $(u, u_t) \in C((0, T), L^2(\Omega) \times H^{-1}(\Omega))$.

It is also well-known that, if $T > 4$ for any initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control $f \in L^2(0, T)$ such that the solution of (2.1) $u \in C((0, T); H_0^2(\Omega)) \cap C^1((0, T); L^2(\Omega)))$ can be driven to any final target. We assume without loss of generality that this final target is the equilibrium, i.e.

$$u(T, x) = u_t(T, x) = 0, \ \ x \in \Omega. \tag{2.2}$$

It is important to note that the control, when it exists, is not unique in general. Among the set of controls, a natural choice is to consider the minimal L^2-norm control which is usually unique.

In this chapter, we are interested in approximating one of the controls $f \in L^2(0, T)$ for which (2.2) is satisfied. More precisely, we follow H.U.M (see Lions, 1988a) and approximate the control that has minimal L^2-norm with respect to a suitable weighted norm, with a smooth weight $\eta(t)$ that is compactly supported bump function in $t \in (0, T)$. This ensures that the control itself is compactly supported and avoids possible singularities at times $t = 0$ and $t = T$.

Here we propose a novel numerical approach to obtain an approximation of the control f employing a spectral collocation method to approximate the solution of system (2.1). In this numerical method solutions are approximated by polynomials of degree N that satisfy the equations in (2.1) at some finite number of ”a prior”, chosen collocation points. The resulting system is finite-dimensional and depends on the so-called discretization parameter N. For fixed initial data (u^0, u^1) and boundary condition $f(t)$ this method is convergent with higher accuracy depending on the smoothness of the data (see Boumimez and Castro, 2024).

We prove the controllability of the associated discrete control problem and how to construct a sequence of discrete controls that converges to a control of the continuous system (2.1). This is achieved by adding an extra discrete boundary control that vanishes as $N \to \infty$, leading to a precise approximation of continuous control. The validity of our result is underpinned by two crucial elements: a uniform observability inequality with respect to N for the corresponding discrete adjoint system and an exhaustive spectral analysis of discrete low frequencies and their convergence to the continuous one. The former ensures the uniform boundedness of discrete controls, while the latter guarantees the convergence of discrete control to its continuous counterpart.

To obtain the uniform observability inequality for the uncontrolled adjoint system we use a novel approach based on the continuous error equation that satisfies the discrete solutions. Such error equations are easily obtained when we approximate solutions by smooth functions in the domain, like polynomials. This error equation is the same 1-d wave equation but with a nonhomogenous second-hand term, known as the error term. Consequently, the observability inequality can be derived using techniques analogous to those of the continuous model, with the sole need to estimate this additional error term. This method offers a notable advantage as it can be readily extended to more diverse equations (elasticity, fluid dynamics, etc.) and higher dimensions, as long as rectangular domains are considered.

The second important advantage of the method is in the convergence rate of the approximation. Here we only prove that convergence holds but the numerical experiments illustrate that one recovers the high accuracy expected by a spectral method, even when non-smooth data are considered. For example, in experiment 2 below we estimated rates of convergence for the L^2-norm of the order $N^{-1.5}$ and $N^{-0.4}$ for two chosen Lipschitz and discontinuous initial data respectively.

The rest of this chapter is organized as follows. In section 2.2, we briefly describe the mathematical background of the continuous wave control problem. In section 2.3, we introduce the discrete control problem obtained by the spectral method and state the main results of the chapter. In section 2.4, we present the proof of the existence of discrete controls for the spectral approximation. In section 2.5, we present the proof of convergence of the discrete control to the continuous one. In section 2.6, we state the matrix formulation of the discrete control problem that we approximate using the finite-dimensional control theory. In section 2.7, we present some numerical experiments. Finally, section 2.8 is devoted to the Appendix contains the main spectral results required for the analysis of the convergence.

2.2 The continuous control problem

In this section, we briefly describe the mathematics background of the variational characterization of the control problem (2.1) that we use later to find their numerical approximation. In particular, we prove that a class of controls can be obtained as minimizers of a quadratic functional defined on a Hilbert space. The results of this section are not new and can be found in Micu and Zuazua, 2004. We include them here for completeness.

For technical reasons, we restrict ourselves to controls which are zero near $t = 0, T$. This affects to the quadratic functional that we define below.

Let us consider the following backwards wave equation,

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0 & \text{in } (t, x) \in (0, T) \times \Omega, \\ \phi(t, 1) = \phi(t, -1) = 0 & \text{in } t \in (0, T), \\ \phi(T, x) = \phi^0, \quad \phi_t(T, x) = \phi^1 & \text{in } x \in \Omega, \end{cases} \tag{2.3}$$

where $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$. We also define the duality product between $L^2(\Omega) \times H^{-1}(\Omega)$ and $H_0^1(\Omega) \times L^2(\Omega)$ by

$$\Big\langle (\phi^0, \phi^1), (u^0, u^1) \Big\rangle = < u^1, \phi^0 >_{H^{-1}, H_0^1} - \int_{-1}^{1} u^0 \phi^1 dx, \tag{2.4}$$

where $< \cdot, \cdot >_{H^{-1}, H_0^1}$ is the usual duality product betweenH_0^1 and its dual space H^{-1}.

The following result provides a variational characterization of the control.

Lemma 2.2.1. *Assume that $T > 0$, and consider some initial data $(u^0,\ u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.*

Any controls f that make the solution of the discrete system (2.1) *satisfy* (2.2) *are solution of,*

$$0 = \int_0^T \phi_x(t,1)f(t)dt - \Big\langle (\phi(0,\cdot), \phi_t(0,\cdot)), (u^0, u^1) \Big\rangle, \tag{2.5}$$

for all $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, where (ϕ, ϕ_t) is the solution of the backwards wave equation (2.3)*. Reciprocally, if f satisfies* (2.5) *then* (2.2) *holds.*

Proof. Multiplying the equation of u in (2.1) by ϕ the solution of (2.3) and integrating by parts in time, one obtains

$$\begin{aligned} 0 = \int_0^T \int_{-1}^1 (u_{tt} - u_{xx})\phi dx dt = \int_0^T \int_{-1}^1 u\phi_{tt} dx dt - \int_0^T \int_{-1}^1 u_{xx}\phi dx dt \\ - \Big\langle (\phi(0,\cdot), \phi_t(0,\cdot)), (u^0, u^1) \Big\rangle . \end{aligned} \tag{2.6}$$

We now integrate by parts in x taking into account that f is control, we have

$$0 = \int_0^T \int_{-1}^1 u(\phi_{tt} - \phi_{xx}) dx dt + \int_0^T f(t)\phi_x(t,1)dt - \Big\langle (\phi(0,\cdot), \phi_t(0,\cdot)), (u^0, u^1) \Big\rangle , \tag{2.7}$$

by using(2.3) and (2.7) we easily find (2.5). □

According to **H.U.M**, one possibility to construct controls f that satisfy the variational condition (2.5) is as minimizers of a particular quadratic functional. We define the following cost functional $J : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$ by

$$J(\phi^0, \phi^1) = \frac{1}{2} \int_0^T \eta(t) \left|\phi_x(t,1)\right|^2 dt - \Big\langle (\phi(0,\cdot), \phi_t(0,\cdot)), (u^0,\ u^1) \Big\rangle , \tag{2.8}$$

where (ϕ, ϕ_t) is the solution of (2.3) with final data $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$. The function $\eta(t)$ is a prescribed smooth function in $(0, T)$ introduced to guarantee that the controls vanish in a neighbourhood of $t = 0, T$. Thus, we consider $\delta > 0$ a small number and $0 \le \eta(t) \le 1$ such that,

$$\eta(t) = \begin{cases} 1 \text{ in } [2\delta, T - 2\delta], \\ 0 \text{ in } [0, \delta] \cup [T - \delta, T]. \end{cases} \tag{2.9}$$

Note that both η and J will depend on this parameter δ but this is not relevant in the rest of the analysis and we will not make explicit this dependence in the notation.

Theorem 2.2.2. *Assume* $(u^0,\ u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *and we suppose that* $(\hat{\phi}^0, \hat{\phi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *is a minimizer of* J. *If* $\hat{\phi}$ *is the corresponding solution of* (2.3) *with final data* $(\hat{\phi}^0, \hat{\phi}^1)$ *then*

$$f(t) = \eta(t)\phi_x(t,1), \tag{2.10}$$

is a control such that the solution of (2.1) *satisfies* (2.2). *Moreover the control* f *is the one with minimal* L^2 *weight norm* $\|f\|^2_{L^2(0,T)} = \int_0^T \eta(t)|\phi_x(t,1)|^2 dt$. *We refer this control as HUM control.*

Proof. If J achieves its minimum at $(\hat{\phi}^0, \hat{\phi}^1)$ its Gateaux derivative in the direction (ϕ^0, ϕ^1) must vanish,i.e.

$$0 = \int_0^T \eta(t)\hat{\phi_x}(t,1)\phi_x^N(t,1)dt - \Big\langle (\phi(0,\cdot), \phi_t(0,\cdot)), (u^0, u^1) \Big\rangle,$$

for all $(\phi^0, \phi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. From Lemma 2.2.1 it follows that (2.10) is control for which (2.1) holds. ☐

Remark 1. *The HUM control is characterized by the following :*

- f *satisfies the variational characterization* (2.5).
- $f(t) = \eta(t)\hat{\phi}_x(t,1)$, *where* $\hat{\phi}$ *is a solution of* (2.3) *with final data* $(\hat{\phi}^0, \hat{\phi}^1)$.

Let us now give a general condition which ensures the existence of a minimizer for J and therefore a control for system (2.1).

The functional J is evidently continuous and convex. Consequently, the existence of a minimizer, and thus a control for system (2.1), is assured once we establish its coercivity. This follows as a consequence of the following lemma.

Lemma 2.2.3. *Given* $T > 4$ *there exists a constant* $C > 0$, *such that the solution of system* (2.3) *satisfy*

$$C \left\|(\phi^0, \phi^1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_0^T |\phi_x(t,1)|^2 \, dt, \tag{2.11}$$

for any final data $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Inequality (2.11) is known as an observability inequality. It roughly says that we can "observe" the energy of the solution, meaning the normal derivative at one extremity for a sufficiently large time.

Proof. We try the classical multiplier technique to recover the observability inequality. Multiplying the equation in (2.3) by $(x+1)\phi_x$ and integrating by parts one obtain (see Lions, 1988a)

$$\begin{aligned} &\int_0^T \int_{-1}^1 (x+1)(\phi_{tt} - \phi_{xx})\phi_x dxdt \\ &= \int_{-1}^1 (x+1)\phi_t\phi_x dx \bigg|_0^T + \frac{1}{2}\int_0^T \int_{-1}^1 (|\phi_t|^2 + |\phi_x|^2) dxdt - \int_0^T |\phi_x(t,1)|^2 \, dt. \end{aligned} \tag{2.12}$$

Thus, if we set $X = \int_{-1}^1 (x+1)\phi_t\phi_x dx \Big|_0^T$ then

$$\frac{1}{2}\int_0^T \int_{-1}^1 (|\phi_t|^2 + |\phi_x|^2) dxdt \leq |X| + \int_0^T |\phi_x(t,1)|^2 \, dt \tag{2.13}$$

Now, define the energy

$$E(t) = \frac{1}{2}(\|\phi_t(t,\cdot)\|^2_{L^2(-1,1)} + \|\phi_x(t,\cdot)\|^2_{L^2(-1,1)}). \tag{2.14}$$

It is easy to see that this energy is conserved, i.e. $E(t) = E(0)$ for all $t > 0$. We just multiply (2.3) by ϕ_t and integrate with respect to time.

We now estimate the first terms in the right-hand side of (2.13). Observe that,

$$\begin{aligned} \left|\int_{-1}^1 (x+1)\phi_t\phi_x dx\right| \leq 2\frac{1}{2}(\|\phi_t\|^2_{L^2(-1,1)} + \|\phi_x\|^2_{L^2(-1,1)}) &\leq (\|\phi_t\|^2_{L^2(-1,1)} + \|\phi_x\|^2_{L^2(-1,1)}) \\ &= 2E(t) = 2E(T). \end{aligned} \tag{2.15}$$

Therefore, $|X| \leq 4E(T)$. It follows from (2.13) that

$$(T-4)\, E(0) \leq \int_0^T |\phi_x(t,1)|^2 \, dt.$$

Then, inequality (2.11) holds as long as $T > 4$. □

Note that the observability that we have defined above in (2.11) is a sufficient condition for the functional J in (2.8) to have a minimizer. According to Theorem 2.2.2 this minimizer gives the control with minimal L^2 weight norm.

2.3 Approximation by the spectral collocation method

In this section, we introduce some notation, the approximate discrete control problem and state the main results of the chapter.

Let N be a natural number and consider $C = \{x_i,\ 0 \leq i \leq N\}$ the Legendre-Gauss-Lobatto (LGL) nodes in Ω that are the roots of

$$(1-x^2)\frac{d}{dx}L^N(x) = 0,$$

where $L^k(x)$ is the k-th Legendre polynomial in $(-1,1)$ (e.g. Szeg, 1939) which are the eigenfunctions of the singular Sturm-Liouville problem

$$((1-x^2)L_x^k(x))_x + k(k+1)L^k(x) = 0,$$

it is normalized and $L^k(\pm 1) = \pm 1$, then for any k:

$$L^k(x) = \frac{1}{2^k}\sum_{l=0}^{[k/2]}(-1)^l \begin{pmatrix} k \\ l \end{pmatrix} \begin{pmatrix} 2k-2l \\ k \end{pmatrix} x^{k-2l}, \tag{2.16}$$

where $[k/2]$ denotes the integral part of $k/2$. We divide $C = C^\Omega \cup C^{Di}$ into interior and boundary nodes, i.e.

$$\begin{aligned} C^\Omega &= C \cap \Omega = \{x_i, i \in I_\Omega\}, \\ C^{Di} &= C \cap \{-1,1\} = \{x_i, i \in I_{Di}\}, \end{aligned}$$

where $I_\Omega,\ I_{Di}$ are the sets of indexes corresponding to the interior and boundary collocation nodes respectively, and we denote $I = I_\Omega \cup I_{Di}$. In Figure 2.1 we show the LGL nodes in $(-1,1)$.

Let $\mathbb{P}_N(\Omega)$ be the space of continuous functions in $\bar{\Omega}$ which are polynomials of degree less

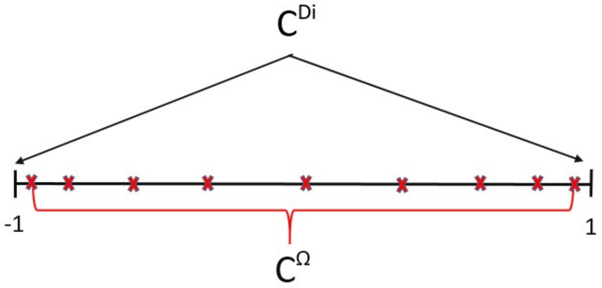


Figure 2.1: The LGL nodes in $(-1, 1)$.

than or equal to N and let $\mathbb{P}_N^{Di}(\Omega)$ be the subspace of $\mathbb{P}_N(\Omega)$ of those functions vanishing on $x = \{-1, 1\}$. We define the following discrete inner product that approximates that of $L^2(\Omega)$:

$$(w, z)_N = \sum_{i \in I} (wz)(t, x_i)\omega_i, \ \ 0 \leq t \leq T. \tag{2.17}$$

Here, ω_i is the discrete weight associated with the one-dimensional LGL quadrature formula (e.g.Canuto et al., 1988, Chapter 2]) expressed as follows:

$$\omega_i = \frac{2}{N(N+1)} \frac{1}{|L^N(x_i)|^2}, \ \ i = 0, ..., N. \tag{2.18}$$

Owing to the exactness of the integration LGL formula, we have

$$(w, z)_N = \int_\Omega wz \ dx \text{ for all } w, z \text{ such that } wz \in \mathbb{P}_{2N-1}(\Omega). \tag{2.19}$$

The symbol $\|\cdot\|_N$ denotes the discrete norm which is defined as $\|z\|_N^2 = (z, z)_N$.

We recall that the discrete norm $\|\cdot\|_N$ is uniformly equivalent to the L^2-norm $|\cdot|_{L^2}$ in $\mathbb{P}_N(\Omega)$ (Canuto et al., 1988, Chapter 9). In other words, there exist two positive constants $C_1 = 1$, and $C_2 = 2 + \dfrac{1}{N}$, such that

$$C_1 \, |p|_{L^2}^2 \leq \|p\|_N^2 \leq C_2 \, |p|_{L^2}^2 \, , \ \forall p \in \mathbb{P}_N(\Omega). \tag{2.20}$$

We denote by Ψ_i the Lagrange polynomials which is 1 at x_i and 0 at all the other collocation nodes. For the commonly used Gauss-Lobatto points one has

$$\Psi_i(x) = \frac{1}{N(N+1)L^N(x_i)} \frac{(1-x^2)L_x^N(x)}{x-x_i}. \tag{2.21}$$

Observe that $\{\Psi_i,\ i \in I\}$ constitutes a basis in $\mathbb{P}_N^{Di}(\Omega)$.

In order to find a numerical approximation of this control f in (2.1) we proceed as follows: first we introduce a discrete version of the control problem (2.1), depending on a discrete parameter $N \longrightarrow \infty$. Then, we prove that this system is controllable for all N with three different controls, $f^N, g_R^N, g_L^N \in L^2(0,T)$, that we can choose in such way that

$$f^N \underset{N\to\infty}{\longrightarrow} f,\ g_R^N \underset{N\to\infty}{\longrightarrow} 0,\ g_L^N \underset{N\to\infty}{\longrightarrow} 0,$$

where f is a control of (2.1). Therefore f^N is a numerical approximation of a continuous control f, while g_R^N, g_L^N can be understood as artificial controls which are only necessary to obtain f^N.

Now, we introduce the following discrete control problem: Given $u^{0,N},\ u^{1,N} \in \mathbb{P}_N^{Di}(\Omega)$ and $T > 0$, find $f^N,\ g_R^N,\ g_L^N \in L^2(0,T)$ such that the solution $u^N \in C^2(0,T;\mathbb{P}_N(\Omega))$ of system:

$$\begin{cases} (u_{tt}^N - u_{xx}^N)(t,x_i) = g_L^N(t)G_L^N(x_i) + g_R^N(t)G_R^N(x_i) & \text{in } (t,x_i) \in (0,T) \times C^\Omega, \\ u^N(t,1) = f^N(t) & \text{in } t \in\ (0,T), \\ u^N(t,-1) = 0 & \text{in } t \in\ (0,T), \\ u^N(0,x_i) = u^{0,N}(x_i), u_t^N(0,x_i) = u^{1,N}(x_i) & \text{for } \ x_i \in C^\Omega \end{cases} \tag{2.22}$$

satisfies

$$u^N(T,x_i) = u_t^N(T,x_i) = 0,\ x_i \in C^\Omega. \tag{2.23}$$

Here $G^N_L, G^N_R \in \mathbb{P}_{N-1}(\Omega)$ are defined by

$$\begin{cases} G^N_L(x_i) = \left(\dfrac{h^L_{xx}}{\sqrt{\omega_0}} - \dfrac{\Psi_{0,x}}{\sqrt{\omega_0}\omega_0}\right)(x_i), \quad G^N_R(x_i) = \left(\dfrac{h^R_{xx}}{\sqrt{\omega_N}} + \dfrac{\Psi_{N,x}}{\sqrt{\omega_N}\omega_N}\right)(x_i) \\ h^L,\ h^R \in \mathbb{P}^{Di}_N(\Omega) \\ h^L(x_i) = \dfrac{1-x_i}{2},\ h^R(x_i) = \dfrac{1+x_i}{2},\ x_i \in C^\Omega. \end{cases} \tag{2.24}$$

Note that $h^L(x) \neq \dfrac{1-x}{2}$ since $h^L \in \mathbb{P}^{Di}_N(\Omega)$ and then $h^L(1) = h^L(-1) = 0$. Something similar can be said about $h^R(x)$.

Remark 2. *Note that the solutions of* (2.22) *are prescribed only at the interior collocation points. System* (2.22) *is a second-order system of ODE with* $(N+1)$ *equations and unknowns, namely the coefficients of the polynomial* u^N*. Comparing the control problems* (2.1) *and* (2.22) *we see that the discrete version has* 2 *extra controls* $g^N_L, g^N_R \in L^2(0,T)$ *that vanishes as* $N \to \infty$*. They are necessary to obtain a bounded sequence of discrete controls* f^N*.*

The main results in this chapter are the following :

Theorem 2.3.1. *Given* $T > 4(2+N^{-1})$ *and* $(u^{0,N},\ u^{1,N}) \in \mathbb{P}^{Di}_N(\Omega) \times \mathbb{P}^{Di}_N(\Omega)$*, there exist controls* $f^N,\ g^N_L,\ g^N_R \in L^2(0,T)$ *such that the solution* u^N *of* (2.22) *satisfies* (2.23).

Theorem 2.3.2. *Given* $(u^0,u^1) \in L^2 \times H^{-1}$*, there exists a sequence* $(u^{0,N},u^{1,N}) \in (\mathbb{P}^{Di}_N(\Omega))^2$ *such that*

$$(u^{0,N},u^{1,N}) \to (u^0,u^1) \text{ in } L^2 \times H^{-1}, \qquad \text{as } N \to \infty.$$

Furthermore, for any $T > 4(2+N^{-1})$*, we can choose the controls* $f^N,\ g^N_L,\ g^N_R \in L^2(0,T)$ *such that the solution* u^N *of* (2.22) *satisfies* (2.23) *and*

$$f^N \to f, \quad g^N_R \to 0, \quad g^N_L \to 0, \qquad \text{as } N \to \infty, \text{ in } L^2(0,T),$$

where f *is a control of the continuous wave equation* (2.1).

Remark 3. *Note that the control time* T *in Theorem 2.3.2 is basically two times the time required in the continuous problem. This is due to the constant* C_2 *in* (2.20) *and probably not optimal, as we illustrate in the experiments below.*

Remark 4. *When* u^0 *and* u^1 *are continuous functions the polynomials* $\tilde{u}^{0,N}, \tilde{u}^{1,N} \in \mathbb{P}^{Di}_N$ *that* *coincides with* u^0 *and* u^1 *at the nodes* $x_i \in C^{\Omega}$ *give a discretization that satisfies*

$$(\tilde{u}^{0,N}, \tilde{u}^{1,N}) \to (u^0, u^1) \text{ in } L^2 \times H^{-1},$$

as $N \to \infty$*. In this case the result in Theorem 2.3.2 is still true when considering* $(u^{0,N}, u^{1,N}) = (\tilde{u}^{0,N}, \tilde{u}^{1,N})$*. Thus, for smooth functions, there exists a constructive way to choose* $(u^{0,N}, u^{1,N})$ *in Theorem 2.3.1.*

2.4 Existence of discrete controls: proof of Theorem 2.3.1

In this section we prove Theorem 2.3.1. We first introduce a variational characterization of discrete HUM controls (2.22) and then prove that a particular discrete control can be obtained as the minimizer of a convex quadratic functional defined on a polynomial space. Finally, we prove the coerciveness of the functional that guarantees the existence of minimizers.

Let us introduce the following bilinear form in $\mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$:

$$\Big\langle (\phi^{0,N}, \phi^{1,N}), (u^{0,N}, u^{1,N}) \Big\rangle_N = (u^{1,N}, \phi^{0,N})_N - (u^{0,N}, \phi^{1,N})_N. \tag{2.25}$$

Lemma 2.4.1. *Assume that* $T > 0$*, and consider some initial data* $(u^{0,N}, u^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$*. Any controls* f^N, g_R^N, g_L^N *that make the solution of the discrete system* (2.22) *satisfy* (2.23) *are solutions of*

$$\begin{aligned} &\int_0^T (\phi_x^N(t,1) - \omega_N \phi_{xx}^N(t,1)) f^N(t)dt + \int_0^T \sqrt{\omega_N}\phi_{xx}^N(t,1) g_R^N(t)dt \\ &\quad + \int_0^T \sqrt{\omega_0}\phi_{xx}^N(t,-1) g_L^N(t)dt - \Big\langle (\phi^N(0,\cdot), \phi_t^N(0,\cdot)), (u^{0,N}, u^{1,N}) \Big\rangle_N = 0, \end{aligned} \tag{2.26}$$

for all $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$*, where* $(\phi^N, \phi_t^N) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$ *is the solution of* *the following collocation backwards wave equation:*

$$\begin{cases} (\phi^N_{tt} - \phi^N_{xx})(t, x_i) = 0 & \text{in } (t, x_i) \in (0, T) \times C^\Omega, \\ \phi^N(t, 1) = \phi^N(t, -1) = 0 & \text{in } t \in (0, T), \\ \phi^N(T, x_i) = \phi^{0,N}(x_i), \quad \phi^N_t(T, x_i) = \phi^{1,N}(x_i) & \text{at } x_i \in C^\Omega. \end{cases} \tag{2.27}$$

Proof. Multiplying the equation of $u^N(t, x_i)$ in (2.22) by $\omega_i \phi^N(t, x_i)$ and adding in $i \in I$, one obtains

$$\int_0^T (u^N_{tt} - u^N_{xx}, \phi^N)_N dt = \int_0^T g^N_L \, (G^N_L, \phi^N)_N dt + \int_0^T g^N_R \, (G^N_R, \phi^N)_N dt. \tag{2.28}$$

We first simplify the left-hand side. Using (2.19), integrating by parts in time and taking into account that f^N, g^N_L and g^N_R are controls, we have

$$\begin{aligned} \int_0^T (u^N_{tt} - u^N_{xx}, \phi^N)_N dt = \int_0^T (u^N, \phi^N_{tt})_N dt - \int_0^T \int_{-1}^1 u^N_{xx} \phi^N dx dt \\ - \left\langle (\phi^N(0,.), \phi^N_t(0,.)), (u^{0,N}, u^{1,N}) \right\rangle_N . \end{aligned} \tag{2.29}$$

We now integrate by parts in x and use again formula (2.19), since the resulting integrand is also a polynomial of degree $2N - 2$

$$\begin{aligned} 0 = \quad & \int_0^T (u^N_{tt} - u^N_{xx}, \phi^N)_N dt = \int_0^T (u^N, \phi^N_{tt} - \phi^N_{xx})_N dt \\ & + \int_0^T f^N(t) \phi^N_x(t, 1) dt - \left\langle (\phi^N(0,.), \phi^N_t(0,.)), (u^{0,N}, u^{1,N}) \right\rangle_N \\ = & \int_0^T f^N(t)(\phi^N_x - \omega_N \phi^N_{xx})(t, 1) dt - \left\langle (\phi^N(0,.), \phi^N_t(0,.)), (u^{0,N}, u^{1,N}) \right\rangle_N . \end{aligned} \tag{2.30}$$

The last equality is a consequence of the first equation in (2.27). Note that an extra term appears on the right-hand side of this expression coming from the fact that the first equation in (2.27) is only true for the interior nodes while the discrete scalar product involves also the boundary nodes.

For the right hand side in (2.28) we use again formula (2.19) and the fact that $G^N_R \phi^N$ is a polynomial of degree $2N-1$

$$\begin{aligned}(G_R^N,\phi^N)_N =&\frac{1}{\sqrt{\omega_N}}\left(h^R_{xx}+\frac{\Psi_{N,x}}{\omega_N},\phi^N\right)_N = \frac{1}{\sqrt{\omega_N}}\int_{-1}^{1}\left(h^R_{xx}+\frac{\Psi_{N,x}}{\omega_N}\right)\phi^N dx\\ &= \frac{1}{\sqrt{\omega_N}}\left(\int_{-1}^{1}h^R\phi^N_{xx}dx-\int_{-1}^{1}\frac{\Psi_N}{\omega_N}\phi^N_x dx\right)\\ &= \frac{1}{\sqrt{\omega_N}}\left(\sum_{i\in I_\Omega}(h^R\phi^N_{xx})(t,x_i)\omega_i-\phi^N_x(t,1)\right).\end{aligned} \tag{2.31}$$

To simplify the first term on the right-hand side we observe that $h^R(x_i)=(1+x_i)/2$ at the interior nodes, i.e. $i\in I_\Omega$. Then,

$$\sum_{i\in I_\Omega}(h^R\phi^N_{xx})(t,x_i)\omega_i=\int_{-1}^{1}\frac{1+x}{2}\phi^N_{xx}\,dx-\omega_N\phi^N_{xx}(t,1)=\phi^N_x(t,1)-\omega_N\phi^N_{xx}(t,1). \tag{2.32}$$

From (2.31)-(2.32), we easily obtain that

$$(G_R^N,\phi^N)_N=-\sqrt{\omega_N}\phi^N_{xx}(t,1). \tag{2.33}$$

Combining (2.28), (2.30) and (2.33) we easily find (2.26). □

According to HUM, one possibility to construct controls f^N, g_R^N, g_L^N that satisfy the variational condition (2.23) is as minimizers of the following cost functional $J^N:\mathbb{P}_N^{Di}(\Omega)\times\mathbb{P}_N^{Di}(\Omega)\longrightarrow\mathbb{R}$ defined by

$$\begin{aligned}J^N(\phi^{0,N},\phi^{1,N})&=\frac{1}{2}\int_0^T\eta(t)\left|\phi^N_x(t,1)-\omega_N\phi^N_{xx}(t,1)\right|^2dt\\ &+\frac{1}{2}\int_0^T\eta(t)\omega_N\left|\phi^N_{xx}(t,1)\right|^2dt+\frac{1}{2}\int_0^T\eta(t)\omega_0\left|\phi^N_{xx}(t,-1)\right|^2dt\\ &-\left\langle(\phi^N(0,\cdot),\phi^N_t(0,\cdot)),(u^{0,N},u^{1,N})\right\rangle_N,\end{aligned} \tag{2.34}$$

where (ϕ^N,ϕ^N_t) is the solution of (2.27) with final data $(\phi^{0,N},\phi^{1,N})\in\mathbb{P}_N^{Di}(\Omega)\times\mathbb{P}_N^{Di}(\Omega)$. The function $\eta(t)$ is prescribed as in (2.9).

Note that J^N will depend on this parameter δ but this is not relevant in the rest of the analysis and we will not make explicit this dependence in the notation.

Theorem 2.4.2. *Assume* $(u^{0,N},u^{1,N})\in\mathbb{P}_N^{Di}(\Omega)\times\mathbb{P}_N^{Di}(\Omega)$ *and that* $(\hat{\phi}^{0,N},\hat{\phi}^{1,N})\in\mathbb{P}_N^{Di}(\Omega)\times$

$\mathbb{P}_N^{Di}(\Omega)$ is a minimizer of J^N. If $\widehat{\phi^N}$ is the corresponding solution of (2.27) with final data $(\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$, then

$$\begin{aligned} &f^N(t) = \eta(t)(\widehat{\phi_x^N}(t,1) - \omega_N \widehat{\phi_{xx}^N}(t,1)), \\ &g_R^N(t) = \eta(t)\sqrt{\omega_N}\widehat{\phi_{xx}^N}(t,1), \qquad g_L^N(t) = \eta(t)\sqrt{\omega_0}\widehat{\phi_{xx}^N}(t,-1) \end{aligned} \tag{2.35}$$

are controls such that the solution of (2.22) satisfies (2.23).

Proof of Theorem 2.4.2. If J^N achieves its minimum at $(\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$ its Gateaux derivative in the direction $(\phi^{0,N}, \phi^{1,N})$ must vanish, i.e.

$$\begin{aligned} 0 = &\int_0^T \eta(t)(\widehat{\phi_x^N}(t,1) - \omega_N\widehat{\phi_{xx}^N}(t,1))(\phi_x^N(t,1) - \omega_N\phi_{xx}^N(t,1))dt \\ &+ \int_0^T \eta(t)\omega_N\widehat{\phi_{xx}^N}(t,1)\phi_{xx}^N(t,1)dt + \int_0^T \eta(t)\omega_0\widehat{\phi_{xx}^N}(t,-1)\phi_{xx}^N(t,-1)dt \\ &- \left\langle (\phi^N(0,\cdot), \phi_t^N(0,\cdot)), (u^{0,N}, u^{1,N}) \right\rangle_N, \end{aligned}$$

for all $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$. From Lemma 2.4.1 it follows that (2.35) are controls for which (2.22) holds. □

The functional J^N is clearly continuous and convex so that the existence of a minimizer (and therefore a discrete control for system (2.22)) is guaranteed as soon as we prove its coercivity. This is a consequence of the following lemma. Note that this also concludes the proof of Theorem 2.3.1.

Lemma 2.4.3. *Given $T > 4(2 + N^{-1})$ there exists a constant $C > 0$, independent of N, such that the solutions of system (2.27) satisfy*

$$\begin{aligned} C\left\|(\phi_x^{0,N}, \phi^{1,N})\right\|_{N\times N}^2 \leq &\int_0^T \left|\phi_x^N(t,1) - \omega_N\phi_{xx}^N(t,1)\right|^2 dt \\ &+ \omega_N \int_0^T \left|\phi_{xx}^N(t,1)\right|^2 dt + \omega_0 \int_0^T \left|\phi_{xx}^N(t,-1)\right|^2 dt, \end{aligned} \tag{2.36}$$

for any final data $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$ and $\eta(t)$ is defined as in (2.9).

Proof. Since $\omega_N < 1$ (see (2.18)) it is enough to prove the following

$$\begin{aligned} C\left\|(\phi_x^{0,N},\phi^{1,N})\right\|_{N\times N}^2 \quad &\leq \quad \int_0^T \left|\phi_x^N(t,1)\right|^2 dt + \omega_N \int_0^T \left|\phi_{xx}^N(t,1)\right|^2 dt \\ &\qquad +\omega_0 \int_0^T \left|\phi_{xx}^N(t,-1)\right|^2 dt. \end{aligned} \tag{2.37}$$

We first observe that the solutions of (2.27) solve the following equivalent system:

$$\begin{cases} \phi_{tt}^N - \phi_{xx}^N = -\phi_{xx}^N(t,-1)\Psi_0 - \phi_{xx}^N(t,1)\Psi_N & \text{in } (t,x)\in(0,T)\times\Omega, \\ \phi^N(t,1) = \phi^N(t,-1) = 0 & \text{in } t\in(0,T), \\ \phi^N(T,x) = \phi^{0,N}, \phi_t^N(T,x) = \phi^{1,N} & \text{in } x\in\Omega. \end{cases} \tag{2.38}$$

In fact, this is easily seen by writing the polynomial $\phi_{tt}^N - \phi_{xx}^N$ in the Lagrangian basis and using system (2.27). Now, we try the classical multiplier technique to recover the observability inequality. Multiplying the equation in (2.38) by $(x+1)\phi_x^N$ and integrating by parts one obtain

$$\begin{aligned} \int_0^T \int_{-1}^1 (x+1)(\phi_{tt}^N - \phi_{xx}^N)\phi_x^N dxdt = -\int_0^T \int_{-1}^1 (x+1)(\phi_{xx}^N(t,-1)\Psi_0 dxdt \\ + \phi_{xx}^N(t,1)\Psi_N)\phi_x^N dxdt. \end{aligned} \tag{2.39}$$

As for the continuous wave equation, (see Lions, 1988a) the left-hand side can be simplified as follows,

$$\begin{aligned} &\int_0^T \int_{-1}^1 (x+1)(\phi_{tt}^N - \phi_{xx}^N)\phi_x^N dxdt \\ &= \int_{-1}^1 (x+1)\phi_t^N \phi_x^N dx\Big|_0^T + \frac{1}{2}\int_0^T \int_{-1}^1 (\left|\phi_t^N\right|^2 + \left|\phi_x^N\right|^2) dxdt - \int_0^T \left|\phi_x^N(t,1)\right|^2 dt. \end{aligned} \tag{2.40}$$

Thus, if we set $X = \int_{-1}^1 (x+1)\phi_t^N \phi_x^N dx\Big|_0^T$ then

$$\begin{aligned} &\frac{1}{2}\int_0^T \int_{-1}^1 (\left|\phi_t^N\right|^2 + \left|\phi_x^N\right|^2) dxdt \leq |X| + \int_0^T \left|\phi_x^N(t,1)\right|^2 dt \\ &+ \left|\int_0^T \phi_{xx}^N(t,-1)\int_{-1}^1 (x+1)\phi_x^N \Psi_0 dxdt\right| + \left|\int_0^T \phi_{xx}^N(t,1)\int_{-1}^1 (x+1)\phi_x^N \Psi_N dxdt\right|. \end{aligned} \tag{2.41}$$

In the rest of this proof, we estimate each one of the terms in this expression.

We start with the left-hand side in (2.41). Define the discrete energy

$$E^N(t) = \frac{1}{2}(\left\|\phi_t^N(t,\cdot)\right\|_N^2 + \left\|\phi_x^N(t,\cdot)\right\|_N^2). \tag{2.42}$$

It is easy to see that this energy is conserved, i.e. $E^N(t) = E^N(0)$ for all $t > 0$. We just multiply (2.27) by $\phi_t^N \omega_i$, add in $i \in I$ and integrate with respect time.

This conservation of energy, together with the norm equivalence in (2.20) gives,

$$\frac{1}{2}\int_0^T \int_{-1}^1 (\left|\phi_t^N\right|^2 + \left|\phi_x^N\right|^2) dxdt \geq \frac{1}{C_2}\int_0^T E^N(t)dxdt = \frac{T}{C_2}E^N(T).$$

We now estimate the terms in the right-hand side of (2.41). We start with X. Observe that,

$$\left|\int_{-1}^1 (x+1)\phi_t^N \phi_x^N dx\right| \leq 2\frac{1}{2}(\left\|\phi_t^N\right\|_{L^2}^2 + \left\|\phi_x^N\right\|_{L^2}^2) \leq (\left\|\phi_t^N\right\|_N^2 + \left\|\phi_x^N\right\|_N^2) = 2E^N(t) = 2E^N(T). \tag{2.43}$$

Therefore $|X| \leq 4E^N(T)$. Let us turn to estimate the third term on the right-hand side in (2.41). Using Young's inequality, for any $\varepsilon > 0$ we can find a sufficient large constant $C_\varepsilon > 0$ such that

$$\begin{aligned} \left|\int_0^T \phi_{xx}^N(t,-1)\int_{-1}^1 (x+1)\phi_x^N \Psi_0 dxdt\right| &\leq \int_0^T \left|\phi_{xx}^N(t,-1)\right| |\Psi_0|_{L^2(\Omega)} \left|\phi_x^N\right|_{L^2(\Omega)} dt \\ &\leq C_\varepsilon |\Psi_0|_{L^2(\Omega)}^2 \left|\phi_{xx}^N(t,-1)\right|_{L^2(0,T)}^2 + \varepsilon \left|\left|\phi_x^N\right|_{L^2(\Omega)}\right|_{L^2(0,T)}^2 . \end{aligned} \tag{2.44}$$

Taking into account the norm equivalence in (2.20), the conservation of the discrete energy proved above and the fact that as $\Psi_0 \in \mathbb{P}_N(\Omega)$, $|\Psi_0|_{L^2(\Omega)}^2 \leq \|\Psi_0\|_N^2 = \omega_0$, we obtain

$$\left|\int_0^T \phi_{xx}^N(t,-1)\int_{-1}^1 (x+1)\phi_x^N \Psi_0 dxdt\right| \leq C_\varepsilon \omega_0 \left|\phi_{xx}^N(t,-1)\right|_{L^2(0,T)}^2 + 2\varepsilon T E^N(T). \tag{2.45}$$

An analogous estimate holds for the last term on the right-hand side in (2.41).

It follows from (2.41), (2.45) and the fact that $\omega_N = \omega_0$,

$$\left(\frac{T}{C_2} - 4 - 4\varepsilon T\right) E_0^N \leq \int_0^T \left|\phi_x^N(t,1)\right|^2 dt + \omega_N C_\varepsilon \left|\phi_{xx}^N(t,1)\right|^2_{L^2(0,T)} + \omega_N C_\varepsilon \left|\phi_{xx}^N(t,-1)\right|^2_{L^2(0,T)}.$$

Then, inequality (2.37) holds as long as $\frac{T}{C_2} - 4\varepsilon T - 4 > 0$. As ε can be chosen arbitrarily small and $C_2 = 2 + 1/N$ we have the condition $T > 4(2 + N^{-1})$. $\square$

Remark 5. *The method we present here to obtain the uniform observability inequality relies on the continuous error equation* (2.38)*, equivalent to* (2.27)*. The observability inequality is derived using the same techniques as in the continuous model (see Lions, 1988a) and we only have to estimate the extra error term appearing in* (2.38)*. Therefore, this can be easily adapted to other equations or systems and higher dimensions where the controllability of the continuous model is known.*

Remark 6. *Estimate* (2.37) *is known as an observability inequality and it is the discrete analogue of* (2.11)*. Roughly speaking it establishes that the energy of the solutions can be bounded by boundary observations, i.e. quantities that depend only on the solution at the boundary of the domain. Note that the uniformity in N of the constant in* (2.37) *is not necessary to prove Theorem 2.3.1, but it is essential to establish the convergence result in Theorem 2.3.2, as we show in the next section.*

Note also that, at least formally, the terms involving $\varphi_{xx}^N(t,\pm 1)$ in the right-hand side of the observability inequality (2.36) *should disappear as $N \to \infty$, since for the backwards wave equation* (2.3) $\varphi_{xx}(t,\pm1) = 0$*. However, if we remove these terms in* (2.36) *the constant C cannot be chosen to be uniform in $N \to \infty$. In fact, for any $T > 0$, we have numerically*

$$\sup_{(\phi^{0,N},\phi^{1,N}) \in \mathbb{P}_N^{Di} \times \mathbb{P}_N^{Di}} \frac{\left\|(\phi_x^{0,N}, \phi^{1,N})\right\|^2_{N\times N}}{\int_0^T \left|\phi_x^N(t,1)\right|^2 dt} \longrightarrow \infty \text{ as } N \longrightarrow \infty. \tag{2.46}$$

Two spectral properties are at the origin of this lack of uniformity. On one hand the high frequency modes associated to the corresponding spectral collocation second-order differential *operator*

$$\begin{cases} \varphi^N_{xx}(x_i) + \lambda^N \varphi^N(x_i) = 0, & \textit{at} \;\; x_i \in C^\Omega, \\ \varphi^N(-1) = \varphi^N(1) = 0. \end{cases} \tag{2.47}$$

The eigenvalues associated to the continuous problem are given by $\lambda_k = (k\pi/2)^2$ *and they satisfy a spectral gap property*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = \pi/2 > 0.$$

However, a numerical dispersion appears in the discrete eigenvalue problem and the analogous spectral gap is only true for the low frequencies. In Figure 2.2 we show the behaviour of the square root of eigenvalues $\sqrt{\lambda^N_k}$ *associated to* (2.47)*.*

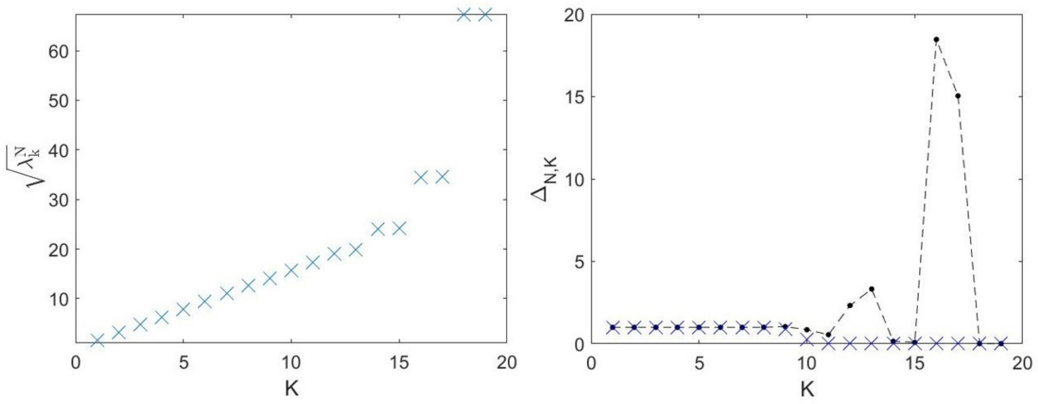


Figure 2.2: The left figure clarify the eigenvalues of (2.47) for $N = 20$ and the right one illustrate the behavior of the ratio (2.48) versus k for $N = 20$ (dots in black), and the reinforced observability (2.49) (crosses in blue).

