Finite Element Method: Applications based on Octave/MATLAB

Part II: Elliptic Problems

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Outline

1D Elliptic Equation: Weak Formulation

Original/Strong Problem

$$au(x) + b\frac{du}{dx} - k\frac{d^2u(x)}{dx^2} = f(x) \quad x \in (c, d), \ +b.c.$$

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Variational/Weak Formulation: Be solution $u \in H^1(D)$ that fulfills

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v is a *test* function. Integrating by parts the second derivative term:

$$a\int_c^d uv \cdot dx + b\int_c^d u'v + k\int_c^d ku'v' = \int_c^d fv + k\left[u'(d)\,v(d) - u'(c)\,v(c)\right]$$

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$$A(u,v) = L(v) \qquad \forall v \in H^1(D)$$

1D Elliptic Equation: Finite Element Method

FEM Formulation: Be solution $u_h \in V_h \subset H^1(D)$ that fulfills

$$A(u_h, v_h) = L(v_h) \qquad \forall v_h \in V_h = \{v_h \in C([c,d]) : v_h|_{K_i} \in P_m(K_i), \quad 1 \leq i \leq N_e\}$$

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Be $u_h(x) = \sum_{j=1}^{N} u_j \phi_j(x)$ and $v_h = \phi_i(x)$ for $1 \leq i \leq N$

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- Applying the **Lax-Milgram** theorem:
  - Depending on b.c.: $\exists! u$ and $\exists! u_h$, solutions for weak formulation and FEM.
  - If $u \in H^1$ is a strong solution $\Rightarrow$ also is a weak solution.
  - If $u \in V_h$ is a strong solution $\Rightarrow$ also is a FEM solution.

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- Being $h < 1$ the mesh size, m the polynomial degree for V_h and u and u_h the solutions for weak and FEM formulations, then $\exists K > 0$ such as

$$\|u - u_h\|_{L^2(D)} \leq K \cdot h^{m+1} \left\| u^{(m+1)} \right\|_{L^2(D)}, \quad \|u' - u_h'\|_{L^2(D)} \leq K \cdot h^{m} \left\| u^{(m+1)} \right\|_{L^2(D)}.$$

1D Elliptic Equation: FEM arrays

$$\sum_{j=1}^{N} u_j A(\phi_j, \phi_i) = L(\phi_i)$$

Linear System $\mathbf{Au} = \mathbf{d}$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$

$$\mathbf{A} = a\mathbf{M} + b\mathbf{C} + k\mathbf{R} \quad \mathbf{d} = \mathbf{Mf}$$

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- **Mass** Matrix: $\mathbf{M} = (m_{ij}) = \int_c^d \phi_j \phi_i$
- **Convection** Matrix: $\mathbf{C} = (c_{ij}) = \int_c^d \phi_j' \phi_i$
- **Stiffness** Matrix: $\mathbf{R} = (r_{ij}) = \int_c^d \phi_j' \phi_i'$

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$$u(c) = g_1 \quad u(d) = g_2$$

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Original Linear System: $\mathbf{Au} = \mathbf{d}$

$$\left.\begin{array}{c} a_{11}u_1 + a_{12}u_2 + \cdots + a_{1N-1}u_{N-1} + a_{1N}u_N = d_1 - k\,(c)\,u'\,(c) \\ a_{21}u_1 + a_{22}u_2 + \cdots + a_{2N-1}u_{N-1} + a_{2N}u_N = d_2 \\ \vdots \\ a_{N-1,1}u_1 + a_{N-1,2}u_2 + \cdots + a_{N-1,N-1}u_{N-1} + a_{N-1,N}u_N = d_{N-1} \\ a_{N1}u_1 + a_{N2}u_2 + \cdots + a_{NN-1}u_{N-1} + a_{NN}u_N = d_N + k\,(d)\,u'\,(d) \end{array}\right\}$$

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a_{21}u_1 + a_{22}u_2 + \cdots + a_{2N-1}u_{N-1} + a_{2N}u_N &= d_2 \\
&\vdots \\
a_{N-1,1}u_1 + a_{N-1,2}u_2 + \cdots + a_{N-1,N-1}u_{N-1} + a_{N-1,N}u_N &= d_{N-1} \\
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\end{aligned}\right\}$$

Modified Linear System $\widetilde{\mathbf{A}}\mathbf{u} = \widetilde{\mathbf{d}}$

$$\left.\begin{aligned}
1u_1 + 0u_2 + \cdots + 0u_{N-1} + 0u_N &= g_1 \\
0u_1 + a_{22}u_2 + \cdots + a_{2N-1}u_{N-1} + 0u_N &= d_2 - a_{21}g_1 - a_{2N}g_2 \\
&\vdots \\
0u_1 + a_{N-1,2}u_2 + \cdots + a_{N-1,N-1}u_{N-1} + 0u_N &= d_{N-1} - a_{N-1,1}g_1 - a_{N-1,N}g_2 \\
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$$\left.\begin{array}{c} (a_{11} - k(c)\,\alpha_1/\beta_1)\,u_1 + a_{12}u_2 + \cdots + a_{1N-1}u_{N-1} + a_{1N}u_N = d_1 - k(c)\,g_1/\beta_1 \\ a_{21}u_1 + a_{22}u_2 + \cdots + a_{2N-1}u_{N-1} + a_{2N}u_N = d_2 \\ \vdots \\ a_{N-1,1}u_1 + a_{N-1,2}u_2 + \cdots + a_{N-1,N-1}u_{N-1} + a_{N-1,N}u_N = d_{N-1} \\ a_{N1}u_1 + a_{N2}u_2 + \cdots + a_{NN-1}u_{N-1} + (a_{NN} + k(d)\,\alpha_2/\beta_2)\,u_N = d_N + k(d)\,g_2/\beta_2 \end{array}\right\}$$

