UNIVERSIDAD POLITÉCNICA DE MADRID
Escuela Técnica Superior de Ingeniería y Sistemas de Telecomunicación

Algebraic models of tonal function

TESIS DOCTORAL

Presentada para optar al título de Doctor por:

David Isaac Fernández del Pozo Esteban
M. A.

Madrid, 2024

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Título: Algebraic models of tonal function

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I want to dedicate this thesis to my parents, Miguel and Mercedes.

Abstract

This thesis presents an algebraic model of tonal function. This model is based on several pillars. The first one is a mathematical formalization of the musical universe, from pure frequencies to voice leading, passing through voice distribution in chords and a distance between notes. This formalization will provide a mathematical and computational basis for the tonal function model that is developed subsequently.

The second pillar is a study of the vertical dimension of harmony. This study will examine dissonances in the vertical musical dimension, particularly the avoid notes. Within this second pillar, concepts such as compact tonalities, open tonalities, and poly-chords will be reviewed; all these structures are contemplated in the general model.

The third pillar is a detailed and in-depth study of the physical characteristics of harmony. Elements such as the energy of voicings, the Weber-Fechner law, the Hungarian algorithm, and voice optimization are incorporated into the model. The Hungarian algorithm and its application to obtaining optimal voice leading is one of the most outstanding contributions of this thesis.

The fourth pillar is a study of the horizontal dimension of harmony. It begins with a very detailed review of the concept of tonal function, and after this review, it is generalized in a way that can explain and predict tonal functions beyond those seen in the period of common practice in classical music. Tonal function categories are expanded and become more subtle and precise. It is within this fourth pillar that the core of the algebraic tonal function model is presented. Parametric and non-parametric cadences are presented here, and the complete process by which the tonal function is obtained is described, going from link matrices to the polynomial criterion. Ultimately, the model reduces the classification of tonal function to the location of the roots of a certain complex polynomial in certain areas. The model is both algebraic and geometric. Within this fourth pillar, we also find a generalization of tonal gravity. One of the virtues of this model is that it does not require the number of voices between two consecutive chords to be the same. This generalization implies an extension of the Hungarian algorithm to infinite arithmetic and this extension to be consistent with the case of constant voices. The model is consistent in this regard.

The fifth pillar is a study of modulation in the light of the previously designed model.

At the end of the thesis, several practical applications are presented, and by this, we mean applications that a composer or performer can directly use without having to go through understanding the model. There are applications for voice leading, progression generation, writing bass lines and melody writing, among others.

The last part of the thesis is a study of cadences through the algebraic model presented here. The most important cadences in each of the seven modes have been studied.

Resumen

En esta tesis se presenta un modelo algebraico de función tonal. Este modelo está basado en varios pilares.

El primero es una formalización matemática del universo musical, desde las frecuencias puras hasta la conducción de voces, pasando por la distribución de voces en acordes y por la distancia entre notas. Esta formalización proporcionará una base matemática y computacional al modelo de la función tonal que se desarrolla a continuación.

El segundo pilar es un estudio de la dimensión vertical de la armonía. En este estudio se examinarán las disonancias en la dimensión musical vertical y en particular, las notas a evitar. Dentro de este segundo pilar se revisarán conceptos tales como las tonalidades compactas, las tonalidades abiertas y los poli-acordes, estructuras musicales todas contempladas en el modelo general.

El tercer pilar es un estudio detallado y profundo de las características físicas de la armonía. Se incorporan al modelo elementos tales como la energía de la distribución de voces de un acorde, la ley de Weber-Fechner, el algoritmo húngaro o la optimización de voces. El algoritmo húngaro y su aplicación a la obtención de la conducción de voces óptima es una de las contribuciones más sobresalientes de esta tesis.

El cuarto pilar es un estudio de la dimensión horizontal de la armonía. Se empieza por una revisión muy profunda del concepto de función tonal y después de dicha revisión, se generaliza de manera que puede explicar y predecir funciones tonales más allá de las que se pueden ver en la música clásica y otras músicas de la práctica común. Las categorías de función tonal se amplían y son más sutiles y precisas. Es dentro de este cuarto pilar, donde se presenta el núcleo del modelo algebraico de función tonal. Aquí se presentan las cadencias paramétricas y no paramétricas y se describe el proceso completo, por el cual se obtiene la función tonal, yendo desde las matrices de enlace hasta el criterio polinomial. En última instancia, el modelo reduce la clasificación de la función tonal a la ubicación de las raíces de un cierto polinomio complejo en ciertas áreas. El modelo es a la vez, algebraico y geométrico. En este cuarto pilar encontramos también una generalización de la gravedad tonal. Una de las virtudes que tiene este modelo es que no requiere que el número de voces entre dos acordes consecutivos sea el mismo. Esta generalización implica una ampliación del algoritmo húngaro a una aritmética infinita y que dicha ampliación sea consistente con el caso de voces constantes. El modelo es consistente en este sentido.

El quinto pilar es un estudio de la modulación a la luz del modelo previamente diseñado. Al final de la tesis se presentan varias aplicaciones prácticas, y por ello queremos decir que son aplicaciones que un compositor o un intérprete pueden usar de manera directa, sin tener que pasar por la comprensión del modelo. Hay aplicaciones a la conducción de voces, a la generación de progresiones, a la escritura de bajos o a la escritura de melodias, entre otras.

La última parte de la tesis es un estudio de las cadencias a través del modelo algebraico presentado aquí. Se han estudiado las cadencias más importantes en cada uno de los modos.

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Abbreviations and Acronyms

UPM Universidad Politécnica de Madrid

HTFM Hypervolumetric tonal function model.

LCAA Local constant amplitude assumption

PQR Primary research question.

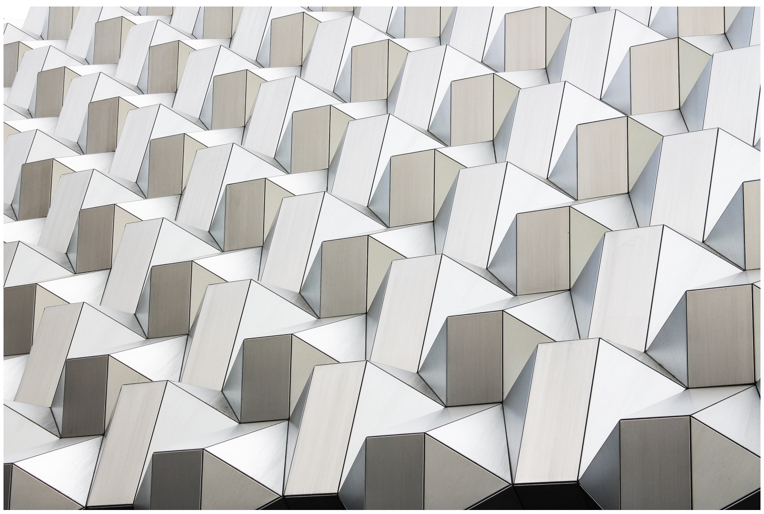
SQR Secondary research question.

VDG Vertical dissonance graph.

WFC Weber-Fechner connection.

Chapter 1

Tonal Function



Silvio Kundt

https://unsplash.com/photos/Fixg8KipOg8

1.1. Tonal function

In music, there is always a need to explain it in the broadest and deepest way possible. Not fully understanding music is a betrayal of its own spirit. Musicians need to deeply understand what they play, not only out of a philosophical need but also for the sake of creating new music. Only is it possible to create out of something it is understood. Since the beginning of the avant-gardes in the 20th century, music has evolved rapidly and has taken stunning and fascinating paths, ranging from jazz and contemporary classical music to pop and rock music all the way up to hip-hop or rap, just to name but a few. There is no longer such a thing as mainstream music. Music is now composed of multiple planets that makes the musical galaxy we live in.

As a matter of fact, understanding each musical planet and its stylistic features is key, but seeking principles that underlie the whole universe is also of significance. Music knowledge and understanding have always been a continuous change of focus, from the minute detail to the broad overview. The presence of these underlying principles in many musics is closely associated to the inherent human's need for music fusion. Consider jazz music, one of the most representative examples of this situation. Jazz music is the result of a formidable melting pot made up by many influences where European harmony and African rhythmic rituals are two of the most important ones; see [Kos00],[Fer93],[SW08] for more on the origin of jazz. Furthermore, as jazz spread around the world, it was exposed to national, regional, and local musical cultures, which in turn gave rise to different styles. Among the underlying principles of music, we may distinguish two categories:

(1) Musical universals, such as the use of discrete pitches, the octave equivalence, the transposability of music, the organization of music into phrases, or arousal factors in emotive expression; these are some of the conserved universals proposed by Brown and Jordania [BJ11].

(2) Stylistic features, such as the use of scales having seven or fewer pitches per octave, the predominance of isometric rhythms; the divisional organization of durational/rhythmic structure, or the use of motivic patterns in melody generation. These features are instances of the predominant patterns described in [BJ11].

The stylistic features we are considering here refers to features that were passed on through cultural contact.

Many music cultures have Western functional harmony as one of their main musical features. Modern functional harmony was the result of the synthetic thinking of two musical schools: the German school led by musicologist and composer Hugo Riemann and the Viennese school driven by Simon Setcher and Arnold Schoenberg. Riemann defined the functions of a chord and the proponents of the Viennese school associated those functions to the degrees of the scale resulting in the modern functional harmony theory. Functional harmony is a term used to describe the relationship between a chord and a tonal center (*Oxford Companion to Music* [Lat22]). Some authors —for example, Piston [Pis50]—extend this definition and add that functional harmony also describes the relationship of a scale degree to a tonal center. Moreover, functional harmony establishes a hierarchy among chords, which results in a classification of chords, normally into tonic, dominant and subdominant chords (subdominant function is also called pre-dominant function by some authors; see [BS08] or [BTB21] and the references therein). Through the **tonal function** chord progressions are classified. Each chord plays a role (a function) either by establishing or otherwise contradicting the tonal center. Tonic chords establish the tonal center, whereas dominant and subdominant chords move away from or return to the tonal center. The classification of chords in turn defines and structures general chord progressions as well as the musical tension/release dialectic. The tonal function, as we will see later on, is closely related to voice leading. Functional harmony started to be used in the Baroque around 1650 and was widely spread throughout all the period of the common practice period (1650–1900) in classical music but also in many other musical cultures such as jazz, folk music, pop music, rock music, or contemporary classical music.

Jazz constituted one of the most important musical revolutions of the 20th century, which still reaches out to today. Especially substantial was the innovation in harmony. By 1910

New Orleans jazz was already established as a combination of brass-band marches, ragtime, and blues [Sch86]. In the 1930s, swing and Gypsy jazz are the more notable styles, where figures such as Duke Ellington, Benny Goodman, Count Basie, Glenn Miller stood out in swing and figures such as Django Reinhardt and Stéphane Grappelli were prominent in gypsy jazz [Sch91]. In part as a reaction to swing, bebop was born in the early-to-mid-1940s. Bebop abandoned the danceable character of swing in favor of a more instrumental style. Musicians now were able to play at faster tempos; this came along with a deeper and more extensive exploration of the musical resources, entailing advanced harmonies, complex syncopation, altered chords, extended chords, chord substitutions, asymmetrical phrasing, and intricate melodies. The main representatives were sax players Charlie Parker, Dexter Gordon, Sonny Rollins; trumpeters Miles Davis, and Dizzy Gillespie; pianists Bud Powell and Thelonious Monk; and drummers Kenny Clarke and Max Roach. As it often happens, the history of music is the history of humanity's reactions to it. Cool jazz was a reaction to the complexity of bebop and already by the end of the 1940s, some jazz musicians introduced calmer, smoother sounds together with long, linear melodic lines [Owe95]; the most outstanding advocates include Chet Baker, Stan Kenton, Stan Getz, and Dave Brubeck. In the mid-1950s, jazz received influences from blues, rhythm and blues, and gospel. The new jazz style product of that blend was hard bop [Gol02], which counted with practitioners of the stature of Charles Mingus, John Coltrane, Miles Davis and again Thelonious Monk (great jazz musicians seem to try more than one style). In the late 1950s, modal jazz and free jazz emerged. The former uses musical modes instead of setting one tonal center across the piece, often modulating among different musical modes to build the accompaniment to melody [Gio21]. Composer George Russell formalized modal jazz in his 1953 book *Lydian Chromatic Concept of Tonal Organization* [Rus01]. Many fine jazz musicians adhered to modal jazz; among them we find Miles Davis (again), Chick Corea, Wayne Shorter, or Larry Young. Free jazz (sometimes called avant-garde jazz), however, tried to break down musical conventions and come up with new forms of expression. In free jazz there are no regular tempos, or regular meters, or strict form; harmonically, tonal centers and usual diatonic chords are discarded in favor of altered dominant chords [CC16]. Figures such as Pharoah Sanders, John Coltrane, and Ornette Coleman played in free jazz style. In the late 1960s and early 1970s, jazz is influenced by rock music, which has been originated in the early 1950s by taking elements from blues and rhythm and blues. The new kind of jazz was named jazz-rock fusion and essentially consisted of combining jazz harmonies and improvisation with rock music, funk, and rhythm and blues (amplification of sound, electric instruments, guitar grooves, characteristic rhythms from rock music) [Gio21]. Charles Lloyd, Keith Jarrett, Jack DeJohnette, and (again) Miles Davis were some of the most widely known practitioners of this style. More than one style of jazz music co-existed through time. Thus, Latin jazz, being the two main sub-styles Afro-Cuban jazz and Afro-Brazilian jazz, developed from the influence of Latin American music on jazz, especially its rhythmic language. That influence began as early as in late 19th century and lasted for several decades. Latin jazz incorporates the clave rhythms and piano montunos into their pieces; see [Was20, SR99, Uri96]. Many of these jazz styles still exist to this day in different re-interpretations.

Most jazz styles have employed functional harmony in very deep and extensive ways. So much so that the very concept of tonal function was often re-defined, extended, and generalized to suit the needs of its dynamic style. Coming back to the underlying principles in music, it can be stated that tonal function is a predominant pattern in jazz music.

After the end of the common practice period, conventionally set at 1900, many music cultures were still using functional harmony, often transforming and extending its grammar and vocabulary. From the end of the Second World War to the present day, functional harmony can be found in music styles such as:

- New Simplicity. This was another reaction to the complexity of post-war serialism and post-tonal movements of the 1950s and 1960s. It started with some German composers such as Hans-Jürgen von Bose or Hans-Christian von Dadelsen (two of the main members of the group of the seven identified by German composer by Aribert Reimann). Musicologist Josiah Fisk used the term to describe the music of Górecki's, John Tavener's and Arvo Pärt's [Fis94, 402], where he states, "in the New Simplicity, the development of ideas in the manner of Western classical music is carefully avoided, the stated goal being the attainment of a simplicity and 'purity' o musical material and character."

- Electronic music [All18], especially in commercial electronic music after the late 1980s (as opposed to electronic music produced by research institutions such as IRCAM or Electroacoustic Music in Sweden.

- Pop music. It originated in the mid-1950s mainly in the United States of American and the United Kingdom and drew on styles such as rock, urban, dance, Latin, and country [FSS01]. Pop music is often associated to commercial music or chart music, but it is not always the case. The harmonic progressions that can be heard in pop music are mostly derived from the application of functional harmony (though different from the classical one); see [HL22] for more information.

- Musical historicism. Apart from its meaning referring to the use of historical sources to inform musical practice in classical music, this term also refers to music composed in past styles by contemporary composers. For example, historicist neo-Baroque composers include Elam Rotem, Federico Maria Sardelli, or Joseph Dillon Ford. As a matter of fact, those composers employ functional harmony techniques in their pieces.

We have made out a strong case for the presence of functional harmony (or tonal function) in many musical cultures and hence a predominant pattern in music. Therefore, it is an underlying principle of music in our context. One of the main research questions looked into in this work is that of providing an operating, general definition of tonal function. In the next section we pose the main questions to be solved in this thesis.

1.2. Research questions about the tonal function

In the previous section, we emphasized the importance of the tonal function in many musical cultures. Strange as it may seem, there are not many formal, precise, and comprehensive definitions of tonal function in the literature. Some of them seem not to fully capture the essence of this musical phenomenon. Some definitions are applicable to very narrow musical contexts and cannot explain harmonic relationships beyond those of the particular style. In our work, we look for comprehensive models of tonal function that can be applicable to a large number of musical contexts (such as jazz or other modern musical cultures). Given that the desired model of tonal function has to be abstract and scalable, it seems that a reasonable way to achieve that is to recourse to mathematical models. To the best of our knowledge, there are very few mathematical models of tonal function. In this tesis, we provide a full mathematical model to define and characterize tonal function that works in disparate musical contexts.

Next step is to lay out the research questions of this thesis. We will formulate the main three research questions —the primary research questions—and for each we will associate secondary research questions, which further elaborate the primary questions.

Our first research question reads as follows:

RESEARCH QUESTION 1: Construct a mathematical model of tonal function that works for diverse and meaningful musical contexts.

This question in turn results in other secondary research questions (SQR from now on below).

Musicology

SRQ1 What musical principles would a model of tonal function be based on so that it has high explanatory power and is at the same time comprehensive? We seek for models not suffering from conceptual overfitting, that is, we reject models that are able to explain very satisfactorily a reduced number of musical cultures. In particular, as we will see later on, models of tonal function based on consonance are rather limited; this is elaborated further in Chapter 3. What are the reasons for said principles to be the foundation of the model? S. Frith and W. Straw [FH20] discuss the ontological and epistemological features of models and contend that "models are vehicles for learning about the world" and among their cognitive functions it can be found understanding and explaining the target being modeled. In our work here, we aspire to creating a model of tonal function from which we learn and explain tonal function itself.

SRQ2 Is the model of tonal function consistent with the music observed in reality? Certainly, the tonal function in Baroque music is not the same as in, say, jazz-rock fusion. We aim for a sufficiently comprehensive model to consistently account for as many as music cultures as possible.

SRQ3 How accurate and natural are the explanations provided by the model when applied to actual music? How functional is the model? In model theory, there is the parsimony principle. It states that models should minimize complexity while maximizing explanatory and predictive power [Ger98, VMW15]. The idea behind parsimonious models stems from Occam's razor or "the law of briefness." We aspire to building parsimonious models that explain tonal function. Chapter 9 show a few practical applications in the form of graphs of several types such as vertical dissonance graphs, progression generation graphs, just to name but a few.

SRQ4 What is the contribution of the model to music analysis? And to musicology in general? What lacunae do the model fill?

Applications to composition

SRQ5 Once the model of tonal function is established, how can it be used in musical composition? How versatile is it? To what extent does the model allow for ease and agility in music composition? Does the model help the composer compose music in an organic manner? Is there any kind of bias in the music composed by following our model of tonal function? In Chapter 9, some compositions will be presented to answer these questions.

Applications to improvisation

SRQ6 In the case of jazz music and other improvisational styles, can the model of tonal function be used to improvise? Does the model provide a natural and straightforward way to improvise?

SRQ7 If the answer to the previous questions are affirmative, how much effort does it demand to learn how to improvise under the model?

Pedagogical applications.

SRQ8 How cognitively demanding is the learning of the model? Is there a special methodology to learn the model both for playing or composing music? Is there specific material to learn how to compose and improvise with the tools provided by the model?

The second primary research question is formulated as follows.

RESEARCH QUESTION 2: Construct a mathematical model of the musical universe so that Research Question 1 can satisfactorily be answered within that model.

The mathematical model

SRQ9 Construct a mathematical model for the following objects in the musical universe: pitch, pitch class, chord, root of a chord, quality of a chord, chord progression, voice leading, arrangement, pitch class distance, distance of a chord progression, cadence, chord classification, and tonal function.

SRQ10 Is the model a computational model? It would be a requirement that the model be computational due to practical reasons (for example, an application to compute tonal functions in real time) and theoretical reasons (epistemological reasons mainly).

SRQ11 The model has to be built according to scientific standards. It has to have the right amount of abstraction, be rigorous in its construction, and follow the parsimony principle. More importantly, once the model has been built, it must produce explanations and predictions that are musically meaningful. There must be a careful assessment of the results output by the model.

Algebraic model. The model proposed in this work is based on algebra (although in some parts it has a geometric component). Why are algebraic models more suitable for the modeling of tonal function than geometric models (such as, for example, those of Tymoczko [Tym11] or Kelley [Kel06]?

SRQ12 Once accepted that algebraic models are satisfactory to explain and understand the tonal function, what are the most suitable algebraic structures to build the model?

Lastly, the third primary research question is posed below.

RESEARCH QUESTION 3: Construct a mathematical model of voice leading and find algorithms to computationally solve the problem of computing the optimal voice leading between two chords.

SQR13 All the previous models of voice leading assume that the number of voices is kept constant throughout the piece. However, in many musical contexts (jazz is a case in point) that is not the case. A powerful model should be able to work out when the number of voices varies.

SQR14 Determining the minimal voice leading for chords of arbitrary cardinality is an open problem (in practice, it is computed by hand given the low number of voices). Is it possible to create a formalization and design an algorithm to solve this problem?

SQR15 If the answer to Question SQR14 was affirmative, what is the running time for algorithm to compute the optimal voice leading? And its space complexity? Is there any lower bound proved for the problem of finding such optimal voice leading? Is it reducible to a problem of known complexity?