We observe that there are two families of eigenvalues, corresponding to even and odd eigenfunctions, that become closer for large frequencies. In particular, we find numerically

$$\sqrt{\lambda^N_{2k+1}} - \sqrt{\lambda^N_{2k}} \sim N^{-1.6}.$$

This lack of a uniform spectral gap for the discrete problem is one of the reasons to have (2.46) *(we refer to Zuazua, 2005 where this is analyzed in detail for other numerical approximations). This phenomena was already observed in Weideman and Trefethen, 1988.*

On the other hand, there is another spectral property of the continuous problem that is lost in the discretization and justifies (2.46)*. When considering particular solutions containing one single discrete eigenfunction* φ_k^N *the left-hand side in* (2.46) *can be interpreted as an observability quotient for the eigenfunctions. This is uniformly bounded for all the eigenfunctions associated to the continuous eigenvalue problem but in the discrete case, this property is lost. In particular, we have numerically*

$$\max_{1\le k\le N} \frac{\sum_{i\in I}\omega_i \left|\varphi_{k,x}^N(x_i)\right|^2}{\left|\varphi_{k,x}^N(1)\right|^2} \sim N^{1.7}. \tag{2.48}$$

It turns out that, when considering the reinforced observation that includes the second derivative in space at the extremes we recover this uniformity,

$$\sup_{1\le k\le N} \frac{\sum_{i\in I}\omega_i \left|\varphi_{k,x}^N(x_i)\right|^2}{\left|\varphi_{k,x}^N(1)\right|^2 + \omega_N \left|\varphi_{k,xx}^N(1)\right|^2 + \omega_0 \left|\varphi_{k,xx}^N(-1)\right|^2} \le C. \tag{2.49}$$

In Figure 2.2, we show numerically that this constant $C \le 1$.

2.5 Convergence of the discrete controls: proof of Theorem 2.3.2

We now prove the convergence result mentioned in Theorem 2.3.2. All along this section, we assume that the hypotheses of the theorem hold. To clarify the exposition we proceed in 3 steps where we first define the sequence $(u^{0,N}, u^{1,N}) \in (\mathbb{P}_N^{Di}(\Omega))^2$ and prove the uniform boundedness of the associated sequence of controls, then characterize their weak limit and finally prove the strong convergence respectively.

Step 1: Uniform bound of the controls. We first state the following result that we prove in the Appendix below.

Lemma 2.5.1. *Given* $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$*, there exists a sequence* $(u^{0,N}, u^{1,N}) \in (\mathbb{P}_N^{Di}(\Omega))^2$ *such that*

$$(u^{0,N}, u^{1,N}) \longrightarrow (u^0, u^1) \text{ in } L^2(\Omega) \times H^{-1}(\Omega). \tag{2.50}$$

On the other hand, for any $(\phi^{0,N},\phi^{1,N})\in\mathbb{P}_N^{Di}(\Omega)\times\mathbb{P}_N^{Di}(\Omega)$ *we have*

$$\left|\Big\langle(\phi^{0,N},\phi^{1,N}),(u^{0,N},u^{1,N})\Big\rangle_N\right|\leq|(u^0,u^1)|_{L^2\times H^{-1}}\|(\phi_x^{0,N},\phi^{1,N})\|_{N\times N}. \tag{2.51}$$

Furthermore, if $(\phi^{0,N},\phi^{1,N})\to(\phi^0,\phi^1)$ *in* $H_0^1(\Omega)\times L^2(\Omega)$ *then,*

$$\Big\langle(\phi^{0,N},\phi^{1,N}),(u^{0,N},u^{1,N})\Big\rangle_N\to\Big\langle(\phi^0,\phi^1),(u^0,u^1)\Big\rangle. \tag{2.52}$$

The result in Lemma 2.5.1 is still true when considering $(u^{0,N},u^{1,N})=(\tilde{u}^{0,N},\tilde{u}^{1,N})$ as defined in Remark 4.

We choose $(u^{0,N},u^{1,N})\in(\mathbb{P}_N^{Di}(\Omega))^2$ as in this lemma. Note that in particular this sequence is uniformly bounded in $L^2\times H^{-1}$ as $N\to\infty$ and the boundedness of the associated controls is a direct consequence of the following result:

Proposition 2.5.1. *Let* f^N, g_L^N, g_R^N *be the controls defined in* (2.35)*. Then, there exists a constant* $M>0$*, independent of* N*, such that*

$$\int_0^T\left|f^N(t)\right|^2dt\ +\int_0^T\left|g_R^N(t)\right|^2dt+\int_0^T\left|g_L^N(t)\right|^2dt\leq M\left|(u^0,u^1)\right|^2_{L^2\times H^{-1}}. \tag{2.53}$$

Proof. As $\hat{\phi}^N$ is the solution of the discrete adjoint system (2.27) associated to the the minimizer $(\hat{\phi}^{0,N},\hat{\phi}^{1,N})$ of J^N in $\mathbb{P}_N^{Di}(\Omega)\times\mathbb{P}_N^{Di}(\Omega)$, we have in particular

$$J^N(\hat{\phi}^{0,N},\hat{\phi}^{1,N})\leq J^N(0,0)=0. \tag{2.54}$$

Then, taking into account the conservation of the discrete energy, defined in (2.42), the approximation in (2.50), estimate (2.51) and the uniform observability inequality in Lemma 2.4.3 we obtain

$$\begin{aligned}\int_0^T\eta(t)\left|\hat{\phi}_x^N(t,1)\ dt-\omega_N\hat{\phi}_{xx}^N(t,1)\right|^2dt+\int_0^T\eta(t)\omega_N\left|\hat{\phi}_{xx}^N(t,-1)\right|^2dt\\+\int_0^T\eta(t)\omega_N\int_0^T\left|\hat{\phi}_{xx}^N(t,1)\right|^2dt\leq C\left|(u^0,u^1)\right|^2_{L^2\times H^{-1}},\end{aligned}$$

which is equivalent to (2.53). □

Step 2: Weak convergence of the control f^N**.** Thanks to the bound (2.53) controls

f^N, g_R^N, g_L^N are uniformly bounded in $L^2(0,T)$ and therefore, there exists a subsequence, still denoted by f^N, g_R^N, g_L^N, such that

$$f^N \underset{N\to\infty}{\rightharpoonup} h,\ g_R^N \underset{N\to\infty}{\rightharpoonup} h_R,\ g_L^N \underset{N\to\infty}{\rightharpoonup} h_L,\ \text{ weakly in }\ L^2(0,T). \tag{2.55}$$

Let us see that $h = f$ where f is the control with minimal L^2-weighted norm of system (2.1), with the weight function $\eta(t)$. In the next step, we show that $h_R = h_L = 0$. This control $f(t)$ can be characterized by the following two properties (see Remark 1):

(P1) f satisfies the variational characterization of controls (2.5).

(P2) $f(t) = \eta(t)\hat{\phi}_x(t,1)$, where $\hat{\phi}$ is a solution of adjoint continue problem (2.3) with final data $(\hat{\phi}^0, \hat{\phi}^1)$.

In what follows we see that h verifies these two properties. We start with the second one. By the boundedness of controls, the estimate (2.36) and the norm equivalence in (2.20), we deduce that $(\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$ is uniformly bounded in $H_0^1 \times L^2$. So, we can extract a sub-sequence, still denoted $(\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$ such that

$$(\hat{\phi}^{0,N}, \hat{\phi}^{1,N}) \rightharpoonup (\hat{\phi}^0, \hat{\phi}^1)\ \textit{weakly in}\ H_0^1(\Omega) \times L^2(\Omega). \tag{2.56}$$

Let $\hat{\phi}^N$ and $\hat{\phi}$ be the solutions of the discrete adjoint system (2.27) and the continuous one (2.3), associated to the final data $(\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$ and $(\hat{\phi}^0, \hat{\phi}^1)$ respectively. The following holds:

$$\hat{\phi}^N \rightharpoonup \hat{\phi},\ \text{weakly-* in }\ L^\infty(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)), \tag{2.57}$$

$$\hat{\phi}_x^N(t,1) - \omega_N \hat{\phi}_{xx}^N(t,1) \rightharpoonup \hat{\phi}_x(t,1),\ \text{weakly in }\ L^2(0,T). \tag{2.58}$$

The convergence result (2.57) is easily deduced from the classical theory of spectral approximation in Canuto et al., 1988 (Section 10.5). Concerning (2.58), we write system (2.27) in a weak form and take as test functions $\psi^N(x) = \frac{1+x}{2} \in \mathbb{P}_N$ and $l(t) \in C_0^1(0,T)$. Multiplying the equations (2.27) by the weights w_i and $\psi^N(x_i)l(t)$, adding in $i \in I$ and integrating in time we easily obtain the following identity,

$$\begin{aligned}
0 &= -\int_0^T (\hat{\phi}^N_{tt}(t,\cdot), \frac{1+x}{2})_N l(t)dt + \int_0^T \int_{-1}^1 \hat{\phi}^N_{xx}(t,x)\frac{1+x}{2} l(t)dt \\
&\quad - \int_0^T \omega_N \hat{\phi}^N_{xx}(t,1))l(t)dt \\
&= \int_0^T \int_{-1}^1 \hat{\phi}^N_t(t,x)\frac{1+x}{2} l_t(t)dt - \frac{1}{2}\int_0^T \int_{-1}^1 \hat{\phi}^N_x(t,x)l(t)dtdx \\
&\quad + \int_0^T (\hat{\phi}^N_x(t,1) - \omega_N \hat{\phi}^N_{xx}(t,1))l(t)dt, \ \forall l(t) \in C^1_0(0,T).
\end{aligned} \tag{2.59}$$

Note that in the second term on the right-hand side, we have used the quadrature formula since the integrand is a polynomial of degree $2N-2$, and this allowed us to integrate by parts in the x variable.

We can pass to the limit in (2.59) thanks to (2.55) and (2.57). Then $\hat{\phi}$ verifies

$$0 = \int_0^T \int_{-1}^1 \hat{\phi}_t(x)\frac{1+x}{2} l_t(t)dxdt - \frac{1}{2}\int_0^T \int_{-1}^1 \hat{\phi}_x(t,x)l(t)dxdt + \int_0^T \tilde{h}(t)l(t)dt, \tag{2.60}$$

for all $l(t) \in C^1_0(0,T)$. Here $\tilde{h}$ is the weak limit of $\hat{\phi}^N_x(t,1) - \omega_N \hat{\phi}^N_{xx}(t,1)$. On the other hand, as $\hat{\phi}$ is a solution of (2.3), it also verifies

$$0 = \int_0^T \int_{-1}^1 \hat{\phi}_t(t,x)\frac{1+x}{2} l_t(t)dxdt - \frac{1}{2}\int_0^T \int_{-1}^1 \hat{\phi}_x(t,x)l(t)dxdt + \int_0^T \hat{\phi}_x(t,1)l(t)dt, \tag{2.61}$$

for all $l(t) \in C^1_0(0,T)$. From (2.60) and (2.61) we finally deduce

$$\int_0^T \tilde{h}(t)l(t)dt = \int_0^T \hat{\phi}_x(t,1)l(t)dt, \ \forall \ l(t) \in C^1_0(0,T),$$

and then $\tilde{h}(t) = \hat{\phi}_x(t,1)$ with $\hat{\phi}$ the solution of (2.3). This finishes the proof of (2.58) and in particular that h satisfies property (P2) above.

Now, we check that the weak limit of $f^N(t)$, $h(t)$, also verifies the first property (P1) above. We need the following lemma that we prove in the appendix below:

Lemma 2.5.2. *Given* $(\phi^0, \phi^1) \in H^1_0(\Omega) \times L^2(\Omega)$*, there exists a sequence* $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}^{Di}_N(\Omega) \times \mathbb{P}^{Di}_N(\Omega)$ *such that*

$$(\phi^{0,N}, \phi^{1,N}) \longrightarrow (\phi^0, \phi^1), \ \textit{in} \ H^1_0(\Omega) \times L^2(\Omega). \tag{2.62}$$

Furthermore, if ϕ^N *is the solution of* (2.27) *with final data* $(\phi^{0,N}, \phi^{1,N})$ *and* ϕ *is the solution of* (2.3) *with final data* (ϕ^0, ϕ^1) *the following holds:*

$$\phi^N \longrightarrow \phi \textit{ in } C((0,T); H_0^1(\Omega)) \cap C^1((0,T); L^2(\Omega)), \tag{2.63}$$

$$\phi_x^N(t,1) \longrightarrow \phi_x(t,1) \textit{ in } L^2(0,T), \tag{2.64}$$

$$\sqrt{w_N}\phi_{xx}^N(t,\pm 1) \longrightarrow 0 \textit{ in } L^2(0,T). \tag{2.65}$$

Given $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, by Lemma 2.5.2 there exists a sequence $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{\partial\Omega}(\Omega)$ such that (2.62) holds. Furthermore, from formula (2.63) we deduce in particular that

$$(\phi^N(0,x), \phi_t^N(0,x)) \longrightarrow (\phi(0,x), \phi_t(0,x)) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega), \tag{2.66}$$

where ϕ^N is the solution of (2.27) with final data $(\phi^{0,N}, \phi^{1,N})$ and ϕ is the solution of (2.3) with final data (ϕ^0, ϕ^1). Now Passing to the limit, as $N \longrightarrow \infty$ in formula (2.26) and taking into account Lemmas 2.5.1 and 2.5.2 we obtain that h satisfies

$$\int_0^T \phi_x(t,1) h dt - \Big\langle (\phi(0,\cdot), \phi_t(0,\cdot)), (u^0, u^1) \Big\rangle = 0, \tag{2.67}$$

$\forall (\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, So h verifies property (P1) above.

Step 3: Strong convergence of the controls. From the lower semi-continuity of the norm with respect to the weak convergence we have

$$|f|_{L^2}^2 + |h_R|_{L^2}^2 + |h_L|_{L^2}^2 \le \liminf_{N\to\infty} |f^N|_{L^2}^2 + |g_R^N|_{L^2}^2 + |g_L^N|_{L^2}^2. \tag{2.68}$$

On the other hand, if we consider formulas (2.26) and (2.5) with $(\phi^{0,N}, \phi^{1,N}) = (\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$

and $(\phi^0, \phi^1) = (\hat{\phi}^0, \hat{\phi}^1)$ respectively, we obtain:

$$\begin{aligned} 0 = \int_0^T \eta(t) \left|\hat{\phi}_x^N(t,1) - \omega_N \hat{\phi}_{xx}^N(t,1)\right|^2 dt + \int_0^T \eta(t) \left|\sqrt{\omega_N}\hat{\phi}_{xx}^N(t,1)\right|^2 dt \\ + \int_0^T \eta(t) \left|\sqrt{\omega_0}\hat{\phi}_{xx}^N(t,-1)\right|^2 dt - \Big\langle (\hat{\phi}^N(\cdot,0), \hat{\phi}_t^N(\cdot,0)), (u^{0,N}, u^{1,N}) \Big\rangle_N, \end{aligned} \tag{2.69}$$

and

$$\int_0^T \eta(t) \left|\hat{\phi}_x(t,1)\right|^2 dt - \Big\langle (\hat{\phi}(0,\cdot), \hat{\phi}_t(0,\cdot)), (u^0, u^1) \Big\rangle = 0. \tag{2.70}$$

The last term in (2.69) converges to the second term in (2.70) and therefore

$$\begin{aligned} \int_0^T \eta(t) \left|\hat{\phi}_x^N(t,1) - \omega_N \hat{\phi}_{xx}^N(t,1)\right|^2 dt + \int_0^T \eta(t) \left|\sqrt{\omega_N}\hat{\phi}_{xx}^N(t,1)\right|^2 dt \\ + \int_0^T \eta(t) \left|\sqrt{\omega_0}\hat{\phi}_{xx}^N(t,-1)\right|^2 dt \longrightarrow \int_0^T \eta(t) \left|\hat{\phi}_x(t,1)\right|^2 dt \; in \; L^2(0,T), \end{aligned} \tag{2.71}$$

as $N \longrightarrow \infty$. Now, taking into account (2.68), (2.71) and the definition of the controls in (2.35) we deduce,

$$|f|_{L^2}^2 + |h_R|_{L^2}^2 + |h_L|_{L^2}^2 \leq |f|_{L^2}^2,$$

and therefore $h_R = h_L = 0$. From (2.71) we also have,

$$|f^N|_{L^2} \to |f|_{L^2}, \quad |h_R^N|_{L^2}^2 \to 0, \quad |h_L^N|_{L^2}^2 \to 0, \tag{2.72}$$

as $N \to \infty$. The strong convergence of the controls $f^N \underset{N\to\infty}{\longrightarrow} f$ in $L^2(0,T)$ is a consequence of the weak convergence stated in Step 2 above and the convergence of the norms stated in (2.72).

2.6 Matrix formulation and finite-dimensional control

In this section, we consider some implementation issues related to the finite-dimensional approximation of boundary control given by (2.22)-(2.23). There are several ways to do that but we follow the more direct approach which consists in writing the collocation control problem into the standard matrix form problem and computing the control using the explicit expression given for the finite-dimensional control theory. This approach requires more computing time but it has a simpler implementation.

We assume $\Omega = (-1, 1)$ and that the control acts at the extreme $x = 1$. Let $N + 1$ be the number of collocation points. The collocation scheme (2.22)-(2.24) can be restated in matrix form as follows:

$$\begin{cases} U_{tt}^N = A^N U^N + B^N F^N(t) \\ U^N(0) = U^{0,N}, U_t^N(0) = U^{1,N}. \end{cases} \tag{2.73}$$

Here we have taken $U^N(t)$ as a $(N-1)$-vector containing the values of the discrete solution at the interior collocation points C^Ω as follows

$$U^N = (u_2^N, u_3^N, ..., u_N^N)^T, \text{where } u_j^N = u^N(t, x_i)\ j = 2, ..., N,\ i = 1, ..., N-1. \tag{2.74}$$

Also the components of the initial data $(U^{0,N}, U^{1,N})$ and the target $(U^N(T), U_t^N(T))$ are written in vector form where

$$(U^{k,N}) = (u_2^{k,N}, u_3^{k,N}, ..., u_N^{k,N})^T,\ U_T^{k,N} = (u_{T,2}^{k,N}, u_{T,3}^{k,N}, ..., u_{T,N}^{k,N})^T,\ k = 0, 1. \tag{2.75}$$

On other hand, $F^N(t) \in \mathcal{M}_{3(N-1)\times 1}$ is the vector of controls and $A^N \in \mathcal{M}_{(N-1)^2}$, $B^N \in \mathcal{M}_{(N-1)\times 3(N-1)}$ are suitable matrixes that we define below.

The matrix A^N in (2.73) is computed from the well-known Legendre differentiation matrix $K^N \in \mathcal{M}_{(N+1)^2}$ that relates a polynomial of degree N with its derivative at the LGL nodes (see Canuto et al., 1988).

$$K_{ji}^N = \begin{cases} \dfrac{L^N(x_j)}{L^N(x_i)} \dfrac{1}{x_j - x_i}, & j \neq i, \\[2ex] \dfrac{N(N+1)}{4}, & j = i = 1, \\[2ex] -\dfrac{N(N+1)}{4}, & j = i = N+1, \\[2ex] 0, & \text{otherwise.} \end{cases} \tag{2.76}$$

In fact,

$$A^N = (K^N)^2(2:N, 2:N), \tag{2.77}$$

(in Matlab notation), i.e. the square of K^N stripped of its first and last rows and columns corresponding to the boundary nodes, where the solution is given either by the boundary condition or the control. The vector $F^N(t)$ contains 3 components corresponding to the 3 controls in (2.22), i.e.

$$F^N = (f^N, g_L^N, g_R^N)^T.$$

where f^N is a column vector with the coefficients of the controls

$$f^N = (f_2^N, f_3^N, ..., f_N^N)^T \in \mathcal{M}_{(N-1)\times 1}.$$

Something similar can be said about g_L^N and g_R^N.

Finally, the matrix $B^N \in \mathcal{M}_{(N-1)\times 3(N-1)}$ corresponds to the contribution of the boundary control accounting for the last columns that affect the values of solution in the boundary and the artificial controls required to obtain the uniform controllability of the discrete system. They are given by the right-hand sides of equilibrium equation (2.22) as follows

$$B^N = ((K^N)^2(2:N, N+1)\ G_r^N(2:N)\ G_l^N(2:N)).$$

Here G_r^N, G_l^N are defined by (2.24) as:

$$\begin{cases} G_r^N = \dfrac{(K^N)^2 h_r}{\sqrt{\omega_N}} + \dfrac{K^N \Psi_N}{\omega_N\sqrt{\omega_N}}, \\ G_l = \dfrac{(K^N)^2 h_l}{\sqrt{\omega_0}} + \dfrac{K^N \Psi_0}{\omega_0\sqrt{\omega_0}}, \\ h_r = (0, \dfrac{1+x(2:N)}{2}, 0)^T,\ h_l = (0, \dfrac{1-x(2:N)}{2}, 0)^T, \\ \Psi_N = (zeros(N,1), 1)^T, \Psi_0 = (1, zeros(N,1))^T, \omega_N = 2/(N(N+1)), \end{cases} \tag{2.78}$$

where $x \in \mathcal{M}_{(N+1)\times 1}$-vector of (LGL) collocation points. To apply the general theory of controllability for finite dimensional systems, we write (2.73) as a first order one

$$\begin{cases} Z_t^N = D^N Z^N + C^N F^N(t), \\ Z^N(0) = Z^{0,N}, \end{cases} \tag{2.79}$$

with $Z^N = \begin{pmatrix} U^N \\ U_t^N \end{pmatrix}$, $D^N = \begin{pmatrix} 0 & I \\ A^N & 0 \end{pmatrix}$, $C^N = \begin{pmatrix} 0 \\ B_t^N \end{pmatrix}$, $Z^{0,N} = \begin{pmatrix} U^{0,N} \\ U^{1,N} \end{pmatrix}$ and the target $Z_T^N = \begin{pmatrix} U_T^{0,N} \\ U_T^{1,N} \end{pmatrix}$.

According to the control theory for finite dimensional systems (see Zuazua, 2014), a control that drives the initial data $Z^{0,N}$ to the target Z_T^N is given by

$$F^N(t) = -\eta(t)(C^N)^T e^{-t(A^N)^T} Q_T^{-1}\Big(e^{-t(A^N)} Z^{0,N} - Z_T^N\Big), \tag{2.80}$$

where Q_T is the average controllability gramiam (see Zuazua, 2014) given by

$$Q_T = \int_0^T \eta(t)(C^N)e^{(T-t)(A^N)}(C^N)^T e^{(T-t)(A^N)^T} dt. \tag{2.81}$$

The fact that $F^N(t)$ in (2.80) is a control for (2.79) can be easily checked. We only have to substitute it in the solution of the system (2.79), given by

$$Z^N(t) = e^{A^N t} Z^{0,N} + \int_0^t e^{A^N(t-s)} C^N F^N(s) ds. \tag{2.82}$$

Moreover, this control $F^N(t)$ is a vector that contains the 3 controls that are obtained from the minimizer of J^N in (2.34).

2.7 Numerical experiments

In this section, we illustrate the results in this chapter approximating the boundary control for the 1-d wave equation. For the time discretization, we use a classical Newmark method with parameters $\gamma = 1/2$, $\beta = 1/4$ and time step $dt = 10^{-2}$. With this choice, the time scheme is second-order accurate (see Raviart and Thomas, 1983).

Experiment 1: We first consider the one-dimensional wave equation with three different types of initial position and velocity (see Figure 2.3). The first one corresponds to a smooth bump that moves to the left-hand side and it is controlled from the right extreme. It is given by $u^0(x) = e^{-10x^2}$, $u^1(x) = -20xe^{-10x^2}$. The second one corresponds to a Lipschitz continuous initial data $u^0(x) = 1 - |x|$, $u^1(x) = 0$. The last one corresponds to a discontinuous initial data $u^0(x) = 2(x+1)_{(-1,0)}$, $u^1(x) = 0$. We take final time $T = 4.4$ with time step $dt = 10^{-2}$.

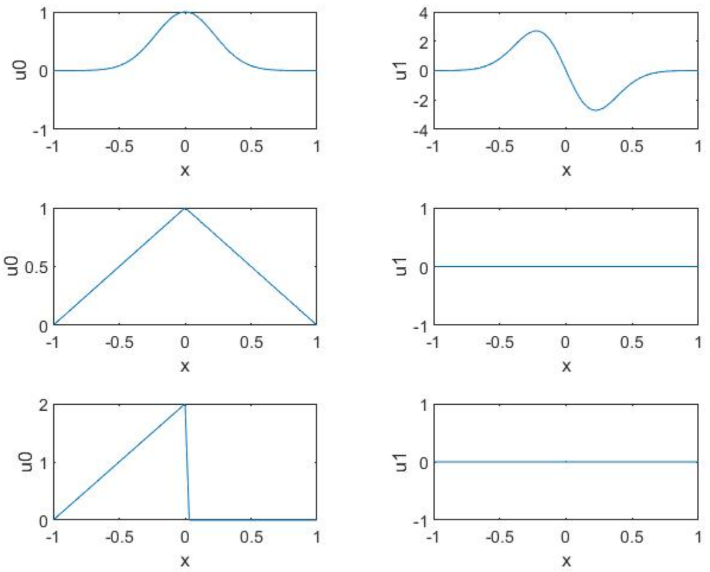


Figure 2.3: Initial data $u^0(x) = e^{-10x^2}$, $u^1(x) = -20xe^{-10x^2}$ (upper one), $u^0(x) = 1-|x|$, $u^1(x) = 0$ (center one) and $u^0(x) = 2(x+1)_{(-1,0)}$, $u^1(x) = 0$ (bottom one).

Note that the time control is only slightly greater than the minimal control time for the continuous wave equation ($T = 4$) and lower than the time given by the discrete control problem in Theorems 2.3.1 and 2.3.2. The good approximation obtained in this case provides numerical evidence that the control time in the mentioned theorems is probably not optimal.

In Table 2.1 and 2.2 we show the behaviour of the norm of the controls when the degree of polynomials N grows. As stated in Theorem 2.3.2, we observe that the boundary control remains bounded while the two artificial controls included in the system (g_L^N and g_R^N) vanish as N grows. The boundary control is plotted In Figure 2.4 and the controlled solutions for these different initial data are plotted In Figure 2.5, 2.6 and 2.7.

Experiment 2: Here we illustrate the rate of convergence of the discrete control to the limit one with two different types of initial data: the first one is given by $u^0(x) = 1-|x|$, $u^1(x) = 0$

N	$\left\|f^N\right\|_{L^2}$	$\left\|g_R^N\right\|_{L^2}$	$\left\|g_L^N\right\|_{L^2}$
20	5.6×10^{-1}	2×10^{-3}	2×10^{-3}
50	5.6×10^{-1}	1×10^{-4}	1×10^{-4}
100	5.6×10^{-1}	1×10^{-6}	1×10^{-6}

Table 2.1: Norm of the null controls when $u^0(x) = e^{-10x^2}, u^1(x) = -20xe^{-10x^2}$ as N grows.

N	$\left\|f^N\right\|_{L^2}$	$\left\|g_R^N\right\|_{L^2}$	$\left\|g_L^N\right\|_{L^2}$
20	5.8×10^{-1}	3×10^{-4}	3×10^{-4}
50	5.8×10^{-1}	4×10^{-4}	4×10^{-4}
100	5.8×10^{-1}	1×10^{-4}	1×10^{-4}
200	5.8×10^{-1}	1×10^{-6}	1×10^{-6}

Table 2.2: Norm of the null controls as N grows for $u^0(x) = 1 - |x|$, $u^1(x) = 0$.

(Lipschitz) and the other considers $u^0(x) = 2(x+1)1_{(-1,0)}$, $u^1(x) = 0$ (discontinuous). The exact control is known in the latter and it is given by $f(t) = -(t-2)\, 1_{(1,3)}(t)$ (see Burman et al., 2023). For the first one, we compare the L^2-norm of the difference between the discrete control when $N = 200$ (that we take as continuous control) and the discrete control as N grows. For the second one we compare directly the approximation with the exact control. The approximate controls are plotted In Figure 2.4 (center and right plots).

In Table 2.3-2.4 we show the error between the discrete control and the limit one. Comparing the error associated to values of $N \in [10, 100]$ we can give a rough estimate of the rate of convergence. More precisely, we observe that $\left|f^N - f\right|_{L^2} \sim N^{-\alpha}$ where α is estimated by the slope of the graphics relating $-\log_{10}\left|f^N - f\right|_{L^2}$ with $\log_{10} N$ (see Figure 2.8).

It is interesting also to compare the results with those obtained in Burman et al., 2023 for the space-time finite element method. While the error in the latter depends linearly on the size of the finite element mesh, for the spectral method the dependence is a power of the polynomial degree N. This power depends on the smoothness of the initial data. Of course, there are other important parameters involved as the complexity of the programming or the time cost that are much more difficult to assess.

N	$\log(\left\|f^N - f^{200}\right\|_{L^2})$	$\log(\left\|g_R^N\right\|_{L^2})$	$\log(\left\|g_L^N\right\|_{L^2})$
10	-1.9	-2.5	-2.5
50	-2.5	-3.3	-3.3
100	-3.0	-3.8	-3.8

Table 2.3: Convergence of the discrete control to the limit as N grows for $u^0(x) = 1-|x|$, $u^1(x) = 0$.

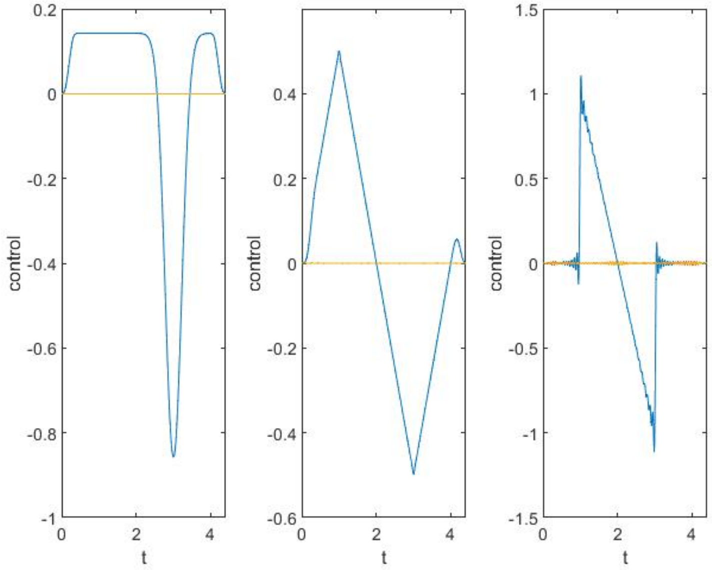


Figure 2.4: Discrete controls f^{100} for different initial data: $u^0(x) = e^{-10x^2},\ u^1(x) = -20xe^{-10x^2}$ (left one), $u^0(x) = 1-|x|,\ u^1(x) = 0$ (center one) and $u^0(x) = 2(x+1)1_{(-1,0)},\ u^1(x) = 0$ (right one).

N	$\log(\left\| f^N - f \right\|_{L^2})$	$\log(\left\| g_R^N \right\|_{L^2})$	$\log(\left\| g_L^N \right\|_{L^2})$
10	-0.4	-0.8	-0.8
50	-0.7	-1.5	-1.5
100	-0.8	-1.7	-1.7

Table 2.4: Convergence of the discrete control to the limit as N grows for $u^0(x) = 2(x+1)1_{(-1,0)}$, $u^1(x) = 0$.

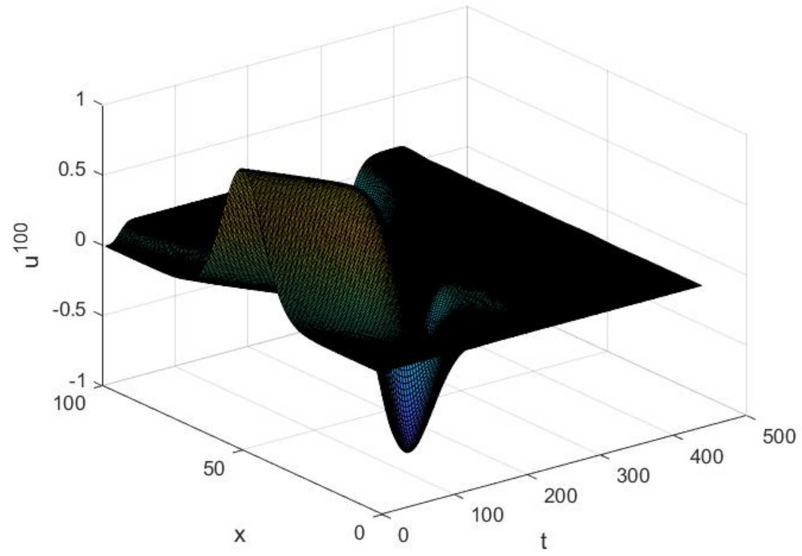


Figure 2.5: Discrete solution u^{100} for smooth initial data $u^0(x) = e^{-10x^2},\ u^1(x) = -20xe^{-10x^2}$.

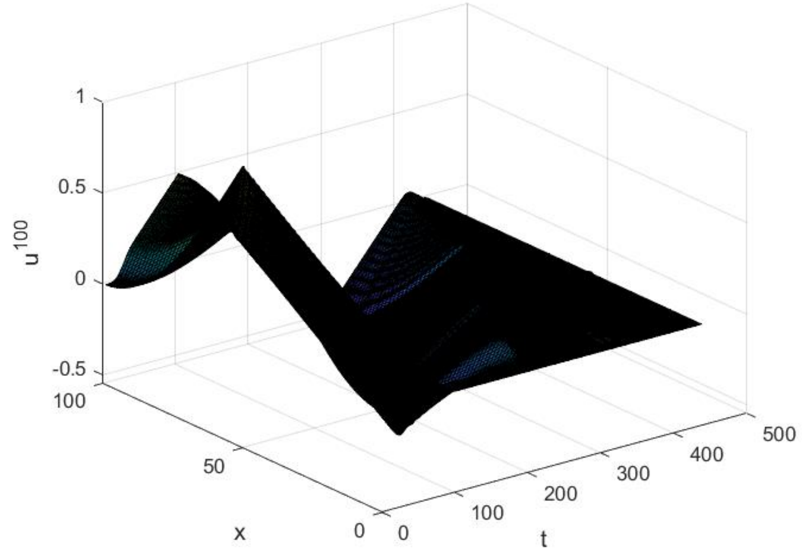


Figure 2.6: Discrete solution u^{100} for lipshitz initial data $u^0(x) = 1 - |x|,\ u^1(x) = 0$.

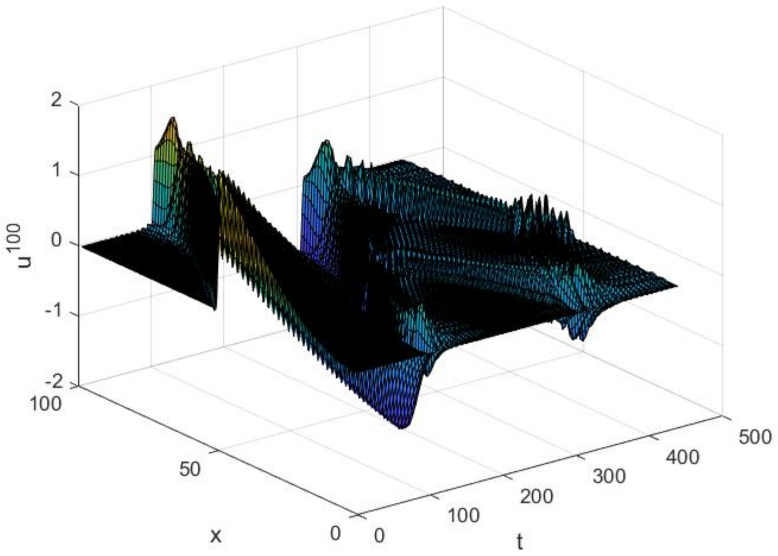


Figure 2.7: Discrete solution u^{100} for discontinuous initial data$u^0(x) = 2(x+1)1_{(-1,0)}$, $u^1(x) = 0$.

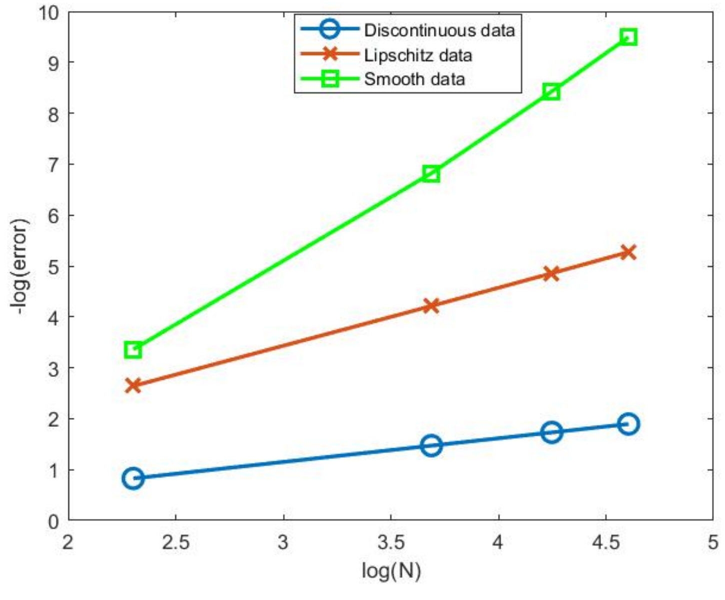


Figure 2.8: The slope of each graphic gives an experimental estimate of the value α for which $\left|f^N - f\right|_{L^2} \sim N^{-\alpha}$. The three graphics correspond to the three initial data considered in experiments 1 and 2. We observe that α is greater for smoother data, as expected in spectral methods.

Chapter 3

Numerical approximation of the boundary control for the wave equation in a square domain with a spectral collocation method

In this chapter, we extend the results of the first chapter to the square domain and show some numerical experiments to illustrate the expected convergence of spectral methods in this case. The results are extended by the separation of variables, following the same strategies used in the previous chapter. First, we introduce a suitable approximate control problem obtained by collocation in the Gauss-Lobatto points and solve this finite-dimensional system in the space of polynomials of degree $\mathbf{N} = (N_1, N_2) \in \mathbb{N}^2$ in space. Then we prove that we can choose a sequence of controls $f^{\mathbf{N}}$ associated to the approximate control problem in such a way that they converge, as $\mathbf{N} \to \infty$, to a control of the continuous wave equation. Since the method uses separation of space variables, our approach is not suitable to tackle generalization to more general domains.

3.1 Problem Statement

Consider the wave equation on a square domain $\Omega = (-1,1)^d \subset \mathbb{R}^d$ $(d = 1,2)$ with a control f acting on one part of the boundary $\Gamma \subset \partial\Omega$ for some time $t \in (0,T)$:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = (0,T) \times \Omega, \\ u = f & \text{on } (0,T) \times \Gamma, \\ u = 0 & \text{on } (0,T) \times \partial\Omega \backslash \Gamma, \\ u(0,\mathbf{x}) = u^0(\mathbf{x}),\ u_t(0,\mathbf{x}) = u^1(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{3.1}$$

Given any $f \in L^2((0,T) \times \Gamma)$ and some initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, problem (3.1) has a unique solution $(u, u_t) \in C((0,T), L^2(\Omega) \times H^{-1}(\Omega))$. It is also well-known that, if $T > T_0$ with T_0 sufficiently large and $\Gamma \subset \partial\Omega$ satisfies some geometric conditions (see Bardos et al., 1992, Lions, 1988a), for any initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control $f \in L^2((0,T) \times \Gamma)$ such that the solution of (3.1) $u \in C((0,T); H_0^2(\Omega)) \cap C^1((0,T); L^2(\Omega)))$ can be driven to any final target. We assume without loss of generality that this final target is the equilibrium, i.e.

$$u(T,\mathbf{x}) = u_t(T,\mathbf{x}) = 0,\ \ \mathbf{x} \in \Omega. \tag{3.2}$$

In particular, this is true in dimension $d = 2$ when Γ is the union of two consecutive sides and $T_0 = 4\sqrt{2}$ (see Lions, 1988a). The main control results of the continuous problem (3.1) is the following theorem.

Theorem 3.1.1. *Given* $T > 4\sqrt{2}$ *and* $(u^0,\ u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$*, there exist a control* $f \in\ L^2(0,T;\Gamma)$*, such that the solution* u *of* (3.1) *satisfies* (3.2).

We follow HUM (see Lions, 1988a) and approximate the control that has minimal L^2-norm with respect to a suitable weighted norm, with a smooth weight $\eta(t)$ that is compactly supported bump function in $t \in (0,T)$. This ensures that the control itself is compactly supported and avoids possible singularities at times $t = 0, T$.

In order to find a numerical approximation of this control f in (3.1) we proceed as follows: first we introduce a discrete version of the control problem (3.1) by collocation, depending on a discrete parameter $\mathbf{N} = (N_1, N_2) \longrightarrow \infty$. Then, we prove that this system is controllable for all $\mathbf{N}$ with five different boundary controls, $f^{\mathbf{N}}, g_k^{\mathbf{N}}, k = 1, ..., 4$ that we can choose in such way that

$$f^{\mathbf{N}} \underset{\mathbf{N}\to\infty}{\longrightarrow} f,\ g_k^{\mathbf{N}} \underset{\mathbf{N}\to\infty}{\longrightarrow} 0,\ k = 1, ..., 4,$$

where f is a control of (3.1). Therefore, $f^{\mathbf{N}}$ is a numerical approximation of a continuous control f, while $g_k^{\mathbf{N}}, k = 1, ..., 4$ can be understood as artificial controls which are only necessary to obtain $f^{\mathbf{N}}$.

The rest of this chapter is organized as follows. In section 3.2, we briefly describe the mathematical background of the continuous wave control problem. In section 3.3, we introduce the discrete control problem obtained by the spectral method and state the main results of the chapter. In section 3.4, we present the proof of the existence of discrete controls for the spectral approximation. In section 3.5, we present the proof of convergence of the discrete control to the continuous one. In section 3.6, we state the matrix formulation of the discrete control problem that we approximate using the finite-dimensional control theory. In section 3.7, we present some numerical experiments. Finally, section 3.8 is devoted to the Appendix containing the main spectral results required for the analysis of the convergence.

3.2 The continuous control problem

In this section, we briefly describe the mathematics background of the variational characterization of the control problem (3.1) that we use later to find their numerical approximation. In particular, we prove that a class of controls can be obtained as minimizers of a quadratic functional defined on a Hilbert space. The results of this section are not new and can be found in Micu and Zuazua, 2004. We include them here for completeness.

To avoid discontinuities between the control at $t = 0$ and the initial data, which is zeros at the boundary, we restrict ourselves to controls which are zero near $t = 0, T$. This affects to the quadratic functional that we define below.

Let us consider the following backwards wave equation,

$$\begin{cases} \phi_{tt} - \Delta\phi = 0 & \text{in } (t, \mathbf{x}) \in (0, T) \times \Omega, \\ \phi = 0 & \text{in } (t, \mathbf{x}) \in (0, T) \times \partial\Omega, \\ \phi(T, \mathbf{x}) = \phi^0, \quad \phi_t(T, \mathbf{x}) = \phi^1 & \text{in } \mathbf{x} \in \Omega, \end{cases} \tag{3.3}$$

where $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$. We also define the duality product between $L^2(\Omega) \times H^{-1}(\Omega)$ and $H_0^1(\Omega) \times L^2(\Omega)$ by

$$\Big\langle(\phi^0, \phi^1), (u^0, u^1)\Big\rangle =< u^1, \phi^0 >_{H^{-1}, H_0^1} - \int_\Omega u^0 \phi^1 d\mathbf{x}, \tag{3.4}$$

where $< \cdot, \cdot >_{H^{-1}, H_0^1}$ is the usual duality product between H_0^1 and its dual space H^{-1}.

The following result provides a variational characterization of the control.

Lemma 3.2.1. *Assume that* $T > 0$*, and consider some initial data* $(u^0,\ u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$*. Any controls* f *that make the solution of the discrete system* (3.1) *satisfy* (3.2) *are solution of,*

$$0 = \int_0^T \int_\Gamma \frac{\partial\phi}{\partial\boldsymbol{\nu}} f d\gamma dt - \Big\langle(\phi(0, \cdot), \phi_t(0, \cdot)), (u^0, u^1)\Big\rangle, \tag{3.5}$$

for all $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, where* (ϕ, ϕ_t) *is the solution of the backwards wave equation* (3.3)*. Reciprocally, if* f *satisfies* (3.5)*, then* (3.2) *holds.*

The proof of Lemma 3.2.1 is easily obtained as in the 1-d case (see proof of Lemma 2.2.1).

According to **H.U.M**, one possibility to construct controls f that satisfy the variational condition (3.5) is as minimizers of a particular quadratic functional. We define the following cost functional $J : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$ by

$$J(\phi^0, \phi^1) = \frac{1}{2} \int_0^T \int_\Gamma \eta(t) \left|\frac{\partial\phi}{\partial\boldsymbol{\nu}}\right|^2 d\gamma dt - \Big\langle(\phi(0, \cdot), \phi_t(0, \cdot)), (u^0,\ u^1)\Big\rangle, \tag{3.6}$$

where (ϕ, ϕ_t) is the solution of (3.3) with final data $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$.The function $\eta(t)$ is chosen to give a compact support control in $(0, T)$ as prescribed in (2.9).