Matrices for 1D Linear Elements

$$\hat{\mathbf{M}} = \begin{pmatrix} \int_{-1}^{1} \hat{\phi}_0^2 & \int_{-1}^{1} \hat{\phi}_0 \hat{\phi}_1 \\ \int_{-1}^{1} \hat{\phi}_0 \hat{\phi}_1 & \int_{-1}^{1} \hat{\phi}_1^2 \end{pmatrix} \hat{\mathbf{C}} = \begin{pmatrix} \int_{-1}^{1} \hat{\phi}_0 \hat{\phi}_0{}' & \int_{-1}^{1} \hat{\phi}_0 \hat{\phi}_1{}' \\ \int_{-1}^{1} \hat{\phi}_1 \hat{\phi}_0{}' & \int_{-1}^{1} \hat{\phi}_1 \hat{\phi}_1{}' \end{pmatrix} \hat{\mathbf{R}} = \begin{pmatrix} \int_{-1}^{1} \hat{\phi}'^2_0 & \int_{-1}^{1} \hat{\phi}'_0 \hat{\phi}'_1 \\ \int_{-1}^{1} \hat{\phi}'_0 \hat{\phi}'_1 & \int_{-1}^{1} \hat{\phi}'^{\,2}_1 \end{pmatrix}$$

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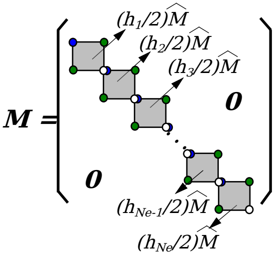

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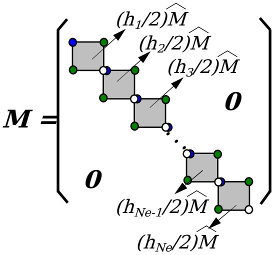

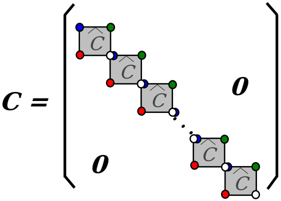

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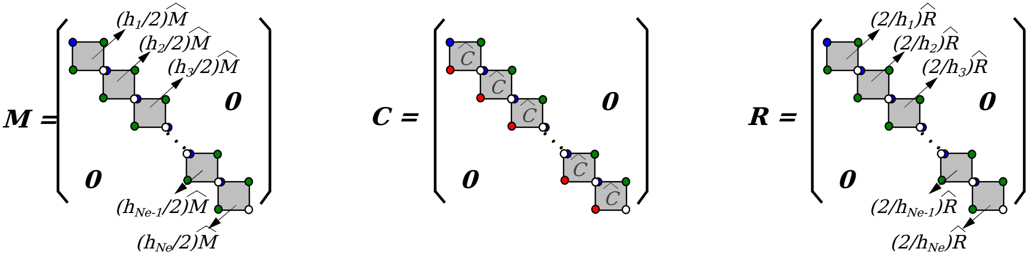

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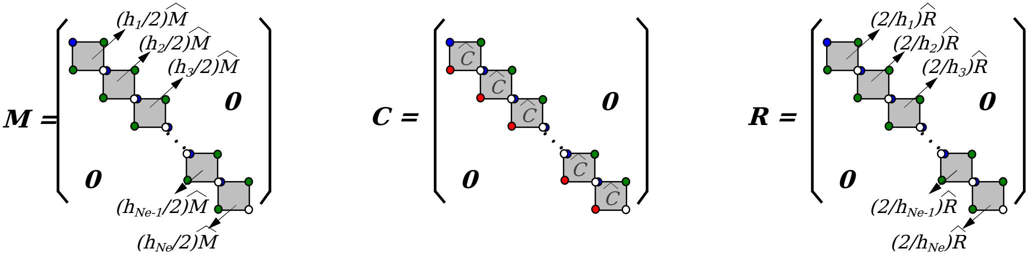

Tridiagonal matrices. $\mathbf{M}$ and $\mathbf{R}$ are **symmetrical**. $\mathbf{C}$ and $\mathbf{R}$ are **singular**.