Although the research questions have been presented in descending order of abstraction and relevance, in the text those questions will be answered in reversed order, that is, in a bottom-up manner. This is because the most basic building blocks have to come first —the algebraic model —to scaffold the rest of the model —the optimal voice leading and the model of tonal function.

1.3. The contribution of this thesis

In this thesis the three primary questions are answered as well as the 14 secondary research questions. First, we present a mathematical model of tonal function. Such model is based on optimal voice leadings and not on consonance (which answers SQR1). The model is applicable to a large and disparate set of musical cultures as shown in Chapters 6 and 9 (SQR2). Furthermore, the model is sufficiently simple as to satisfy the parsimony principle. It is grounded in voice leading principles. The classification of a chord with respect to a given tonal center is made in terms of how many and in which direction the voices in the optimal voice leading move. Thus, the model requires a low number of parameters to work (which takes care of SQR3). The model renders itself to deep musical analysis, among other reasons because it allows the study of chords with a variable number of voices (SQR4); in Chapter 9 applications of the model to music analysis can be examined. In that very chapter, applications to composition are discussed (SQR5). The possibilities for improvising by having in mind the tonal function at all times are evident (SQR6 and SQR7 partially). As for the pedagogical applications, we briefly show some directions on how to implement a curriculum to teach models of tonal function in Chapter 9 (SQR8). The model outputs a formula to compute the tonal function of a chord with respect to a tonal center; we cannot help but give in and reproduce it below given its beauty.

$$H_E(\lambda) = \Delta(s^{\sigma(L_E^{BC} - W)} + P - \lambda I d_{\max\{dim(X), dim(Y)\}})$$

The meaning of the formula is explained in Chapter 5 entitled *The horizontal dimension of harmony.*

Chapters 2 to 8 lay out our model of tonal function (SQR9, SQR11). The model is a computational one; in fact, we present the Nabla application, an application for computing optimal voice leadings based on our model. Our algebraic model is based on equivalence classes, vector spaces, endomorphism, matrix theory, optimization theory (the Hungarian algorithm), and characteristic polynomials (SQR12).

In Chapter 5, we solve the problem of computing an optimal voice leading between two chords (SQR13). Our solution works for a variable number of voices of the chords (SQR14). In our solution, we used the Hungarian algorithm, a combinatorial optimization algorithm that solves the assignment problem in polynomial time (SQR15).

This thesis contains seven appendixes, from Appendix A to G in which the most relevant chord progression are studied in the light of our model. For each pair of chords presented there, our algebraic model is applied and thus their tonal function is determined. In Chapter refchapter-16, actual and practical applications of the model are presented (many of which are based on the tonal functions studied in the appendixes). These applications are one of the most important contributions of this thesis. Through the applications, any composer or player can perceive and understand the versatility and depth of the model.

1.4. The organization of this thesis

Chapter 1 starts off with a discussion on the concept of tonal function. Its definition is framed in the context of this work. It is also discussed the importance and pervasiveness of such concept in many music cultures, especially jazz music and other modern styles. After that, the research questions dealt with in this thesis are presented. The importance of the work laid out here is argued for. Primary and secondary research questions are listed. Next section is the contribution of the thesis, section in which the answers to the research questions are put forward. The first chapter closes with this section. The rest of the thesis is organized as follows.

- In Chapter 2 contains the formalization of the musical universe. In this chapter the main musical concepts are formalized in mathematical terms. The notation introduced here will be serve for later formalization and description of our results.
- In Chapter 3, entitled *The Vertical Dimension of Harmony*, consonance and dissonance are examined in vertical harmony, that is, when the sounds are played simultaneously. The concept of avoid note will act as the main concept for dissonance.
- Physical characteristics of harmony are studied in Chapter 4. This study is necessary to provide physical and psychoacoustic foundation to our results.
- Chapter 5 is the core of the thesis. In it, the main concepts and results along with their formalization are presented there. The tonal function is studied there in terms of voice leadings. Out of the concept of link matrix, the matrix modeling the voice leading between two consecutive chords, a method to obtain the optimal link is examined. Such method is based on the Hungarian algorithm, an optimization algorithm by Harold Kuhn [Kuh55]. When run over the link matrix, the algorithm produces the optimal link. From the optimal link, a polynomial is associated. The position of the roots of the polynomial characterizes the motion of the voices within the chord progression. That motion determines in turn the tonal function of said progression.
- The concept of tonality is exhaustively examined in Chapter 6; furthermore, a generalization of tonality is put forward for discussion.
- Modulation is addressed in Chapter 7. Tonal function can be generalized to modulation. In this chapter, the techniques to do so are presented.
- Global tonal functions are studied in Chapter 8. In particular, tonal graphs are analyzed. These kinds of graphs constitute the main practical results of this thesis.

- Chapter 9 contains the practical applications of the methods and results of this thesis.
- This thesis comes to an end in Chapter 10, where conclusions and future work are discussed.
- In Appendices A to G, a thorough study of the tonal functions for the main chord progressions in the seven modes is presented. In that study, the methods and results obtain in the previous chapters are now applied to its fullest.

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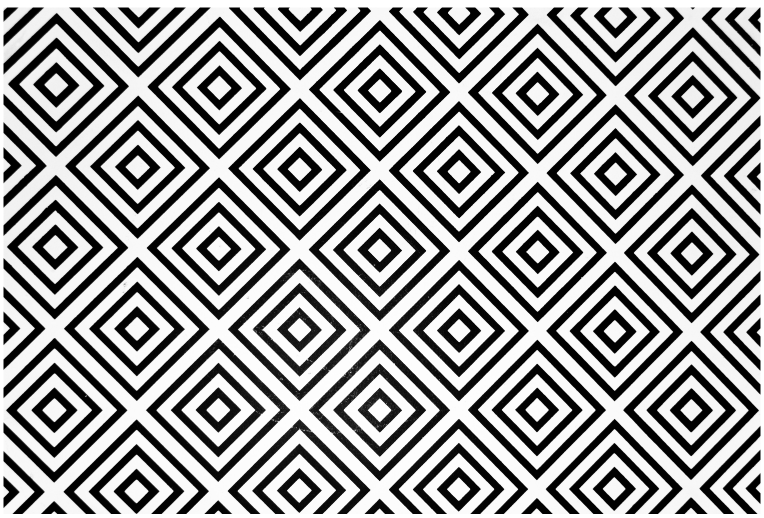
Chapter 2

The Mathematical Formalization of the Musical Universe



Akshar Dave

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2.1. Previous work on mathematical formalization

Voice leading is the art and science of how to connect chords to one another. Rules on how to connect chords are not immutable and have changed over time. Common-practice principles in the Renaissance impose that voices should be smooth and independent. These two principles translate into specific rules such as: (1) Each voice should move the shortest distance possible; (2) Voice crossing should be avoided except to create melodic interest; (3) Parallel fifths and octaves should also be avoided (so that melodic independence is favored). Later on, these rules are relaxed and during the Baroque era outer voices grow in importance and they move more by leaps whereas inner voices move stepwise. From the 19th century on, many composers do not follow the common-practice part-writing rules, although they tend to follow more abstract voice leading principles. In the 20th century, Lewin [Lew87] introduced Neo-Riemannian theory, which is based on the idea of connecting chords according to some definition of harmonic proximity [Coh12]. This notion of harmonic proximity requires, in turn, a notion of a distance between chords. It is natural then to introduce at this point some mathematical formalism to address the question of how to measure the distance between two chords; Geometry and algebra have been used to formalize music, especially voice leadings and distance functions; see the work of Tymoczko [Tym06], [Tym09], [Tym11], Hall and Tymocko [HT12], Wixey and Sturman [WS16], and Derfler [Der10], just to name but a few. Many distance functions have been defined to quantify the distance between two chords. Tymoczko, in his paper *Three conceptions of musical distance* [Tym09], proposes three definitions. The first is based on geometric spaces and consists of a collection of continuous quotient spaces or orbifolds; these quotient spaces are built by considering musical objects that are invariant under certain operations (the OPTIC operations). The second one is founded on acoustics and gives rise to the Tonnetz and tuning lattices. The third one is based on the interval content of a chord, where chords that share the same interval content are represented by the same point (see [Qui06] and the references therein). In [HT12], general properties of the distances between chords are studied by looking at the concept of submajorization, a notion that reflects to what extent components of vectors are spread out (here vectors represent chords). In [WS16], the authors look into distances from a graph-theoretic standpoint in order to overcome certain drawbacks found in the geometric approach. Once the idea of harmonic proximity has been established, the goal is to produce the most efficient voice leadings between the chords of a progression. Here comes the concept of parsimony, also called minimal voice leading. Minimal voice leadings have the property that minimizes the voice leadings between chords as measured by the chosen distance (harmonic proximity).

This work focuses on the pedagogical aspects of voice leading in jazz music, a style where voice leading is also an important feature. We would like to provide composers with a tool to understand and write voice leadings by following criteria that are at the same time systematic and musically meaningful. The function that measures the harmonic proximity is just the number of half-tones between the notes of a chord, and the parsimonious or minimal voice leadings will be obtained by minimizing certain sums, as we will see shortly. On the mathematical side, we offer the musician a minimal but meaningful mathematical formalization of voice leadings so that musical concepts are still recognizable in the formalization. The structure of this paper is as follows. We start by introducing some definitions, which will help build the formal framework (the musical universe). In Section 3, we study metric spaces in music and introduce the nabla distance, which is the distance to measure the size of a voice leading. Section 4 contains the pedagogical applications of the nabla methodology.

2.2. The musical universe

We begin by defining the space of frequencies. In principle, it would be enough for our purposes to consider the set of audible frequencies, say, the interval $(20, 2 \cdot 10^4)$, when measured in Hz. However, for completeness we will consider the space of frequencies Φ as the real line (it is closed under product and sum of frequencies). Let x, y be two pitches described by their frequencies. We write $x \sim y$ if and only if $x = 2^k \cdot y$, for some integer k. Recall that two pitches are an octave apart when the quotient of the highest frequence to the lowest is 2. This relation identifies all the pitches that are apart any number of octaves as just one pitch.

From now on, we assume we are in the presence of the equal temperament. Given a fixed pitch class $[k]$, we define the circle of fifths $LC_k/{\sim}$ as the set $LC_k/\sim = \left\{[k], \left[k \cdot 2^{\frac{7}{12}}\right], \left[k \cdot 2^{\frac{14}{12}}\right], \left[k \cdot 2^{\frac{21}{12}}\right], \ldots, \left[k \cdot 2^{\frac{77}{12}}\right]\right\}$. This definition is illustrated in Figure 2.1. The pitch class of A was chosen as the base and then the circle of fifths is built up from it by multiplying the previous pitch by $2^{\frac{7}{12}}$, the distance of a fifth in terms of frequency. If we consider the first seven pitches in their order (the diameter shown in the Figure), they form the Lydian scale (A-B-C♯-D♯-E-F♯-G♯).

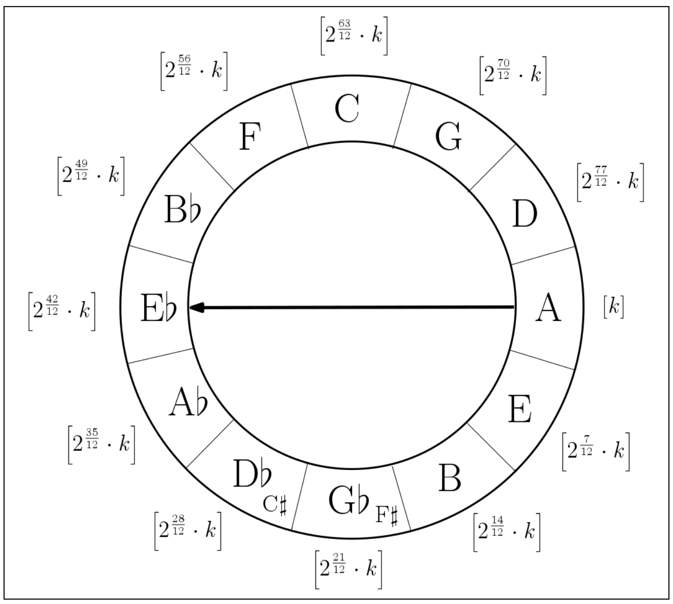

Figure 2.1: The circle of fifths

A **chord** X is a subset of pitch classes in $(LC_k/{\sim})$. In Western tonal music, some chords are described by a **root** and a **quality**. A chord is an unordered collection of pitches. The root is usually the lowest pitch in any voicing of the chord whereas the quality refers to labels given to chords. For example, a dominant seventh chord of C is the chord composed by C, E, G and B♭. The root of this chord is C and the quality dominant seventh. This label tells us that the first three notes form a major triad and that B must flat so that there is minor seventh between C and B. The quality of a chord is indicated by several symbols (m or lowercase for minor chords, + for augmented chords, etc.).

A **chord progression** is a sequence of chords. As such, chords in a progression are presented in a given order, which is the order they appear in time. A suitable way to deal with chord progressions is by considering the matrix of classes. If $P \in \mathcal{M}_{n\times m}(LC_k/\sim)$ is a chord progression of length m, then each chord is a vector of n notes and there are m chords in the progression. We can arrange the notes of the chord progression in a matrix as follows.

$$P = \begin{pmatrix} [\theta_{11}] & \dots & [\theta_{1m}] \\ \vdots & \ddots & \vdots \\ [\theta_{n1}] & \dots & [\theta_{nm}] \end{pmatrix} \tag{2.1}$$

To fix ideas, let consider the 2-note chord progression {E, C} to {F, E}, which from now on will be notated as {E, C}$\Longrightarrow${F, E}. Its matrix representation is $P = \begin{pmatrix} [E] & [F] \\ [C] & [E] \end{pmatrix}$.

Another way to denote progressions is as follows. Let $P \in \mathcal{M}_{n\times m}(LC_k/\sim)$ be a progression; then a progression class is formed by an ordered list of chords $X_1, \dots, X_m$. The notation

$$[P] = (|_{j=1}^{m} X_j) \tag{2.2}$$

refers to the list of chords, where the symbol $|_{j=1}^{m}$ is used indicate the order of the chords. This notation will be employed throughout in Chapters 3 (section on compact tonality) and 5 (section on tonal function labels).

A **link** E of two chords is a progression where in their respective classes there are only two chords. A link class $[E]$ is then defined as:

$$[E] = \left(\Big|_{j=1}^{2} X_j\right) = (X_1|X_2) \tag{2.3}$$

From now on, the **dimension of a chord** is the number of classes in X. The reason to use the word dimension is that a voicing can be thought of as a vector in the space of frequencies. Bear in mind that when we say that a link has two chords, we mean that its class (all the equivalent links with it) has two chords.

When we study a link we use one of the kind of links that contain the chords under study.

$$E \subset [E] = \left(\Big|_{j=1}^{2} X_j\right) = (X_1|X_2) \tag{2.4}$$

Next, and independently of the chord dimension, we write it as $E = (X|Y)$, where X is the **antecedent chord** and Y is the **consequent chord**.

As an abuse of notation and when the context is clear, we sometimes omit the class and just write $E = (X|Y)$ instead of $[E] = (X|Y)$. On other occasions, subscripts and superscripts will be used to show that the chord appears in a particular arrangement, thus following the classical music notation for chords in root position. Superscript r is used to show that a chord is written in root position when read bottom up top in its corresponding column in the link.

The subscript is also used with one of the classes of $LC_k/\sim$ to use the Roman numeral notation. As we will see later on, this is a consequence of the fact that the tonal function is invariant by transposition; this result will formally be stated in the **static function theorem** (see Section 4.2.1 for further details). To identify two chords in the key of D, for example, Roman numerals, qualities of chords, and the superscript r is used as shown below.

$$E = (I_d^r - |II_d^r -) = \begin{pmatrix} A & B \\ F & G \\ D & E \end{pmatrix}$$

Let Φ^+ be the set of positive frequencies. A **voicing** of a chord is a mapping $\psi(X)$ from $\mathcal{M}_{n\times 1}(LC_k/\sim)$ to $\mathcal{M}_{n\times 1}(\Phi^+)$. The mapping takes a given class to a note. Indeed,

$$\psi\left(\begin{pmatrix} [\theta_{1j}] \\ \vdots \\ [\theta_{nj}] \end{pmatrix}\right) = \begin{pmatrix} \phi_{1j} \\ \vdots \\ \phi_{nj} \end{pmatrix} \tag{2.5}$$

where $\phi_{ij} \in [\theta_{ij}]$, for some i in $\{1, \ldots, n\}$ and some j in $\{1, \ldots, m\}$. Following with the previous example, a voicing for the chord progression could be (among other possibilities) $\psi\left(\begin{pmatrix} [C] \\ [E] \end{pmatrix}\right) = \begin{pmatrix} C_4 \\ E_4 \end{pmatrix}$. For ease of reading, we will notate the frequencies by their standard names instead of their numerical values. Therefore, we will write A_4 instead of 440 Hz.

An **arrangement** of a chord progression is the mapping defining which notes of the chords are chosen for the voice leading. Formally, it is a mapping $A : \mathcal{M}_{n\times m}(LC_k/\sim) \to \mathcal{M}_{n\times m}(\Phi^+)$ written as

$$A\left(\begin{pmatrix} [\theta_{11}] & \ldots & [\theta_{1m}] \\ \vdots & \ddots & \vdots \\ [\theta_{n1}] & \ldots & [\theta_{nm}] \end{pmatrix}\right) = \begin{pmatrix} \phi_{11} & \ldots & \phi_{1m} \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \ldots & \phi_{nm} \end{pmatrix} \tag{2.6}$$

where $\phi_{ij} \in [\theta_{ij}]$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. From now on, arrangements will be notated as $(\phi_1, \ldots, \phi_m) \longrightarrow (\phi'_1, \ldots, \phi'_m)$, that is, as bijections between sequences of notes; compare this notation to that of chord progressions above. Figure 2.2 illustrate the correspondence between the chords of a progression and the associated matrices.

For the chord progression {E, C}$\Longrightarrow${D, G}, A could take on the form, among others, of $A\left(\begin{pmatrix} [C] & [G] \\ [E] & [D] \end{pmatrix}\right) = \begin{pmatrix} C4 & G4 \\ E4 & D4 \end{pmatrix}$

Each voicing $\psi(X)$ gives place to a vector space called the **color** of the voicing $\psi(X)$.

Given an arrangement, we can consider the direct sum of the colors of each chord in the arrangement. If the dimension of the direct sum of colors is 1, then the arrangement is called a **monochromatic arrangement**. Otherwise, the arrangement is called **polychromatic**.

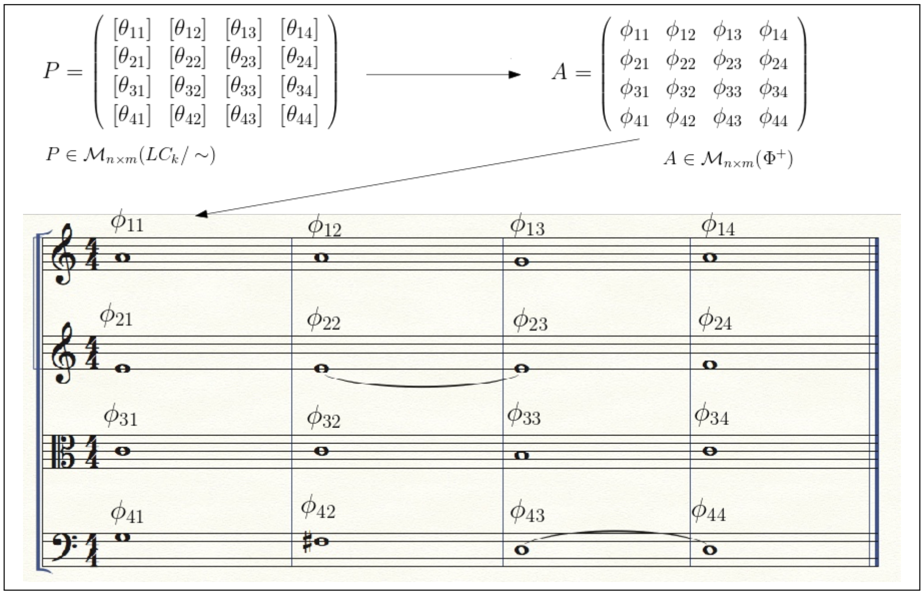

Figure 2.2: An arrangement

$$A = \begin{pmatrix} a & \lambda_1 a & \lambda_2 a & \lambda_3 a & \lambda_4 a & \lambda_5 a & \lambda_6 a & \lambda_7 a \\ b & \lambda_1 b & \lambda_2 b & \lambda_3 b & \lambda_4 b & \lambda_5 b & \lambda_6 b & \lambda_7 b \\ c & \lambda_1 c & \lambda_2 c & \lambda_3 c & \lambda_4 c & \lambda_5 c & \lambda_6 c & \lambda_7 c \\ d & \lambda_1 d & \lambda_2 d & \lambda_3 d & \lambda_4 d & \lambda_5 d & \lambda_6 d & \lambda_7 d \end{pmatrix}$$

Figure 2.3: A monochromatic arrangement (taken from Pozo's *Pi Suite*)

2.3. The nabla distance

It is possible to endow the musical space with a metric. The idea is to measure the distance between two notes and what follows is just a formalization of what our ears do in a natural way all the time; see [Deu92] for more information on the cognitive aspects of music. We define a metric $\Delta : (\Phi^+)^2 \to \mathbb{R}$ as an integral.

$$\Delta(\alpha, \beta) = \left| \int_{\alpha}^{\beta} \frac{\Omega}{\phi} d\phi \right| \tag{2.7}$$

where α and β are frequencies and Ω is a constant such that $\left| \int_1^2 \frac{\Omega}{\phi} d\phi \right| = 12$; see [Ben06] for a relationship between this constant and the definition of cents. By working out the integral above, this distance can be expressed as $\Delta(\alpha, \beta) = \left| \Omega \ln \left(\frac{\beta}{\alpha} \right) \right|$. The value of the constant is $\Omega = 12 \cdot |\log_2(e)|$, which indicates that the octave is divided into 12 equal half-tones. This Δ function does hold the three properties of a metric, namely: positivity, $\Delta(\alpha, \beta) \geq 0$; symmetry, $\Delta(\alpha, \beta) = \Delta(\beta, \alpha)$; and the triangle inequality $\Delta(\alpha, \beta) \leq \Delta(\alpha, \gamma) + \Delta(\gamma, \beta)$. This metric receives the name of **delta metric**.

