Finite Element Method: Applications based on Octave/MATLAB

Part III: Parabolic and Hyperbolic Problems

José M. Chaquet

Simulation and Technology Department, Industria de Turbo Propulsores S.A.U., Madrid

Departamento de Matemática Aplicada a la Ingeniería Industrial, Universidad Politécnica de Madrid

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Outline

1D Parabolic Equation: *Crank-Nicolson*

Original Problem: $\frac{\partial u}{\partial t} + au + b\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2} = f(x,t) \quad x \in D = (c,d)\,, t \in (0,T)\ + b.c. + i.c.$

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Time Integration: *Trapezoidal* **Rule (Second Order)**

$$u^{n+1} = u^n + \frac{1}{2}\left(-a\Delta t \cdot u^n - b\Delta t\frac{\partial u^n}{\partial x} + k\Delta t\frac{\partial^2 u^n}{\partial x^2} + \Delta t \cdot f^n\right) +$$
$$+\frac{1}{2}\left(-a\Delta t \cdot u^{n+1} - b\Delta t\frac{\partial u^{n+1}}{\partial x} + k\Delta t\frac{\partial^2 u^{n+1}}{\partial x^2} + \Delta t \cdot f^{n+1}\right) + b.c.$$

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Variational/Weak Formulation: Be solution $u^{n+1} \in H^1(D)$ that fulfills for $\forall v \in H^1(D)$

$$\left(1+\frac{a}{2}\Delta t\right)\int_c^d u^{n+1}v + \frac{b}{2}\Delta t\int_c^d \frac{\partial u^{n+1}}{\partial x}v + \frac{k}{2}\Delta t\int_c^d \frac{\partial u^{n+1}}{\partial x}\frac{\partial v}{\partial x} = \int_c^d u^n v - \frac{a}{2}\Delta t\int_c^d u^n v - \\ -\frac{b}{2}\Delta t\int_c^d \frac{\partial u^n}{\partial x}v - \frac{k}{2}\Delta t\int_c^d \frac{\partial u^n}{\partial x}\frac{\partial v}{\partial x} + \frac{\Delta t}{2}\int_c^d \left(f^n + f^{n+1}\right)v + \frac{\Delta t}{2}\left[k\left(\frac{\partial u^n}{\partial x} + \frac{\partial u^{n+1}}{\partial x}\right)v\right]_c^d$$

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$$-\frac{b}{2}\Delta t\int_c^d \frac{\partial u^n}{\partial x}v - \frac{k}{2}\Delta t\int_c^d \frac{\partial u^n}{\partial x}\frac{\partial v}{\partial x} + \frac{\Delta t}{2}\int_c^d \left(f^n + f^{n+1}\right)v + \frac{\Delta t}{2}\left[k\left(\frac{\partial u^n}{\partial x} + \frac{\partial u^{n+1}}{\partial x}\right)v\right]_c^d$$

FEM: Be $u_h^{n+1} \in V_h \Rightarrow u_h^{n+1}(x) = \sum_{j=1}^N u_j\phi_j(x)$ and $v_h = \phi_i(x)$ for $1 \leq i \leq N$

$$\mathbf{A}\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n + \frac{\Delta t}{2}\left(\mathbf{d}^{n+1} + \mathbf{d}^n\right)$$

$$\mathbf{A} \equiv \left(1+\frac{a\Delta t}{2}\right)\mathbf{M} + \frac{b\Delta t}{2}\mathbf{C} + \frac{k\Delta t}{2}\mathbf{R} \qquad \mathbf{B} \equiv \left(1-\frac{a\Delta t}{2}\right)\mathbf{M} - \frac{b\Delta t}{2}\mathbf{C} - \frac{k\Delta t}{2}\mathbf{R}$$
$$\mathbf{d}^n = \mathbf{M}\mathbf{f}^n \qquad \mathbf{d}^{n+1} = \mathbf{M}\mathbf{f}^{n+1}$$

