Finite Element Method: Applications based on Octave/MATLAB

Part IV: Structural Mechanics (Elasticity)

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February 2024

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Outline

2D Steady Elasticity: Strong Formulation

Get the displacement field $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ for a 2D domain $D \subset \mathbb{R}^2$ with $\mathit{\Gamma}_D, \mathit{\Gamma}_N \subset \partial D$

$$\left.\begin{array}{c} -\nabla \cdot \left(\boldsymbol{\sigma}\left(\mathbf{u}\right)\right) = \mathbf{f} \\ \left.\mathbf{u}\right|_{\mathit{\Gamma}_D} = \mathbf{g}_D \\ \left.\boldsymbol{\sigma}\left(\mathbf{u}\right)\cdot\mathbf{n}\right|_{\mathit{\Gamma}_N} = \mathbf{g}_N \end{array}\right\}$$

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Stress tensor ($\left[N/m^2\right]$) defined with *Lamé* (μ, λ) coefficients and the **strain** tensor

$$\boldsymbol{\sigma}(\mathbf{u}) = \lambda \nabla \cdot (\mathbf{u})\,\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right)$$

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Lamé coefficients related with *Young* E modulus and *Poisson* ν coefficient for 2D problems

$$\lambda = \frac{E\nu}{(1+\nu)(1-\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

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For 3D problems

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

Variational Formulation

Be $V = \left\{ v \in H^1(D) : v|_{\Gamma_D} = 0 \right\}$ and $\mathbf{v} = (v_1, v_2) \in V \times V$ a test function

$$-\int_D \langle \nabla \cdot (\boldsymbol{\sigma}(\mathbf{u})), \mathbf{v} \rangle = \int_D \langle \mathbf{f}, \mathbf{v} \rangle$$

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Divergence theorem (tensor flavor): $\int_D \langle \nabla \cdot (\mathbf{T}), \mathbf{v} \rangle + \int_D \mathbf{T} : (\nabla \mathbf{v}) = \int_{\partial D} \langle \mathbf{v}, \mathbf{T}\mathbf{n} \rangle$

Variational or weak formulation: find $\mathbf{u} \in V \times V$ such as $\forall \mathbf{v} \in V \times V$

$$\int_D (\lambda (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}) + 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})) = \int_D \langle \mathbf{f}, \mathbf{v} \rangle + \int_{\Gamma_N} \langle \mathbf{g}_N, \mathbf{v} \rangle$$

$$A(\mathbf{u}, \mathbf{v}) = L(\mathbf{v})$$

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Korn inequality $\Rightarrow$ A operator is *coercitive* $\Rightarrow$ *Lax-Milgram* theorem $\Rightarrow$ $\exists! \mathbf{u}$ weak solution

FEM formulation (I)

Find $\mathbf{u}_h \in V_h \times V_h$ such as $\forall \mathbf{v}_h \in V_h \times V_h$

$$A\left(\mathbf{u}_h, \mathbf{v}_h\right) = L\left(\mathbf{v}_h\right)$$

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$\left\{\phi_i\left(x,y\right)\right\}_{1\leq i\leq N}$ base of V_h

$$\mathbf{u}_h\left(x,y\right) = u_{1h}\mathbf{e}_1 + u_{2h}\mathbf{e}_2 = \sum_{j=1}^{N} u_{1j}\phi_j\left(x,y\right)\mathbf{e}_1 + \sum_{j=1}^{N} u_{2j}\phi_j\left(x,y\right)\mathbf{e}_2$$

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$\left\{\mathbf{\Phi}_i\left(x,y\right)\right\}_{1 \le i \le 2N}$ base of $V_h \times V_h$ such as $\mathbf{u}_h = \sum_{j=1}^{2N} u_j \mathbf{\Phi}_j$

$$u_j = \begin{cases} u_{1j} & 1 \le j \le N \\ u_{2j-N} & N+1 \le j \le 2N \end{cases} \qquad \mathbf{\Phi}_j = \begin{cases} \phi_j \mathbf{e}_1 & 1 \le j \le N \\ \phi_{j-N}\mathbf{e}_2 & N+1 \le j \le 2N \end{cases}$$

FEM formulation (II)

$$A\left(\mathbf{u},\mathbf{v}\right) = (\lambda+2\mu)\int\limits_D \frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + (\lambda+2\mu)\int\limits_D \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} + \lambda\int\limits_D \frac{\partial u_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} +$$

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$1 \le k \le N$, calling $i = k$ then $\mathbf{\Phi}_k = (\phi_i, 0)$

$$A(\mathbf{u}_h, \mathbf{\Phi}_k) = (\lambda+2\mu)\sum_{j=1}^{N} u_{1j}\int\limits_D \frac{\partial\phi_j}{\partial x_1}\frac{\partial\phi_i}{\partial x_1} + \lambda\sum_{j=1}^{N} u_{2j}\int\limits_D \frac{\partial\phi_j}{\partial x_2}\frac{\partial\phi_i}{\partial x_1} + \mu\sum_{j=1}^{N} u_{1j}\int\limits_D \frac{\partial\phi_j}{\partial x_2}\frac{\partial\phi_i}{\partial x_2} + \mu\sum_{j=1}^{N} u_{2j}\int\limits_D \frac{\partial\phi_j}{\partial x_1}\frac{\partial\phi_i}{\partial x_2}$$

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$N+1 \le k \le 2N$, calling $i = k - N$ then $\mathbf{\Phi}_k = (0, \phi_i)$

$$A(\mathbf{u}_h,\mathbf{\Phi}_k) = (\lambda+2\mu)\sum_{j=1}^N u_{2j}\int_D \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} + \lambda\sum_{j=1}^N u_{1j}\int_D \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} + \mu\sum_{j=1}^N u_{1j}\int_D \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} + \mu\sum_{j=1}^N u_{2j}\int_D \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1}$$

Linear System

$$\mathbf{R}^{11} = \left(r^{11}_{ij} \equiv \int_D \frac{\partial \phi_i}{\partial x_1} \frac{\partial \phi_j}{\partial x_1} \right) \quad \mathbf{R}^{12} = \left(r^{12}_{ij} \equiv \int_D \frac{\partial \phi_i}{\partial x_2} \frac{\partial \phi_j}{\partial x_1} \right) \quad \mathbf{R}^{21} = \left(r^{21}_{ij} \equiv \int_D \frac{\partial \phi_i}{\partial x_1} \frac{\partial \phi_j}{\partial x_2} \right) \quad \mathbf{R}^{22} = \left(r^{12}_{ij} \equiv \int_D \frac{\partial \phi_i}{\partial x_2} \frac{\partial \phi_j}{\partial x_2} \right)$$

$$\mathbf{M} = \left(\int_D \phi_i \phi_j \right)_{ij} \quad \mathbf{M}_\partial = \left(\int_{\partial D} \phi_i \phi_j \right)_{ij}$$

$$\mathbf{A}\mathbf{u} = \mathbf{b}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ u_{N+1} \\ \vdots \\ u_{2N} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \\ b_{N+1} \\ \vdots \\ b_{2N} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}$$

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$$\mathbf{A}=\begin{pmatrix}(\lambda+2\mu)\,\mathbf{R}^{11}+\mu\mathbf{R}^{22} & \lambda\mathbf{R}^{21}+\mu\mathbf{R}^{12}\\ \lambda\mathbf{R}^{12}+\mu\mathbf{R}^{21} & (\lambda+2\mu)\,\mathbf{R}^{22}+\mu\mathbf{R}^{11}\end{pmatrix}\equiv\begin{pmatrix}\mathbf{A}^{11} & \mathbf{A}^{12}\\ \mathbf{A}^{21} & \mathbf{A}^{22}\end{pmatrix}$$

$$b_k=L\left(\mathbf{\Phi}_k\right)=\begin{cases}\int_D f_1\phi_i+\int_{\Gamma_N} g_{N1}\phi_i & 1\le k\le N,\,(j=k)\\ \int_D f_2\phi_i+\int_{\Gamma_N} g_{N2}\phi_i & N+1\le k\le 2N,\,(j=k-N)\end{cases}\Rightarrow \mathbf{b}=\begin{pmatrix}\mathbf{M}\mathbf{f}_1+\mathbf{M}_\partial\mathbf{g}_{N1}\\ \mathbf{M}\mathbf{f}_2+\mathbf{M}_\partial\mathbf{g}_{N2}\end{pmatrix}$$