The pair (Φ^+, Δ) is called the **musical metric space**. This metric can be extended to the spaces of pitch classes by just taking the minimum of the elements in each pitch class. For two classes $[\theta], [\tau]$ in $(LC_k/\sim)$, we define $\tilde{\Delta}$ as $\tilde{\Delta}([\theta], [\tau]) = \min\{\Delta(\alpha, \beta) \,|\, \alpha \in [\theta], \beta \in [\tau]\}$. See the work [For73] of Forte for more information on distance functions. For example, $\Delta(C5, E4) = 8$ and $\Delta(C4, E5) = 16$, but $\tilde{\Delta}([C], [E]) = \min\{\Delta(\alpha, \beta) \,|\, \alpha \in [C], \beta \in [E]\} = 4$. Notice that the maximum value the distance $\tilde{\Delta}$ can take is 6.

A useful result, known as the **color theorem**, states that, if $\lambda \in \mathbb{R}, \lambda \neq 0$, then

$$\left| \Omega \int_{\alpha}^{\beta} \frac{1}{\phi} d\phi \right| = \left| \Omega \int_{\lambda\alpha}^{\lambda\beta} \frac{1}{\phi} d\phi \right| \tag{2.8}$$

Let $P \in \mathcal{M}_{n \times m}(LC_k/\sim)$ be a chord progression such that $P = ([p_{ij}])$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Consider σ, an element in the symmetric group $\mathcal{S}_n$ defined over the set of indices $\{1, 2, \ldots, n\}$. Then, we define $E(P)$, the **extension** of P, as those matrices $B = ([b_{ij}])$ in $\mathcal{M}_{n \times m}(LC_k/\sim)$ such the following two conditions hold:

(1) For some values of j, $[p_{ij}] = [b_{ij}]$, for all $i = 1, \ldots, m$;

(2) For the rest of values of j, $[p_{ij}] = [b_{\sigma_k(i)j}]$, for all $i = 1, \ldots, n$, where σ_k is a permutation in $\mathcal{S}_n$.

These conditions state that a column in B is either the same column in P or a permutation of some column of P. $E(P)$ is the set of such matrices. Consider again the matrix associated to the chord progression {E, C}$\Longrightarrow${F, A}. Then, the extension of P is

$$E(P) = \left\{ \begin{pmatrix} [C] & [A] \\ [E] & [F] \end{pmatrix}, \begin{pmatrix} [C] & [F] \\ [E] & [A] \end{pmatrix}, \begin{pmatrix} [E] & [A] \\ [C] & [F] \end{pmatrix}, \begin{pmatrix} [E] & [F] \\ [C] & [A] \end{pmatrix} \right\}$$

Next, we need to define the distance that a voice travels through a given chord progression. We will use the symbol $\tilde{\nabla}$ to define the **distance** of a **chord progression** P. Then, $\tilde{\nabla}(P)$ is defined as follows: $\tilde{\nabla}(P) = \sum_{i=1}^{n} \sum_{j=1}^{m-1} \tilde{\Delta}([\theta_{ij}], [\theta_{i(j+1)}])$. This distance receives the name of **nabla distance**.

The value of $\tilde{\nabla}(P)$ is the sum of all the distances between consecutive notes of a voice over all voices in the chord progression.

The operator nabla can also be defined for the set $E(P)$ as follows: $\tilde{\nabla}(E(P)) = \left\{ \tilde{\nabla}(B) \,|\, B \in E(P) \right\}$. Notice that $\tilde{\nabla}(P)$ is a real value and $\tilde{\nabla}(E(P))$ a set of values. Let us compute $\tilde{\nabla}(P)$ for the chord progression {E, C}$\Longrightarrow${F, A}. Indeed, $\tilde{\nabla}(P) = \sum_{i=1}^{n} \sum_{j=1}^{m-1} \tilde{\Delta}([\theta_{ij}], [\theta_{i(j+1)}]) = \tilde{\Delta}([E], [F]) + \tilde{\Delta}([C], [A]) = 1 + 3 = 4$. Actually, we don't need to consider all the matrices in $E(P)$ to compute $\tilde{\nabla}(E(P))$. It is enough to choose those where the first column is not rearranged. The nabla distances of the matrices in $E(P)$ are

$$\tilde{\nabla}\left(\begin{pmatrix} [C] & [A] \\ [E] & [F] \end{pmatrix} \right) = 1 + 3 = 4, \quad \tilde{\nabla} = \left(\begin{pmatrix} [C] & [F] \\ [E] & [A] \end{pmatrix} \right) = 5 + 5 = 10,$$

The nabla value of the extension of P is $\tilde{\nabla}(E(P)) = \left\{\tilde{\nabla}(B) \,|\, B \in E(P)\right\} = \{4, 10\}$.

A **chord progression** is said to be **optimal** if $\tilde{\nabla}(P) = \min\left\{\tilde{\nabla}(E(P))\right\}$. In our example, the chord progression {E, C}$\Longrightarrow${F, A} was optimal as the nabla distance attained the minimum at that progression. The optimal chord progression will be denoted by P^0 (we will come back to it in Chapters 4 and 5).

Analogously, the nabla distance can be defined for arrangements; it will be notated by ∇ (without tilde). If $A = (\phi_{ij}) \in \mathcal{M}_{n\times m}(\Phi^+)$ is an arrangement, then the formal definition of ∇ is $\nabla(A) = \sum_{i=1}^{n} \sum_{j=1}^{m-1} \Delta(\phi_{ij}, \phi_{i(j+1)})$.

An **arrangement** A is said to be **optimal** if $\nabla(A) = \tilde{\nabla}(P_A)$, where P_A is the chord progression associated to A. Let us consider two arrangements associated to the chord progression {E, C}$\Longrightarrow${F, E}, say, A_1:(E4, C4)$\longrightarrow$(F4, E4) and A_2:(E4, C4)$\longrightarrow$(F5, E5). Let us find which one is optimal by computing their nabla distances. We have $\nabla(A_1) = \Delta(E4, F4) + \Delta(C4, E4) = 1 + 4 = 5$ and $\nabla(A_2) = \Delta(E4, F5) + \Delta(C4, E5) = 13 + 16 = 29$. Therefore, the first arrangement is the optimal one.

Let us work out a larger example, with three voices and three chords in the progression. In the example below, we have removed the square brackets to simplify the notation as it is clear we are speaking of pitch classes. Since the extension of P is composed by all matrices that are a result of permutating the classes in the columns of P, we can apply a sequence of permutations (the σ's below) to obtain a sequence of chord progressions reaching the minimum value.

$$P = \begin{pmatrix} A & D & G \\ F & B & E \\ D & G & C \end{pmatrix} \xrightarrow{\sigma_1 : D \leftrightarrow G} P_1 = \begin{pmatrix} A & G & G \\ F & B & E \\ D & D & C \end{pmatrix} \xrightarrow{\sigma_2 : G \leftrightarrow B} P_2 = \begin{pmatrix} A & B & G \\ F & G & E \\ D & D & C \end{pmatrix}$$

$$\tilde{\nabla}(P) = 31 \qquad \tilde{\nabla}(P_1) = 15 \qquad \tilde{\nabla}(P_2) = 13$$

$$P_2 = \begin{pmatrix} A & B & G \\ F & G & E \\ D & D & C \end{pmatrix} \xrightarrow{\sigma_3 : G \leftrightarrow E} P_3 = \begin{pmatrix} A & B & E \\ F & G & G \\ D & D & C \end{pmatrix} \xrightarrow{\sigma_4 : C \leftrightarrow E} P_4 = \begin{pmatrix} A & B & C \\ F & G & G \\ D & D & E \end{pmatrix}$$

$$\tilde{\nabla}(P_2) = 13 \qquad \tilde{\nabla}(P_3) = 11 \qquad \tilde{\nabla}(P_4) = 7$$

In this case, P_4 is the chord progression with minimum $\tilde{\nabla}$ distance.

The computation of the minimum of the nabla distance is equivalent to minimize the sums of the weights for all voices in the chord progression, where the weights are given by the distance between the notes. We are looking for a global minimum, which may not need to be unique. Given that the weights are positive, it suffices to find the minimum of the nabla distance between each pair of consecutive chords to obtain the global minimum. Indeed, if an optimal chord progression had two consecutive chords where the distance between them was not minimal, then by changing the link between those chords to the minimum value, the global minimum would also decrease. That would contradict the fact that we had reach the global minimum in the first place.

In Chapter 5, we will present a polynomial-time algorithm to compute the nabla distance. The algorithm is called the Hungarian algorithm and was published by Harold Kuhn [Kuh55] in 1955.

It is possible to abstract up even more and define progression classes where two progressions are said to be equivalent if their chords are the same. This will be done in full detail in Chapter 3.

2.4. The Nabla application

In this section we show how to use the idea of the nabla distance to teach voice-leadings in jazz music. Voice-leadings, contrary to what it might seem, are common in jazz music and part of a proper performance practice. In order to help the interested musician to understand and use the nabla approach to part-writing, we wrote an application, the ∇ app (read the Nabla application), which, from a sequence of chords input by the user, computes the optimal chord progression. The application is already available on Apple store and the interface is in Spanish, although it will be translated into English very soon. This application can be used to illustrate concepts of mathematical theory in the classroom. It may help the music student to familiarize themselves with mathematical formalization (all the concepts found in Section 2). Also, it allows the teacher to take a hands-on approach to part-writing in jazz or classical music.

Figure 2.4: The nabla application.

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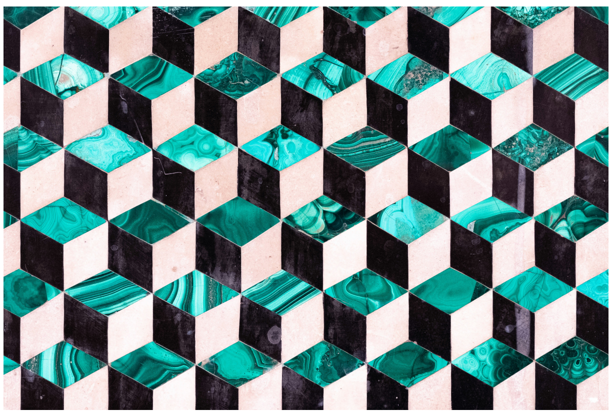
Chapter 3

The Vertical Dimension of Harmony



Mike Hindle

https://unsplash.com/photos/a-group-of-colorful-blocks-pc0Z_tiMAJ4

3.1. Consonance and dissonance

Consonance and dissonance are two phenomena of importance in the organization of Western music. In fact, creating new dissonances or consider old dissonances as modern consonances have been driving forces of new musical styles throughout the history of music. The study of how sounds are related when played simultaneously is referred here as the **vertical dimension of harmony**.

It is common for music students to study consonance and dissonance in the first few years of their training. History, however, has taught us that those two concepts are not carved in stone. They have changed throughout history and furthermore they are dependent on particular styles of music. In our experience as musicians, there is no a fixed, absolute criterion that establishes which intervals are consonant or dissonant, sometimes not even a gradient of consonance among intervals.

A historical context for the semantic problems associated with consonance and dissonance is suggested (though not pursued very far) by Paul Hindemith in the *Craft of Musical Composition* [Hin37], when he says:

> The two concepts have never been completely explained, and for a thousand years the definitions have varied. At first thirds were dissonant; later they became consonant. A distinction was made between perfect and imperfect consonances. The wide use of seventh-chords has made the major second and the minor seventh almost consonant to our ears. The situation of the fourth has never been cleared up. Theorists, basing their reasoning on acoustical phenomena, have repeatedly come to conclusions wholly at variance with those of practical musicians.

Consonance can be defined in acoustics or psychophysiology as those sounds presenting a great coincidence of harmonic partials; conversely, dissonance would be given by sounds presenting beating and roughness due to low coincidence of harmonic partials. While this definition may seem objetive, more often than not it is subjective, conventional, cultural, and style- or period-dependent. For instance, a major second, such as the notes C and D played together, would be considered dissonant in the music of Bach in the 1700s. However, the same interval may sound consonant in the context of a piece by Claude Debussy from the early 1900s or an atonal contemporary piece. For a good account of the history of dissonance and consonance, see Tenney [Ten88].

3.2. Avoid notes

In the Western music tradition, dissonance acts as an organizing musical principle. Composers make use of dissonance in musical composition with many purposes such as to create movement, elicit change, rising and falling action, a sense of urgency, and build tension followed by a period of release. Furthermore, dissonances are used to provoke emotions in music. Classifying chords according to its dissonance is a fundamental block in the general harmony theory in Wester music. In our research, we adopt a definition based on the concept of **avoid note**. Nettle and Graf [NG97] provide the following definition for an avoid note: The pitch or pitches of a scale which are not used harmonically because they will destabilize the sound of the chord. It is implied that avoid notes form dissonances with the notes of the sounding chord; hence, the importance of dissonance is now discernible. This is a general definition that allows for quite a few notes to become avoid notes. For instance, over a major chord, a perfect fourth is an avoid note.

In modern harmony, the avoid notes are those a half step above the chord tone. According to Nettle and Graf, the avoid notes of the seven modes are in Table 3.1.

Scale degree	Chord	Mode	Avoid note	Available tensions	Jazz tensions
I△	Cmaj7	Ionian	First and fourth scale steps, C and F	9, 13	9, 13
II-7	Dm7	Dorian	No avoid note	9, 11, 13	9, 11, 13
III-7	Em7	Phrygian	F and C	11	♭9, 11, b13
IV△	Fmaj7	Lydian	First scale step, F	9, ♯11, 13	9, ♯11, 13
V7	G7	Mixolydian	Fourth scale step, C	9, 13	9, 13
VI-7	Am7	Aeolian	Sixth scale step, F	9, 11	9, 11, b13
VIIø	Bø7	Locrian	Second scale step, C	11	♭9, 11, b13

Table 3.1: Avoid notes for modes of the C major scale

Our definition of avoid note will be more general than the one given previously. A note is said to be a **generalized avoid note (♭9)** if it forms a minor ninth with any of the notes of the sounding chord. Notice that by ♭9th here it is referred module octave, that is, a minor second can be considered as an avoid note, too. For brevity, we will simply refer to a generalized avoid note as an avoid note. We will call **available tensions** to those notes that do not form a ♭9th with notes from the underlying chord. **Jazz tensions** are those notes that do not form a ♭9th with the guide tone.

3.3. Vertical dissonance graphs

Since chords sound perceptually equivalent by octave transposition, any specific representation of the chords in terms of voicings is discarded. We define a graph —called the **vertical dissonance graph** (VDG)— that shows all the intervalic distances among the notes of the chord. In Figure 3.1, such graph is shown for chord $C\Delta(\sharp 11)$.

Assuming the notation introduced in Chapter 2, Δ denotes the distance in half-tones between two notes. Refer to Figure 3.2; it shows the chord $C\Delta(\sharp 11) = \{C, E, G, B, F\sharp\}$ and its VDG. There two potential edges that could lead to avoid notes, marked as ① in the figure. The ascending interval $G - F\sharp$ holds that $\Delta(G, F\sharp) = 1$, but it does not form an ascending ♭9th because $F\sharp$ will sound on top of the voicing. On the other hand, for the ascending interval $B - C$, $\Delta(C, B)$ equals 1, and this time a ♭9th is formed (as a minor second).

In jazz music these relationships are studied within the well-known scale-chord theory; see [Lev11], [Ber13], [Ter17], and [MH13]. In that theory, the objective is to establish a relationship (not always bijective) between each scale and each chord. Let $\mathcal{S} \subset \wp(LC_k/\sim)$ be a scale and let $\mathcal{C} \subset \wp(LC_k/\sim)$ be a chord. The scale is said to be **compatible** with the chord, and it is denoted by $\mathcal{S} \triangleq \mathcal{C}$, if the union of both vertical dissonance graphs has no ♭9th interval. A given chord $\mathcal{C}$ can be framed on two or more different scales, say $\mathcal{S}$ and $\mathcal{S}^*$. Usually, the scale of greater cardinality is usually established for reference for both chords.

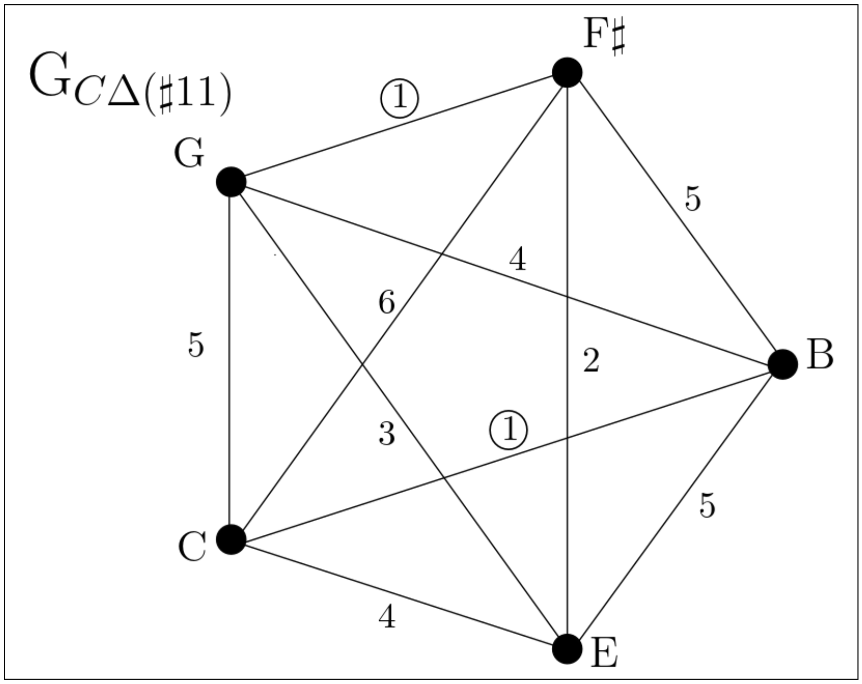


Figure 3.1: Vertical dissonance graph

Thus, the following implication holds:

$$\mathcal{S} \triangleq \mathcal{C} \text{ and } \mathcal{S}^* \subset \mathcal{S} \Longrightarrow \mathcal{S}^* \triangleq \mathcal{C}$$

Let $G_{\mathcal{S}\cup\mathcal{C}}$ be the union graph of both graphs for chord $\mathcal{C}$ and scale $\mathcal{S}$. As established in Chapter 2, the distance between two note classes is taken as the minimum distance between elements of the classes. Therefore, we now have to take into account the sign of the metric between a class of the chord $\mathcal{C}^\varepsilon$ and a class of the scale $\mathcal{S}^\varepsilon$, where ε denotes any edge in the union graph. The σ function is called **sign recovery** and marks the direction in which the distance Δ is attained. For a scale to be **compatible** with a chord, the sign recovery distances between classes have to be different from 1 for every edge ε. More formally,

$$\mathcal{S} \triangleq \mathcal{C} \Longleftrightarrow \sigma(\Delta(\mathcal{C}^\varepsilon, \mathcal{S}^\varepsilon)) \neq 1, \text{for all edges } \varepsilon \text{ in } G_{\mathcal{S}\cup\mathcal{C}}$$

where $\sigma()$ denotes the sign recovery function.

Sign recovery is fundamental because the distances of ♭9th and seventh are the same (because of the absolute value in the distance formula), but have different harmonic role since their frequencies in the spectrum are different. Thus, it is necessary to distinguish which of the two will be higher by recovering the sign and assuming that in general any frequency in an $\mathcal{S}$ class will sound higher than a frequency in a $\mathcal{C}$ class.

As an example, let us find whether there are avoid notes in the half-diminished with respect to the scale C Locrian, that is, we want to determine if C Locrian forms ♭9th intervals with the half-diminished chord in the same key. The VDG in Figure 3.2 shows all the distances between the notes; notice that the graph shown below is the union graph of C Locrian and the half-diminished chord. There are two edges ε^1 and ε^2 that are candidates to generate the ♭9th as the distance associated to those edges is 1.