Theorem 3.2.2. *Assume* $(u^0,\ u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *and we suppose that* $(\hat{\phi}^0, \hat{\phi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *is a minimizer of* J*. If* $\hat{\phi}$ *is the corresponding solution of* (3.3) *with final data* $(\hat{\phi}^0, \hat{\phi}^1)$

then

$$f(t,\mathbf{x}) = \eta(t)\frac{\partial\phi}{\partial\boldsymbol{\nu}}\bigg|_{\Gamma} \tag{3.7}$$

is a control such that the solution of (3.1) *satisfies* (3.2)*, which is the one with minimal* L^2 *weighted norm.*

The proof of Theorem 3.2.2 is easily obtained as in the 1-d case (see proof of Theorem 3.2.2). Let us now give a general condition which ensures the existence of a minimizer for J and, therefore, a control result for system (3.1).

The functional J is evidently continuous and convex. Consequently, the existence of a minimizer, and thus discrete controls for system (3.1), are assured once we establish its coercivity. This follows as a consequence of the following lemma.

Lemma 3.2.3. *Given* $T > 4\sqrt{2}$*, there exists a constant* $C > 0$*, such that the solution of system* (3.3) *satisfy*

$$C\left\|(\phi^0,\phi^1)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_0^T\int_\Gamma \left|\frac{\partial\phi}{\partial\boldsymbol{\nu}}\right|^2 d\gamma dt, \tag{3.8}$$

for any final data $(\phi^0,\phi^1) \in H_0^1(\Omega)\times L^2(\Omega)$*.*

The proof of Lemma 3.2.3 is easily obtained as in the 1-d case (see proof of Lemma 2.2.3). Note that the observability that we have defined above in (3.8) is a sufficient condition for the controllability of problem (3.1)and then the functional J defined by (3.6) has an unique minimizer $(\hat{\phi}^0,\hat{\phi}^1) \in H_0^1(\Omega)\times L^2(\Omega)$. Hence, a unique control $f \in L^2(0,T;\Gamma)$.

3.3 Approximation by the spectral collocation method

In this section, we introduce some notation, the approximate discrete control problem and state the main results of the chapter.

Here, we assume that the control acts in the right and top sides $\Gamma = \Gamma_1\cup\Gamma_2 = \{(1,x_2)\in\mathbb{R}^2 : -1\leq x_2\leq 1\}\cup\{(x_1,1)\in\mathbb{R}^2 : -1\leq x_1\leq 1\}$. The rest of the boundary is then the left and down sides $\partial\Omega\backslash\Gamma = \Gamma_3\cup\Gamma_4 = \{(-1,x_2)\in\mathbb{R}^2 : -1\leq x_2\leq 1\}\cup\{(x_1,-1)\in\mathbb{R}^2 : -1\leq x_1\leq 1\}$ (see Figure 3.1).

We also consider the discrete parameter $\mathbf{N} = (N_1, N_2) \in \mathbb{N} \times \mathbb{N}$ and the Legendre Gauss-Lobatto nodes in each variable $C = \{\mathbf{x_i} = (x_{1i_1}, x_{2i_2}), (0,0) \leq \mathbf{i} = (i_1, i_2) \leq (N_1, N_2)\}$ that are the roots of

$$(1 - x_1^2)\frac{d}{dx_1}L^{N_1}(x_1)(1 - x_2^2)\frac{d}{dx_2}L^{N_2}(x_2),$$

where L^k is the k-th Legendre polynomial in $(-1, 1)$ defined as in (2.16). Note that C (and some other quantities defined below) depends on $\mathbf{N}$, but we do not make explicit this dependence in the notation to simplify. In Figure 3.1 we show the LGL nodes in $(-1, 1)^2$.

We also set $C = C^{\Omega} \cup C^{\partial\Omega}$ such that $C^{\partial\Omega} = C^{\Gamma} \cup C^{\partial\Omega\setminus\Gamma}$ and $\mathbf{I} = \mathbf{I}_{\Omega} \cup \mathbf{I}_{\partial\Omega}$ such that $\mathbf{I}_{\partial\Omega} = \cup_{k=1}^{4}\mathbf{I}_k$ where $\mathbf{I}_k$ is the set of indexes corresponding to the collocation nodes on the boundary Γ_k.

Let $\mathbb{P}_{\mathbf{N}}(\Omega)$ be the space of continuous functions in $\bar{\Omega}$ which are polynomials of degree less than or equal to N_1 (respectively N_2) in the x_1-variable (respectively in the x_2-variable), and let $\mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$ be the subspace of $\mathbb{P}_{\mathbf{N}}(\Omega)$ of those functions vanishing on $\partial\Omega$. The set $\mathbb{P}_{\mathbf{N}}(\Gamma)$ (resp $\mathbb{P}_{\mathbf{N}}(\Gamma_k), k = 1, ..., 4$) is the restriction of $\mathbb{P}_{\mathbf{N}}(\Omega)$ to Γ (resp to $\Gamma_k, k = 1, ..., 4$).

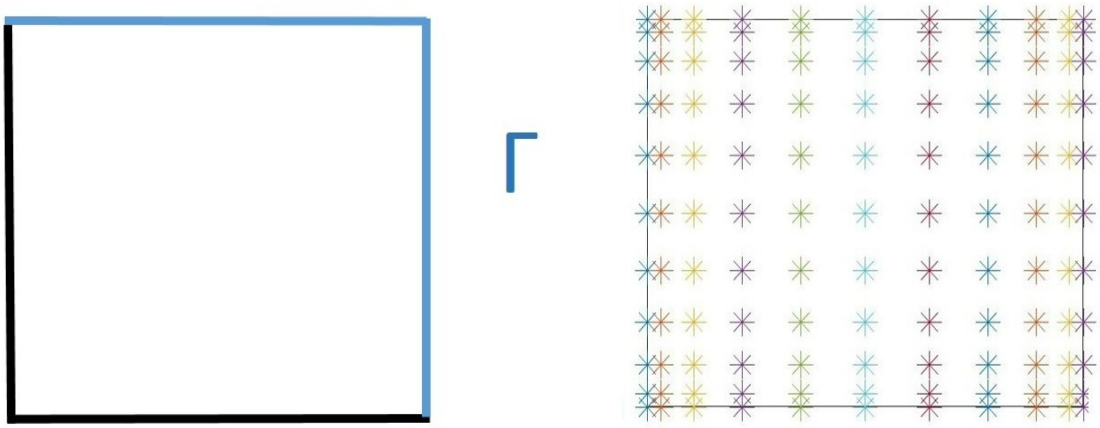


Figure 3.1: The left figure clarify the square domain and the boundary Γ when we act a control in problem (3.1) and the right one illustrate the LGL nodes $(\mathbf{x_i})$ in $(-1, 1)^2$ and on the boundaries (crosses).

We define the following discrete inner product that approximates that of $L^2(\Omega)$:

$$(w, z)_{\mathbf{N}} = \sum_{\mathbf{i} \in \mathbf{I}}(wz)(t, \mathbf{x_i})\omega_{\mathbf{i}} = \sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}(wz)(t, x_{1i_1}, x_{2i_2})\omega_{i_1}^{x_1}\omega_{i_2}^{x_2}, \quad 0 \leq t \leq T, \tag{3.9}$$

where $\omega_{i_1}^{x_1}, \omega_{i_2}^{x_2}$ are the discrete weights associated with the one-dimensional LGL quadrature formula in each one of the variables expressed as in (2.18). As in one-dimensional owing to the exactness of the integration LGL formula, we have

$$(w,z)_{\mathbf{N}} = \int_\Omega wz \; d\mathbf{x} \text{ for all } w, z \text{ such that } wz \in \mathbb{P}_{2\mathbf{N}-1}(\Omega). \tag{3.10}$$

Note that as in (2.20), the discrete norm $\|\cdot\|_{\mathbf{N}}$ is uniformly equivalent to the L^2-norm $|\cdot|_{L^2}$ in $\mathbb{P}_{\mathbf{N}}(\Omega)$ with constant $C_1 = 1$ and $C_2 = (2 + N_1^{-1})(2 + N_2^{-1})$:

$$C_1 \left|p\right|_{L^2}^2 \leq \|p\|_{\mathbf{N}}^2 \leq C_2 \left|p\right|_{L^2}^2, \; \forall p \in \mathbb{P}_{\mathbf{N}}(\Omega). \tag{3.11}$$

Now, we introduce the following discrete control problem: Given $u^{0,\mathbf{N}}, u^{1,\mathbf{N}} \in \mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$ and $T > 0$, find $f^{\mathbf{N}} \in L^2(0,T;\mathbb{P}_{\mathbf{N}}(\Gamma))$, $g_k^{\mathbf{N}} \in L^2(0,T;\mathbb{P}_{\mathbf{N}}(\Gamma_k))$, $k = 1, ..., 4$, such that the solution $u^{\mathbf{N}} \in C^\infty(L^2(0,T);\mathbb{P}_{\mathbf{N}}(\Omega))$ of the following system:

$$\begin{cases} \left(u_{tt}^{\mathbf{N}} - \Delta u^{\mathbf{N}}\right)(t,\mathbf{x_i}) = \sum_{k=1}^4 g_k^{\mathbf{N}}(t,\mathbf{x_i}) G_k^{\mathbf{N}}(\mathbf{x_i}) & \text{in } (t,\mathbf{x_i}) \in (0,T) \times C^\Omega, \\ u^{\mathbf{N}}(t,\mathbf{x_i}) = f^{\mathbf{N}}(t,\mathbf{x_i}) & \text{in } (t,\mathbf{x_i}) \in (0,T) \times C^\Gamma, \\ u^{\mathbf{N}}(t,\mathbf{x_i}) = 0 & \text{in } (t,\mathbf{x_i}) \in (0,T) \times C^{\partial\Omega\setminus\Gamma}, \\ u^{\mathbf{N}}(0,\mathbf{x_i}) = u^{0,\mathbf{N}}(\mathbf{x_i}), \quad u_t^{\mathbf{N}}(0,\mathbf{x_i}) = u^{1,\mathbf{N}}(\mathbf{x_i}) & \text{in } \mathbf{x_i} \in C^\Omega \end{cases} \tag{3.12}$$

satisfies

$$u^{\mathbf{N}}(T,\mathbf{x_i}) = u_t^{\mathbf{N}}(T,\mathbf{x_i}) = 0, \; \mathbf{x_i} \in C^\Omega. \tag{3.13}$$

Here, $G_k^{\mathbf{N}}, k = 1, .., 4$ are defined as in the 1-d case (see (2.24)). For example, for the left and right boundaries, Γ_3 and Γ_1, the functions $G_3^{\mathbf{N}}$ and $G_1^{\mathbf{N}}$ depend only on the x_1 variable and coincide with $G^L(x_1)$ and $G^R(x_1)$ defined in (2.24) associated to polynomials of degree N_1 in x_1. Analogously, for the bottom and top boundaries, Γ_4 and Γ_2, the functions $G_4^{\mathbf{N}}$ and $G_2^{\mathbf{N}}$ depend only on the x_2 variable and coincide again with $G^L(x_2)$ and $G^R(x_2)$ defined in (2.24), this time associated to polynomials of degree N_2 in x_2.

Remark 7. *Note that system* (3.12) *is obtained from* (3.1) *by collocation and therefore the equation in* (3.12) *is prescribed only at the interior collocation points. Assuming* $f^{\mathbf{N}}$, $g_k^{\mathbf{N}}$, $k = 1, .., 4$ *are known, system* (3.12) *is a second-order system of ODE with* $(N_1 + 1)(N_2 + 1)$

equations and unknowns, namely the coefficients of the polynomial $u^{\mathbf{N}}$*.*

Comparing the control problems (3.1) *and* (3.12)*, we see that the discrete version has* 4 *extra controls* $g_k^{\mathbf{N}} \in L^2(0, T : \Gamma_k)$*,* $k = 1, ..., 4$*. As we show below, they are necessary to obtain a bounded sequence of discrete controls* $f^{\mathbf{N}}$*.*

We now state the main results of this chapter.

Theorem 3.3.1. *Given* $T > 4\sqrt{2}(2+N_1^{-1})(2+N_2^{-1})$ *and* $(u^{0,\mathbf{N}},\ u^{1,\mathbf{N}}) \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^2$*, there exist controls* $f^{\mathbf{N}} \in\ L^2(0, T; \mathbb{P}_{\mathbf{N}}(\Gamma))$*,* $g_k^{\mathbf{N}} \in\ L^2(0, T; \mathbb{P}_{\mathbf{N}}(\Gamma_k))$*,* $k = 1, ..., 4$*, such that the solution* $u^{\mathbf{N}}$ *of* (3.12) *satisfies* (3.13)*.*

Theorem 3.3.2. *Given* $(u^0, u^1) \in L^2 \times H^{-1}$*, there exists a sequence* $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^2$ *such that*

$$(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \to (u^0, u^1) \text{ in } L^2 \times H^{-1}.$$

Furthermore, for any $T > 4\sqrt{2}(2 + N_1^{-1})(2 + N_2^{-1})$*, we can choose the controls* $f^{\mathbf{N}}, g_k^{\mathbf{N}}$*,* $k = 1, ..., 4$*, such that the solution* $u^{\mathbf{N}}$ *of* (3.12) *satisfies* (3.13) *and*

$$f^{\mathbf{N}} \longrightarrow f \text{ in } L^2(0, T; \Gamma), \quad g_k^{\mathbf{N}} \longrightarrow 0 \text{ in } L^2(0, T; \Gamma_k), \tag{3.14}$$

where f *is a control of the continuous wave equation* (3.1)*.*

Remark 8. *Note that the control time* T *in Theorem 3.3.2 is basically four times the time required in the continuous problem. This is due to the constant* C_2 *in* (3.11) *and probably not optimal, as we illustrate in the experiments below.*

Remark 9. *When* u^0 *and* u^1 *are continuous functions the polynomials* $\tilde{u}^{0,\mathbf{N}}, \tilde{u}^{1,\mathbf{N}} \in \mathbb{P}_{\mathbf{N}}^{Di}$ *that coincides with* u^0 *and* u^1 *at the nodes* $\mathbf{x_i} \in C^{\Omega}$ *give a discretization that satisfies*

$$(\tilde{u}^{0,\mathbf{N}}, \tilde{u}^{1,\mathbf{N}}) \to (u^0, u^1) \text{ in } L^2 \times H^{-1},$$

as $\mathbf{N} \to \infty$*. In this case, the result in Theorem 3.3.2 is still true when considering* $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) = (\tilde{u}^{0,\mathbf{N}}, \tilde{u}^{1,\mathbf{N}})$*. Thus, for smooth functions, there exists a constructive way to choose* $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}})$ *in Theorem 3.3.1.*

The proofs of these two results follow closely the one-dimensional case. They are based on a suitable variational characterization of the controls and the uniform observability inequality for a corresponding adjoint system.

3.4 Existence of discrete controls: proof of Theorem 3.3.1

In this section, we prove Theorem 3.3.1. As we did above in the previous chapter, first, we establish a variational characterization of the discrete HUM controls using equation (3.12). Subsequently, we prove that a specific discrete control can be derived as the minimizer of a convex quadratic functional defined within a polynomial space. Lastly, we establish the coerciveness of this function, ensuring the existence of minimizers.

Let us introduce the following bilinear form in $(\mathbb{P}_{\mathbf{N}}^{Di}(\Omega)) \times (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))$:

$$\left\langle (\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \right\rangle_{\mathbf{N}} = (u^{1,\mathbf{N}}, \phi^{0,\mathbf{N}})_{\mathbf{N}} - (u^{0,\mathbf{N}}, \phi^{1,\mathbf{N}})_{\mathbf{N}}. \tag{3.15}$$

Lemma 3.4.1. *Assume that $T > 0$, and consider some initial data $(u^{0,\mathbf{N}},\ u^{1,\mathbf{N}}) \in \mathbb{P}_{\mathbf{N}}^{Di}(\Omega) \times \mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$. Any controls $f^{\mathbf{N}}, g_k^{\mathbf{N}}, k = 1, ..., 4$ that make the solution of the discrete system* (3.12) *satisfy* (3.13) *are solution of*

$$\int_0^T \sum_{\mathbf{i}\in\mathbf{I_1}\cup\mathbf{I_2}} \left(f^{\mathbf{N}} \left(\frac{\partial \phi^{\mathbf{N}}}{\partial \boldsymbol{\nu}} - \omega_{\mathbf{N}}^{\xi_1} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right) \right)(t,\mathbf{x_i})\omega_{\mathbf{i}}^{\xi_2}\, dt + \sum_{k=1}^{4} \int_0^T \sum_{\mathbf{i}\in\mathbf{I_k}} \left(g_k^{\mathbf{N}} \sqrt{\omega_{\mathbf{N}}^{\xi_1}} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right)(t,\mathbf{x_i})\omega_{\mathbf{i}}^{\xi_2}\, dt \\ - \left\langle (\phi^{\mathbf{N}}(0,\cdot), \phi_t^{\mathbf{N}}(0,\cdot)), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \right\rangle_{\mathbf{N}} = 0, \tag{3.16}$$

for all $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \in \mathbb{P}_{\mathbf{N}}^{Di}(\Omega) \times \mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$, where $(\phi^{\mathbf{N}}, \phi_t^{\mathbf{N}}) \in \mathbb{P}_{\mathbf{N}}^{Di}(\Omega) \times \mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$ is the solution of the following collocation approximation of the adjoint system:

$$\begin{cases} (\phi_{tt}^{\mathbf{N}} - \Delta \phi^{\mathbf{N}})(t, \mathbf{x_i}) = 0 & \textit{in } (t, \mathbf{x_i}) \in (0, T) \times C^{\Omega}, \\ \phi^{\mathbf{N}}(t, \mathbf{x_i}) = 0 & \textit{in } (t, \mathbf{x_i}) \in (0, T) \times C^{\partial\Omega}, \\ \phi^{\mathbf{N}}(T, \mathbf{x_i}) = \phi^{0,\mathbf{N}}, \phi_t^{\mathbf{N}}(T, \mathbf{x_i}) = \phi^{1,\mathbf{N}} & \textit{in } \mathbf{x_i} \in C^{\Omega}. \end{cases} \tag{3.17}$$

Here, $\omega_{\mathbf{N}}^{\xi_1}$ *and* $\omega_{\mathbf{i}}^{\xi_2}$ *are defined by*

$$\begin{cases} \omega_{\mathbf{N}}^{\xi_1} = \omega_{N_1}^{x_1}, \quad \omega_{\mathbf{i}}^{\xi_2} = \omega_{i_2}^{x_2} \quad \mathit{if}\ \mathbf{i} \in \mathbf{I}_1 \cup \mathbf{I}_3, \\ \omega_{\mathbf{N}}^{\xi_1} = \omega_{N_2}^{x_2}, \quad \omega_{\mathbf{i}}^{\xi_2} = \omega_{i_1}^{x_1} \quad \mathit{if}\ \mathbf{i} \in \mathbf{I}_2 \cup \mathbf{I}_4. \end{cases} \tag{3.18}$$

Proof of Lemma 3.4.1. Multiplying the equation of $u^{\mathbf{N}}(t, \mathbf{x_i})$ in (3.12) by $\omega_{\mathbf{i}}\phi^{\mathbf{N}}(t, \mathbf{x_i})$ and adding in $\mathbf{i} \in \mathbf{I}$, one obtains

$$\int_0^T (u_{tt}^{\mathbf{N}} - \Delta u^{\mathbf{N}}, \phi^{\mathbf{N}})_{\mathbf{N}} dt = \sum_{k=1}^{4} \int_0^T (G_k^{\mathbf{N}}, \phi^{\mathbf{N}})_{\mathbf{N}} dt. \tag{3.19}$$

We first simplify the left-hand side. Using (3.10), integrating by parts in time and taking into account that $f^{\mathbf{N}}, g_k^{\mathbf{N}}, k = 1, .., 4$ are controls we have

$$\begin{aligned} \int_0^T (u_{tt}^{\mathbf{N}} - \Delta u^{\mathbf{N}}, \phi^{\mathbf{N}})_{\mathbf{N}} dt = & \int_0^T (u^{\mathbf{N}}, \phi_{tt}^{\mathbf{N}})_{\mathbf{N}} dt - \int_0^T (\Delta u^{\mathbf{N}}, \phi^{\mathbf{N}})_{\mathbf{N}} dt \\ & - \left\langle (\phi^{\mathbf{N}}(0,\cdot), \phi_t^{\mathbf{N}}(0,\cdot)), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \right\rangle_{\mathbf{N}}. \end{aligned} \tag{3.20}$$

Now, we estimate the second term on the right-hand side. We use separation of variable and (3.9), then we proceed to perform spatial integration by parts and once again use formula (2.19), as the resulting integrand remains a polynomial of degree $2N_1 - 2$ (or $2N_2 - 2$) in the x_1 (or x_2) variable:

$$\begin{aligned} 0 &= \int_0^T (u_{tt}^{\mathbf{N}} - \Delta u^{\mathbf{N}}, \phi^{\mathbf{N}})_{\mathbf{N}} dt = \int_0^T (u^{\mathbf{N}}, \phi_{tt}^{\mathbf{N}} - \Delta \phi^{\mathbf{N}})_{\mathbf{N}} dt \\ &+ \int_0^T \sum_{\mathbf{i} \in \mathbf{I}_1 \cup \mathbf{I}_2} f^{\mathbf{N}} \left(\frac{\partial \phi^{\mathbf{N}}}{\partial \boldsymbol{\nu}} \right) (t, \mathbf{x_i}) \omega_{\mathbf{i}} dt - \left\langle (\phi^{\mathbf{N}}(0,\cdot), \phi_t^{\mathbf{N}}(0,\cdot)), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \right\rangle_{\mathbf{N}} \\ &= \int_0^T \sum_{\mathbf{i} \in \mathbf{I}_1 \cup \mathbf{I}_2} f^{\mathbf{N}} \left(\frac{\partial \phi^{\mathbf{N}}}{\partial \boldsymbol{\nu}} - \omega_{\mathbf{N}}^{\xi_1} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right) (t, \mathbf{x_i}) \omega_{\mathbf{i}}^{\xi_2} dt - \left\langle (\phi^{\mathbf{N}}(0,\cdot), \phi_t^{\mathbf{N}}(0,\cdot)), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \right\rangle_{\mathbf{N}}. \end{aligned} \tag{3.21}$$

The final equality arises as a result of the first equation in (3.17). Note that the term multiplied by $\omega_{\mathbf{N}}^{\xi_1}$ which appears in the right-hand side of this expression is coming from the fact that the first equation in (3.17) holds exclusively for the interior nodes, whereas the discrete scalar product involves both the interior and boundary nodes.

For the right-hand side in (3.19), it is enough to estimate the term which acts as one part of the boundary since the others are similar. For example $k = 1$ which corresponds to Γ_1, we write $\mathbf{x_i} = (x_{1i_1}, x_{2i_2}) \in C^{\Omega}$, $\mathbf{x_i}^1 = (1, x_{2i_2}) \in \Gamma_1$. We use formula (3.10) and the fact that $G_1^{\mathbf{N}}\phi^{\mathbf{N}}$ is a polynomial of degree $2N_1 - 1$ in the x_1-variable,

$$\begin{aligned}
(G_1^{\mathbf{N}}, \phi^{\mathbf{N}})_{\mathbf{N}} &= \frac{1}{\sqrt{\omega^{x_1}_{N_1}}}\left(\left(h^R_{x_1x_1} + \frac{\Psi^{x_1}_{N_1,x_1}}{\omega^{x_1}_{N_1}}\right)_N g_1^{\mathbf{N}}(t, \mathbf{x_i}^1), \phi^{\mathbf{N}}\right)_{\mathbf{N}} \\
&= \sum_{\mathbf{i}\in\mathbf{I_1}} \left(g_1^{\mathbf{N}} \int_{-1}^{1} \frac{1}{\sqrt{\omega^{x_1}_{N_1}}} \left(h^R_{x_1x_1} + \frac{\Psi^{x_1}_{N_1,x_1}}{\omega^{x_1}_{N_1}}\right) \phi^{\mathbf{N}} dx_1\right)(t, \mathbf{x_i}^1)\omega^{x_2}_{i_2} \\
&= \sum_{\mathbf{i}\in\mathbf{I_1}} \left(g_1^{\mathbf{N}} \frac{1}{\sqrt{\omega^{x_1}_{N_1}}}\left(\int_{-1}^{1} h^R \phi^{\mathbf{N}}_{x_1x_1} dx_1 - \int_{-1}^{1} \frac{\Psi^{x_1}_{N_1}}{\omega^{x_1}_{N_1}} \phi^{\mathbf{N}}_{x_1} dx_1\right)\right)(t, \mathbf{x_i}^1)\omega^{x_2}_{i_2} \\
&= \sum_{\mathbf{i}\in\mathbf{I_1}} \left(g_1^{\mathbf{N}}(t, \mathbf{x_i}^1) \frac{1}{\sqrt{\omega^{x_1}_{N_1}}}\left(\sum_{k=1}^{N_1-1} (h^R \phi^{\mathbf{N}}_{x_1x_1})(t, \mathbf{x_i})\omega^{x_1}_{i_1} - \phi^{\mathbf{N}}_{x_1}(t, \mathbf{x_i}^1)\right)\right)\omega^{x_2}_{i_2}.
\end{aligned} \tag{3.22}$$

To simplify the second term in this right-hand side, we use the fact that $h^R(x_{1i_1}) = \dfrac{1 + x_{1i_1}}{2}$ at the interior nodes in $(-1, 1)$, i.e $i_1 = 1, N_1 - 1$. Then, we have

$$\begin{aligned}
\sum_{i_1=1}^{N_1-1} (h^R \phi^{\mathbf{N}}_{x_1x_1})(t, \mathbf{x_i})\omega^{x_1}_{i_1} &= \int_{-1}^{1} \frac{1 + x_1}{2} \phi^{\mathbf{N}}_{x_1x_1} dx_1 - \omega^{x_1}_{N_1} \phi^{\mathbf{N}}_{x_1x_1}(t, \mathbf{x_i}^1) \\
&= \phi^{\mathbf{N}}_{x_1}(t, \mathbf{x_i}^1) - \omega^{x_1}_{N_1} \phi^{\mathbf{N}}_{x_1x_1}(t, \mathbf{x_i}^1).
\end{aligned} \tag{3.23}$$

From (3.22)-(3.23), we easily obtain

$$\begin{aligned}
(G_1^{\mathbf{N}}, \phi^{\mathbf{N}})_{\mathbf{N}} &= -\sum_{\mathbf{i}\in\mathbf{I_1}} \left(g_1^{\mathbf{N}} \sqrt{\omega^{x_1}_{N_1}} \phi^{\mathbf{N}}_{x_1x_1}\right)(t, \mathbf{x_i}^1)\omega^{x_2}_{i_2} \\
&= -\sum_{\mathbf{i}\in\mathbf{I_1}} \left(g_1^{\mathbf{N}} \sqrt{\omega^{x_1}_{N_1}} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right)(t, 1, x_{2i_2})\omega^{x_2}_{i_2}.
\end{aligned} \tag{3.24}$$

Combining (3.19), (3.21) and (3.24), (3.18), we easily find (3.16). □

According to **H.U.M**, one possibility to construct controls $f^{\mathbf{N}}, g_k^{\mathbf{N}}, k = 1, ..., 4$ that satisfy the variational condition (3.13) is as minimizers of a particular quadratic functional. We define the following cost functional $J^{\mathbf{N}} : \mathbb{P}^{Di}_{\mathbf{N}}(\Omega) \times \mathbb{P}_{\mathbf{N}}(\Omega) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} J^{\mathbf{N}}(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) &= -\left\langle (\phi^{\mathbf{N}}(0,\cdot), \phi^{\mathbf{N}}_t(0,\cdot)), (u^{0,\mathbf{N}},\ u^{1,\mathbf{N}}) \right\rangle_{\mathbf{N}} \\ &+ \frac{1}{2}\int_0^T \eta(t) \sum_{\mathbf{i}\in \mathbf{I}_1\cup\mathbf{I}_2} \left| \frac{\partial \phi^{\mathbf{N}}}{\partial \boldsymbol{\nu}} - \omega^{\xi_1}_{\mathbf{N}} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right|^2 (t, \mathbf{x_i}) \omega^{\xi_2}_{\mathbf{i}} dt \\ &+ \sum_{k=1}^{4} \frac{1}{2}\int_0^T \eta(t) \sum_{\mathbf{i}\in \mathbf{I}_k} \omega^{\xi_1}_{\mathbf{N}} \left| \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right|^2 (t, \mathbf{x_i}) \omega^{\xi_2}_{\mathbf{i}} dt, \end{aligned} \tag{3.25}$$

where $(\phi^{\mathbf{N}}, \phi^{\mathbf{N}}_t)$ is the solution of (3.17) with final data $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \in \mathbb{P}^{Di}_{\mathbf{N}}(\Omega) \times \mathbb{P}^{Di}_{\mathbf{N}}(\Omega)$. The function $\eta(t)$ is a prescribed as in (2.9).

Theorem 3.4.2. *Assume* $(u^{0,\mathbf{N}},\ u^{1,\mathbf{N}}) \in \mathbb{P}^{Di}_{\mathbf{N}}(\Omega) \times \mathbb{P}^{Di}_{\mathbf{N}}(\Omega)$ *and we suppose that* $(\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}}) \in \mathbb{P}^{Di}_{\mathbf{N}}(\Omega) \times \mathbb{P}^{Di}_{\mathbf{N}}(\Omega)$ *is a minimizer of* $J^{\mathbf{N}}$*. If* $\hat{\phi}^{\mathbf{N}}$ *is the corresponding solution of* (3.17) *with final data* $(\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}})$*, then*

$$\begin{aligned} f^{\mathbf{N}}(t, \mathbf{x_i}) &= \eta(t) \left(\frac{\partial \phi^{\mathbf{N}}}{\partial \boldsymbol{\nu}} - \omega^{\xi_1}_{\mathbf{N}} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right) \Bigg|_{\mathbf{I}_1\cup\mathbf{I}_2} \\ g^{\mathbf{N}}_k(t, \mathbf{x_i}) &= \eta(t) \sqrt{\omega^{\xi_1}_{\mathbf{N}}} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \Bigg|_{\mathbf{I}_k}, k = 1, ..., 4 \end{aligned} \tag{3.26}$$

are controls such that the solution of (3.12) *satisfies* (3.13).

Proof. If $J^{\mathbf{N}}$ achieves its minimum at $(\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}})$ its Gateaux derivative in the direction $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}})$ must vanish, i.e.

$$\begin{aligned} 0 = &\int_0^T \eta(t) \sum_{\mathbf{i}\in \mathbf{I}_1\cup\mathbf{I}_2} \left(\frac{\partial \hat{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} - \omega^{\xi_1}_{\mathbf{N}} \frac{\partial^2 \hat{\phi}^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right) \left(\frac{\partial \phi^{\mathbf{N}}}{\partial \boldsymbol{\nu}} - \omega^{\xi_1}_{\mathbf{N}} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right) (t, \mathbf{x_i}) \omega^{\xi_2}_{\mathbf{i}} dt \\ &+ \sum_{k=1}^{4} \int_0^T \eta(t) \sum_{\mathbf{i}\in \mathbf{I}_k} \omega^{\xi_1}_{\mathbf{N}} \left(\frac{\partial^2 \hat{\phi}^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right) (t, \mathbf{x_i}) \omega^{\xi_2}_{\mathbf{i}} dt - \left\langle (\phi^{\mathbf{N}}(0,\cdot), \phi^{\mathbf{N}}_t(0,\cdot)), (u^{0,\mathbf{N}},\ u^{1,\mathbf{N}}) \right\rangle_{\mathbf{N}}, \end{aligned}$$

for all $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \in \mathbb{P}^{Di}_{\mathbf{N}}(\Omega) \times \mathbb{P}^{Di}_{\mathbf{N}}(\Omega)$. From Lemma 3.4.1 it follows that (3.26) are controls for which (3.12) holds. □

The functional $J^{\mathbf{N}}$ is evidently continuous and convex. Consequently, the existence of a minimizer, and thus discrete controls for system (3.12), are assured once we establish its coercivity. This follows as a consequence of the following lemma.

Lemma 3.4.3. *Given $T > 4\sqrt{2}(2+N_1^{-1})(2+N_2^{-1})$, there exists a constant $C > 0$, independent of $\mathbf{N}$, such that the solution of system (3.17) satisfy*

$$\begin{aligned} C\left\|(\nabla\phi^{0,\mathbf{N}},\phi^{1,\mathbf{N}})\right\|^2_{\mathbf{N}\times\mathbf{N}} \le & \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_1\cup\mathbf{I}_2}\left(\frac{\partial\phi^{\mathbf{N}}}{\partial\boldsymbol{\nu}} - \omega_{\mathbf{N}}^{\xi_1}\frac{\partial^2\phi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}\right)^2(t,\mathbf{x_i})\omega_{\mathbf{i}}^{\xi_2}dt \\ & + \int_0^T\sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}}\omega_{\mathbf{N}}^{\xi_1}\left(\frac{\partial^2\phi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}\right)^2(t,\mathbf{x_i})\omega_{\mathbf{i}}^{\xi_2}dt, \end{aligned} \tag{3.27}$$

for any final data $(\phi^{0,\mathbf{N}},\phi^{1,\mathbf{N}}) \in \mathbb{P}_{\mathbf{N}}^{Di}(\Omega)\times\mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$.

Proof. Since $\omega_{\mathbf{N}}^{\xi_1} < 1$, from (3.18) and (2.18), it is enough to prove the following

$$\begin{aligned} C\left\|(\nabla\phi^{0,\mathbf{N}},\phi^{1,\mathbf{N}})\right\|^2_{\mathbf{N}\times\mathbf{N}} \le & \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_1\cup\mathbf{I}_2}\left(\frac{\partial\phi^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)^2(t,\mathbf{x_i})\omega_{\mathbf{i}}^{\xi_2}dt \\ & + \int_0^T\sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}}\omega_{\mathbf{N}}^{\xi_1}\left(\frac{\partial^2\phi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}\right)^2(t,\mathbf{x_i})\omega_{\mathbf{i}}^{\xi_2}dt, \end{aligned} \tag{3.28}$$

As in the one-dimensional case, the proof of Lemma 3.4.3 is based on the associated error equation, equivalent to (3.17) and given by,

$$\begin{cases} \phi^{\mathbf{N}}_{tt} - \Delta\phi^{\mathbf{N}} = -\sum_{\mathbf{i}\in I_{\partial\Omega}}\dfrac{\partial^2\phi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}(t,\mathbf{x_i})\Psi_{i_1}^{x_1}\Psi_{i_2}^{x_2} & \text{in } Q=(0,T)\times\Omega, \\ \phi^{\mathbf{N}} = 0 & \text{on } (0,T)\times\ \partial\Omega, \\ \phi^{\mathbf{N}}(T,\mathbf{x}) = \phi^{0,\mathbf{N}}, \phi^{\mathbf{N}}_t(T,\mathbf{x}) = \phi^{1,\mathbf{N}} & \text{in } \Omega, \end{cases} \tag{3.29}$$

where $\mathbf{i} = (i_1, i_2)$ and $\Psi_{i_1}^{x_1}(x_1)$ (respectively $\Psi_{i_2}^{x_2}(x_2)$) are the Lagrange polynomial which is 1 at x_{1i_1} (respectively at x_{2i_2}) and 0 at all the other collocation points. For this error equation, we can apply the classical multipliers technique (see Lions Lions, 1988a). The extra terms coming from the right-hand side in (3.29) are estimated following the same idea in the 1-d case.

First, given $\mathbf{x^0} = (-1,-1)\in\mathbb{R}^2$ and $\mathbf{m}(\mathbf{x})\in\mathbb{R}^2$ with components $m_j(\mathbf{x}) = x_j - x_j^0,\ j = 1,2$. It is clear that for all $\mathbf{x}\in\partial\Omega\backslash\Gamma : \mathbf{m}\cdot\boldsymbol{\nu} = 0$. Let us set $X = \int_\Omega \phi_t^{\mathbf{N}} m_j \dfrac{\partial\phi^{\mathbf{N}}}{\partial x_j}d\mathbf{x}\Big|_0^T$ and $Y = \int_\Omega\phi_t^{\mathbf{N}}\phi^{\mathbf{N}}d\mathbf{x}\Big|_0^T$. Multiplying the equation (3.29) by $m_j\dfrac{\partial\phi^{\mathbf{N}}}{\partial x_j}$ (here the repeated index j stands for the sum in $j = 1,2$) and integrating by parts, the left-hand side can be simplified as follows

$$X+\frac{1}{2}Y-\int_0^T\int_{\partial\Omega}\frac{\boldsymbol{m}\cdot\boldsymbol{\nu}}{2}\left|\frac{\partial\phi^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right|^2 d\gamma dt+\frac{1}{2}\int_0^T\int_\Omega(\left|\phi_t^{\mathbf{N}}\right|^2+\left|\nabla\phi^{\mathbf{N}}\right|^2)d\mathbf{x}dt.$$

Using the fact that $\boldsymbol{m}\cdot\boldsymbol{\nu}=0$ on $\partial\Omega\backslash\Gamma$ and $\sup_{\mathbf{x}\in\Gamma}\ m_j\nu_j=2\sqrt{2}$, we obtain

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega\left(\left|\phi_t^{\mathbf{N}}\right|^2+\left|\nabla\phi^{\mathbf{N}}\right|^2\right)d\mathbf{x}dt &\leq \left|X+\frac{1}{2}Y\right|+\frac{2\sqrt{2}}{2}\int_0^T\int_\Gamma\left(\left|\frac{\partial\phi^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right|^2\right)d\gamma dt\\ &+\left|\int_0^T\int_\Omega\left(\sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}}\left(\frac{\partial^2\phi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}\right)(t,\mathbf{x_i})\Psi_{i_1}^{x_1}\Psi_{i_2}^{x_2}\right)\left(m_j\frac{\partial\phi^{\mathbf{N}}}{\partial x_j}\right)d\mathbf{x}dt\right|.\end{aligned}\tag{3.30}$$

We now estimate each one of the terms in this expression. We start with the left-hand side in (3.30). Define the discrete energy

$$E^{\mathbf{N}}(t)=\frac{1}{2}\left(\left\|\phi_t^{\mathbf{N}}(t,\cdot)\right\|_{\mathbf{N}^2}^2+\left\|\nabla\phi^{\mathbf{N}}(t,\cdot)\right\|_{\mathbf{N}^2}^2\right).\tag{3.31}$$

This energy is conserved, i.e. $E^{\mathbf{N}}(t)=E^{\mathbf{N}}(0)$ for all $t>0$. As usual, this is obtained just multiplying (3.17) by $\phi_t^{\mathbf{N}}\omega_{\mathbf{i}}$, adding in $\mathbf{i}\in\mathbf{I}$ and integrating with respect to time and using the quadrature formula (3.10), since the integrated is a polynomial of degree $2\mathbf{N}-1$.

The norm equivalence, together with the conservation of the discrete energy gives

$$\frac{1}{2}\int_0^T\int_\Omega\left(\left|\phi_t^{\mathbf{N}}\right|^2+\left|\nabla\phi^{\mathbf{N}}\right|^2\right)d\mathbf{x}dt\geq\frac{1}{C_2}\int_0^T E^{\mathbf{N}}(t)d\mathbf{x}dt=\frac{T}{C_2}E^{\mathbf{N}}(0).$$

We now estimate the terms in the right hand side of (3.30). We start with $\left|X+\frac{1}{2}Y\right|$. This is a quantity evaluated at the times $t=0,T$. This quantity is easily estimated by the discrete energy which is conserved in time and the equivalence of the norm. In particular, we find

$$\left|X+\frac{1}{2}Y\right|\leq 4\sqrt{2}E^{\mathbf{N}}(0).\tag{3.32}$$

Concerning the second term in the right hand side of (3.30), it is enough to consider one of the 4 faces of the domain $\Omega=(-1,1)^2\subset\mathbb{R}^2$. We focus on Γ_1. For $\mathbf{i}\in\mathbf{I}_1$, the Lagrangian basis can be written as $\Psi^{x_1}_{i_1}\Psi^{x_2}_{i_2} = \Psi^{x_1}_{N_1}\Psi^{x_2}_{i_2}$. Therefore, for $\mathbf{x} \in \Gamma_1$, $\mathbf{x} = (1, x_2)$, and we have

$$\begin{aligned}
&\sum_{\mathbf{i}\in\mathbf{I}_1} \left(\frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right)(t, \mathbf{x_i})\Psi^{x_1}_{i_1}(x_1)\Psi^{x_2}_{i_2}(x_2) \\
&\qquad = \Psi^{x_1}_{N_1} \sum_{\mathbf{i}\in\mathbf{I}_1} \left(\frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right)(t, \mathbf{x_i})\Psi^{x_2}_{i_2}(x_2) \\
&\qquad = \Psi^{x_1}_{N_1}(x_1) \left(\frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right)(t, 1, x_2).
\end{aligned} \tag{3.33}$$

We now replace this in the last term on the right-hand side in (3.30). Using the Young's inequality, we have

$$\begin{aligned}
&\left|\int_0^T \int_\Omega \left(\left(\frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right)(t, 1, x_2)\Psi^{x_1}_{N_1}(x_1)\right)\left(m_j \frac{\partial \phi^{\mathbf{N}}}{\partial x_j}\right)(\mathbf{x}) d\mathbf{x} dt\right| \\
&\leq \int_0^T C_\varepsilon \int_{-1}^1 \int_{-1}^1 \left|\left(\frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right)(t, 1, x_2)\right|^2 \left|\Psi^{x_1}_{N_1}(x_1)\right|^2 dx_1 dx_2 dt \\
&+ \int_0^T \varepsilon \int_\Omega \left|m_j \frac{\partial \phi^{\mathbf{N}}}{\partial x_j}\right|^2 d\mathbf{x} dt \leq C_\varepsilon |\Psi^{x_1}_{N_1}|^2_{L^2(-1,1)} \int_0^T \int_{\gamma_1} \left|\frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right|^2 d\gamma dt \\
&+ \varepsilon(\sup_{\mathbf{x}\in\Gamma} m_j \nu_j)^2 \int_0^T \int_\Omega |\nabla \phi^{\mathbf{N}}|^2 d\mathbf{x} dt.
\end{aligned} \tag{3.34}$$

Taking into account the norm equivalence in (2.20), $\left|\Psi^{x_1}_{N_1}\right|^2_{L^2(-1,1)} \leq \left\|\Psi^{x_1}_{N_1}\right\|^2_{N_1} = \omega^{x_1}_{N_1}$, the conservation of the discrete energy proved above in (3.31), the fact that $sup_{\mathbf{x}\in\Gamma}\ m_j \nu_j = 2\sqrt{2}$ and (3.34), we obtain

$$\begin{aligned}
&\left|\int_0^T \int_\Omega \left(\sum_{\mathbf{i}\in\mathbf{I}_{\Gamma_1}} \left(\frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right)(t, \mathbf{x_i})(\Psi^{x_1}_{N_1}\Psi^{x_2}_{m})(\mathbf{x})\right)\left(m_j \frac{\partial \phi^{\mathbf{N}}}{\partial x_j}\right)(\mathbf{x}) d\mathbf{x} dt\right| \\
&\leq C_\varepsilon \omega^{x_1}_{N_1} \int_0^T \int_{\Gamma_1} \left|\frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right|^2 d\gamma dt + 16\varepsilon T E^{\mathbf{N}}(0).
\end{aligned} \tag{3.35}$$

An analogous estimate holds for the other 3 terms of the boundary in the right-hand side of (3.30). It follows from (3.30)-(3.35) and the fact that $\omega^{x_1}_{N_1} = \omega^{x_1}_0$ and $\omega^{x_2}_{N_2} = \omega^{x_2}_0$ (see Canuto et al., 1988, Chapter 2) that

$$\begin{aligned}\left(\frac{T}{C_2} - 4\sqrt{2} - 64\varepsilon T\right) E^{\mathbf{N}}(0) \leq \sqrt{2} \int_0^T \int_\Gamma \left|\frac{\partial \phi^{\mathbf{N}}}{\partial \boldsymbol{\nu}}\right|^2 d\gamma dt \\ + C_\varepsilon \int_0^T \int_{\partial\Omega} \omega_{\mathbf{N}}^{\xi_1} \left|\frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}}\right|^2 d\gamma dt.\end{aligned} \tag{3.36}$$

Here, we can replace the integrals in boundary Γ (resp. $\partial\Omega$) by the sum in $\mathbf{i} \in \mathbf{I}_1 \cup \mathbf{I}_2$ (resp. $\mathbf{i} \in \mathbf{I}_{\partial\Omega}$) this is a consequence of the equivalence of the discrete L^2-norm in (2.20) for polynomials of degree $2N_1$ (resp. $2N_2$) in x_1-variable (resp. x_2). Inequality (3.27) holds as long as $\dfrac{T}{C_2} - 64\varepsilon T - 4\sqrt{2} > 0$. As ε can be chosen arbitrarily small and $C_2 = (2 + N_1^{-1})(2 + N_2^{-1})$, we have the condition $T > 4\sqrt{2}(2 + N_1^{-1})(2 + N_2^{-1})$. □

Remark 10. *The method we present here to obtain the uniform observability inequality relies on the continuous error equation* (3.29)*, equivalent to* (3.17)*. The observability inequality is derived using the same techniques as in the continuous model (see Lions, 1988a) and we only have to estimate the extra error term appearing in* (3.29).