Dirichlet Boundary Conditions

$$u^{n+1}(c) = g_1^{n+1} \quad u^{n+1}(d) = g_2^{n+1}$$

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$$\left.\begin{array}{c}
a_{11}u_1^{n+1} + a_{12}u_2^{n+1} + \cdots + a_{1N-1}u_{N-1}^{n+1} + a_{1N}u_N^{n+1} = e_1 - \Delta t \frac{k}{2}\left(\frac{\partial u^{n}(c)}{\partial x} + \frac{\partial u^{n+1}(c)}{\partial x}\right) \\
a_{21}u_1^{n+1} + a_{22}u_2^{n+1} + \cdots + a_{2N-1}u_{N-1}^{n+1} + a_{2N}u_N^{n+1} = e_2 \\
\vdots \\
a_{N-1,1}u_1^{n+1} + a_{N-1,2}u_2^{n+1} + \cdots + a_{N-1,N-1}u_{N-1}^{n+1} + a_{N-1,N}u_N^{n+1} = e_{N-1} \\
a_{N1}u_1^{n+1} + a_{N2}u_2^{n+1} + \cdots + a_{NN-1}u_{N-1}^{n+1} + a_{NN}u_N^{n+1} = e_N + \Delta t \frac{k}{2}\left(\frac{\partial u^{n}(d)}{\partial x} + \frac{\partial u^{n+1}(d)}{\partial x}\right)
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\end{array}\right\}$$

Modified Linear System $\widetilde{\mathbf{A}}\mathbf{u}^{n+1} = \widetilde{\mathbf{e}}$

$$\left.\begin{array}{c}
1u_1^{n+1} + 0u_2^{n+1} + \cdots + 0u_{N-1}^{n+1} + 0u_N^{n+1} = g_1^{n+1} \\
0u_1^{n+1} + a_{22}u_2^{n+1} + \cdots + a_{2N-1}u_{N-1}^{n+1} + 0u_N^{n+1} = e_2 - a_{21}g_1^{n+1} - a_{2N}g_2^{n+1} \\
\vdots \\
0u_1^{n+1} + a_{N-1,2}u_2^{n+1} + \cdots + a_{N-1,N-1}u_{N-1}^{n+1} + 0u_N^{n+1} = e_{N-1} - a_{N-1,1}g_1^{n+1} - a_{N-1,N}g_2^{n+1} \\
0u_1^{n+1} + 0u_2^{n+1} + \cdots + 0u_{N-1}^{n+1} + 1u_N^{n+1} = g_2^{n+1}
\end{array}\right\}$$

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$$\alpha_1^{n+1} u^{n+1}(c) + \beta_1^{n+1} \frac{\partial u^{n+1}(c)}{\partial x} = g_1^{n+1} \quad \alpha_2^{n+1} u^{n+1}(d) + \beta_2^{n+1} \frac{\partial u^{n+1}(d)}{\partial x} = g_2^{n+1}$$

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Modified Linear System $\widetilde{\mathbf{A}}\mathbf{u}^{n+1} = \widetilde{\mathbf{e}}$

$$\left.\begin{aligned}
\left(a_{11} - \Delta t \frac{k}{2}\frac{\alpha_1^{n+1}}{\beta_1^{n+1}}\right)u_1^{n+1} + a_{12}u_2^{n+1} \cdots a_{1N-1}u_{N-1}^{n+1} + a_{1N}u_N^{n+1} &= e_1 + \Delta t \frac{k}{2}\frac{\alpha_1^n}{\beta_1^n} - \Delta t \frac{k}{2}\left(\frac{g_1^n}{\beta_1^n} + \frac{g_1^{n+1}}{\beta_1^{n+1}}\right)\\
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a_{N1}u_1^{n+1} + a_{N2}u_2^{n+1} \cdots a_{NN-1}u_{N-1}^{n+1} + \left(a_{NN} + \Delta t \frac{k}{2}\frac{\alpha_2^{n+1}}{\beta_2^{n+1}}\right)u_N^{n+1} &= e_N - \Delta t \frac{k}{2}\frac{\alpha_N^n}{\beta_N^n} + \Delta t \frac{k}{2}\left(\frac{g_N^n}{\beta_N^n} + \frac{g_N^{n+1}}{\beta_N^{n+1}}\right)
\end{aligned}\right\}$$