Boundary Conditions

Newmann: $\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}|_{\Gamma_N} = \mathbf{g}_N$

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Dirichlet $\mathbf{u}|_{\Gamma_D} = \mathbf{g}_D$

Being k index for nodes $\mathbf{x} \in \Gamma_D$ the linear system turns to $\mathbf{A}'\mathbf{u} = \mathbf{b}'$

$$\left(\mathbf{b}_1'\right)_i = \begin{cases} g_{D1,k} & i = k \\ (\mathbf{b}_1)_i - g_{D1,k} \sum_{l=1, l\neq k}^{N} \left(\mathbf{A}^{11}\right)_{lk} - g_{D2,k} \sum_{l=1, l\neq k}^{N} \left(\mathbf{A}^{12}\right)_{lk} & i \neq k \end{cases}$$

$$\left(\mathbf{b}_2'\right)_i = \begin{cases} g_{D2,k} & i = k \\ (\mathbf{b}_2)_i - g_{D1,k} \sum_{l=1, l\neq k}^{N} \left(\mathbf{A}^{21}\right)_{lk} - g_{D2,k} \sum_{l=1, l\neq k}^{N} \left(\mathbf{A}^{22}\right)_{lk} & i \neq k \end{cases}$$

$$\left(\mathbf{A}^{11'}\right)_{ij} = \begin{cases} \delta_{ik}\delta_{jk} & i = k \text{ or } j = k \\ \left(\mathbf{A}^{11}\right)_{ij} & i \neq k \text{ and } j \neq k \end{cases}, \quad \left(\mathbf{A}^{12'}\right)_{ij} = \begin{cases} 0 & i = k \text{ or } j = k \\ \left(\mathbf{A}^{12}\right)_{ij} & i \neq k \text{ and } j \neq k \end{cases},$$

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Strain and Stress Tensors

Once the elasticity problem is solved, the displacements are known: $\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}$

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Von Mises **σ_{VM} yield criterion:**

$$2\sigma_{VM}^2 = (\hat{\sigma}_1 - \hat{\sigma}_2)^2 + (\hat{\sigma}_2 - \hat{\sigma}_3)^2 + (\hat{\sigma}_3 - \hat{\sigma}_1)^2 \overset{2D}{\Rightarrow} \boldsymbol{\sigma^2}_{VM} = \hat{\sigma}_1^2 + \hat{\sigma}_2^2 - \hat{\sigma}_1\hat{\sigma}_2$$

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Once the elasticity problem is solved, the displacements are known: $\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}$

Strain tensor:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \qquad \left.\begin{array}{r} \mathbf{M}\varepsilon_{11} = \mathbf{C}_x \mathbf{u}_1 \\ \mathbf{M}\varepsilon_{22} = \mathbf{C}_y \mathbf{u}_2 \\ \mathbf{M}\varepsilon_{12} = \mathbf{M}\varepsilon_{21} = \mathbf{C}_x \mathbf{u}_2 + \mathbf{C}_y \mathbf{u}_1 \end{array}\right\}$$

Stress tensor:

$$\boldsymbol{\sigma}(\mathbf{u}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \qquad \left.\begin{array}{c} \sigma_{11} = (\lambda + 2\mu)\,\varepsilon_{11} + \lambda\varepsilon_{22} \\ \sigma_{22} = \lambda\varepsilon_{11} + (\lambda + 2\mu)\,\varepsilon_{22} \\ \sigma_{12} = \sigma_{21} = 2\mu\varepsilon_{12} \end{array}\right\}$$

Von Mises σ_{VM} yield criterion:

$$2\sigma_{VM}^2 = (\hat{\sigma}_1 - \hat{\sigma}_2)^2 + (\hat{\sigma}_2 - \hat{\sigma}_3)^2 + (\hat{\sigma}_3 - \hat{\sigma}_1)^2 \overset{2D}{\Rightarrow} \boldsymbol{\sigma^2}_{VM} = \hat{\sigma}_1^2 + \hat{\sigma}_2^2 - \hat{\sigma}_1\hat{\sigma}_2$$

$$\left.\begin{array}{l} \hat{\sigma}_1 = \frac{1}{2}\left(\sigma_{11} + \sigma_{22} + \sqrt{(\sigma_{11} + \sigma_{22})^2 + 4\sigma_{12}^2}\right) \\ \hat{\sigma}_2 = \frac{1}{2}\left(\sigma_{11} + \sigma_{22} - \sqrt{(\sigma_{11} + \sigma_{22})^2 + 4\sigma_{12}^2}\right) \end{array}\right\} \Rightarrow \sigma_{VM} = \sqrt{\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2}$$

Octave Code: Arrays and *Neumann* Conditions

```
% Get matrices
[M R11 C] = fem_mrc(x, y, tri, [1 0;0 0], 0);
[M R12 C] = fem_mrc(x, y, tri, [0 1;0 0], 0);
[M R21 Cx] = fem_mrc(x, y, tri, [0 0;1 0], [1;0]);
[M R22 Cy] = fem_mrc(x, y, tri, [0 0;0 1], [0;1]);
A11 = (lambda+2*mu)*R11 + mu*R22;
A12 = lambda*R21 + mu*R12;
A21 = lambda*R12 + mu*R21;
A22 = mu*R11 + (lambda+2*mu)*R22;
vect_b1 = vect_b2 = 0*x;
% Neumann condition
[AR DR1] = fem_robin(x, y, tri, gn1, 0*x);
[AR DR2] = fem_robin(x, y, tri, gn2, 0*x);
```

Octave Code: *Dirichlet* Conditions. Displacements.

```
% Dirichlet condition
for k=1:length(bcd)
   vect_b1 -= (defx*A11(:,bcd(k)) + defy*A12(:,bcd(k)));
   vect_b2 -= (defx*A21(:,bcd(k)) + defy*A22(:,bcd(k)));
end
A11(bcd,:)  = 0; A11(:,bcd) = 0;
A12(bcd,:)  = 0; A12(:,bcd) = 0;
A21(bcd,:)  = 0; A21(:,bcd) = 0;
A22(bcd,:)  = 0; A22(:,bcd) = 0;
for i=1:length(bcd)
   A11(bcd(i),bcd(i)) = A22(bcd(i),bcd(i)) = 1;
   A12(bcd(i),bcd(i)) = A21(bcd(i),bcd(i)) = 0;
end
vect_b1(bcd) = gd1; vect_b2(bcd) = gd2;
```

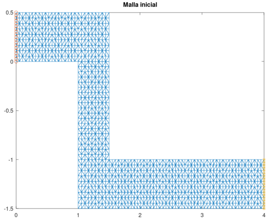
```
% Solve displacements
uh =[A11 A12; A21 A22]\[vect_b1; vect_b2];
N = length(x);
uh1 = uh(1:N);
uh2 = uh(N+[1:N]);
```

Octave Code: *Von Mises*

```
%Strain Tensor
e11 = M\(Cx*uh1);
e22 = M\(Cy*uh2);
e12 = M\(Cx*uh2+Cy*uh1);
%Stress Tensor
sig11 = (lambda+2*mu)*e11 + lambda*e22;
sig22 = lambda*e11 + (lambda+2*mu)*e22;
sig12 = 2*mu*e12;
%Von Mises
sigvm = sqrt(sig11.^2 + sig22.^2 - sig11.*sig22 + 3*sig12.^2);
[maxsig imax] = max(sigvm);
printf("Max.  von Mises (%f MPa) at (%f, %f)\n", maxsig/1e6, x(imax), y(imax));
```

Problem 5.8: `elas8.m`

The 1 centimeter thick steel plate in the figure is supported on a horizontal platform and held at its left end (marked with red dots in the figure), leaving the rest of the sheet free on the horizontal plane. It is subjected to a displacement of the right end (orange crosses) of 20 centimeters to the right of the figure and 40 centimeters downward. Determine the most likely area where a crack will appear due to material failure.