Example 2.6: `p2_6lineal.m`

$$\begin{cases} u - 0.01u'' = 1, \quad x \in (0,1) \\ u(0) = 0; u'(1) = 1 \end{cases}$$

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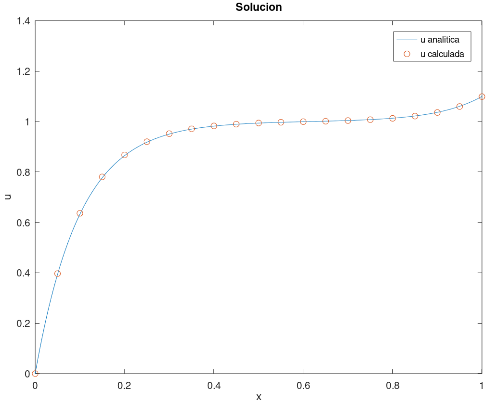

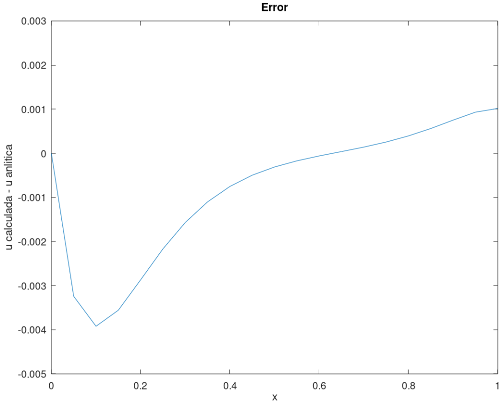

$$N_e = 20, N = 21 \Rightarrow \max|Error| = 4 \cdot 10^{-3}$$

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$$\hat{\phi}_0(\hat{x}) = -\hat{x}(1-\hat{x})/2 \qquad \hat{\phi}_1(\hat{x}) = \hat{x}(1-\hat{x}) \qquad \hat{\phi}_2(\hat{x}) = \hat{x}(1+\hat{x})/2$$

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$$\hat{\mathbf{M}} = \begin{pmatrix} \int_{-1}^{1} \hat{\phi}_0^2 & \int_{-1}^{1} \hat{\phi}_0\hat{\phi}_1 & \int_{-1}^{1} \hat{\phi}_0\hat{\phi}_2 \\ \int_{-1}^{1} \hat{\phi}_1\hat{\phi}_0 & \int_{-1}^{1} \hat{\phi}_1^2 & \int_{-1}^{1} \hat{\phi}_0\hat{\phi}_2 \\ \int_{-1}^{1} \hat{\phi}_2\hat{\phi}_0 & \int_{-1}^{1} \hat{\phi}_2\hat{\phi}_0 & \int_{-1}^{1} \hat{\phi}_2^2 \end{pmatrix} \hat{\mathbf{R}} = \begin{pmatrix} \int_{-1}^{1} \hat{\phi}'^2_0 & \int_{-1}^{1} \hat{\phi}'_0\hat{\phi}'_1 & \int_{-1}^{1} \hat{\phi}'_0\hat{\phi}'_2 \\ \int_{-1}^{1} \hat{\phi}'_1\hat{\phi}'_0 & \int_{-1}^{1} \hat{\phi}'^2_1 & \int_{-1}^{1} \hat{\phi}'_0\hat{\phi}'_2 \\ \int_{-1}^{1} \hat{\phi}'_2\hat{\phi}'_0 & \int_{-1}^{1} \hat{\phi}'_2\hat{\phi}'_0 & \int_{-1}^{1} \hat{\phi}'^2_2 \end{pmatrix}$$

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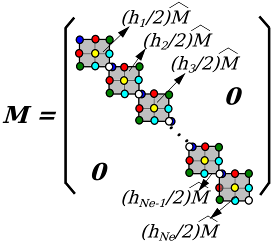
$$\hat{\mathbf{M}} = \begin{pmatrix} \int_{-1}^{1}\hat{\phi}_0^2 & \int_{-1}^{1}\hat{\phi}_0\hat{\phi}_1 & \int_{-1}^{1}\hat{\phi}_0\hat{\phi}_2 \\ \int_{-1}^{1}\hat{\phi}_1\hat{\phi}_0 & \int_{-1}^{1}\hat{\phi}_1^2 & \int_{-1}^{1}\hat{\phi}_0\hat{\phi}_2 \\ \int_{-1}^{1}\hat{\phi}_2\hat{\phi}_0 & \int_{-1}^{1}\hat{\phi}_2\hat{\phi}_0 & \int_{-1}^{1}\hat{\phi}_2^2 \end{pmatrix} \quad \hat{\mathbf{R}} = \begin{pmatrix} \int_{-1}^{1}\hat{\phi}'^2_0 & \int_{-1}^{1}\hat{\phi}'_0\hat{\phi}'_1 & \int_{-1}^{1}\hat{\phi}'_0\hat{\phi}'_2 \\ \int_{-1}^{1}\hat{\phi}'_1\hat{\phi}'_0 & \int_{-1}^{1}\hat{\phi}'^2_1 & \int_{-1}^{1}\hat{\phi}'_0\hat{\phi}'_2 \\ \int_{-1}^{1}\hat{\phi}'_2\hat{\phi}'_0 & \int_{-1}^{1}\hat{\phi}'_2\hat{\phi}'_0 & \int_{-1}^{1}\hat{\phi}'^2_2 \end{pmatrix}$$



Pentadiagonal matrices. M and R are **symmetrical**. C and R are **singular**.