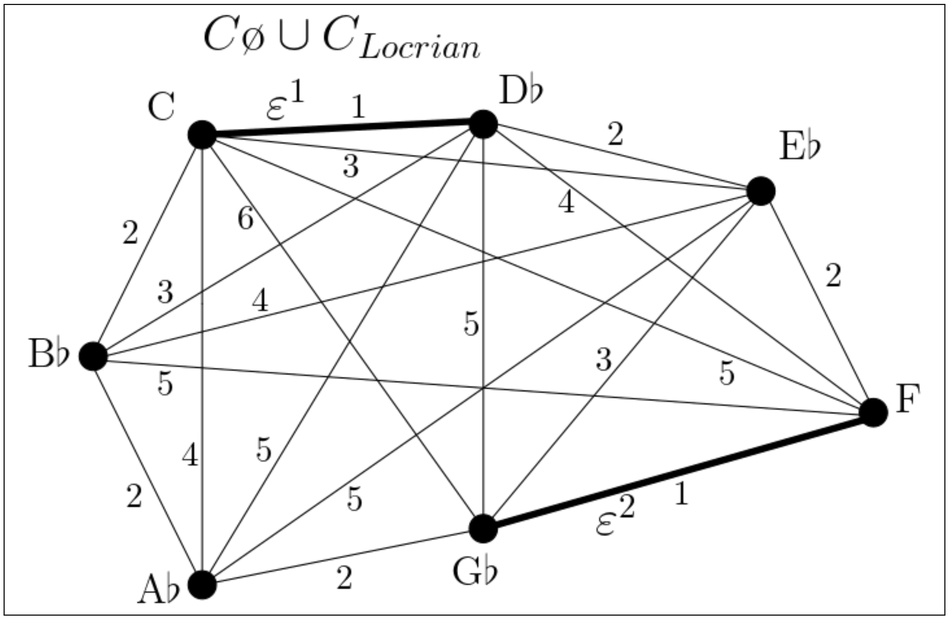


Figure 3.2: Vertical dissonance graph for C Locrian and the half-diminished chord on C

We compute the distance between the note $[C]$ in the chord $\mathcal{C}$ (not to be confused with note C) and $D\flat$ in the scale $\mathcal{S}$, which is the C Locrian scale. The order is important as the sign recovery function outputs different values if its arguments are given in a different order:

$$\sigma(\Delta(\mathcal{C}^{\varepsilon^1}, \mathcal{S}^{\varepsilon^1})) \neq \sigma(\Delta(\mathcal{S}^{\varepsilon^1}, \mathcal{C}^{\varepsilon^1}))$$

Taking the class belonging to the first chord and that of the scale yields,

$$\sigma(\Delta(\mathcal{C}^{\varepsilon^1}, \mathcal{S}^{\varepsilon^1})) = \sigma(\Delta(C, D\flat)) = 1$$

Since the sign recovery takes value 1, C Locrian is a scale with avoid notes since class $[D\flat]$ forms a ♭9th when the scale is played on top of $C\varnothing 7$. This is the reason why the Locrian ♮2 scale is used instead of the Locrian scale over a half-diminished chord.

In the case of the edge ε^2, the sign recovery function outputs -1. Indeed,

$$\sigma(\Delta(\mathcal{C}^{\varepsilon^2}, \mathcal{S}^{\varepsilon^2})) = \sigma(\Delta(G\flat, F)) = -1$$

The interval formed in this case is a major seventh; therefore, F is not an avoid note.

Let us give another example, now with chord C∆ and C Lydian. The VDG $G_{C\triangle \cup C_{\text{Lydian}}}$ is shown in Figure 3.3.

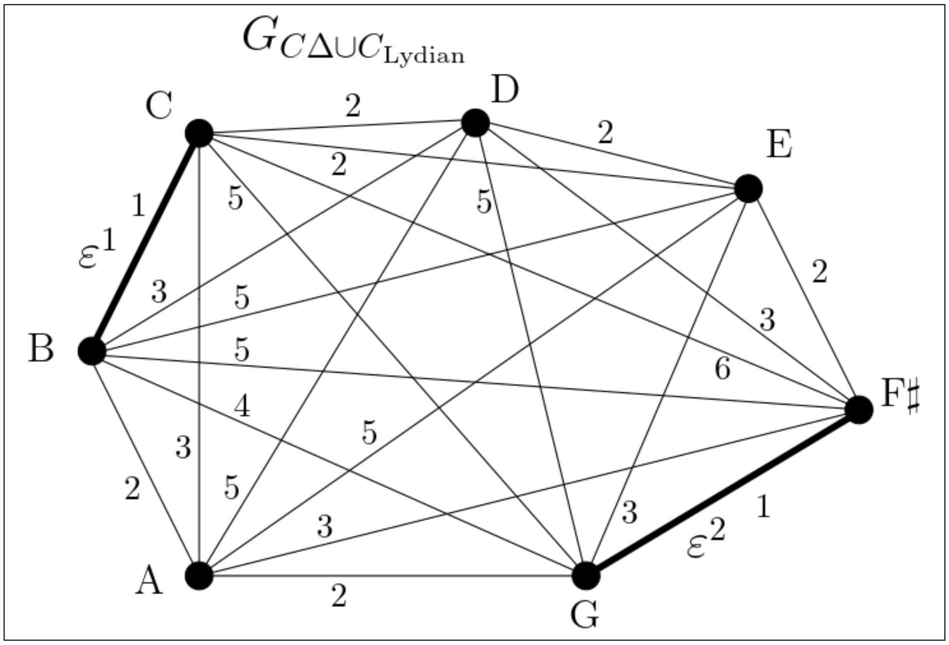


Figure 3.3: VDG for $G_{C\cup C_{\text{Lydian}}}$

Two edges with weight 1 appear on the graph, $\varepsilon^1 = (B, C)$ and $\varepsilon^2 = (G, F\sharp)$. Taking the sign recovery, it yields,

$$\sigma(\Delta(\mathcal{C}^{\varepsilon^1}, \mathcal{S}^{\varepsilon^1})) = \sigma(\Delta(B, C)) = 1, \quad \sigma(\Delta(\mathcal{C}^{\varepsilon^2}, \mathcal{S}^{\varepsilon^2})) = \sigma(\Delta(G, F\sharp) = -1$$

Therefore there are notes forming a ♭9th, in this case the tonic.

Vertical harmonic thinking is paramount to music improvisation, especially in music traditions such as jazz. The graph presented here allows to effectively model the scale-chord relationship.

Let us now consider another example, the minor pentatonic scale on E,
$E-P = \{[E], [G], [A], [B], [D]\}$, whose graph (when it is played over $E-7$) is denoted by $G_{E-7\cup E-P}$. In Figure 3.4, its VDG is shown.

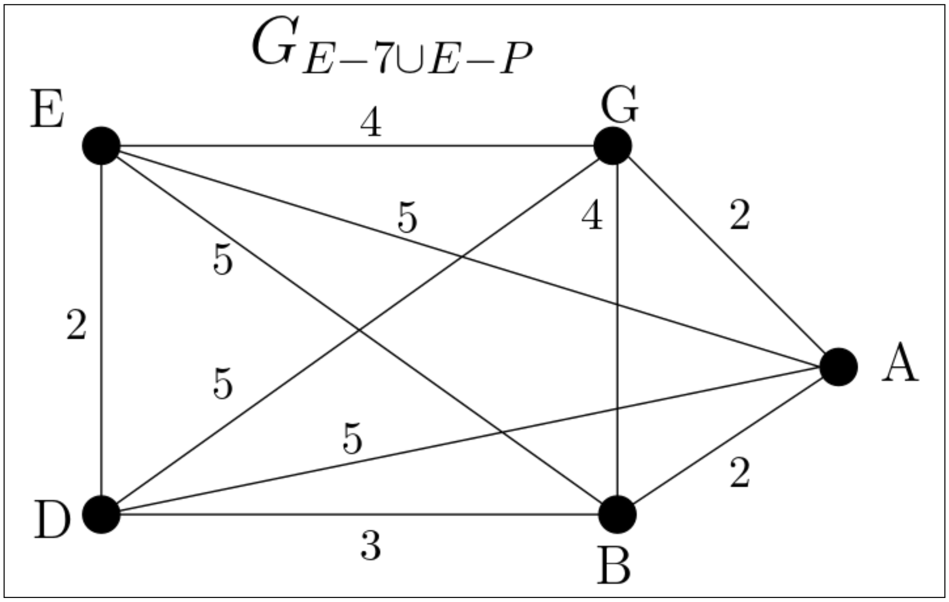


Figure 3.4: VDG for the minor pentatonic scale

A vertical dissonance graph is said to be **stable** if the sign recovery function does not take value 1 for any edge on the VDG. In other words, a VDG is stable if $\sigma(\Delta(\varepsilon)) \neq 1$ for all edges ε. In the case of the pentatonic scale, we observe that its vertical dissonance graph is always stable.

The formalization presented so far allows us to study the internal structure of only one chord. Later on in this work, we will study the horizontal dimension of chord progressions. In practice, musicians learn patterns and gestures associated to chords. That allows them to develop the right musical "flexibility", in other words, acquire the ability to improvise in many ways on a given chord. This skill is acquired in a similar way as in martial arts, that is, by studying the movements in slow motion and repeating them many times until it becomes second nature. For example, if we want to develop flexibility on F Lydian, we can play chords from the following set $\{F\Delta, G7, A{-}7, B\varnothing 7, C\Delta, D{-}7, E{-}7\}$ on a bass in F by using the 24 permutations of each chord.

You can also do the same with other chords (triads, ambi-chords, fourths, etc.) to achieve a good command of improvisation on each chord. This process requires hours of practice and can be extended almost to infinity because these chords can be worked in many different ways, varying the exercises rhythmically, adding approximation notes, and also in different timbres, articulations, among other possibilities.

Dan Haerle in his 1975 book *Scales for Jazz Improvisation* [Hae75] compiled all the scales that are harmonically appropriate for each chord type. We reproduce his tables here for completeness (see next pages). The asterisk in the tables indicate the absence of avoid notes. However, Haerle made some mistakes in his tables, which are indicated by $(*)$ in our tables below. Furthermore, Haerle uses the term *appropriate* to refer to some of the scales that can be played over the given chord types. However, there are more usable scales that those listed by Haerle.

Chord type	**Appropriate scale form**
Major 7th	Major Lydian$^{(*)}$ Minor pentatonic on the 3rd* Minor pentatonic on the 7th* 6th mode, harmonic minor
Major 6-9	Major pentatonic *
Major 7th, ♭5	Lydian$^{(*)}$ Minor pentatonic on the 7th*
Major 7th, ♯5	Lydian augmented$^{(*)}$ Augmented 3rd mode, harmonic minor
Major triad, sus 4	Major Augmented Minor pentatonic on the 2nd*

Table 3.2: Guide to scale choice: major family

Chord type	**Appropriate scale form**
Minor 7th, tonic (I) function	Dorian* Aeloian Minor pentatonic* Minor pentatonic on the 5th* Blues Phrygian 4th mode, harmonic minor
Minor 6th or 6-9	Dorian(*) Melodic minor* Minor pentatonic(*) Blues
Minor 7th, supertonic (II) function	Dorian* Minor pentatonic* Blues Whole step-half step diminished

Table 3.3: Guide to scale choice: minor family

Chord type	**Appropriate scale form**
Dominant 7th, unaltered	Mixolydian Lydian, $\flat$7* Major pentatonic*
Dominant 7th, $\flat$7, or $\sharp$11	Lydian, $\flat 7^{(*)}$
Dominant 7th, $\flat$5, $\sharp$5, or both	Whole tone*
Dominant 7th, $\flat$9	Half step-Whole step diminished$^{(*)}$ 5th Mode, Harmonic minor
Dominant 7th, $\sharp$9	Half step-Whole step diminished$^{(*)}$ Dorian Blues Minor pentatonic
Dominant 7th, $\flat$9 and $\sharp$9	Half step-Whole step diminished
Dominant 7th, altered 5th and 9th (any combination)	Super Locrian$^{(*)}$ Minor pentatonic on the $\flat$3rd$^{(*)}$ Major pentatonic on the $\flat$5th$^{(*)}$
Dominant 7th, sus 4	Mixolydian Minor pentatonic on the 2nd* Major pentatonic on the 5th$^{(*)}$

Table 3.4: Guide to scale choice: dominant family

Chord type	**Appropriate scale form**
Half-diminished (minor 7th, ♭5)	Locrian Locrian, ♮2* 2nd mode, harmonic minor Whole step-half step diminished

Table 3.5: Guide to scale choice: half-diminished family

Chord type	**Appropriate scale form**
Diminished 7th	Whole step-half step diminished 7th mode, harmonic minor

Table 3.6: Guide to scale choice: diminished family

3.4. Tonality and progressions

3.4.1. Compact tonality

The definition of **tonality** is well established in music theory. Tonality infiltrated the unchallenged realm of modality for near a century and it replaced modality with a new musical organization. The realm of tonality spanned from the late 1600s to the early 1900s and in Western music this organization establish an iron monopoly. Many definitions of tonality were provided since its origin, which was not a sudden change but a slow evolution from diverse forms of modality. For example, in [Kos13], pages 454–455, we read that tonality is the arrangement of pitches and/or chords of a musical work in a hierarchy of perceived relations, stabilities, attractions and directionality. This definition encompasses all the main characteristics of tonality as we know it. We may state that tonality is an organized selection of tones such that one of the tones —the **tonic** —becomes the central point for the remaining tones; see [BS03]. This definition can be mainly applied to music from the common period. New styles of music such as jazz or modal music require a more general definition of tonality. In this work, we will define such a general definition.

We now offer a definition of tonality in mathematical terms. Let us first look at the circle of fifths and select the n notes for the tonality, where typical values of n are natural numbers from 5 to 8, 7 being the commonest in practice. The selections of notes may be made by taking arbitrary set of notes or by selecting consecutive notes from the circle of fifths. For example, if we take the first seven notes of the circle of fifths (see Figure 3.5 on next page), we build the Lydian mode C-D-E-F♯-G-A-B, which corresponds to the tonality of G major or E minor Aeloian.

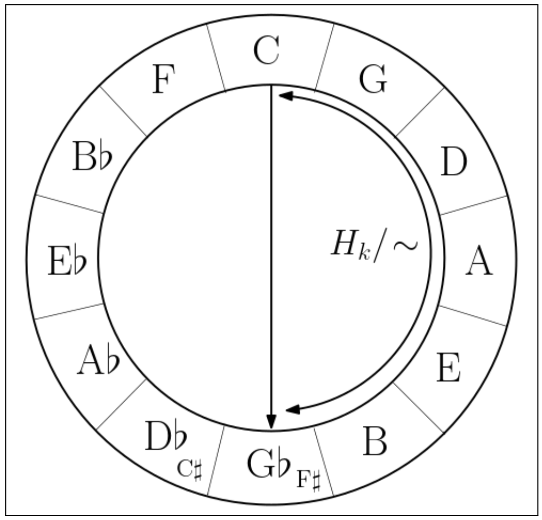


Figure 3.5: The circle of fifths

For another, if starting at C we take every other note, we will obtain the whole-tone scale, C-D-E-F♯-A♭-B♭, which is not formed by consecutive notes. Four our purposes, we will only consider tonalities derived from the selection of consecutive notes. More formally, given a starting note $[\theta]$ and a span of n notes, a compact tonality of n notes has the following mathematical expression:

$$H_k/\!\sim = \bigcup_{i=0}^{6} [s^{7i}\theta] \tag{3.1}$$

where $s = 2^{\frac{1}{12}}$ is the division of the octave in 12 half-tones and k is the concert pitch (the note from which the whole system is built upon). A **compact tonality** is a tonality composed of 7 consecutive notes taken from the circle of fifths; this term is used to refer both to the tonality determined by those notes as well as the scale they form.

Let $P \in \mathcal{M}_{n\times m}(LC_k/\sim)$ be now a progression, and its progression class written as

$$[P] = (|_{j=1}^{m} X_j) = (X_1 \mid X_2 \mid \ldots \mid X_{m-1} \mid X_m)$$

(this notation was introduced in Section 2.2), then he **tonality of a progression** is the tonality given by all the notes present in the progression. If all the notes in the progression form a compact tonality the progression is said to be **compact**. Let $[P] = (|_{j=1}^{m} X_j)$ be a progression class and τ a fixed 7-note span on the circle of fifths. The progression P^τ will be a **compact progression** if the following expression holds,

$$P^\tau = \{P \subset \mathcal{M}_{n\times m}(LC_k/\sim)/\forall X_j \in [P],\, X_j \subset H_k/\!\sim = \tau\} \tag{3.2}$$

that is, the tonality τ is made up of the notes found in the m chords of progression P, each chord having n or less notes, such that all the chords are found in $H_k/\sim$.

In practice, the definition of compact tonality is not as strict as its mathematical counterpart. The mathematical definition requires that all the classes of a progression stay in some span $H_k/\!\sim$. However, in actual music we observe that some notes are not in the span given by the tonality, but still we deem the progression to be in a compact tonality. Frequencies outside of the span may appear and they correspond to passing notes and other non-harmonic notes such as neighbor notes. Some musicians sometimes speak of a **pure compact progression** when the passage meets the mathematical definition above and use the term compact progression for the case a certain level of chromaticism is allowed.

3.4.2. Open tonality

In many styles of music the concept of tonality is not restricted to a span of 7 notes on the circle of fifths. Traditionally, tonality has been getting closer to that span for reasons related to vertical dissonance, mainly musicians chose that span size to keep the number of minor ninths (♭9) as low as possible. As the need for modulation and harmonic motion was growing in the musical practice, especially from the end of nineteenth century on, composers started to use and accept non-compact tonalities. Tonalities that are not compact receive the name of **open tonalities**. Examples of open tonalities are given by the whole-tone scale, the melodic minor scale, the harmonic major scale, among others.

Let $\mathcal{O}_k/\sim$ be an arbitrary subset of n notes taken from the circle of fifths. If $\mathcal{O}$ is an open tonality, similarly to the expression above, we can write an open progression as

$$P^{\mathcal{O}} = \{P \subset \mathcal{M}_{n\times m}(LC_k/\sim)/\forall X_j \in [P],\, X_j \subset \mathcal{O}_k/\sim = \mathcal{O}\} \tag{3.3}$$

Open tonality is a concept that is commonly used in jazz music and here we are going to use it as a bridge to define the global tonal function in a general way; see Chapter 6. Thus we will use the concept of open tonality both as a set of classes and as a progression whose chords are contained in such a set of classes.

3.5. Poly-chords

3.5.1. Definition of poly-chords

A very natural context in which to examine vertical dissonance is that of **poly-chords**. Poly-chords, in spite of its name, usually consists of the superposition of two chords. Dan Haerle [Hae82] defines them as follows.

> The term polychord literally means many (poly) chords. In actual practice, a polychord is usually a combination of only two chords which creates a more complex sound.

Although the use of poly-chords is mainly found in modern musical traditions such as jazz music or avant-garde music, early 20th century composers such as Igor Stravinsky, Darius Milhaud [Mar64], or Béla Bartók [OM15]. In his chapter *The Vertical Dimension in Twentieth Century Music* [Rei75] in the book *Aspects of Twentieth-Century Music*, Reisberg analyzes the use of poly-chords in the work of Stravinsky. For example, the so-called **Petrushka chord** is a voicing of the poly-chord formed by two major triads. In Figure 3.6, we can see the voicing of that poly-chord as the first inversion of C over the second inversion of $F\sharp$.

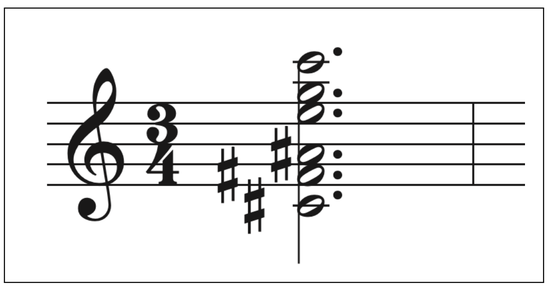


Figure 3.6: The Petruska chord voicing

Poly-chords have also been used in jazz music, for example, by Dave Brucker. See the paper by Mark McFarland [McF09] and the references therein for a thorough account of the use poly-chords in jazz music.

Poly-chords are often found in the context of **polytonality** [Pis91, Mar64, Bak93]. The popularization of polytonality (bitonality, actually) is credited to Stravinsky's *The Rite of Spring*, where this composer overlaps two tonalities for a substantial amount of time. Apart from the above-mentioned composers, other composers such as Casella, Aaron Copland, or Benjamin Britten made use of polytonality in their works. In his 2002 article *Stravinsky and the Octatonic: A Reconsideration* [Tym02], Tymoczko makes three claims about polytonality:

(1) For polytonality to be perceived as such what is required is that we be able to hear two different notes (which may be instances of the same pitch class), in two different auditory streams, as having different tonal functions.