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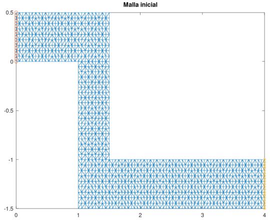

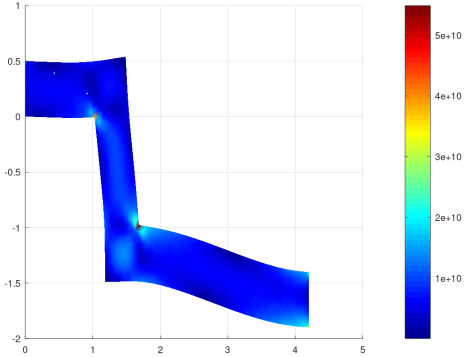

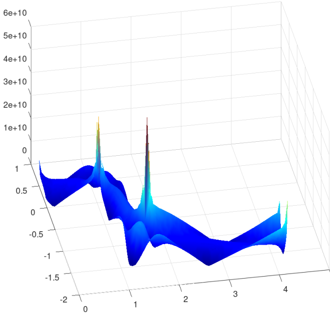

Outline

Thermal Expansion Coefficient

Temperature field affects to the stress tensor

$$\boldsymbol{\sigma}^{*}(\mathbf{u}) = \lambda \nabla \cdot (\mathbf{u})\,\mathbf{I} + 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) - \beta \left(T - T_{ref}\right)\mathbf{I}$$

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$$\sigma^{*}_{ij}(\mathbf{u}) = \lambda \nabla \cdot (\mathbf{u})\, \delta_{ij} + 2\mu \varepsilon_{ij}(\mathbf{u}) - \delta_{ij} \beta (T - T_{ref}) = \sigma_{ij}(\mathbf{u}) - \delta_{ij} \beta (T - T_{ref})$$

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Temperature field affects to the stress tensor

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$$\sigma^*_{ij} (\mathbf{u}) = \lambda \nabla \cdot (\mathbf{u}) \delta_{ij} + 2\mu \varepsilon_{ij} (\mathbf{u}) - \delta_{ij} \beta (T - T_{ref}) = \sigma_{ij} (\mathbf{u}) - \delta_{ij} \beta (T - T_{ref})$$

β coefficient is related to the thermal expansion $\alpha = \frac{1}{A} \frac{\partial A}{\partial T}$ for 2D problems

$$\beta = E \frac{1 + 3\nu}{2 (1 - \nu^2)} \alpha$$

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$$\sigma^*_{ij}(\mathbf{u}) = \lambda \nabla \cdot (\mathbf{u}) \delta_{ij} + 2\mu \varepsilon_{ij}(\mathbf{u}) - \delta_{ij} \beta (T - T_{ref}) = \sigma_{ij}(\mathbf{u}) - \delta_{ij} \beta (T - T_{ref})$$

β coefficient is related to the thermal expansion $\alpha = \frac{1}{A}\frac{\partial A}{\partial T}$ for 2D problems

$$\beta = E \frac{1 + 3\nu}{2(1 - \nu^2)} \alpha$$

For 3D problems being $\alpha = \frac{1}{V}\frac{\partial V}{\partial T}$

$$\beta = E \frac{1}{1 - 2\nu} \alpha$$

Thermo-Mechanical Problem

Get the displacement field $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ for a 2D domain $D \subset \mathbb{R}^2$ with $\Gamma_D, \Gamma_N \subset \partial D$

$$\left.\begin{aligned} -\nabla \cdot (\boldsymbol{\sigma}(\mathbf{u})) &= \mathbf{f} - \beta \mathbf{I} \nabla T \\ \mathbf{u}|_{\Gamma_D} &= \mathbf{g}_D \\ \boldsymbol{\sigma}^*(\mathbf{u}) \cdot \mathbf{n}|_{\Gamma_N} &= \mathbf{g}_N + \beta (T - T_{ref})\,\mathbf{n} \end{aligned}\right\}$$

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Same FEM formulation as Elastic Problem except $\mathbf{b}$ array

$$b'_n = L(\Phi_n) = \begin{cases} \int_D \left(f_1 - \beta \frac{\partial T}{\partial x_1}\right) \phi_i + \int_{\Gamma_N} \left(g_{N1} + \beta (T - T_{ref}) n_1\right) \phi_i & 1 \leq n \leq N, (j = n) \\ \int_D \left(f_2 - \beta \frac{\partial T}{\partial x_2}\right) \phi_i + \int_{\Gamma_N} \left(g_{N2} + \beta (T - T_{ref}) n_2\right) \phi_i & N + 1 \leq n \leq 2N, (j = n - N) \end{cases}$$

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Inverse thermal problem: find field T such as the following problem has as solution $\mathbf{u} = \mathbf{0}$:

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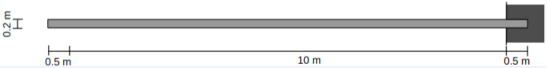
Physical meaning if field T fulfills the heat equation: $-k\Delta T = f$

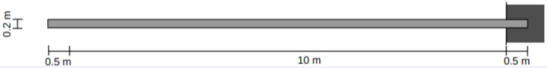
termomecanico1.m



$$\left.\begin{array}{r}-\nabla\cdot(\boldsymbol{\sigma}(\mathbf{u}))=\rho\mathbf{g}-\beta\mathbf{I}\nabla T\\ \mathbf{u}|_{\Gamma_D}=\mathbf{0}\\ \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}|_{\partial D-\Gamma_D}=\beta\left(T-T_{ref}\right)\mathbf{n}\end{array}\right\}$$

$$E=2.1\cdot 10^{11}\,Pa,\ \nu=0.2$$
$$\rho=7900\ kg/m^3$$
$$\mathbf{g}=-9.8\mathbf{e}_2\ m/s$$
$$T_{ref}=15\,^oC\quad \alpha=1.15\cdot 10^{-5}\ {}^oC^{-1}\quad T=\left\{\begin{array}{c}15\,^oC\\ 1000\,^oC\end{array}\right.$$

termomecanico1.m



$$
\left.\begin{aligned}
-\nabla\cdot(\boldsymbol{\sigma}(\mathbf{u})) &= \rho\mathbf{g}{\color{red}-\beta\mathbf{I}\nabla T}\\
\mathbf{u}|_{\Gamma_D} &= \mathbf{0}\\
\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}|_{\partial D-\Gamma_D} &= {\color{red}\beta\left(T-T_{ref}\right)}\,\mathbf{n}
\end{aligned}\right\}
$$

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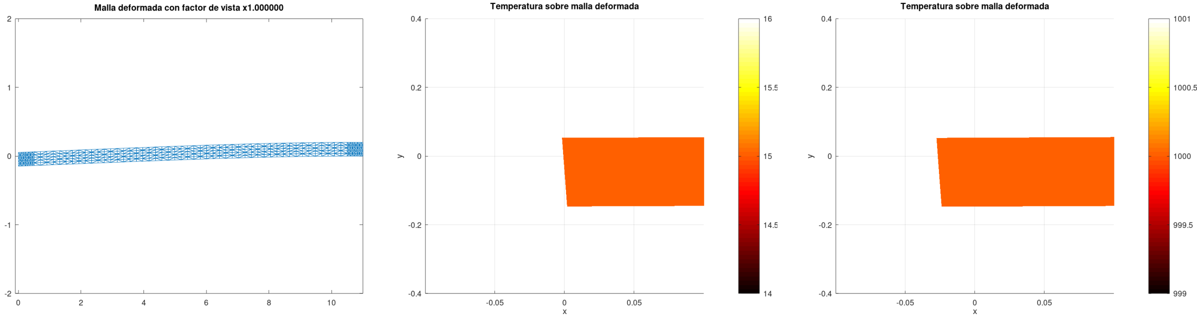

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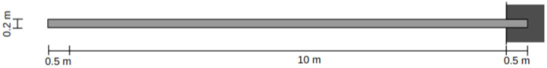

$$\left.\begin{array}{c} -\nabla \cdot (\boldsymbol{\sigma}(\mathbf{u})) = \rho \mathbf{g} - \beta \mathbf{I} \nabla T \\ \mathbf{u}|_{\Gamma_D} = \mathbf{0} \\ \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}|_{\partial D - \Gamma_D} = \beta (T - T_{ref})\, \mathbf{n} \end{array}\right\}$$