3.5 Convergence of the discrete controls: proof of Theorem 3.3.2

In order to establish the convergence result as stated in Theorem 3.3.2, we assume throughout this section that the theorem's hypotheses hold. For clarity the exposition we proceed in three steps where we first define the sequence $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^2$ and prove the uniform boundedness of the associated sequence of controls. Next, we characterize their weak limit and finally, we prove the strong convergence.

Step 1: Uniform bound of the controls. As above, in the one-dimensional case, we first state the following result that we prove in the Appendix below.

Lemma 3.5.1. *Given* $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$*, there exists a sequence* $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^2$ *such that*

$$(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \longrightarrow (u^0, u^1) \text{ in } L^2(\Omega) \times H^{-1}(\Omega). \tag{3.37}$$

On the other hand, for any $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \in \mathbb{P}^{Di}_{\mathbf{N}}(\Omega) \times \mathbb{P}^{Di}_{\mathbf{N}}(\Omega)$*, we have*

$$\left| \Big\langle (\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \Big\rangle_{\mathbf{N}} \right| \leq |(u^0, u^1)|_{L^2 \times H^{-1}} \| (\nabla \phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \|_{\mathbf{N} \times \mathbf{N}}. \tag{3.38}$$

Furthermore, if $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \to (\phi^0, \phi^1)$ *in* $H^1_0(\Omega) \times L^2(\Omega)$ *then,*

$$\Big\langle (\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \Big\rangle_{\mathbf{N}} \to \Big\langle (\phi^0, \phi^1), (u^0, u^1) \Big\rangle. \tag{3.39}$$

The result in Lemma 2.5.1 is still true when considering $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) = (\tilde{u}^{0,\mathbf{N}}, \tilde{u}^{1,\mathbf{N}})$ as defined in Remark 9.

We choose $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \in (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^2$ as in this lemma. Note that, in particular, this sequence is uniformly bounded in $L^2 \times H^{-1}$ as $\boldsymbol{N} \to \infty$ and the boundedness of the associated controls is a direct consequence of the following result:

Proposition 3.5.1. *Let* $f^{\mathbf{N}}$, $g^{\mathbf{N}}_k, k = 1, ..., 4$ *be the controls defined in* (3.26)*. Then, there exists a constant* $M > 0$*, independent of* $\mathbf{N}$*, such that*

$$\int_0^T \sum_{\mathbf{i} \in \mathbf{I}_1 \cup \mathbf{I}_2} \left| f^{\mathbf{N}} \right|^2 (t, \mathbf{x_i}) \omega^{\xi_2}_{\mathbf{i}} dt \ + \sum_{k=1}^{4} \int_0^T \sum_{\mathbf{i} \in \mathbf{I}_k} \left| g^{\mathbf{N}}_k \right|^2 (t, \mathbf{x_i}) \omega^{\xi_2}_{\mathbf{i}} dt \leq M \left| (u^0, u^1) \right|^2_{L^2 \times H^{-1}}. \tag{3.40}$$

Proof. As $\hat{\phi}^{\mathbf{N}}$ is the solution of the discrete adjoint system (3.17) associated to the the minimizer $(\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}})$ of $J^{\mathbf{N}}$ in $\mathbb{P}^{Di}_{\mathbf{N}}(\Omega) \times \mathbb{P}^{Di}_{\mathbf{N}}(\Omega)$, we have in particular

$$J^{\mathbf{N}}(\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}}) \leq J^{\mathbf{N}}(0,0) = 0. \tag{3.41}$$

Then, taking into account the conservation of the discrete energy, defined in (3.31), the approximation in (3.37), estimate (3.38) and the uniform observability inequality in Lemma 3.4.3, we obtain

$$\int_0^T \sum_{\mathbf{i} \in \mathbf{I}_1 \cup \mathbf{I}_2} \eta(t) \left| \frac{\partial \phi^{\mathbf{N}}}{\partial \boldsymbol{\nu}} - \omega^{\xi_1}_{\mathbf{N}} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right|^2 (t, \mathbf{x_i}) \omega^{\xi_2}_{\mathbf{N}} dt + \int_0^T \sum_{\mathbf{i} \in \mathbf{I}_{\partial\Omega}} \eta(t) \omega^{\xi_1}_{\mathbf{N}} \left| \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \right|^2 (t, \mathbf{x_i}) \omega^{\xi_2}_{\mathbf{N}} dt$$
$$\leq C \left| (u^0, u^1) \right|^2_{L^2 \times H^{-1}},$$

which is equivalent to (3.40). □

Step 2: Weak convergence of the control $f^{\mathbf{N}}$. Thanks to the bound (3.40), controls $f^{\mathbf{N}}, g_k^{\mathbf{N}}, k = 1, ..., 4$ are uniformly bounded in $L^2(0,T;\Gamma), L^2(0,T;\Gamma_k), k = 1, ..., 4$, respectively and therefore, there exists a subsequence, still denoted by $f^{\mathbf{N}}, g_k^{\mathbf{N}}, k = 1, ..., 4$ such that

$$f^{\mathbf{N}} \underset{\boldsymbol{N}\to\infty}{\rightharpoonup} h, \quad \text{weakly in} \quad L^2(0,T;\Gamma), \tag{3.42}$$

$$g_k^{\mathbf{N}} \underset{\boldsymbol{N}\to\infty}{\rightharpoonup} h_k, \quad \text{weakly in} \quad L^2(0,T;\Gamma_k), k = 1, ..., 4. \tag{3.43}$$

Let us see that $h = f$ where f is the control with minimal L^2- weighted norm of system (3.1), with the weight function $\eta(t)$. In the next step, we show that $h_k = 0, k = 1, ..., 4$. This control f can be characterized by the following two properties:

(P1) f satisfies the variational characterization of controls in (3.5).

(P2) $f(t) = \eta(t)\dfrac{\partial\hat{\phi}}{\partial\boldsymbol{\nu}}$, where $\hat{\phi}$ is a solution of adjoint continue problem (3.3) with final data $(\hat{\phi}^0, \hat{\phi}^1)$.

In what follows, we see that h verifies these two properties. We start with the second one. By the boundedness of controls, the estimate (3.27) and the norm equivalence, we deduce that $(\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}})$ is uniformly bounded in $H_0^1(\Omega) \times L^2(\Omega)$. So, we can extract a sub-sequence, still denoted $(\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}})$ such that

$$(\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}}) \rightharpoonup (\hat{\phi}^0, \hat{\phi}^1) \text{ weakly in } H_0^1(\Omega) \times L^2(\Omega). \tag{3.44}$$

Let $\hat{\phi}^{\mathbf{N}}$ and $\hat{\phi}$ be the solutions of the discrete adjoint system (3.17) and the continuous one (3.3), associated to the final data $(\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}})$ and $(\hat{\phi}^0, \hat{\phi}^1)$ respectively. The following holds:

$$\hat{\phi}^{\mathbf{N}} \rightharpoonup \hat{\phi} \text{ weakly-* in } L^\infty(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)), \tag{3.45}$$

$$\frac{\partial\hat{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} - \omega_{\mathbf{N}}^{\xi_1}\frac{\partial^2\hat{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} \rightharpoonup \frac{\partial\hat{\phi}}{\partial\boldsymbol{\nu}} \text{ weakly in } L^2(0,T;\Gamma). \tag{3.46}$$

The convergence result (3.45) is easily deduced from the classical theory of spectral approximation in Canuto et al., 1988 (Section 10.5). Concerning (3.46), we write system (3.17) in a weak form and take as test functions $\psi(\mathbf{x}) = \frac{1+x_1}{2}\frac{1+x_2}{2} \in \mathbb{P}_{\mathbf{N}}(\Omega)$ and $l(t) \in C_0^1(0,T)$. Multiplying the equations (3.17) by the weights $\omega_{\mathbf{i}}$ and $\psi^{\mathbf{N}}(\mathbf{x}_{\mathbf{i}})l(t)$, adding in $\mathbf{i} \in \mathbf{I}$ and integrating in time, we easily obtain the following identity,

$$\begin{aligned} 0 = &-\int_0^T (\hat{\phi}^{\mathbf{N}}_{tt}(t,\cdot),\psi)_{\mathbf{N}} l(t)dt + \int_0^T \int_\Omega \Delta\hat{\phi}^{\mathbf{N}}(t,\mathbf{x})\psi l(t)d\mathbf{x}dt \\ &- \int_0^T \int_\Gamma \omega^{\xi_1}_N \frac{\partial^2 \hat{\phi}^{\mathbf{N}}}{\partial^2 \nu}\psi l(t)d\gamma dt \\ = &\int_0^T \int_\Omega \hat{\phi}^{\mathbf{N}}_t(t,\mathbf{x})\psi^{\mathbf{N}} l_t(t)d\mathbf{x}dt - \int_0^T \int_\Omega \nabla\hat{\phi}^N(t,\mathbf{x})\nabla\psi l(t)d\mathbf{x}dt \\ &+ \int_0^T \int_\Gamma \left(\frac{\partial\hat{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} - \omega^{\xi_1}_N \frac{\partial^2\hat{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} \right) \psi l(t)d\gamma dt. \end{aligned} \tag{3.47}$$

Note that in the second term (resp. the third term) on the right-hand side, we have used the quadrature formula since the integrand is a polynomial of degree $2\mathbf{N}-1$ (resp. $2N_1 - 1$ or $2N_2 - 1$ depend on the direction), and this allowed us to integrate by parts in the x_1, x_2 variables.

We can pass to the limit in (3.47) thanks to (3.42) and (3.45). Then, $\hat{\phi}$ verifies

$$0 = \int_0^T \int_\Omega \hat{\phi}_t(t,\mathbf{x})\psi l_t(t)d\mathbf{x}dt - \int_0^T \int_\Omega \nabla\hat{\phi}(t,\mathbf{x})\nabla\psi l(t)d\mathbf{x}dt + \int_0^T \int_\Gamma \tilde{h}\psi l(t)d\gamma dt, \tag{3.48}$$

for all $l(t) \in C^1_0(0,T)$. Here, $\tilde{h}$ is the weak limit of $\dfrac{\partial\hat{\phi}}{\partial\boldsymbol{\nu}} - \omega^{\xi_1}_{\mathbf{N}}\dfrac{\partial^2\hat{\phi}}{\partial^2\boldsymbol{\nu}}$. On the other hand, as $\hat{\phi}$ is a solution of (3.3), it also verifies

$$0 = \int_0^T \int_\Omega \hat{\phi}_t(t,\mathbf{x})\psi l_t(t)d\mathbf{x}dt - \int_0^T \int_\Omega \nabla\hat{\phi}(t,\mathbf{x})\nabla\psi l(t)d\mathbf{x}dt + \int_0^T \int_\Gamma \frac{\partial\hat{\phi}}{\partial\boldsymbol{\nu}}\psi l(t)d\gamma dt, \tag{3.49}$$

for all $l(t) \in C^1_0(0,T)$. From (3.48) and (3.49) we finally deduce

$$\int_0^T \int_\Gamma \tilde{h}\psi l(t)d\gamma dt = \int_0^T \int_\Gamma \frac{\partial\hat{\phi}}{\partial\boldsymbol{\nu}}\psi l(t)d\gamma dt, \ \forall \ l(t) \in C^1_0(0,T)$$

and, then $\tilde{h} = \dfrac{\partial\hat{\phi}}{\partial\boldsymbol{\nu}}$ with $\hat{\phi}$ the solution of (3.3). This finishes the proof of (3.46) and, in particular, implies that h satisfies property (P2) above.

Now, we check that the weak limit of $f^{\mathbf{N}}$, h, also verifies the first property (P1) above. We need the following lemma that we prove in the appendix below:

Lemma 3.5.2. *Given* $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, there exists a sequence* $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \in \mathbb{P}_{\mathbf{N}}^{Di}(\Omega) \times \mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$ *such that*

$$(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \longrightarrow (\phi^0, \phi^1) \text{ in } H_0^1(\Omega) \times L^2(\Omega). \tag{3.50}$$

Furthermore, if $\phi^{\mathbf{N}}$ *is the solution of* (3.17) *with final data* $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}})$ *and* ϕ *is the solution of* (3.3) *with final data* (ϕ^0, ϕ^1) *the following holds:*

$$\phi^{\mathbf{N}} \longrightarrow \phi \text{ in } C((0,T); H_0^1(\Omega)) \cap C^1((0,T); L^2(\Omega)), \tag{3.51}$$

$$\frac{\partial \phi^{\mathbf{N}}}{\partial \boldsymbol{\nu}} \longrightarrow \frac{\partial \hat{\phi}}{\partial \boldsymbol{\nu}} \text{ in } L^2(0,T;\Gamma), \tag{3.52}$$

$$\sqrt{\omega_{\mathbf{N}}^{\xi_1}} \frac{\partial^2 \phi^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} \longrightarrow 0 \text{ in } L^2(0,T;\Gamma_k), k = 1, ..., 4. \tag{3.53}$$

Given $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, by Lemma 3.5.2 there exists a sequence $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \in \mathbb{P}_{\mathbf{N}}^{Di}(\Omega) \times \mathbb{P}_{\mathbf{N}}^{\partial\Omega}(\Omega)$ such that (3.50) holds. Furthermore, from formula (3.51) we deduce, in particular, that

$$(\phi^{\mathbf{N}}(0, \mathbf{x}), \phi_t^{\mathbf{N}}(0, \mathbf{x})) \longrightarrow (\phi(0, \mathbf{x}), \phi_t(0, \mathbf{x})) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega), \tag{3.54}$$

where $\phi^{\mathbf{N}}$ is the solution of (3.17) with final data $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}})$ and ϕ is the solution of (3.3) with final data (ϕ^0, ϕ^1). Now, passing to the limit, as $\mathbf{N} \longrightarrow \infty$ in formula (3.16) and taking into account Lemmas 3.5.1 and 3.5.2, we obtain that h satisfies

$$\int_0^T \int_\Gamma \frac{\partial \phi}{\partial \boldsymbol{\nu}} h d\gamma dt - \Big\langle (\phi(0, \cdot), \phi_t(0, \cdot)), (u^0, u^1) \Big\rangle = 0, \tag{3.55}$$

$\forall (\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, So h verifies property (P1) above.

Step 3: Strong convergence of the controls. From the lower semi-continuity of the norm with respect to the weak convergence we have

$$|f|^2_{L^2(0,T;\Gamma)} + \sum_{k=1}^{4} |h_k|^2_{L^2(0,T;\Gamma_k)} \le \liminf_{\mathbf{N}\to\infty} |f^{\mathbf{N}}|^2_{L^2(0,T;\Gamma)} + \sum_{k=1}^{4} |g_k^{\mathbf{N}}|^2_{L^2(0,T;\Gamma_k)}. \tag{3.56}$$

On the other hand, if we consider formulas (3.16) and (3.5) with $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) = (\hat{\phi}^{0,\mathbf{N}}, \hat{\phi}^{1,\mathbf{N}})$ and $(\phi^0, \phi^1) = (\hat{\phi}^0, \hat{\phi}^1)$ respectively, we obtain

$$\begin{aligned} 0 = \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_1\cup\mathbf{I}_2} \eta(t) \left| \frac{\partial\hat{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} - \omega_{\mathbf{N}}^{\xi_1} \frac{\partial^2\hat{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} \right|^2 (t, \mathbf{x_i}) \omega_{\mathbf{i}}^{\xi_2} dt + \sum_{k=1}^{4} \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_k} \eta(t) \omega_{\mathbf{N}}^{\xi_1} \left| \frac{\partial\hat{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} \right|^2 (t, \mathbf{x_i}) \omega_{\mathbf{i}}^{\xi_2} dt \\ - \left\langle (\hat{\phi}^{\mathbf{N}}(\cdot,0), \hat{\phi}_t^{\mathbf{N}}(\cdot,0)), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \right\rangle_{\mathbf{N}} \end{aligned} \tag{3.57}$$

and

$$\int_0^T \int_\Gamma \eta(t) \left| \frac{\partial\hat{\phi}}{\partial\boldsymbol{\nu}} \right|^2 d\gamma dt - \left\langle (\hat{\phi}(0,\cdot), \hat{\phi}_t(0,\cdot)), (u^0, u^1) \right\rangle = 0. \tag{3.58}$$

The last term in (3.57) converges to the second term in (3.58) and therefore

$$\begin{aligned} \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_1\cup\mathbf{I}_2} \eta(t) \left| \frac{\partial\hat{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} - \omega_{\mathbf{N}}^{\xi_1} \frac{\partial^2\hat{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} \right|^2 (t, \mathbf{x_i}) \omega_{\mathbf{i}}^{\xi_2} dt + \sum_{k=1}^{4} \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_k} \eta(t) \omega_{\mathbf{N}}^{\xi_1} \left| \frac{\partial\hat{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} \right|^2 (t, \mathbf{x_i}) \omega_{\mathbf{i}}^{\xi_2} dt \\ \longrightarrow \int_0^T \int_\Gamma \eta(t) \left| \frac{\partial\hat{\phi}}{\partial\boldsymbol{\nu}} \right|^2 d\gamma dt, \end{aligned} \tag{3.59}$$

as $\mathbf{N} \longrightarrow \infty$. Now, taking into account (3.56), (3.59) and the norm equivalence in (2.20), the definition of the controls in (3.26), we deduce

$$|f|^2_{L^2(0,T;\Gamma)} + \sum_{k=1}^{4} |h_k|^2_{L^2(0,T;\Gamma_k)} \le |f|^2_{L^2(0,T;\Gamma)}$$

and, therefore, $h_k = 0, k = 1, ..., 4$. From (3.59), we also have

$$|f^{\mathbf{N}}|_{L^2(0,T;\Gamma)} \to |f|_{L^2(0,T;\Gamma)}, \quad |h_k^{\mathbf{N}}|^2_{L^2(0,T;\Gamma_k)} \to 0, \ k = 1, ..., 4, \tag{3.60}$$

as $\mathbf{N} \to \infty$. The strong convergence of the controls $f^{\mathbf{N}} \underset{\mathbf{N}\to\infty}{\longrightarrow} f$ in $L^2(0,T;\Gamma)$ is a consequence of the weak convergence stated in Step 2 above and the convergence of the norms stated in

(3.60).

3.6 Matrix formulation and finite-dimensional control

In this section, we consider as in Chapter 2 (section 2.6), some implementation issues related to the finite-dimensional approximation of boundary control given by (3.12)-(3.13). We write the collocation control problem into the standard matrix form problem and compute the control using the explicit expression given for the finite-dimensional control theory.

We assume $\Omega = (-1,1)^2$ and that the control acts in $\Gamma = \{1\} \times (-1,1) \cup (-1,1) \times \{1\}$. Let $(N+1)^2$ be the number of collocation points. A straightforward computation allows us to write system (3.12)-(3.13) in the matrix form (2.73) for suitable matrixes $A^{\mathbf{N}}$, $B^{\mathbf{N}}$ and a column vector of the controls $F^{\mathbf{N}}$, where $U^{\mathbf{N}}$ is a vector containing the values of the discrete solution at the interior collocation points as follow:

$$U^{\mathbf{N}} = (u_{22}^N, u_{23}^N, ..., u_{2N}^N, u_{32}^N, u_{33}^N, ..., u_{3N}^N, ..., u_{N2}^N, u_{N3}^N, ..., u_{NN}^N)^T, \text{ where } u_{jl}^N = u^{\mathbf{N}}(t, x_{1i_1}, x_{2i_2}),$$

where $j, l = 1, ..., N+1,\ i_1 = 0, ..., N_1$ *and* $i_2 = 0, ..., N_2$. Also, the components of the initial data $(U^{0,\mathbf{N}}, U^{1,\mathbf{N}})$ and the target $(U^{\mathbf{N}}(T), U_t^{\mathbf{N}}(T))$ are written in vector form, where

$$U^{k,\mathbf{N}} = (u_{22}^{k,N}, ..., u_{NN}^{k,N})^T,\ U_T^{k,\mathbf{N}} = (u_{T,22}^{k,N}, ..., u_{T,NN}^{k,N})^T, k = 0, 1$$

and

$$A^{\mathbf{N}} = \text{kron}((K^N)^2(2:N, 2:N), I) + \text{kron}(I, (K^N)^2(2:N, 2:N)),$$

where (K^N) define as (2.76) and the vector $\mathbf{F}^{\mathbf{N}}$ contains 5 components corresponding to the 5 controls in (3.12), i.e.

$$\mathbf{F}^{\mathbf{N}} = (f^{\mathbf{N}}, g_1^{\mathbf{N}}, g_2^{\mathbf{N}}, g_3^{\mathbf{N}}, g_4^{\mathbf{N}})^T,$$

where $\mathbf{f}^{\mathbf{N}}$ is a column vector with the coefficients of the controls

$$\mathbf{f}^{\mathbf{N}} = (f_{R,2}^{\mathbf{N}}, ..., f_{R,N}^{\mathbf{N}}, f_{T,2}^{\mathbf{N}}, ..., f_{T,N}^{\mathbf{N}})^T \in \mathcal{M}_{2(N-1)\times 1}.$$

Something similar can be said about $g_k^{\mathbf{N}}, k = 1, .., 4$. As we said above in 1-d, the matrix $B^{\mathbf{N}} \in \mathcal{M}_{(N-1)^2 \times 6(N-1)}$ corresponds to the contribution of the boundary control accounting for some rows and columns that affect the values of solution in the boundary and they are determined by the right-hand sides of the equilibrium equation (3.12).

The rest is similar to Section 2.6 chapter 2.

3.7 Numerical experiments

In this section, we illustrate the results in this chapter approximating the boundary control for the 2-d wave equation in the square. For the time discretization, we use a classical Newmark method with parameters $\gamma = 1/2$, $\beta = 1/4$ and time step $dt = 10^{-2}$. With this choice, the time scheme is second-order accurate (see Raviart and Thomas, 1983).

Experiment 1: We consider a two dimensional square domain $(-1,1)^2$. The control acts on the two sides $\{1\} \times (-1,1) \cup (-1,1) \times \{1\}$ in the time interval $t \in (0, 4.4)$, with two different types of initial data (see Figure 3.2 and 3.3). We consider the degree of polynomial is $\mathbf{N} = (80, 80)$ in the x_1, x_2-variable, respectively. The first one given by a bump function $u^0 = e^{-10x_1^2}e^{-10x_2^2}$, $u^1 = (-20x_1e^{-10x_1^2})(-20x_2e^{-10x_2^2})$. The second one corresponds to a Lipschitz continuous initial data $u^0 = (1-|x_1|)(1-|x_2|)$, $u^1 = 0$ with time step $dt = 10^{-2}$.

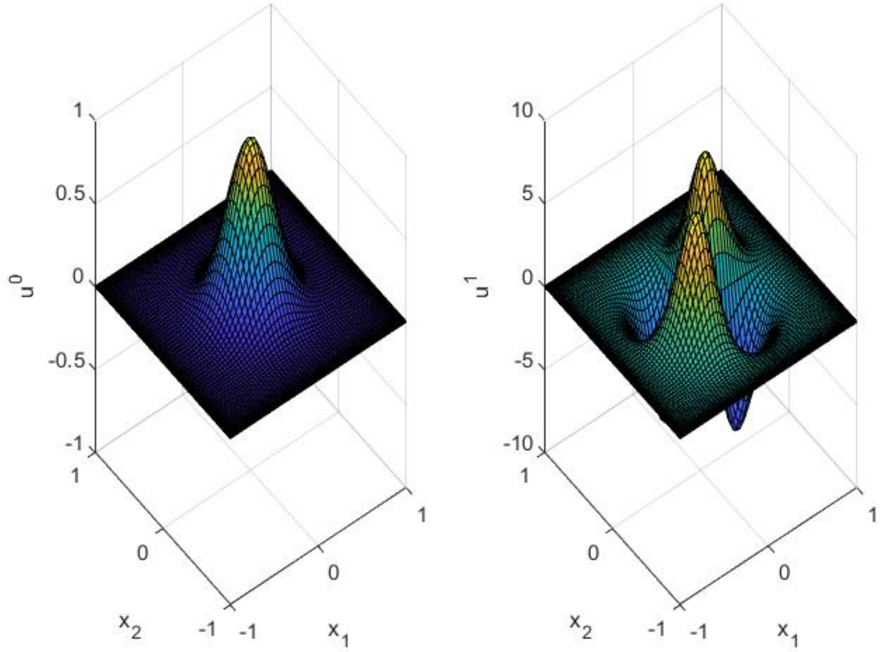


Figure 3.2: The smooth initial data $u^0 = e^{-10x_1^2}e^{-10x_2^2}$, $u^1 = (-20x_1e^{-10x_1^2})(-20x_2e^{-10x_2^2})$.

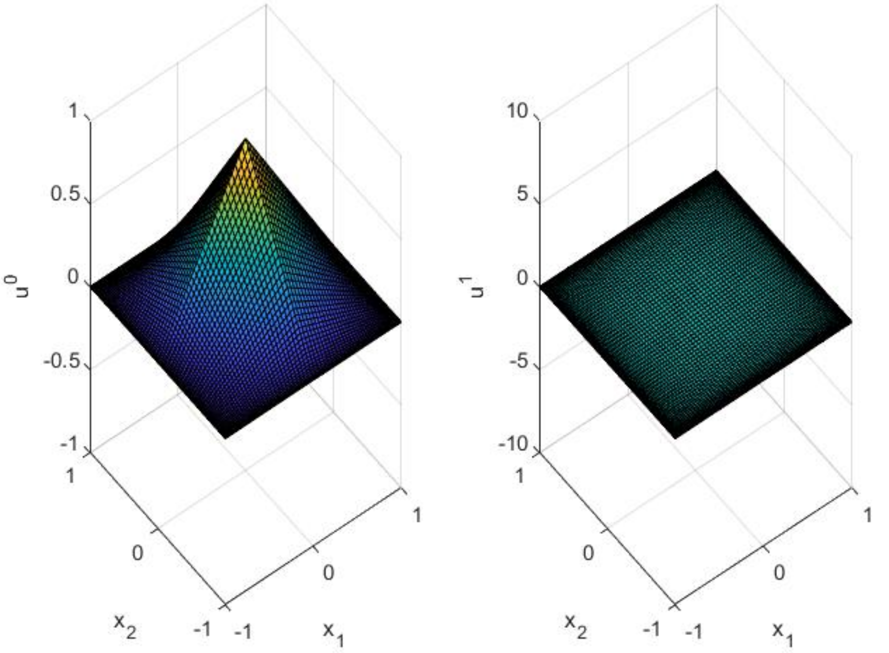


Figure 3.3: The Lipschitz continuous initial data $u^0 = (1 - |x_1|)(1 - |x_2|)$, $u^1 = 0$.

As in 1-d the time control is lower than the time given by the discrete control problem in Theorems 3.3.1 and 3.3.2, and also lower than the control time for the continuous wave equation ($T = 4\sqrt{2}$) (Lions, 1988a). However, the initial data is almost compactly supported in the disc $|\mathbf{x}| < 1/2$ inside the domain and this makes this special data controllable for the chosen time.

In Figure 3.4, we have drawn the behaviour of the norm of control acting on the two sides of the square during the time since the other controls are of the order 10^{-5}. As in 1-d in Tables 3.1 and 3.2 we show the behaviour of the norm for the controls when the degree of polynomials $\mathbf{N}$ grows. As stated in Theorem 3.3.2, we observe that the boundary control remains bounded while the four artificial controls included in the system ($g_k^{\mathbf{N}}$, $k = 1, ..., 4$) vanish as N grows. The controlled solutions for these controls are plotted in Figures 3.5 and 3.6.

Experiment 2: Here, we illustrate the rate of convergence of the discrete control to the limit one with two different types of initial data given above. We compare the L^2-norm of the difference between the discrete control when $\mathbf{N} = (80, 80)$ (that we take as continuous control) and the discrete control as $\mathbf{N}$ grows. The approximate controls are plotted in Figure 3.4.

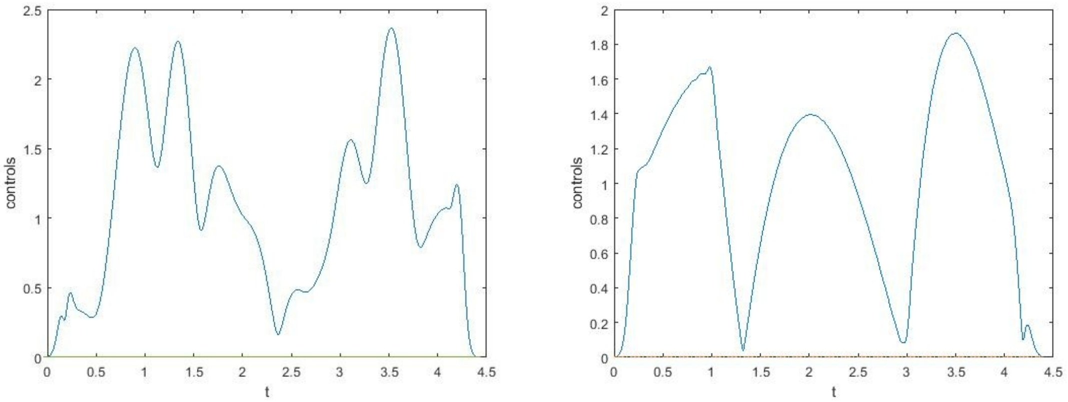


Figure 3.4: The behavior of control $f^{\mathbf{N}}$ during the time for $\mathbf{N} = (80, 80)$ and different initial data:$u^0 = e^{-10x_1^2}e^{-10x_2^2}$, $u^1 = (-20x_1e^{-10x_1^2})(-20x_2e^{-10x_2^2})$ (left one), $u^0 = (1 - |x_1|)(1 - |x_2|)$, $u^1 = 0$ (right one).

$\mathbf{N}$	$\left\|f^{\mathbf{N}}\right\|_{L^2(0,T;\Gamma)}$	$\left\|g_1^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}$	$\left\|g_2^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}$	$\left\|g_3^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}$	$\left\|g_4^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}$
$(20, 20)$	7.2×10^{-1}	8.5×10^{-3}	8.2×10^{-3}	8.5×10^{-3}	8×10^{-3}
$(50, 50)$	7.2×10^{-1}	7.4×10^{-4}	7.4×10^{-4}	7.4×10^{-4}	7.4×10^{-4}
$(80, 80)$	7.2×10^{-1}	6.4×10^{-5}	6.4×10^{-5}	6.4×10^{-5}	6.4×10^{-5}

Table 3.1: Norm of the controls for $u^0(x) = e^{-10(x_1^2+x_2^2)}$, $u^1(x) = (20^2x_1x_2)\, e^{-10(x_1^2+x_2^2)}$ as $\mathbf{N}$ grows.

In Tables 3.3 and 3.4, we show the error between the discrete control and the limit one. Comparing the error associated to values of $\mathbf{N} = (N, N) \in [10, 50]^2$ we can give a rough estimate of the rate of convergence. More precisely, we observe that $\left|f^{\mathbf{N}} - f\right|_{L^2(0,T;\Gamma)} \sim N^{-\alpha}$, where α is estimated by the slope of the graphics relating $-\log_{10}\left|f^{\mathbf{N}} - f\right|_{L^2(0,T;\Gamma)}$ with $\log_{10} N$ (see Figure 3.7).

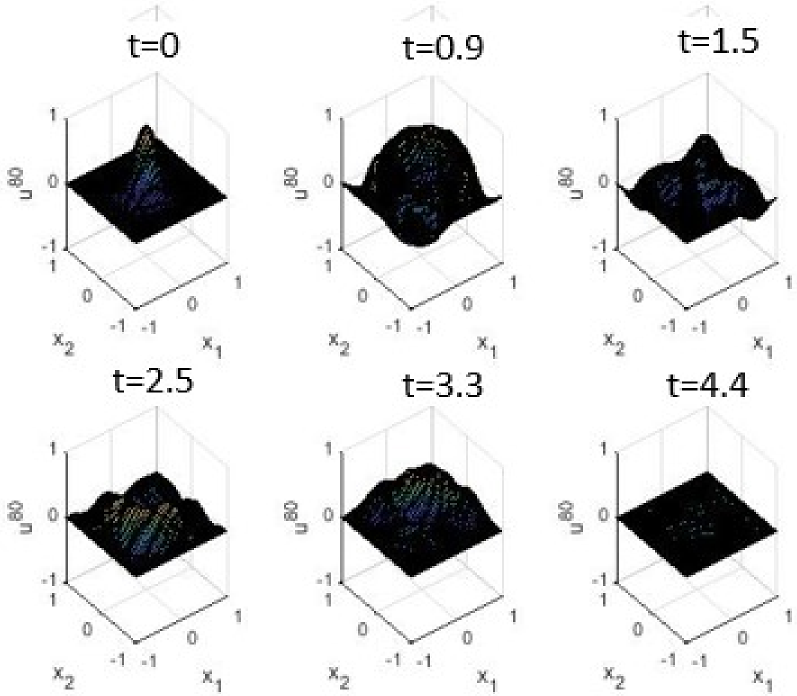


Figure 3.5: The behavior of controlled solution $u^{\mathbf{80}}$ during the time for $\mathbf{N} = (80, 80)$ with initial data:$u^0 = e^{-10x_1^2}e^{-10x_2^2}$ and $u^1 = (-20x_1e^{-10x_1^2})(-20x_2e^{-10x_2^2})$.

$\mathbf{N}$	$\left\|f^{\mathbf{N}}\right\|_{L^2(0,T;\Gamma)}$	$\left\|g_1^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}$	$\left\|g_2^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}$	$\left\|g_3^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}$	$\left\|g_4^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}$
$(20, 20)$	6.6×10^{-1}	3.9×10^{-3}	3.9×10^{-3}	3.9×10^{-3}	3.9×10^{-3}
$(50, 50)$	6.5×10^{-1}	3.4×10^{-4}	3.4×10^{-4}	3.4×10^{-4}	3.4×10^{-4}
$(80, 80)$	6.5×10^{-1}	2.3×10^{-5}	2.3×10^{-5}	2.3×10^{-5}	2.3×10^{-5}

Table 3.2: Norm of the controls for $u^0(x) = (1 - |x_1|)(1 - |x_2|)$, $u^1(x) = 0$ as $\mathbf{N}$ grows.

N	$\log(\left\|f^{\mathbf{N}} - f^{\mathbf{80}}\right\|_{L^2(0,T;\Gamma)})$	$\log(\left\|g_j^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}), j = 1, ..., 4$
10	-0.7	-0.9
30	-1.1	-1.7
50	-1.3	-2.1

Table 3.3: Convergence of the discrete control to the limit as N grows for $u^0(x) = (1-|x_1|)(1-|x_2|)$, $u^1(x) = 0$.

N	$\log(\left\|f^{\mathbf{N}} - f^{\mathbf{80}}\right\|_{L^2(0,T;\Gamma)})$	$\log(\left\|g_j^{\mathbf{N}}\right\|_{L^2(0,T;\partial\Omega)}), j = 1, ..., 4$
10	-2.7	-3.2
30	-3.1	-3.8
50	-3.9	-4.3

Table 3.4: Convergence of the discrete control to the limit as N grows for $u^0 = e^{-10x_1^2}e^{-10x_2^2}$, $u^1 = (-20x_1e^{-10x_1^2})(-20x_2e^{-10x_2^2})$.

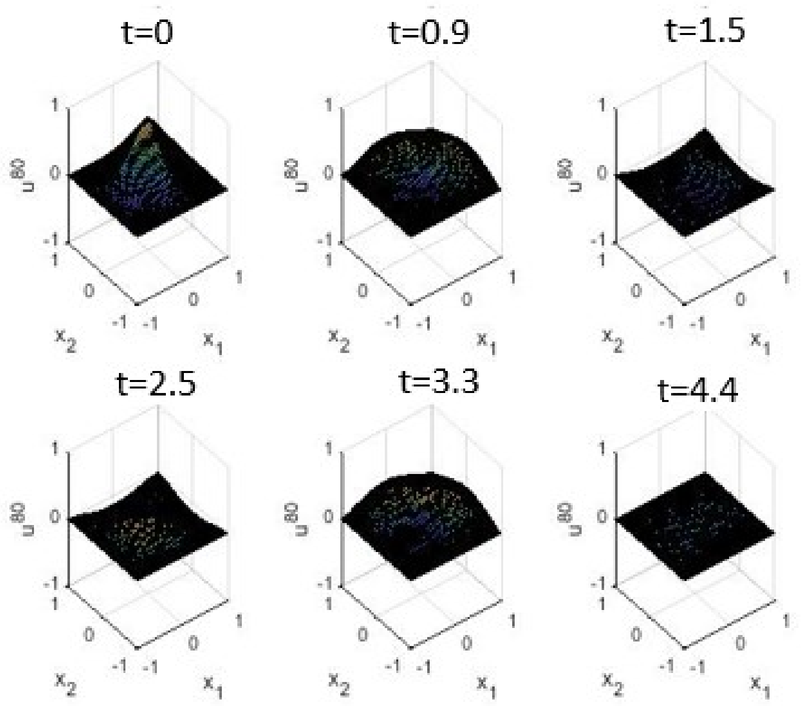


Figure 3.6: The behavior of controlled solution $u^{\mathbf{80}}$ during the time for $\mathbf{N} = (80, 80)$ with initial data: $u^0 = (1 - |x_1|)(1 - |x_2|)$ and $u^1 = 0$.

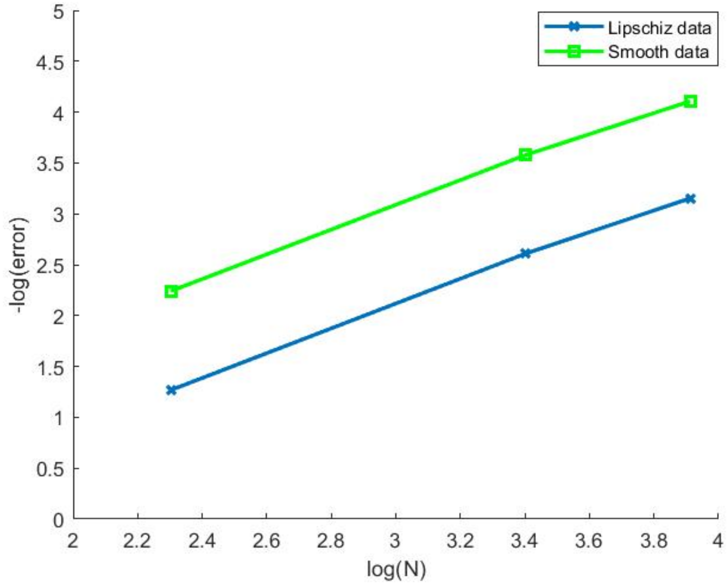


Figure 3.7: The slope of each graphic gives an experimental estimate of the value α for which $|f^{\mathbf{N}} - f^{\mathbf{80}}|_{L^2} \sim N^{-\alpha}$. The two graphics correspond to the two initial data considered in the experiments. We observe that α is greater for smoother data, as expected in spectral methods.

Chapter 4

Numerical approximation of the boundary control for the linear elasticity system with a spectral collocation method

In this chapter, we show how to extend the numerical approximation of the control problem in previous chapters to the elasticity system in higher dimensions ($d \geq 2$). More precisely, the control that drives the system from any initial data to rest in a given finite time $T > 0$. We propose the same spectral collocation in the previous chapters, based on the polynomial approximation of the solutions with degree $\mathbf{N}$. Here, we prove the uniform bounded of the discrete controls as $\mathbf{N} \to \infty$. However, convergence of the discrete controls to the continuous one is much more difficult in this case and we just have found numerical evidence. In fact, if we try to apply the general theory established in the previous chapters to recover the convergence of controls, some important hypotheses related to the convergence of the solutions of discrete adjoint systems must be checked. Such convergent results are not standard with the usual numerical analysis techniques and require further investigation. Here, we focus on the implementation details for a particular square domain with a Dirichlet control acting on two adjacent edges.

4.1 Problem statement

We consider the free vibrations of a d-dimensional homogeneous and isotropic elastic body occupying by a bounded domain $\Omega = (-1,1)^d \subset \mathbb{R}^d, (d \geq 2)$ with boundary $\partial\Omega$:

$$\begin{cases} \left(\mathbf{u}_{tt} - \Delta^*\mathbf{u}\right)(t,\mathbf{x}) = \mathbf{0}, & (t,\mathbf{x}) \in (0,T) \times \Omega, \\ \mathbf{u}(t,\mathbf{x}) = \mathbf{f}(t,\mathbf{x})\chi_\Gamma(\mathbf{x}), & (t,\mathbf{x}) \in (0,T) \times \partial\Omega, \\ (\mathbf{u}(0,\mathbf{x}), \mathbf{u}_t(0,\mathbf{x})) = (\mathbf{u^0}(\mathbf{x}), \mathbf{u^1}(\mathbf{x})), & \mathbf{x} \in \Omega, \end{cases} \tag{4.1}$$

where $\mathbf{u}(t,\mathbf{x})$ is the displacement of the material point $\mathbf{x}$ at time t, $\Delta^* = \mu\Delta + (\lambda+\mu)\nabla\text{div}$, $\lambda, \mu > 0$ are Lamé parameters that are assumed to be constant and $\mathbf{f}$ is the control acting on one part of the boundary $\Gamma \subset \partial\Omega$. Here, χ_Γ is the characteristic function of the set Γ. The fact that we only consider rectangular domains is due to the particular spectral method that we analyze below. On the other hand, the choice of Γ considered here is somehow optimal in the sense that the observability inequality fails for any smaller open subset of Γ.

Given some initial data $(\mathbf{u^0}(\mathbf{x}), \mathbf{u^1}(\mathbf{x}))$ and a time horizon T we are interested in the problem of computing the control $\mathbf{f}$ such that the solution $\mathbf{u}$ of (4.1) satisfies the null controllability condition.

$$\mathbf{u}(T,\mathbf{x}) = \mathbf{u}_t(T,\mathbf{x}) = \mathbf{0}, \;\; \mathbf{x} \in \Omega. \tag{4.2}$$

It is well-known that, under some smoothness conditions on the initial data and for a sufficiently large time T, a control $\mathbf{f}$ exists (Alabau and Komornik, 1999 and Lions, 1988a). Moreover, among all the controls, the one with minimal L^2-norm is unique. Here, we give an algorithm to approximate this control numerically. We follow a variational approach to characterize the controls as minimizers of a suitable function and a Galerkin method to obtain finite dimensional approximations of these minima. In particular, this requires a finite-dimensional discrete approximation of the elasticity system (4.1). This problem was also addressed in Bottois and Cîndea, 2023 using the space-time finite element approach mentioned above. It is also worth mentioning reference Font and Periago, 2010 where the authors find numerical approximations of these boundary controls, in some particular cases, using the so-called Russell's method that exploit the energy dissipation of the solutions of the elasticity system in an extended domain.

The novelty here is that we are able to prove a uniform bounded for the discrete controls by adding extra discrete boundary controls that vanish as $\mathbf{N} \to \infty$. This provides an accurate approximation of the continuous control. The result relies on a uniform observability inequality for the associated discrete adjoint system which allows us to obtain the uniform boundedness of discrete controls.

As in the previous chapters, for the wave equation, the method we present here to obtain the uniform observability inequality is new and considers, instead of the discrete collocation system, the equivalent continuous error equation associated to the polynomial approximation (see Gottlieb and Lustman, 1983). This error equation is the same elasticity equation but with a nonhomogeneous second-hand term, known as the error term. Therefore, the observability inequality can be derived using the same techniques as in the continuous model and we only have to estimate this extra error term.

In order to find a numerical approximation of this control f in (4.1) we proceed as follows: first we introduce a discrete version of the control problem (4.1), depending on a discrete parameter $\mathbf{N} \longrightarrow \infty$. Then, we prove that this system is controllable for all $\mathbf{N} \in \mathbb{N}^d$ with $2d+1$ different controls, $\mathbf{f}^{\mathbf{N}}$, $\mathbf{g}_j^{\mathbf{N}}$, $j = 1, .., 2d$, where $\mathbf{f}$ is control of (2.1). Therefore, $\mathbf{f}^{\mathbf{N}}$ is a numerical approximation of a continuous control $\mathbf{f}$, while $\mathbf{g}_j^{\mathbf{N}}, j = 1, .., 2d$ can be understood as artificial controls which are only necessary to obtain $\mathbf{f}^{\mathbf{N}}$. In fact, the existence of discrete controls and their uniform bound (with respect to $\mathbf{N}$) in terms of the initial data is a direct consequence of the observability results.