Problem 3.8: `p3_8.m`

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - 10^{-2}\dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), & (x,t) \in D = (-1,4) \times (0,100), \\ \dfrac{\partial u}{\partial x}(-1,t) = -6; u(4,t) = 30, & t \in (0,100), \\ u(x,0) = 20, & x \in D, \end{cases}$$

$$f(x,t) = 50\exp\left(-\frac{x^2}{0.01} - (t-20)^2\right)$$

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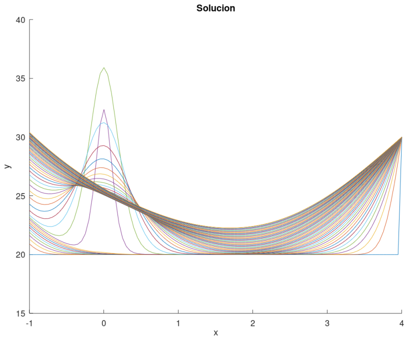

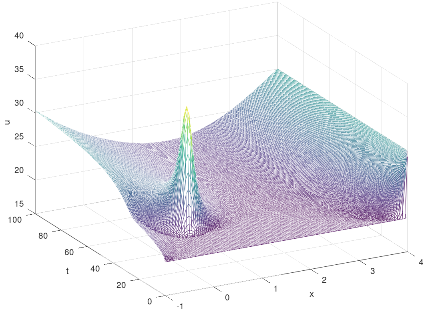

Example 3.1: `e3_1.m`

$$\left.\begin{array}{l} c_p\rho\frac{\partial u}{\partial t} = k\Delta u + f(x,y,t)\,, \quad \textbf{en } D\times(0,T) \\ u\mid_{\Gamma_1} = u_0 + 5t/T \\ \langle k\nabla u, \mathbf{n}\rangle \mid_{\Gamma_2} = h(x,y,t)\left(u_f(t) - u\right) \\ \langle k\nabla u, \mathbf{n}\rangle \mid_{\partial D-(\Gamma_1\cup\Gamma_2)} = 0 \\ u(x,y,0) = u_0 \end{array}\right\}$$

$$\begin{array}{l} D = (-1,1)\times(0,1)\,, \\ T = 100\ s, \\ \Gamma_1 = \{(x,y)\in\partial D : y = 0\}\,, \\ \Gamma_2 = \{(x,y)\in\partial D : y = 1\}\,, \\ h(x,y,t) = 5000\left(2 - x^2\right)\ W/\left(m^2K\right), \end{array}$$

$$\begin{array}{l} u_0 = 200\ K, \\ u_f(t) = u_0 - 10t/T\ K, \\ k = 300\ W/(m\cdot K)\,, \\ c_p = 450\ J/(kg\cdot K)\,, \\ \rho = 7800\ kg/m^3, \end{array}$$

$$f(x,y,t) = \left\{\begin{array}{cc} 10^6\cdot(T-2t)/T & |x|\le 0.5\ \wedge\ y\in[0.25,0.75]\ \wedge\ t<T/2 \\ 0 & otherwise \end{array}\right.$$

Example 3.1: `e3_1.m`

$$\left.\begin{array}{l} c_p \rho \frac{\partial u}{\partial t} = k\Delta u + f(x,y,t), \quad \text{en } D \times (0,T) \\ u\,|_{\Gamma_1} = u_0 + 5t/T \\ \langle k\nabla u, \mathbf{n}\rangle\,|_{\Gamma_2} = h(x,y,t)\,(u_f(t) - u) \\ \langle k\nabla u, \mathbf{n}\rangle\,|_{\partial D - (\Gamma_1 \cup \Gamma_2)} = 0 \\ u(x,y,0) = u_0 \end{array}\right\}$$

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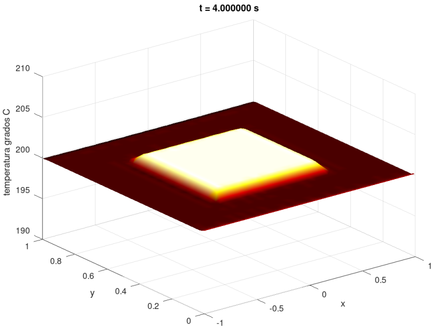
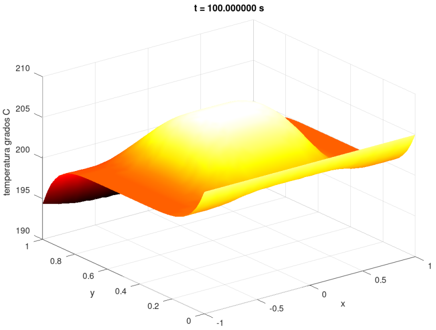
$$\begin{array}{l} u_0 = 200\ K, \\ u_f(t) = u_0 - 10t/T\ K, \\ k = 300\ W/(m\cdot K), \\ c_p = 450\ J/(kg\cdot K), \\ \rho = 7800\ kg/m^3, \end{array}$$