Example 2.6: `p2_6cuadratico.m`

$$\begin{cases} u - 0.01u'' = 1, \quad x \in (0,1) \\ u\left(0\right) = 0; u'\left(1\right) = 1 \end{cases}$$

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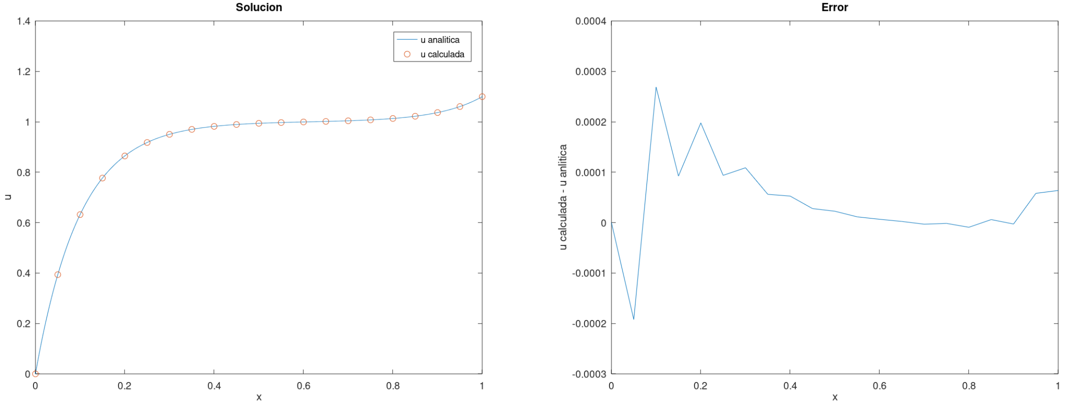

$$N_e = 10, N = 21 \Rightarrow \max|Error| = 2 \cdot 10^{-4}$$

Outline

2D Problem: Weak Formulation

Original/Strong Problem

$$au + (\mathbf{b} \cdot \nabla)\, u - \nabla \cdot (k \nabla u) = f \quad x \in D \subset \mathbb{R}^2, \; +b.c.$$

2D Problem: Weak Formulation

Original/Strong Problem

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Variational/Weak Formulation: Be solution $u \in H^1(D)$ that fulfills

$$\int_D (au + (\mathbf{b} \cdot \nabla)\, u - \nabla \cdot (k \nabla u))\, v = \int_D f v \qquad \forall v \in H^1(D)$$

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$$\int_D (au + (\mathbf{b} \cdot \nabla)\, u - \nabla \cdot (k \nabla u))\, v = \int_D fv \qquad \forall v \in H^1(D)$$

v are the test functions. Using the divergence (or Gauss's) theorem:

$$auv + \int_D \langle \mathbf{b}, \nabla u \rangle\, v + \int_D k \langle \nabla u, \nabla v \rangle = \int_D fv + \int_{\partial D} v \langle k \nabla u, \mathbf{n} \rangle$$

Variational/Weak Formulation: Be solution $u \in H^1(D)$ that fulfills

$$A(u, v) = L(v) \qquad \forall v \in H^1(D)$$

2D Elliptic Equation: Finite Element Method

Be $u_h\left(x\right)=\sum_{j=1}^{N}u_j\phi_j\left(x\right)$ and $v_h=\phi_i\left(x\right)$ for $1\leq i\leq N$

$$\sum_{j=1}^{N}A\left(\phi_j,\phi_i\right)u_j=L\left(\phi_i\right)\quad\forall i=1,...,N$$

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$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \quad \mathbf{b}_x = \begin{bmatrix} b_{x1} \\ \vdots \\ b_{xN} \end{bmatrix} \quad \mathbf{b}_y = \begin{bmatrix} b_{y1} \\ \vdots \\ b_{yN} \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$

$$\mathbf{A} = a\mathbf{M} + \mathbf{C}_x \mathbf{b}_x + \mathbf{C}_y \mathbf{b}_y + k\mathbf{R} \quad \mathbf{d} = \mathbf{M}\mathbf{f}$$

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- **Mass** matrix: $\mathbf{M} = (m_{ij}) = \int_D \phi_j \phi_i$
- **Convection** matrices: $\mathbf{C}_x = (c_{xij}) = \int_D \frac{\partial \phi_j}{\partial x} \phi_i$ and $\mathbf{C}_y = (c_{yij}) = \int_D \frac{\partial \phi_j}{\partial y} \phi_i$
- **Stiffness** matrix: $\mathbf{R} = (r_{ij}) = \int_D \langle \nabla \phi_j, \nabla \phi_i \rangle$

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`fem_mrc` works for linear and quadratic grids

```
[M R Cx] = fem_mrc(x, y, tri, 0, [1;0]);
[M R Cy] = fem_mrc(x, y, tri, 1, [0;1]);
A = a*M + Cx*bx + Cy*by + k*R;
```

