Finite Element Method: Applications based on Octave/MATLAB

Part V: Computational Fluid Dynamics

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Outline

Stokes Problem

Simplification of full *Navier-Stokes* equations:

- Imcompressible flow ($\partial \rho = 0$).
- Dominance of viscous effects ($\nu \gg 1$).
- Steady state ($\partial / \partial t = 0$)

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Find velocity $\mathbf{u}$ and pressure p fields in a domain D

$$\left.\begin{array}{c} -\nu \Delta \mathbf{u} = -\frac{1}{\rho_0} \nabla p + \mathbf{b} \\ \nabla \cdot (\mathbf{u}) = 0 \end{array}\right\} + b.c.\, (\mathbf{u})$$

where $\mathbf{b}$ is a volumetric force field and ν the kinematic viscosity (m^2/s).

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Variational formulation: Find $\mathbf{u} \in H^1(D)^2$ and $p \in H^1(D)$ such as $\forall \mathbf{v} \in H^1(D)^2$ and $\forall q \in H^1(D)$

$$\left.\begin{array}{c} -\nu \int_D \langle \nabla \cdot (\nabla \mathbf{u})\,, \mathbf{v} \rangle + \frac{1}{\rho_0} \int_D \langle \nabla p, \mathbf{v} \rangle = \int_D \langle \mathbf{b}, \mathbf{v} \rangle \\ \int_D (\nabla \cdot (\mathbf{u}))\, q = 0 \end{array}\right\}$$

Solution: Existence and Uniqueness

Using that $\nabla \cdot (\mathbf{I}p) = \nabla p$ and the divergence theorem, we define the following operators:

$$\left.\begin{array}{c} a\left(\mathbf{u}, \mathbf{v}\right) = \nu \int_D \nabla \mathbf{u} \colon \nabla \mathbf{v} \\ b\left(p, \mathbf{v}\right) = -\frac{1}{\rho_0} \int_D p\mathbf{I} \colon \nabla \mathbf{v} \\ L\left(\mathbf{v}\right) = \int_D \langle \mathbf{b}, \mathbf{v} \rangle + \nu \int_{\partial D} \langle \mathbf{v}, \left(\nabla \mathbf{u}\right) \mathbf{n} \rangle - \frac{1}{\rho_0} \int_{\partial D} p\mathbf{I} \colon \nabla \mathbf{v} \end{array}\right\}$$

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Variational formulation: $\left(\mathbf{u},p\right)\in H^1\left(D\right)^2\times L^2_0\left(D\right)$ with $L^2_0\left(D\right)=\left\{q\in L^2\left(D\right):\int_D q=0\right\}$ such

$$\left.\begin{array}{cc} a\left(\mathbf{u},\mathbf{v}\right)+b\left(p,\mathbf{v}\right)=L\left(\mathbf{v}\right) & \forall\mathbf{v}\in H^1\left(D\right)^2 \\ b\left(q,\mathbf{u}\right)=0 & \forall q\in L^2_0\left(D\right) \end{array}\right\}$$

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Variational formulation: $\left(\mathbf{u},p\right) \in H^1\left(D\right)^2 \times L_0^2\left(D\right)$ with $L_0^2\left(D\right) = \left\{q \in L^2\left(D\right) : \int_D q = 0\right\}$ such

$$\left.\begin{array}{lr} a\left(\mathbf{u},\mathbf{v}\right) + b\left(p,\mathbf{v}\right) = L\left(\mathbf{v}\right) & \forall \mathbf{v} \in H^1\left(D\right)^2 \\ b\left(q,\mathbf{u}\right) = 0 & \forall q \in L_0^2\left(D\right) \end{array}\right\}$$

$\exists! \left(\mathbf{u},p\right) \in H^1\left(D\right)^2 \times L_0^2\left(D\right)$ weak solution. Being $V = \left\{\mathbf{v} \in H^1\left(D\right) : \nabla\cdot\mathbf{v} = 0\right\}$:

- Find $\mathbf{u} \in V$ such as $a\left(\mathbf{u},\mathbf{v}\right) = L\left(\mathbf{v}\right) \quad \forall \mathbf{v} \in V$: *Lax-Milgram* theorem.
- Find $p \in L_0^2$ such as $b\left(p,\mathbf{v}\right) = L\left(\mathbf{v}\right) - a\left(\mathbf{u},\mathbf{v}\right) \quad \forall \mathbf{v} \in H^1\left(D\right)^2$: *Inf-Sup* condition.

Stokes: FEM Formulation

Taylor-Hood **Finite Element Space: Quadratic $\{\phi_i\}_{1\le i\le N}$ and linear base $\{\psi_i\}_{1\le i\le M}$**

$$V_h = \left\{ v_h \in C\left(\overline{D}\right) : v_h|_{K_i} \in P_2\left(K_i\right), 1 \le i \le N_e \right\},$$
$$M_h = \left\{ q_h \in C\left(\overline{D}\right) : q_h|_{K_i} \in P_1\left(K_i\right), 1 \le i \le N_e \right\}.$$

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$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N} u_{1j}\phi_j \\ \sum_{j=1}^{N} u_{2j}\phi_j \end{pmatrix} \qquad p = \sum_{j=1}^{M} p_j\psi_j$$

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$$
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\end{aligned}
$$

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$$

$$
\left.
\begin{array}{c}
\nu \sum_{j=1}^{N} u_{1j} \left(\int_D \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x} + \int_D \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y} \right) - \frac{1}{\rho_0} \sum_{k=1}^{M} p_k \int_D \psi_k \frac{\partial \phi_i}{\partial x} = \sum_{j=1}^{N} b_{1j} \int_D \phi_j \phi_i \quad 1 \le i \le N \\
\nu \sum_{j=1}^{N} u_{2j} \left(\int_D \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x} + \int_D \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y} \right) - \frac{1}{\rho_0} \sum_{k=1}^{M} p_k \int_D \psi_k \frac{\partial \phi_i}{\partial y} = \sum_{j=1}^{N} b_{2j} \int_D \phi_j \phi_i \quad 1 \le i \le N \\
\sum_{j=1}^{N} u_{1j} \int_D \frac{\partial \phi_j}{\partial x}\psi_i + \sum_{j=1}^{N} u_{2j} \int_D \frac{\partial \phi_j}{\partial y}\psi_i = 0 \quad 1 \le i \le M
\end{array}
\right\}
$$

Stokes: Linear System

Unknown vector: $\mathbf{s} = \begin{pmatrix} u_{11} & \cdots & u_{1N} & u_{21} & \cdots & u_{2N} & p_1 & \cdots & p_M \end{pmatrix}^t$

