Finite Element Method: Applications based on Octave/MATLAB

Part I: Introduction

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Outline

1. Course Description
2. Previous Concepts
   - Partial Differential Equation
   - Functional Spaces L^2 and H^1
   - Finite Element Spaces
   - Numeric Integration
   - 2D Mesh Generation

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  - Wave equation (**Acoustics**).
  - Structural Mechanics (**Elasticity**).
  - Navier-Stokes equation (**CFD**).
  - **Coupled** problems: thermo-mechanical, energy equation, inverse design, ...
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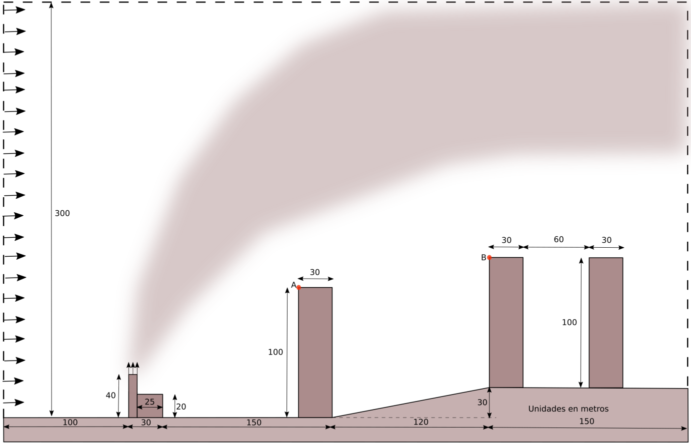
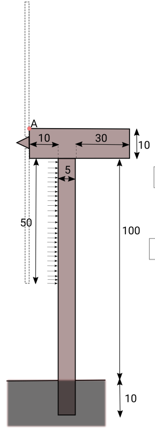
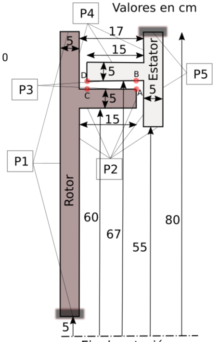
  - **Orthotropic** media.
- Resources:
  - **"Técnicas Numéricas Avanzadas"** notes, ETSII-UPM.
  - **Octave** scripts and examples: Pre/Post and solving in the **same environment**.
  - **Proposed** works/projects.
  - This and more presentations.

Some Proposed Works (I)

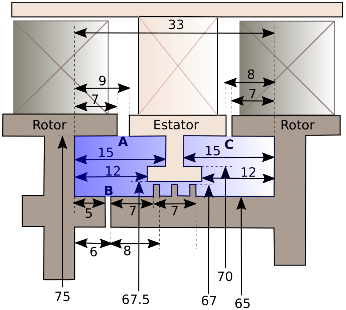




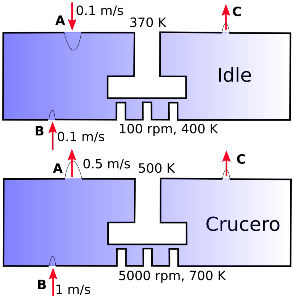
Valores geométricos en metros

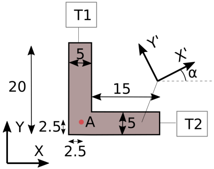


Some Proposed Works (II)

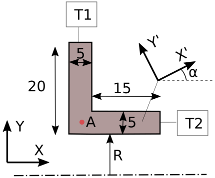


Valores en cm



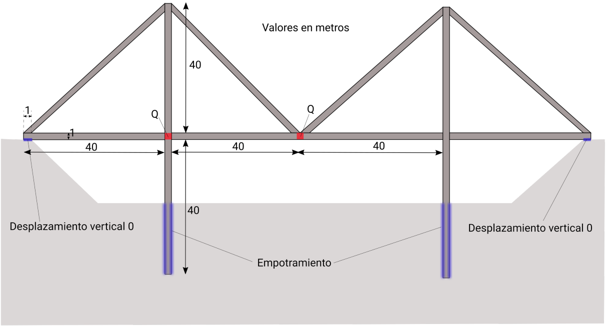


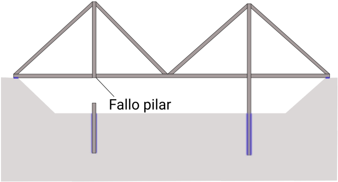
Geometría plana
Valores geométricos en cm

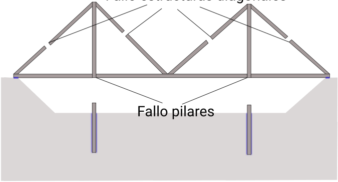


Geometría axilsimétrica
Valores geométricos en cm

Some Proposed Works (III)