The most important advantage of the method is the convergence result. This is illustrated when the numerical approximation of the boundary control for the simpler wave equation is considered (chapters 2 and 3) but it is more difficult to prove in elasticity. In fact, if we try to apply the general theory established in Ervedoza and Zuazua, 2012 to recover the convergence of controls $f^{\mathbf{N}} \to f$ in $(L^2(0,T;\Gamma))$ and the numerical control $g_j^{\mathbf{N}}, j = 1, ..., 2d$ vanishes as $\mathbf{N} \to \infty$, some important hypotheses related with the convergence of the solutions of discrete adjoint system must be checked. Such convergent results are not standard with the usual numerical analysis techniques and require further investigation (see Remark 14).

The rest of this chapter is organized as follows. In section 4.2, we briefly describe the mathematical background of the continuous elasticity control problem. In section 4.3, we introduce the discrete control problem obtained by the spectral method and state the main result of the chapter. In section 4.4, we present the proof of the existence of discrete controls for the spectral approximation. Finally, in section 4.5, we state the matrix formulation of the discrete control problem that we approximate using the finite-dimensional control theory and present some numerical experiments and numerical results.

4.2 Variational characterization of the continuous control problem

In this section, we briefly describe the mathematics background of the variational characterization of the control problem (4.1) that we use later to find their numerical approximation. In particular, we prove that a class of controls can be obtained as minimizers of a quadratic functional defined on a Hilbert space. The results of this section are not new and can be found in (Alabau and Komornik, 1999 and Lions, 1988a). We include them here for completeness.

For technical reasons, we restrict ourselves to controls which are zero near $t = 0, T$. This affects to the quadratic functional that we define below.

Let us consider the following backwards wave equation,

$$\begin{cases} \left(\boldsymbol{\phi}_{tt} - \Delta^* \boldsymbol{\phi}\right)(t, \mathbf{x}) = \mathbf{0}, & (t, \mathbf{x}) \in (0, T) \times \Omega, \\ \boldsymbol{\phi}(t, \mathbf{x}) = \mathbf{0}, & (t, \mathbf{x}) \in (0, T) \times \partial\Omega, \\ (\boldsymbol{\phi}(0, \mathbf{x}), \boldsymbol{\phi}_t(0, \mathbf{x})) = (\boldsymbol{\phi}^{\mathbf{0}}(\mathbf{x}), \boldsymbol{\phi}^{\mathbf{1}}(\mathbf{x})), & \mathbf{x} \in \Omega, \end{cases} \tag{4.3}$$

where $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d$. We also define the duality product between $(L^2(\Omega))^d \times (H^{-1}(\Omega))^d$ and $(H_0^1(\Omega))^d \times (L^2(\Omega))^d$ by

$$\left\langle (\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}), (\mathbf{u}^{\mathbf{0}}, \mathbf{u}^{\mathbf{1}}) \right\rangle = < \mathbf{u}^{\mathbf{1}}, \boldsymbol{\phi}^{\mathbf{0}} >_{(H^{-1})^d, (H_0^1)^d} - \int_\Omega \mathbf{u}^{\mathbf{0}} \cdot \boldsymbol{\phi}^{\mathbf{1}} d\mathbf{x}, \tag{4.4}$$

where $< \cdot, \cdot >_{(H^{-1})^d, (H_0^1)^d}$ is the usual duality product between $(H_0^1)^d(\Omega)$ and its dual space $(H^{-1})^d(\Omega)$.

The following result provides a variational characterization of the control.

Lemma 4.2.1. *Assume that* $T > 0$, *and consider some initial data* $(\mathbf{u}^{\mathbf{0}},\ \mathbf{u}^{\mathbf{1}}) \in (L^2(\Omega))^d \times (H^{-1}(\Omega))^d$. *Any controls* $\mathbf{f}$ *that make the solution of the discrete system* (4.1) *satisfy* (4.2) *are solution of*

$$0 = \int_0^T \int_\Gamma (\mu \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\nu}} + (\lambda + \mu)\boldsymbol{\nu}\, div \boldsymbol{\phi})\mathbf{f}\, d\gamma dt - \Big\langle (\boldsymbol{\phi}(0,\cdot), \boldsymbol{\phi}_t(0,\cdot)), (\mathbf{u^0}, \mathbf{u^1}) \Big\rangle, \tag{4.5}$$

for all $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d$*, where* $(\boldsymbol{\phi}, \boldsymbol{\phi}_t)$ *is the solution of the backwards wave equation* (4.3)*. Reciprocally, if* $\mathbf{f}$ *satisfies* (4.5)*, then* (4.2) *holds.*

According to **H.U.M**, one possibility to construct controls $\mathbf{f}$ that satisfy the variational condition (4.5) is as minimizers of a particular quadratic functional. We define the following cost functional $J : (H_0^1(\Omega))^d \times (L^2(\Omega))^d \longrightarrow \mathbb{R}$ by

$$J(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) = \frac{1}{2} \int_0^T \int_\Gamma \eta(t) \left| \mu \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\nu}} + (\lambda + \mu)\boldsymbol{\nu} \operatorname{div} \boldsymbol{\phi} \right|^2 d\gamma dt - \Big\langle (\boldsymbol{\phi}(0,\cdot), \boldsymbol{\phi}_t(0,\cdot)), (\mathbf{u^0},\ \mathbf{u^1}) \Big\rangle, \tag{4.6}$$

where $(\boldsymbol{\phi}, \boldsymbol{\phi}_t)$ is the solution of (4.3) with final data $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d$. The function $\eta(t)$ is chosen to give a compact support control in $(0, T)$ as prescribed in (2.9).

Theorem 4.2.2. *Assume* $(\mathbf{u^0},\ \mathbf{u^1}) \in (L^2(\Omega))^d \times (H^{-1}(\Omega))^d$ *and we suppose that* $(\hat{\boldsymbol{\phi}}^{\mathbf{0}}, \hat{\boldsymbol{\phi}}^{\mathbf{1}}) \in (H_0^1(\Omega))^d \times (L^2(\Omega))^d$ *is a minimizer of* J*. If* $\hat{\boldsymbol{\phi}}$ *is the corresponding solution of* (4.3) *with final data* $(\hat{\boldsymbol{\phi}}^{\mathbf{0}}, \hat{\boldsymbol{\phi}}^{\mathbf{1}})$*, then*

$$\mathbf{f}(t, \mathbf{x}) = \eta(t)(\mu \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\nu}} + (\lambda + \mu)\boldsymbol{\nu}\, div \boldsymbol{\phi}), \quad x \in \Gamma,\ t > 0 \tag{4.7}$$

is a control such that the solution of (4.1) *satisfies* (4.2)*.*

Let us now give a general condition which ensures the existence of a minimizer for J and, therefore, a control for system (4.1).

The functional J is evidently continuous and convex. Consequently, the existence of a minimizer, and thus discrete controls for system (4.1), are assured once we establish its coercivity. This follows as a consequence of the following lemma.

Lemma 4.2.3. *Given sufficiently large time* $T > \dfrac{4\sqrt{d}}{\sqrt{\mu}}$*, there exists a constant* $C > 0$*, such* *that the solution of system* (4.3) *satisfy*

$$C\left|(\boldsymbol{\phi}^{\mathbf{0}},\boldsymbol{\phi}^{\mathbf{1}})\right|^2_{(H_0^1)^d\times(L^2)^d}\leq\int_0^T\int_\Gamma\left|\mu\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{\nu}}+(\lambda+\mu)\boldsymbol{\nu}\,div\boldsymbol{\phi}\right|^2d\gamma dt, \tag{4.8}$$

for any initial data $(\boldsymbol{\phi}^{\mathbf{0}},\boldsymbol{\phi}^{\mathbf{1}})$.

Note that the observability that we have defined above in (4.8) is a sufficient condition for the controllability of problem (4.1) and then the functional J defined by (4.6) has an unique minimizer $(\hat{\boldsymbol{\phi}}^{\mathbf{0}},\hat{\boldsymbol{\phi}}^{\mathbf{1}})\in(H_0^1(\Omega))^d\times(L^2(\Omega))^d$. Hence, a unique control $\mathbf{f}\in(L^2(0,T;\Gamma))^d$.

4.3 Approximation by the spectral collocation method

In this section, we introduce some notation, the approximate discrete control problem and state the main results of the chapter.

Let $\mathbf{N}=(N,...,N)\in\mathbb{N}^d$ be the number of collocation points in each variable x_j that we assume to be the same. We can consider more general situations with different numbers of points in each dimension, but this is not relevant in our analysis and would make the notation more involved.

We also consider $C=\{\mathbf{P_i}=(x_1^{k_1},...,x_d^{k_d}),(0,...,0)\leq\mathbf{i}=(k_1,...,k_d)\leq(N,...,N)\}$ the Legendre-Gauss-Lobatto (LGL) nodes in Ω that are the roots of the polynomial

$$\prod_{j=1}^d(1-x_j^2)\partial_{x_j}L^N(x_j),$$

where L^k is the k-th Legendre polynomial in $(-1,1)$ defined as (2.16). We divide C into interior and boundary nodes, i.e. $C=C^\Omega\cup C^{\partial\Omega}$ where $C^\Omega=C\cap\Omega=\{\mathbf{P_i},\mathbf{i}\in\mathbf{I}_\Omega\}$ and $C^{\partial\Omega}=C\cap\partial\Omega=\{\mathbf{P_i},\mathbf{i}\in\mathbf{I}_{\partial\Omega}\}$, and $\mathbf{I}_\Omega$, $\mathbf{I}_{\partial\Omega}$ are the sets of indexes corresponding to the interior and boundary collocation nodes, respectively. We denote $\mathbf{I}=\mathbf{I}_\Omega\cup\mathbf{I}_{\partial\Omega}$.

Let $\mathbb{P}_{\mathbf{N}}(\Omega)$ be the space of polynomials of degree at most N in the x_j-variable, $j=1,...,d$ and let $\mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$ be the subspace of $\mathbb{P}_{\mathbf{N}}(\Omega)$ of those vanishing on the boundary $\partial\Omega$.

We define the following discrete inner product that approximates the $L^2(\Omega)$ one:

$$(w,z)_{\mathbf{N}} = \sum_{\mathbf{i}\in\mathbf{I}}(wz)(t,\mathbf{P_i})\omega_{\mathbf{i}},\ 0 \leq t \leq T. \tag{4.9}$$

Here, $\omega_{\mathbf{i}} = \prod_{j=1}^{d}\omega_{k_j}$, where ω_{k_j} is the discrete weight associated with the 1-d Legendre-Gauss-Lobato (LGL) quadrature formula defined as (2.18) (e.g.Canuto et al., 1988, Chap. 2). Owing to the exactness of this quadrature we have

$$(w,z)_{\mathbf{N}} = \int_{\Omega} wz\ d\mathbf{x} \text{ for all } w,z \text{ such that } wz \in \mathbb{P}_{\mathbf{2N-1}}(\Omega), \quad \mathbf{x} = (x_1,...,x_d). \tag{4.10}$$

Moreover, the discrete norm $\|\cdot\|_{\mathbf{N}} = \sqrt{(z,z)_{\mathbf{N}}}$ is uniformly equivalent to the $|\cdot|_{L^2}$ −norm in $\mathbb{P}_{\mathbf{N}}(\Omega)$ (Canuto et al., 1988, Chapter 9). In fact, for the constants $C_1 = 1$, and $C_2 = (2+N^{-1})^d$,

$$C_1\,|p|_{L^2}^2 \leq \|p\|_{\mathbf{N}}^2 \leq C_2\,|p|_{L^2}^2\,,\ \forall p \in \mathbb{P}_{\mathbf{N}}(\Omega). \tag{4.11}$$

We denote by $\Psi_{k_j}(x_j), j = 1,...,d,\ k_j = 0,...,N$ the Lagrange polynomial which is 1 at $x_j^{k_j}$ and 0 at all the other collocation points. Note that $\{\Psi_{\mathbf{i}}(\mathbf{x}) = \prod_{j=1}^{d}\Psi_{k_j}(x_j),\ \mathbf{i} = (k_1,...,k_d) \in \mathbf{I}\}$ constitutes a basis in $\mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$.

Our main objective is to approximate the control $\mathbf{f}$. Let us denote $\partial\Omega = \cup_j^{2d}\Gamma_j$, where $\Gamma_j = \{\mathbf{x} \in \overline{\Omega},\ \text{s.t. } x_j = 1\}$ and $\Gamma_{d+j} = \{\mathbf{x} \in \overline{\Omega},\ \text{s.t. } x_j = -1\},\ j = 1,...d$. The set $\mathbb{P}_{\mathbf{N}}(\Gamma)$ (resp $\mathbb{P}_{\mathbf{N}}(\Gamma_j), j = 1,...,2d$) is the restriction of $\mathbb{P}_{\mathbf{N}}(\Omega)$ to Γ (resp to $\Gamma_j,\ j = 1,...,2d$) and $\mathbf{I}_\Gamma$ is the restriction of $\mathbf{I}$ to the boundary Γ.

We introduce the following discrete control problem: given $\mathbf{u^{0,N}},\ \mathbf{u^{1,N}} \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d$, find $\mathbf{f^N} \in (L^2(0,T;\mathbb{P}_{\mathbf{N}}(\Gamma)))^d$, $\mathbf{g}_j^{\mathbf{N}} \in (L^2(0,T;\mathbb{P}_{\mathbf{N}}(\Gamma_j)))^d, j = 1,...,2d$, such that the solution $\mathbf{u^N} \in (C^\infty((0,T);\mathbb{P}_{\mathbf{N}}(\Omega)))^d$ of system

$$\begin{cases} \left(\mathbf{u}_{tt}^{\mathbf{N}} - \Delta^*\mathbf{u^N}\right)(t,\mathbf{P_i}) = \sum_{j=1}^{2d} G_j^{\mathbf{N}}(\mathbf{P_i}) & \text{in } (t,\mathbf{P_i}) \in (0,T)\times C^{\Omega}, \\ \mathbf{u^N}(t,\mathbf{P_i}) = \mathbf{f^N}\chi_\Gamma(t,\mathbf{P_i}) & \text{on } (t,\mathbf{P_i}) \in (0,T)\times C^{\partial\Omega}, \\ \mathbf{u^N}(0,\mathbf{P_i}) = \mathbf{u^{0,N}}(\mathbf{P_i}), \mathbf{u}_t^{\mathbf{N}}(0,\mathbf{P_i}) = \mathbf{u^{1,N}}(\mathbf{P_i}) & \text{in } \mathbf{P_i} \in C^{\Omega} \end{cases} \tag{4.12}$$

satisfies

$$\mathbf{u}^{\mathbf{N}}(T,\mathbf{P_i}) = \mathbf{u}_t^{\mathbf{N}}(T,\mathbf{P_i}) = \mathbf{0}, \text{ at } \mathbf{P_i} \in C^{\Omega}. \tag{4.13}$$

Note that (4.12) is a second-order system of ODE with $d(N+1)^d$ equations and unknowns, namely the coefficients of the polynomial $\mathbf{u}^{\mathbf{N}}$. Here, $G_j^{\mathbf{N}} \in (\mathbb{P}_{\mathbf{N}}(\Omega))^d$ depend on $g_j^{\mathbf{N}}$, $j = 1, ..., 2d$. For example, for $j=1$ and $j=d+1$, which corresponds to Γ_1, Γ_{d+1} respectively, we write $\mathbf{P_i} = (x_1^{k_1}, x_2^{k_2}, ..., x_d^{k_d}) \in C^{\Omega}$, $\mathbf{P}_{\mathbf{i}}^1 = (1, x_2^{k_2}, ..., x_d^{k_d}) \in \Gamma_1$ and $\mathbf{P}_{\mathbf{i}}^2 = (-1, x_2^{k_2}, ..., x_d^{k_d}) \in \Gamma_{d+1}$,

$$\begin{cases} G_1^{\mathbf{N}}(\mathbf{P_i}) = A_1 \tilde{h}_1(x_1^{k_1}) g_1^{\mathbf{N}}(t,\mathbf{P}_{\mathbf{i}}^1),\ G_{d+1}^{\mathbf{N}}(\mathbf{P_i}) = A_1 \tilde{h}_2(x_1^{k_1}) g_{d+1}^{\mathbf{N}}(t,\mathbf{P}_{\mathbf{i}}^2), \\ A_j \in \mathcal{M}_{d\times d} \text{ diagonal with components } \ a_{kk} = \mu + (\mu+\lambda)\delta_{kj}, \\ \tilde{h}_1(s) = \dfrac{1}{\sqrt{\omega_N}} \left(h_{ss}^1 + \dfrac{\Psi_{N,s}}{\omega_N} \right)(s),\ \tilde{h}_2(s) = \dfrac{1}{\sqrt{\omega_0}} \left(h_{ss}^2 - \dfrac{\Psi_{0,s}}{\omega_0} \right)(s), \\ h^1, h^2 \in \mathbb{P}_N^{Di}(-1,1),\ h^1(s^k) = \dfrac{1+s^k}{2}, h^2(s^k) = \dfrac{1-s^k}{2},\ s^k \in C^{(-1,1)}. \end{cases} \tag{4.14}$$

The main results in this chapter are the following theorems:

Theorem 4.3.1. *Given* $T > \dfrac{4\sqrt{d}(2+N^{-1})^d}{\sqrt{\mu}}$ *and* $(\mathbf{u}^{\mathbf{0},\mathbf{N}}, \mathbf{u}^{\mathbf{1},\mathbf{N}}) \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d \times (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d$, *there exist controls* $\mathbf{f}^{\mathbf{N}} \in (L^2(0,T;\mathbb{P}_{\mathbf{N}}(\Gamma)))^d$, $\mathbf{g}_j^{\mathbf{N}} \in (L^2(0,T;\mathbb{P}_{\mathbf{N}}(\Gamma_j)))^d$. $j = 1,..,2d$ *such that the solution* $\mathbf{u}^{\mathbf{N}}$ *of* (4.12) *satisfies* (4.13).

Theorem 4.3.2. *Given* $(\mathbf{u}^{\mathbf{0}}, \mathbf{u}^{\mathbf{1}}) \in (L^2(\Omega))^d \times (H^{-1}(\Omega))^d$, *there exists a sequence* $(\mathbf{u}^{\mathbf{0},\mathbf{N}}, \mathbf{u}^{\mathbf{1},\mathbf{N}}) \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d \times (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d$ *such that*

$$(\mathbf{u}^{\mathbf{0},\mathbf{N}}, \mathbf{u}^{\mathbf{1},\mathbf{N}}) \longrightarrow (\mathbf{u}^{\mathbf{0}}, \mathbf{u}^{\mathbf{1}}) \text{ in } (L^2(\Omega))^d \times (H^{-1}(\Omega))^d \tag{4.15}$$

and there exists a constant $M > 0$, *independent of* $\mathbf{N}$, *such that*

$$\int_0^T \sum_{\mathbf{i}\in \mathbf{I}_\Gamma} \left|\mathbf{f}^{\mathbf{N}}\right|^2 (t,\mathbf{P_i})\omega_{\mathbf{i}} dt \ + \sum_{j=1}^{2d} \int_0^T \sum_{\mathbf{i}\in \mathbf{I}_j} \left|\mathbf{g}_j^{\mathbf{N}}\right|^2 (t,\mathbf{P_i})\omega_{\mathbf{i}} dt \leq M \left|(\mathbf{u}^{\mathbf{0}}, \mathbf{u}^{\mathbf{1}})\right|^2_{((L^2)^d \times (H^{-1})^d}. \tag{4.16}$$

Remark 11. *Note that the control time* T *in Theorem 4.3.1 is* $(2+N^{-1})^d$ *times the time required in the continuous problem. This is due to the constant* C_2 *in* (4.11) *and probably not optimal, as we illustrate in the experiments below.*

Remark 12. *When* $\mathbf{u^0}$ *and* $\mathbf{u^1}$ *are continuous functions the polynomials* $\mathbf{\tilde{u}^{0,N}}, \mathbf{\tilde{u}^{1,N}} \in (\mathbb{P}^{Di}_{\mathbf{N}})^d$ *that coincides with* $\mathbf{u^0}$ *and* $\mathbf{u^1}$ *at the nodes* $\mathbf{P_i} \in C^{\Omega}$ *give a discretization that satisfies*

$$(\mathbf{\tilde{u}^{0,N}}, \mathbf{\tilde{u}^{1,N}}) \to (\mathbf{u^0}, \mathbf{u^1}) \text{ in } (L^2(\Omega))^d \times (H^{-1})(\Omega)^d,$$

as $\mathbf{N} \to \infty$*. In particular,*

$$\|(\mathbf{\tilde{u}^{0,N}}, \mathbf{\tilde{u}^{1,N}})\|_{\mathbf{N}^d \times \mathbf{N}^d} \leq |(\mathbf{u^0}, \mathbf{u^1})|_{(L^2)^d \times (H^{-1})^d}.$$

In this case, the result in Theorem 3.3.2 is still true when considering $(\mathbf{u^{0,N}}, \mathbf{u^{1,N}}) = (\mathbf{\tilde{u}}^{0,N}, \mathbf{\tilde{u}^{1,N}})$*. Thus, for smooth functions there exist a constructive way to choose* $(\mathbf{u^{0,N}}, \mathbf{u^{1,N}})$ *in Theorem 3.3.1.*

Remark 13. *In context with the wave equation in Chapter 2, here we are not able to prove the convergence of the controls to the continuous one.*

The proofs of these results follow closely the previous chapters. It is based on a suitable variational characterization of the controls and the uniform observability inequality for a corresponding adjoint system.

4.4 Existence of discrete controls: proof of Theorem 4.3.1

In this section, we state the proof of this result is based on a suitable variational characterization of discrete HUM control problem (4.12). In particular, we prove that a class of discrete controls can be obtained as minimizers of a quadratic functional defined on a polynomial space. Finally, the coerciveness of the functional that guarantees the existence of minimizers. Let us introduce the following bilinear form in $(\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d \times (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d$:

$$\Big\langle (\boldsymbol{\phi}^{\mathbf{0,N}}, \boldsymbol{\phi}^{\mathbf{1,N}}), (\mathbf{u^{0,N}}, \mathbf{u^{1,N}}) \Big\rangle_{\mathbf{N}} = (\mathbf{u^{1,N}}, \boldsymbol{\phi}^{\mathbf{0,N}})_{\mathbf{N}} - (\mathbf{u^{0,N}}, \boldsymbol{\phi}^{\mathbf{1,N}})_{\mathbf{N}}. \tag{4.17}$$

Lemma 4.4.1. *Assume that* $T > 0$*, and consider some initial data* $(\mathbf{u^{0,N}},\ \mathbf{u^{1,N}}) \in (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d \times (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d$*. Any controls* $\mathbf{f^N},\ \mathbf{g}^{\mathbf{N}}_j, j = 1, ..., 2d$ *that make the solution of the discrete system*

(4.12) *satisfy* (4.13) *are solution of*

$$\begin{aligned}\int_0^T \sum_{\mathbf{i}\in\mathbf{I}_\Gamma} &\left(\mathbf{f}^{\mathbf{N}}\left(\mu\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}+(\lambda+\mu)\boldsymbol{\nu}\,div\boldsymbol{\phi}^{\mathbf{N}}-\omega_N\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial div\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)\right)\right)(t,\mathbf{P_i})\omega_{\mathbf{i}}dt\\ &+\sum_{j=1}^{2d}\int_0^T\sum_{\mathbf{i}\in\mathbf{I}_j}\left(\mathbf{g}_j^{\mathbf{N}}\sqrt{\omega_N}\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial div\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)\right)(t,\mathbf{P_i})\omega_{\mathbf{i}}dt\\ &-\left\langle(\boldsymbol{\phi}^{\mathbf{N}}(0,\cdot),\boldsymbol{\phi}_t^{\mathbf{N}}(0,\cdot)),(\mathbf{u}^{\mathbf{0},\mathbf{N}},\ \mathbf{u}^{\mathbf{1},\mathbf{N}})\right\rangle_{\mathbf{N}}=0,\end{aligned} \tag{4.18}$$

for all $(\boldsymbol{\phi}^{\mathbf{0},\mathbf{N}},\boldsymbol{\phi}^{\mathbf{1},\mathbf{N}})\in(\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d\times(\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d$, *where* $(\boldsymbol{\phi}^{\mathbf{N}},\boldsymbol{\phi}_t^{\mathbf{N}})\in(\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d\times(\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d$ *is the solution of the following collocation approximation of adjoint system:*

$$\begin{cases}(\boldsymbol{\phi}_{tt}^{\mathbf{N}}-\Delta^*\boldsymbol{\phi}^{\mathbf{N}})(t,\mathbf{P_i})=\mathbf{0}, & (t,\mathbf{P_i})\in(0,T)\times C^{\Omega},\\ \boldsymbol{\phi}^{\mathbf{N}}(t,\mathbf{P_i})=\mathbf{0}, & (t,\mathbf{P_i}),\in(0,T)\times C^{\partial\Omega},\\ \boldsymbol{\phi}^{\mathbf{N}}(0,\cdot)=\boldsymbol{\phi}^{\mathbf{0},\mathbf{N}},\boldsymbol{\phi}_t^{\mathbf{N}}(0,\cdot)=\boldsymbol{\phi}^{\mathbf{1},\mathbf{N}}.\end{cases} \tag{4.19}$$

Proof. Multiplying the equation of $\mathbf{u}^{\mathbf{N}}(t,\mathbf{P_i})$ in (4.12) by $\omega_{\mathbf{i}}\boldsymbol{\phi}^{\mathbf{N}}(t,\mathbf{P_i})$ and adding in $\mathbf{i}\in\mathbf{I}$, one obtains

$$\int_0^T(\mathbf{u}_{tt}^{\mathbf{N}}-\Delta^*\mathbf{u}^{\mathbf{N}},\boldsymbol{\phi}^{\mathbf{N}})_{\mathbf{N}}dt=\sum_{j=1}^{2d}\int_0^T(G_j^{\mathbf{N}},\boldsymbol{\phi}^{\mathbf{N}})_{\mathbf{N}}dt. \tag{4.20}$$

We first simplify the left-hand side. Using (4.10), integrating by parts in time and taking into account that $\mathbf{f}^{\mathbf{N}},\ \mathbf{g}_j^{\mathbf{N}}, j=1,..,2d$ are controls, we have

$$\begin{aligned}\int_0^T(\mathbf{u}_{tt}^{\mathbf{N}}-\Delta^*\mathbf{u}^{\mathbf{N}},\boldsymbol{\phi}^{\mathbf{N}})_{\mathbf{N}}dt=\int_0^T(\mathbf{u}^{\mathbf{N}},\boldsymbol{\phi}_{tt}^{\mathbf{N}})_{\mathbf{N}}dt&-\int_0^T(\Delta^*\mathbf{u}^{\mathbf{N}},\boldsymbol{\phi}^{\mathbf{N}})_{\mathbf{N}}dt\\ &-\left\langle(\boldsymbol{\phi}^{\mathbf{N}}(0,\cdot),\boldsymbol{\phi}_t^{\mathbf{N}}(0,\cdot)),(\mathbf{u}^{\mathbf{0},\mathbf{N}},\mathbf{u}^{\mathbf{1},\mathbf{N}})\right\rangle_{\mathbf{N}}.\end{aligned} \tag{4.21}$$

In the rest of this proof, we estimate the second term on the right-hand side of this expression. By definition of elasticity operator it is enough to estimate the following $\int_0^T(\Delta\mathbf{u}^{\mathbf{N}},\boldsymbol{\phi}^{\mathbf{N}})_{\mathbf{N}}dt$ and $\int_0^T(\nabla\mathrm{div}\mathbf{u}^{\mathbf{N}},\boldsymbol{\phi}^{\mathbf{N}})_{\mathbf{N}}dt$ based on separation of variable.

Now, using formula (4.9) and integrate by parts in space since the resulting integrand is also a polynomial of degree $2N-2$ in the x_j-variable, $j=1,...,d$, we have

$$\begin{aligned}
&\int_0^T (\Delta\mathbf{u^N},\boldsymbol{\phi}^\mathbf{N})_\mathbf{N}dt = \int_0^T (\mathbf{u^N},\Delta\boldsymbol{\phi}^\mathbf{N})_\mathbf{N}dt + \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_\Gamma}\Big(\mathbf{f^N}\frac{\partial\boldsymbol{\phi}^\mathbf{N}}{\partial\boldsymbol{\nu}}\Big)(t,\mathbf{P_i})\omega_\mathbf{i}dt\\
&\int_0^T (\nabla\mathrm{div}\mathbf{u^N},\boldsymbol{\phi}^\mathbf{N})_\mathbf{N}dt = \int_0^T (\mathbf{u^N},\nabla\mathrm{div}\boldsymbol{\phi}^\mathbf{N})_\mathbf{N}dt + \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_\Gamma}\Big(\mathbf{f^N}\boldsymbol{\nu}\mathrm{div}\boldsymbol{\phi}^\mathbf{N}\Big)(t,\mathbf{P_i})\omega_\mathbf{i}dt.
\end{aligned}\tag{4.22}$$

From (4.21) and (4.22), we easily obtain

$$\begin{aligned}
0 &= \int_0^T (\mathbf{u}_{tt}^\mathbf{N}-\Delta^*\mathbf{u^N},\boldsymbol{\phi}^\mathbf{N})_\mathbf{N}dt = \int_0^T (\mathbf{u^N},\boldsymbol{\phi}_{tt}^\mathbf{N}-\Delta^*\boldsymbol{\phi}^N)_\mathbf{N}dt\\
&+\int_0^T \sum_{\mathbf{i}\in\mathbf{I}_\Gamma}\mathbf{f^N}\left(\mu\frac{\partial\boldsymbol{\phi}^\mathbf{N}}{\partial\boldsymbol{\nu}}+(\lambda+\mu)\boldsymbol{\nu}\mathrm{div}\boldsymbol{\phi}^\mathbf{N}\right)(t,\mathbf{P_i})\omega_\mathbf{i}dt - \Big\langle(\boldsymbol{\phi}^\mathbf{N}(0,\cdot),\boldsymbol{\phi}_t^\mathbf{N}(0,\cdot)),(\mathbf{u^{0,N}},\mathbf{u^{1,N}})\Big\rangle_\mathbf{N}\\
&=\int_0^T \sum_{\mathbf{i}\in\mathbf{I}_\Gamma}\left(\mathbf{f^N}\left(\mu\frac{\partial\boldsymbol{\phi}^\mathbf{N}}{\partial\boldsymbol{\nu}}+(\lambda+\mu)\boldsymbol{\nu}\mathrm{div}\boldsymbol{\phi}^\mathbf{N}-\omega_N\left(\mu\frac{\partial^2\boldsymbol{\phi}^\mathbf{N}}{\partial^2\boldsymbol{\nu}}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^\mathbf{N}}{\partial\boldsymbol{\nu}}\right)\right)\right)(t,\mathbf{P_i})\omega_\mathbf{i}dt\\
&\quad-\Big\langle(\boldsymbol{\phi}^\mathbf{N}(0,\cdot),\boldsymbol{\phi}_t^\mathbf{N}(0,\cdot)),(\mathbf{u^{0,N}},\mathbf{u^{1,N}})\Big\rangle_\mathbf{N}.
\end{aligned}\tag{4.23}$$

The last equality is a consequence of the first equation in (4.19). Note that the term multiplied by ω_N which appears in the right-hand side of this expression coming from the fact that the first equation in (4.19) is only true for the interior nodes while the discrete scalar product involves also the boundary nodes.

For the right hand side in (4.20) we use again formula (4.10) and the fact that is a polynomial of degree $2N-1$ in the x_j-variable, $j=1,...,d$. It is enough to estimate it on one parts of boundary since the others are similar. For example for $j=1$ which corresponds to

Γ_1, we write $\mathbf{P_i} = (x_1^{k_1}, x_2^{k_2}, ..., x_d^{k_d}) \in C^{\Omega}$, $\mathbf{P}_{\mathbf{i}}^1 = (1, x_2^{k_2}, ..., x_d^{k_d}) \in \Gamma_1$ and $\omega_{\mathbf{i}}^1 = \prod_{j=2}^{d} \omega_{k_j}$,

$$\begin{aligned}
(G_1^{\mathbf{N}}, \boldsymbol{\phi}^{\boldsymbol{N}})_{\mathbf{N}} &= (A_1 \tilde{h}_1(x_1^{k_1}) g_1^{\mathbf{N}}(t, \mathbf{P}_{\mathbf{i}}^1), \boldsymbol{\phi}^{\boldsymbol{N}})_{\mathbf{N}} \\
&= \sum_{\mathbf{i} \in \mathbf{I_1}} \left(A_1 g_1^{\mathbf{N}} \int_{-1}^{1} \tilde{h}_1(x_1^{k_1}) \boldsymbol{\phi}^{\boldsymbol{N}} dx_1 \right)(t, \mathbf{P}_{\mathbf{i}}^1) \omega_{\mathbf{i}}^1 \\
&= \sum_{\mathbf{i} \in \mathbf{I_1}} \left(A_1 g_1^{\mathbf{N}} \int_{-1}^{1} \frac{1}{\sqrt{\omega_N}} \left(h_{x_1 x_1}^1 + \frac{\Psi_{N,x_1}}{\omega_N} \right) \boldsymbol{\phi}^{\boldsymbol{N}} dx_1 \right)(t, \mathbf{P}_{\mathbf{i}}^1) \omega_{\mathbf{i}}^1 \\
&= \sum_{\mathbf{i} \in \mathbf{I_1}} \left(A_1 g_1^{\mathbf{N}} \frac{1}{\sqrt{\omega_N}} \left(\int_{-1}^{1} h^1 \boldsymbol{\phi}_{x_1 x_1}^{\boldsymbol{N}} dx_1 + \int_{-1}^{1} \frac{\Psi_N}{\omega_N} \right) \boldsymbol{\phi}_{x_1}^{\boldsymbol{N}} dx_1 \right)(t, \mathbf{P}_{\mathbf{i}}^1) \omega_{\mathbf{i}}^1 \\
&= \sum_{\mathbf{i} \in \mathbf{I_1}} \left(A_1 g_1^{\mathbf{N}}(t, \mathbf{P}_{\mathbf{i}}^1) \frac{1}{\sqrt{\omega_N}} \left(\sum_{k_1=1}^{N} (h^1 \boldsymbol{\phi}_{x_1 x_1}^{\boldsymbol{N}})(t, \mathbf{P_i}) \omega_{k_1} + \boldsymbol{\phi}_{x_1}^{\boldsymbol{N}}(t, \mathbf{P}_{\mathbf{i}}^{\mathbf{1}}) \right) \right) \omega_{\mathbf{i}}^1.
\end{aligned} \tag{4.24}$$

To simplify the first term on the right-hand side, we observe that $\tilde{h}_1(x_1) = \dfrac{1 + x_1}{2}$ at the interior nodes in $(-1, 1)$. Then,

$$\begin{aligned}
\sum_{k_1=1}^{N} (h^1 \boldsymbol{\phi}_{x_1 x_1}^{\boldsymbol{N}})(t, \mathbf{P_i}) \omega_{k_1} &= \int_{-1}^{1} \frac{1 + x_1}{2} \boldsymbol{\phi}_{x_1 x_1}^{\boldsymbol{N}} dx_1 - \omega_N \boldsymbol{\phi}_{x_1 x_1}^{\boldsymbol{N}}(t, \mathbf{P}_{\mathbf{i}}^{\mathbf{1}}) \\
&= \boldsymbol{\phi}_{x_1}^{\boldsymbol{N}}(t, \mathbf{P}_{\mathbf{i}}^{\mathbf{1}}) - \omega_N \boldsymbol{\phi}_{x_1 x_1}^{\boldsymbol{N}}(t, \mathbf{P}_{\mathbf{i}}^{\mathbf{1}}).
\end{aligned} \tag{4.25}$$

From (4.24)-(4.25) and definition of A_1 in (4.14), we easily obtain

$$\begin{aligned}
(G_1^{\mathbf{N}}, \boldsymbol{\phi}^{\boldsymbol{N}})_{\mathbf{N}} &= - \sum_{\mathbf{i} \in \mathbf{I_1}} \left(g_1^{\mathbf{N}} \omega_N A_1 \boldsymbol{\phi}_{x_1 x_1}^{\boldsymbol{N}} \right)(t, \mathbf{P}_{\mathbf{i}}^{\mathbf{1}}) \omega_{\mathbf{i}}^1 \\
&= - \sum_{\mathbf{i} \in \mathbf{I_1}} \left(g_1^{\mathbf{N}} \sqrt{\omega_N} \left(\mu \frac{\partial^2 \boldsymbol{\phi}^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} + (\lambda + \mu) \boldsymbol{\nu} \frac{\partial \mathrm{div} \boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} \right) \right)(t, \mathbf{P}_{\mathbf{i}}^{\mathbf{1}}) \omega_{\mathbf{i}}^1.
\end{aligned} \tag{4.26}$$

Combining (4.20), (4.23) and (4.26), the symmetry of $A_j, j = 1, ..., 2d$ in (4.14), we easily find (4.18). □

According to **H.U.M**, one possibility to construct controls $\mathbf{f}^{\mathbf{N}}$, $\mathbf{g}_j^{\mathbf{N}}$, $j = 1, ..., 2d$ that satisfy the variational condition (4.13) is as minimizers of a particular quadratic functional.

We define the following cost functional $J^{\mathbf{N}} : (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d \times (\mathbb{P}_{\mathbf{N}}(\Omega))^d \longrightarrow \mathbb{R}$ by

$$\begin{aligned}
&J^{\mathbf{N}}(\boldsymbol{\phi}^{\mathbf{0},\mathbf{N}}, \boldsymbol{\phi}^{\mathbf{1},\mathbf{N}}) = -\left\langle (\boldsymbol{\phi}^{\mathbf{N}}(0,\cdot), \boldsymbol{\phi}^{\mathbf{N}}_t(0,\cdot)), (\mathbf{u}^{\mathbf{0},\mathbf{N}},\ \mathbf{u}^{\mathbf{1},\mathbf{N}}) \right\rangle_{\mathbf{N}} \\
&+ \frac{1}{2}\int_0^T \eta(t) \sum_{\mathbf{i}\in\mathbf{I}_\Gamma} \left| \mu\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}} - \omega_{\mathbf{N}}\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)\right|^2 (t,\mathbf{P}_{\mathbf{i}})\omega_{\mathbf{i}}dt \\
&+ \frac{1}{2}\int_0^T \eta(t) \sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}} \omega_{\mathbf{N}} \left| \mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right|^2 (t,\mathbf{P}_{\mathbf{i}})\omega_{\mathbf{i}}dt,
\end{aligned} \tag{4.27}$$

where $(\boldsymbol{\phi}^{\mathbf{N}}, \boldsymbol{\phi}^{\mathbf{N}}_t)$ is the solution of (4.19) with final data $(\boldsymbol{\phi}^{\mathbf{0},\mathbf{N}}, \boldsymbol{\phi}^{\mathbf{1},\mathbf{N}}) \in (\mathbb{P}^{Di}_N(\Omega))^d \times (\mathbb{P}_{\mathbf{N}}(\Omega))^d$. The function $\eta(t)$ is chosen to give a compact support control in $(0,T)$ as prescribed in (2.9).

Theorem 4.4.2. *Assume* $(\mathbf{u}^{\mathbf{0},\mathbf{N}},\ \mathbf{u}^{\mathbf{1},\mathbf{N}}) \in (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d \times (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d$ *and we suppose that* $(\hat{\boldsymbol{\phi}}^{\mathbf{0},\mathbf{N}}, \hat{\boldsymbol{\phi}}^{\mathbf{1},\mathbf{N}}) \in (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d \times (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d$ *is a minimizer of* $J^{\mathbf{N}}$. *If* $\hat{\boldsymbol{\phi}^{\mathbf{N}}}$ *is the corresponding solution of* (4.19) *with final data* $(\hat{\boldsymbol{\phi}}^{\mathbf{0},\mathbf{N}}, \hat{\boldsymbol{\phi}}^{\mathbf{1},\mathbf{N}})$*, then*

$$\begin{aligned}
&\mathbf{f}^{\mathbf{N}}(t,\mathbf{P}_{\mathbf{i}}) = \eta(t)\left(\mu\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\,div\boldsymbol{\phi}^{\mathbf{N}} - \omega_{\mathbf{N}}\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial div\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)\right)\Bigg|_{\mathbf{I}_\Gamma} \\
&g^{\mathbf{N}}_j(t,\mathbf{P}_{\mathbf{i}}) = \eta(t)\sqrt{\omega_{\mathbf{N}}}\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial div\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)\Bigg|_{\mathbf{I}_j},\ j = 1,...,2d
\end{aligned} \tag{4.28}$$

are controls such that the solution of (4.12) *satisfies* (4.13).

Proof. If $J^{\mathbf{N}}$ achieves its minimum at $(\hat{\boldsymbol{\phi}}^{\mathbf{0},\mathbf{N}}, \hat{\boldsymbol{\phi}}^{\mathbf{1},\mathbf{N}})$ its Gateaux derivative in the direction $(\boldsymbol{\phi}^{\mathbf{0},\mathbf{N}}, \boldsymbol{\phi}^{\mathbf{1},\mathbf{N}})$ must vanish, i.e.

$$\begin{aligned}
0 = &\int_0^T \sum_{\mathbf{i}\in\mathbf{I}_\Gamma} \left(\mu\frac{\partial\hat{\boldsymbol{\phi}}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\mathrm{div}\hat{\boldsymbol{\phi}}^{\mathbf{N}} - \omega_N\left(\mu\frac{\partial^2\hat{\boldsymbol{\phi}}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\hat{\boldsymbol{\phi}}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)\right) \\
&\left(\mu\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}} - \omega_N\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)\right)(t,\mathbf{P}_{\mathbf{i}})\omega_{\mathbf{i}}dt \\
&+ \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}} \omega_N\left(\mu\frac{\partial^2\hat{\boldsymbol{\phi}}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\hat{\boldsymbol{\phi}}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)(t,\mathbf{P}_{\mathbf{i}})\omega_{\mathbf{i}}dt \\
&- \left\langle (\boldsymbol{\phi}^{\mathbf{N}}(0,\cdot), \boldsymbol{\phi}^{\mathbf{N}}_t(0,\cdot)), (\mathbf{u}^{\mathbf{0},\mathbf{N}},\ \mathbf{u}^{\mathbf{1},\mathbf{N}}) \right\rangle_{\mathbf{N}},
\end{aligned}$$

for all $(\boldsymbol{\phi}^{\mathbf{0},\mathbf{N}}, \boldsymbol{\phi}^{\mathbf{1},\mathbf{N}}) \in (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d \times (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d$. From Lemma 4.4.1 it follows that (4.28) are controls for which (4.12) holds. $\square$

Functional $J^{\mathbf{N}}$ is clearly continuous and convex so that the existence of a minimizer (and therefore a discrete control for system (4.12)) is guaranteed as soon as we prove its coercivity. This is a consequence of the following lemma.

Lemma 4.4.3. *System* (4.19) *is uniformly observable from the boundary* Γ *in time* $T > \dfrac{4\sqrt{d}(2+N^{-1})^d}{\sqrt{\mu}}$. *More precisely, there exist two constants* $C_1, C_2 > 0$, *independent of* $\mathbf{N}$, *such that*

$$C\left|(\nabla\boldsymbol{\phi}^{\mathbf{0},\mathbf{N}},\boldsymbol{\phi}^{\mathbf{1},\mathbf{N}})\right|^2_{N^d\times N^d} \leq \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}} \omega_{\mathbf{N}}\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\, \mathit{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)^2(t,\mathbf{P_i})\omega_{\mathbf{i}}dt$$
$$+\int_0^T \sum_{\mathbf{i}\in\mathbf{I}_{\Gamma}}\left(\mu\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\,\mathit{div}\boldsymbol{\phi}^{\mathbf{N}} - \omega_{\mathbf{N}}\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\, \mathit{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)\right)^2(t,\mathbf{P_i})\omega_{\mathbf{i}}dt, \tag{4.29}$$

for all initial data $(\boldsymbol{\phi}^{\mathbf{0},\mathbf{N}},\boldsymbol{\phi}^{\mathbf{1},\mathbf{N}}) \in (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d \times (\mathbb{P}^{Di}_{\mathbf{N}}(\Omega))^d$.