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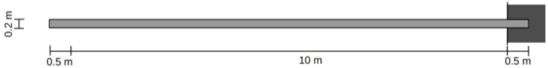
termomecanico1.m



$$\left.\begin{array}{c} -\nabla\cdot(\boldsymbol{\sigma}(\mathbf{u})) = \rho\mathbf{g} - \beta\mathbf{I}\nabla T \\ \mathbf{u}|_{\Gamma_D} = \mathbf{0} \\ \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}|_{\partial D-\Gamma_D} = \beta\left(T - T_{ref}\right)\mathbf{n} \end{array}\right\}$$

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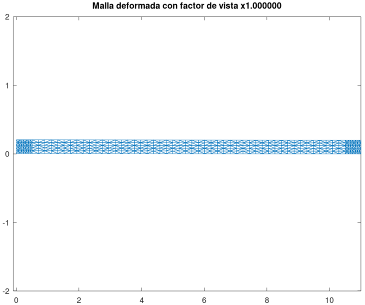

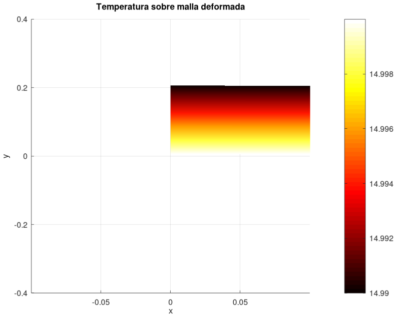

Outline

Dynamic Elasticity

Get the displacement field $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ for a 2D domain $D \subset \mathbb{R}^2$ with $\Gamma_D, \Gamma_N \subset \partial D$

$$\left.\begin{array}{c} \rho\mathbf{u}_{tt} + \frac{b}{2}\mathbf{u}_t - \nabla\cdot(\boldsymbol{\sigma}(\mathbf{u})) = \mathbf{f} \\ \mathbf{u}|_{\Gamma_D} = \mathbf{g}_D \quad t \in (0,T) \\ \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}|_{\Gamma_N} = \mathbf{g}_N \quad t \in (0,T) \\ \mathbf{u}(\mathrm{t}=0) = \mathbf{u}_0 \\ \mathbf{u}_{\mathrm{t}}(\mathrm{t}=0) = \mathbf{v}_0 \end{array}\right\}$$

Dynamic Elasticity

Get the displacement field $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ for a 2D domain $D \subset \mathbb{R}^2$ with $\Gamma_D, \Gamma_N \subset \partial D$

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***Newmark* scheme**

$$\left.\begin{array}{c} \rho\mathbf{a}^{n+1} + \frac{b}{2}\mathbf{v}^{n+1} - \nabla\cdot\left(\boldsymbol{\sigma}\left(\mathbf{u}^{n+1}\right)\right) = \mathbf{f}^{n+1} \quad en \quad D \\ \mathbf{u}^{n+1}\big|_{\Gamma_D} = \mathbf{g}_D^{n+1} \\ \boldsymbol{\sigma}\left(\mathbf{u}^{n+1}\right)\cdot\mathbf{n}\big|_{\Gamma_N} = \mathbf{g}_N^{n+1} \end{array}\right\}$$

Dynamic Elasticity

Get the displacement field $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ for a 2D domain $D \subset \mathbb{R}^2$ with $\mathit{\Gamma}_D, \mathit{\Gamma}_N \subset \partial D$

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Newmark scheme

$$\left.\begin{array}{c} \rho\mathbf{a}^{n+1} + \frac{b}{2}\mathbf{v}^{n+1} - \nabla\cdot\left(\boldsymbol{\sigma}\left(\mathbf{u}^{n+1}\right)\right) = \mathbf{f}^{n+1} \quad en \quad D \\ \mathbf{u}^{n+1}\big|_{\mathit{\Gamma}_D} = \mathbf{g}_D^{n+1} \\ \boldsymbol{\sigma}\left(\mathbf{u}^{n+1}\right)\cdot\mathbf{n}\big|_{\mathit{\Gamma}_N} = \mathbf{g}_N^{n+1} \end{array}\right\}$$

$$\left(\rho + \frac{b}{2}\Delta t\gamma\right)\mathbf{u}^{n+1} - \Delta t^2\beta\nabla\cdot\left(\boldsymbol{\sigma}\left(\mathbf{u}^{n+1}\right)\right) = \Delta t^2\beta\mathbf{f}^{n+1} + \left(\rho + \frac{b}{2}\Delta t\gamma\right)\mathbf{u}^n +$$
$$\Delta t\left[\left(\rho + \frac{b}{2}\Delta t\gamma\right) - \frac{b}{2}\Delta t\beta\right]\mathbf{v}^n + \frac{1}{2}\Delta t^2\left[\left(\rho + \frac{b}{2}\Delta t\gamma\right)(1-2\beta) - b\Delta t\beta(1-\gamma)\right]\mathbf{a}^n$$

Dynamic Elasticity

Get the displacement field $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ for a 2D domain $D \subset \mathbb{R}^2$ with $\Gamma_D, \Gamma_N \subset \partial D$

$$\left.\begin{array}{c} \rho\mathbf{u}_{tt} + \frac{b}{2}\mathbf{u}_t - \nabla\cdot(\boldsymbol{\sigma}(\mathbf{u})) = \mathbf{f} \\ \mathbf{u}|_{\Gamma_D} = \mathbf{g}_D \quad t\in(0,T) \\ \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}|_{\Gamma_N} = \mathbf{g}_N \quad t\in(0,T) \\ \mathbf{u}(\mathrm{t}=0) = \mathbf{u}_0 \\ \mathbf{u}_\mathrm{t}(\mathrm{t}=0) = \mathbf{v}_0 \end{array}\right\}$$

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$$\left.\begin{array}{c} \rho\mathbf{a}^{n+1} + \frac{b}{2}\mathbf{v}^{n+1} - \nabla\cdot\left(\boldsymbol{\sigma}\left(\mathbf{u}^{n+1}\right)\right) = \mathbf{f}^{n+1} \quad en \quad D \\ \mathbf{u}^{n+1}\big|_{\Gamma_D} = \mathbf{g}_D^{n+1} \\ \boldsymbol{\sigma}\left(\mathbf{u}^{n+1}\right)\cdot\mathbf{n}\big|_{\Gamma_N} = \mathbf{g}_N^{n+1} \end{array}\right\}$$

$$\left(\rho + \tfrac{b}{2}\Delta t\gamma\right)\mathbf{u}^{n+1} - \Delta t^2\beta\nabla\cdot\left(\boldsymbol{\sigma}\left(\mathbf{u}^{n+1}\right)\right) = \Delta t^2\beta\mathbf{f}^{n+1} + \left(\rho + \tfrac{b}{2}\Delta t\gamma\right)\mathbf{u}^n +$$
$$\Delta t\left[\left(\rho + \tfrac{b}{2}\Delta t\gamma\right) - \tfrac{b}{2}\Delta t\beta\right]\mathbf{v}^n + \tfrac{1}{2}\Delta t^2\left[\left(\rho + \tfrac{b}{2}\Delta t\gamma\right)(1-2\beta) - b\Delta t\beta(1-\gamma)\right]\mathbf{a}^n$$

For the first instant we have to compute $\mathbf{a}^0$

$$\mathbf{a}^0 = \frac{1}{\rho}\left[\mathbf{f}^0 - \frac{b}{2}\mathbf{v}^0 + \nabla\cdot\left(\boldsymbol{\sigma}\left(\mathbf{u}^0\right)\right)\right]$$

Dynamic Elasticity: FEM formulation

$$\mathbf{u}^n = \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \\ u_{N+1}^n \\ \vdots \\ u_{2N}^n \end{pmatrix} \equiv \begin{pmatrix} \mathbf{u}_1^n \\ \mathbf{u}_2^n \end{pmatrix}, \quad \mathbf{v}^n = \begin{pmatrix} v_1^n \\ \vdots \\ v_N^n \\ v_{N+1}^n \\ \vdots \\ v_{2N}^n \end{pmatrix} \equiv \begin{pmatrix} \mathbf{u}_1^n \\ \mathbf{u}_2^n \end{pmatrix}, \quad \mathbf{a}^n = \begin{pmatrix} a_1^n \\ \vdots \\ a_N^n \\ a_{N+1}^n \\ \vdots \\ a_{2N}^n \end{pmatrix} \equiv \begin{pmatrix} \mathbf{a}_1^n \\ \mathbf{a}_2^n \end{pmatrix}$$

$$\overline{\mathbf{M}} = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix}, \qquad \overline{\mathbf{M}}_\partial = \begin{pmatrix} \mathbf{M}_\partial & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_\partial \end{pmatrix}$$