Content

- Part I: Introduction
- Part II: Elliptic problems
- Part III: Parabolic and Hyperbolic problems
- Part IV: Structural Mechanics (Elasticity)
- Part V: Computational Fluid Dynamics (Incompressible Flows)

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Linear Second Order Partial Differential Equation

Be x_i independent variables, $u\left(x_1, \cdots, x_n\right)$ the unknown function and coefficients a_{ij}, b_k, g and f not depending on u:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k} + g \cdot u = f$$

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Be $\mathbf{A} = (a_{ij})$ the **symmetric** matrix ($\Rightarrow$eigenvalues $\lambda_k \in \mathbb{R}$) formed with the second derivatives coefficients. Equation is classified as:

- *Elliptic:* All λ_k have the same sign. None zero.
- *Parabolic:* All λ_k have the same sign, except only one $\lambda_i = 0$.
- *Hyperbolic:* All λ_k have the same sign, except only one λ_i with opposite sign.

Sobolev Functional Spaces

Be the open set $D \subset \mathbb{R}^d$. Then we define the following spaces:

$$L^2(D) = \left\{ f : D \to \mathbb{R} : \int_D |f|^2 < \infty \right\}$$
$$H^1(D) = \left\{ f : D \to \mathbb{R} : f \in L^2(D) \quad and \quad \frac{\partial f}{\partial x_i} \in L^2(D), 1 \leq i \leq d \right\}$$
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$$\langle f, g \rangle_{L^2(D)} = \int_D fg.$$
$$\langle f, g \rangle_{H^1(D)} = \int_D fg + \sum_{i=1}^d \int_D \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} = \langle f, g \rangle_{L^2(D)} + \langle \nabla f, \nabla g \rangle_{L^2(D)}.$$

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And the associated **norms** $\|\|$:

$$\|f\|_{L^2(D)} = \left(\int_D |f|^2 \right)^{1/2} \qquad \|f\|_{H^1(D)} = \left(\|f\|_{L^2(D)}^2 + \|\nabla f\|_{L^2(D)}^2 \right)^{1/2}$$

1D Linear Finite Element Space

Mesh or partition $D_h = \{K_i\}_{1 \le i \le n}$

Be the interval $[c, d]$. Be *nodes* $\{x_i\}_{i=1,2,\cdots,n+1} \subset [c, d]$ with $c = x_1 < \cdots < x_{n+1} = d$. Be $N_e = n$ *elements* $K_i = [x_i, x_{i+1}]$ such as

$$[c, d] = \cup_{i=1}^{n} K_i,$$
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Length of each element $h_i = x_{i+1} - x_i$ and $h = \max_{1 \le i \le n} h_i$. If $h_i = h$ for all $1 \le i \le N_e \Rightarrow$ uniform mesh with $h = (d - c)/n$.

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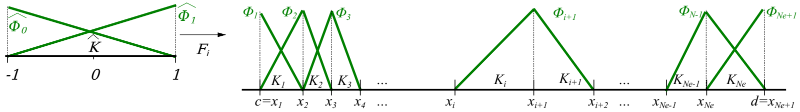

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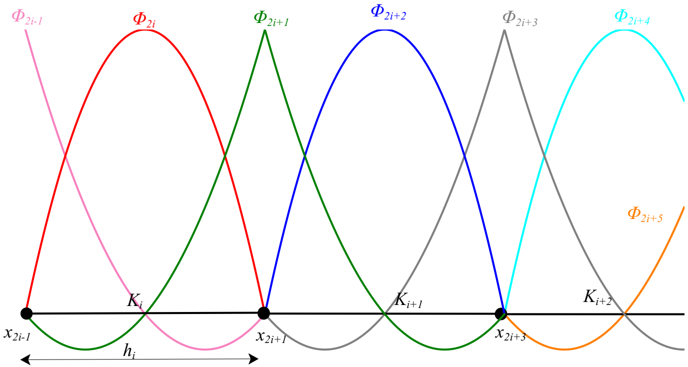

2D Finite Element Spaces

Be the open set $D \subset \mathbb{R}^2$ with polygonal boundary ∂D. $D_h = \{T_i\}_{1 \leq i \leq N_e}$ is a **conformal** mesh if

$$D \cup \partial D = \cup_{i=1}^{N_e} T_i, \qquad \forall i \neq j \Rightarrow T_i \cap T_j = \begin{cases} (x_k, y_k) & or \\ \Gamma_{ij} & or \\ \emptyset \end{cases}$$

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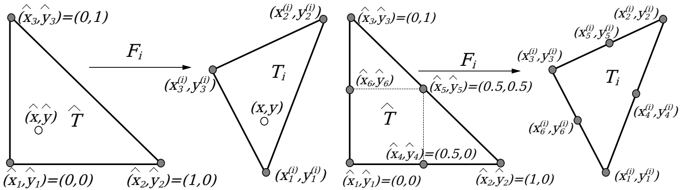

Numeric Integration: Newton-Côtes

$f : [a, b] \to \mathbb{R},\ a \leq x_0 < x_1 < \cdots < x_n \leq b,\ f(x_i) := f_i$

$$\int_a^b f(x)\,dx = Q[f] + E[f] \qquad Q[f] := \sum_{i=0}^{n} \omega_i f_i$$

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Degree of accuracy m for quadrature Q:

- $\forall p_n$ polynomial of grade $n \leq m \Rightarrow E[p_n] = 0$.
- $\exists p_{m+1}$ such as $E[p_{m+1}] \neq 0$.