2D Boundary Condition: Dirichlet

Same as 1D problem

```
front = find(x==-1 | x==1 | y==0 | y==1);
A0 = A;
A0(front,:)=0;
A0(:,front)=0;
for j=1;length(front)
 A0(front(j),front(j)) = 1;
end
d(front) = g;
for k=1:length(front)
 d -= g*A(:,front(k)); %Note :  -> efficient
end
% Solve linear system
uh = A0\d;
```

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end
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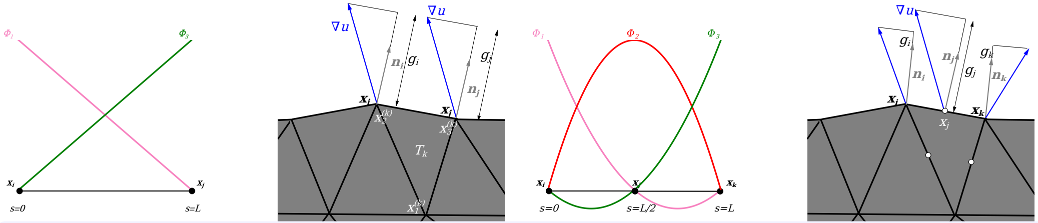
uh = A0\d;
```

Not efficient version

```
for i=1:length(x)
 for k=1:length(front)
  d(i) -= g*A(i,front(k)); %Note i -> NOT efficient
 end
end
```

2D Boundary Condition: *Neumann* and *Robin*



Neumann 2D: $\langle \nabla u, \mathbf{n} \rangle = g\,(x, y) \quad (x, y) \in \Gamma \subset \partial D$

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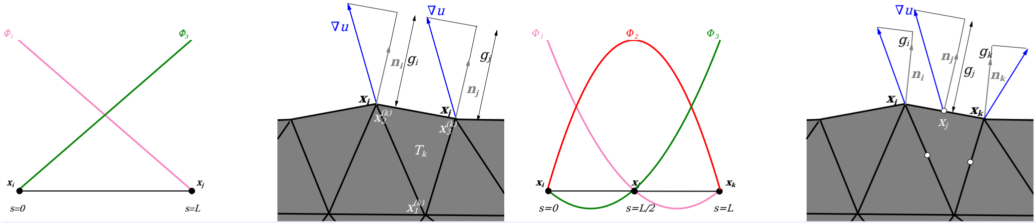

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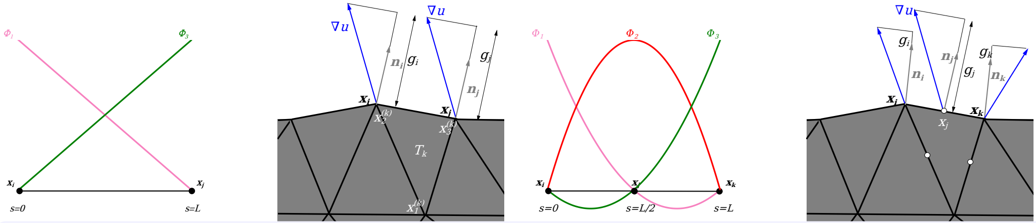

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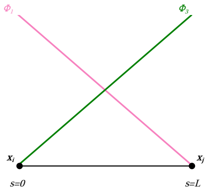
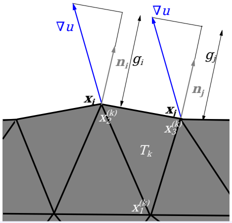
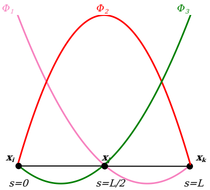
Convective **2D** $=$ *Robin* with $\alpha = htc, \qquad g = htc \cdot u_{fluid}$:

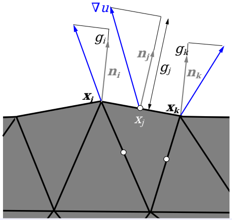
$$\langle k \nabla u, \mathbf{n} \rangle = htc\,(u_{fluid} - u) \quad (x, y) \in \Gamma \subset \partial D$$

2D Boundary Condition: *Neumann* and *Robin*









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fem_robin Octave function