Linear system $\mathbf{A}\mathbf{s} = \mathbf{b}$

$$\begin{pmatrix} \nu\mathbf{R} & \mathbf{0} & -\frac{1}{\rho_0}\mathbf{S}_1 \\ \mathbf{0} & \nu\mathbf{R} & -\frac{1}{\rho_0}\mathbf{S}_2 \\ \mathbf{S}_1^t & \mathbf{S}_2^t & \mathbf{0} \end{pmatrix} \mathbf{s} = \begin{pmatrix} \mathbf{M}\mathbf{b}_1 \\ \mathbf{M}\mathbf{b}_2 \\ \mathbf{0} \end{pmatrix}$$

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$$(\mathbf{S}_1)_{ij} = \int_D \frac{\partial \phi_i}{\partial x}\psi_j \qquad (\mathbf{S}_2)_{ij} = \int_D \frac{\partial \phi_i}{\partial y}\psi_j \qquad (\mathbf{R})_{ij} = \int_D \langle \nabla\phi_i, \nabla\phi_j \rangle \qquad (\mathbf{M})_{ij} = \int_D \phi_i\phi_j$$

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Boundary conditions ($\mu = \nu/\rho$ dynamic viscosity)

- Free flow condition: $\int_{\Gamma_N} (\mu \nabla \mathbf{u} - p\mathbf{I})\, \mathbf{n} = \mathbf{0}$
- *Dirichlet* condition (**mass conservation** must be fulfilled!): $\mathbf{u}|_{\Gamma_D} = \mathbf{u}_0$.
  - Inlet or outlet flow: $\langle \mathbf{u}_0, \mathbf{n} \rangle < 0$ or $\langle \mathbf{u}_0, \mathbf{n} \rangle > 0$
  - Wall (non viscous or no slip condition): $\langle \mathbf{u}_0, \mathbf{n} \rangle = 0$ or $\mathbf{u}_0 = 0$
  - Confined fluid: $\langle \mathbf{u}_0, \mathbf{n} \rangle = 0$ in all the boundary ∂D.
    - Pressure field is not unique ⇒ imposed $p = p_0$ in a node

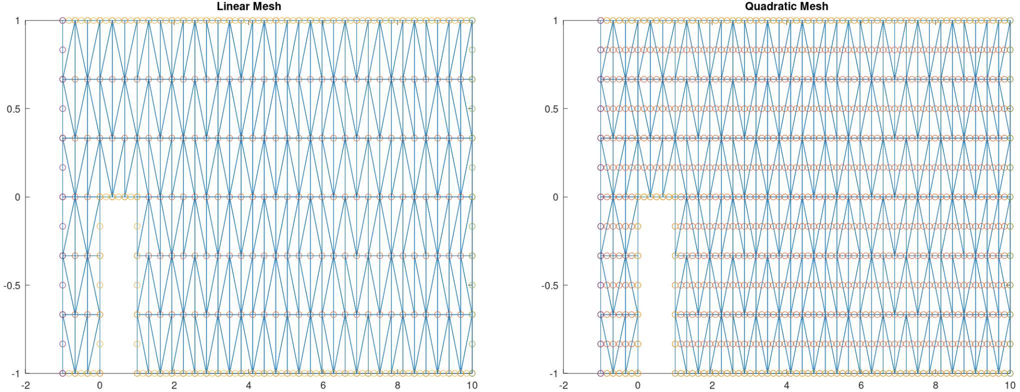
Stokes: Octave Code

```
% Get matrices
[M R C] = fem_mrc(x2, y2, tri2, 1, 0);
[S1 S2] = fem_stokes(x2, y2, tri2);
[n m] = size(S1);
A = [R 0*R -S1; 0*R R -S2; S1' S2' zeros(m)];
b = [M*f1; M*f2; zeros(m,1)];
% Dirichlet condition (efficient)
for k=1:length(bc2)
   v = uin(x2(bc2(k)),y2(bc2(k)));
   b -= v(1)*A(:,bc2(k));
   b -= v(2)*A(:,bc2(k)+n);
end
bc12_ = [bc1, bc2, bc1+N, bc2+n];
b(bc12_) = 0;
b(bc2) = uin(x2(bc2),y2(bc2))(:,1);
b(bc2+n) = uin(x2(bc2),y2(bc2))(:,2);
A(bc12_,:)  = 0; A(:,bc12_) = 0;
for i=bc12_
   A(i,i) = 1;
end;
% Solve
uh = A\b;
```

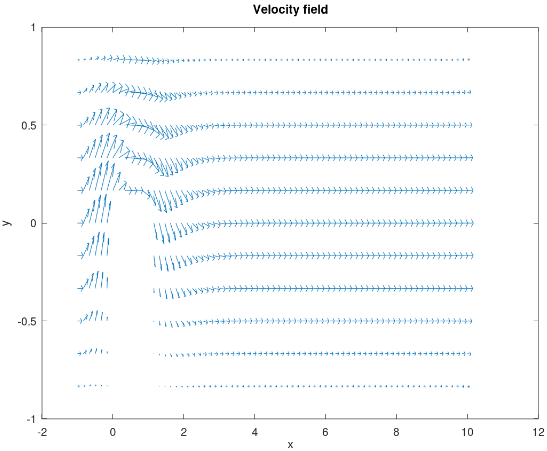
Example 6.4: `stokes4.m`

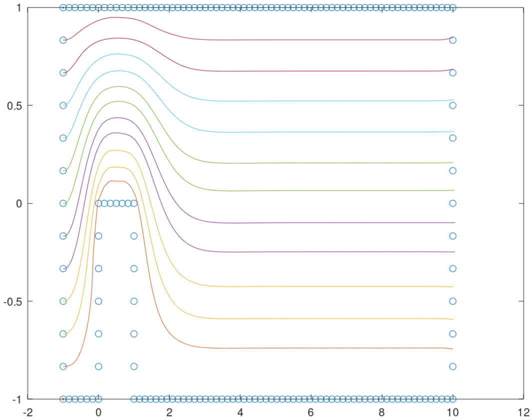
$D = (-1, 10) \times (-1, 1) - (0, 1) \times (-1, 0)$,

$$\left.\begin{array}{c} -\Delta \mathbf{u} + \nabla p = \mathbf{0} \\ \nabla \cdot (\mathbf{u}) = \mathbf{0} \end{array}\right\} \quad \left.\begin{array}{c} \mathbf{u}|_{\Gamma_1} = \mathbf{0}, \quad \mathbf{u}|_{\Gamma_2} = \left(1 - y^2\right)\mathbf{e}_1 \\ \left(\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p \cdot \mathbf{n}\right)_{\Gamma_3} = \mathbf{0} \end{array}\right\}$$