Dynamic Elasticity: FEM formulation

$$\mathbf{u}^n = \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \\ u_{N+1}^n \\ \vdots \\ u_{2N}^n \end{pmatrix} \equiv \begin{pmatrix} \mathbf{u}_1^n \\ \mathbf{u}_2^n \end{pmatrix}, \quad \mathbf{v}^n = \begin{pmatrix} v_1^n \\ \vdots \\ v_N^n \\ v_{N+1}^n \\ \vdots \\ v_{2N}^n \end{pmatrix} \equiv \begin{pmatrix} \mathbf{u}_1^n \\ \mathbf{u}_2^n \end{pmatrix}, \quad \mathbf{a}^n = \begin{pmatrix} a_1^n \\ \vdots \\ a_N^n \\ a_{N+1}^n \\ \vdots \\ a_{2N}^n \end{pmatrix} \equiv \begin{pmatrix} \mathbf{a}_1^n \\ \mathbf{a}_2^n \end{pmatrix}$$

$$\overline{\mathbf{M}} = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix}, \qquad \overline{\mathbf{M}}_\partial = \begin{pmatrix} \mathbf{M}_\partial & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_\partial \end{pmatrix}$$

Acceleration first instant

$$\overline{\mathbf{M}}\mathbf{a}^0 = \frac{1}{\rho}\left[\overline{\mathbf{M}}\left(\mathbf{f}^0 - \frac{b}{2}\mathbf{v}^0\right) + \overline{\mathbf{M}}_\partial \mathbf{g}_N^0 - \mathbf{A}\mathbf{u}^0\right]$$

Dynamic Elasticity: FEM formulation

Next instants

$$\overline{\mathbf{A}}\mathbf{u}^{n+1} = \overline{\mathbf{M}}\mathbf{b} + \Delta t^2 \beta \overline{\mathbf{M}}_{\partial} \mathbf{g}_N^n$$

Dynamic Elasticity: FEM formulation

Next instants

$$\overline{\mathbf{A}}\mathbf{u}^{n+1} = \overline{\mathbf{M}}\mathbf{b} + \Delta t^2 \beta \overline{\mathbf{M}}_{\partial} \mathbf{g}_N^n$$

$$\overline{\mathbf{A}} = \left(\rho + \tfrac{b}{2}\Delta t \gamma\right) \overline{\mathbf{M}} + \Delta t^2 \beta \mathbf{A}$$

$$\mathbf{b} = \Delta t^2 \beta \mathbf{f}^{n+1} + \left(\rho + \tfrac{b}{2}\Delta t \gamma\right) \mathbf{u}^n + \Delta t \left[\rho + \tfrac{b}{2}\Delta t \left(\gamma - \beta\right)\right] \mathbf{v}^n +$$
$$+ \tfrac{1}{2}\Delta t^2 \left[\left(\rho + \tfrac{b}{2}\Delta t \gamma\right)\left(1 - 2\beta\right) - b \Delta t \beta \left(1 - \gamma\right)\right] \mathbf{a}^n$$

Dynamic Elasticity: FEM formulation

Next instants

$$\overline{\mathbf{A}}\mathbf{u}^{n+1} = \overline{\mathbf{M}}\mathbf{b} + \Delta t^2 \beta \overline{\mathbf{M}}_{\partial}\mathbf{g}_N^n$$

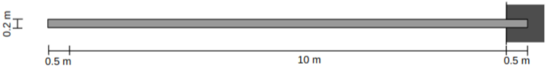
$$\overline{\mathbf{A}} = \left(\rho + \tfrac{b}{2}\Delta t \gamma\right)\overline{\mathbf{M}} + \Delta t^2 \beta \mathbf{A}$$
$$\mathbf{b} = \Delta t^2 \beta \mathbf{f}^{n+1} + \left(\rho + \tfrac{b}{2}\Delta t\gamma\right)\mathbf{u}^n + \Delta t\left[\rho + \tfrac{b}{2}\Delta t\left(\gamma - \beta\right)\right]\mathbf{v}^n +$$
$$+\tfrac{1}{2}\Delta t^2\left[\left(\rho + \tfrac{b}{2}\Delta t\gamma\right)\left(1 - 2\beta\right) - b\Delta t\beta\left(1-\gamma\right)\right]\mathbf{a}^n$$

After getting displacements $\mathbf{u}^{n+1}$ we must obtain $\mathbf{v}^{n+1}$ and $\mathbf{a}^{n+1}$

$$\mathbf{v}^{n+1} = \mathbf{v}^n + \Delta t\left(\gamma \mathbf{a}^{n+1} + \left(1-\gamma\right)\mathbf{a}^n\right)$$

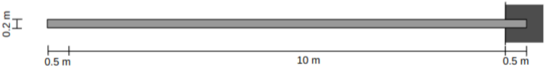
$$\mathbf{a}^{n+1} = \frac{1}{\beta \Delta t^2}\left[\mathbf{u}^{n+1} - \mathbf{u}^n - \Delta t \mathbf{v}^n - \frac{1}{2}\Delta t^2\left(1 - 2\beta\right)\mathbf{a}^n\right]$$

Problem 5.6: `elas6.m`

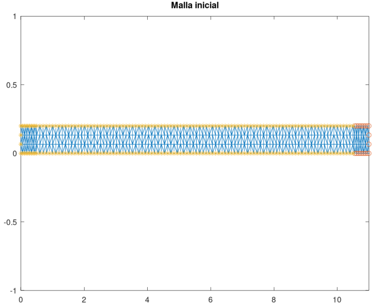


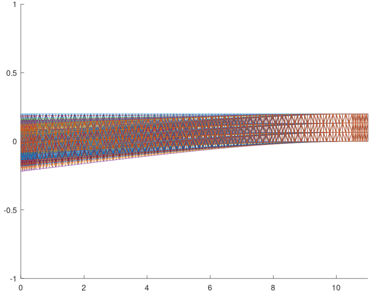
$$
\left.\begin{array}{c}
\rho \mathbf{u}_{tt} + \frac{b}{2}\mathbf{u}_t - \nabla \cdot (\boldsymbol{\sigma}(\mathbf{u})) = \rho \mathbf{g} \\
\mathbf{u}|_{\Gamma_D} = \mathbf{0} \\
\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}|_{\partial D - \Gamma_D} = \mathbf{0} \\
\mathbf{u}(\mathrm{t}=0) = \mathbf{u}_{\mathrm{t}}(\mathrm{t}=0) = \mathbf{0}
\end{array}\right\}
\begin{array}{c}
E = 2.1 \cdot 10^{11} Pa, \ \nu = 0.2 \\
\rho = 7900 \ kg/m^3 \\
\mathbf{g} = -9.8\mathbf{e}_2 \ m/s \\
b = 10^4 Pa \cdot s/m
\end{array}
$$

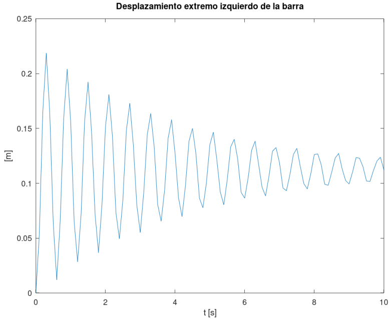
Problem 5.6: `elas6.m`



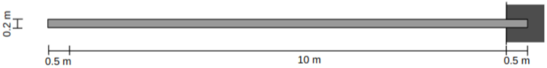
$$\left.\begin{array}{c}\rho \mathbf{u}_{tt} + \frac{b}{2}\mathbf{u}_t - \nabla\cdot(\boldsymbol{\sigma}(\mathbf{u})) = \rho\mathbf{g} \\ \mathbf{u}|_{\Gamma_D} = \mathbf{0} \\ \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}|_{\partial D-\Gamma_D} = \mathbf{0} \\ \mathbf{u}(\mathrm{t}=0) = \mathbf{u}_{\mathrm{t}}(\mathrm{t}=0) = \mathbf{0}\end{array}\right\} \quad \begin{array}{c} E = 2.1\cdot 10^{11} Pa,\ \nu = 0.2 \\ \rho = 7900\ kg/m^3 \\ \mathbf{g} = -9.8\mathbf{e}_2\ m/s \\ b = 10^4 Pa\cdot s/m \end{array}$$