$$f(x,y,t) = \begin{cases} 10^6\cdot(T-2t)/T & |x| \le 0.5 \wedge y \in [0.25, 0.75] \wedge t < T/2 \\ 0 & otherwise \end{cases}$$





Outline

1. Parabolic Problem
   - Crank-Nicolson Scheme
   - Boundary Conditions
   - Examples

2. Wave Equation
   - Newmark Scheme
   - Boundary Conditions
   - Energy Conservation
   - Examples

Wave Equation: *Newmark* Scheme

Original problem: $u(\mathbf{x},t)$ defined in $D \subset \mathbb{R}^d$:

$$\left.\begin{array}{c} u_{tt} + 2k u_t - c^2 \Delta u = f \quad \mathbf{x} \in D, t \in (0,T) + b.c. \\ i.c. : \left\{ \begin{array}{c} u(\mathbf{x},0) = u_0(\mathbf{x}) \\ u_t(\mathbf{x},0) = v_0(\mathbf{x}) \end{array} \right. \end{array}\right\}$$

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Defining $v(\mathbf{x},t) = u_t(\mathbf{x},t)$ and $a(\mathbf{x},t) = u_{tt}(\mathbf{x},t) = v_t(\mathbf{x},t)$ for $t = t_{n+1}$

$$a^{n+1} + 2kv^{n+1} - c^2\Delta u^{n+1} = f^{n+1} \qquad (*)$$

Wave Equation: *Newmark* Scheme

Original problem: $u(\mathbf{x},t)$ defined in $D \subset \mathbb{R}^d$:

$$\left.\begin{array}{c} u_{tt} + 2ku_t - c^2\Delta u = f \quad \mathbf{x} \in D, t \in (0,T) + b.c. \\ i.c. : \left\{ \begin{array}{l} u(\mathbf{x},0) = u_0(\mathbf{x}) \\ u_t(\mathbf{x},0) = v_0(\mathbf{x}) \end{array} \right. \end{array}\right\}$$

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$$a^{n+1} + 2kv^{n+1} - c^2\Delta u^{n+1} = f^{n+1} \qquad (*)$$

Defining γ and β parameters

$$\left.\begin{array}{r} v^{n+1} = v^n + \Delta t\left(\gamma a^{n+1} + (1-\gamma)a^n\right) \\ u^{n+1} = u^n + \Delta t v^n + \frac{\Delta t^2}{2}\left(2\beta a^{n+1} + (1-2\beta)a^n\right) \end{array}\right\} \quad (**)$$

Wave Equation: *Newmark* Scheme

Original problem: $u(\mathbf{x},t)$ defined in $D \subset \mathbb{R}^d$:

$$\left.\begin{array}{c} u_{tt} + 2ku_t - c^2\Delta u = f \quad \mathbf{x} \in D, t \in (0,T) + b.c. \\ i.c. : \left\{ \begin{array}{c} u(\mathbf{x},0) = u_0(\mathbf{x}) \\ u_t(\mathbf{x},0) = v_0(\mathbf{x}) \end{array} \right. \end{array}\right\}$$

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Defining γ and β parameters

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***Newmark* method**

- $\gamma = \beta = 0$: Explicit Euler Scheme.
- $\gamma = 1$ and $\beta = 1/2$: Implicit Euler Scheme.
- $\gamma = 1/2$ and $\beta = 1/4$: Trapezoidal Scheme. **Unconditionally stable.**
- $\gamma = 1/2$ and $\beta = 0$: Centered Differences. Conditionally stable if $\Delta t < h/c$.