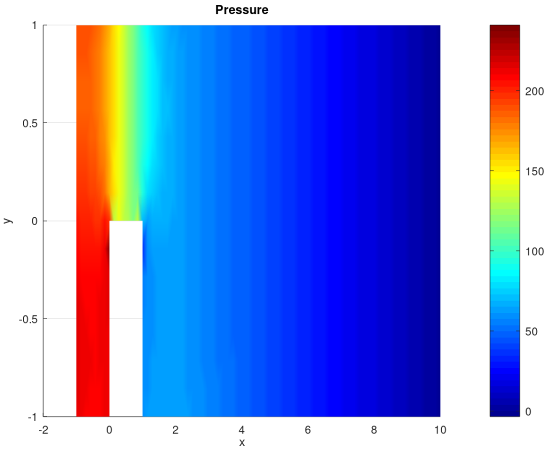


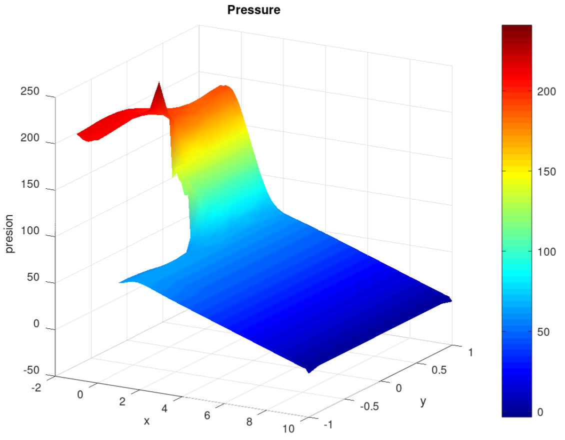
Example 6.4: Velocity



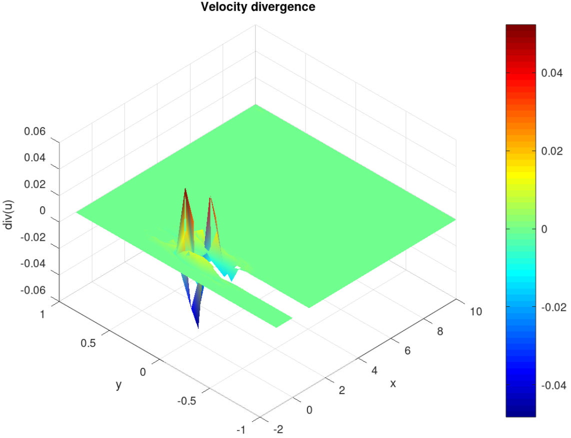


Example 6.4: Pressure

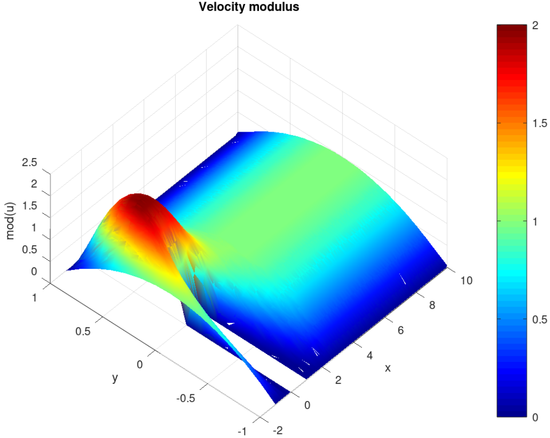




Example 6.4: Check $\nabla \cdot (\mathbf{u})$



`trisurf(tri2,x2,y2,(Cx*uh1+Cy*uh2)\M2);`



`trisurf(tri2,x2,y2,hypot(uh1,uh2));`

Example 6.4: $\dot{m}_{in} + \dot{m}_{out} = \int_{\Gamma_2 \cup \Gamma_3} \langle \mathbf{u}, \mathbf{n} \rangle = -\int_{-1}^{1} \left(1 - y^2\right) dy + \int_{\Gamma_2} u_1 = -\frac{4}{3} + \int_{\Gamma_2} u_1 = 0$

```
Gin = 0*x2; Gin(bc2) =-uh1(bc2); %Integrate inlet velocity
[AR Min] = fem_robin(x2, y2, tri2, Gin, 0*x2); mIn = sum(Min);
Gou = 0*x2; Gou(bc3) = uh1(bc3); %Integrate outlet velocity
[AR Mou] = fem_robin(x2, y2, tri2, Gou, 0*x2); mOu = sum(Mou)
printf("Inlet and outlet mass flow:  %f %f (error = %f)\n", mIn, mOu, mIn+mOu);
```

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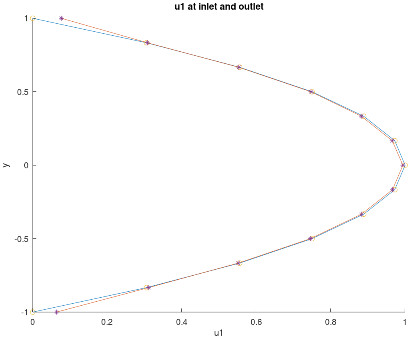

```
> Inlet and outlet mass flow:  -1.333333 1.335880 (error = 0.002547)
```

Axisymmetric Stokes Problem

Stokes problem in cylindrical coordinates

$$\left.\begin{array}{c}
-\nu\left[\frac{\partial^2 u_x}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)\right]=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+b_x\\
-\nu\left[\frac{\partial^2 u_r}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}\right]=-\frac{1}{\rho_0}\frac{\partial p}{\partial r}+b_r\\
-\nu\left[\frac{\partial^2 u_\theta}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}\right]=0\\
\frac{\partial u_x}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)=0
\end{array}\right\}+b.c$$

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Stokes problem in cylindrical coordinates

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Equation for u_θ: uncoupled elliptical problem

$$\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}=0$$

Axisymmetric Stokes Problem

Stokes problem in cylindrical coordinates

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-\nu\left[\frac{\partial^2 u_\theta}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}\right]=0 \\
\frac{\partial u_x}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(r u_r)=0
\end{array}\right\} + b.c$$

Equation for u_θ: uncoupled elliptical problem

$$\frac{\partial^2 u_\theta}{\partial x^2}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}=0$$

Variational formulation for u_x, u_r and p

$$\left.\begin{array}{c}
\nu\iiint_D \nabla\mathbf{u}:\nabla\mathbf{v}-\frac{1}{\rho_0}\iiint_D p\mathbf{I}:\nabla\mathbf{v}=\iiint_D\langle\mathbf{b},\mathbf{v}\rangle \\
\iiint_D(\nabla\cdot(\mathbf{u}))\,q=0
\end{array}\right\}$$