Proof. Since $\omega_{\mathbf{N}} < 1$, from (2.18) and the equivalence of the discrete L^2-norm for polynomials of degree N_1 (resp. N_2) in x_1-variable (resp. x_2), it is enough to prove the following

$$\begin{aligned} C_1\left|(\boldsymbol{\phi}^{\mathbf{0},\mathbf{N}},\boldsymbol{\phi}^{\mathbf{1},\mathbf{N}})\right|^2_{(H^1_0)^d\times(L^2)^d} &\leq \int_0^T\int_\Gamma\left|\mu\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}\right|^2 d\gamma dt \\ &\quad + C_2\int_0^T\int_{\partial\Omega}\left|\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right|^2 d\gamma dt. \end{aligned} \tag{4.30}$$

The main idea is to observe that the solutions of (4.19) solve the following equivalent continuous system,

$$\begin{cases} (\boldsymbol{\phi}^{\mathbf{N}}_{tt} - \Delta^*\boldsymbol{\phi}^{\mathbf{N}})(t,\mathbf{x}) = -\sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}}\Delta^*\boldsymbol{\phi}^{\mathbf{N}}(t,\mathbf{P_i})\Psi_{\mathbf{i}}(\mathbf{x}), & (t,\mathbf{x})\in(0,T)\times\Omega, \\ \boldsymbol{\phi}^{\mathbf{N}}(t,\mathbf{x}) = \mathbf{0}, & (t,\mathbf{x})\in(0,T)\times\partial\Omega, \\ \boldsymbol{\phi}^{\mathbf{N}}(0,\mathbf{x}) = \boldsymbol{\phi}^{\mathbf{0},\mathbf{N}}, \boldsymbol{\phi}^{\mathbf{N}}_t(0,\mathbf{x}) = \boldsymbol{\phi}^{\mathbf{1},\mathbf{N}}, & \mathbf{x}\in\Omega. \end{cases} \tag{4.31}$$

In fact, this is easily seen by writing $\boldsymbol{\phi}^{\mathbf{N}}$ and $\boldsymbol{\phi}^{\mathbf{N}}_{tt} - \Delta^*\boldsymbol{\phi}^{\mathbf{N}}$ in the Lagrangian basis, since they are polynomial of degree N in each variable, and using system (4.19).

Note that system (4.31) is a perturbation of the original adjoint system so that we can adapt the continuous multipliers proof in Lions Lions, 1988a, estimating the extra non-homogeneous right hand side in (4.31).

Given $\mathbf{x^0} = (-1, ..., -1) \in \mathbb{R}^d$ and $\mathbf{m}(\mathbf{x}) \in \mathbb{R}^d$ with components $m_j(\mathbf{x}) = x_j - x_j^0,\ j = 1, ..., d$. It is clear that for all $\mathbf{x} \in \partial\Omega\backslash\Gamma$ we have $\mathbf{m}\boldsymbol{\cdot}\boldsymbol{\nu} = \mathbf{0}$. Let us set $X = \int_\Omega \boldsymbol{\phi}_t^{\mathbf{N}} \boldsymbol{\cdot} m_j \dfrac{\partial \boldsymbol{\phi}^{\mathbf{N}}}{\partial x_j} d\mathbf{x}\Big|_0^T$ and $Y = \int_\Omega \boldsymbol{\phi}_t^{\mathbf{N}} \boldsymbol{\cdot} \boldsymbol{\phi}^{\mathbf{N}} d\mathbf{x}\Big|_0^T$. Multiplying scalarly the first vector equation in (4.31) by the vector $m_j \dfrac{\partial \boldsymbol{\phi}^{\mathbf{N}}}{\partial x_j}$ (here the repeated index j stands for the sum in $j = 1, ..., d$) and integrating by parts, the left-hand side can be simplified as follows:

$$\begin{aligned} &X + \frac{d-1}{2} Y - \int_0^T \int_{\partial\Omega} \frac{\boldsymbol{m}\boldsymbol{\cdot}\boldsymbol{\nu}}{2} \left(\mu \left| \frac{\partial \boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} \right|^2 + (\lambda+\mu) \left| \mathrm{div} \boldsymbol{\phi}^{\mathbf{N}} \right|^2 \right) d\gamma dt \\ &+ \frac{1}{2} \int_0^T \int_\Omega (\left| \boldsymbol{\phi}_t^{\mathbf{N}} \right|^2 + \mu \left| \nabla \boldsymbol{\phi}^{\mathbf{N}} \right|^2 + (\lambda+\mu) \left| \mathrm{div} \boldsymbol{\phi}^{\mathbf{N}} \right|^2) d\mathbf{x} dt. \end{aligned}$$

Using the fact that $\boldsymbol{m}\boldsymbol{\cdot}\boldsymbol{\nu} = 0$ on $\partial\Omega\backslash\Gamma$ and $\sup_{\mathbf{x}\in\Gamma}\ m_j\nu_j = 2\sqrt{d}$, we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^T \int_\Omega \left(\left| \boldsymbol{\phi}_t^{\mathbf{N}} \right|^2 + \mu \left| \nabla \boldsymbol{\phi}^{\mathbf{N}} \right|^2 + (\lambda+\mu) \left| \mathrm{div} \boldsymbol{\phi}^{\mathbf{N}} \right|^2 \right) d\mathbf{x} dt \le \left| X + \frac{d-1}{2} Y \right| \\ &+ \frac{2\sqrt{d}}{2} \int_0^T \int_\Gamma \left(\mu \left| \frac{\partial \boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} \right|^2 + (\lambda+\mu) \left| \mathrm{div} \boldsymbol{\phi}^{\mathbf{N}} \right|^2 \right) d\gamma dt \qquad (4.32) \\ &+ \left| \int_0^T \int_\Omega \left(\sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}} \left(\mu \frac{\partial^2 \boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}^2} + (\lambda+\mu) \boldsymbol{\nu} \frac{\partial \mathrm{div} \boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} \right) (t, \mathbf{P_i}) \Psi_{\mathbf{i}}(\mathbf{x}) \right) \boldsymbol{\cdot} \left(m_j \frac{\partial \boldsymbol{\phi}^{\mathbf{N}}}{\partial x_j} \right) d\mathbf{x} dt \right|. \end{aligned}$$

We now estimate each one of the terms in this expression. We start with the left-hand side in (4.32). Define the discrete energy

$$E^{\mathbf{N}}(t) = \frac{1}{2} \left(\left\| \boldsymbol{\phi}_t^{\mathbf{N}}(t,\cdot) \right\|_{\mathbf{N}^d}^2 + \mu \left\| \nabla \boldsymbol{\phi}^{\mathbf{N}}(t,\cdot) \right\|_{\mathbf{N}^d}^2 + (\lambda+\mu) \left\| \mathrm{div} \boldsymbol{\phi}^{\mathbf{N}}(t,\cdot) \right\|_{\mathbf{N}^d}^2 \right). \qquad (4.33)$$

This energy is conserved, i.e. $E^{\mathbf{N}}(t) = E^{\mathbf{N}}(0)$ for all $t > 0$. As usual, this is obtained just multiplying (4.19) by $\boldsymbol{\phi}_t^{\mathbf{N}} \omega_{\mathbf{i}}$, adding in $\mathbf{i} \in \mathbf{I}$ and integrating with respect to time. It is worth noting that the corresponding conservation of energy for the continuous system requires an integration by parts in space. In order to do the same, here we have to transform the sum in $\mathbf{i} \in \mathbf{I}$ into a $\mathbf{x} \in \Omega$-integral. This can be done here since the integrand is a polynomial of degree $\mathbf{2N-1}$ and the quadrature formula (4.10) is exact for such polynomials.

The norm equivalence in (4.11), together with the conservation of the discrete energy gives

$$\frac{1}{2}\int_0^T\int_\Omega\left(\left|\boldsymbol{\phi}_t^{\mathbf{N}}\right|^2+\mu\left|\nabla\boldsymbol{\phi}^{\mathbf{N}}\right|^2+(\lambda+\mu)\left|\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}\right|^2\right)d\mathbf{x}dt\geq\frac{1}{C_2}\int_0^T E^{\mathbf{N}}(t)d\mathbf{x}dt=\frac{T}{C_2}E^{\mathbf{N}}(0).$$

We now estimate the terms in the right-hand side of (4.32). We start with $\left|X+\frac{d-1}{2}Y\right|$. This is a quantity evaluated at the times $t=0,T$. This quantity is easily estimated by the discrete energy (using the equivalence of the norm in (4.11)) which is conserved in time. In particular, we find

$$\left|X+\frac{d-1}{2}Y\right|\leq\frac{4\sqrt{d}}{\sqrt{\mu}}E^{\mathbf{N}}(0). \tag{4.34}$$

Concerning the second term in the right hand side of (4.32), we use the fact that on the boundary $\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}=\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}$, to obtain (see Lions, 1988a)

$$\int_0^T\int_\Gamma\left(\mu\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}+(\lambda+\mu)\boldsymbol{\nu}\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}\right)^2d\gamma dt\geq\mu\int_0^T\int_\Gamma\left(\mu\left|\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right|^2+(\lambda+\mu)\left|\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}\right|^2\right)d\gamma dt. \tag{4.35}$$

Finally, we turn to estimate the last term on the right-hand side in (4.32). It is enough to consider one of the $2d$ faces of the domain $\Omega=(-1,1)^d\subset\mathbb{R}^d$. We focus on $\Gamma_1=\{\mathbf{x}\in\overline{\Omega},\ \text{s.t. } x_1=1\}$. Let us denote $\mathbf{I}_{\Gamma_1}$ the set of indexes corresponding to the collocation nodes on the boundary Γ_1. For $\mathbf{i}\in\mathbf{I}_{\Gamma_1}$, the Lagrangian basis can be written as $\Psi_{\mathbf{i}}(\mathbf{x})=\Psi_N(x_1)\Psi'_{\mathbf{i}}(\mathbf{x}')$ where $\mathbf{x}'=(x_2,...,x_d)\in\mathbb{R}^{d-1}$ and $\Psi'_{\mathbf{i}}(\mathbf{x}')=\prod_{j=2}^d\Psi_{k_j}(x_j)$. Therefore, for $\mathbf{x}\in\boldsymbol{\Gamma_1}$, $\mathbf{x}=(1,\mathbf{x}')$, and we have

$$\begin{aligned}&\sum_{\mathbf{i}\in\mathbf{I}_{\Gamma_1}}\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)(t,\mathbf{P_i})\Psi_{\mathbf{i}}(\mathbf{x})\\&\quad=\Psi_N(x_1)\sum_{\mathbf{i}\in\mathbf{I}_{\Gamma_1}}\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)(t,\mathbf{P_i})\Psi'_{\mathbf{i}}(\mathbf{x}')\\&\quad=\Psi_N(x_1)\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)(t,1,\mathbf{x}').\end{aligned}$$

We now replace this in the last term on the right-hand side in (4.32). Using the Young's inequality, we can write

$$\begin{aligned}
&\left|\int_0^T\int_\Omega\left(\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)(t,1,\mathbf{x}')\Psi_N(x_1)\right)\cdot\left(m_j\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial x_j}\right)(\mathbf{x})d\mathbf{x}dt\right|\\
&\le\int_0^T C_\varepsilon\int_{-1}^1\int_{[-1,1]^{d-1}}\left|\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)(t,1,\mathbf{x}')\right|^2|\Psi_N(x_1)|^2\,dx_1d\mathbf{x}'dt\\
&+\int_0^T\varepsilon\int_\Omega\left|m_j\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial x_j}\right|^2d\mathbf{x}dt\le C_\varepsilon|\Psi_N|^2_{L^2(-1,1)}\int_0^T\int_{\Gamma_1}\left|\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right|^2d\gamma dt\\
&+\varepsilon(\sup_{\mathbf{x}\in\Gamma}m_j\nu_j)^2\int_0^T\int_\Omega|\nabla\boldsymbol{\phi}^{\mathbf{N}}|^2d\mathbf{x}dt.
\end{aligned}\tag{4.36}$$

Taking into account the norm equivalence in (4.11), $|\Psi_N|^2_{L^2(-1,1)}\le\|\Psi_N\|^2_N=\omega_N$, the conservation of the discrete energy proved above in (4.33), the fact that $sup_{\mathbf{x}\in\Gamma}\ m_j\nu_j=2\sqrt{d}$ and (4.36), we obtain

$$\begin{aligned}
&\left|\int_0^T\int_\Omega\left(\sum_{\mathbf{i}\in\mathbf{I}_{\Gamma_1}}\left(\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right)(t,\mathbf{P_i})\Psi_{\mathbf{i}}(\mathbf{x})\right)\cdot\left(m_j\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial x_j}\right)(\mathbf{x})d\mathbf{x}dt\right|\\
&\le C_\varepsilon\omega_N\int_0^T\int_{\Gamma_1}\left|\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right|^2d\gamma dt+\frac{8d}{\mu}\varepsilon TE^{\mathbf{N}}(0).
\end{aligned}\tag{4.37}$$

An analogous estimate holds for the other $2d-1$ terms of the boundary in the right-hand side of (4.32). It follows from (4.32)-(4.37) and the fact that $\omega_N=\omega_0=\dfrac{2}{N(N+1)}$ (see Canuto et al., 1988, Chapter 2)

$$\begin{aligned}
&\left(\frac{T}{C_2}-\frac{4\sqrt{d}}{\sqrt{\mu}}-\frac{16d^2\varepsilon T}{\mu}\right)E^{\mathbf{N}}(0)\le\frac{\sqrt{d}}{\mu}\int_0^T\int_\Gamma\left|\mu\frac{\partial\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}+(\lambda+\mu)\boldsymbol{\nu}\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}\right|^2d\gamma dt\\
&+\frac{4dC_\varepsilon}{N^2}\int_0^T\int_{\partial\Omega}\left|\mu\frac{\partial^2\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}^2}+(\lambda+\mu)\boldsymbol{\nu}\frac{\partial\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial\boldsymbol{\nu}}\right|^2d\gamma dt.
\end{aligned}\tag{4.38}$$

The fact that we can replace the discrete energy $E^{\mathbf{N}}(0)$, by the $(H_0^1)^d\times(L^2)^d$ norm of the initial data is a consequence of the equivalence of the discrete L^2-norm in (4.11) for polynomials of degree $\mathbf{N}$. Inequality (4.30) holds as long as $\dfrac{T}{C_2}-\dfrac{16d^2T\varepsilon}{\mu}-\dfrac{4\sqrt{d}}{\sqrt{\mu}}>0$. As ε can be chosen arbitrarily small and $C_2=(2+N^{-1})^d$, we have the condition $T>\dfrac{4\sqrt{d}(2+N^{-1})^d}{\sqrt{\mu}}$. □

4.5 Boundedness of discrete controls: proof of Theorem 4.3.2

In order to establish the bounded result as stated in Theorem 3.3.2, first we assume throughout this section that the theorem's hypothesis holds. We choose $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^2$ as in this Remark. Note that in particular this sequence is uniformly bounded in $(L^2)^d \times (H^{-1})^d$ as $\mathbf{N} \to \infty$.

Proof. As $\hat{\boldsymbol{\phi}}^{\mathbf{N}}$ is the solution of the discrete adjoint system (4.19) associated to the the minimizer $(\hat{\boldsymbol{\phi}}^{\mathbf{0},\mathbf{N}}, \hat{\boldsymbol{\phi}}^{\mathbf{1},\mathbf{N}})$ of $J^{\mathbf{N}}$ in $(\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d \times (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^d$, we have in particular

$$J^{\mathbf{N}}(\hat{\boldsymbol{\phi}}^{\mathbf{0},\mathbf{N}}, \hat{\boldsymbol{\phi}}^{\mathbf{1},\mathbf{N}}) \leq J^{\mathbf{N}}(\mathbf{0}, \mathbf{0}) = 0. \tag{4.39}$$

Then, taking into account the conservation of the discrete energy, defined in (4.33), the approximation in Remark 12, in particular for $(\mathbf{u}^{\mathbf{0},\mathbf{N}}, \mathbf{u}^{\mathbf{1},\mathbf{N}})$ and the uniform observability inequality in Lemma 4.4.3, we obtain

$$\begin{aligned} &\int_0^T \sum_{\mathbf{i}\in\mathbf{I}_\Gamma} \eta(t) \left| \mu \frac{\partial \boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\mathrm{div}\boldsymbol{\phi}^{\mathbf{N}} - \omega_{\mathbf{N}} \left(\mu \frac{\partial^2 \boldsymbol{\phi}^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu} \frac{\partial \mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} \right) \right|^2 (t, \mathbf{P}_{\mathbf{i}}) \omega_{\mathbf{i}} dt \\ &+ \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}} \eta(t)\omega_{\mathbf{N}} \left| \left(\mu \frac{\partial^2 \boldsymbol{\phi}^{\mathbf{N}}}{\partial^2 \boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu} \frac{\partial \mathrm{div}\boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} \right) \right|^2 (t, \mathbf{P}_{\mathbf{i}}) \omega_{\mathbf{i}} dt \leq M \left| (\mathbf{u}^{\mathbf{0}}, \mathbf{u}^{\mathbf{1}}) \right|^2_{(L^2)^d \times (H^{-1})^d}, \end{aligned}$$

which is equivalent to (4.16). □

Remark 14. *Comparing the control problems* (4.1) *and* (4.12)*, we see that the discrete version* (4.12) *has* $2d$ *extra controls* $\mathbf{g}_j^{\mathbf{N}}, j = 1, .., 2d$ *depending on* Γ_j*, that together can be interpreted as a single control in the whole boundary* $\mathbf{g}^{\mathbf{N}} \in (L^2(0,T;\mathbb{P}_{\mathbf{N}}(\partial\Omega)))^d$*. This "numerical" boundary control is associated with the last term in the inequality* (4.30) *and it is necessary to obtain a bounded sequence of discrete controls* $\mathbf{f}^{\mathbf{N}}$ *as* $\mathbf{N} \to \infty$*.*

In fact, the existence of discrete controls and their uniform bound (with respect to N*) in terms of the initial data is a direct consequence of the observability result established in Lemma 4.4.3. As we show above in Chapter 2 in the case of the wave equation we go further and prove that, under some technical hypotheses,* $f^{\mathbf{N}} \to f$ *in* $(L^2(0,T;\Gamma))$ *and the numerical* *control* $g^{\mathbf{N}}$ *vanishes as* $\mathbf{N} \to \infty$. *For the elasticity system such convergence result is more difficult to prove. In fact, if we try to apply the general theory established in Ervedoza and Zuazua, 2012 to recover the convergence of controls, some important hypotheses related to the convergence of the solutions of* (4.3) *must be checked.*

In particular, it should be established that for a convergent sequence of discrete initial data $(\boldsymbol{\phi}^{\mathbf{0},\boldsymbol{N}}, \boldsymbol{\phi}^{\mathbf{1},\boldsymbol{N}}) \to (\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}})$ *in* $(H_0^1(\Omega))^d \times (L^2(\Omega))^d$ *the right hand side in* (4.30) *converges to the right hand side in* (4.8). *More precisely, we have*

$$\begin{aligned}
&\mu\frac{\partial \boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\, div\boldsymbol{\phi}^{\mathbf{N}} \to \mu\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\nu}} + (\lambda+\mu)\boldsymbol{\nu}\, div\boldsymbol{\phi} \quad \text{in } L^2(0,T;\Gamma),\\
&\mu\frac{\partial^2 \boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}^2} + (\lambda+\mu)\boldsymbol{\nu}\frac{\partial\, div\boldsymbol{\phi}^{\mathbf{N}}}{\partial \boldsymbol{\nu}} \to \mathbf{0} \quad \text{in } L^2(0,T;\partial\Omega).
\end{aligned}$$

Such convergent results are not standard with the usual numerical analysis techniques and require further investigation.

4.6 Matrix formulation and numerical experiment

In this section, we consider some implementation issues related to the finite-dimensional approximation of boundary control given by (4.12)-(4.13) in a $2d$ dimensional and present a numerical experiment which gives numerical evidence of the convergence. There are several ways to do that, but we follow the more direct approach as in the seconde chapter (section 2.6) which consists of writing the collocation control problem into the standard form

$$\begin{cases}
\mathbf{U^N}_{tt} = A^{\mathbf{N}}\mathbf{U^N} + B^{\mathbf{N}}\mathbf{F^N},\\
\mathbf{U^N}(0) = \mathbf{U^{0,N}}, \mathbf{U}_t^{\mathbf{N}}(0) = \mathbf{U^{1,N}}.
\end{cases} \tag{4.40}$$

Here, we have taken $\mathbf{U^N}(t) = (u_1^{\mathbf{N}}, u_2^{\mathbf{N}})$ is a $2(N-1)^2$-vector containing the values of the solution at the interior collocation points as follows:

$$\mathbf{U^N} = (u_{122}^{\mathbf{N}}, u_{123}^{\mathbf{N}}, ..., u_{12N}^{\mathbf{N}}, ..., u_{1N2}^{\mathbf{N}}, u_{1N3}^{\mathbf{N}}, ..., u_{1NN}^{N}, u_{222}^{N}, u_{223}^{\mathbf{N}}, ..., u_{2N2}^{\mathbf{N}}, u_{2N3}^{\mathbf{N}}, ..., u_{2NN}^{\mathbf{N}})^T,$$

$\mathbf{F^N}(t) \in \mathcal{M}_{p\times 1}$ is the vector of controls and $A^{\mathbf{N}} \in \mathcal{M}_{(2(N-1)^2)^2}$, $B^{\mathbf{N}} \in \mathcal{M}_{2(N-1)^2\times p}$ are suitable matrixes that we define below. Once written in this form, the control can be computed explicitly by an integral expression in time (Zuazua, 2014) that we approximate by quadrature as similar to the 1-d case.

The matrix $A^{\mathbf{N}}$ in (4.40) is computed from the well-known Legendre differentiation matrix $K^N \in \mathcal{M}_{(N+1)^2}$ as in (2.16) that relates a polynomial of degree N with its derivative at the LGL nodes (see Canuto et al., 1988). In fact, we have

$$A^{\mathbf{N}} = \begin{pmatrix} (\lambda+2\mu)\,\mathrm{kron}(D^2, I) + \mu\,\mathrm{kron}(I, D^2) & (\lambda+\mu)\,\mathrm{kron}(D, I)\,\mathrm{kron}(I, D) \\ (\lambda+\mu)\,\mathrm{kron}(D, I)\mathrm{kron}(I, D) & \mu\,\mathrm{kron}(D^2, I) + (\lambda+2\mu)\,\mathrm{kron}(I, D^2) \end{pmatrix},$$

where $D = K(2:N, 2:N)$, $D^2 = (K^N)^2(2:N, 2:N)$ (in Matlab notation), i.e. K^N and the square of K^N stripped of its first and last rows and columns corresponding to the boundary nodes, where the solution is given either by the boundary condition or the control. The vector $\mathbf{F}^{\mathbf{N}}(t)$ contains 5 components corresponding to the 5 controls in (4.12), i.e.

$$\mathbf{F}^{\mathbf{N}} = (\mathbf{f}^{\mathbf{N}}, \mathbf{g}_{\mathbf{1}}^{\mathbf{N}}, \mathbf{g}_{\mathbf{3}}^{\mathbf{N}}, \mathbf{g}_{\mathbf{2}}^{\mathbf{N}}, \mathbf{g}_{\mathbf{4}}^{\mathbf{N}})^T,$$

where $\mathbf{f}^{\mathbf{N}}$ is a column vector with the coefficients of the controls

$$\mathbf{f}^{\mathbf{N}} = (f_{12}^{R,\mathbf{N}}, ..., f_{1N}^{R,\mathbf{N}}, f_{22}^{R,\mathbf{N}}, ..., f_{2N}^{R,\mathbf{N}}, f_{12}^{T,\mathbf{N}}, ..., f_{1N}^{T,\mathbf{N}}, f_{22}^{T,\mathbf{N}}, ..., f_{2N}^{T,\mathbf{N}})^T \in \mathcal{M}_{4(N-1)\times 1}.$$

Something similar can be said about $\mathbf{g}_j^{\mathbf{N}}, j = 1, .., 4$.

Finally, the matrix $B^N \in \mathcal{M}_{2(N-1)^2\times 12(N-1)}$ corresponds to the contribution of the boundary control accounting for some rows and columns that affect the values of solution in the boundary and the artificial controls required to obtain the uniform controllability of the discrete system. They are given by the right-hand sides of equilibrium equation (4.12).

The rest is similar to Section 2.6 chapter 2. Here we present numerical evidences of the convergence of the discrete control to the continuous one. For the time discretization, we use a classical Newmark method with parameters $\gamma = 1/2$, $\beta = 1/4$ and time step $dt = 10^{-2}$. With this choice, the time scheme is second order accurate (see Raviart and Thomas, 1983). We consider the following initial conditions, $\mathbf{u}^{\mathbf{0}}(x_1, x_2) = (0.2\sin(\pi(x_1+1)/2)\sin(\pi(x_2+1)/2), 0.2\sin(\pi(x_1+1)/2)\sin(\pi(x_2+1)/2))$ and $\mathbf{u}^{\mathbf{1}}(x_1, x_2) = (0, 0)$. We take $\lambda = 0.5$, $\mu = 4$ and final time $T = 3$.

Note that the time control is only slightly greater than the minimal control time for the continuous elasticity system $T > \dfrac{4\sqrt{2}}{\sqrt{\mu}}$ (see Lions, 1988a) and lower than the time given by the uniform discrete observability inequality in Theorem 4.3.1 and Lemma 4.4.3 (which is probably not optimal).

$\mathbf{N}$	$\left\lvert \mathbf{f^N} \right\rvert_{(L^2(0,T;\Gamma))^2}$	$\left\lvert \mathbf{g^N} \right\rvert_{(L^2(0,T;\partial\Omega))^2}$
$(10,10)$	1.6×10^{-1}	4.8×10^{-3}
$(20,20)$	2.2×10^{-1}	2.5×10^{-3}
$(40,40)$	2.5×10^{-1}	8.2×10^{-4}

Table 4.1: Norm of the discrete controls.

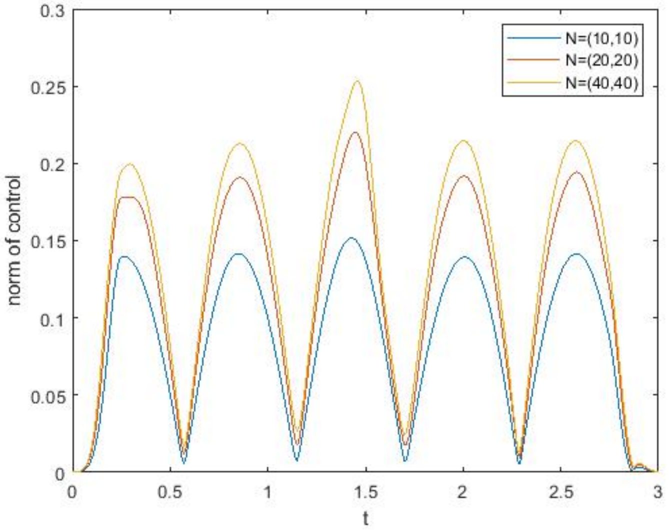


Figure 4.1: Time behavior of $|\mathbf{f^N}|_{(L^2(\Gamma))^2}$.

In Figure 4.1 we show the behaviour of the norm of control $\mathbf{f^N}$ in time for different values of $\mathbf{N}$.

Table 4.1 illustrates the behavior of the norm for the controls when the degree of the polynomials $\mathbf{N}$ grows. We observe that the boundary control $\mathbf{f^N}$ remains bounded and should converge to a continuous control $\mathbf{f}$, while the numerical artificial control $\mathbf{g^N}$ vanishes as $\mathbf{N} \to \infty$.

We can also illustrate the rate of convergence of the discrete control to the limit one with the initial data given above. We compare the L^2-norm of the difference between the discrete control when $\mathbf{N} = (40,40)$ (that we take as continuous control) and the discrete control as N

grows.

In Table 4.2 we show the error between the discrete control and the limit one. Comparing the error associated to values of $\mathbf{N} = (N, N) \in [10, 30] \times [10, 30]$, we can give a rough estimate of the rate of convergence. More precisely, we observe that $\left|\mathbf{f}^{\mathbf{N}} - \mathbf{f}\right|_{(L^2(0,T;\Gamma))^2} \sim N^{-\alpha}$, where α is estimated by the slope of the graphics relating $-\log_{10}\left|\mathbf{f}^{\mathbf{N}} - \mathbf{f}\right|_{(L^2(0,T;\Gamma))^2}$ with $\log_{10} N$ (see Figure 4.2).

Figure 4.3 shows the animation of the state of the norm of the solution $\mathbf{u}(t, x_1, x_2)$ at different times and in the form of a deformed mesh for $\mathbf{N} = (40, 40)$.

N	$\log(\left\|\mathbf{f}^{\mathbf{N}} - \mathbf{f}^{\mathbf{40}}\right\|_{(L^2(0,T;\Gamma))^2})$	$\log(\left\|\mathbf{g}^{\mathbf{N}}\right\|_{(L^2(0,T;\partial\Omega))^2})$
(10,10)	-1.2	-5.3
(20,20)	-1.5	-5.6
(30,30)	-1.7	-7.1

Table 4.2: Convergence of the discrete control to the limit as **N** grows.

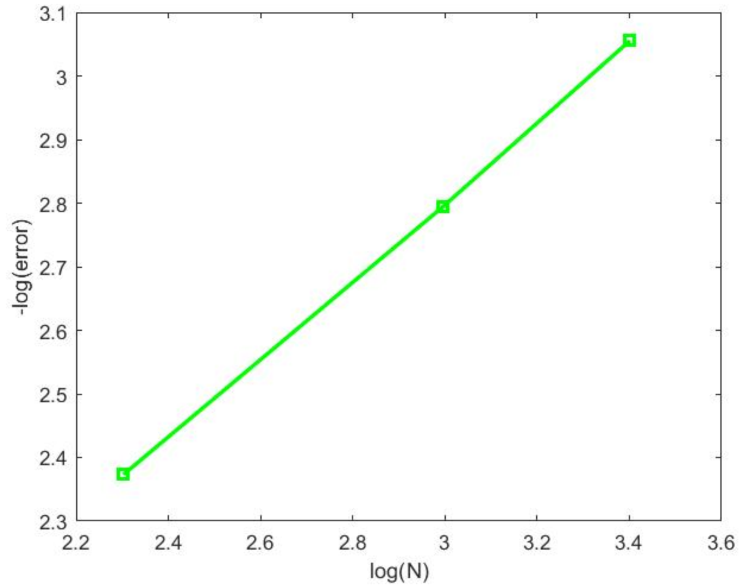


Figure 4.2: The slope of the graphic gives an experimental estimate of the value α for which $\left|\mathbf{f}^{\mathbf{N}} - \mathbf{f}^{\mathbf{40}}\right|_{(L^2(0,T;\Gamma))^2} \sim N^{-\alpha}$. We observe that α is increase as **N** grows for smoother data, as expected in spectral methods.

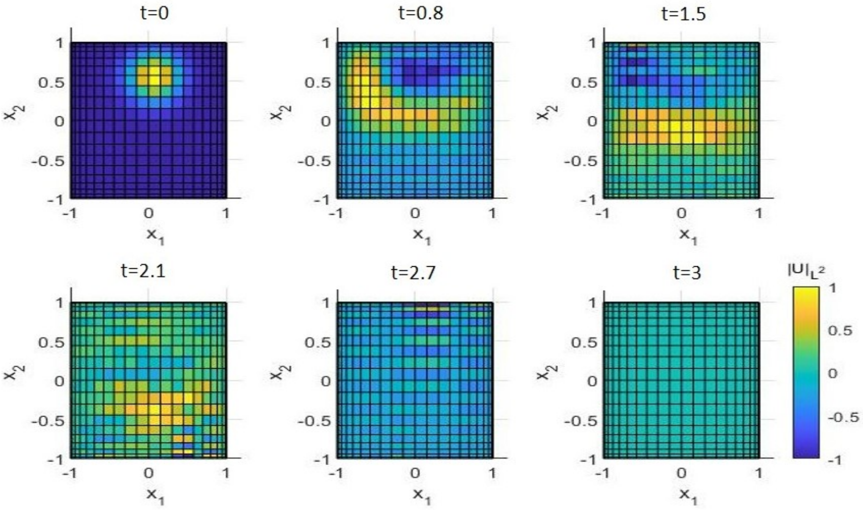


Figure 4.3: Behavior of norm of solution.

Chapter 5

Spectral approximation of inverse source problems from observability

In this chapter we consider inverse source problems for evolution problems whose stability can be derived with a boundary observability inequality. This is the case when we are interested in recovering the nonhomogeneous term in the wave equation from boundary data of a single solution for sufficiently large time, for example. Based on a discretization adapted to spectral methods, we derive a finite dimensional minimization problem whose minimizers converge to the continuous source term, as the discretization parameter N goes to infinity. This provides an efficient algorithm to approximate numerically this type of inverse problems.

As an application, we show how to reconstruct a source term in the wave equation with two different spectral methods: the polynomial collocation method and a projection method.

5.1 Problem statement

Let X, Y be Hilbert space and $\mathcal{D}(A) \subset X$. Let $A : \mathcal{D}(A) \to X$ be a generator of a strongly continuous group in X and let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an observation operator of the following problem

$$\begin{cases} z_t(t) = Az(t) + \lambda(t)f, & t \in (0,T), \\ z(0) = z_0, & \\ y(t) = Cz(t), & t \in (0,T), \end{cases} \tag{5.1}$$

where $T > 0$, $z_0 \in \mathcal{D}(A)$ and $\lambda \in H^1(0,T)$ with $\lambda(0) \neq 0$ are known.

We are interested in the following inverse source problem (ISP): Given z_0 and λ, find $f \in X$ from the observation $y(t)$ in $t \in (0,T)$ for a sufficiently large time T.

As usual in inverse problems, three important aspects are of interest:

1. **Uniqueness**, i.e. the observation $y(t)$, for $t \in (0,T)$, characterizes the source term f.
2. **Stability**, i.e. two close observations cannot be associated with arbitrarily large source terms.
3. **Reconstruction**, i.e. to describe numerical algorithms to recover the source f from the observation.

The key ingredient to establish the uniqueness an stability of this inverse problem is the following property:

Definition 5.1.1. *The pair (A, C) is exactly observable in time T if there exists a positive constant k_T such that,*

$$\int_0^T \|Cx(t)\|_Y^2 dt \geq k_T \|x_0\|_X^2, \quad x_0 \in X, \tag{5.2}$$

where $x(t)$ is the solution of the homogeneous problem

$$\begin{cases} x_t(t) = Ax(t), & t \in (0,T), \\ x(0) = x_0. & \end{cases} \tag{5.3}$$

In fact, the following result holds:

Theorem 5.1.1 (Alves et al., 2009)**.** *Let X, Y be Hilbert spaces and assume that the pair (A, C) is exactly observable in some time $T_0 > 0$. For every $z_0 \in \mathcal{D}(A)$, $T \geq T_0$, $\lambda \in H^1(0, T)$, $\lambda(0) \neq 0$, the map $f \mapsto y$ is one-to-one from X to $H^1((0, T); Y)$ and there exists a positive constant c_T such that*

$$\|f^1 - f^2\|_X \leq c_T \|y^1 - y^2\|_{H^1((0,T);Y)}, \ \forall f^1, f^2 \in X, \tag{5.4}$$

where y^1,y^2 are the observation corresponding to the sources f^1,f^2 in (5.1)*.*

Here we are interested in the practical reconstruction of the ISP introduced above. To this end we consider the following approach:

1. Introduce a finite-dimensional approximation of the inverse problem (5.1) coming from a spectral method, depending on a discretization parameter $N \to \infty$.
2. Prove the stability of the discrete system for each N.
3. Introduce a discrete algorithm for the reconstruction of the discrete source term.
4. Prove the convergence of the discrete source term to the continuous one.

One of the main ingredients to carry out this program is a discrete version of the inequality (5.2) for the associated discrete operators that guarantees the stability of the discrete ISP. But this is not enough to establish the convergence of the discrete source terms to the continuous one. Uniformity of the constant k_T, with respect to N, in (5.2) will be also required.

As an example of application, we consider the wave equation. For the continuous wave equation, the stability of the inverse source problem has been studied by several authors (see for example Puel and Yamamoto, 1995, Komornik and Yamamoto, 2002, Komornik and Yamamoto, 2005, Nicaise and Zaïr, 2003, Nicaise and Zair, 2004).

The rest of the chapter is divided into four more sections. In the second section, we give the main results. In the third and fourth sections, we prove the stability and convergence results respectively. In the fifth section, we show how to apply this result to obtain convergent approximations of the discrete source for the wave system with two different numerical approaches. Finally, we state the matrix formulation of the discrete inverse source problem that we approximate using the finite-dimensional ISP and present some numerical experiments and numerical results.

5.2 Main results

In this section, we focus on the practical reconstruction of f from the observation $y(t)$ in $t \in (0,T)$ when (5.2) holds. To do that, we introduce a finite-dimensional approximation of the inverse problem (5.1) coming from a spectral method.

Let us consider a family of finite dimensional sub-spaces $X_N \subset X$ and $Y_N \subset Y$, indexed by the parameter $N = \mathrm{dim} X_N \to \infty$, with the norms $\|\cdot\|_{X_N}$ (resp $\|\cdot\|_{Y_N}$) equivalent to the norm of X (resp. Y) with constants independent of N.

We approximate the continuous model (5.1) by the sequence of finite-dimensional systems,

$$\begin{cases} z_t^N(t) = A^N z^N(t) + \lambda(t) f^N, & t \in (0,T), \\ z^N(0) = z_0^N, \\ y^N(t) = C^N z^N(t), & t \in (0,T), \end{cases} \tag{5.5}$$

where $A^N : X_N \to X_N$ and $C^N : X_N \to Y_N$ are two sequences of finite-dimensional linear operators approximating A and C respectively, in a sense that will be made explicit below.

Now we can state the discrete inverse source problem (ISP)^N: Given z_0^N and λ, find $f^N \in X^N$ from the observation $y^N(t)$ in $t \in (0,T)$ for a sufficiently large time T.

Here again the key ingredient to establish the uniqueness and stability for the (ISP)^N is the exact observability of (A^N, C^N) in the sense of Definition 5.1.1. However, in this case a uniformity with respect to N will play an important roll. We make precise this notion in the following definition.

Definition 5.2.1. *The sequence of pairs* $\{(A^N, C^N)\}_N$ *are said to be* **uniformly exactly observable** *in time* T *if there exists a positive constant* k_T*, independent of* N*, such that*

$$\int_0^T \|C^N x^N(t)\|_{Y^N}^2 dt \geq k_T \|x_0^N\|_{X^N}^2, \quad x_0^N \in X^N, \tag{5.6}$$

where $x^N(t)$ *is the solution of the homogeneous problem*

$$\begin{cases} x_t^N(t) = A^N x^N(t), & t \in (0,T), \\ x^N(0) = x_0^N. \end{cases} \tag{5.7}$$

The first result in this chapter is the uniform stability of the $(\mathrm{ISP})^N$:

Theorem 5.2.1. *Assume that the family* $\{(A^N, C^N)\}_N$ *is uniformly exactly observable in time* $T_0 > 0$ *(see Definition 5.2.1). Let* $T \geq T_0$, $\lambda \in H^1((0,T))$, $\lambda(0) \neq 0$ *and* $z_0^N \in X_N$*. The map* $f^N \mapsto y^N$ *is one-to-one from* X^N *to* $H^1((0,T);Y^N)$ *and there exists a positive constant* c_T *such that*

$$\|f^{1,\mathbf{N}} - f^{2,\mathbf{N}}\|_X \leq c_T \|y^{1,\mathbf{N}} - y^{2,\mathbf{N}}\|_{H^1((0,T);Y)}, \ \forall f^{1,\mathbf{N}}, f^{2,\mathbf{N}} \in X_{\mathbf{N}}, \tag{5.8}$$

where $y^{1,\mathbf{N}}$,$y^{2,\mathbf{N}}$ *are observation corresponding to the sources* $f^{1,\mathbf{N}}$,$f^{2,\mathbf{N}}$ *of* (5.5)*.*

This is basically a finite dimensional version of Theorem 5.1.1, stated in Alves et al., 2009. The proof is the same and we only have to check that we can choose the constant c_T uniform with respect to N, as long as the sequence of pairs $\{(A^N, C^N)\}_N$ are uniformly exactly observable. We give a proof in the next section for completeness.

To relate the (ISP) with the discrete $(\mathrm{ISP})^N$ a consistency hypotheses is required. We consider the following:

(H) There exist two projection operators $P_N : X \to X_N$ and $Q_N : Y \to Y_N$ such that for all $x_0 \in X$ and $y \in Y$, we have

$$P_N x_0 \to x_0 \text{ in } \mathcal{D}(A), \quad \text{as } N \to \infty \tag{5.9}$$

and

$$Q_N y \to y \text{ in } Y, \quad \text{as } N \to \infty. \tag{5.10}$$

Moreover, if $x_0 \in D(A)$ and x, x^N are the solutions of (5.3) and (5.7) associated to the initial data x_0 and $x_0^N = P_{\mathbf{N}} x_0$, respectively, then

$$x^N \to x \text{ in } C^0([0,T];\mathcal{D}(A)) \cap C^1((0,T);X), \quad \text{as } N \to \infty, \tag{5.11}$$

$$C^N(x^N) \to Cx \text{ in } L^2((0,T);Y), \quad \text{as } N \to \infty. \tag{5.12}$$

We now focus on the reconstruction. Assume that we want to find approximations of a continuous source $f^* \in X$ from its continuous observation $y^*(t)$ for $t \in (0,T)$, obtained with initial data $z_0 \in X$ (to simplify we choose $z_0 = 0$) in (5.1). We introduce a discrete cost function $J^N : X_N \longrightarrow \mathbb{R}$ by

$$J^N(f^N) = \frac{1}{2} \left\| Q_N y^*(t) - y^N(t) \right\|^2_{H^1((0,T);Y_{\mathbf{N}})}, \tag{5.13}$$

where $y^N \in H^1((0,T);Y_N)$ is the observations corresponding the source term $f^N \in X_N$, with initial data $z_0^N = 0$ and $Q_{\mathbf{N}} y^* \in H^1((0,T);Y_N)$ is the projection of $y^* \in H^1((0,T);Y)$, introduced in the consistency hypotheses (H) above (see (5.10)). As we show in the following result, the minimization of J^N provides an approximation of f^*.

Let us introduce some notation before. Consider the operator $\Psi_T^N \in \mathcal{L}(X_{\mathbf{N}}, L^2((0,T);Y_N))$ for each $T > 0$ by

$$(\Psi_T^N x_0^N)(t) = C^N x^N(t) \text{ for } t \in (0,T), \tag{5.14}$$

where $x^N(t)$ is the solution of the homogeneous system (5.7) with initial data $x_0^N \in X_N$.

Analogously we define $\Psi_T \in \mathcal{L}(X, L^2((0,T);Y))$ for each $T > 0$ by

$$(\Psi_T x_0)(t) = Cx(t) \text{ for } t \in (0,T), \tag{5.15}$$

where $x(t)$, the solution of the homogeneous system (5.3) with initial data $x_0 \in X$.

With this notation, condition (5.12) in hypotheses (H) is equivalent to

$$\Psi_T^N(P_N x_0) \to \Psi_T x_0 \text{ in } L^2(0,T;Y). \tag{5.16}$$

Theorem 5.2.2. *Assume that the consistency hypotheses (H) in* (5.10)-(5.12) *holds and that the family* $\{(A^N, C^N)\}_N$ *is uniformly exactly observable in time* $T_0 > 0$ *(see Definition 5.2.1). Let* $f^* \in X$ *be a source term of problem* (5.1) *with observation* $y^* \in H^1((0,T);Y)$ *when* $z_0 = 0$*. For any* $T \geq T_0$*, the functional* J^N *in* (5.13) *has a unique minimizer* $f^N_{min} \in X_N$*. Moreover, there exists a constant* $C_T > 0$ *such that*

$$\begin{aligned} \|f^N_{min} - f^*\|_X \leq C_T \Big(&\|\Psi_T f^* - \Psi_T^N P_N f^*\|_{L^2((0,T);Y)} \\ &+ \|Q_N y^* - y^*\|_{H^1_L((0,T);Y)} + \|P_N f^* - f^*\|_X \Big). \end{aligned} \tag{5.17}$$

In particular,

$$f^N_{min} \to f^* \text{ in } X, \text{ as } N \to \infty. \tag{5.18}$$

This result motivates a natural algorithm to approximate the source term f^* for the continuous observation y^*, by solving the discrete minimization problem (5.13). We show in the experiments below the efficiency of this algorithm in a system governed by the wave equation.

5.3 Stability: proof of Theorem 5.2.1

We follow closely the proof of Theorem 5.1.1 for the infinite dimensional setting in Alves et al., 2009.

We start recalling the following proposition in Alves et al., 2009.

Proposition 5.3.1. *Let $T > 0$, let Y be a Hilbert space and let $\lambda \in H^1((0,T))$ with $\lambda(0) \neq 0$. Let $S : L^2((0,T);Y) \to H^1((0,T);Y)$ be defined by*

$$(Sg)(t) = \int_0^t \lambda(t-s)g(s)ds. \tag{5.19}$$

Then S is an isomorphism from $L^2((0,T);Y)$ onto $H^1((0,T);Y)$.