resonancia2.m



$$
\left.\begin{array}{c}
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\mathbf{u}|_{\Gamma_D} = \mathbf{0} \\
\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}|_{\partial D - \Gamma_D} = 0.5\cdot 10^5\left(1-\sin\left(2\pi t/0.6\right)\right)\mathbf{e}_2 \\
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\begin{array}{c}
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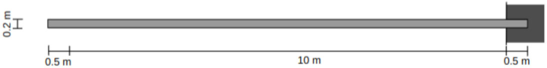
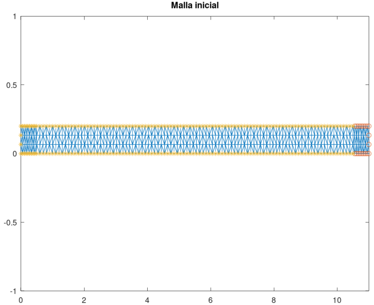
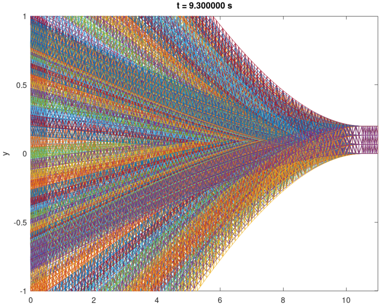
\end{array}
$$

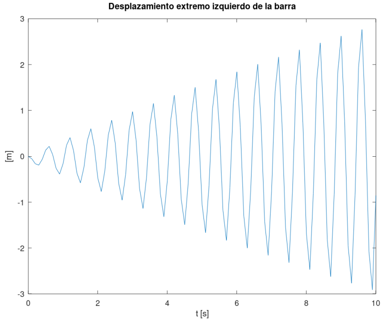
resonancia2.m



$$\left.\begin{array}{c} \rho \mathbf{u}_{tt} + \frac{b}{2}\mathbf{u}_t - \nabla \cdot (\boldsymbol{\sigma}(\mathbf{u})) = \rho \mathbf{g} \\ \mathbf{u}|_{\Gamma_D} = \mathbf{0} \\ \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}|_{\partial D - \Gamma_D} = 0.5 \cdot 10^5 \left(1 - \sin(2\pi t/0.6)\right) \mathbf{e}_2 \\ \mathbf{u}(\mathrm{t}=0) = \mathbf{u}_{\mathrm{t}}(\mathrm{t}=0) = \mathbf{0} \end{array}\right\} \begin{array}{l} E = 2.1 \cdot 10^{11} Pa,\ \nu = 0.2 \\ \rho = 7900\ kg/m^3 \\ \mathbf{g} = -9.8\mathbf{e}_2\ m/s \\ b = 10^4 Pa \cdot s/m \end{array}$$







Outline

Axisymmetric Elastic Problem

The tensors continue to be symmetrical, but necessary to take into account θ component

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xr} & 0 \\ \varepsilon_{xr} & \varepsilon_{rr} & 0 \\ 0 & 0 & \varepsilon_{\theta\theta} \end{pmatrix} \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xr} & 0 \\ \sigma_{xr} & \sigma_{rr} & 0 \\ 0 & 0 & \sigma_{\theta\theta} \end{pmatrix}$$

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In *Voigt* notation

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{xr} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_r}{\partial r} \\ \frac{u_r}{r} \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x}\right) \end{pmatrix} \quad \begin{pmatrix} \sigma_{xx} \\ \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{xr} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{xr} \end{pmatrix}$$

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$\nabla \cdot (\boldsymbol{\sigma}) = -\mathbf{f}$ in cylindrical coordinates

$$\left.\begin{aligned} (\lambda+2\mu)\frac{\partial^2 u_x}{\partial x^2} + \mu\frac{\partial^2 u_x}{\partial r^2} + (\lambda+\mu)\frac{\partial^2 u_r}{\partial x \partial r} + \frac{\mu}{r}\frac{\partial u_x}{\partial r} + \frac{\lambda+\mu}{r}\frac{\partial u_r}{\partial x} = -f_x \\ (\lambda+2\mu)\frac{\partial^2 u_r}{\partial r^2} + \mu\frac{\partial^2 u_r}{\partial x^2} + (\lambda+\mu)\frac{\partial^2 u_x}{\partial x \partial r} + \frac{\lambda}{r}\frac{\partial u_x}{\partial x} + 2\frac{\lambda+\mu}{r}\frac{\partial u_r}{\partial x} = -f_r \end{aligned}\right\}$$

FEM Formulation: $\mathbf{A}\mathbf{u} = \mathbf{b}$

$$\iiint_D \boldsymbol{\sigma}(\mathbf{u}) : \nabla \mathbf{v} = \iiint_D \langle \mathbf{f}, \mathbf{v} \rangle + \iint_{\partial D} \langle \mathbf{g}_N, \mathbf{v} \rangle$$

$$\begin{pmatrix} (\lambda + 2\mu)\,\mathbf{R}_r^{xx} + \mu \mathbf{R}_r^{rr} & \lambda \left(\mathbf{R}_r^{rx} + \mathbf{C}_r^{0x}\right) + \mu \mathbf{R}_r^{xr} \\ \lambda \left(\mathbf{R}_r^{xr} + \mathbf{C}_r^{x0}\right) + \mu \mathbf{R}_r^{rx} & (\lambda + 2\mu)\left(\mathbf{R}_r^{rr} + \mathbf{M}^0\right) + \mu \mathbf{R}^{xx} + \lambda \left(\mathbf{C}_r^{0r} + \mathbf{C}_r^{r0}\right) \end{pmatrix} \mathbf{u} = \begin{pmatrix} \mathbf{M}_r \mathbf{f}_1 + \mathbf{M}_{\partial r} \mathbf{g}_{N1} \\ \mathbf{M}_r \mathbf{f}_2 + \mathbf{M}_{\partial r} \mathbf{g}_{N1} \end{pmatrix}$$

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$$\mathbf{R}_r^{xx} = \left(r_{ij}^{rx} \equiv 2\pi \iint_\Omega \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} r \cdot dx \cdot dr\right) \qquad \mathbf{R}_r^{xr} = \left(r_{ij}^{rx} \equiv 2\pi \iint_\Omega \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial r} r \cdot dx \cdot dr\right)$$

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$$\mathbf{C}_r^{0x} = \left(c_{ij}^{0x} \equiv 2\pi \iint_\Omega \phi_j \frac{\partial \phi_i}{\partial x} dx \cdot dr\right) \qquad \mathbf{C}_r^{0r} = \left(c_{ij}^{0r} \equiv 2\pi \iint_\Omega \phi_j \frac{\partial \phi_i}{\partial r} dx \cdot dr\right)$$

$$\mathbf{C}_r^{x0} = \left(c_{ij}^{0x} \equiv 2\pi \iint_\Omega \frac{\partial \phi_j}{\partial x} \phi_i dx \cdot dr\right) \qquad \mathbf{C}_r^{r0} = \left(c_{ij}^{r0} \equiv 2\pi \iint_\Omega \frac{\partial \phi_j}{\partial r} \phi_i dx \cdot dr\right)$$

$$\mathbf{M}_r = \left(m_{ij} \equiv 2\pi \iint_\Omega \phi_j \phi_i r \cdot dx \cdot dr\right) \qquad \mathbf{M}^0 = \left(m_{ij}^0 \equiv 2\pi \iint_\Omega \frac{1}{r} \phi_j \phi_i dx \cdot dr\right)$$