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Newton-Côtes quadratures for $\int_a^b f$

- *Rectangle* $(m = 0)$: $Q[f] = (b - a) f_0$ or $Q[f] = (b - a) f_1$
- *Trapezoid* $(m = 1)$: $Q[f] = (b - a)(f_0 + f_1)/2$
- *Simpson* $(m = 3)$: $Q[f] = (b - a)(f_0 + 4f_1 + f_2)/6$

Numeric Integration: Gauss-Legendre 1D

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(a + \frac{b-a}{2}(t+1)\right)\frac{b-a}{2}dt = \int_{-1}^{1} g(t)\,dt = \sum_{i=0}^{n} \omega_i g(x_i) + E[g]$$

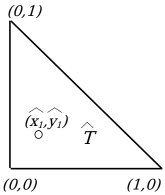
m	x_i	ω_i	$E[f] = E[g]$
1	0	2	$\frac{g''(c)}{3}$
3	$\pm 1/\sqrt{3}$	1	$\frac{g^{(4)}(c)}{135}$
5	0 $\pm\sqrt{3/5}$	8/9 5/9	$\frac{g^{(6)}(c)}{15750}$
7	$\pm\sqrt{\frac{1}{7}\left(3-2\sqrt{\frac{6}{5}}\right)}$ $\pm\sqrt{\frac{1}{7}\left(3+2\sqrt{\frac{6}{5}}\right)}$	0.6521451548625461 0.3478548451374538	$\frac{g^{(8)}(c)}{3472875}$
9	0 ± 0.5384693101056831 ± 0.9061798459386640	0.5688888888888889 0.4786286704993665 0.2369268850561891	$\frac{g^{(10)}(c)}{1237732650}$

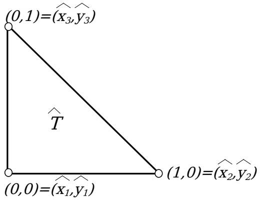
$$-1 < c < 1$$

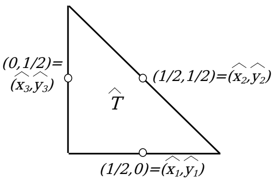
Numeric Integration: Gauss-Legendre 2D

$$\int\int_{T_i} f(x,y)\,dxdy = \int\int_{F_i^{-1}(T_i)} f(F_i(\hat{x},\hat{y}))\,|\mathbf{J}_{F_i}(\hat{x},\hat{y})|\,d\hat{x}d\hat{y} = \int\int_{\hat{K}} \hat{g}^{(i)}(\hat{x},\hat{y})\,d\hat{x}d\hat{y}$$

- One point ($m=1$): $Q[\hat{g}] = (\hat{g}(1/3,1/3))/2$
- Three points ($m=1$): $Q[\hat{g}] = (\hat{g}(0,0)+\hat{g}(1,0)+\hat{g}(0,1))/6$
- Three points ($m=2$): $Q[\hat{g}] = (\hat{g}(1/2,0)+\hat{g}(1/2,1/2)+\hat{g}(0,1/2))/6$



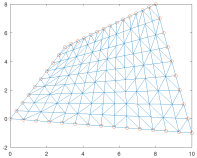


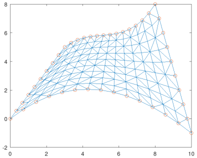


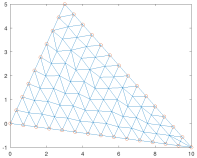
2D mesh Generation

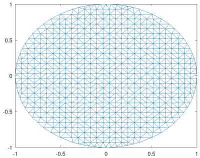
Primitives

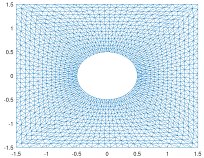
`mesh_block`, `mesh_block_hermite`, `mesh_block_curve`, `mesh_triangle`, `mesh_circle`, `mesh_hole`, `mesh_profile`