(2) There is a considerable amount of music that contains several relatively independent musical streams, each of which, would suggest a different tonal center if heard in isolation. In other words, this music is polytonal in construction if not actually perceived as such.

(3) Multiple tonal centers can be heard at once. In its broadest sense, the term "tonality" embraces not just Western functional tonality of the sort that developed around 1600, but a broad range of music—including much modal music, contemporary rock music, and much non-Western music—in which a single pitch is heard as having priority over the others.

As we will propose very soon, the standard definition of poly-chords can be extended by considering the overlapping of more complex musical objects than just the overlapping of triads. For example, two ambichords stacked one on top another will also be considered as a poly-chord. Therefore, the definition of poly-chord is generalized.

3.5.2. Basic structures in poly-chords

In order to describe poly-chords in all its generality, we will consider two musical structures. By analogy with the piano, they will be called left-hand and right-hand structures. The left-hand structure corresponds with the structure on the lowest register and the right-hand structure the highest. As we said earlier, our goal is to generalize the concept of poly-chords conceived as just two chords overlapping to two more general musical structures overlapping. This generalization will yield to a greater variety in term of voicings. In Table 3.7, the **basic structures** for the left-hand and right-hand are shown. These basic structures are: **triads**, **seventh chords**, **stacked fourths**, **ambichords**, **bass lines**, and **solos**. The set of all basic structures is denoted by $\mathcal{B}$.

Left hand	**Right hand**
Triad	Triad
Tetrad	Tetrad
Stacked fourths	Stacked fourths
Ambichords	Ambichords
Bass	Bass
Improvised solo	Improvised solo

Table 3.7: Basic structures for two hands

Any two of the basic structures can be combined to make up a new structure to be called a poly-chord. The only requirement to be fulfilled for two basic structures to form a poly-chord is that they do not form an interval of a minor ninth (the avoid note) between them. When the VDG of the poly-chord is computed, those edges with weight equal to 1 are potential candidates to avoid notes.

If we examine the distance between two classes of basic structures, say J, W, we find that the nabla distance between them remains the same; hence, we can write

$$\Delta(J, W) = \Delta(W, J) \tag{3.4}$$

Recall that the sign recovery provides the direction in which the delta distance is attained (page 26). Bearing this in mind, then there is a property of symmetry that arises when applying the sigma function. Such property is given by the expression

$$\sigma(J, W) = -\sigma(W, J) \tag{3.5}$$

We will consider structures that are strictly contained in a Lydian tonal center, although this restriction can be relaxed later on. In other words, we will study basic structures whose notes belong to a compact tonality. Let us name the basic structures $\mathcal{B}$ as follows: triads (T), tetrads (C), stacked fourths (Q), ambichords (A), bass (B) and solo improvisation (S). Thus, in a strictly diatonic way, we would have the following list of common sets in jazz music to create our poly-chords. Obviously, more complex structures can be thought of (with higher number of voices or different intervalic structure, for example), but we have chosen these structures for familiarity and frequency of use as well as because they fit the tradition of jazz and classical music.

If a tonality τ is fixed, say the C Lydian mode, to be denoted by C_L, then the basic structures associated to that tonality $\mathcal{B}^\tau$ can be written as families:

$$\begin{aligned}
T^\tau &= \{G, A-, B-, C, D, E-, \#Fo\} \subset \wp(C_L)\\
C^\tau &= \{G\triangle, A-7, B-7, C\triangle, D7, E-7, \#Fo\} \subset \wp(C_L)\\
Q^\tau &= \{Dsus4, Esus4, \#Fsus4, Asus4, Bsus4\} \subset \wp(C_L)\\
A^\tau &= \{Csus4, Dsus4, Esus4, Gsus4, Asus4\} \subset \wp(C_L)\\
B^\tau &= \{C_L\} \subset \wp(C_L)\\
S^\tau &= \{C_L\} \subset \wp(C_L)
\end{aligned}$$

3.5.3. A mathematical model of poly-chords

We defined (generalized) poly-chords as two structures being played at the same time, which may be of different nature (an ambichord over a set of stacked fourths, for example). We will denote a poly-chord by the fraction $\frac{U}{I}$ where U is the basic structure placed on top of I, that is, U is higher in pitch than I. A **form** $\mathcal{F}$ is a set of poly-chords where the upper and lower structures belong to two families $\mathcal{U}$ and $\mathcal{I}$ of basic structures (possibly different families). In mathematical notation,

$$\mathcal{F} = \frac{\mathcal{U}}{\mathcal{I}} = \left\{\frac{U}{I} \,/\, U \in \mathcal{U}, I \in \mathcal{I} \text{ and } \mathcal{I}, \mathcal{U} \in \mathcal{B}\right\}$$

If one family is contained in the power set of an open tonality, then we say that the form is open $\mathcal{F}^{\mathcal{O}}$. Given two tonalities $\mathcal{O}_1, \mathcal{O}_2$, at least one being open, the definition for an **open form** $\mathcal{F}^{\mathcal{O}}$ will be:

$$\mathcal{F}^{\mathcal{O}} = \frac{\mathcal{U}}{\mathcal{I}} = \left\{\frac{U}{I} \,/\, U \in \mathcal{U} \subset \wp(\mathcal{O}_1), I \in \mathcal{I} \subset \wp(\mathcal{O}_2)\right\}$$

A form is said to be **compact** if the upper and lower families of basic structures are strictly contained in the power set of same compact tonality. This is equivalent to say that they belong to a fixed half of the circle of fifths. Its formal definition is as follows.

$$\mathcal{F}^{\mathcal{C}} = \frac{\mathcal{U}}{\mathcal{I}} = \left\{ \frac{U}{I} / U \in \mathcal{U} \subset \wp(H_k/\sim), I \in \mathcal{I} \subset \wp(H_k/\sim) \right\}$$

A **combination of forms** $\mathcal{X}$ is a set of forms. For example, if the lower structure is set as triads, we obtain the following.

$$\mathcal{X}_1 = \left\{ \frac{T}{T}, \frac{C}{T}, \frac{F}{T}, \frac{A}{T}, \frac{S}{T} \right\}$$

A second combination could be given by setting the lower structure to tetrads:

$$\mathcal{X}_2 = \left\{ \frac{T}{C}, \frac{C}{C}, \frac{Q}{C}, \frac{A}{C}, \frac{S}{C} \right\}$$

Let us show by means of an example how to compute a compact form. First, we fix the compact tonality τ; choose, for instance, the Lydian scale starting at C (as a major scale, it is G major). Then we compute the family of triads:

$$T^{\tau} = \{G, A-, B-, C, D, E-, \#F\circ\}$$

To build a form $\mathcal{F}$ from τ, we just have to combine elements from the upper and lower families, in this case only triads. Let T^{τ} be the family of triads associated to τ. The symbol $\frac{T^{\tau}}{T^{\tau}}$ denotes all the poly-chords composed of two diatonic triads in $\wp(\tau)$. Therefore, we can write:

$$\frac{T^{\tau}}{T^{\tau}} = \left\{ \frac{G}{G}, \frac{A-}{G}, \frac{B-}{G}, \frac{C}{G}, \frac{D}{G}, \frac{E-}{G}, \frac{\#F\circ}{G}, \ldots, \frac{G}{A}, \frac{A-}{A}, \frac{B-}{A}, \frac{C}{A}, \frac{D}{A}, \frac{E-}{A}, \frac{\#F\circ}{A}, \ldots, \frac{\#F\circ}{\#F\circ} \right\}$$

Once the form $\frac{T^{\tau}}{T^{\tau}}$ is computed, all poly-chords that contain a minor ninth are discarded. The rest of the dissonances remain and are not reason to be discarded. To locate the minor ninths, we will use the vertical dissonance graph, which connects all the classes of the poly-chord having the minimum distance between classes as its weighted edges. For example, if we take the poly-chord $\frac{D}{G}$ contained in form $\frac{T^{\tau}}{T^{\tau}}$, we observe that there is an edge of weight 1 between two classes. That edge corresponds to a seventh interval; hence, the VDG is stable.Having clarified the poly-chord, we take some voicings keeping the classes of said chords separated by register.

Given a form $\mathcal{F}$, said form is said to be a **clean form**—and it will be denoted by $\mathcal{C}(\mathcal{F})$—to the set of poly-chords in $\mathcal{F}$ whose vertical dissonance graph does not have minor ninths. We call the **inverted form** $\mathcal{F}^{-1}$ the set of poly-chords obtained by exchanging the upper and lower families. A clean form is of great use in composition since it is a palette of poly-chords that has the right mathematical properties to generate color. For example, in the case of the a compact tonality it holds that if $\mathcal{F} = \frac{T^{\tau}}{Q^{\tau}}$ if and only $\mathcal{F}^{-1} = \frac{Q^{\tau}}{T^{\tau}}$. However, if one of the tonalities is open, then we observe that $\mathcal{F} = \frac{Q^{\mathcal{O}}}{A^{\tau}}$ if and only $\mathcal{F}^{-1} = \frac{A^{\tau}}{Q^{\mathcal{O}}}$.

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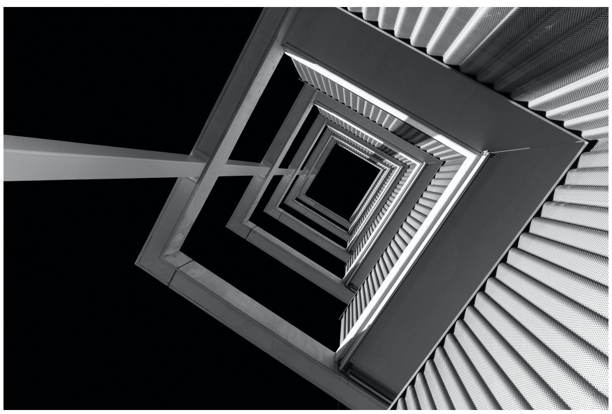
Chapter 4

Physical Characteristics of Harmony



Ricardo Gómez

https://unsplash.com/photos/eI9_PANcq-w

4.1. The hypervolumetric tonal function model

Music—an undisputed instance of complex and multi-dimensional phenomena—has vigorously been studied by many other scientific disciplines apart from music itself. During antiquity, the Greeks especially studied music from a mathematical and physical standpoint. They found out the relation between frequencies and ratios, which allowed them to define several tuning systems. The musical system of ancient Greece evolved over a period of more than 500 years from simple scales of tetrachords, or divisions of the perfect fourth, into several complex systems including tetrachords and octaves, as well as octave scales divided into seven to thirteen intervals; see Murray Barbour [Bar51] and David Benson [Ben06] for excellent accounts on tuning systems and in particular the Greek systems. At the Renaissance, that study was reawakened and many significant advances took place. It was taken even further during the Scientific Revolution, where some music theorists and scientists started to study perceptual aspects of music, endeavor that had not been undertaken before. Hermann Helmholtz's book *Sensations of Tone* [Hel54] constitute a very serious and deep research of the the sensation of tone and perception of sound. His studies are endowed with modern methodology and empiricism in the physiology of perception. Thorough and illuminating references and material for the history of music study from a mathematical and physical standpoint can be found in [Ten88] (fine exposition both from a pedagogical and technical perspective), [WW14] (already a classical text, epitome of clarity), [BM19] (the textbook for the course on physics of music at Yale University) and [TM21] (here historical and philosophical aspects are addressed on top of classical core content).

The work of pioneers at the end of the nineteenth century such as Felix Savart, Hermann Helmholtz, and Karl Koenig, who were interested in the nature of the sense organs and higher-order processes, prompted the incorporation of psychology into the study of music. In the period of 1860 to 1960, music is mainly studied from empirical point of view. Psychologist carried out experiments on subjects to understand how perception and cognition of music takes place in human's brain. However, psychologists of music at that time would ignore the work of musicologists; unfortunately, the converse was also true. Prominent figures of this period are Géza Révész, Albert Wellek, and Carl Seashore. The two former focused on perception on pitch and the latter on performance and education; see this entry [DGS+01] of the *Grove music online dictionary* by Diana Deutsch and other authors.

From the 1960s to the present day, the field of psychology of music grew along with cognitive science and become more open and collaborative with the other disciplines focused on the study of music. The scope of the field broadened tremendously, including research in areas such as music perception (not only pitch and rhythm but harmony, melody, timbre as well), musical universals, music enculturation, development and aptitude, music performance, and affective responses to music, just to name but a few. Important breakthroughs came with Leonard Meyer's *Emotion and meaning in music*, published as early as 1956, as well as Lundi's foundational 1967 book *An Objective Psychology of Music* [Lun67].

From the 1980s, the body of psychology of music has grown exponentially both in terms of depth and scope. By the end of the 80s, several works summarized the main body of knowledge. It is worth referencing the following works: the book by Rudolf Radocy and David Boyle [RB06] *Psychological Foundations of Musical Behavior*, in which a holistic analysis of music is presented (the first edition was in 1988 and today is a well known textbook); the 1987 book *The Developmental Psychology of Music* [Har87] by David Heargreaves, where the author researches the psychological basis of musical development in children and adults; the book *Music cognition* [DH86] by Dowling and Hardwood, which soon would become a dependable reference for music experts and researchers in the field of music cognition; the book *The musical mind: the cognitive psychology of music* [Slo86] by John Sloboda, where author examines the cognitive nature of music by by surveying the growing experimental literature on the subject at the time; and finally, Richard Parncutt's book [Par89] on the psychoacoustical foundations of harmony. In the 1990s and 2000s other authors continued the compilation of knowledge in the field; outstanding examples of those efforts would be Krumhansl [Kru90] Diana Deutsch [Deu92, Deu12], John Sloboda [Slo05] (again, such a prolific author), David Temperley [Tem01], and David Huron [Hur06].

The study of music from different disciplines lead to the consolidation of a new branch of musicology, namely, **systematic musicology**. This new discipline flourished in the late 1990s and early 2000s, but it had been maturing for a few decades back at different pace and momentum. Musicologist Richard Parncutt provided a broad definition [Par07] that reflects the interdisciplinary nature of this field.

> Systematic musicology is an umbrella term, used mainly in Central Europe, for subdisciplines of musicology that are primarily concerned with music in general, rather than specific manifestations of music. This article aims to explain the concept in English to international music scholars. Scientific systematic musicology (or scientific musicology) is primarily empirical and data-oriented; it involves empirical psychology and sociology, acoustics, physiology, neurosciences, cognitive sciences, and computing and technology.

Indeed, the study of music was pursued by and from many disciplines, in particular, analysis of music. There were epistemological and foundational consequences for the field of musicology, especially from traditional musicological (that is, that of a more sociological and historical flavor and methodology). For example, Anja Volk and Aline Honingh in their paper entitled *Mathematical and computational approaches to music: challenges in an interdisciplinary enterprise* [VH12] discuss and analyze the interdisciplinary challenges, benefits, and failures of those approaches (the discussion in the paper arose from a lively panel discussion at the third International Conference on Mathematics and Computation in Music 2011 in Paris). Similar concerns have been expressed in other disciplines.

Another way of thinking of music and its analysis was brought by the advent of computers and computing (in the modern sense of the word). The computer is a finite machine and the input to problems has to be given in finite form or at least characterized in finite form. Many musical phenomena are finite in nature or can be discretized (that is, described in finite terms). For example, frequencies are continuous in principle; however, music deals with notes, which are taken from a finite set, twelve notes in the case of equal temperament. Duration of notes is another musical objects thought of and worked with in a finite manner. Very soon computers and computation played a dominant and key role in many disciplines, including systematic musicology. The type of thinking derived from this new paradigm was called **computational thinking**. Many descriptions and characterizations of computational thinking. Authoritative computer scientist Alfred Aho [Aho11] claims that,

> Computational thinking refers to the thought processes involved in formulating problems so their solutions can be represented as computational steps and algorithms.

Bear in mind when reading the above that the expression "computational steps" implies that the problem and its solution are discrete in nature.

J. M. Wing [Win08] contend that computational thinking is a set of problem-solving methods and skills that involve expressing problems and their solutions in ways that a computer could also execute. Those skills are associated to abstraction, analysis and pattern recognition. P. J. Denning and M. Tedre [DT19] make emphasis on the automation of processes apart from the above characteristics.

Computational models and methods for the study and processing of music were incorporated into the mainstream research apace and steadly. In industry, computational models were created so that music could efficiently processed. Problems in industry such as music recommendation, speech and music recognition, source separation, music similarity, computer-generated music, music emotion recognition, or more recently artificial intelligence processing of music, among others, required robust computational models; see the following references for information on those problems: [KS15], [KSG16], [RRS15], [OVS+10], [Tem01], [BWHW21], [YHC11], [Mir21]. Probably, the work that most showed the possibilities of computational thinking at the beginning was *A Generative Theory of Tonal Music* [LJ83] by F. Lerdahl and R. Jackendoff. On page 1, we read that their theory is, "a formal description of the musical intuitions of a listener who is experienced in a musical idiom." Here by formal description the authors are actually providing a computational model. Their formal description is based on four hierarchies, namely, grouping structure, metrical structure, time-span reduction, and prolongational reduction, all of which are of computational nature.

Hypervolumetric tonal function is a new model of harmonic analysis and composition, whose presentation is the main goal of this thesis. The goals of the model were presented in Chapter 1, Section 1.2 in the form of three main research questions. The first question was to construct a mathematical model of tonal function that works for diverse and meaningful musical contexts. The third question was to construct a mathematical model of voice leading and find algorithms to computationally solve the problem of computing the optimal voice leading between two chords. As stated in the historical at the outset, modern music theory has been tackled with through a melting pot of disciplines. For our purposes and also by the nature of the problem we are dealing with here, we will use results and methodologies from mathematics, physics, and cognition. Mathematics makes us focus our intellect and soul on the dimensions of space and time in music. Physics provides a description of sound, especially in pitch and rhythm, and also constitutes a link to music itself, a source of constant checking with reality. We would not want to construct a beautiful, abstract, internally consistent model that would bear a poor, flawed relation with reality. Music can deeply be understood from meaningful models. Music is the space where we understand the real meaning of time through the study of rhythm, form, and harmony.

The hypervolumetric tonal function model (HTFM from now on) is a model that combines physical, mathematical, and musical elements into a whole. The goals set when designing the HTFM derived from the research questions were posed in Chapter 1, Section 1.2. Said questions were of two kinds in nature, questions seeking explanation of tonal function (SQR1–SQR4) and questions seeking to provide an efficient compositional method (SQR5–SQR7). In the next few chapters, we will try to answer those questions satisfactorily. On next page, there is a mental map (Figure 4.1) showing the main elements of the HTFM.

Mathematical formalization The formalization of the musical universe—the association of mathematical objects to musical objects —was carried out in Chapter 2. Concepts such as the Φ, $LC_k/\sim$, matrices of chord progressions, voicings as mappings from $\mathcal{M}_{n\times 1}(LC_k/\sim)$ to $\mathcal{M}_{n\times 1}(\Phi^+)$, arrangements, and the nabla distance.

Physical elements of musical voicing This is presented in Section 4.2. Several elements are examined: energy of a voicing, prime harmonics, the energy in the harmonic series, and the $\Delta^{\mathbb{E}}$ energy (Section 4.2.4).

The Weber-Fechner law In Section 4.3, the Weber-Fechner law is reviewed and reinter-

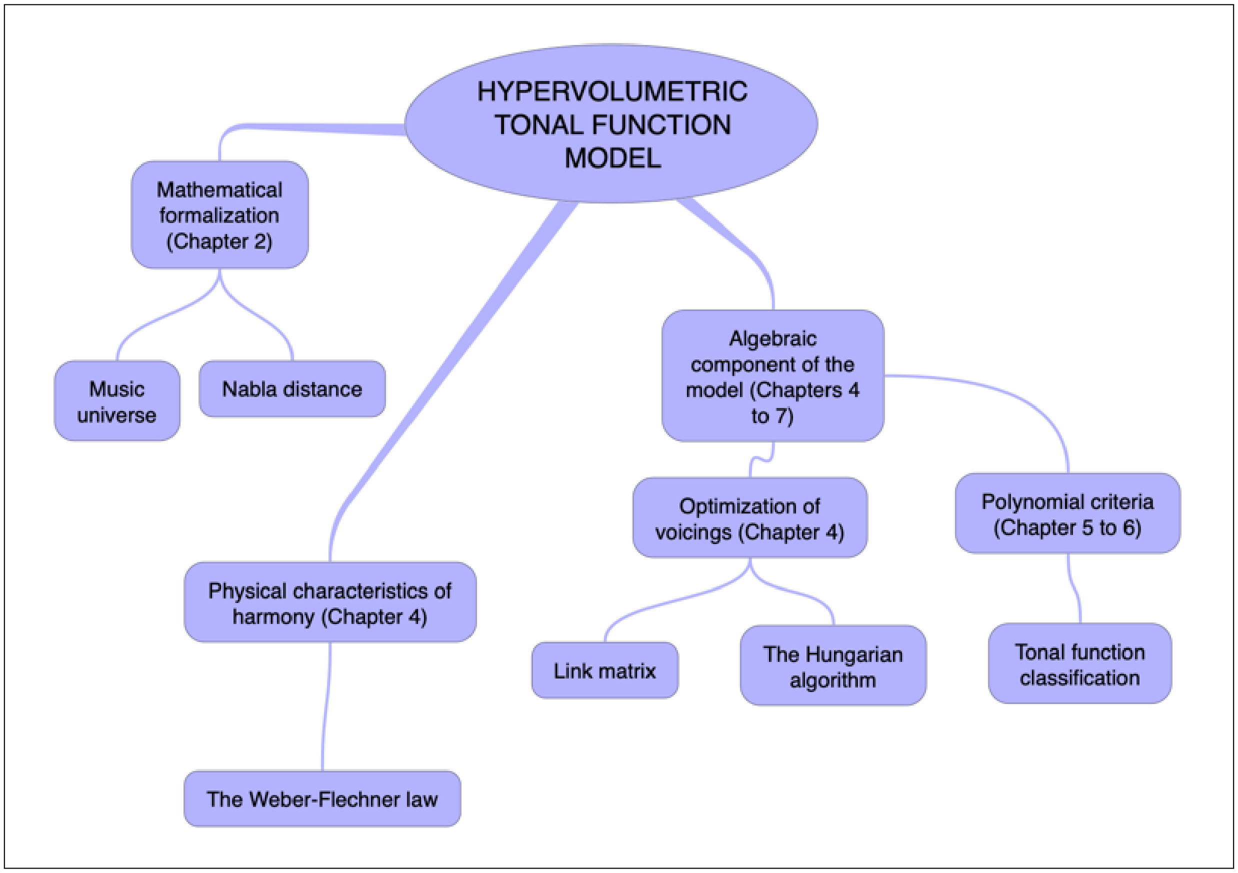


Figure 4.1: The hypervolumetric tonal function model

preted in Section 4.3 in the context of our mathematical formalization.