By linearity it is enough to prove that there exists a positive constant c_T such that

$$\|f^N\|_X \leq c_T \|y^N\|_{H^1_L((0,T);Y)}, \ \forall f^N \in X_{\mathbf{N}}, \tag{5.20}$$

where y^N is the observation associated to the source term f^N with zero initial data in (5.5).

By Duhamel formula and (5.14), the observation y^N can be expressed as follows

$$y^N(t) = \int_0^t \lambda(t-s)\Psi^N_T f^N(s)ds = [(S \circ \Psi^N_T)f^N](t), \tag{5.21}$$

where S is defined in (5.19). Since $\lambda(0) \neq 0$, S is an isomorphism from $L^2((0,T);Y_N)$ onto $H^1((0,T);Y_N)$ and $\Psi^N_T \in \mathcal{L}(X_{\mathbf{N}}, L^2((0,T);Y_N))$. Therefore, using the equivalence of the norm in Y_N there exist $c, M > 0$ such that

$$\|y^N\|_{H^1_L((0,T;Y)} \geq c\|y^N\|_{H^1_L((0,T;Y_N)} \geq cM\|\Psi_T^N f^N\|_{L^2((0,T);Y_N)}. \tag{5.22}$$

From (5.14), the equivalence of the norm in X_N and the uniform exact observability of (A^N, C^N), we deduce that there exists $c' > 0$ such that

$$\|\Psi_T^N f^N\|_{L^2((0,T);Y_N)} \geq k_T\|f^N\|_{X_N} \geq k_T c'\|f^N\|_X. \tag{5.23}$$

Combining the two above inequalities we deduce (5.20).

5.4 Convergence: proof Theorem 5.2.2

Note that J^N is continuous, convex and coercive, due to the stability result in Theorem 5.2.1. Therefore, there exists a minimizer $f^N_{min} \in X_N$. It remains to prove (5.18).

We denote by $\overline{y^N}$ and y^N_{min} the observations associated to the sources $P_N f^*$ and f^N_{min} respectively. We have

$$\|f^N_{min} - f^*\|_X \leq \|f^N_{min} - P_N f^*\|_X + \|P_N f^* - f^*\|_X. \tag{5.24}$$

The second term in the right hand side of (5.24) is already in (5.17). Then we only have to estimate the first one. By (5.13), the stability results in (5.20) and the equivalence of the norm of Y and Y_N (with constants $c_1,\ c_2$), we get

$$\begin{aligned}\frac{1}{c_T^2}\|f^N_{min} - P_N f^*\|^2_X &\leq \|y^N_{min} - \overline{y^N}\|^2_{H^1_L((0,T);Y)} \leq c_1^2\|y^N_{min} - \overline{y^N}\|^2_{H^1_L((0,T);Y_N)} \\ &\leq 2c_1^2\|Q_N y^* - y^N_{min}\|^2_{H^1_L((0,T);Y_N)} + 2c_1^2\|Q_N y^* - \overline{y^N}\|^2_{H^1((0,T);Y_N)} \\ &= 4c_1^2 J^N(f^N_{min}) + 4c_1^2 J^N(P_N f^*) \leq 8c_1^2 J^N(P_N f^*) \\ &= 4c_1^2\|Q_N y^* - \overline{y^N}\|^2_{H^1((0,T);Y_N)} \leq 4c_1^2 c_2^2\|Q_N y^* - \overline{y^N}\|^2_{H^1((0,T);Y)}.\end{aligned} \tag{5.25}$$

Here we have used the fact that $\tilde{f^N}$ is the minimizer of J^N. On the other hand, we have

$$\|Q_N y^* - \overline{y^N}\|_{H^1((0,T);Y)} \leq \|Q_N y^* - y^*\|_{H^1((0,T);Y)} + \|y^* - \overline{y^N}\|_{H^1((0,T);Y)} \tag{5.26}$$

and we only have to estimate the second term in (5.26), since the first one is in (5.17).

By using Duhamel formula for the solutions associated to the source terms f^*, $P_N f^*$, respectively, we get

$$\left\| y^* - \overline{y^N} \right\|_{H^1((0,T);Y)} = \left\| \int_0^t \lambda(t-s)(\Psi_T f^* - \Psi_T^N P_N f^*)(s) ds \right\|_{H^1((0,T);Y)}$$
$$= \left\| S \circ (\Psi_T f^* - \Psi_T^N P_N f^*) \right\|_{H^1((0,T);Y)} .$$

Since $Y_N \subset Y$ and S is an isomorphism from $L^2((0,T);Y)$ onto $H^1((0,T);Y)$, there exist $M > 0$ such that

$$\| y^* - \overline{y^N} \|_{H^1((0,T);Y)} \leq M \| \Psi_T f^* - \Psi_T^N P_N f^* \|_{L^2((0,T);Y)}. \tag{5.27}$$

Combining (5.24)-(5.27) we easily deduce (5.17).

5.5 Application for the wave equation

In this section we illustrate the results with numerical experiments. We consider a simple straightforward example of a wave equation within a square domain with observation in two adjacent sides. To clarify the exposition we divide this section in three more subsections where we show how to set the problem in the abstract setting described above and two different numerical approaches: the first one based on a projection method of the ISP onto the finite-dimensional subspace formed by the first eigenfunctions of the Laplacian, while the second one uses the spectral collocation method described in the previous chapters. We state and prove uniqueness, stability and convergence results for both approaches.

5.5.1 Continuous ISP for the wave equation

Let $\Omega = (-1,1)^2$ with boundary $\partial\Omega$ and consider the system,

$$\begin{cases} \theta_{tt}(t,x) - \Delta\theta(t,x) = \lambda(t) g(x) & \text{in } Q = (0,T) \times \Omega, \\ \theta(x,t) = 0 & \text{on } (0,T) \times \partial\Omega, \\ (\theta(0,\mathbf{x}), \theta_t(0,\mathbf{x})) = (0,0) & \text{in } \Omega, \end{cases} \tag{5.28}$$

where $(\theta^0, \theta^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $\lambda \in H^1(0,T)$ with $\lambda(0) \neq 0$ and $f \in L^2(\Omega)$.

Let Γ be an open nonempty subset of $\partial\Omega$ and $T > 0$. We are interested in the following ISP: Find $g \in L^2(\Omega)$ from $\partial_n \theta(t,x)$ on $(0,T) \times \Gamma$.

To state this ISP in the abstract framework introduced in the previous section we define,

$$X = L^2(\Omega) \times H^{-1}(\Omega), \quad \mathcal{D}(A) = H_0^1(\Omega) \times L^2(\Omega),\ Y = L^2(\Gamma), \tag{5.29}$$

$$A \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \Delta\phi \end{pmatrix},\ C \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \frac{\partial \phi}{\partial \nu}|_\Gamma,\ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{D}(A), \quad f = \begin{pmatrix} 0 \\ g \end{pmatrix}. \tag{5.30}$$

Therefore the ISP in (5.28) can be written in the general form (5.5).

It is well known that (A, C) is exactly observable when Γ consists in two adjacent sides of the square and $T > T_0 = 4\sqrt{2}$ (see Lions, 1988a). Consequently, we can apply Theorem 5.1.1 and the stability of the inverse source problem holds.

Remark 15. *Note that, if we consider* $Y = L^2(\Gamma) \times L^2(\partial\Omega)$ *and* $C \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi}{\partial \nu}|_\Gamma \\ 0|_{\partial\Omega} \end{pmatrix}$*, we can define a map* $g \mapsto y = (\frac{\partial \theta}{\partial \nu}|_\Gamma, 0|_{\partial\Omega})^T$ *is one to one from* $L^2(\Omega)$ *to* $H^1((0,T), L^2(\Gamma) \times L^2(\partial\Omega))$ *and get the same stability results. This version will be useful when considering the spectral collocation approximation below.*

5.5.2 Projection method

Let us consider $\{\omega_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$ constituted by eigenfunctions of the Laplace operator with Dirichlet boundary conditions, $-\Delta_D$, and $\{\lambda_k\}_{k\in\mathbb{N}}$ be corresponding eigenvalues. We assume that they are ordered increasingly, i.e. $0 < \lambda_1^2 < \lambda_2^2 \leq \lambda_3^2 < \ldots$. We define the scaled spaces $H^\alpha(\Omega)$ as

$$H^\alpha = \left\{ \sum_{k=1}^{\infty} c_k \omega_k(\mathbf{x}),\ \text{with} \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |c_k|^2 < \infty \right\}.$$

Note that $H^0 = L^2(\Omega)$ and $H^1 = H_0^1(\Omega)$. We also introduce the finite dimensional space X^N generated by the first N eigenfunctions, i.e.

$$S^N = \left\{ \sum_{k=1}^{N} c_k \omega_k(\mathbf{x}), \text{ with } c_k \in \mathbb{R} \right\} \subset H^\alpha, \ \alpha = 0, 1$$

and the set S_Γ^N is the restriction of S^N to Γ, the projection operator

$$P^N : L^2(\Omega) \longrightarrow S^N,$$

defined by

$$P^N u = \sum_{k=1}^{N} u_k \omega_k(\mathbf{x}) \quad \text{for } u = \sum_{k=1}^{\infty} u_k \omega_k(\mathbf{x}).$$

Clearly, this operator can be extended to the analogous projection in H_0^1 that we still denote by P^N. Let us set

$$X^N = S^N \times S^N, \ Y^N = S_\Gamma^N. \tag{5.31}$$

Now, we apply the projection operator to system (5.28). Taking into account that the Laplace operator conmmutes with P^N and leave invariant the subspace S^N we easily obtain the finite dimensional system

$$\begin{cases} \theta_{tt}^N - D^N \theta^N = \lambda(t) g^N & \text{in } Q, \\ \theta^N \in S^N & \text{in } (0, T), \\ (\theta^N(0, \mathbf{x}), \theta_t^N(0, \mathbf{x})) = (0, 0) & \text{in } \Omega, \end{cases} \tag{5.32}$$

where $D^N = P^N \Delta : S^N \to S^N$. Let us also consider

$$A^N \begin{pmatrix} \phi^N \\ \psi^N \end{pmatrix} = \begin{pmatrix} \psi^N \\ \Delta \phi^N \end{pmatrix}, \ C^N \begin{pmatrix} \phi^N \\ \psi^N \end{pmatrix} = \frac{\partial \phi^N}{\partial \nu} |_\Gamma, \ \begin{pmatrix} \phi^N \\ \psi^N \end{pmatrix} \in X^N. \tag{5.33}$$

It is easy to check that the pair (A^N, C^N) is uniformly exactly observable when (A, C) is exactly observable and therefore we can apply Theorem 5.2.1. On the other hand, hypotheses (H) in (5.10)-(5.12) is also verified (see Appendix C) and therefore the result of Theorem 5.2.2.

5.5.3 Spectral collocation method

Here we use the same notation introduced in Chapter 3, Section 4.3 in a particular case when $\mathbf{N} = (N, N) \in \mathbb{N}^2$ be the number of collocation points in each variable x_j that we assume to be the same. Let us set

$$X^{\mathbf{N}} = \mathbb{P}^{Di}_{\mathbf{N}}(\Omega) \times \mathbb{P}^{Di}_{\mathbf{N}}(\Omega),\ Y^{\mathbf{N}} = \mathbb{P}_{\mathbf{N}}(\Gamma) \times \mathbb{P}_{\mathbf{N}}(\partial\Omega), \tag{5.34}$$

and consider the following discrete inverse problem obtained from spectral collocation (see chapter 3):

$$\begin{cases} (\theta^{\mathbf{N}}_{tt} - \Delta\theta^{\mathbf{N}})(t, \mathbf{x_i}) = \lambda(t) g^{\mathbf{N}}(\mathbf{x_i}) & \text{in } (t, \mathbf{x_i}) \in (0, T) \times C^{\Omega}, \\ \theta^{\mathbf{N}}(t, \mathbf{x_i}) = 0 & \text{in } (t, \mathbf{x_i}) \in (0, T) \times C^{\partial\Omega}, \\ (\theta^{\mathbf{N}}(0, \mathbf{x_i}), \theta^{\mathbf{N}}_t(0, \mathbf{x_i})) = (\theta^{0,\mathbf{N}}, \theta^{1,\mathbf{N}}) & \text{in } \mathbf{x_i} \in\ C^{\Omega}, \end{cases} \tag{5.35}$$

where $(\theta^{0,\mathbf{N}}, \theta^{1,\mathbf{N}}) \in X^{\mathbf{N}}$. More precisely, we observe that the solutions of (5.35) solve the following equivalent continuous system:

$$\begin{cases} \theta^{\mathbf{N}}_{tt} - \Delta\theta^{\mathbf{N}} = -\sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}} \dfrac{\partial^2\theta^{\mathbf{N}}}{\partial^2\nu}(t, \mathbf{x_i})\Psi_{\mathbf{i}} + \lambda(t) g^{\mathbf{N}} & \text{in } (0, T) \times \Omega, \\ \theta^{\mathbf{N}} = 0 & \text{on } (0, T) \times \partial\Omega, \\ (\theta^{\mathbf{N}}(0, \mathbf{x}), \theta^{\mathbf{N}}_t(0, \mathbf{x})) = (\theta^{0,\mathbf{N}}, \theta^{1,\mathbf{N}}) & \text{in } \Omega. \end{cases} \tag{5.36}$$

On the other hand, let us also consider

$$A^{\mathbf{N}} \begin{pmatrix} \phi^{\mathbf{N}} \\ \psi^{\mathbf{N}} \end{pmatrix} = \begin{pmatrix} \psi^{\mathbf{N}} \\ \Delta\phi^{\mathbf{N}} - \sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}} \dfrac{\partial^2\phi^{\mathbf{N}}}{\partial^2\nu}(t, \mathbf{x_i})\Psi_{\mathbf{i}} \end{pmatrix},\ C^{\mathbf{N}} \begin{pmatrix} \phi^{\mathbf{N}} \\ \psi^{\mathbf{N}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial\phi^{\mathbf{N}}}{\partial\nu} - \omega_N \dfrac{\partial^2\phi^{\mathbf{N}}}{\partial^2\nu}|_{\mathbf{I}_{\Gamma}} \\ \sqrt{\omega_N}\dfrac{\partial^2\phi^{\mathbf{N}}}{\partial^2\nu}|_{\mathbf{I}_{\partial\Omega}} \end{pmatrix}, \tag{5.37}$$

for all $\begin{pmatrix} \phi^{\mathbf{N}} \\ \psi^{\mathbf{N}} \end{pmatrix} \in X^{\mathbf{N}}$.

The pair (A^N, C^N) is uniformly exactly observable (see Lemma 3.4.3) and, therefore, we can apply Theorem 5.2.1. On the other hand, hypotheses (H) in (5.10)-(5.12) is also verified (see Lemma 3.5.2) and, therefore, the result of Theorem 5.2.2.

5.6 Matrix formulation and finite-dimensional ISP for wave equation

In this section, we consider some implementation issues related to the finite-dimensional approximation of ISP for waves equation given by (5.32) and (5.35). We follow the more direct approach which consists in writing the discrete ISP into the standard matrix form problem and computing the gradient of functional J^N in (5.13). The Discrete scheme can be restated in matrix form as follows:

$$\begin{cases} \Theta^N_{tt} = A^N\Theta^N + \lambda(t)G^N \\ \Theta^N(0) = \Theta^{0,N},\ \Theta^N_t(0) = \Theta^{1,N}, \\ Z^N = C^N\Theta^N. \end{cases} \tag{5.38}$$

In an inverse source problem, you typically aim to determine the source or cause of a given phenomenon based on observed data. This often involves minimizing a cost function that quantifies the mismatch between observed and predicted data. The gradient descent method is a common approach to minimize such cost functions iteratively. Here's a general steps of how you can use the gradient descent method to minimize a cost function in an inverse source problem:

- **Step 1:** Define the Cost Function J^N in (5.13) that measures the mismatch between observed data observed and predicted data based on the current estimate of the source parameters as follows:

  $$J^N(G^N) = \frac{1}{2}\int_0^T \rho(t)\|C^N Q_N \Theta_t - C^N\Theta^N_t\|^2_{Y^N} dt, \tag{5.39}$$

  where T is the observation time and ρ is a suitable positive bump function, compactly supported in $(0, T)$, chosen to avoid possible numerical instabilities at times $t = 0, T$.

- **Step 2:** Compute the gradient of the cost function. First, we derive the Gateaux derivative of problem (5.38) by linearization:

$$\begin{cases} \delta\Theta_{tt}^N = A^N \delta\Theta^N + \lambda(t)\delta G^N \\ \delta\Theta^N(0) = 0,\ \delta\Theta_t^N(0) = 0, \\ \delta Z^N = C^N \delta\Theta^N, \end{cases} \tag{5.40}$$

and introduce the adjoint system:

$$\begin{cases} P_{tt}^N = A^{N*}P^N + C^{N*}(\rho(t)(Q_N Z_t - Z_t^N))_t \\ P^N(T) = 0,\ P_t^N(T) = 0. \end{cases} \tag{5.41}$$

Multiplying (5.40) by the solution of (5.41) we find,

$$\begin{aligned} 0 &= \int_0^T \left\langle \delta\Theta_{tt}^N, P^N \right\rangle_{X^N} dt - \int_0^T \left\langle A^N\delta\Theta^N, P^N \right\rangle_{X^N} dt - \int_0^T \lambda(t) \left\langle \delta G^N, P^N \right\rangle_{X^N} dt \\ &= -\int_0^T \left\langle \delta\Theta_t^N, P_t^N \right\rangle_{X^N} dt - \int_0^T \left\langle \delta\Theta^N, A^{N*}P^N \right\rangle_{X^N} dt - \int_0^T \lambda(t) \left\langle \delta G^N, P^N \right\rangle_{X^N} dt \\ &= \int_0^T \left\langle \delta\Theta^N, P_{tt}^N \right\rangle_{X^N} dt - \int_0^T \left\langle \delta\Theta^N, A^{N*}P^N \right\rangle_{X^N} dt - \int_0^T \lambda(t) \left\langle \delta G^N, P^N \right\rangle_{X^N} dt \\ &= \int_0^T \left\langle \delta\Theta^N, C^{N*}(\rho(t)(Q_N Z_t - Z_t^N))_t \right\rangle_{X^N} dt - \int_0^T \lambda(t) \left\langle \delta G^N, P^N \right\rangle_{X^N} dt. \end{aligned}$$

Hence,

$$\int_0^T \left\langle \delta\Theta^N, C^{N*}(\rho(t)(Q_N Z_t - Z_t^N))_t \right\rangle_{X^N} dt = \int_0^T \lambda(t) \left\langle \delta G^N, P^N \right\rangle_{X^N} dt. \tag{5.42}$$

Now, we can compute the derivative of the functional,

$$\begin{aligned} \delta J^N(G^N) &= -\int_0^T \rho(t) \left\langle \delta Z_t^N, Q_N Z_t - Z_t^N \right\rangle_{Y^N} dt \\ &= -\int_0^T \left\langle \delta Z^N, (\rho(t)(Q_N Z_t - Z_t^N))_t \right\rangle_{Y^N} dt \\ &= -\int_0^T \left\langle C^N\delta\Theta^N, (\rho(t)(Q_N Z_t - Z_t^N))_t \right\rangle_{Y^N} dt \\ &= -\int_0^T \left\langle \delta\Theta^N, C^{N*}(\rho(t)(Q_N Z_t - Z_t^N))_t \right\rangle_{X^N} dt. \end{aligned} \tag{5.43}$$

Finally by (5.42) and (5.43),

$$\delta J^{\mathbf{N}}(G^N) = -\int_0^T \left\langle \delta G^N, \lambda(t) P^N \right\rangle_{X^N} dt. \tag{5.44}$$

Therefore $\nabla J^N(G^N) = -\lambda P^N$.

- **Step 3:** Update the parameters using the gradient descent update rule:

  $$F^N_{new} = F^N_{old} - \alpha \nabla J^N(F_{old}),$$

  where α is the step size.

- **Step 4:** Repeat steps 3 and 4 until convergence criteria are met, usually the change in parameter values between iterations falls below a predefined threshold.

We now give explicit formulas for the matrixes in the one dimensional case, i.e. we assume $\Omega = (-1, 1)$. For the matrix formulation, we explore the two distinct numerical approaches: projection and collocation methods.

- **Projection:** Here the eigenfunctions of the Laplace operator in this domain are given by $w_i(x) = \sin(i\pi(x+1)/2)$ with $i \in \mathbf{N}$ and the corresponding eigenvalues are $\mu_i^2 = \frac{i^2\pi^2}{4}$. Note that w_i are normalized in the $L^2(-1,1)$ norm. In particular, in this case, X^N is the subspace generated by those eigenfunctions w_i with $i \leq N$ and the dimension of X^N is therefore N.

  Let us denote by $\{\Theta_i^N(t)\}_{i=1}^N$ the components of Θ^N in the basis of eigenfunctions of the Laplace operator. The system can be written in matrix form as (5.38) where $A^N \in \mathcal{M}_{N^2}, C^N \in \mathcal{M}_{N\times 1}$ are suitable matrixes. such that

  $$A^N = diag(\mu_1, \mu_2, ..., \mu_N),\ C^N = \left(-\tfrac{\pi}{2}, \pi, \tfrac{N\pi}{2}(-1)^N\right)^T,$$

  and G^N is a line vector with the coefficients of the source term at the points

  $$G^N = (g_1^N, g_2^N, ..., g_N^N) \in \mathcal{M}_{1\times N}.$$

- **Collocation:** Here we have taken $\Theta^N(t)$ as a $(N-1)$-vector containing the values of the discrete solution at the interior collocation points as in (2.74).

Also the components of the initial data $(\Theta^{0,N}, \Theta^{1,N})$ and the target $(\Theta^N(T), \Theta_t^N(T))$ are written in vector form as in (2.75). The system can be written in matrix form as (5.38) where $A^N \in \mathcal{M}_{(N-1)^2}, C^N \in \mathcal{M}_{(N-1)\times 3}$ are suitable matrixes. such that A^N is defined in the second chapter as in (2.77), G^N is a line vector with the coefficients of the source term at the interior points

$$G^N = (g_2^N, g_3^N, ..., g_N^N) \in \mathcal{M}_{1\times(N-1)},$$

and $C^N \in \mathcal{M}_{(N-1)\times 3}$ corresponds to the observation accounting for the last columns that affect the values of solution in the boundary which can easily be deduced by (2.36) as follows

$$C^N = \begin{pmatrix} K^N(N+1, 2:N) - \omega_N (K^N)^2(N+1, 2:N) \\ \sqrt{\omega}_N (K^N)^2(N+1, 2:N) \\ \sqrt{\omega}_N (K^N)^2(1, 2:N) \end{pmatrix}^T.$$

Here K is defined by (2.76).

5.7 Numerical experiments

In this section, we illustrate the results in this chapter approximating the one-dimensional ISP with projection and collocation methods. In both cases, we use the minimizer of a cost function for the finite-dimensional systems obtained by the projection and collocation method described above. For the time discretization, we use a classical Newmark method with parameters $\gamma = 1/2$, $\beta = 1/4$ and time step $dt = 10^{-2}$.

Experiment 1: We first consider the one-dimensional wave equation using two different approaches, the projection and the spectral collocation methods with zeros initial position and velocity. We take as final time $T = 4.4$, the exact source term $f(x) = 1_{(\frac{-1}{2}, \frac{1}{2})} \cos(\pi x)$ and the intensity $\lambda(t) = 1$ and, then, we illustrate the rate of convergence of the discrete source term to the exact one with these approaches.

Note that $N = 100$ refers to the number of Fourier coefficients utilized in the projection method and also denotes the degree of polynomials employed in the collocation approach.

In Figure 5.1 we show the behaviour of the exact source term and the approximate one.

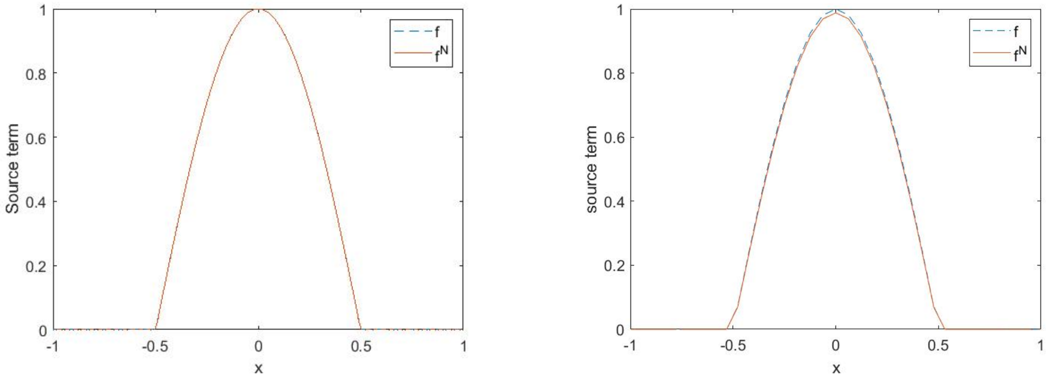


Figure 5.1: The behavior of Inverse source term f and the approximate one f^N as $N = 100$ with two different approaches projection method (left one), collocation method (right one).

In table 5.1-5.2 we show the error between the discrete source and the exact one. Comparing the error associated to values of $N \in [10, 100]$ we can give a rough estimate of the rate of convergence. More precisely, we observe that $\left|f^N - f\right|_{L^2} \sim N^{-\alpha}$ where α is estimated by the slope of the graphics relating $-\log\left|f^N - f\right|_{L^2}$ with $\log N$ (see Figure 5.2).

N	log10($\left\|f^N - f\right\|_{L^2(-1,1)}$)
10	-0.37
50	-2.7
100	-3.9

Table 5.1: Convergence of the discrete source term to the exact as N grows by the projection method.

N	log10($\left\|f^N - f\right\|_{L^2(-1,1)}$)
10	-0.4
50	-2.9
100	-4.1

Table 5.2: Convergence of the discrete source term to the exact as N grows by the collocation method.

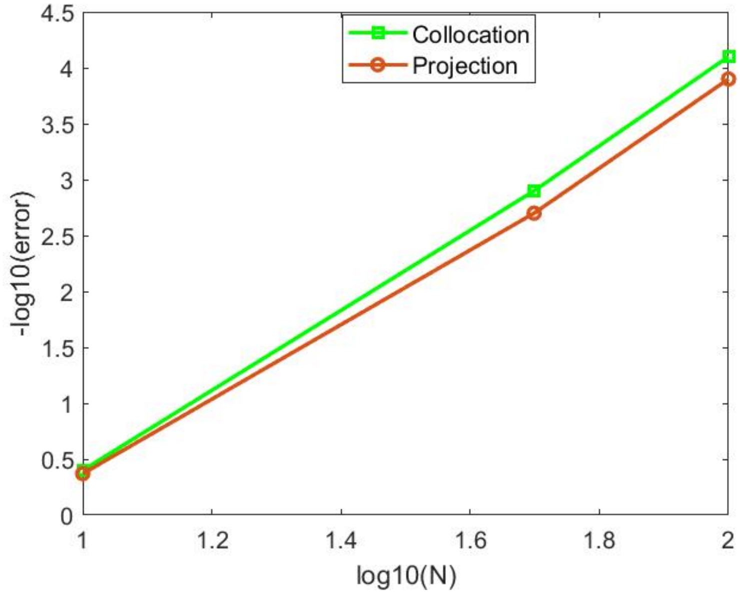


Figure 5.2: The slope of each graphic gives an experimental estimate of the value α for which $\left|f^N - f\right|_{L^2} \sim N^{-\alpha}$ which is approximately $\alpha = 1.95$ in the projection and $\alpha = 2.05$ in the collocation for this example. Of course, other examples could have different rates.

Chapter 6

Conclusions

This thesis has investigated the use of spectral methods to address numerical challenges in control and inverse problems associated with the wave equation and elasticity systems, with a focus on boundary control scenarios common in practical applications. We began by recognizing that boundary controllability depends on observability inequalities, and then set out to build an effective mechanism for discretizing control problems while maintaining convergence to continuous equivalents. This endeavour required the formulation of discrete observability inequalities with uniformity regarding discretization parameters, which was a historically difficult challenge. Our work into the spectral collocation approach effectively handled this issue. Unlike conventional techniques, spectral collocation provided high-order accuracy, allowing for good approximations even with coarse discretizations, which was especially useful in higher dimensions and vector problems such as elasticity. The spectral collocation approach proved to be an effective numerical approximation technique, albeit its use was restricted to rectangular domains because of boundary condition complications. By means of extensive testing and analysis, we have demonstrated the effectiveness and adaptability of our method in a variety of dimensions and systems, highlighting the importance of spectral collocation techniques in effectively and precisely solving control and inverse source problems. As such, our study contributes to the development of computational techniques in applied mathematics and engineering by providing a strong basis for future research and use of spectral methods in solving complicated dynamical systems and real-world engineering difficulties.

Chapter 7

Future research lines

The findings of this thesis open up promising avenues for future research, as advancements in computational techniques continue to shape the landscape of applied mathematics and engineering. Building upon the insights gained from our investigation into spectral methods for control and inverse problems, several intriguing research directions emerge for further exploration and refinement. In this section, we outline some potential future research lines that hold promise for extending the scope and impact of our work as:

- Extend the method's applicability beyond rectangular geometries to more general domains.
- Rigorously establishing convergence results and rates for the spectral collocation method applied to the elasticity system. This requires to address challenging issues for the eigenfunctions of the elasticity system.
- One of the main advantages of spectral methods is the convergence rate. Here we only prove convergence, but we also give numerical evidence of the fact that the convergence rate will depend on the regularity of the data, as expected. An analytical proof of this result would be an important contribution.
- Exploring the rate of convergence in the context of the inverse source problem.
- Provide numerical evidence of the convergence of the discrete source to a source of the limit wave system in a square domain when approximating the problem with two different numerical approaches: the spectral collocation method and the projection

method. In this case, we can compare the results obtained from each approach.

- Extend the method to other control problems associated to equations where spectral methods are widely used, as in fluid dynamics.
- Analyze control problems with variable coefficients and systems.

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Appendix

Appendix A

In this section we give a proof of Lemmas 2.5.1 and 2.5.2. The proofs rely on a careful spectral analysis that we address first. Let $(\lambda_k, \varphi_k)_{k\in\mathbb{N}}$ be the eigenvalues and the eigenfunctions associated to the Laplace equation:

$$\begin{cases}\varphi_{xx} + \lambda\varphi = 0 & \text{in } \Omega = (-1,1),\\ \varphi(1) = \varphi(-1) = 0.\end{cases} \tag{1}$$

The eigenvalues are simple and can be computed explicitly ($\lambda_k = (k\pi/2)^2$, $k \in \mathbb{N}$), while the associated eigenfunctions $\{\varphi_k\}_{k\in N}$ constitutes an orthogonal basis in $L^2(-1,1)$.

Associated to the collocation numerical approximation of the wave equation, we introduce the following discrete eigenvalue problem:

$$\begin{cases}\varphi^N_{xx}(x_i) + \lambda^N\varphi^N(x_i) = 0, & \text{at } \; x_i \in C^{\Omega},\\ \varphi^N(-1) = \varphi^N(1) = 0.\end{cases} \tag{2}$$

It is known that this eigenvalue problem admits $N-1$ eigenvalues which are simple and real numbers (see Vandeven, 1990 and Weideman and Trefethen, 1988). We assume they are written in increasing order, i.e. $0 < \lambda^N_1 < \lambda^N_2 < ... < \lambda^N_{N-1}$. The associated eigenfunctions $\{\varphi^N_k\}_{k=1}^{N-1}$ constitutes an orthogonal basis in $\mathbb{P}^{Di}_N$ with the discrete scalar product $(\cdot,\cdot)_N$.

From now on, we assume that both φ^N_k and φ_k are normalized in the L^2-norm.

The following Theorem states the spectral approximation results that we need.

Theorem .0.1. *Let $m \geq 2$, $\alpha \in (0, 2/\pi)$ and $r(N) = \alpha N^{\frac{1}{8}}$. Then, for any $k \in \{1, ..., r(N)\}$ there exists constants C_α, C, independent of N, such that the following estimates hold:*

$$\left|\sqrt{\lambda^N_k} - \sqrt{\lambda_k}\right| \leq C_\alpha p(\alpha)^{N^{2/3}}, \; 0 < p(\alpha) < 1, \tag{3}$$

$$\left|\varphi^N_k - \varphi_k\right|_{L^2} \leq CN^{-3/4}, \tag{4}$$

$$\left|\varphi_k^N - \varphi_k\right|_{H_0^1}^2 \leq C\left(p(\alpha)^{N^{2/3}} + N^{-1}\right), \tag{5}$$

$$\left|\sqrt{w_0^N}\varphi_{k,xx}^N(\pm 1)\right| \leq CN^{-1/2}, \tag{6}$$

$$\left|\varphi_{k,x}^N(1) - \varphi_{k,x}(1)\right| \leq C\left(p(\alpha)^{N^{2/3}} + N^{-1}\right). \tag{7}$$

Proof. **Step 1: Estimate** (3)**.** This is a direct consequence of the following estimate proved in Vandeven, 1990:

$$\left|\lambda_k^N - \lambda_k\right| \leq C_\alpha p(\alpha)^{N^{2/3}}. \tag{8}$$

Step 2: Estimate (4)**.** We follow the idea in Raviart and Thomas, 1983 (Lemma 6.4-3) where a related result is obtained for the Galerkin approximation. Let us introduce the variational characterizations of both (1) and (2):

$$a(\varphi, v) = \lambda(\varphi, v), \quad \forall v \in H_0^1(\Omega), \tag{9}$$

where $a(\cdot,\cdot)$ is the bi-linear form defined by $a(u,v) = (u_x, v_x), \forall u, v \in H_0^1$, and $(\cdot,\cdot)$ denotes the scalar product in $L^2(-1,1)$, and

$$a(\varphi^N, v^N) = \lambda^N(\varphi^N, v^N)_N, \qquad \forall v^N \in \mathbb{P}_N^{Di}(\Omega). \tag{10}$$

Note that the bi-linear form $a(\cdot,\cdot)$ is the same both in the continuous and the discrete formulation. In fact, (10) is deduced from (2) multiplying the equations by $\omega_i^N v^N(x_i)$ and adding in $i \in I$, i.e.

$$\begin{aligned} 0 &= \sum_{i\in I}\varphi_{xx}^N(x_i)\omega_i^N v^N(x_i) + \sum_{i\in I}\lambda^N\varphi^N(x_i)\omega_i^N v^N(x_i) = \int_{-1}^1 \varphi_{xx}^N v^N\, dx + \lambda^N(\varphi^N, v^N)_N \\ &= -\int_{-1}^1 \varphi_x^N v_x^N\, dx + \lambda^N(\varphi^N, v^N)_N = -a(\varphi^N, v^N) + \lambda^N(\varphi^N, v^N)_N. \end{aligned}$$

The main difference with the case treated in Raviart and Thomas, 1983 is that here the right-hand side in the variational characterization (10) makes appear the discrete scalar product $(\cdot,\cdot)_N$, instead of the L^2 one, and this introduces some technical details. Let us define

$$\varrho_{k,N} = \max_{i\in I_\Omega, i\neq k} \frac{\lambda_k}{|\lambda_i^N - \lambda_k|},\ I_\Omega = \{1, ..., N-1\} \tag{11}$$

and the orthogonal projection $\Pi_N \varphi_k \in \mathbb{P}_N^{Di}$ characterized by

$$a(\Pi_N\varphi_k - \varphi_k, v^N) = 0, \qquad \forall v^N \in \mathbb{P}_N^{Di}(\Omega). \tag{12}$$

We now write $\Pi_N\varphi_k$ in the basis φ_i^N. If we normalize φ_i^N in such a way that $|\varphi_i^N|_{L^2} = 1$, then

$$\Pi_N\varphi_k = \sum_{i\in I_\Omega} \frac{(\Pi_N\varphi_k, \varphi_i^N)_N}{\|\varphi_i^N\|_N^2}\varphi_i^N. \tag{13}$$

If we denote,

$$v_k^N = \frac{(\Pi_N\varphi_k, \varphi_k^N)_N}{\|\varphi_k^N\|_N^2}\varphi_k^N. \tag{14}$$

Then,

$$\left|\Pi_N\varphi_k - v_k^N\right|_{L^2}^2 = \left|\sum_{i\in I_\Omega, i\neq k} \frac{(\Pi_N\varphi_k, \varphi_i^N)_N}{\|\varphi_i^N\|_N^2}\varphi_i^N\right|_{L^2}^2 \leq 2\sum_{i\in I, i\neq k} \frac{\left|(\Pi_N\varphi_k, \varphi_i^N)_N\right|^2}{\|\varphi_i^N\|_N^2}. \tag{15}$$

From (10),(9) and (12) we easily obtain

$$(\lambda_i^N - \lambda_k)(\Pi_N\varphi_k, \varphi_i^N)_N = \lambda_k(\varphi_k - \Pi_N\varphi_k, \varphi_i^N)_N + \lambda_k\bigg((\varphi_k, \varphi_i^N) - (\varphi_k, \varphi_i^N)_N\bigg).$$

Therefore, using (11) we get

$$\left|(\Pi_N\varphi_k, \varphi_i^N)_N\right|^2 \leq 2\varrho_{k,N}^2 \left|(\varphi_k - \Pi_N\varphi_k, \varphi_i^N)_N\right|^2 + 2\varrho_{k,N}^2 \left|(\varphi_k, \varphi_i^N) - (\varphi_k, \varphi_i^N)_N\right|^2, \tag{16}$$

that we now substitute in (15). Taking into account the norm equivalence in (2.20), we easily deduce

$$\left|\Pi_N\varphi_k - v_k^N\right|_{L^2} \leq \sqrt{6}\varrho_{k,N}\left(\|\varphi_k - \Pi_N\varphi_k\|_N + \sum_{i\in I_\Omega, i\neq k}\left|(\varphi_k, \varphi_i^N) - (\varphi_k, \varphi_i^N)_N\right|\right). \tag{17}$$

The idea now is to replace the $\|\cdot\|_N$ norm on the right-hand side by the $|\cdot|_{L^2}$ norm. In fact, these two norms are equivalent for polynomials in $\mathbb{P}_N$ so we first replace φ_k by a polynomial. Let's define the interpolation $I_N : H_0^1 \longrightarrow \mathbb{P}_N^{Di}$ as follow

$$\begin{cases} I_N\varphi_k \in \mathbb{P}_N^{Di}, \\ I_N\varphi_k(x_i) = \varphi_k(x_i), \ \forall i \in I. \end{cases} \tag{18}$$

Note that,

$$\|\varphi_k - \Pi_N\varphi_k\|_N = \|I_N\varphi_k - \Pi_N\varphi_k\|_N \leq \sqrt{3}\,|I_N\varphi_k - \Pi_N\varphi_k|_{L^2}\,. \tag{19}$$

Therefore, substituting in (17) and then adding and subtracting φ_k we obtain,

$$\begin{aligned}\left|\Pi_N\varphi_k - v_k^N\right|_{L^2} \leq \sqrt{6}\varrho_{k,N}\Bigg(&\sqrt{3}\,|I_N\varphi_k - \varphi_k|_{L^2} + \sqrt{3}\,|\varphi_k - \Pi_N\varphi_k|_{L^2} \\ &+ \sum_{i\in I_\Omega, i\neq k}\left|(\varphi_k, \varphi_i^N) - (\varphi_k, \varphi_i^N)_N\right|\Bigg).\end{aligned} \tag{20}$$

Furthermore, (14) and $\left|\varphi_k^N\right|_{L^2} = |\varphi_k|_{L^2} = 1$ give

$$|\varphi_k|_{L^2} - \left|\varphi_k - v_k^N\right|_{L^2} \leq \left|v_k^N\right|_{L^2} \leq |\varphi_k|_{L^2} + \left|\varphi_k - v_k^N\right|_{L^2}$$

and

$$\left|\frac{\left|(\Pi_N\varphi_k, \varphi_k^N)_N\right|}{\|\varphi_k^N\|_N^2} - 1\right| \leq \left|\varphi_k - v_k^N\right|_{L^2}. \tag{21}$$

As it is always possible to choose the eigenfunctions φ_k^N such that $(\Pi_N\varphi_k, \varphi_k^N)_N \geq 0$, we obtain

$$\left|v_k^N - \varphi_k^N\right|_{L^2} \leq |\varphi_k - \Pi_N\varphi_k|_{L^2} + \left|\Pi_N\varphi_k - v_k^N\right|_{L^2}. \tag{22}$$

Finally, from (20) and (22)

$$\begin{aligned}\left|\varphi_k^N-\varphi_k\right|_{L^2} &\leq 2(1+3\sqrt{3}\varrho_{k,N})\left|\varphi_k-\Pi_N\varphi_k\right|_{L^2}+2(3\sqrt{2}\varrho_{k,N})\left|\varphi_k-I_N\varphi_k\right|_{L^2}\\ &+2(\sqrt{6}\varrho_{k,N})\left(\sum_{i\in I_\Omega, i\neq k}\left|(\varphi_k,\varphi_i^N)-(\varphi_k,\varphi_i^N)_N\right|\right).\end{aligned}$$

We recall that, if $v \in H_0^m$ for some $m \geq 1$ and $v^N \in \mathbb{P}_N^{Di}$, then there exist constants $c, c_1 > 0$ such that (see Canuto et al., 1988, [chapter(9)])

$$|v-\Pi_N v|_{L^2} \leq cN^{-m}|v|_{H^m} \text{ and } |v-I_N v|_{L^2} \leq c_1 N^{1/2-m}|v|_{H^m}. \tag{23}$$

On the other hand, there exists a constant $c_2 > 0$ such that (see Cividini et al., 1993 [estimation (3.22)])

$$|(v,v^N)_N-(v,v^N)| \leq c_2 N^{-m}|v|_{H^m}\left|v^N\right|_{L^2}. \tag{24}$$

From the classical projection results for spectral methods in (23) and (24) when $m = 2$ and the fact that $|\varphi_k|_{H^2} \leq c\lambda_k \leq cN^{1/4}$ (since by hypotheses $k \leq cN^{1/8}$) we deduce that there exists a constant $C > 0$ such that

$$\left|\varphi_k^N-\varphi_k\right|_{L^2} \leq CN^{-3/4}, \quad k \leq r(N). \tag{25}$$

Step 3: Estimate (5)**.** From (9), (10) and the norm equivalence in (2.20) we can write

$$a(\varphi_k^N-\varphi_k,\varphi_k^N-\varphi_k) \leq 3\lambda_k^N+\lambda_k-2\lambda_k(\varphi_k,\varphi_k^N).$$

We also have

$$\left|\varphi_k^N-\varphi_k\right|_{L^2}^2 = 2(1-(\varphi_k,\varphi_k^N))$$

and, then

$$a(\varphi_k^N-\varphi_k,\varphi_k^N-\varphi_k) \leq 3\left(\lambda_k^N-\lambda_K+\lambda_K\left|\varphi_k^N-\varphi_k\right|_{L^2}^2\right).$$

From the coercivity of the bi-linear form a in (9), (4), (8) and the fact that $\lambda_k \leq M_\alpha N^{\frac{1}{4}}$, we can deduce, there exists a constant $C_1 > 0$ such that

$$\left|\varphi_k^N - \varphi_k\right|^2_{H_0^1} \leq C_1 \left(p(\alpha)^{N^{2/3}} + N^{-1}\right).$$

Step 4: Estimate (6). It is enough to prove the estimate at $x = 1$, since the other one is similar. First, we observe that we can write the discrete eigenvalue problem (2) in the following equivalent form:

$$\begin{cases} \varphi^N_{k,xx} + \lambda^N_k \varphi^N_k = \varphi^N_{k,xx}(-1)\Psi^N_0 + \varphi^N_{k,xx}(1)\Psi^N_N \quad \text{ in } x \in \Omega, \\ \varphi^N_k(1) = \varphi^N_k(-1) = 0, \end{cases} \tag{26}$$

where $\Psi^N_0 \in \mathbb{P}_N(\Omega)$ (resp. Ψ^N_N) is the Lagrangian polynomial which is 1 at $x = -1$ (resp. $x = 1$) and 0 at the rest of quadrature points in $C^\Omega = C^{\Omega,N}$. At this point, we make explicit the dependence on N of the set of quadrature points by writing $C^{\Omega,N}$, since we consider different sets below.