FEM Formulation: $\mathbf{Au} = \mathbf{b}$

$$\iiint_D \boldsymbol{\sigma}(\mathbf{u}) : \nabla \mathbf{v} = \iiint_D \langle \mathbf{f}, \mathbf{v} \rangle + \iint_{\partial D} \langle \mathbf{g}_N, \mathbf{v} \rangle$$

$$\begin{pmatrix} (\lambda + 2\mu)\,\mathbf{R}_r^{xx} + \mu \mathbf{R}_r^{rr} & \lambda\left(\mathbf{R}_r^{rx} + \mathbf{C}_r^{0x}\right) + \mu \mathbf{R}_r^{xr} \\ \lambda\left(\mathbf{R}_r^{xr} + \mathbf{C}_r^{x0}\right) + \mu \mathbf{R}_r^{rx} & (\lambda + 2\mu)\left(\mathbf{R}_r^{rr} + \mathbf{M}^0\right) + \mu \mathbf{R}^{xx} + \lambda\left(\mathbf{C}_r^{0r} + \mathbf{C}_r^{r0}\right) \end{pmatrix} \mathbf{u} = \begin{pmatrix} \mathbf{M}_r \mathbf{f}_1 + \mathbf{M}_{\partial r} \mathbf{g}_{N1} \\ \mathbf{M}_r \mathbf{f}_2 + \mathbf{M}_{\partial r} \mathbf{g}_{N1} \end{pmatrix}$$

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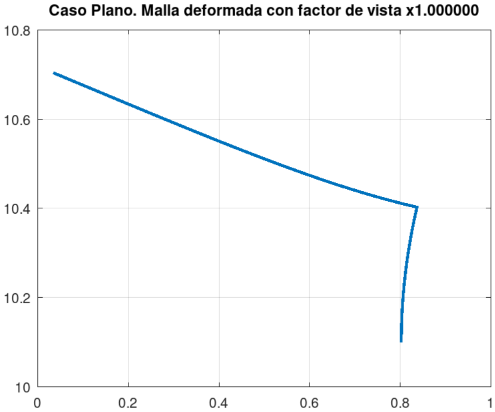
$$\mathbf{C}_r^{x0} = \left(c_{ij}^{0x} \equiv 2\pi \iint_\Omega \frac{\partial \phi_j}{\partial x} \phi_i dx \cdot dr\right) \qquad \mathbf{C}_r^{r0} = \left(c_{ij}^{r0} \equiv 2\pi \iint_\Omega \frac{\partial \phi_j}{\partial r} \phi_i dx \cdot dr\right)$$

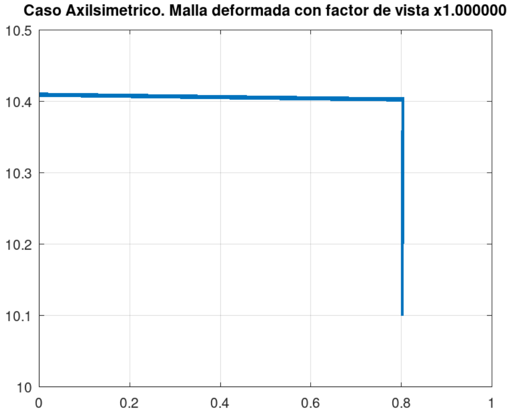
$$\mathbf{M}_r = \left(m_{ij} \equiv 2\pi \iint_\Omega \phi_j \phi_i r \cdot dx \cdot dr\right) \qquad \mathbf{M}^0 = \left(m_{ij}^0 \equiv 2\pi \iint_\Omega \frac{1}{r} \phi_j \phi_i dx \cdot dr\right)$$

```
[M M0 R11 C] = fem_mrc_axis(x, y, tri, [1 0;0 0], 0);
[M M0 R12 C] = fem_mrc_axis(x, y, tri, [0 1;0 0], 0);
[M M0 R21 Cx] = fem_mrc_axis(x, y, tri, [0 0;1 0], [1./y 0*x]);
[M M0 R22 Cr] = fem_mrc_axis(x, y, tri, [0 0;0 1], [0*x 1./y]);
A11 = (lambda+2*mu)*R11 + mu*R22; A12 = lambda*R21 + mu*R12 + lambda*Cx';
A21 = lambda*R12 + mu*R21 + lambda*Cx;
A22 = (lambda+2*mu)*(R22 + M0) + mu*R11 + lambda*(Cr'+Cr);
A = [A11 A12; A21 A22];
```

Example: Plane-Axisymmetric Comparison

Calculate the deformations of a plane case and its axisymmetric equivalent. It is a 5 cm steel plate. The horizontal section supports a pressure of 0.2 bar. The boundary at lowest coordinate has zero displacement.





Orthotropic Media (I)

2D problem, 4 parameters: E_1, E_2, ν_{12} (*Young* modulus and *Poisson* ratio) and G_{12} (shear modulus)

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \equiv \mathbf{D}^{-1} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$

As $\mathbf{D}^{-1}$ is symmetric: $\nu_{21}E_1 = \nu_{12}E_2$. Also $\mathbf{D}$ is symmetric.

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As $\mathbf{D}^{-1}$ is symmetric: $\nu_{21}E_1 = \nu_{12}E_2$. Also $\mathbf{D}$ is symmetric.

The generalization of *Lamé* parameters turns

$$\boldsymbol{\sigma}(\mathbf{u}) = \nabla \cdot (\mathbf{u}) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + 2 \begin{pmatrix} \sqrt{\mu_1} & 0 \\ 0 & \sqrt{\mu_2} \end{pmatrix} \boldsymbol{\varepsilon}(\mathbf{u}) \begin{pmatrix} \sqrt{\mu_1} & 0 \\ 0 & \sqrt{\mu_2} \end{pmatrix} \Rightarrow$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \begin{pmatrix} (\lambda_1 + 2\mu_1)\,\varepsilon_{11} + \lambda_1\varepsilon_{22} & 2\sqrt{\mu_1\mu_2}\varepsilon_{12} \\ 2\sqrt{\mu_1\mu_2}\varepsilon_{12} & (\lambda_2 + 2\mu_2)\,\varepsilon_{22} + \lambda_2\varepsilon_{11} \end{pmatrix}$$

Orthotropic Media (I)

2D problem, 4 parameters: E_1, E_2, ν_{12} (*Young* modulus and *Poisson* ratio) and G_{12} (shear modulus)

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \equiv \mathbf{D}^{-1} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$

As $\mathbf{D}^{-1}$ is symmetric: $\nu_{21}E_1 = \nu_{12}E_2$. Also $\mathbf{D}$ is symmetric.

The generalization of *Lamé* parameters turns

$$\boldsymbol{\sigma}(\mathbf{u}) = \nabla\cdot(\mathbf{u}) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + 2 \begin{pmatrix} \sqrt{\mu_1} & 0 \\ 0 & \sqrt{\mu_2} \end{pmatrix} \boldsymbol{\varepsilon}(\mathbf{u}) \begin{pmatrix} \sqrt{\mu_1} & 0 \\ 0 & \sqrt{\mu_2} \end{pmatrix} \Rightarrow$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \begin{pmatrix} (\lambda_1 + 2\mu_1)\,\varepsilon_{11} + \lambda_1\varepsilon_{22} & 2\sqrt{\mu_1\mu_2}\varepsilon_{12} \\ 2\sqrt{\mu_1\mu_2}\varepsilon_{12} & (\lambda_2 + 2\mu_2)\,\varepsilon_{22} + \lambda_2\varepsilon_{11} \end{pmatrix}$$

$$\lambda_1 = E_1\frac{\nu_{21}}{1-\nu_{12}\nu_{21}} \qquad \lambda_2 = E_2\frac{\nu_{12}}{1-\nu_{12}\nu_{21}}$$
$$\mu_1 = \frac{E_1}{2}\frac{1-\nu_{21}}{1-\nu_{12}\nu_{21}} \qquad \mu_2 = \frac{E_2}{2}\frac{1-\nu_{12}}{1-\nu_{12}\nu_{21}}$$

Orthotropic Media (I)

2D problem, 4 parameters: E_1, E_2, ν_{12} (*Young* modulus and *Poisson* ratio) and G_{12} (shear modulus)

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \equiv \mathbf{D}^{-1} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$

As $\mathbf{D}^{-1}$ is symmetric: $\nu_{21}E_1 = \nu_{12}E_2$. Also $\mathbf{D}$ is symmetric.