Axisymmetric Stokes Problem

FEM formulation for u_x, u_r and p

$$\begin{pmatrix} \nu \mathbf{R} & \mathbf{0} & -\frac{1}{\rho_0}\mathbf{S}_\mathrm{x} \\ \mathbf{0} & \nu\left(\mathbf{R}+\mathbf{M}_{rr}\right) & -\frac{1}{\rho_0}\left(\mathbf{S}_r+\mathbf{M}_r\right) \\ \left(\mathbf{S}_x\right)^t & \left(\mathbf{S}_r+\mathbf{M}_r\right)^t & \mathbf{0} \end{pmatrix} \mathbf{s} = \begin{pmatrix} \mathbf{M}\mathbf{b}_1 \\ \mathbf{M}\mathbf{b}_2 \\ \mathbf{0} \end{pmatrix}$$

$$\left(\mathbf{S}_x\right)_{ij} = 2\pi \iint_\Omega \frac{\partial \phi_i}{\partial x}\psi_j r \cdot dx \cdot dr \qquad \left(\mathbf{S}_r\right)_{ij} = 2\pi \iint_\Omega \frac{\partial \phi_i}{\partial y}\psi_j r \cdot dx \cdot dr$$

$$\left(\mathbf{M}_r\right)_{ij} = 2\pi \iint_\Omega \phi_i\psi_j \cdot dx \cdot dr \qquad \left(\mathbf{M}_r\right)_{ij} = 2\pi \iint_\Omega \frac{1}{r}\phi_i\psi_j \cdot dx \cdot dr$$

Axisymmetric Stokes Problem

FEM formulation for u_x, u_r and p

$$\begin{pmatrix} \nu \mathbf{R} & \mathbf{0} & -\frac{1}{\rho_0}\mathbf{S}_x \\ \mathbf{0} & \nu\left(\mathbf{R}+\mathbf{M}_{rr}\right) & -\frac{1}{\rho_0}\left(\mathbf{S}_r+\mathbf{M}_r\right) \\ \left(\mathbf{S}_x\right)^t & \left(\mathbf{S}_r+\mathbf{M}_r\right)^t & \mathbf{0} \end{pmatrix} \mathbf{s} = \begin{pmatrix} \mathbf{M}\mathbf{b}_1 \\ \mathbf{M}\mathbf{b}_2 \\ \mathbf{0} \end{pmatrix}$$

$$\left(\mathbf{S}_x\right)_{ij} = 2\pi \iint_\Omega \frac{\partial \phi_i}{\partial x}\psi_j r \cdot dx \cdot dr \qquad \left(\mathbf{S}_r\right)_{ij} = 2\pi \iint_\Omega \frac{\partial \phi_i}{\partial y}\psi_j r \cdot dx \cdot dr$$

$$\left(\mathbf{M}_r\right)_{ij} = 2\pi \iint_\Omega \phi_i\psi_j \cdot dx \cdot dr \qquad \left(\mathbf{M}_r\right)_{ij} = 2\pi \iint_\Omega \frac{1}{r}\phi_i\psi_j \cdot dx \cdot dr$$

Flow angles postprocessing

- **Circumferential** angle: $\alpha = \arctan\left(\frac{u_\theta}{\sqrt{u_x^2+u_r^2}}\right)$
- **Radial** angle: $\beta = \arctan\left(\frac{u_r}{u_x}\right)$

Outline

1. Stokes Problem
   - Formulation
   - Example
   - Axisymmetric Problem
2. Method of Characteristics
   - Characteristics Computation
   - Application to the Diffusion-Convection Equation
3. Navier-Stokes Equations
   - Incompressible Flow
   - Postprocessing
   - Energy Equation

Material Derivative

Material derivative of a scalar field (*Lagrangian* approach) $\varphi \equiv \varphi\left(x_1, x_2, t\right)$

$$\varphi\left(t\right) \equiv \varphi\left(x_1\left(t\right), x_2\left(t\right), t\right)$$

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$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial \varphi}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial \varphi}{\partial t}$$

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Material derivative of a vector field $\mathbf{w} \equiv w_i\left(x_1, x_2, t\right)\mathbf{e}_i$ (Einstein notation)

$$\frac{D\mathbf{w}}{Dt} = \frac{\partial\mathbf{w}}{\partial t} + \left(\mathbf{u}\cdot\nabla\right)\mathbf{w} = \left(\frac{\partial w_i}{\partial t} + u_j\frac{\partial w_i}{\partial x_j}\right)\mathbf{e}_i$$

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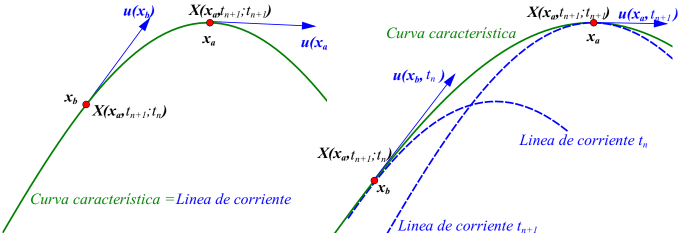
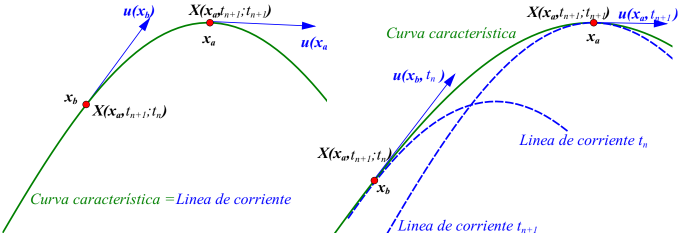


$\mathbf{u}(\mathbf{x},t)\in\mathbb{R}^d$, being $t_{n-1}\le t\le t_{n+1}$ with $t_j=j\Delta t$, the characteristic curve $\mathbf{X}(\mathbf{x},t_{n+1};t)\in\mathbb{R}^d$ is

$$\left.\begin{array}{c}\mathbf{X}'(\mathbf{x},t_{n+1};t)=\mathbf{u}(\mathbf{X}(\mathbf{x},t_{n+1};t),t)\\ \mathbf{X}(\mathbf{x},t_{n+1};t_{n+1})=\mathbf{x}\end{array}\right\}$$



$\mathbf{u}(\mathbf{x},t) \in \mathbb{R}^d$, being $t_{n-1} \leq t \leq t_{n+1}$ with $t_j = j\Delta t$, the **characteristic** curve $\mathbf{X}(\mathbf{x},t_{n+1};t) \in \mathbb{R}^d$ is

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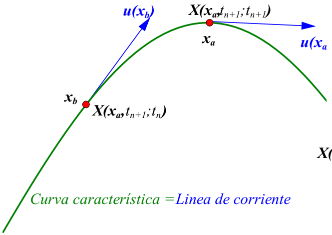
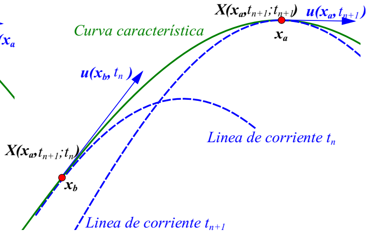
Material derivative on a characteristic curve

$$\frac{D}{Dt}\varphi(\mathbf{X}(\mathbf{x},t_{n+1};t),t) = \frac{d}{dt}\varphi(\mathbf{X}(\mathbf{x},t_{n+1};t),t)$$