It is easy to see that the eigenfunctions associated to (26) are either even or odd. We focus on the case of even eigenfunctions since the other one is similar. In this case $\varphi^N_{k,xx}(-1) = \varphi^N_{k,xx}(1)$. Multiplying (26) by $\Psi^{N-1}_{N-1} \in \mathbb{P}_{N-1}(\Omega)$ the Lagrangian polynomial which is 1 at $x = 1$ and 0 at all other collocation points in $C^{\Omega,N-1}$, one has

$$\varphi^N_{k,xx}(1)\omega^{N-1}_{N-1} + \lambda^N_k \int_{-1}^1 \varphi^N_k \Psi^{N-1}_{N-1} dx = \varphi^N_{k,xx}(1)\omega^N_N. \tag{27}$$

Note that in the first term on the left-hand side, we have used the quadrature formula with nodes in $C^{\Omega,N-1}$, which is exact for polynomials of degree $2(N-1)-1$. In fact, the term inside the integral is a polynomial of degree $2N-3$ and by hypotheses Ψ^{N-1}_{N-1} is 1 at $x = 1$ and 0 at the other quadrature nodes. On the right-hand side, we have used the quadrature formula with nodes in $C^{\Omega,N}$ which is also exact since the integrated is a polynomial of degree

$2N-1$. Therefore, we can write

$$\begin{aligned}\left|\sqrt{\omega_N^N}\varphi_{k,xx}^N(1)\right| &= \left|\frac{\sqrt{\omega_N^N}\lambda_k^N\int_{-1}^1\varphi_k^N\Psi_{N-1}^{N-1}dx}{(w_{N-1}^{N-1}-w_N^N)}\right| \\ &= \left|\frac{\sqrt{\omega_N^N}\lambda_k^N\int_{-1}^1(\varphi_k^N-P_{N-2}\varphi_k^N)\Psi_{N-1}^{N-1}dx}{(\omega_{N-1}^{N-1}-\omega_N^N)}\right|,\end{aligned} \tag{28}$$

where $P_{N-2}\varphi_k^N$ is the orthogonal projection of φ_k^N in $\mathbb{P}_{N-2}^{Di}(\Omega)$ with respect to the L^2-scalar product. The last equality comes from the fact that $\int_{-1}^1 P_{N-2}\varphi_k^N\Psi_{N-1}^{N-1}dx=0$ by the quadrature formula with nodes in $C^{\Omega,N-1}$.

Now using Cauchy schwarz inequality in (28) and taking into account the norm equivalent in (2.20), as $\Psi_{N-1}^{N-1}\in\mathbb{P}_{N-1}(\Omega)$,$\left|\Psi_{N-1}^{N-1}\right|_{L^2}\le\left\|\Psi_{N-1}^{N-1}\right\|_{N-1}=\sqrt{\omega_{N-1}^{N-1}}$ and $\omega_{N-1}^{N-1}=\dfrac{2}{(N-1)N}$,$\omega_N^N=\dfrac{2}{N(N+1)}$, from (23) and the fact that $\lambda_k^N\le M_\alpha N^{\frac14}$ we obtain, there exists a constant $c_3>0$ such that

$$\begin{aligned}\left|\sqrt{\omega_N^N}\varphi_{k,xx}^N(1)\right| &\le \frac{\sqrt{\omega_N^N}\sqrt{\omega_{N-1}^{N-1}}\lambda_k^N\left|\varphi_k^N-P_{N-2}\varphi_k^N\right|_{L^2}}{\left|(\omega_{N-1}^{N-1}-\omega_N^N)\right|} \\ &\le c_3N^{5/4}(N-2)^{-m}\left|\varphi_k^N\right|_{H^m}.\end{aligned} \tag{29}$$

To estimate this last term we use the following result:

Lemma .0.2. *Assume that $m=2$, then there exists a constant $M_2>0$ such that*

$$\left|\varphi_k^N\right|_{H^2}^2\le M_2|\lambda_k^N|^2(1+w_N^N\left|\varphi_{k,xx}^N(1)\right|^2). \tag{30}$$

From Lemma .0.2, the fact that $\lambda_k^N\le M_\alpha N^{\frac14}$ and (29) we easily deduce estimate (6).

We now prove Lemma .0.2.

Proof. Multiply (26) by φ_k^N and integrating by parts one has

$$\int_{-1}^1\left|\varphi_{k,x}^N\right|^2dx=\lambda_k^N\int_{-1}^1\left|\varphi_k^N\right|^2dx-2\varphi_{k,xx}^N(1)\int_{-1}^1\Psi_N^N\varphi_k^Ndx.$$

The last equality comes from the fact that $\int_{-1}^1\Psi_0^N\varphi_k^Ndx=\int_{-1}^1\Psi_N^N\varphi_k^Ndx$. Now, multiplying and dividing the second term on the right-hand side by $\sqrt{\omega_N^N}$, first using Young's inequality and then Cauchy Schwarz inequality we obtain

$$\left|\varphi_{k,x}^N\right|_{L^2}^2 \leq \left(\lambda_k^N + \frac{\left|\Psi_N^N\right|_{L^2}^2}{\omega_N^N}\right)\left|\varphi_k^N\right|_{L^2}^2 + \omega_N^N \left|\varphi_{k,xx}^N(1)\right|^2. \tag{31}$$

On the other hand multiply (26) by $\varphi_{k,xx}^N$ and integrating by parts one obtains

$$\left|\varphi_{k,xx}^N\right|_{L^2}^2 = \lambda_k^N \left|\varphi_{k,x}^N\right|_{L^2}^2 + 2\omega_N^N \left|\varphi_{k,xx}^N(1)\right|^2. \tag{32}$$

Here on the right-hand side we have used the quadrature formula with nodes in C^N since the integrated is a polynomial of degree $2N-2$ and by hypotheses Ψ_0^N (resp. Ψ_N^N) is 1 at $x=-1$ (resp. $x=1$) and 0 at the rest of quadrature points in C^N and the fact that φ_k^N is even. Finally from (31),(32) and the normalization of the eigenfunctions $\left|\varphi_k^N\right|_{L^2} = 1$ we easily obtain (30). ☐

Step 5: Estimate (7). Multiplying the equation of φ_k^N in (2) by $\frac{1+x_i}{2}\omega_i^N$ and adding in $i \in I$, one obtains

$$\begin{aligned} 0 &= \sum_{i\in I} \varphi_{k,xx}^N(x_i)\frac{1+x_i}{2}\omega_i^N + \lambda_k^N \sum_{i\in I} \varphi_k^N(x_i)\frac{1+x_i}{2}\omega_i^N - \varphi_{k,xx}^N(1)\omega_N^N \\ &= \varphi_{k,x}^N(1) + \lambda_k^N \int_{-1}^1 \varphi_k^N \frac{1+x}{2}dx - \varphi_{k,xx}^N(1)\omega_N^N. \end{aligned} \tag{33}$$

Note that in the first and second terms on the right-hand side, we have used the quadrature formula. In particular, this allowed us to integrate by parts in the first term.

Now multiplying the equation in (1) by $\frac{1+x}{2}$ and integrating by parts one has,

$$\varphi_{k,x}(1) + \lambda_K \int_{-1}^1 \varphi_k \frac{1+x}{2}dx = 0. \tag{34}$$

Combining (33) and (34), we obtain

$$\varphi_{k,x}^N(1) - \varphi_{k,x}(1) = \lambda_k \int_{-1}^1 \varphi_k \frac{1+x}{2}dx - \lambda_k^N \int_{-1}^1 \varphi_k^N \frac{1+x}{2}dx + \varphi_{k,xx}^N(1)w_N^N.$$

Here, the right hand side is easily estimated using (4), (8) and the normalization of the eigenfunctions $\left|\varphi_k^N\right|_{L^2} = 1$. This gives (7) and concludes the proof of the theorem. □

We now move to give a proof of Lemma 2.5.2 and Lemma 2.5.1.

Proof of Lemma 2.5.2. Define $\mu_k = sign(k)\sqrt{\lambda_{|k|}}$ and $\Phi_k = (\varphi_{|k|}/(i\mu_k), \varphi_{|k|})/\sqrt{2}$ for $k \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. Note that $\{\Phi_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis in $H_0^1 \times L^2$. Thus, given $(\phi^0, \phi^1) \in H_0^1 \times L^2$ we can write

$$(\phi^0, \phi^1) = \sum_{k\in\mathbb{Z}^*} a_k \Phi_k, \quad |(\phi^0, \phi^1)|^2_{H_0^1\times L^2} = \sum_{k\in\mathbb{Z}^*} |a_k|^2 < \infty, \tag{35}$$

for some Fourier coefficients $a_k \in \mathbb{C}$. Analogously, we define $\mu_k^N = sign(k)\sqrt{\lambda_{|k|}^N}$ and $\Phi_k^N = (\varphi_{|k|}^N/(i\mu_k^N), \varphi_{|k|}^N)/\sqrt{2}$ for $|k| \leq N,\ k \neq 0$. Again, $\{\Phi_k^N\}_{|k|\leq N}$ is an orthonormal basis of $H_0^1 \times \mathbb{P}_N$, where the scalar product in $\mathbb{P}_N$ is the discrete inner product $(\cdot,\cdot)_N$. Let us consider

$$(\phi^{0,N}, \phi^{1,N}) = \sum_{|k|\leq r(N)} a_k \Phi_k^N. \tag{36}$$

From the convergence results in Theorem .0.1, we have

$$\begin{aligned} |(\phi^0, \phi^1) - (\phi^{0,N}, \phi^{1,N})|^2_{H_0^1\times L^2} &\leq \sup_{|k|\leq r(N)} \left(|\Phi_k - \Phi_k^N|^2_{H_0^1\times L^2}\right) \sum_{|k|\leq r(N)} |a_k|^2 \\ &+ \sum_{|k|>r(N)} |a_k|^2 \to 0, \quad \text{as } N \to \infty. \end{aligned} \tag{37}$$

This concludes the proof of (2.62). Moreover, the solution of the continuous wave equation (2.3) is given by

$$(\phi(t,x), \phi_t(t,x)) = \sum_{k\in\mathbb{Z}^*} a_k e^{i\mu_k t} \Phi_k, \tag{38}$$

while the one associated to (2.27) with initial data $(\phi^{0,N}, \phi^{1,N})$ is given by

$$(\phi^N(t,x), \phi_t^N(t,x)) = \sum_{|k|\leq r(N)} a_k e^{i\mu_k^N t} \Phi_k^N. \tag{39}$$

Again, the uniform convergence of the low frequencies stated in Theorem .0.1 allows us to obtain (2.63)-(2.65).

Proof of Lemma 2.5.1. We follow the idea in the proof of Lemma 2.5.2. Let us define $\hat{\Phi}_k = (i\mu_k)\Phi_k$ for $k \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$ where Φ_k, where introduced at the beginning of

Lemma 2.5.2. Note that $\{\hat{\Phi}_k\}_{k\in\mathbb{Z}}$ is now an orthonormal basis in $L^2 \times H^{-1}$. Thus, given $(u^0, u^1) \in L^2 \times H^{-1}$ we can write

$$(u^0, u^1) = \sum_{k\in\mathbb{Z}^*} \hat{b}_k \hat{\Phi}_k, \quad |(u^0, u^1)|^2_{L^2\times H^{-1}} = \sum_{k\in\mathbb{Z}^*} |\hat{b}_k|^2 < \infty, \tag{40}$$

for some Fourier coefficients $\hat{b}_k \in \mathbb{C}$. Analogously, we define $\hat{\Phi}^N_k = i\mu^N_k \Phi^N_k$ for $|k| \leq N,\ k \neq 0$. Let us consider

$$(u^{0,N}, u^{1,N}) = \sum_{|k|\leq r(N)} \hat{b}_k \hat{\Phi}^N_k. \tag{41}$$

Note that if (u^0, u^1) are continuous functions, the sequence that we choose $(u^{0,N}, u^{1,N})$ are the polynomial which coincides with the value of (u^0, u^1) at the collocation points.

Arguing as in (37) the convergence result in (2.50) can be reduced to prove

$$\sup_{|k|\leq r(N)} \left(|\hat{\Phi}_k - \hat{\Phi}^N_k|^2_{L^2\times H^{-1}}\right) \to 0, \qquad \text{as } N \to \infty.$$

Note that,

$$|\hat{\Phi}_k - \hat{\Phi}^N_k|^2_{L^2\times H^{-1}} = |\varphi_{|k|} - \varphi^N_{|k|}|^2_{L^2} + |i\mu_k\varphi_{|k|} - i\mu^N_k \varphi^N_{|k|}|^2_{H^{-1}}.$$

The first term here can be estimated uniformly for $|k| \leq r(N)$ by Theorem .0.1 and converges to zero as $N \to \infty$. Concerning the second term, we use the fact that the eigenfunctions φ_k (resp. $\varphi^N_{|k|}$) satisfy (1) (resp. (26)) together with the isometry of the Laplacian between H^1_0 and H^{-1}. Therefore,

$$|i\mu_k\varphi_{|k|} - i\mu^N_k \varphi^N_{|k|}|^2_{H^{-1}} = \left| \frac{\varphi_{|k|,xx}}{\mu_k} - \frac{\varphi^N_{|k|,xx} - \varphi^N_{|k|,xx}(-1)\Psi^N_0 - \varphi^N_{|k|,xx}(1)\Psi^N_N}{\mu^N_k} \right|^2_{H^{-1}} \tag{42}$$

$$\leq \left| \frac{\varphi_{|k|}}{\mu_k} - \frac{\varphi^N_{|k|}}{\mu^N_k} \right|^2_{H^1_0} + \frac{|\varphi^N_{|k|,xx}(-1)|^2}{\lambda^N_k} |\Psi^N_0|^2_{H^{-1}} + \frac{|\varphi^N_{|k|,xx}(1)|^2}{\lambda^N_k} |\Psi^N_N|^2_{H^{-1}}, \tag{43}$$

that converges uniformly to zero for $|k| \leq r(N)$ as a consequence of Theorem .0.1 and the uniform bound of $|\Psi^N_0|_{L^2}$ and $|\Psi^N_N|_{L^2}$.

We now prove (2.51). Observe that, $\{\Phi^N_k\}_{|k|\leq N}$ is orthonormal in $\mathbb{P}^N \times \mathbb{P}^N$ with the scalar product

$$\Big((v^{0,N},v^{1,N}),(w^{0,N},w^{1,N})\Big)^*_N=(v^{0,N}_x,w^{0,N}_x)_N+(v^{1,N},v^{1,N})_N,$$

whose associated norm is equivalent to the usual norm in $H^1_0\times L^2$.

Therefore, if write any $(\phi^{0,N},\phi^{1,N})$ as $\sum_{|k|\le N}a^N_k\Phi^N_k$ and by the orthogonality of the eigenfunctions φ^N_k with respect to the discrete scalar product $(\cdot,\cdot)_N$ and the duality product (2.25), we have

$$\begin{aligned}&\Big|\Big\langle(\varphi^{0,N},\varphi^{1,N}),(u^{0,N},u^{1,N})\Big\rangle_N\Big|=\Bigg|\Big\langle\sum_{|k|\le r(N)}\hat{b}_k\hat{\Phi}^N_k,\sum_{|k|\le N}a^N_k\Phi^N_k\Big\rangle_N\Bigg|\\&\le\Bigg|\sum_{|k|\le r(N)}\hat{b}_ka^N_k\Bigg|\le\Bigg(\sum_{|k|\le r(N)}|\hat{b}_k|^2\Bigg)^{1/2}\Bigg(\sum_{|k|\le r(N)}|a^N_k|^2\Bigg)^{1/2}\\&\le|(u^0,u^1)|_{L^2\times H^{-1}}\|(\varphi^{0,N}_x,\varphi^{1,N})\|_{N\times N}.\end{aligned}$$

Finally, we prove (2.52). We assume now that $(\phi^{0,N},\phi^{1,N})\to(\phi^0,\phi^1)$ in $H^1_0\times L^2$ that we write as $(\phi^0,\phi^1)=\sum_{k\in\mathbb{Z}^*}a_k\Phi_k$. We have,

$$\begin{aligned}&\Big\langle(\varphi^{0,N},\varphi^{1,N}),(u^{0,N},u^{1,N})\Big\rangle_N=\Big\langle\sum_{|k|\le r(N)}\hat{b}_k\hat{\Phi}^N_k,\sum_{|k|\le N}a^N_k\Phi^N_k\Big\rangle_N,\\&\Big\langle(\varphi^0,\varphi^1),(u^0,u^1)\Big\rangle=\Big\langle\sum_{k\in\mathbb{Z}^*}\hat{b}_k\hat{\Phi}_k,\sum_{k\in\mathbb{Z}^*}a_k\Phi_k\Big\rangle.\end{aligned}$$

The convergence results in Theorem .0.1 allow to prove the estimate

$$|a^N_k-a_k|\le CN^{-1/4},\quad |k|\le r(N),$$

and, using the strong convergence $(\phi^{0,N},\phi^{1,N})\to(\phi^0,\phi^1)$,

$$\Big\langle\sum_{|k|\le r(N)}\hat{b}_k\hat{\Phi}^N_k,\sum_{|k|\le N}a^N_k\Phi^N_k\Big\rangle_N\to\Big\langle\sum_{k\in\mathbb{Z}^*}\hat{b}_k\hat{\Phi}_k,\sum_{k\in\mathbb{Z}^*}a_k\Phi_k\Big\rangle.$$

This concludes the proof of (2.52).

Appendix B

In this section we give a proof of Lemmas 3.5.1 and 3.5.2. The proofs rely on a careful spectral analysis that we address first. Let consider $(\lambda_{km}, \varphi_{km})_{(k,m)\in\mathbb{N}^2}$ be the eigenvalues and the eigenfunctions associated to the Laplace equation,

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{in } \Omega = (-1,1)^2, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{44}$$

According to Courant-Hilber Courant and Hilbert, 1957 (Chapter 5) and by separation of variable, the eigenvalues can be computed explicitly ($\lambda_{km} = \lambda_k + \lambda_m = \dfrac{k^2+m^2}{2^2}\pi^2$, $(k,m)\in\mathbb{N}^2$), such that λ_k, λ_m are the eigenvalues associated to the one-dimensional Laplace equation (1) in x_1 (resp. x_2) direction while the associated eigenfunctions $\{\varphi_{km}(\mathbf{x}) = \varphi_k(x_1)\varphi_m(x_2)\}_{(k,m)\in N_1\times N_2}$ constitutes an orthogonal basis in $L^2(\Omega)$. Here, $\{\varphi_k(x_1)\}_{k\in N_1}$ (resp. $\{\varphi_m(x_2)\}_{m\in N_2}$) constitutes an orthogonal basis in $L^2(-1,1)$.

Associated to the collocation numerical approximation of the wave equation, we introduce the following discrete eigenvalue problem:

$$\begin{cases} (\Delta\varphi^{\mathbf{N}} + \lambda^{\mathbf{N}}\varphi^{\mathbf{N}})(\mathbf{x_i}) = 0 & \text{at } \mathbf{x_i}\in C^{\Omega}, \\ \varphi^{\mathbf{N}}(\mathbf{x_i}) = 0 & \text{at } \mathbf{x_i}\in C^{\partial\Omega}. \end{cases} \tag{45}$$

By using the separation of variable, we note that the eigenvalue problem (45) admits $(N_1 - 1)\times(N_2-1)$ eigenvalues, where $\lambda^{\mathbf{N}}_{km} = \lambda^{N_1}_k + \lambda^{N_2}_m$ such that $\lambda^{N_1}_k, \lambda^{N_2}_m$ are the eigenvalues associated to the Laplace equation (2) in x_1 (resp.x_2) direction. The associated eigenfunctions $\{\varphi^{\mathbf{N}}_{km}(\mathbf{x}) = \varphi^{N_1}_k(x_1)\varphi^{N_2}_m(x_2)\}^{(N_1-1)\times(N_2-1)}_{(k,m)=(1,1)}$ constitutes an orthogonal basis in $\mathbb{P}^{Di}_{\mathbf{N}}(\Omega)$ with the discrete scalar product $(\cdot,\cdot)_{\mathbf{N}}$ such that the associated eigenfunctions $\{\varphi^{N_1}_k(x_1)\}^{N_1-1}_{k=1}$ (resp.$\{\varphi^{N_2}_m(x_2)\}^{N_2-1}_{m=1}$) constitutes an orthogonal basis in $\mathbb{P}^{Di}_{N_1}(-1,1)$ (resp.$\mathbb{P}^{Di}_{N_2}(-1,1)$) with the discrete scalar product $(\cdot,\cdot)_{N_1}$ in x_1 (resp. $(\cdot,\cdot)_{N_2}$), see (2.17).

We observe that we can write the discrete eigenvalue problem (45) in the following equivalent form:

$$\begin{cases}\Delta\varphi^{\mathbf{N}} + \lambda^{\mathbf{N}}\varphi^{\mathbf{N}}_{km} = -\sum_{\mathbf{i}\in I_{\partial\Omega}} \dfrac{\partial^2\varphi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}(t,\mathbf{x_i})\Psi^{x_1}_{i_1}\Psi^{x_2}_{i_2} & \text{in } \mathbf{x}\in\Omega,\\ \varphi^{\mathbf{N}} = 0 & \text{on } \partial\Omega,\end{cases} \tag{46}$$

where $\mathbf{i} = (i_1, i_2)$ and $\Psi^{x_1}_{i_1}(x_1)$ (resp. $\Psi^{x_2}_{i_2}(x_2)$) are the Lagrange polynomial which are 1 at x_{1i_1} (resp. at x_{2i_2}) and 0 at all the other collocation points. In fact, this is easily seen by writing the polynomial $\Delta\varphi^{\mathbf{N}} + \lambda^{\mathbf{N}}\varphi^{\mathbf{N}}$ in the previous Lagrangian basis and using system (45).

Proof of Lemma 3.5.2.

As we did above in the proof of Lemma 2.5.2, we define $\Phi_{km} = (\varphi_{|k|}\varphi_{|m|}/(i\mu_k\mu_m), \varphi_{|k|}\varphi_{|m|})/\sqrt{2}$ for $(k,m)\in(\mathbb{Z}^*)^2 = (\mathbb{Z}\backslash\{0\})^2$. Note that $\{\Phi_{km}\}_{km\in\mathbb{Z}^2}$ is an orthonormal basis in $H^1_0\times L^2$. Thus, given $(\phi^0,\phi^1)\in H^1_0\times L^2$ we can write

$$(\phi^0,\phi^1) = \sum_{(k,m)\in(\mathbb{Z}^*)^2} a_{km}\Phi_{km}, \quad |(\phi^0,\phi^1)|^2_{H^1_0\times L^2} = \sum_{(k,m)\in(\mathbb{Z}^*)^2}|a_{km}|^2 < \infty, \tag{47}$$

for some Fourier coefficients $a_{km}\in\mathbb{C}$.

Analogously, we define $\Phi^{\mathbf{N}}_{km} = (\varphi^{N_1}_{|k|}\varphi^{N_2}_{|m|}/(i\mu^{N_1}_k\mu^{N_2}_m), \varphi^{N_1}_{|k|}\varphi^{N_2}_{|m|})/\sqrt{2}$ for $(|k|,|m|)\le(N_1,N_2)$, $k,m\neq 0$. Again, $\{\Phi^{\mathbf{N}}_{km}\}_{(|k|,|m|)\le(N_1,N_2)}$ is an orthonormal basis of in $\mathbb{P}^{\mathbf{N}}\times\mathbb{P}^{\mathbf{N}}$ with the scalar product,

$$\Big((v^{0,\mathbf{N}},v^{1,\mathbf{N}}),(w^{0,\mathbf{N}},w^{1,\mathbf{N}})\Big)^*_{\mathbf{N}} = (\nabla v^{0,\mathbf{N}},\nabla w^{0,\mathbf{N}})_{\mathbf{N}} + (v^{1,\mathbf{N}},w^{1,\mathbf{N}})_{\mathbf{N}},$$

whose associated norm is equivalent to the usual norm in $H^1_0\times L^2$. Let us consider

$$(\phi^{0,\mathbf{N}},\phi^{1,\mathbf{N}}) = \sum_{(|k|,|m|)\le(r(N_1),r(N_2))} a_{km}\Phi^{\mathbf{N}}_{km}. \tag{48}$$

From definition of $\Phi_{km},\Phi^{\mathbf{N}}_{km}$, we have

$$|\Phi_{km}-\Phi^{\mathbf{N}}_{km}|^2_{H^1_0\times L^2} = \left|\Big(\varphi_{|k|}\varphi_{|m|}/(i\mu_k\mu_m) - \varphi^{N_1}_{|k|}\varphi^{N_2}_{|m|}/(i\mu^{N_1}_k\mu^{N_2}_m), \varphi_{|k|}\varphi_{|m|} - \varphi^{N_1}_{|k|}\varphi^{N_2}_{|m|}\Big)/\sqrt{2}\right|^2_{H^1_0\times L^2}.$$

Now, adding and subtracting $\varphi^{N_1}_{|k|}\varphi_{|m|}/(i\mu^{N_1}_k\mu_m)$ in the first component and $\varphi^{N_1}_{|k|}\varphi_{|m|}$in the second component, we get

$$|\Phi_{km}-\Phi^{\mathbf{N}}_{km}|^2_{H^1_0\times L^2} = \left|\left(\frac{\varphi_{|m|}}{\mu_m}\Big(\frac{\varphi_{|k|}}{i\mu_k}-\frac{\varphi^{N_1}_{|k|}}{i\mu^{N_1}_k}\Big)+\frac{\varphi^{N_1}_{|k|}}{\mu^{N_1}_k}\Big(\frac{\varphi_{|m|}}{i\mu_m}-\frac{\varphi^{N_2}_{|m|}}{i\mu^{N_2}_m}\Big),\varphi_{|m|}\Big(\varphi_{|k|}-\varphi^{N_1}_{|k|}\Big)+\varphi^{N_1}_{|k|}\Big(\varphi_{|m|}-\varphi^{N_2}_{|m|}\Big)\right)/\sqrt{2}\right|^2_{H^1_0\times L^2}. \tag{49}$$

Therefore, from (49) and the convergence results in Theorem .0.1, we have

$$|(\phi^0,\phi^1)-(\phi^{0,\mathbf{N}},\phi^{1,\mathbf{N}})|^2_{H^1_0\times L^2} \leq \sup_{(|k|,|m|)\leq(r(N_1),r(N_2))}\Big(|\Phi_{km}-\Phi^{\mathbf{N}}_{km}|^2_{H^1_0\times L^2}\Big)\sum_{(|k|,|m|)\leq(r(N_1),r(N_2))}|a_{km}|^2 + \sum_{(|k|,|m|)>(r(N_1),r(N_2))}|a_{km}|^2 \to 0, \quad \text{as } \mathbf{N}\to\infty. \tag{50}$$

This concludes the proof of (3.50). Moreover, the solution of the continuous wave equation (3.3) is given by

$$(\phi(t,x),\phi_t(t,x)) = \sum_{(k,m)\in(\mathbb{Z}^*)^2} a_{km}e^{i\sqrt{\lambda_{km}}t}\Phi_{km}, \tag{51}$$

while the one associated to (3.17) with initial data $(\phi^{0,\mathbf{N}},\phi^{1,\mathbf{N}})$ is given by

$$(\phi(t,x),\phi^{\mathbf{N}}_t(t,x)) = \sum_{(|k|,|m|)\leq(r(N_1),r(N_2))} a_{km}e^{i\sqrt{\lambda^{\mathbf{N}}_{km}}t}\Phi^{\mathbf{N}}_{km}. \tag{52}$$

Again, (49) and the uniform convergence of the low frequencies stated in Theorem .0.1 allows us to obtain (3.51)-(3.53).

Proof of Lemma 3.5.1. We follow the idea in the proof of Lemma 3.5.2. Let us define $\hat{\Phi}_{km} = (i\mu_k\mu_m)\Phi_{km}$ for $k,m\in\mathbb{Z}^*=\mathbb{Z}\backslash\{0\}$, where Φ_{km} where introduced at the beginning of proof of the Lemma 3.5.2. Note that $\{\hat{\Phi}_{km}\}_{(k,m)\in\mathbb{Z}^2}$ is now an orthonormal basis in $L^2\times H^{-1}$. Thus, given $(u^0,u^1)\in L^2\times H^{-1}$ we can write

$$(u^0,u^1) = \sum_{(k,m)\in(\mathbb{Z}^*)^2}\hat{b}_{km}\hat{\Phi}_{km}, \quad |(u^0,u^1)|^2_{L^2\times H^{-1}} = \sum_{(k,m)\in(\mathbb{Z}^*)^2}|\hat{b}_{km}|^2<\infty, \tag{53}$$

for some Fourier coefficients $\hat{b}_{km}\in\mathbb{C}$. Analogously, we define $\hat{\Phi}^{\mathbf{N}}_{km} = i\mu^{N_1}_k\mu^{N_2}_m\Phi^{\mathbf{N}}_{km}$ for

$(|k|,|m|) \leq (N_1, N_2),\ k, m \neq 0$. Let us consider

$$(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) = \sum_{(|k|,|m|)\leq(r(N_1),r(N_2))} \hat{b}_{km}\hat{\Phi}^{\mathbf{N}}_{km}. \tag{54}$$

Note that if (u^0, u^1) are continuous functions, the sequence that we choose $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}})$ are the polynomial which coincides with the value of (u^0, u^1) at the collocation points.

Arguing as in (50), the convergence result in (3.37) can be reduced to prove

$$\sup_{(|k|,|m|)\leq(r(N_1),r(N_2))} \left(|\hat{\Phi}_{km} - \hat{\Phi}^{\mathbf{N}}_{km}|^2_{L^2\times H^{-1}}\right) \to 0, \quad \text{as } \mathbf{N} \to \infty.$$

Note that

$$|\hat{\Phi}_{km} - \hat{\Phi}^{\mathbf{N}}_{km}|^2_{L^2\times H^{-1}} = |\varphi_{|km|} - \varphi^{\mathbf{N}}_{|km|}|^2_{L^2} + |i\mu_k\mu_m\varphi_{|km|} - i\mu_k^{N_1}\mu_m^{N_2}\varphi^{\mathbf{N}}_{|km|}|^2_{H^{-1}}.$$

The first term here can be estimated uniformly for $(|k|,|m|) \leq (r(N_1), r(N_2))$ by (49), Theorem .0.1 and converges to zero as $N_1, N_2 \to \infty$. Concerning the second term, we use the fact that the eigenfunctions φ_{km} (resp. $\varphi^{\mathbf{N}}_{|km|}$) satisfy (44) (resp. (46)) together with the isometry of the Laplacian between H^1_0 and H^{-1}. Therefore, we have

$$\begin{aligned} |i\mu_k\mu_m\varphi_{|km|} - i\mu_k^{N_1}\mu_m^{N_2}\varphi^{\mathbf{N}}_{|km|}|^2_{H^{-1}} &= \left| \frac{\Delta\varphi_{|km|}}{\mu_k\mu_m} - \frac{\Delta\varphi^{\mathbf{N}}_{|km|} - \sum_{\mathbf{i}\in I_{\partial\Omega}} \frac{\partial^2\varphi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}(t,\mathbf{x}_\mathbf{i})\Psi^{x_1}_{i_1}\Psi^{x_2}_{i_2}}{\mu_k^{N_1}\mu_m^{N_2}} \right|^2_{H^{-1}} \\ &\leq \left| \frac{\Delta\varphi_{|km|}}{\mu_k\mu_m} - \frac{\Delta\varphi^{\mathbf{N}}_{|km|}}{\mu_k^{N_1}\mu_m^{N_2}} \right|^2_{H^{-1}} + \sum_{s=1}^{4} \frac{\left| \frac{\partial^2\varphi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}|_{\Gamma_s}\Psi^{\xi_1}_{\mathbf{N}} \right|^2_{H^{-1}}}{\lambda_k^{N_1}\lambda_m^{N_2}}. \end{aligned} \tag{55}$$

Note that the last term in (55) came from (3.33) and the fact that $\sum_{\mathbf{i}\in I_{\partial\Omega}} \frac{\partial^2\varphi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}(t,\mathbf{x}_\mathbf{i})\Psi^{x_1}_{i_1}\Psi^{x_2}_{i_2} = \sum_{s=1}^4 \frac{\partial^2\varphi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}|_{\Gamma_s}\Psi^{\xi_1}_{\mathbf{N}}$ (see (3.33)), where $\Psi^{\xi_1}_{\mathbf{N}}$ is defined by

$$\begin{cases} \Psi^{\xi_1}_{\mathbf{N}} = \Psi^{x_1}_{N_1} \quad \text{on } \Gamma_1,\ \Psi^{\xi_1}_{\mathbf{N}} = \Psi^{x_1}_{0} \quad \text{on } \Gamma_3, \\ \Psi^{\xi_1}_{\mathbf{N}} = \Psi^{x_2}_{N_2} \quad \text{on } \Gamma_2,\ \Psi^{\xi_1}_{\mathbf{N}} = \Psi^{x_2}_{0} \quad \text{on } \Gamma_4. \end{cases} \tag{56}$$

Concerning this term, it is enough to consider one of the 4 faces of the domain $\Omega = (-1,1)^2 \subset$

$\mathbb{R}^2$. We focus on Γ_1. By using the Poincaré inequality, we find

$$\left|\frac{\partial^2\varphi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}|_{\Gamma_1}\Psi^{x_1}_{N_1}\right|^2_{H^{-1}} = \sup_{v\in H^1_0}\left|\left\langle\frac{\partial^2\varphi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}\right|_{\Gamma_1}\Psi^{x_1}_{N_1}, v\right\rangle_{H^{-1},H^1_0}\right|^2 \leq \left|\frac{\partial^2\varphi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}|_{\Gamma_1}\Psi^{x_1}_{N_1}\right|^2_{L^2}. \tag{57}$$

Duo to the fact that $|\varphi^{N_2}_{|m|}|_{L^2(-1,1)} = 1$, we find

$$\left|\frac{\partial^2\varphi^{\mathbf{N}}}{\partial^2\boldsymbol{\nu}}|_{\Gamma_1}\Psi^{x_1}_{N_1}\right|^2_{L^2} = \left|\varphi^{N_2}_{|m|}(x_2)\varphi^{N_1}_{|k|,x_1x_1}(1)\Psi^{x_1}_{N_1}(x_1)\right|^2_{L^2}. \tag{58}$$

On the other hand, the first term on the right-hand side of (55) can be estimated as follows by using integration by parts, the Cauchy Schwarz and the Poincaré inequality,

$$\begin{aligned}\left|\frac{\Delta\varphi_{|km|}}{\mu_k\mu_m} - \frac{\Delta\varphi^{\mathbf{N}}_{|km|}}{\mu_k^{N_1}\mu_m^{N_2}}\right|^2_{H^{-1}} &= \sup_{v\in H^1_0}\frac{\left|\left\langle\Delta\varphi_{|km|}/(i\mu_k\mu_m) - \Delta\varphi^{N}_{|km|}/(i\mu_k^{N_1}\mu_m^{N_2}), v\right\rangle_{H^{-1},H^1_0}\right|^2}{|v|^2_{H^1_0}}\\ &\leq \left|\frac{\varphi_{|km|}}{\mu_k\mu_m} - \frac{\varphi^{\mathbf{N}}_{|km|}}{\mu_k^{N_1}\mu_m^{N_2}}\right|^2_{H^1_0}\\ &\leq \left|\frac{\varphi_{|k|}}{\mu_k}\Big(\frac{\varphi_{|m|}}{\mu_m} - \frac{\varphi^{N_2}_{|m|}}{\mu_m^{N_2}}\Big)\right|^2_{H^1_0} + \left|\frac{\varphi^{N_2}_{|m|}}{\mu_m^{N_2}}\Big(\frac{\varphi_{|k|}}{\mu_k} - \frac{\varphi^{N_1}_{|k|}}{\mu_k^{N_1}}\Big)\right|^2_{H^1_0}.\end{aligned} \tag{59}$$

Finally, that converges in (55) uniformly to zero for $(|k|,|m|) \leq (r(N_1), r(N_2))$ as a consequence of(58), (59) and Theorem .0.1, the uniform bound of $|\Psi^{x_1}_{N_1}|_{L^2}$,$|\Psi^{x_1}_0|_{L^2}$ and $|\Psi^{x_2}_{N_2}|_{L^2}$,$|\Psi^{x_2}_0|_{L^2}$.

We now prove (3.38), if write any $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}})$ as $\sum_{(|k|,|m|)\leq(N_1,N_2)} a^{\mathbf{N}}_{km}\Phi^{\mathbf{N}}_{km}$ and by the orthogonality of the eigenfunctions $\varphi^{\mathbf{N}}_{km}$ with respect to the discrete scalar product $(\cdot,\cdot)_{\mathbf{N}}$ and the duality product (3.15), we have

$$\begin{aligned}&\left|\left\langle(\varphi^{0,\mathbf{N}}, \varphi^{1,\mathbf{N}}), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}})\right\rangle_{\mathbf{N}}\right| = \left|\Big\langle\sum_{(|k|,|m|)=(1,1)}^{(r(N_1),r(N_2))}\hat{b}_{km}\hat{\Phi}^{\mathbf{N}}_{km}, \sum_{(|k|,|m|)=(1,1)}^{(N_1,N_2)} a^{\mathbf{N}}_{km}\Phi^{\mathbf{N}}_{km}\Big\rangle_{\mathbf{N}}\right|\\ &\leq \left|\sum_{(|k|,|m|)=(1,1)}^{(r(N_1),r(N_2))}\hat{b}_{km}a^{\mathbf{N}}_{km}\right| \leq \left(\sum_{(|k|,|m|)=(1,1)}^{(r(N_1),r(N_2))}|\hat{b}_{km}|^2\right)^{1/2}\left(\sum_{(|k|,|m|)=(1,1)}^{(r(N_1),r(N_2))}|a^{\mathbf{N}}_{km}|^2\right)^{1/2}\\ &\leq |(u^0,u^1)|_{L^2\times H^{-1}}\|(\nabla\varphi^{0,\mathbf{N}}, \varphi^{1,\mathbf{N}})\|_{\mathbf{N}\times\mathbf{N}}.\end{aligned}$$

Finally, we prove (3.39). We assume now that $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \to (\phi^0, \phi^1)$ in $H_0^1 \times L^2$ that we write as $(\phi^0, \phi^1) = \sum_{(k,m)\in(\mathbb{Z}^*)^2} a_{km}\Phi_{km}$. We have

$$\Big\langle(\varphi^{0,\mathbf{N}}, \varphi^{1,\mathbf{N}}), (u^{0,\mathbf{N}}, u^{1,\mathbf{N}})\Big\rangle_{\mathbf{N}} = \Big\langle \sum_{(|k|,|m|)=(1,1)}^{(r(N_1),r(N_2))} \hat{b}_{km}\hat{\Phi}_{km}^{\mathbf{N}}, \sum_{(|k|,|m|)=(1,1)}^{(N_1,N_2)} a_{km}^{\mathbf{N}}\Phi_{km}^{\mathbf{N}}\Big\rangle_{\mathbf{N}},$$

$$\Big\langle(\varphi^0, \varphi^1), (u^0, u^1)\Big\rangle = \Big\langle \sum_{(k,m)\in(\mathbb{Z}^*)^2} \hat{b}_{km}\hat{\Phi}_{km}, \sum_{(k,m)\in(\mathbb{Z}^*)^2} a_{km}\Phi_{km}\Big\rangle.$$

The convergence results in Theorem .0.1 allow to prove the estimate

$$|a_{km}^{\mathbf{N}} - a_{km}| \leq C\; min(N_1, N_2)^{-1/4}, \quad (|k|, |m|) \leq (r(N_1), r(N_2))$$

and, using the strong convergence $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \to (\phi^0, \phi^1)$, we get

$$\Big\langle \sum_{(|k|,|m|)=(1,1)}^{(r(N_1),r(N_2))} \hat{b}_{km}\hat{\Phi}_k^N, \sum_{(|k|,|m|)=(1,1)}^{(N_1,N_2)} a_{km}^{\mathbf{N}}\Phi_{km}^{\mathbf{N}}\Big\rangle_{\mathbf{N}} \to \Big\langle \sum_{(k,m)\in(\mathbb{Z}^*)^2} \hat{b}_{km}\hat{\Phi}_{km}, \sum_{(k,m)\in(\mathbb{Z}^*)^2} a_{km}\Phi_{km}\Big\rangle.$$

This concludes the proof of (3.39).

Appendix C

In this section, we give a technical result used in Section 5.5.3. We prove that the hypothesis **H** introduced in Section 5.2 holds when considering the projection method described in Section 5.5.2. In particular, the following holds:

Lemma .0.3. *Given* $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, there exists a sequence of* $(\varphi^{0,N}, \varphi^{1,N}) \in S^N \times S^N$ *such that,*

$$(\varphi^{0,N}, \varphi^{1,N}) \to (\varphi^0, \varphi^1), \ \textit{in}\ H_0^1(\Omega) \times L^2(\Omega), \quad \textit{as}\ N \to \infty. \tag{60}$$

Furthermore, if φ *is the solution of*

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \textit{in}\ Q \\ \varphi = 0 & \textit{on}\ (0,T) \times \partial\Omega \\ (\varphi(0,\mathbf{x}), \varphi_t(0,\mathbf{x})) = (\varphi^0, \varphi^1) & \textit{in}\ \Omega, \end{cases} \tag{61}$$

with associated observation $y = \dfrac{\partial\varphi}{\partial\nu}|_\Gamma$*, and* φ^N *is the solution of*

$$\begin{cases} \varphi_{tt}^N - \Delta\varphi^N = 0 & \textit{in}\ Q \\ \varphi^{\mathbf{N}} \in S^N & \textit{for}\ \ t \in (0,T) \\ (\varphi^N(0,\mathbf{x}), \varphi_t^N(0,\mathbf{x})) = (\varphi^{0,N}, \varphi^{1,N}) & \textit{in}\ \in\ \Omega, \end{cases} \tag{62}$$

with associated observation $y^N = \dfrac{\partial\varphi^N}{\partial\nu}|_\Gamma$*, the following holds:*

$$\varphi^N \to \varphi, \ \textit{in}\ C^0((0,T); H_0^1(\Omega)) \cap C^1((0,T); H_0^1(\Omega)), \quad \textit{as}\ N \to \infty, \tag{63}$$

$$\frac{\partial\varphi^N}{\partial\nu} \to \frac{\partial\varphi}{\partial\nu}, \ \textit{in}\ L^2((0,T);\Gamma), \quad \textit{as}\ N \to \infty. \tag{64}$$

Proof. First, let us consider $\{\varphi_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$ constituted by eigenfunctions of (44) and $\{\lambda_k\}_{(k)\in\mathbb{N}^2}$ be corresponding eigenvalues. We define the scaled spaces $H^\alpha(\Omega)$ as

$$H^\alpha = \left\{ \sum_{k=1}^{N} c_k \varphi_k(\mathbf{x}),\ \text{with} \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |c_k|^2 < \infty \right\}.$$

Note that $H^0 = L^2(\Omega)$ and $H^1 = H_0^1(\Omega)$. We also introduce the finite-dimensional space S^N generated by the first N eigenfunctions, i.e.

$$S^N = \left\{ \sum_{k=1}^{N} c_k \varphi_k(\mathbf{x}),\ \text{with} c_k \in \mathbb{R} \right\} \subset H^\alpha,\ \alpha = 0, 1,$$

On the other hand, we assume that $\mu_k = \sqrt{\lambda_k}$ and $\Phi_k = (\varphi_k/\mu_k, \varphi_k)/\sqrt{2}$ for $k \in \mathbb{N}\backslash\{0\}$. Note that Φ_k is an orthonormal basis in $H_0^1 \times L^2(\Omega)$. We consider the projection operator

$$P^N : H^\alpha(\Omega) \longrightarrow S^N,$$

and define

$$(\phi^{0,N}, \phi^{1,N}) = (P^N \phi^0, P^N \phi^1) = \sum_{k \leq N} a_k \Phi_k. \tag{65}$$

Then, by Parseval identity

$$|(\phi^0, \phi^1) - (\phi^{0,N}, \phi^{1,N})|^2_{H_0^1 \times L^2} = \sum_{|k| > N} |a_k|^2 \to 0, \quad \text{as } N \to \infty.$$

This concludes the proof of (60). Moreover, the solution of the adjoint wave equation (61) is given by

$$(\phi(t, \mathbf{x}), \phi_t(t, \mathbf{x})) = \sum_{k \in \mathbb{Z}^*} a_k e^{i\mu_k t} \Phi_k, \tag{66}$$

where $\mu_{|k|} = \mu_k$ and $\Phi_k = (\varphi_{|k|}/\mu_k, \varphi_{|k|})/\sqrt{2}$ for $k \in \mathbb{Z}^*$, while the one associated to (62) with initial data $(\phi^{0,N}, \phi^{1,N})$ is given by

$$(\phi^N(t, \mathbf{x}), \phi_t^N(t, \mathbf{x})) = \sum_{|k| \leq N} a_k e^{i\mu_k t} \Phi_k. \tag{67}$$

The proof of (63) is then straightforward and (64) holds in view of the continuity of the L^2-norm of the normal derivatives for H_0^1 functions.