```
[AR dR] = fem_robin(x, y, tri, G, ALPHA);
d += dR; A += AR; uh = A\d;
```

Gradient Projection in V_h

∇u_h estimation

$$\nabla u_h = \frac{\partial u_h}{\partial x}\mathbf{e}_1 + \frac{\partial u_h}{\partial y}\mathbf{e}_2 \simeq u_{hx}\mathbf{e}_1 + u_{hy}\mathbf{e}_2$$

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Be $u_h = \sum_{j=1}^{N} u_j\phi_j$, $u_{hx} = \sum_{j=1}^{N} u_{xj}\phi_j$ and $u_{hy} = \sum_{j=1}^{N} u_{yj}\phi_j$.
Then we applied the FEM method, so for all $v_h \in V_h$ we seek u_{hx} and u_{hy} that

$$\langle u_{hx}, v_h\rangle_{L^2} = \left\langle \frac{\partial u_h}{\partial x}, v_h \right\rangle_{L^2} \quad \langle u_{hy}, v_h\rangle_{L^2} = \left\langle \frac{\partial u_h}{\partial y}, v_h \right\rangle_{L^2}$$

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Selecting $v_h = \phi_i$ for $1 \leq i \leq N$

$$\sum_{j=1}^{N} u_{xj} \int_D \phi_j\phi_i = \sum_{j=1}^{N} u_j \int_D \frac{\partial \phi_j}{\partial x}\phi_i \qquad \sum_{j=1}^{N} u_{yj} \int_D \phi_j\phi_i = \sum_{j=1}^{N} u_j \int_D \frac{\partial \phi_j}{\partial y}\phi_i$$

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which turn in two linear systems

$$\mathbf{M}\mathbf{u}_{hx} = \mathbf{C}_x\mathbf{u}_h$$
$$\mathbf{M}\mathbf{u}_{hy} = \mathbf{C}_y\mathbf{u}_h$$

Area and Perimeter Computation

Computation of domain area D: as $\sum_{i=1}^{N} \phi_i = 1$

$$Area_D = \iint_D dxdy = \iint_D \sum_{i=1}^{N} \phi_i \sum_{j=1}^{N} \phi_j dxdy = \sum_{i,j=1}^{N} \iint_D \phi_i \phi_j dxdy = \sum_{i,j=1}^{N} m_{ij}$$

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Example: approximation for π

```
[x y tri] = mesh_circle(0, 0, 1, 100); [M R C] = fem_mrc(x, y, tri, 0, 0);
my_pi = sum(M*(x*0+1)); printf("Area = %f Error = %f\n", my_pi, my_pi-pi);
```

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Computation of domain perimeter ∂D: being $N_{\partial D}$ the number of nodes in ∂D

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Example: approximation for π