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Backward Differentiation Formula (second order)

$$\frac{D}{Dt}\varphi(\mathbf{X}(\mathbf{x},t_{n+1};t_{n+1}),t_{n+1}) \approx \frac{3\varphi^{n+1}(\mathbf{x}) - 4\varphi^{n*}(\mathbf{x}) + \varphi^{(n-1)**}(\mathbf{x})}{2\Delta t}$$

$$\varphi^{n}(\mathbf{x}) = \varphi(\mathbf{x},t_n), \qquad \varphi^{n*}(\mathbf{x}) = \varphi(\mathbf{X}(\mathbf{x},t_{n+1};t_n),t_n), \qquad \varphi^{(n-1)**}(\mathbf{x}) = \varphi(\mathbf{X}(\mathbf{x},t_{n+1};t_{n-1}),t_{n-1})$$

Characteristic Curve Computation

Given nodes $\{\mathbf{x}_i\}_{1 \leq i \leq N}$ and a velocity field $\mathbf{u}(\mathbf{x}_i, t_n)$ we integrate backwards in time

Explicit Euler: no accurate

$$\mathbf{x}_i^* := \mathbf{X}(\mathbf{x}_i, t_{n+1}; t_n) \approx \mathbf{x}_i - \Delta t \cdot \mathbf{u}(\mathbf{x}_i, t_n)$$

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4th order Runge-Kutta

$$\left.\begin{array}{c}
\mathbf{f}_i^1 = \mathbf{u}\left(\mathbf{x}_i,t_{n+1}\right)\Delta t \\
\mathbf{f}_i^2 = \mathbf{u}\left(\mathbf{x}_i-\frac{1}{2}\mathbf{f}_i^1,t_{n+1}-\frac{1}{2}\Delta t\right)\Delta t \\
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Space interpolation: `fem_interpol`

To find i (best triangle), use `tsearch`. If *NaN*, use `dsearch`. $m = 3$ (linear) or m=6 (quadratic mesh).

$$\mathbf{u}\left(\mathbf{x}_0\right) = \sum_{j=1}^{m} \mathbf{u}_h\left(\mathbf{x}_j^{(i)}\right)\hat{\phi}_j\left(F_i^{-1}\left(\mathbf{x}_0\right)\right)$$

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Time extrapolation: $\varphi_{h,i} = \varphi_{h,i}^n + \frac{t-t_n}{\Delta t}\left(\varphi_{h,i}^n - \varphi_{h,i}^{n-1}\right)$

Diffusion-Convection Equation

Method of characteristics

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$$\sum_{j=1}^{N} w_j^{n+1} \left(\tfrac{3}{2} \int_D \phi_i \phi_j + \nu \Delta t \int_D \langle \nabla \phi_i, \nabla \phi_j \rangle \right) = \sum_{j=1}^{N} \left(2w_j^{n*} - \tfrac{1}{2} w_j^{(n-1)**} + \Delta t f_j^{n+1} \right) \int_D \phi_i \phi_j$$

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Implicit Euler

$$\left[\mathbf{M} + \Delta t\left(\mathbf{C}_1\mathbf{u}_1^{n+1} + \mathbf{C}_2\mathbf{u}_2^{n+1} + \nu\mathbf{R}\right)\right]\mathbf{w}^{n+1} = \mathbf{M}\left(\mathbf{w}^{n} + \Delta t\mathbf{f}^{n+1}\right)$$

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Crank-Nicolson

$$\left[2\mathbf{M} + \Delta t\left(\mathbf{C}_1\mathbf{u}_1^{n+1} + \mathbf{C}_2\mathbf{u}_2^{n+1} + \nu\mathbf{R}\right)\right]\mathbf{w}^{n+1} = \left[2\mathbf{M} - \Delta t\left(\mathbf{C}_1\mathbf{u}_1^{n} + \mathbf{C}_2\mathbf{u}_2^{n} + \nu\mathbf{R}\right)\right]\mathbf{w}^{n} + \Delta t\mathbf{M}\left(\mathbf{f}^{n+1} + \mathbf{f}^{n}\right)$$

Example 7.2: `carac2.m`

$D = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$, $u_0(x,y) = \exp\left(-\frac{(x-0.5)^2+y^2}{0.01}\right)$, $\mathbf{b} = -y\mathbf{e}_1 + x\mathbf{e}_2$, $k = 6.2070 \times 10^{-4}$

$$\begin{cases} \dfrac{\partial u}{\partial t} + (\mathbf{b}\cdot\nabla)\, u - k\Delta u = 0, & D \times (0, 2\pi)\,, \\ u\,|_{\partial D} = 0, & t \in (0, 2\pi)\,, \\ u\,(x, y, 0) = u_0\,(x, y)\,, & (x, y) \in D, \end{cases}$$

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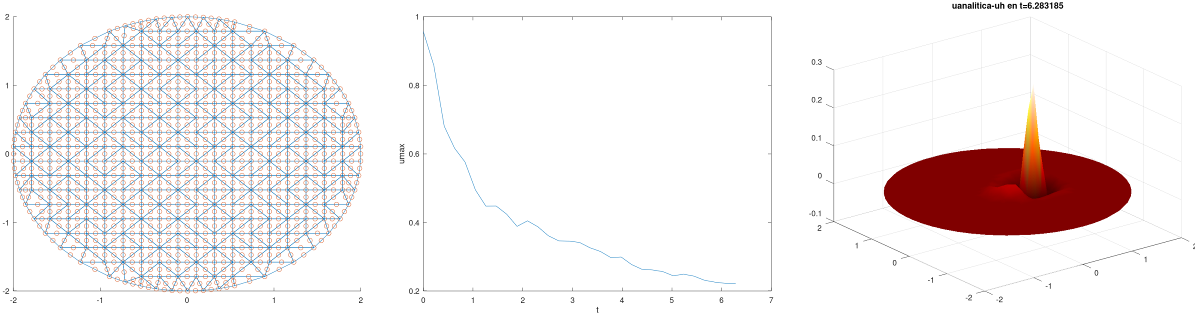

$m = 2, N_e = 562, N = 1177, \Delta t = 2\pi/30$: High numerical **diffusivity**

Outline

1. Stokes Problem
    - Formulation
    - Example
    - Axisymmetric Problem
2. Method of Characteristics
    - Characteristics Computation
    - Application to the Diffusion-Convection Equation
3. Navier-Stokes Equations
    - Incompressible Flow
    - Postprocessing
    - Energy Equation

Navier-Stokes Equation

Find velocity $\mathbf{u}$, pressure p and density ρ fields in a domain D

$$\left.\begin{array}{r}\frac{D\rho}{Dt}+\rho\nabla\cdot(\mathbf{u})=0\\ \rho\frac{D\mathbf{u}}{Dt}=-\nabla p+\nabla\cdot(\boldsymbol{\sigma}(\mathbf{u}))+\rho\mathbf{g}\end{array}\right\}+b.c.(\mathbf{u},\rho)+i.c.(\mathbf{u},\rho)$$