The generalization of *Lamé* parameters turns

$$\boldsymbol{\sigma}(\mathbf{u}) = \nabla \cdot (\mathbf{u}) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + 2 \begin{pmatrix} \sqrt{\mu_1} & 0 \\ 0 & \sqrt{\mu_2} \end{pmatrix} \boldsymbol{\varepsilon}(\mathbf{u}) \begin{pmatrix} \sqrt{\mu_1} & 0 \\ 0 & \sqrt{\mu_2} \end{pmatrix} \Rightarrow$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \begin{pmatrix} (\lambda_1 + 2\mu_1)\,\varepsilon_{11} + \lambda_1\varepsilon_{22} & 2\sqrt{\mu_1\mu_2}\varepsilon_{12} \\ 2\sqrt{\mu_1\mu_2}\varepsilon_{12} & (\lambda_2 + 2\mu_2)\,\varepsilon_{22} + \lambda_2\varepsilon_{11} \end{pmatrix}$$

$$\lambda_1 = E_1 \frac{\nu_{21}}{1 - \nu_{12}\nu_{21}} \qquad \lambda_2 = E_2 \frac{\nu_{12}}{1 - \nu_{12}\nu_{21}}$$

$$\mu_1 = \frac{E_1}{2} \frac{1 - \nu_{21}}{1 - \nu_{12}\nu_{21}} \qquad \mu_2 = \frac{E_2}{2} \frac{1 - \nu_{12}}{1 - \nu_{12}\nu_{21}}$$

- ***Iso-orthotropic*** material: orthotropic material with $E_1 = E_2$ and $\nu_{12} = \nu_{21}$.
- An *iso-orthotropic* material with $G = E/2\,(1 + \nu)$ is ***isotropic***.

Orthotropic Media (II)

Local axis parallel to main axis

$$\left.\begin{array}{l}(\lambda_1+2\mu_1)\dfrac{\partial^2 u_1}{\partial x_1^2}+\sqrt{\mu_1\mu_2}\dfrac{\partial^2 u_1}{\partial x_2^2}+\left(\lambda_1+\sqrt{\mu_1\mu_2}\right)\dfrac{\partial^2 u_2}{\partial x_1\partial x_2}=-f_1\\(\lambda_2+2\mu_2)\dfrac{\partial^2 u_2}{\partial x_2^2}+\sqrt{\mu_1\mu_2}\dfrac{\partial^2 u_2}{\partial x_1^2}+\left(\lambda_2+\sqrt{\mu_1\mu_2}\right)\dfrac{\partial^2 u_1}{\partial x_1\partial x_2}=-f_2\end{array}\right\}$$

Orthotropic Media (II)

Local axis parallel to main axis

$$\left.\begin{array}{l}(\lambda_1 + 2\mu_1)\dfrac{\partial^2 u_1}{\partial x_1^2} + \sqrt{\mu_1\mu_2}\dfrac{\partial^2 u_1}{\partial x_2^2} + (\lambda_1 + \sqrt{\mu_1\mu_2})\dfrac{\partial^2 u_2}{\partial x_1 \partial x_2} = -f_1 \\ (\lambda_2 + 2\mu_2)\dfrac{\partial^2 u_2}{\partial x_2^2} + \sqrt{\mu_1\mu_2}\dfrac{\partial^2 u_2}{\partial x_1^2} + (\lambda_2 + \sqrt{\mu_1\mu_2})\dfrac{\partial^2 u_1}{\partial x_1 \partial x_2} = -f_2\end{array}\right\}$$

FEM formulation: $\mathbf{A}\mathbf{u} = \mathbf{b}$

$$\boldsymbol{\sigma}(\mathbf{u}) : \nabla(\mathbf{v}) = \nabla \cdot (\mathbf{u}) \left\langle \nabla \mathbf{v}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right\rangle + 2\left[\begin{pmatrix} \sqrt{\mu_1} & 0 \\ 0 & \sqrt{\mu_2} \end{pmatrix} \boldsymbol{\varepsilon}(\mathbf{u})\right] : \left[\boldsymbol{\varepsilon}(\mathbf{v}) \begin{pmatrix} \sqrt{\mu_1} & 0 \\ 0 & \sqrt{\mu_2} \end{pmatrix}\right]$$

$$\mathbf{A} = \begin{pmatrix} (\lambda_1 + 2\mu_1)\mathbf{R}^{11} + \sqrt{\mu_1\mu_2}\mathbf{R}^{22} & \lambda_1 \mathbf{R}^{21} + \sqrt{\mu_1\mu_2}\mathbf{R}^{12} \\ \lambda_2 \mathbf{R}^{12} + \sqrt{\mu_1\mu_2}\mathbf{R}^{21} & (\lambda_2 + 2\mu_2)\mathbf{R}^{22} + \sqrt{\mu_1\mu_2}\mathbf{R}^{11} \end{pmatrix}$$

Orthotropic Media (II)

Local axis parallel to main axis

$$\left.\begin{array}{l}(\lambda_1+2\mu_1)\dfrac{\partial^2 u_1}{\partial x_1^2}+\sqrt{\mu_1\mu_2}\dfrac{\partial^2 u_1}{\partial x_2^2}+(\lambda_1+\sqrt{\mu_1\mu_2})\dfrac{\partial^2 u_2}{\partial x_1\partial x_2}=-f_1\\ (\lambda_2+2\mu_2)\dfrac{\partial^2 u_2}{\partial x_2^2}+\sqrt{\mu_1\mu_2}\dfrac{\partial^2 u_2}{\partial x_1^2}+(\lambda_2+\sqrt{\mu_1\mu_2})\dfrac{\partial^2 u_1}{\partial x_1\partial x_2}=-f_2\end{array}\right\}$$

FEM formulation: $\mathbf{Au}=\mathbf{b}$

$$\boldsymbol{\sigma}(\mathbf{u}):\nabla(\mathbf{v})=\nabla\cdot(\mathbf{u})\left\langle\nabla\mathbf{v},\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}\right\rangle+2\left[\begin{pmatrix}\sqrt{\mu_1}&0\\0&\sqrt{\mu_2}\end{pmatrix}\boldsymbol{\varepsilon}(\mathbf{u})\right]:\left[\boldsymbol{\varepsilon}(\mathbf{v})\begin{pmatrix}\sqrt{\mu_1}&0\\0&\sqrt{\mu_2}\end{pmatrix}\right]$$

$$\mathbf{A}=\begin{pmatrix}(\lambda_1+2\mu_1)\mathbf{R}^{11}+\sqrt{\mu_1\mu_2}\mathbf{R}^{22} & \lambda_1\mathbf{R}^{21}+\sqrt{\mu_1\mu_2}\mathbf{R}^{12}\\ \lambda_2\mathbf{R}^{12}+\sqrt{\mu_1\mu_2}\mathbf{R}^{21} & (\lambda_2+2\mu_2)\mathbf{R}^{22}+\sqrt{\mu_1\mu_2}\mathbf{R}^{11}\end{pmatrix}$$

Local axis rotated α. Being $\mathbf{G}$ the basis change matrix:

$$\boldsymbol{\sigma}(\mathbf{u})=\nabla\cdot(\mathbf{u})\,\mathbf{N}(\lambda_1,\lambda_2)+2\mathbf{N}(\sqrt{\mu_1},\sqrt{\mu_2})\,\boldsymbol{\varepsilon}(\mathbf{u})\,\mathbf{N}(\sqrt{\mu_1},\sqrt{\mu_2})$$

$$\mathbf{N}(a,b)=\mathbf{G}\begin{pmatrix}a&0\\0&b\end{pmatrix}\mathbf{G}^t=\begin{pmatrix}a\cos^2(\alpha)+b\sin^2(\alpha) & (b-a)\sin(\alpha)\cos(\alpha)\\ (b-a)\sin(\alpha)\cos(\alpha) & a\sin^2(\alpha)+b\cos^2(\alpha)\end{pmatrix}$$

Challenge

Unsteady thermo-mechanical problem on ...

- an **axisymmetric** geometry,
- **orthotropic** material and
- **variable local base**

$$\left.\begin{array}{c} \rho \mathbf{u}_{tt} + \frac{b}{2}\mathbf{u}_t - \nabla \cdot (\boldsymbol{\sigma}(\mathbf{u})) = \mathbf{f} - \beta \mathbf{I} \nabla T \\ \mathbf{u}|_{\Gamma_D} = \mathbf{g}_D \\ \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}|_{\Gamma_N} = \mathbf{g}_N + \beta (T - T_{ref}) \mathbf{n} \\ \mathbf{u}(\mathrm{t} = 0) = \mathbf{u}_0 \\ \mathbf{u}_\mathrm{t}(\mathrm{t} = 0) = \mathbf{v}_0 \end{array}\right\}$$

End Part IV

Thank you! Questions?