Hungarian algorithm The nabla distance induces a matrix, called the **link matrix**, consisting of all the distances between two consecutive chords. Upon this matrix, the **Hungarian algorithm** finds the optimal voice assignment between the two chords. The distance between the chords given by such an assignment is precisely the nabla distance.

Polynomial criterion As we will see later on, it is possible to define an endomorphism, $C_{\mathbb{E}} : \Phi^n \longrightarrow \Phi^n$ that transform a voicing into another one; this endomorphism is called **cadence endomorphism**. Let $P_{C_{\mathbb{E}}}(\lambda)$ be the characteristic polynomial associated to this endomorphism. The position of the roots of this polynomial with respect to 1 will determine the tonal function (1 is said to be the **stabilizer** of the **Mersenne group**). A tonal function is said to be **polarized** if the roots of $P_{C_{\mathbb{E}}}(\lambda)$ are all on the same side with respect to 1. This is the same as assuming that in the optimal link, the voices move in the same direction. In Chapters A to G, the polynomial criterion is exhaustively used to study tonal function in the most common links in all seven modes. The polynomial criterion is put forward in full detail in Chapter 5.

Tonal function classification Most commonly, tonal functions are polarized and it is less common to find links whose associated hypervolumetric tonal function (its polynomial) is non-polarized. As we will see later on, this model allows the modeling of tonal function in musical traditions other than the Western as well as in other tuning systems (this statement stems from the fact that motion of voices and their optimization do not depend on the style or the tuning system).

We carry on with the same philosophy of trying to understand this intuition of how tonal centers and chords resolve or "gravitate" towards one another (or when they do not). The HTFM determines in what cases that convergence (gravitation) takes place and when it does not.

4.2. Physical elements of musical voicing

4.2.1. The energy of a voicing

Let us start by considering a voicing $\psi(X)$ of n notes. Each note when played produces a series of additional sounds known as the **harmonic series**; the sounds themselves are also called **partials**. When the notes are produced by **acoustic instruments** (as opposed to electric or electronic means), the harmonic series is called the **natural harmonic series** (often we will refer to it as simply the harmonic series). In Figure 4.2, the harmonic series corresponding to the first twenty harmonics of C_2 are shown. The note itself is considered to be the first harmonic, also known as the **fundamental frequency**.

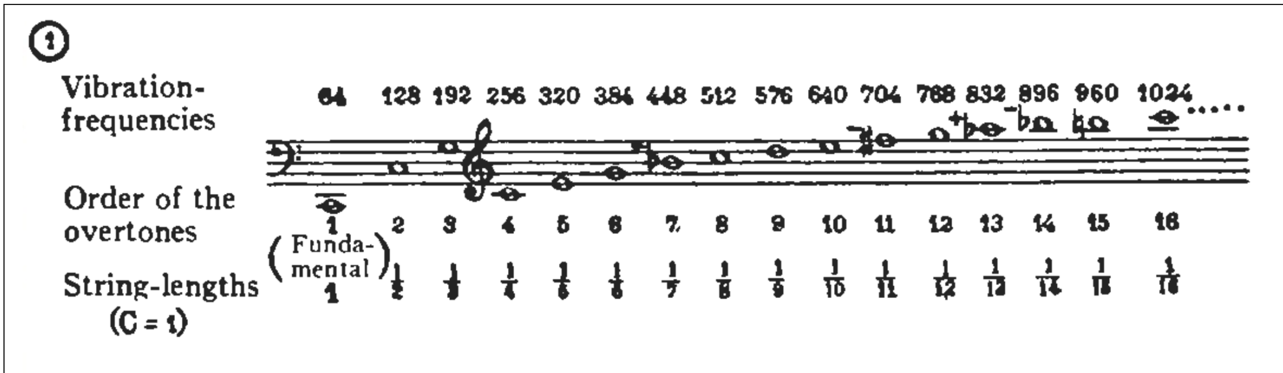


Figure 4.2: The harmonic series (figure taken from [Hin37])

The relative **amplitudes** of these various partials —at a given moment and as they change in time— determine the **color** or **timbre** of the sound. The simplest set of harmonics (also called **spectrum**) is formed by a sine wave with only the fundamental and no other partials. Most sounds, however, contain many partials of the fundamental, and the fundamental pitch can be deduced by the ear (even when it is not explicitly a member of the spectrum). **Artificial harmonics** can be generated by electric or electronic means; see [Ben06, Keu97, Laz21] for more information on this topic.

Returning to the voicing $\psi(X)$, each fundamental frequency $f_j \in \psi(X)$ comes along with a set of harmonics so that we can represent the note along with its harmonics. Let a_{ij} be the amplitude of each harmonic. In general, notes may have a different number of harmonics $\hbar_{ij} f_j$, for $1 \leq i \leq h$ and $1 \leq j \leq n$, where h is the maximum number of harmonics found in the voicing. Moreover, some harmonics may not be present in certain notes. In that case, the value of the harmonic will be zero. It is also true that the if the amplitude of a particular harmonic is zero, then that harmonic will not sound. Therefore, a harmonic having amplitude zero is equivalent to that harmonic not being present. Because of this fact, we can consider that in the voicing $\psi(X)$ every note has the maximum number h of harmonics. When a harmonic is not present, its amplitude will be set to zero. By using this fact, we can arrange the frequencies of the voicing in a matrix as follows.

$$F(\psi(X)) = \begin{pmatrix} f_1 & f_2 & \dots & f_n \\ \hbar_{21} f_1 & \hbar_{22} f_2 & \dots & \hbar_{2n} f_n \\ \dots & \dots & \dots & \dots \\ \hbar_{h1} f_1 & \hbar_{h2} f_2 & \dots & \hbar_{hn} f_n \end{pmatrix} \tag{4.1}$$

For a frequency matrix where all partials follow the natural harmonic series, two consecutive rows are always proportional to an rational number. Moreover, by the properties of the harmonic series, the ratio of any row to the first row is the number of the harmonic. That matrix will have this form.

$$F(\psi(X)) = \begin{pmatrix} f_1 & f_2 & \dots & f_n \\ 2f_1 & 2f_2 & \dots & 2f_n \\ \dots & \dots & \dots & \dots \\ hf_1 & hf_2 & \dots & hf_n \end{pmatrix}$$

Next, we consider the energy transported by a one-dimensional wave along a string. Let us assume that μ is the linear mass density of the string, a is the amplitude, and ω the angular frequency. Then, for a one wavelength λ, the energy E is given by the following formula (see [BM19, Bur10] for its context and derivation):

$$E = \frac{1}{2}\omega^2 \mu a^2 \lambda \tag{4.2}$$

Taking into account that the angular frequency holds the equality $\omega = 2\pi \cdot f$, then substituting this equality in the previous equation yields to

$$E = 2\pi^2 f^2 \mu a^2 \lambda = f^2 \cdot a^2 \cdot (2\pi^2 \mu \lambda) \tag{4.3}$$

The term $2\pi^2\mu\lambda$ will be summarized by a constant $\chi = 2\pi^2\mu\lambda$. The final equation for the energy is

$$E = \chi \cdot f^2 \cdot a^2 \tag{4.4}$$

By applying the previous formula to energy, we next define a matrix $E(\psi(X))$ representing the energy of each harmonic. This is called the **energy matrix** of a voicing. This $h \times n$ matrix has the form below.

$$E(\psi(X)) = \chi \cdot \begin{pmatrix} a_{11}^2 f_1^2 & a_{12}^2 (f_2)^2 & \dots & a_{1n}^2 (f_n)^2 \\ a_{21}^2 (\hbar_{21} f_1)^2 & a_{22}^2 (\hbar_{22} f_2)^2 & \dots & a_{2n}^2 (\hbar_{2n} f_n)^2 \\ \dots & \dots & \dots & \dots \\ a_{h1}^2 (\hbar_{h1} f_1)^2 & a_{h2}^2 (\hbar_{h2} f_2)^2 & \dots & a_{hn}^2 (\hbar_{hn} f_n)^2 \end{pmatrix} \tag{4.5}$$

where f_j are the frequencies of the notes in $\psi(X)$, a_{ij} their amplitudes, and χ the constant defined above. Note that in the previous matrix we assume that the timbre is constant for each voice.

Let us now define the **frequency layer** of a voicing. Given the frequency matrix of a voicing $F(\psi(X))$, we define the i-th layer K_i as the $h \times n$ matrix

$$K_i = \begin{cases} \hbar_{kj} f_j, & \text{for } j = 1, \ldots, n \text{ and } k = i \\ \\ 0, & \text{for } j = 1, \ldots, n \text{ and } k \neq i \end{cases} \tag{4.6}$$

The layer K_i actually consists of the null matrix except in the i-th row, which contains the i-th row of matrix $F(\psi(X))$. Layers allow us to write the following equation.

$$F(\psi(X)) = \sum_{i=1}^{h} K_i \tag{4.7}$$

In the model we have built forward so far, the assumption that the timbre is constant as well as the amplitude was made. Our model works assuming this fact at the moment of change of tonal center. Let us define a **matrix of amplitudes** $A = \sum_{i=1}^{h} A_i$ in a similar way to layer K_i where A_i is defined as

$$A_i = \begin{cases} a_{kj}, & \text{for } j = 1, \ldots, n \text{ and } k = i \\ \\ 0, & \text{for } j = 1, \ldots, n \text{ and } k \neq i \end{cases} \tag{4.8}$$

This matrix is simply the layer of the matrix of amplitudes in Equation 4.4. By applying the definition of layer to the energy matrix and using matrices A_i, the energy can be written as a Hadamard product as follows.

$$E(\psi(X)) = \chi \cdot \sum_{i=1}^{h} A_i^{\odot^2} \odot K_i^{\odot^2} \tag{4.9}$$

where $K_i^{\odot^2}$ is the matrix formed by taking the squares of each entry in K_i (Hadamard notation). Let Λ_i be the **energy layer** associated to the energy matrix $E(\psi(X))$. Thus, the energy matrix can be decomposed into layers.

$$\sum_{i=1}^{h} \Lambda_i = \chi \cdot \sum_{i=1}^{h} A_i^{\odot^2} \odot K_i^{\odot^2} \tag{4.10}$$

We have established a meaningful relationship between the frequency matrix and the energy matrix. This is a fundamental step to be able to study the musical phenomenon of tonal function as a whole.

4.2.2. Prime harmonics

Since music —as many other branches of human thought—can be analyzed by using numbers and basic algebra, has meaningful relationships to prime numbers. Prime numbers can be found in areas of music such as: frequency ratios and intervals; time signatures and rhythms; tuning, intonation, scales; resonance, damping, and instrument construction; acoustics. Excellent works exploring in depth the relationship between music and prime numbers are Marcus du Satoy's *The music of the primes* [DS03] or Benson's *A mathematical offering* [Ben06] (also the references therein). Composers have even composed music based on prime numbers (see the work of Ben Johnston [Fon01] for example).

Prime harmonics are defined as the partials in the harmonic series whose position is a prime number. Referring again to Figure 4.2, the prime harmonics are partials 2,3,5, 7, 11, 13, 17, and 19. Prime harmonics create special tension when considered vertically, that is, when playing two or more notes simultaneous. How does that tension arise? Suppose a sound is produced with its partials following the distribution of the harmonic series. The timbre of this sound is the end result of the overlapping of those partials. Some of these partials may match some partials of the original harmonics series, the ones given by the fundamental note. For example, consider a C_1 note; stay on Figure 4.2 for reference. A note whose fundamental is the 3rd partial, an E_2 note, of an harmonic series will have its own third partial, a D_4 note, coincident with the 9th partial of the original series. For one, it can readily be proven that all further odd composite partials of the original series will show up in the spectra of notes built on lower partials. For another, prime-numbered partials of the original series do not occur in notes built on lower partials. Composite partials exhibit a redundancy of which prime harmonics lack. This results in a kind of qualitative and musical difference between primes and composites as well as the notes built upon them.

The cognitive confirmation of the special musical perception of prime harmonics has not been addressed in depth in the literature yet. Llorenç Balsach, in a 1997 paper [Bal97], examines the question and states that prime harmonics are heard differently. Here is what he contends.

> The reason is that harmonics that are multiples of a prime harmonic have less importance to the auditory system than those who occupy prime position, since they are heard and understood as generated by them. Just as the fundamental is the synthesis of all its harmonics, the prime harmonics are the synthesis of the rest of non-prime harmonics (their own harmonics).

Unfortunately, no empirical confirmation is provided by this author, nor is it found in the literature to the best of our efforts. However, it seems to be a reasonable consensus among composers and musicians over the fact that prime harmonics at least create some kind of tension with respect to composite harmonics. We will assume that assumption here, too.

Prime harmonics are essential to understand a mode on its own. For discussion purpose, assume we study the relationship between a mode played in the middle register of the piano and and a bass line. By changing the fundamentals of the bass line through the notes of a scale, we will see how the mode changes. To do so, we need to introduce the concept of **stability** of a mode. This refers to the **number of prime harmonics** present in the scale by identifying the harmonics with scale degrees. We will illustrate in the tables below. In

Tables 4.1 and 4.2 the conversion of harmonics to scale degrees (denoted by h below) are shown; the difference between the tempered tuning and the tuning of the harmonics has been omitted (check Figure 4.2 again).

h	1	2	3	4	5	6	7	8	9	10
d	I	I	V	I	III	V	$\flat VII$	I	II	III

Table 4.1: Harmonic numbers and scale degrees from $h = 1$ to $h = 10$

h	11	12	13	14	15	16	17	18	19	20
d	$\flat V$	V	$\flat VI$	$\flat VII$	VII	I	$\flat II$	II	$\flat III$	III

Table 4.2: Harmonic numbers and scale degrees from $h = 11$ to $h = 20$

If we select the fundamental plus the prime harmonics 3, 5, 7, 11, 13, 17 and 19 and consider the scale degrees associated to them, then we form a scale with scales degrees I, V, III, ♭VII, #IV, ♭VI, ♭II, and ♭III. This scale is precisely an **altered scale** with and added perfect fifth. Harmonically speaking, this set of notes is precisely a poly-chord composed of a D♭sus triad in the upper structure and a **Hendrix's chord** (a dominant 7#9 chord) in the lower structure. The order of appearance of the prime harmonics in scale degree notation is I,V,III,♭VII,#IV,♭VI,♭II and ♭III.

Stability should not be confused with **brightness**. Brightness of a mode refers to the number of avoid notes (minor ninths or ♭9ths) in the scale. In mathematical terms, this is equivalent to the number of edges with delta metric equal to 1 found in the VDG (Chapter 3). The more minor ninths are in the scale, the brighter the mode is. Thus, the Lydian mode is brighter than the Ionian mode but it is less stable.

In Table 4.3 on next page, we can see the modes derived from the Lydian mode and the Lydian augmented mode in the order in which they were arranged by Ron Miller in his modal harmony book [Mil00] (`nat` stands for natural).

Modes are arranged in decreasing degree of brightness. We believe any experienced composer would have sorted them in the same order. Lydian modes are the brightest, followed by the Ionian mode, followed by two types of Mixolydian modes, and finally the entire palette of minor modes is found.

Table 4.3 is meant to illustrate the difference between brightness and stability. For example, in the comparison between the Ionian and the Lydian mode, it is clear that the Ionian mode has been used much more in the history of music, for, despite being less bright, it is more stable. The Ionian mode possesses three prime harmonics while in the Lydian mode there are four prime harmonics.

The same reasoning applies to the Mixolydian #11 mode and pure Mixolydian. The former is not a prevailing mode in musical contexts such as blues or fusion music genres whereas the latter is. Again, the pure Mixolydian mode is more stable than the Mixolydian #11 mode.

Mode	I	V	III	bVII	♯ IV	♭VI	♭II	♭III
`Lydian aug`	⊙		⊙		⊙	⊙		
`Lydian`	⊙	⊙	⊙		⊙			
`Ionian`	⊙	⊙	⊙					
`Mixolydian #11`	⊙	⊙	⊙	⊙	⊙			
`Mixolydian`	⊙	⊙	⊙	⊙				
`Dorian nat 7`	⊙	⊙						⊙
`Dorian`	⊙	⊙		⊙				⊙
`Aeolian`	⊙	⊙		⊙		⊙		⊙
`Aeolian flat 5`	⊙			⊙	⊙	⊙		⊙
`Prygian nat 6`	⊙	⊙		⊙			⊙	⊙
`Phrygian ♭6`	⊙	⊙		⊙		⊙		⊙
`Locrian`	⊙			⊙	⊙	⊙	⊙	⊙
`Locrian ♭4`	⊙		⊙	⊙	⊙	⊙	⊙	⊙

Table 4.3: Brightness and stability in musical modes

In the case of the Dorian mode, in jazz music it is far more common than the Aeolian mode as the Aeolian mode is considered too melancholic. The Dorian mode with its natural seventh is often used as a central chord in ballads.

Aeolian modes are used following established rules for chord scale relationships and rarely appear in jazz music as main chords; they are more common in classical music where the Aeolian mode is the minor mode par excellence. Phrygian modes are generally less stable and in other music such as flamenco or world music they have a central role. The Locrian and Locrian ♭4 (altered scale) modes are used to generate instability and in resolution sections, although some musicians also use them to superimpose those modes on various tonal centers.

4.2.3. Proportional energy in the harmonic series

When examining Equation 4.4, we notice that the energy of each partial is proportional to the square of the frequency as long as the amplitude, the wavelength, and the linear mass density are kept constant. Assuming the notation in Equations 4.1–4.4 of Section 4.2.1, let us now study the case where the set of harmonics $\hbar_{ij}, i = 1, \ldots, h, j = 1, \ldots, n$ consists of the harmonic series. This means that with respect to the fundamental frequency f_1 the i-th partial has frequency if_1. We will solely focus our attention on the special case where the amplitudes of the partials are given by the sequence $\frac{a}{i}, i \in \mathbb{N}, i > 0$, where a is a positive real constant. The sum of this sequence is the **mathematical harmonic series**, $c \sum_{i=1}^{\infty} \frac{1}{i}$.

Timbre of a sound is formed by the combination of its partials. Thus, if the harmonics of the timbre of the voicing follow the mathematical harmonic series, then the energy of the voicing is proportional to that of their fundamental frequencies. It is evident that there is a connection between the distribution of the energy of a voicing and Riemann zeta function [Ivi12], $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s}$; however, we will not follow that research path since this work deals with tonal functions and not timbre.

Let $E(\psi(X) = \chi \cdot \sum_{i=1}^{h} A_i^{\odot^2} \odot K_i^{\odot^2}$ as defined above. By applying Equation 4.10, we can write

$$\sum_{i=1}^{h} \Lambda_i = \chi \cdot \sum_{i=1}^{h} A_i^{\odot^2} \odot K_i^{\odot^2}$$

As a consequence, the energy matrix $E(\psi(X)$ can be decomposed into layers as follows.

$$E(\psi(X)) = \sum_{i=1}^{h} \Lambda_i \tag{4.11}$$

where Λ_i are the energy layers.

Given a note f, we define its **vector of amplitudes** Γ as the amplitudes associated to that note assuming a maximum number h of harmonics. It will be denoted by

$$\Gamma = (a_1, a_2, \ldots, a_h) \tag{4.12}$$

Let us consider a voicing $\psi(X)$. We define the **energy of a layer** by the following formula:

$$E(\Lambda_i) = \chi \cdot \sum_{j=1}^{n} f_j^2 \cdot \hbar_{ij}^2 \cdot a_{ij}^2 \tag{4.13}$$

where i here refers to the i-th layer.