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Stress tensor $\boldsymbol{\sigma}$ in function of *Lamé* μ, λ parameters

$$\boldsymbol{\sigma}(\mathbf{u}) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + (\lambda\nabla\cdot(\mathbf{u}))$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^t\right)$$

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$$\boldsymbol{\sigma}\left(\mathbf{u}\right)=2\mu\boldsymbol{\varepsilon}\left(\mathbf{u}\right)+\left(\lambda\nabla\cdot\left(\mathbf{u}\right)\right)$$
$$\boldsymbol{\varepsilon}\left(\mathbf{u}\right)=\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^{t}\right)$$

Incompressible flow: $\nabla\cdot\left(\mathbf{u}\right)=0\Rightarrow D\rho/Dt=0\Rightarrow\rho=\rho_0$

$$\left.\begin{array}{c}\frac{\partial\mathbf{u}}{\partial t}+\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}-\nu\Delta\mathbf{u}=-\frac{1}{\rho_0}\nabla p+\mathbf{g}\\ \nabla\cdot\left(\mathbf{u}\right)=0\end{array}\right\}+b.c.\left(\mathbf{u}\right)+i.c.\left(\mathbf{u}\right)$$

- Kinematic viscosity: $\nu=\mu/\rho \quad \left(m^2/s\right)$
- Term $\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}$ is nonlinear

Incompressible Flow

Solving on the characteristic curves we obtain a Stokes problem for each t_{n+1} (linear!)

$$\left.\begin{array}{c} \dfrac{3\mathbf{u}^{n+1}-4\mathbf{u}^{n*}+\mathbf{u}^{(n-1)**}}{2\Delta t}-\nu\Delta\mathbf{u}^{n+1}+\frac{1}{\rho_0}\nabla p^{n+1}=\mathbf{g}^{n+1} \\ \nabla\cdot\left(\mathbf{u}^{n+1}\right)=0 \end{array}\right\}$$

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FEM formulation with the same functional base as Stokes

$$\left(\begin{array}{ccc} 3\mathbf{M} + 2\nu\Delta t\mathbf{R} & \mathbf{0} & -\frac{2\Delta t}{\rho_0}\mathbf{S}_1 \\ \mathbf{0} & 3\mathbf{M} + 2\nu\Delta t\mathbf{R} & -\frac{2\Delta t}{\rho_0}\mathbf{S}_2 \\ \mathbf{S}_1^t & \mathbf{S}_2^t & \mathbf{0} \end{array}\right) \mathbf{s}^{n+1} = \left(\begin{array}{c} \mathbf{M}\left(2\Delta t \mathbf{g}_1^{n+1} + 4\mathbf{u}_1^{n*} - \mathbf{u}_1^{(n-1)**}\right) \\ \mathbf{M}\left(2\Delta t \mathbf{g}_2^{n+1} + 4\mathbf{u}_2^{n*} - \mathbf{u}_2^{(n-1)**}\right) \\ \mathbf{0} \end{array}\right)$$

Incompressible Flow

Solving on the characteristic curves we obtain a Stokes problem for each t_{n+1} (**linear**!)

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$$\begin{pmatrix} 3\mathbf{M}+2\nu\Delta t\mathbf{R} & \mathbf{0} & -\frac{2\Delta t}{\rho_0}\mathbf{S}_1\\ \mathbf{0} & 3\mathbf{M}+2\nu\Delta t\mathbf{R} & -\frac{2\Delta t}{\rho_0}\mathbf{S}_2\\ \mathbf{S}_1^t & \mathbf{S}_2^t & \mathbf{0}\end{pmatrix}\mathbf{s}^{n+1}=\begin{pmatrix}\mathbf{M}\left(2\Delta t\mathbf{g}_1^{n+1}+4\mathbf{u}_1^{n*}-\mathbf{u}_1^{(n-1)**}\right)\\ \mathbf{M}\left(2\Delta t\mathbf{g}_2^{n+1}+4\mathbf{u}_2^{n*}-\mathbf{u}_2^{(n-1)**}\right)\\ \mathbf{0}\end{pmatrix}$$

For steady solutions, compute residuals and stop when below a tolerance

$$res_u^n=\sqrt{\sum_{i=1}^{N}\left(u_{1,i}^n-u_{1,i}^{n-1}\right)^2+\left(u_{2,i}^n-u_{2,i}^{n-1}\right)^2}\qquad res_p^n=\sqrt{\sum_{i=1}^{M}\left(p_i^n-p_i^{n-1}\right)^2}$$

Example 8.6: Flat Plate `placaplana.m`

Solve the incompressible flow over a 1 m flat plate with air and 1 m/s velocity at the leading edge .

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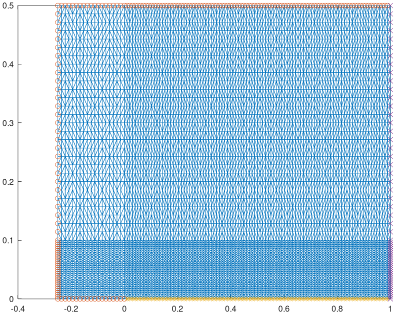

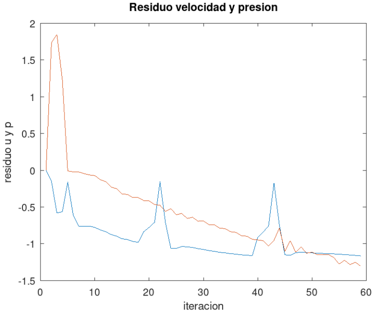

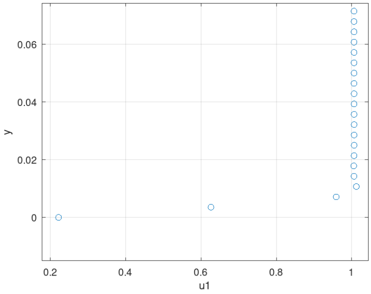

$$m = 2, N_e = 3248, N = 6669,\ \Delta t = 0.01s,\ t_{end} = 1s$$

Reynolds and Mach Numbers

Characteristic length L_0 and velocity U_0

$$\hat{\mathbf{u}} = \mathbf{u}/U_0, \quad \hat{p} = p/\left(\rho_0 U_0^2\right), \quad \hat{\mathbf{x}} = \mathbf{x}/L_0, \quad \hat{t} = tU_0/L_0, \quad \hat{\mathbf{g}} = \mathbf{g}/g.$$

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Dimensionless *Navier-Stokes*

$$\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \left(\hat{\mathbf{u}} \cdot \hat{\nabla}\right) \hat{\mathbf{u}} - \frac{1}{R_e} \hat{\Delta} \hat{\mathbf{u}} = -\hat{\nabla} \hat{p} + \frac{1}{Fr^2} \hat{\mathbf{b}}$$

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Reynolds and *Froude* numbers

$$R_e = U_0 L_0 \rho_0 / \mu \qquad F_r = U_0 / \sqrt{L_0 g}$$

R_e is one of the most important numbers in fluid mechanics. Defined as the ratio between the **inertial** (or convective) and the **viscous** forces. It allows predicting whether the flow will be **laminar** or **turbulent**.