Wave Equation: Finite Element Method

$$\xi u^{n+1} - c^2 \beta \Delta t^2 \Delta u^{n+1} = \xi u^n + \mu \Delta t \cdot v^n + \lambda \Delta t^2 a^n + \beta \Delta t^2 f^{n+1}$$

$$\left.\begin{array}{c} \xi = 1 + 2k\gamma\Delta t \\ \mu = 1 + 2k\left(\gamma - \beta\right)\Delta t \\ \lambda = \xi\left(\frac{1}{2} - \beta\right) - 2k\beta\left(1 - \gamma\right) \end{array}\right\}$$

Wave Equation: Finite Element Method

$$\xi u^{n+1} - c^2\beta\Delta t^2 \Delta u^{n+1} = \xi u^n + \mu\Delta t \cdot v^n + \lambda\Delta t^2 a^n + \beta\Delta t^2 f^{n+1}$$

$$\left.\begin{aligned} \xi &= 1 + 2k\gamma\Delta t \\ \mu &= 1 + 2k\left(\gamma - \beta\right)\Delta t \\ \lambda &= \xi\left(\tfrac{1}{2} - \beta\right) - 2k\beta\left(1 - \gamma\right) \end{aligned}\right\}$$

Assuming $u^{n+1} = \sum_{j=1}^{N} u_j^{n+1}\phi_j \in V_h$ we use FEM approach

$$\left(\xi\mathbf{M} + \beta c^2\Delta t^2\mathbf{R}\right)\mathbf{u}^{n+1} = \mathbf{M}\left(\xi\mathbf{u}^n + \mu\Delta t\mathbf{v}^n + \lambda\Delta t^2\mathbf{a}^n + \beta\Delta t^2\mathbf{f}^{n+1}\right),$$

Wave Equation: Finite Element Method

$$\xi u^{n+1} - c^2\beta\Delta t^2 \Delta u^{n+1} = \xi u^n + \mu\Delta t \cdot v^n + \lambda\Delta t^2 a^n + \beta\Delta t^2 f^{n+1}$$

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At first instant we know $\mathbf{u}^0$ and $\mathbf{v}^0$. We must compute acceleration from (*):

$$\mathbf{M}\mathbf{a}^0 = -c^2\mathbf{R}\mathbf{u}^0 - 2k\mathbf{M}\mathbf{v}^0 + \mathbf{M}\mathbf{f}^0$$

Wave Equation: Finite Element Method

$$\xi u^{n+1} - c^2\beta\Delta t^2\Delta u^{n+1} = \xi u^n + \mu\Delta t \cdot v^n + \lambda\Delta t^2 a^n + \beta\Delta t^2 f^{n+1}$$

$$\left.\begin{array}{c} \xi = 1 + 2k\gamma\Delta t \\ \mu = 1 + 2k(\gamma-\beta)\Delta t \\ \lambda = \xi\left(\frac{1}{2}-\beta\right) - 2k\beta(1-\gamma) \end{array}\right\}$$

Assuming $u^{n+1} = \sum_{j=1}^{N} u_j^{n+1}\phi_j \in V_h$ we use FEM approach

$$\left(\xi\mathbf{M} + \beta c^2\Delta t^2\mathbf{R}\right)\mathbf{u}^{n+1} = \mathbf{M}\left(\xi\mathbf{u}^n + \mu\Delta t\mathbf{v}^n + \lambda\Delta t^2\mathbf{a}^n + \beta\Delta t^2\mathbf{f}^{n+1}\right),$$

At first instant we know $\mathbf{u}^0$ and $\mathbf{v}^0$. We must compute acceleration from (*):

$$\mathbf{M}\mathbf{a}^0 = -c^2\mathbf{R}\mathbf{u}^0 - 2k\mathbf{M}\mathbf{v}^0 + \mathbf{M}\mathbf{f}^0$$

Once instant $n+1$ is solved, we obtain $\mathbf{v}^{n+1}$ and $\mathbf{a}^{n+1}$ from (**)

$$\mathbf{a}^{n+1} = \left(1 - 1/(2\beta)\right)\mathbf{a}^n + \left(\mathbf{u}^{n+1} - \mathbf{u}^n - \Delta t\mathbf{v}^n\right)/\left(\beta\Delta t^2\right)$$
$$\mathbf{v}^{n+1} = \mathbf{v}^n + \Delta t\left(\gamma\mathbf{a}^{n+1} + (1-\gamma)\mathbf{a}^n\right)$$

Wave Equation: Boundary Conditions

Dirichlet

Imposed as in Elliptic and Parabolic problems.