Let assume that the vector of amplitudes for all layers is equal

$$\Gamma = \left(a, \frac{a}{2}, \ldots, \frac{a}{h}\right) \tag{4.14}$$

Next, we will prove that on this assumption that, given two layers Λ_c, Λ_q where $1 \leq c \leq h, 1 \leq q \leq h, c \neq q$, the energies take the same value, that is,

$$\Gamma = \left(a, \frac{a}{2}, \ldots, \frac{a}{h}\right) \Longleftrightarrow E(\Lambda_c) = E(\Lambda_q)$$

By the assumption, Γ is constant for each note. Therefore, $\hbar_{cj} = c$ for all $j = 1, \ldots, n$. Furthermore, it also holds that $a_{cj} = \dfrac{1}{c}$ for all $j = 1, \ldots, n$. We next substitute these two values into the equation of the energy layer (Equation 4.13).

$$E(\Lambda_c) = \chi \cdot \sum_{j=1}^{n} f_j^2 \cdot \hbar_{cj}^2 \cdot a_{cj}^2 = \chi \cdot \sum_{j=1}^{n} f_j^2 \cdot c^2 \cdot \frac{1}{c^2} = \chi \cdot \sum_{j=1}^{n} f_j^2$$

A similar line of reasoning proves the case for $E(\Lambda_q)$. Therefore, $E(\Lambda_c) = E(\Lambda_q)$.

This is a local property that describes the behavior of the layers in an energy matrix under the assumption made above (Equation 4.14). This property isn't sufficient to describe the energy from a voicing to another and thus establish metrics between those voicings. As we will need to generalize the tonal functions to all timbres, we will apply the Fundamental Theorem 4.3.1.

4.2.4. Metrics on the energy space

In this section, we will establish a connection between the delta metric Δ and the energy metric. To that end, we will exploit the formula that describes energy in terms of frequency (Equations 4.4 and 4.9). The energy of a n-note voicing can be thought of as a vector in vector space $\mathbb{E}^n$, where $\mathbb{E}$ denotes the space of energy (which is isomorphic to $\mathbb{R}$). In the following, we will only consider **pure sounds**, also called **simple sounds** or **simple frequencies**; they are sounds formed by just the fundamental, without the other partials. Sounds with more than one partial are called **complex sounds**. Let α, β be two simple frequencies. The number of semitones between them is given by the following integral (see Chapter 2, Equation 2.7):

$$\Delta(\alpha, \beta) = \left| \Omega \int_{\alpha}^{\beta} \frac{d\phi}{\phi} \right|$$

The natural generalization from a metric on the space of frequencies to a space of energy is as follows. Given two energies E_α, E_β associated to frequencies α, β, we establish

$$\Delta^{\mathbb{E}}(E_\alpha, E_\beta) = \left| \Omega \int_{E_\alpha}^{E_\beta} \frac{d\phi}{\phi} \right| \tag{4.15}$$

Let us introduce the **residue** γ to better relate the frequency and energy metrics. The purpose of the residue is to describe the difference between the frequency metric and the energy metric. It will be defined as

$$\gamma = \Delta(\alpha, \beta) - \frac{1}{2}\Delta^{\mathbb{E}}(E_\alpha, E_\beta) \tag{4.16}$$

The following computations will show why the residue is defined this precise way. The energies of two sinusoids α, β are given by $E_\alpha = \chi \cdot a_\alpha^2 f_\alpha^2, E_\beta = \chi \cdot a_\beta^2 f_\beta^2$. By solving for the frequencies and substituting in the delta metric formula, it yields the following expression (recall that Ω is a constant associated to the tuning; see Chapter 2).

$$\begin{aligned}
\Delta(\alpha, \beta) =& \Delta\left(\sqrt{E_\alpha/\chi a_\alpha^2}, \sqrt{E_\beta/\chi a_\beta^2}\right) = \left| \Omega \int_{\sqrt{E_\alpha/\chi a_\alpha^2}}^{\sqrt{E_\beta/\chi a_\beta^2}} \frac{d\phi}{\phi} \right| \\
&= \Omega \left| \left(\ln\left(\sqrt{E_\beta/\chi a_\beta^2}\right) - \ln\left(\sqrt{E_\alpha/\chi a_\alpha^2}\right)\right) \right| = \left| \frac{1}{2}\Omega \ln\left(\frac{E_\beta}{E_\alpha}\right) + \Omega \ln\left(\frac{a_\alpha}{a_\beta}\right) \right|
\end{aligned}$$

The final expression for the residue is then:

$$\gamma = \begin{cases} \frac{1}{2}\Omega \ln\left(\frac{E_\beta}{E_\alpha}\right) + \Omega \ln\left(\frac{a_\alpha}{a_\beta}\right) - \dfrac{1}{2}\Delta^{\mathbb{E}}(E_\alpha, E_\beta) & \text{if } \frac{1}{2}\Omega \ln\left(\frac{E_\beta}{E_\alpha}\right) + \Omega \ln\left(\frac{a_\alpha}{a_\beta}\right) \geq 0 \\ \\ -\frac{1}{2}\Omega \ln\left(\frac{E_\beta}{E_\alpha}\right) - \Omega \ln\left(\frac{a_\alpha}{a_\beta}\right) - \frac{1}{2}\Delta^{\mathbb{E}}(E_\alpha, E_\beta) & \text{if } \frac{1}{2}\Omega \ln\left(\frac{E_\beta}{E_\alpha}\right) + \Omega \ln\left(\frac{a_\alpha}{a_\beta}\right) < 0 \end{cases} \tag{4.17}$$

By Equation 4.17, the frequency metric is then decomposed into two summands, one is half the energy metric plus the residue, which takes two forms depending on the value of the ratio $\frac{a_\alpha}{a_\beta}$ and the value of the frequencies. Equation 4.17 will be the base for further derivations in the upcoming sections.

4.3. The Weber-Fechner law

4.3.1. The fundamental theorem

The **Weber-Fechner law** is a set of laws in the field of psychophysics that describe the relation between the actual change in a physical stimulus and the perceived change. This includes stimuli to all senses: vision, hearing, taste, touch, and smell. **Ernst Weber** (1795-1878) was in the field of psychophysics. **Theodor Fechner** (1801-1887) was his student and both undertook the study of human's perception of physical magnitudes. The law states that the subjective sensation of physical stimulus is proportional to the logarithm of the stimulus intensity. For the purpose of this research, we are interested in this law when applied to sound. Mathematically speaking, if S_1 is the initial stimulus intensity, S_2 the stimulus after the change, and p the perception of that change in stimulus, then p is given by

$$p = k \ln\left(\frac{S_2}{S_1}\right) \tag{4.18}$$

where k is a constant depending on the particular sense. See [Jea69] for a good exposition on the Weber-Fechner laws and their derivation.

However, Weber-Fechner's law does not quite hold for loudness. It is a fair approximation for higher intensities, but not for lower amplitudes; see [Yos00] and the references therein. Notwithstanding, assuming that the Weber-Fechner law works for all the range of amplitudes is enough for our model and research purposes.

Since the formula for the energy metric has already been introduced, we can combine it with the Weber-Fechner law to quantify the level of perception between two notes in terms of their energy. This will provide further insight into our model. We kindly remind the reader the energy matrix $E(\psi(X))$ (Equation 4.9).

$$E(\psi(X)) = \chi \cdot \begin{pmatrix} a_{11}^2 f_1^2 & a_{12}^2 (f_2)^2 & \dots & a_{1n}^2 (f_n)^2 \\ a_{21}^2 (\hbar_{21} f_1)^2 & a_{22}^2 (\hbar_{22} f_2)^2 & \dots & a_{2n}^2 (\hbar_{2n} f_n)^2 \\ \dots & \dots & \dots & \dots \\ a_{h1}^2 (\hbar_{h1} f_1)^2 & a_{h2}^2 (\hbar_{h2} f_2)^2 & \dots & a_{hn}^2 (\hbar_{hn} f_n)^2 \end{pmatrix}$$

where n is the number of notes of the voicing $\psi(X)$, h is the maximum number of harmonics found in $\psi(X)$, f_j are the frequencies, a_{ij}^2 are the amplitudes, $\hbar_{ij}$ the sequence of harmonics.

For the following reasoning, we will assume that the timbre is constant between the initial and final voicings. Having assumed this, in the previous matrix we will now take the vector of amplitudes Γ.

$$\Gamma = (a_1, a_2, \dots, a_h) \tag{4.19}$$

The **note stimulus** S^Γ associated to frequency f with amplitude vector Γ is defined as the sum of the energy of its harmonics.

$$S^\Gamma = \chi \sum_{i=1}^{h} (if)^2 (a_i)^2 \tag{4.20}$$

Next, let p^Γ be the **perceived change** associated with two notes with fundamental frequencies, say f, f^*, S_1^Γ and S_2^Γ, respectively. The natural way of defining p^Γ is by applying the Weber-Fechner law to the sum in Equation 4.20.

$$p^\Gamma = \Omega \ln \left(\frac{S_2^\Gamma}{S_1^\Gamma} \right) \tag{4.21}$$

Here S_1^Γ is the initial stimulus and S_2^Γ the final stimulus. The presence of constant Ω is owed to the nature of the sense, in this case, musical temperament.

Substituting Equation 4.20, the stimulus, into the equation of the perceived change Equation 4.21, we obtain

$$\begin{aligned} p^\Gamma =& \Omega \ln \left(\frac{S_2^\Gamma}{S_1^\Gamma} \right) = \Omega \ln \left(\frac{\chi \ \sum_{i=1}^{h} (f^* i)^2 (a_i)^2}{\chi \sum_{i=1}^{h} (f i)^2 (a_i)^2} \right) \\ &= \Omega \ln \left(\frac{(f^*)^2 \chi \sum_{i=1}^{h} i^2 (a_i)^2}{f^2 \chi \sum_{i=1}^{h} i^2 (a_i)^2} \right) = \Omega \ln \left(\frac{(f^*)^2}{f^2} \right) \\ &= 2\Omega \ln \left(\frac{f^*}{f} \right) \end{aligned}$$

The previous expression is stating that, under the hypothesis made above, the perceived change of energy in a voicing is reduced to the perceived change between its fundamental frequencies even if the notes have arbitrary timbre. This result is called the **fundamental theorem**. If the perceived change in frequency is denoted by p, then this theorem can succinctly be summarized in the following equality:

$$p^\Gamma = 2p \tag{4.22}$$

4.4. Optimization of voice leading: the Hungarian algorithm

4.4.1. The link matrix

Given a chord X, there always exists an optimal voice leading linking X to Y. This optimal voice leading yields a **link matrix** $L = (\Delta_{ij})$, for $i, j = 1, \ldots, n$, where n is the number of voices and Δ_{ij} are the delta metrics between all the classes of X and Y. For now, the number of voices will be constant; later on, that constraint will be relaxed. Let the number of voices n be equal to 4 and $X = \{G, B, D, F\}, Y = \{C, E, G, B\}$ both in $\wp(LC_k/\sim)$ be two chords, a G dominant seventh and a C major seventh chord, respectively. Suppose we wish to determine the tonal function of $V7$; in order to do so, we have to examine the chord progression $E = V7 \longrightarrow I\Delta$[1]. The corresponding progression matrix along with the link matrix are shown below.

$$E = \begin{pmatrix} F & B \\ D & G \\ B & E \\ G & C \end{pmatrix} \quad L = \begin{pmatrix} 6 & 2 & 1 & 5 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \end{pmatrix}$$

4.4.2. The Hungarian algorithm

We now apply a combinatorial optimization algorithm to L matrix. The algorithm is called the **Hungarian method** [Kuh55], which was developed by American mathematician Harold Kuhn in 1955. The reason Kuhn gave the name "Hungarian algorithm" after Hungarian mathematicians **Dénes König** and **Jenő Egerváry** who a couple of decades earlier had put forward some crucial ideas for solving the problem. This algorithm solves the assignment problem in polynomial time. The most efficient implementations of the Hungarian algoritm runs in $O(n^3)$ worst-case time where n is the size of the input; see Jonker-Volgenant algorithm [JV88]. The reason to apply it here is because the optimization of a voice leading between two chords can be interpreted as an assignment problems of agents and tasks; it can furthermore be interpreted as a matching problem in weighted bipartite graphs in which the sum of the edges is minimum. Here follows a description of the Hungarian algorithm and an example of it; the input in our case is the link matrix $L = (\Delta_{ij})$.

[1] Δ here denotes the major seventh chord, not the delta distance. The symbol is the same, but the context determines its meaning without confusion.

Description of the Hungarian Algorithm

Step 1. For each row i in L, $i = 1, \ldots, n$, select the minimum value $\min_{j=1,\ldots,n}\{\Delta_{ij}\}$ and subtract it from each element in said row. This will make the minimum entry in each row equal to 0.

Step 2. For each column j in L, $j = 1, \ldots, n$, select the minimum value $\min_{i=1,\ldots,n}\{\Delta_{ij}\}$ and subtract it from each element in said column. At this point, the minimum value of each row and column is equal to 0.

Step 3. Draw lines through the rows and columns containing a 0 entry so the the minimum number of lines is drawn. The maximum amount of lines that can be drawn is n, the number of chord's voices.

Step 4. If in Step 3 n lines were drawn, then an optimal assignment is possible. Select the 0 entry for each zero and replace it with the original values. This assignment yields the optimum assignment of voice leadings between the two chords. Exit the algorithm. Otherwise, if fewer than n lines were drawn, then go to Step 5.

Step 5. This step is divided into three cases: Find the entries in the matrix not covered by any line and compute minimum entry. Then, (1) Subtract the minimum entry from those entries that are not crossed out; (2) Find the entries in the matrix covered by one line. Leave them unchanged; (3) Find the entries in the matrix covered by two lines. Sum the minimum entry to each entry. Then, go back to Step 3.

The output of the Hungarian algorithm on matrix L is a new matrix L^B, which is a boxed matrix. Applying this algorithm to the previous chord progression, we arrive at the matrix shown on the next page

$$\begin{pmatrix} 6 & 2 & 1 & 5 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 1}} \begin{pmatrix} 5 & 1 & \boxed{0} & 4 \\ 1 & 3 & \boxed{0} & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 2}} \begin{pmatrix} 5 & 1 & \boxed{0} & 4 \\ 1 & 3 & \boxed{0} & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix}$$

$$\xrightarrow{\text{Steps 3}} \begin{pmatrix} 5 & 1 & \boxed{0} & 4 \\ 1 & 3 & 0 & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 4}} \begin{pmatrix} - & - & \boxed{0} & - \\ - & - & - & \boxed{0} \\ \boxed{1} & - & - & - \\ - & \boxed{2} & - & - \end{pmatrix}$$

In Step 3, we obtained 4 zeros, which is the number of voices; of the two possible zeros in column 3, the horizontal line correspond to the first one. After Step 4 is executed, the optimal voice leading corresponds to those zeros. In this case, that voice leading is $B \longrightarrow B, G \longrightarrow G, F \longrightarrow E$ and $D \longrightarrow C$.

In general, the solution to this optimization problem need **not be unique** and when the case the Hungarian algorithm returns all of the solutions. The correctness of the algorithm ensures that the order and the particular zeros chosen in Step 3 always lead to an optimal solution.

Out of the solutions output by the Hungarian algorithm, an optimal link E^0 between the X chord and the Y chord is constructed. The notes of the voice leading can be rearranged so that the optimal values for the voice leading appear on the main diagonal of the associated link matrix. Such link matrix is denoted by L^*. Carrying on with our example, the rearrangement of the voice leading and the link matrix are

$$E^0 = \begin{pmatrix} F & E \\ D & C \\ B & B \\ G & G \end{pmatrix}; \quad L^* = \begin{pmatrix} \boxed{1} & 5 & 6 & 2 \\ 2 & \boxed{2} & 3 & 5 \\ 5 & 1 & \boxed{0} & 4 \\ 3 & 5 & 4 & \boxed{0} \end{pmatrix}$$

Notice that the rearrangement is not unique. It is straightforward to conclude that the trace of L^* is precisely $\nabla(\psi(E^0))$, that is, the distance of the optimal voice leading.

4.4.3. Weber-Fechner connection with the Hungarian algorithm

In this section, we will examine the connection between the optimization method provided by the Hungarian algorithm and the change in perception when considering the motion between two given tonal centers. If in Equation 4.15 the energy term is worked out for, then it yields

$$\frac{1}{2}\Delta^{\mathbb{E}}(E_\alpha, E_\beta) = \Delta(\alpha, \beta) - \gamma$$

When a whole n-note voicing is taken into account, we obtain the sum over all the notes in the voicing. Let α_j, β_j the frequencies of the voicing. The sum over all the notes will be

$$\frac{1}{2}\sum_{j=1}^{n}\Delta^{\mathbb{E}}(E_{\alpha_j}, E_{\beta_j}) = \sum_{j=1}^{n}\Delta(\alpha_j, \beta_j) - \sum_{j=1}^{n}\gamma_j$$

Next, we will generalize Equation 4.15, the energy metric associated to two fixed pure frequencies, to the energy metric associated to **complex sounds**. Given two frequencies α, β, let $\Gamma = (a_1, a_2, \ldots, a_h)$ be the set of amplitudes; $E_{N(\alpha)}$ and $E_{N(\beta)}$ the **energies** of notes with fundamental frequencies α, β, respectively, and h the maximum number of harmonics in the voicing (as in Equation 4.4). Holding the assumption that the timbre is constant, the natural generalization is provided by Equation 4.23 below.

$$\Delta^{\mathbb{E}}_{\Gamma}(E_{N(\alpha)}, E_{N(\beta)}) = \left| \Omega \int_{\chi\sum_{i=1}^{h} a_i^2 i^2 \alpha^2}^{\chi\sum_{i=1}^{h} a_i^2 i^2 \beta^2} \frac{d\phi}{\phi} \right| \tag{4.23}$$

At this point, the fundamental theorem (Equation 4.22) can be applied to the case of complex sounds. For one, we have $|p^\Gamma|$:

$$|\ p^\Gamma\ | = \left| \Omega \ln\left(\frac{E_{N(\beta)}}{E_{N(\alpha)}}\right) \right| = \left| \Omega \int_{\chi\sum_{i=1}^{h} a_i^2 i^2 \alpha^2}^{\chi\sum_{i=1}^{h} a_i^2 i^2 \beta^2} \frac{d\phi}{\phi} \right| = \Delta^{\mathbb{E}}_{\Gamma}(E_{N(\alpha)}, E_{N(\beta)})$$

For another, $|p| = \Delta^{\mathbb{E}}(E_\alpha, E_\beta)$. By the fundamental theorem, it holds $| p^\Gamma | = 2|p|$. Hence,

$$\Delta_\Gamma^{\mathbb{E}}(E_{N(\alpha)}, E_{N(\beta)}) = 2\Delta^{\mathbb{E}}(E_\alpha, E_\beta) \tag{4.24}$$

This equation relates the energy metric of a complex sound to that of a simple sound. The following reasonings uses Equation 4.16, the definition of a residue γ given in Section 4.2.4. Thus, by using that equation along with the fundamental theorem, we can write:

$$(1/2)\Delta_\Gamma^{\mathbb{E}}(E_{N(\alpha)}, E_{N(\beta)}) = \Delta^{\mathbb{E}}(E_\alpha, E_\beta) = 2\left(\Delta(\alpha,\beta) - \gamma\right) \tag{4.25}$$

The fundamental theorem states that, when the residue is zero, the energy metric of the fundamentals is proportional to the energy metric of the whole sounds, including their partials. This equality is true as long as the timbre between both voicings is kept constant.

Moreover, if we consider the whole voicing and take sums on both sides of the equation, as we did in Equation 4.13, the reader will notice that we obtain a nabla distance on the right side as shown below.

$$1/2\sum_{j=1}^{n}\Delta_\Gamma^{\mathbb{E}}(E_{N(\alpha_j)}, E_{N(\beta_j)}) = \sum_{j=1}^{n}\Delta^{\mathbb{E}}(E_{\alpha_j}, E_{\beta_j}) = 2\left(\sum_{j=1}^{n}\Delta(\alpha_j,\beta_j) - \sum_{j=1}^{n}\gamma_j\right) \tag{4.26}$$

We know that for two tonal centers X and Y in $\wp(LC_k/\sim)$, the minimum distance between them occurs at the minimum of the nabla distance function over all the voicings moving between two given chords. A very natural question to pose is what will happen to the change of perception when that minimum is reached. We will see that sums will be converted into into sets of sums, which represent all the possibilities of motion between voices. From elementary combinatorics, we know that the cardinal of the set of all possible pairings between two given chords with a maximum number of n notes is $n!$.