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Mach number, being sound velocity $a = \sqrt{(\partial P/\partial \rho)_s}$

$$M = u/a$$

M allows us to assess **compressibility** effects. For $M < 0.3$, little error is made constant ρ. At higher Mach numbers, the incompressible flow hypothesis is no longer valid. For $M \simeq 1$, discontinuities such as **shock waves** may appear. For ideal gas model the sound velocity depends on the temperature: $a = \sqrt{\gamma R_g T}$.

Postprocessing: Conservation Laws

Mass continuity with `fem_robin`

$$\int_{\partial D} \langle \mathbf{u}, \mathbf{n} \rangle = 0 \qquad \implies -\int_{Inlets} \rho \langle \mathbf{u}, \mathbf{n} \rangle = \int_{Outlets} \rho \langle \mathbf{u}, \mathbf{n} \rangle$$

Postprocessing: Conservation Laws

Mass continuity with `fem_robin`

$$\int_{\partial D} \langle \mathbf{u}, \mathbf{n} \rangle = 0 \qquad \Longrightarrow -\int_{Inlets} \rho \langle \mathbf{u}, \mathbf{n} \rangle = \int_{Outlets} \rho \langle \mathbf{u}, \mathbf{n} \rangle$$

Momentum conservation with `fem_robin`

$$\int\limits_{Inlets} p\mathbf{n} + \int\limits_{Outlets} p\mathbf{n} = \int\limits_{Walls} (-p\mathbf{I} + \boldsymbol{\sigma})\, \mathbf{n}$$

Postprocessing: Streamlines

Selecting one or several seeds $\mathbf{x}_0$: $\quad \mathbf{x}_n = \mathbf{x}_{n-1} + \mathbf{u}\left(\mathbf{x}_{n-1}\right)\Delta\tau \qquad \Delta\tau < h_{min} / \max\left(|\mathbf{u}|\right)$

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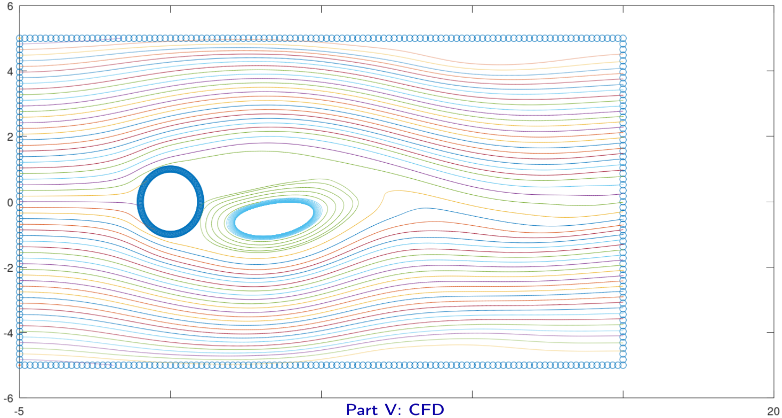
```
[XS YS S] = fem_streamline(x2, y2, tri1, U1, U2, x2(bc1), y2(bc1));
[XS2 YS2 S2] = fem_streamline(x2, y2, tri1, U1, U2, 3, 0);
[XS3 YS3 S3] = fem_streamline(x2, y2, tri1, -U1, -U2, 3, 0);
```



Postprocessing: Trajectories and Traces

A **trajectory** is a characteristic curve: $\mathbf{x}_n = \mathbf{x}_{n-1} + \mathbf{u}\left(\mathbf{x}_{n-1}, t_{n-1}\right) \Delta t$
A **trace** joins the particles that have passed through the same point at some instant prior to a given one.
Both are **streamlines** for steady velocity field.

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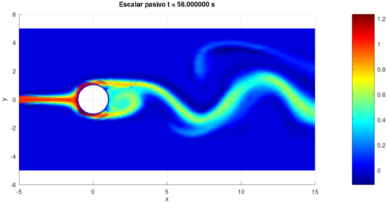
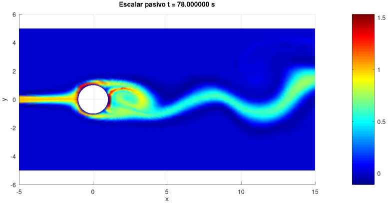
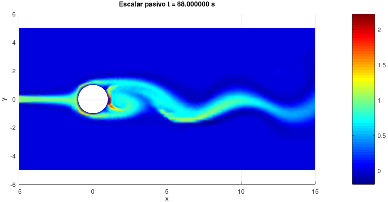
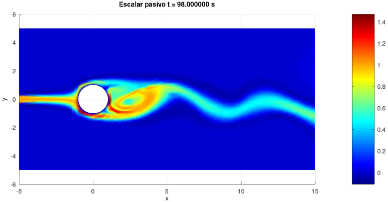
Traces computation: solve a convection-diffusion equation. $\frac{\partial w}{\partial t} + \left(\mathbf{u}\cdot\nabla\right)w - k\Delta w = 0$


Escalar pasivo t = 58.000000 s


Escalar pasivo t = 78.000000 s


Escalar pasivo t = 88.000000 s


Escalar pasivo t = 98.000000 s

Postprocessing: C_p, C_L, C_D and $\boldsymbol{\omega}$

Pressure coefficient: $C_p = (p - p_\infty) \,/\, \left(\frac{1}{2}\rho u_\infty\right)$

Non-dimensionalization of *Bernoulli* equation: neglecting viscous effects, $p_0 = p + \rho \left|\mathbf{u}\right|^2 /2$ is conserved in streamlines. Then $C_p \simeq 1 - (u/u_\infty)^2$. At stagnation points ($u \simeq 0$), pressure coefficient $C_p \simeq 1$.