Neumann

Only **homogeneous** conditions have physical meaning.

Robin

No physical sense.

Wave Equation: Energy Conservation

Let be no friction neither source term. The variational formulation turns:

$$\int_D u_{tt} v + c^2 \int_D \langle \nabla u, \nabla v \rangle = 0 \; \forall v \in H_0^1(D)$$

Wave Equation: Energy Conservation

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$$\int_D u_{tt} v + c^2 \int_D \langle \nabla u, \nabla v \rangle = 0 \; \forall v \in H_0^1(D)$$

Be $v = u_t$:

$$\int_D u_{tt} u_t + c^2 \int_D \langle \nabla u, \nabla u_t \rangle = 0 \Rightarrow \int_D \frac{1}{2} \frac{\partial}{\partial t} (u_t)^2 + c^2 \int_D \frac{1}{2} \frac{\partial}{\partial t} (\nabla u)^2 = 0 \Rightarrow$$

$$\frac{1}{2} \frac{d}{dt} \int_D |u_t|^2 + c^2 \frac{1}{2} \frac{d}{dt} \int_D |\nabla u|^2 = 0 \Rightarrow \frac{d}{dt} \left(\frac{1}{2} \rho \left\| u_t \right\|^2 + \frac{c^2}{2} \rho \left\| \nabla u \right\|^2 \right) = \frac{d}{dt} (E_c + E_p) = 0$$

Wave Equation: Energy Conservation

Let be no friction neither source term. The variational formulation turns:

$$\int_D u_{tt} v + c^2 \int_D \langle \nabla u, \nabla v \rangle = 0 \; \forall v \in H_0^1(D)$$

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$$\frac{1}{2} \frac{d}{dt} \int_D |u_t|^2 + c^2 \frac{1}{2} \frac{d}{dt} \int_D |\nabla u|^2 = 0 \Rightarrow \frac{d}{dt} \left(\frac{1}{2} \rho \|u_t\|^2 + \frac{c^2}{2} \rho \|\nabla u\|^2 \right) = \frac{d}{dt} (E_c + E_p) = 0$$

Conditions to preserve total energy:

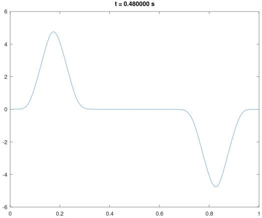
- No friction, no source term: $k = f = 0$
- Boundary conditions:
  - **Steady *Dirichlet*** conditions or ...
  - **Homogeneous *Neumann*** conditions.

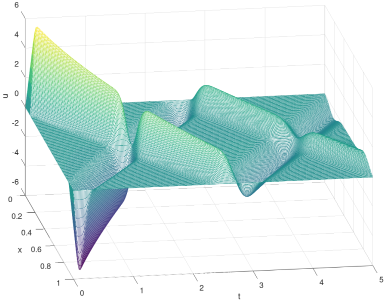
Problem 4.11: `wave1D.m`

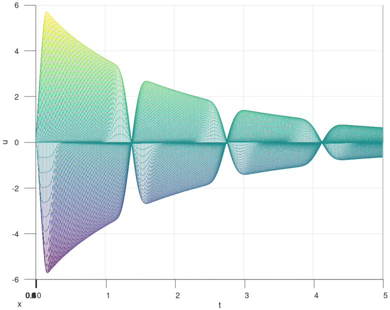
$$\left.\begin{array}{c} u_{tt} + 2ku_t - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \; x \in (0,1) \;\; t \in (0,5) \\ u(0,t) = u(1,t) = 0 \; t \in (0,5) \\ u(x,0) = 0 \; x \in (0,1) \\ u_t(x,0) = v(x) \;\; x \in (0,1) \end{array}\right\}, c^2 = 0.2, k = 0.5, v(x) = \begin{cases} 50 & x \leq 0.1 \\ 0 & x \in (0.1, 0.9) \\ -50 & x \geq 0.9 \end{cases}$$