```
[x y tri] = mesh_circle(0, 0, 0.5, 100); g = 0*x; g(mesh_boundary(tri)) = 1;
[Ar Dr] = fem_robin(x, y, tri, g, 0*x); my_pi = sum(Dr);
printf("Perimeter = %f Error = %f\n", my_pi, my_pi-pi);
```

Example 2.11: `p2_11.m`

$D = (0,1) \times (0,1)$, $\Gamma_1 = \left\{(x,y) \in \partial D : x + y < \frac{1}{2}\right\}$, $\Gamma_2 = \partial\Omega - \Gamma_1$, $\mathbf{b} = (2,1)$

$$\begin{cases} \langle \mathbf{b}, \nabla u \rangle - 0.1\Delta u = 0, & (x,y) \in D \\ u \mid_{\Gamma_1} = 1, u \mid_{\Gamma_2} = 0 \end{cases}$$

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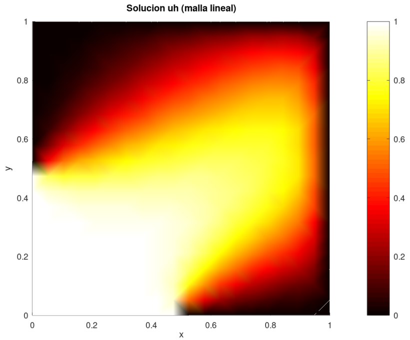
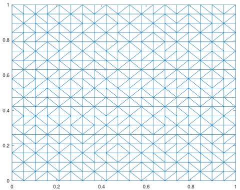
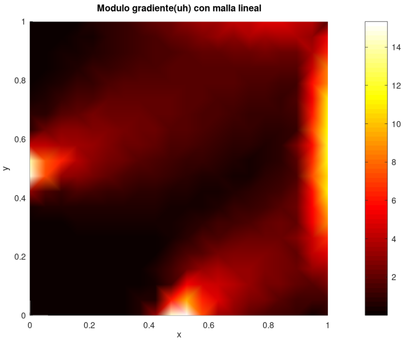




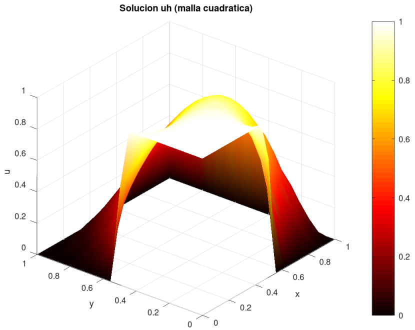


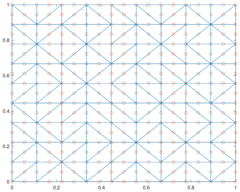
$$m = 1, N_e = 722, N = 400$$

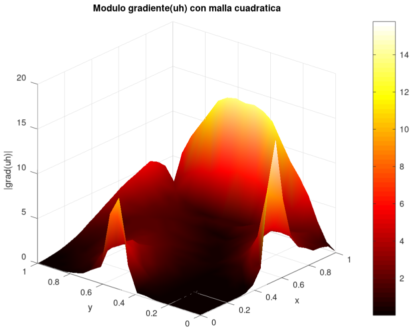
Example 2.11: `p2_11_cuadratico.m`

$D = (0,1) \times (0,1)$, $\Gamma_1 = \left\{(x,y) \in \partial D : x + y < \frac{1}{2}\right\}$, $\Gamma_2 = \partial\Omega - \Gamma_1$, $\mathbf{b} = (2,1)$

$$\begin{cases} \langle \mathbf{b}, \nabla u \rangle - 0.1\Delta u = 0, & (x,y) \in D \\ u\,|_{\Gamma_1} = 1, u\,|_{\Gamma_2} = 0 \end{cases}$$







$m = 2, N_e = 162, N = 361$

Outline

Axisymmetric Problem

$D \subset \mathbb{R}^3, u\,(x, r, \theta)$ such as $\partial u / \partial \theta = 0$.

$$au + (\mathbf{b} \cdot \nabla)\, u - \nabla \cdot (k \nabla u) = f$$

$$au + b_x \frac{\partial u}{\partial x} + \left(b_r - \frac{k}{r} \right) \frac{\partial u}{\partial r} - k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} \right) = f$$

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First Method: be Ω the 2D cut of D on a meridional plane

$$(a\mathbf{M} + \mathbf{C} + k\mathbf{R}) \mathbf{u} = \mathbf{M}\mathbf{f},$$

$$(\mathbf{M})_{ij} = \iint_\Omega \phi_j \phi_i \qquad (\mathbf{R})_{ij} = \iint_\Omega \langle \nabla \phi_j, \nabla \phi_i \rangle \qquad (\mathbf{C})_{ij} = \iint_\Omega \left[b_{x,j} \frac{\partial \phi_j}{\partial x} \phi_i + \left(b_r - \frac{k}{r} \right)_j \frac{\partial \phi_j}{\partial r} \phi_i \right]$$

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Second Method:

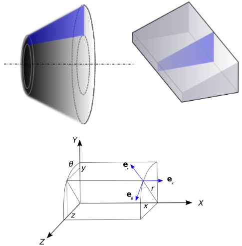
$$\left(a\mathbf{M}_r + \mathbf{C}_r + k\mathbf{R}_r\right) \mathbf{u} = \mathbf{M}_r \mathbf{f},$$

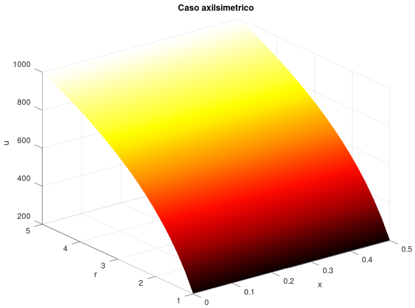
$$(\mathbf{M}_r)_{ij} = \iint_\Omega \phi_j \phi_i r \qquad (\mathbf{R}_r)_{ij} = \iint_\Omega \langle \nabla \phi_j, \nabla \phi_i \rangle r \qquad (\mathbf{C}_r)_{ij} = \iint_\Omega \left(b_{x,j} \frac{\partial \phi_j}{\partial x} \phi_i + b_{r,j} \frac{\partial \phi_j}{\partial r} \phi_i \right) r$$

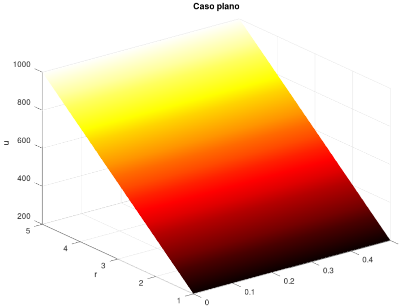
Axisymmetric/Plane Comparison

$\Omega = (0, 1/2) \times (1, 5)$, $\Gamma_1 = \{(x, r) \in \partial\Omega : r = 1\}$, $\Gamma_2 = \{(x, r) \in \partial\Omega : r = 5\}$, $\Gamma_3 = \partial\Omega - \Gamma_1 - \Gamma_2$

$$\begin{cases} \Delta u = 0, \\ u \,|_{\Gamma_1} = 200, \quad u \,|_{\Gamma_2} = 1000, \quad \langle \nabla u, \mathbf{n} \rangle \,|_{\Gamma_3} = 0 \end{cases}$$







Orthotropic Media

Thermal conductivity is a second order tensor