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Lift and drag coefficients: $C_L = F_L / \left(\frac{1}{2}\rho u^2 c \cdot s\right)$ $C_D = F_D / \left(\frac{1}{2}\rho u^2 c \cdot s\right)$

Non-dimensionalization of *lift* and *drag* forces ($c \equiv$ chord, $s \equiv$ thickness).

$$F_L = s \int\limits_{\partial\Omega} \langle (p\mathbf{I} - \boldsymbol{\sigma})\,\mathbf{n}, \mathbf{e}_2 \rangle \qquad F_D = s \int\limits_{\partial\Omega} \langle (p\mathbf{I} - \boldsymbol{\sigma})\,\mathbf{n}, \mathbf{e}_1 \rangle \qquad \boldsymbol{\sigma} = \mu \begin{pmatrix} 2\partial u_1/\partial x_1 & \partial u_1/\partial x_2 \\ \partial u_2/\partial x_1 & 2\partial u_2/\partial x_2 \end{pmatrix}$$

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Vorticity: $\boldsymbol{\omega} = \nabla \times \mathbf{u} = 2\boldsymbol{\Omega}$

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ u_1 & u_2 & 0 \end{vmatrix} = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\mathbf{e}_3$$

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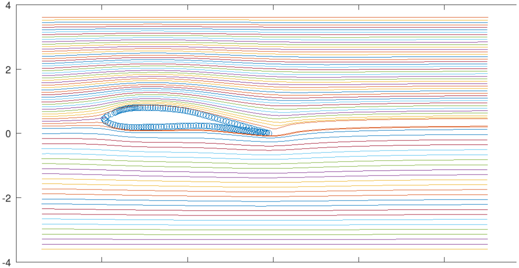
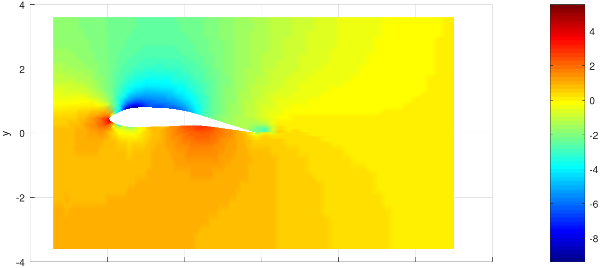
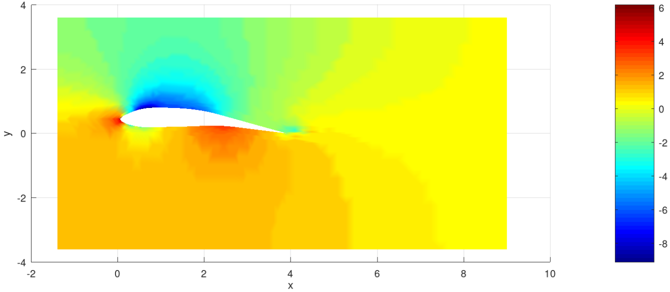
Circulation: $\Gamma = \oint_{\partial S} \langle \mathbf{u}, \mathbf{t} \rangle$

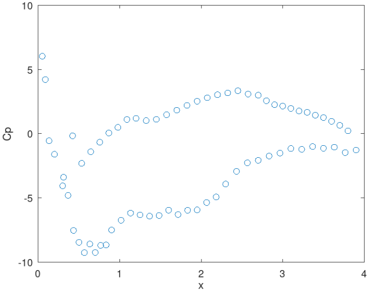
$$\Gamma = \oint\limits_{\partial S} \langle \mathbf{u}, \mathbf{t} \rangle = \iint\limits_{S} \langle \boldsymbol{\omega}, \mathbf{n} \rangle$$

Postprocessing: Aerodynamic Profile




Presion


Coeficiente de presion para ang. de ataque = 0.000000º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Energy Equation

Internal energy e: $\frac{De}{Dt} + p\,(\nabla \cdot \mathbf{u}) - Q = \nabla \cdot (k\nabla T) + \mu\mathit{\Phi}$

Viscous dissipation function: $\Phi = 2\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 - \frac{1}{3}\,(\nabla \cdot \mathbf{u})^2\right] + \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2$

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For incompressible and adiabatic flows

$$\frac{\partial T}{\partial t} + \mathbf{u}\nabla T = \frac{k}{c_v \rho_0}\Delta T + \frac{\nu}{c_v}\Phi$$

$$\Phi = 2\left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\left(\frac{\partial u_2}{\partial x_2}\right)^2 + 2\frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2$$

If $\partial k / \partial T = 0$, *Navier-Stokes* and *Energy* equations are **uncoupled**.

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Nusselt and *Prandtl* numbers

N_u and P_r numbers: $N_u = hL/k$ $\qquad P_r = \mu c_p / k$

$N_u =$ convection/conduction effects. Depends on $P_r =$ viscosity/diffusivity ratio.

Energy Equation

Internal energy e: $\frac{De}{Dt} + p\,(\nabla \cdot \mathbf{u}) - Q = \nabla \cdot (k \nabla T) + \mu \mathit{\Phi}$

Viscous dissipation function: $\Phi = 2\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 - \frac{1}{3}\,(\nabla \cdot \mathbf{u})^2\right] + \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2$

For incompressible and adiabatic flows

$$\frac{\partial T}{\partial t} + \mathbf{u}\nabla T = \frac{k}{c_v \rho_0}\Delta T + \frac{\nu}{c_v}\Phi$$

$$\Phi = 2\left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\left(\frac{\partial u_2}{\partial x_2}\right)^2 + 2\frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2$$

If $\partial k / \partial T = 0$, *Navier-Stokes* and *Energy* equations are **uncoupled**.

Nusselt and *Prandtl* numbers

N_u and P_r numbers: $N_u = hL/k$ $\qquad P_r = \mu c_p / k$

N_u = convection/conduction effects. Depends on P_r = viscosity/diffusivity ratio.

Heat Transfer Coefficient (HTC) h

$$h = \left.\frac{\langle k \nabla T, \mathbf{n}\rangle}{T - T_{ref}}\right|_{\partial\Omega} \qquad T_{ref} = T_0|_\infty = T + \left.\frac{|\mathbf{u}|^2}{2c_p}\right|_\infty$$

Example 9.4: `placaplanaEner.m`

Solve the energy equation for problem 8.6 (flat plate) assuming an inlet temperature of 300K and that the plate is in isothermal conditions, maintaining a temperature of 400K. Obtain the variation of the heat transfer coefficient h along the plate. Show the result as a function of the Reynolds number.

Example 9.4: `placaplanaEner.m`

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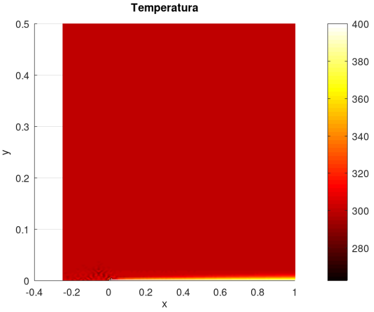

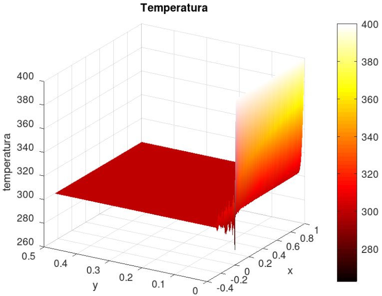

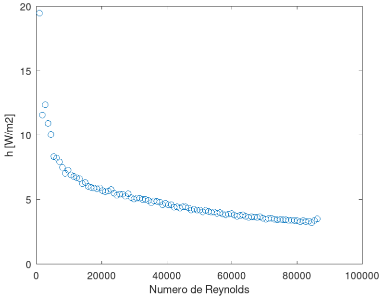

End Part V

Thank you! Questions?