Problem 4.11: `wave1D.m`

$$\left.\begin{array}{c} u_{tt} + 2ku_t - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \; x \in (0,1) \;\; t \in (0,5) \\ u(0,t) = u(1,t) = 0 \; t \in (0,5) \\ u(x,0) = 0 \; x \in (0,1) \\ u_t(x,0) = v(x) \;\; x \in (0,1) \end{array}\right\}, c^2 = 0.2, k = 0.5, v(x) = \begin{cases} 50 & x \leq 0.1 \\ 0 & x \in (0.1, 0.9) \\ -50 & x \geq 0.9 \end{cases}$$





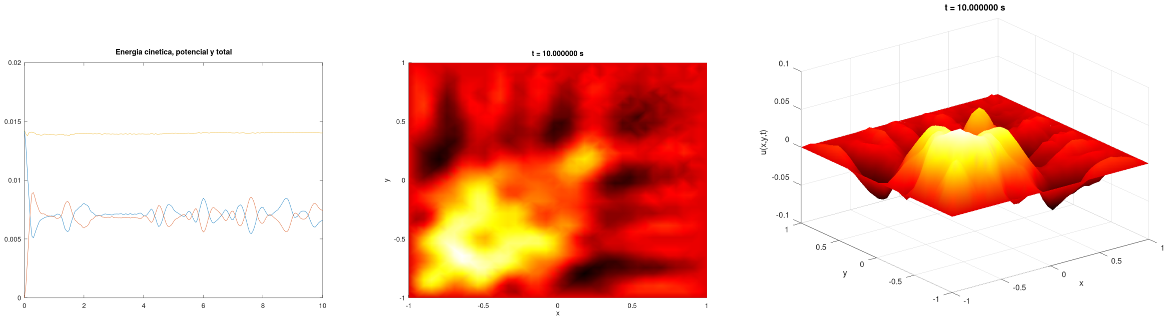


Problem 4.6: `ph_6.m`

$$\left.\begin{array}{c} u_{tt} - 0.1\Delta u = 0 \; \mathbf{x} \in D = (-1,1) \times (-1,1) \;\; t \in (0,20) \\ u|_{\partial D} = 0 \; \forall t \\ u\left(\mathbf{x},0\right) = 0 \; \forall \mathbf{x} \in D \\ u_t\left(\mathbf{x},0\right) = v_0 \; \forall \mathbf{x} \in D \end{array}\right\}, v_0\left(x,y\right) = \left\{\begin{array}{cc} 1 & \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 < \frac{1}{100} \\ 0 & otherwise \end{array}\right.$$

Problem 4.6: `ph_6.m`

$$\left.\begin{array}{c} u_{tt}-0.1\Delta u=0\ \mathbf{x}\in D=(-1,1)\times(-1,1)\ t\in(0,20) \\ u|_{\partial D}=0\ \forall t \\ u(\mathbf{x},0)=0\ \forall \mathbf{x}\in D \\ u_t(\mathbf{x},0)=v_0\ \forall \mathbf{x}\in D \end{array}\right\},\ v_0(x,y)=\left\{\begin{array}{cc} 1 & \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2<\frac{1}{100} \\ 0 & otherwise \end{array}\right.$$