$$\mathbf{K} = \mathbf{G} \begin{pmatrix} k_{11}^* & 0 \\ 0 & k_{22}^* \end{pmatrix} \mathbf{G}^t = \begin{pmatrix} k_{11}^* \cos^2(\alpha) + k_{22}^* \sin^2(\alpha) & (k_{22}^* - k_{11}^*) \sin(\alpha) \cos(\alpha) \\ (k_{22}^* - k_{11}^*) \sin(\alpha) \cos(\alpha) & k_{11}^* \sin^2(\alpha) + k_{22}^* \cos^2(\alpha) \end{pmatrix} \equiv \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

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Stiffness matrix can be computed straightforward

$$(\mathbf{R})_{ij} = \int\limits_{D} \langle \mathbf{K}\nabla\phi_i, \nabla\phi_j \rangle .$$

$$\mathbf{R} = k_{11}\mathbf{R}^{11} + k_{12}\left(\mathbf{R}^{12} + \mathbf{R}^{21}\right) + k_{22}\mathbf{R}^{22}.$$

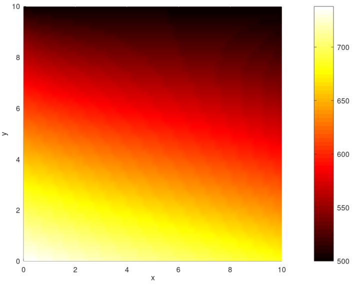
$$\mathbf{R}^{11} = \left(r_{ij}^{11} \equiv \int\limits_{D} \frac{\partial\phi_i}{\partial x_1}\frac{\partial\phi_j}{\partial x_1}\right), \qquad \mathbf{R}^{12} = \left(r_{ij}^{12} \equiv \int\limits_{D} \frac{\partial\phi_i}{\partial x_2}\frac{\partial\phi_j}{\partial x_1}\right),$$

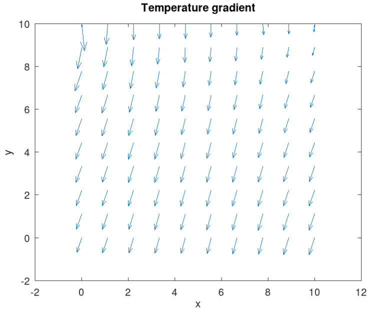
$$\mathbf{R}^{21} = \left(r_{ij}^{21} \equiv \int\limits_{D} \frac{\partial\phi_i}{\partial x_1}\frac{\partial\phi_j}{\partial x_2}\right), \qquad \mathbf{R}^{22} = \left(r_{ij}^{12} \equiv \int\limits_{D} \frac{\partial\phi_i}{\partial x_2}\frac{\partial\phi_j}{\partial x_2}\right).$$

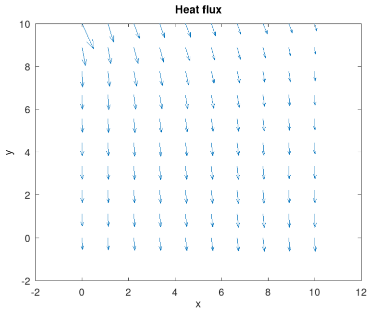
Orthotropic Material: Example

$D = (0,10) \times (10,20)\, mm$, $\Gamma_1 = \{(x,y) \in \partial D : y = 10\}$, $\Gamma_2 = \{(x,y) \in \partial D : y = 20\}$, $\Gamma_3 = \partial D - \Gamma_1 - \Gamma_2$

$$\left.\begin{array}{l} \mathbf{K}\Delta u = 0, \\ \langle \mathbf{K}u, \mathbf{n}\rangle\,|_{\Gamma_1} = h\left(u_f - u\right), \\ u\,|_{\Gamma_2} = 500\ K, \\ \langle \mathbf{K}\nabla u, \mathbf{n}\rangle\,|_{\Gamma_3} = 0 \end{array}\right\}, \quad \left.\begin{array}{c} \mathbf{K}^* = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix} mW/\left(K \cdot mm\right) \\ \alpha = \pi/4 \end{array}\right\}, \quad \left.\begin{array}{c} h = 1\, mW/\left(K \cdot mm^2\right) \\ u_f = 1000\ K \end{array}\right\}$$







Challenge

Thermal problem on ...

- an **axisymmetric** geometry,
- **orthotropic** material and
- **variable local base**

$$au + \nabla \cdot \left(\mathbf{K}\left(\mathbf{x}\right) \nabla u \right) = f$$

$$\mathbf{K}\left(\mathbf{x}\right) = \mathbf{G}\left(\mathbf{x}\right) \begin{pmatrix} k_{11}^* & 0 \\ 0 & k_{22}^* \end{pmatrix} \mathbf{G}\left(\mathbf{x}\right)^t =$$

$$= \begin{pmatrix} k_{11}^* \cos^2\left(\alpha\left(\mathbf{x}\right)\right) + k_{22}^* \sin^2\left(\alpha\left(\mathbf{x}\right)\right) & \left(k_{22}^* - k_{11}^*\right) \sin\left(\alpha\left(\mathbf{x}\right)\right) \cos\left(\alpha\left(\mathbf{x}\right)\right) \\ \left(k_{22}^* - k_{11}^*\right) \sin\left(\alpha\left(\mathbf{x}\right)\right) \cos\left(\alpha\left(\mathbf{x}\right)\right) & k_{11}^* \sin^2\left(\alpha\left(\mathbf{x}\right)\right) + k_{22}^* \cos^2\left(\alpha\left(\mathbf{x}\right)\right) \end{pmatrix} \equiv \begin{pmatrix} k_{11}\left(\mathbf{x}\right) & k_{12}\left(\mathbf{x}\right) \\ k_{21}\left(\mathbf{x}\right) & k_{22}\left(\mathbf{x}\right) \end{pmatrix}$$

End Part II

Thank you! Questions?