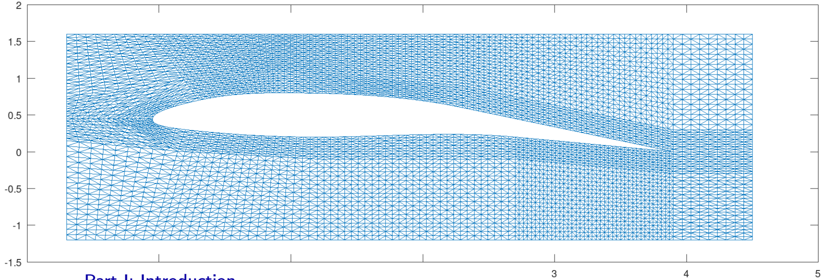






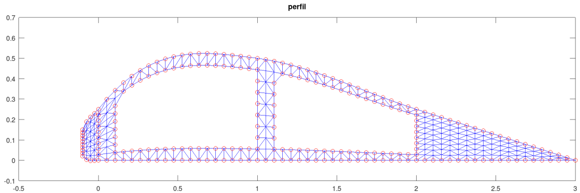


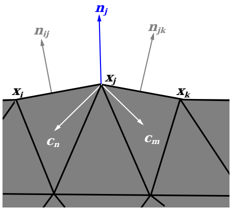




2D mesh Generation

Utilities: `mesh_meld`, `mesh_boundary`, `mesh_normals`





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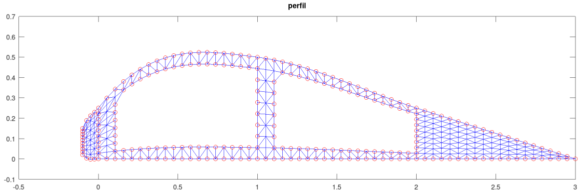
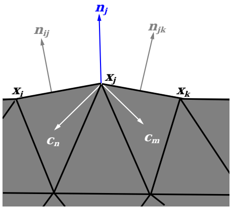
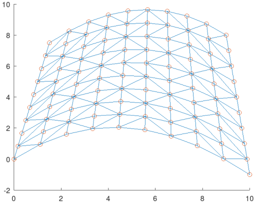
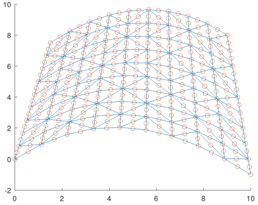
Utilities: mesh_meld, mesh_boundary, mesh_normals

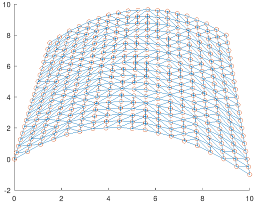




Quadratic mesh: mesh_quadratic, mesh_q2l





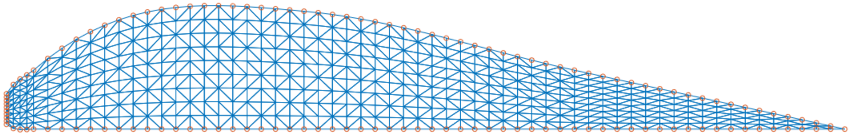


2D mesh Generation: Example

```
[x y tri] = mesh_block_hermite([0 0 1 1],[0 0.25 0.5 0],[0 1 -0.1 0],10,20);
[x1 y1 tri1] = mesh_block_hermite([1 1 2 2],[0 0.5 0.25 0],[0 -0.1 -0.25 0],10,20);
[x y tri] = mesh_meld(x,y,tri,x1,y1,tri1);
[x1 y1 tri1] = mesh_triangle([2 2 3],[0 0.25 0],10,20);
[x y tri] = mesh_meld(x,y,tri,x1,y1,tri1);
[x1 y1 tri1] = mesh_block_hermite([-0.1 -0.1 0 0],[0.02 0.15 0.25 0],[-1 2 1 0],10,5);
[x y tri] = mesh_meld(x,y,tri,x1,y1,tri1);
f = mesh_boundary(tri);
figure 1; triplot(tri, x, y); hold on; plot(x(f),y(d),'o');
```

2D mesh Generation: Example

```
[x y tri] = mesh_block_hermite([0 0 1 1],[0 0.25 0.5 0],[0 1 -0.1 0],10,20);
[x1 y1 tri1] = mesh_block_hermite([1 1 2 2],[0 0.5 0.25 0],[0 -0.1 -0.25 0],10,20);
[x y tri] = mesh_meld(x,y,tri,x1,y1,tri1);
[x1 y1 tri1] = mesh_triangle([2 2 3],[0 0.25 0],10,20);
[x y tri] = mesh_meld(x,y,tri,x1,y1,tri1);
[x1 y1 tri1] = mesh_block_hermite([-0.1 -0.1 0 0],[0.02 0.15 0.25 0],[-1 2 1 0],10,5);
[x y tri] = mesh_meld(x,y,tri,x1,y1,tri1);
f = mesh_boundary(tri);
figure 1; triplot(tri, x, y); hold on; plot(x(f),y(d),'o');
```



End Part I

Thank you! Questions?