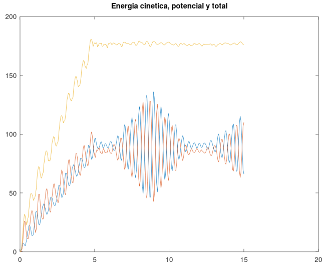


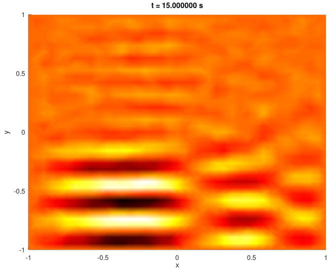
Problem 4.8: `ph_8.m`

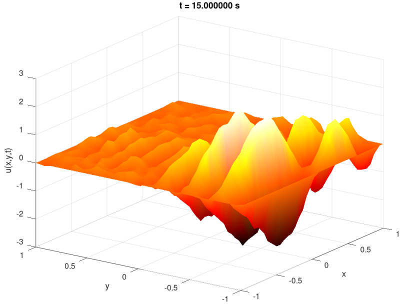
$$\left.\begin{array}{c} u_{tt}+2ku_t-0.1\Delta u=0\ \mathbf{x}\in D \\ u|_{\partial D}=u_d\ \forall t \\ u(\mathbf{x},0)=0\ \forall\mathbf{x}\in D \\ u_t(\mathbf{x},0)=0\ \forall\mathbf{x}\in D \end{array}\right\} \quad \begin{array}{c} D=(-1,1)\times(-1,1)\ \ t\in(0,20) \\ u_d(x,y,t)=\begin{cases} x\sin(2\pi t) & x<0,\ y=-1,\ t<5 \\ 0 & otherwise \end{cases} \end{array}$$

Problem 4.8: `ph_8.m`

$$\left.\begin{array}{c} u_{tt} + 2ku_t - 0.1\Delta u = 0 \;\mathbf{x} \in D \\ u|_{\partial D} = u_d \;\forall t \\ u(\mathbf{x}, 0) = 0 \;\forall \mathbf{x} \in D \\ u_t(\mathbf{x}, 0) = 0 \;\forall \mathbf{x} \in D \end{array}\right\} \quad \begin{array}{c} D = (-1,1) \times (-1,1) \;\; t \in (0,20) \\ u_d(x,y,t) = \begin{cases} x \sin(2\pi t) & x < 0,\; y = -1,\; t < 5 \\ 0 & otherwise \end{cases} \end{array}$$





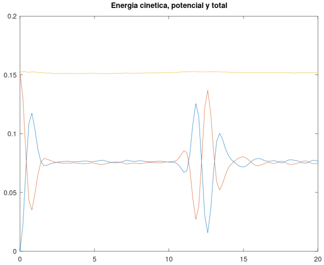


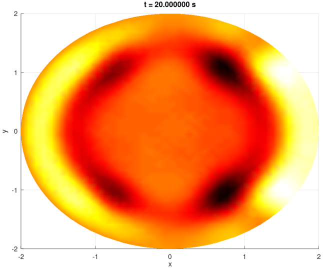
Problem 4.9: `ph_9.m`

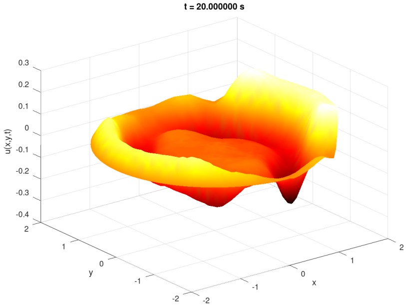
$$\left.\begin{array}{c} u_{tt} - c^2 \Delta u = 0 \; \mathbf{x} \in D \\ \left. \langle c^2 \nabla u, \mathbf{n} \rangle \right|_{\Gamma_n} = 0 \; \forall t \\ \left. u \right|_{\partial D - \Gamma_n} = 0 \; \forall \mathbf{x} \in D \\ u_t \left(\mathbf{x}, 0 \right) = u_0 \; \forall \mathbf{x} \in D \end{array} \right\} \quad \begin{array}{c} D = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4 \right\} \\ \mathit{\Gamma}_n = \left\{ (x, y) \in \partial D : x > 0 \right\} \\ u_0 \left(x, y \right) = \exp \left(-\frac{x^2 + y^2}{0.1} \right) \end{array}$$

Problem 4.9: `ph_9.m`

$$
\left.
\begin{array}{r}
u_{tt} - c^2 \Delta u = 0 \ \mathbf{x} \in D \\
\left.\langle c^2 \nabla u, \mathbf{n} \rangle\right|_{\Gamma_n} = 0 \ \forall t \\
\left. u \right|_{\partial D - \Gamma_n} = 0 \ \forall \mathbf{x} \in D \\
u_t \left(\mathbf{x}, 0 \right) = u_0 \ \forall \mathbf{x} \in D
\end{array}
\right\}
\quad
\begin{array}{c}
D = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4 \} \\
\Gamma_n = \{ (x, y) \in \partial D : x > 0 \} \\
u_0 \left(x, y \right) = \exp \left(-\frac{x^2 + y^2}{0.1} \right)
\end{array}
$$







End Part III

Thank you! Questions?