Our next goal is to select the minimum value over the total number of $n!$ selections. We denote by σ_i each of those selections and further assume that the voicings are **closed** (all notes move to the closest note without changing octaves). By using Equations 4.22 and 4.23, we can write the following.

$$\begin{aligned}\bigcup_{i=1}^{n!}\left\{1/2\sum_{j=1}^{n}\Delta_\Gamma^{\mathbb{E}}(E_{N(\sigma_i(\alpha_j))}, E_{N(\sigma_i(\beta_j))})\right\} &= \bigcup_{i=1}^{n!}\left\{\sum_{j=1}^{n}\Delta^{\mathbb{E}}(E_{\sigma_i(\alpha_j)}, E_{\sigma_i(\beta_j)})\right\} \\ &= \bigcup_{i=1}^{n!}\left\{2\left(\sum_{j=1}^{n}\Delta(\sigma_i(\alpha_j),\sigma_i(\beta_j)) - \sum_{j=1}^{n}\sigma_i(\gamma_j)\right)\right\}\end{aligned}$$

Therefore, we are in the presence of $n!$ measurements of the change of perception between two consecutive chords (tonal centers) when both are sounding with an identical harmonic distribution Γ. With this in mind, we set out to compute the minimum.

$$\begin{aligned}
&\min\left\{\bigcup_{i=1}^{n!}\left\{1/2\sum_{j=1}^{n}\Delta_{\Gamma}^{\mathbb{E}}(E_{N(\sigma_i(\alpha_j))},E_{N(\sigma_i(\beta_j))})\right\}\right\} \\
&=\min\left\{\bigcup_{i=1}^{n!}\left\{\sum_{j=1}^{n}\Delta^{\mathbb{E}}(E_{\sigma_i(\alpha_j)},E_{\sigma_i(\beta_j)})\right\}\right\} \\
&=\min\left\{\bigcup_{i=1}^{n!}\left\{\left(2\left(\sum_{j=1}^{n}\Delta(\sigma_i(\alpha_j),\sigma_i(\beta_j))-\sum_{j=1}^{n}\sigma_i(\gamma_j)\right)\right\}\right)\right\}
\end{aligned}$$

At this point of our reasoning, we will make an assumption to be called the **local constant amplitude assumption** (LCAA). We will assume that at the time of change of tonal center the amplitudes are constant. Consequently, the residues are equal to 0 as follows from Equations 4.16 and 4.17. Therefore, in the previous expression the terms $\sigma_i(\gamma_j)$ are all null, which yields the equality below.

$$\min\left\{\bigcup_{i=1}^{n!}\left\{1/2\sum_{j=1}^{n}\Delta_{\Gamma}^{\mathbb{E}}(E_{N(\sigma_i(\alpha_j))},E_{N(\sigma_i(\beta_j))})\right\}\right\}=\min\left\{\bigcup_{i=1}^{n!}\left\{2\left(\sum_{j=1}^{n}\Delta(\sigma_i(\alpha_j),\sigma_i(\beta_j))\right)\right\}\right\} \tag{4.27}$$

The distance between classes of notes is given in terms of distance between notes (see Chapter 2, Section 2.3). Hence, when finding the minimum distance, we can use that fact and write

$$\min\left\{\bigcup_{i=1}^{n!}\left\{\sum_{j=1}^{n}\Delta(\sigma_i(\alpha_j),\sigma_i(\beta_j))\right\}\right\}=\min\left\{\bigcup_{i=1}^{n!}\left\{\left(\sum_{j=1}^{n}\Delta([\sigma_i(\alpha_j)],[\sigma_i(\beta_j)])\right)\right\}\right\}$$

We now substitute the left-hand side of the previous equality with Equation 4.27.

$$\min\left\{\bigcup_{i=1}^{n!}\left\{1/2\sum_{j=1}^{n}\Delta_{\Gamma}^{\mathbb{E}}(E_{N(\sigma_i(\alpha_j))},E_{N(\sigma_i(\beta_j))})\right\}\right\}=\min\left\{\bigcup_{i=1}^{n!}\left\{2\left(\sum_{j=1}^{n}\Delta([\sigma_i(\alpha_j)],[\sigma_i(\beta_j)])\right)\right\}\right\} \tag{4.28}$$

Bear in mind that all the reasoning made so far was carried out under the assumption that the voicing is closed. This is a sufficient condition to guarantee that the metrics taken between frequencies coincide with the minimum integrals between classes of notes. By taking into account the equality $\left|p^{\Gamma}\right|=\Delta_{\Gamma}^{\mathbb{E}}$ and finding the minimum value in Equation 4.27, we obtain:

$$\begin{aligned}
\min\left(\frac{|\ p^{\Gamma}\ |}{2}\right)&=\min\left\{\bigcup_{i=1}^{n!}\left\{1/2\sum_{j=1}^{n}\sigma_i(|\ p_j^{\Gamma}\ |)\right\}\right\}; \\
\min\left(\frac{\Delta_{\Gamma}^{\mathbb{E}}}{2}\right)&=\min\left\{\bigcup_{i=1}^{n!}\left\{1/2\sum_{j=1}^{n}\Delta_{\Gamma}^{\mathbb{E}}(E_{N(\sigma_i(\alpha_j))},E_{N(\sigma_i(\beta_j))})\right\}\right\}
\end{aligned}$$

In this way we arrive at the following equality:

$$\min\left\{\bigcup_{i=1}^{n!}\left\{1/2\sum_{j=1}^{n}\sigma_i(\mid p_j^{\Gamma}\mid)\right\}\right\} = \min\left\{\bigcup_{i=1}^{n!}\left\{1/2\sum_{j=1}^{n}\Delta_{\Gamma}^{\mathbb{E}}(E_{N(\sigma_i(\alpha_j))}, E_{N(\sigma_i(\beta_j))})\right\}\right\} \tag{4.29}$$

Looking at the value $\min\left\{\bigcup_{i=1}^{n!}\left\{\left(\sum_{j=1}^{n}\Delta([\sigma_i(\alpha_j)],[\sigma_i(\beta_j)])\right)\right\}\right\}$, we know that this minimum value is attained at a certain optimum voicing E^0. Said nabla value is given by the minimum value of the distances between classes of the chords of E^0. Therefore, we have

$$\nabla(E^0) = \min\left\{\bigcup_{i=1}^{n!}\sum_{j=1}^{n}\Delta([\sigma_i(\alpha_j)],[\sigma_i(\beta_j)])\right\}$$

Plugging the previous equality into Equation 4.28, we arrive at

$$2\nabla(E^0) = \min\left\{\bigcup_{i=1}^{n!}\left\{\frac{1}{2}\sum_{j=1}^{n}\Delta_{\Gamma}^{\mathbb{E}}(E_{N(\sigma_i(\alpha_j))}, E_{N(\sigma_i(\beta_j))})\right\}\right\} \tag{4.30}$$

Finally, by using Equation 4.29 on the previous equation, we obtain

$$2\nabla(E^0) = \min\left\{\bigcup_{i=1}^{n!}\frac{1}{2}\sum_{j=1}^{n}\sigma_i(\mid p_j^{\Gamma}\mid)\right\} \tag{4.31}$$

Thus, we conclude that the minimization output by the Hungarian algorithm equals the minimization of the change in perception between two tonal centers. The formula below summarizes this fact.

$$\min(\mid p^{\Gamma}\mid) = 4\nabla(E^0) \tag{4.32}$$

The result obtained after all the above reasoning and computation is known as the **Weber-Fechner connection** (WFC from here on). It establishes that, under the LCAA, the minimum of the change in perception can be obtained from the Hungarian algorithm. When the dimensions of the chords are different, the Weber-Fechner connection still holds. The same computations can be repeated, but this time it is necessary to use an extended real number set $\overset{\infty}{\underset{\infty}{\mathbb{R}}}$ endowed with a special arithmetic. This extension will be put forward and discussed in Sections 5.6.1, 5.6.2, and 5.9.

It is an open problem to design a mathematical model with non-constant amplitudes. This matter is discussed in the chapter devoted to conclusions (Chapter 10, Section 10.2).

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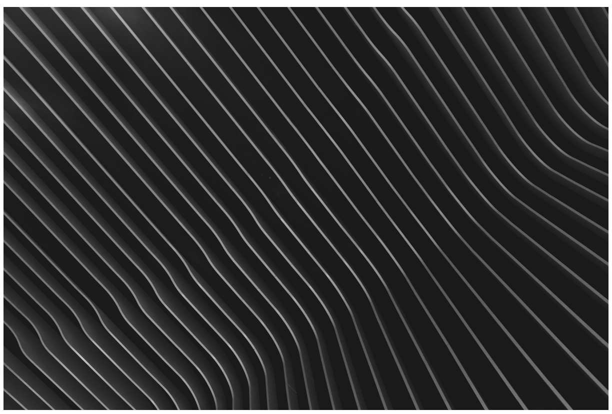
Chapter 5

The Horizontal Dimension of Harmony



Adrien Olichon

https://unsplash.com/photos/gOdavfpH-3s

5.1. What is tonal function?

Several models —both mathematical and musical—have been presented by a number of authors in the past few years. The concept of tonal function was first introduced by **Hugo Riemannn** in his 1893 work *Vereinfachte Harmonielehre* [Rie96] (*Harmony simplified* in its English version). According to Hyer [Hye01], Riemannn borrowed the term from mathematics. Riemannn coined the names for the three main tonal functions, **tonic**, **dominant**, and **subdominant functions**. His theory soon gained popularity and was much used in composition, pedagogy of harmony, and music analysis. About a half century later, the Viennese school of harmony, led by figures such as Simon Sechter and Heinrich Schenker and later Arnold Schoenberg, created the scalar degree theory, which put more emphasis on the use of Roman numeral, the figure bass, and the relation of chords to the tonal center within a harmonic progression; see Watson[Was85].

Riemann's theory of harmony has undergone many extensions and revisions since its initial formulation, which have guaranteed its use in today's musical practice. For example, authors such as Lewin [Lew87], Hyer [Hye95], and Cohn [Coh97, Coh98a] have formulated a **neo-Riemannian theory** where the operations between chords originally defined by Riemann are extended and refined to a larger set of operations resulting in a definition of tonal function capable of dealing with music from the extended common practice. See the excellent *Oxford Handbook of Neo-Riemannian Music Theories* [GR11] by Gollin and Rehding for a survey on those extensions of Riemann's theory.

Strange as it may seem, there are not many formal and precise definitions of tonal function in the literature and yet some of them seem not to fully capture the essence of this musical phenomenon; in particular, there are very few mathematical definitions of tonal function. Riemann's definition is laid out in terms of a theory of consonance and dissonance and also in terms of operations on chords [Rie96]. The resulting theory of harmony as we know it today is a mixture of Riemann's theory and the theories by the Viennese school. However, the definition of tonal function seems to be vague in many cases. For instance, in the classical textbook *Harmony* by Walter Piston [Pis50] —a textbook used by many generations of harmony students—, we find the following definition of tonal function.

> Tonality, then, is not merely a matter of using just the tones of a particular scale. It is more a process of setting forth the organized relationship of these tones to one among them which is to be the tonal center. Each scale degree has its part in the scheme of tonality, its tonal function.

This definition is somewhat vague and similar ones can be found in other textbooks on harmony. About the adequacy of the very term "function", Kopp [Kop55] for example contends that, "the meaning of the word has proved adaptable to support a wide variety of statements concerning harmony." and discuss the semantics of the word "function." In this work, we are developing a definition of tonal function based on optimal voice leading.

A more complete definition is given in the project *Open Music Theory* [BMS+21], which in turns follows Quinn's definition of tonal function [Qui05]. The definition given therein classifies chords on scalar degrees I, III−, and VI− as tonic chords; II− and IV as subdominant chords; and V and VII∘ as dominant chords. The characteristics of the scale degrees are then added to his definition. Tonic's characteristic scale degrees are 1, 3, 5, 6, and 7, (which result from take the scale degrees of the tonic's chords), subdominant's characteristic scale degrees are 1,2, 3, 4, and 6, and, finally, dominant ones are 2, 4, 5, 6, and 7. Some authors do not agree with this chord classification and expand it by assigning VI to subdominant and III to dominant.

Quinn further refines this chord classification and introduces the categories of functional triggers, functional associates, and functional dissonances; see Table 5.1. These categories were created based on the degree of stability of the chord notes with respect to the tonal center; other authors such as Harrison [Har94] or Cohn [Coh98b] have proposed similar classifications. Cohn [Coh98b] can be considered as a precedent of the work presented here (his sum-class transformations, for example).

Function	Triggers	Associates	Dissonances
T	1 and 3	5 and 6	5 (if 6 present) and 7
SD	4 and 6	1 and 2	1 (if 2 present) and 3
D	5 and 7	2	4 and 6

Table 5.1: Scalar degree characteristics of tonal function [Qui05].

Tonal function has been studied from many standpoints. Carpenter [Car83] examined how a musical idea can be shaped through tonal function; Caplin [Cap83] in turn looked into the historical perspectives on the relation between tonal function and metrical accent; Dudeque [Dud97] investigated Schoenberg's concept of tonal function; Agustín-Aquino and Mazzola [AAM20] presented a theory of modulation that includes a definition of tonal function.

In jazz music —one of the musical styles where harmony is central—, it seems that a conceptual, comprehensive definition of tonal function also lacks. Certainly, there are many definitions in the literature and textbooks, but most of them establish the tonal function without elucidating its roots. In his 2004 paper [Sto14] *Jazz harmony: a progress report*, Stover [Sto14] reviews three important jazz textbooks, namely, Mulholland and Hojnacki's *The Berklee Book of Jazz Harmony* [MH13], David Berkman *The Jazz Harmony Book* [Ber13], and Dariusz Terefenko's *Jazz Theory: From Basic to Advanced Study* [Ter14]. In the first book, tonal function is described as "the relationship of a chord to its tonal center" (page 1). In Berkman's book, the three classical tonal functions are first described and then a series of concentric circles are built out of this core, from which all the rest of chords are classified. Moving through these circles is done through chord transformations (this patently resonates with neo-Riemannian theory). In the third book, Terefenko provides a definition of tonal function in terms of hierarchy of chords (page 54), but again that hierarchy is described in detailed, but not its derivation. Worth mentioning is the book by Russell [Rus59] *The Lydian Chromatic Concept of Tonal Organization* where the author describes the tonal function as an interaction of harmonic forces and tonal gravity (page 3 and following). Russell's theory implies a deeper and more comprehensive of tonal function.

5.2. Tonal function labels

For the purposes of our research, we have taken on George Russell's concept of **tonal gravity** developed in his famous book *The Lydian Chromatic Concept of Tonal Organization* [Rus01]. This book had a great impact and laid the foundations for modal jazz. Miles Davis especially absorbed Russell's ideas and put them into practice in the 1950s and 1960s. His album *Kind of Blue* is the best-selling jazz instrumental album of all time (according to the *Best of Jazz* organization[1]) and in it Davis revealed modality as an entirely new creative tool in jazz improvisation. Davis had been searching for new horizons in melodic improvisation and found in modal jazz the tools to achieve his main underlying goal: melodic freedom. In **modal jazz**, the rhythm section plays a static mode for an extended period of time while the soloists improvise based on mode associated with that chord [Boo10]. These two features, static harmonies and improvisation based on modes, fulfilled the expectations of melodic freedom held by many jazz musicians in the late 1950s.

Partly, Russell's ideas on tonal gravity were motivated by Davis's quest for ways to relate melody and chords. As Nisenson [Nis00] describes it,

> We used to have sessions together. He was interested in chords, and I was interested in chords. We would sit at the piano and play chords for each other. He'd play a chord and I'd say, "Ooh, that's a killer." And then I would play a chord. At one of these sessions I asked Miles what he was looking to accomplish. He told me, "I want to learn all the chord changes. How can I go about doing this?" And I thought about that. I didn't challenge it. At times Miles could be very definite, but at other times he could be really obscure. I just said to myself, "He already knows the changes. What could he need?" Even then Miles was noted for outlining each change, identifying it with the melody. In other words, he wouldn't have even needed the piano player, because Miles' melody was dictating what the chords were. He wanted a new way to relate to chords.

Russell was looking for that new way to relate chords and scales. His idea was to use a stack of perfect fifths. He stacked a sequence of seven consecutive fifths and then re-arranged in stepwise order. The corresponding scale is the **Lydian mode**, C-D-E-F♯-G-A-B. Russell noted that the perfect fifth generated the strongest feeling of tonicization [Boo10] and, thus, his construction of the Lydian chromatic and the concept of tonal gravity. With this construction, Russell departed from the traditional concept of harmonic motion. Western music is traditionally governed by harmonic progressions. Every chord is classified in two categories: either it is a tonic chord or it is a part of a chord progression towards the tonic. Russell objected that this approach forces improvisers to focus on the chord progressions rather than on the chords themselves. For more information about how Russell transitioned from the Ionian mode to the Lydian mode, see his own work ([Boo10, Nov21, Moo88] are also good expositions).

Nisenson [Nis00] documents another encounter between Russell and Davis. By then, Russell had developed and thought out his theory and presented it to Davis. The rest is history.

[1] See https://bestofjazz.org/what-is-the-best-selling-jazz-album-of-all-time/

> One night Miles and I had dinner together, and we had a very serious discussion about modes. At the time, Miles was seriously looking for musicians to replace some of the guys in the band. So we sat down at the piano and played chords. I played a chord for him, and he asked me where I got it. I tried to show him where the chord came from. And he got very interested, because, by that time, I could translate any chord in terms of the Concept, and I could show Miles what its parent scale was; the scale formed a unity with the chord. Then Miles understood it. He saw that in the Concept there was an objective explanation for the chord. He saw that the traditional music overlooked any explanation for the chord. Unity was not a factor. When musicians are talking about harmony, they mean progressional harmony. They were ignorant—and still are— about a vertical concept. The Lydian Concept is based on the unity of chord and scale. That night, when Miles saw how he could use the Concept, he said that if Bird were alive, this would kill him. And it was just what Miles needed for the direction his music was taking.

In our model, tonal function will be determined by the motion of the voices of a chord progression. Following the discussion above, we will try to capture the idea of tonal gravity in terms of the voice motion and transform it into tonal function. Given two chords, the HTFM first computes the optimal link between those chords. The manner in which is done consists of feeding the Hungarian algorithm with the link matrix of the link. The Hungarian algorithm returns the optimal link (or links) as shown in Section 4.4, which turns out to match the minimization of energy (as we proved in Section 4.4.3). The procedure to determine the tonal function carries on by associating a polynomial to the link and studying its roots (see the mind map of the HTFM shown in Figure 4.1). Before entering into details about that, we need discuss the tonal functions themselves, in particular, we need to assign **labels** to the tonal functions that we can found in music practice.

We do not want to confuse the mathematical object called the **hypervolumetric tonal function** with the term **tonal function** in the musical sense. For one, a hypervolumetric tonal function is a polynomial whose roots inform us of the motion of the voices in an optimal link. For another, the musical tonal function just refers to the usual tonal functions in harmony. The tonal function will be denoted by an integral with the nabla symbol on it. If E is a link between two chords X, Y denoted by $E = (X|Y)$ (see Section 2.2, Equation 2.2 for a reminder), then for the tonic, dominant, and subdominant tonal functions, we set the following **labels**.

$$\int\!\!\!\!\nabla_{(X|Y)} = T \tag{5.1}$$

$$\int\!\!\!\!\nabla_{(X|Y)} = D \tag{5.2}$$

$$\int\!\!\!\!\nabla_{(X|Y)} = S \tag{5.3}$$

However, in the HTFM, the situation is more nuanced. As we will see soon in this chapter, the tonal function will be determined by the position of the roots of a certain **polynomial** with respect to 1; this polynomial is called the **characteristic polynomial** of the link. For example, if all roots are less than 1, then the tonal function will be set as dominant. However, there are more possibilities for the positions of the roots, so there are more tonal functions that the traditionally defined in music theory or found in practice. The classical tonal functions (tonic, dominant, and subdominant) correspond to the case in which all the roots lie on the same half-plane, either on $\{x \geq 1\}$ or $\{x \leq 1\}$. These half-planes will be called **poles**. In this case, we say that the tonal function is **polarized** and the roots of the polynomial are said to be polarized, too. However, other possibilities may arise. For example, what is the tonal function when all roots but one are on the same half-plane? What musical tonal function is derived from such situation? When the roots of the polynomial are not polarized, then the tonal function is said to be **non-polarized**. Therefore, we will extend define a wider set of labels $\pounds$ and write the general equation

$$\not{\int}_G = \mathcal{L} \tag{5.4}$$

Specifically, the set of labels in this work is defined as

$$\pounds = \{I, N_{md}, N_{msd}, T_c, T_d, S, D\} \tag{5.5}$$

Symbol I stands for the identity; this happens when both X, Y in $(X|Y)$ are the same chords. The polynomial has all its roots on 1. Labels N_{md}, N_{msd} stand for **non-polarized mixed definite** and **non-polarized mixed semidefinite**. Labels T_c, T_d, S, D respectively stand for **convergent tonic**, **divergent tonic**, **subdominant**, and **dominant**; this classification will be given by the position of the polarized roots. The tonal function label may be not unique. Indeed, the tonal functions are determined by the optimal motion of voices in the link; sometimes, there is more than optimal motion. When this situation arises, we speak of **dual tonal function**; we will elaborate further on this later on in this chapter.

Thus, for a link, once we have used the calculations of the hypervolumetric tonal function, we will write the tonal function that is associated with said link. That is to say that a tonal function is seen as an application that goes from the set of links $\mathcal{E}$ to the power set of labels, for there may be more than one label assigned to a particular link.

$$\not{\int} : \mathcal{E} \longrightarrow \mathcal{P}(\pounds)$$

$$\not{\int}_G = \mathcal{L} \subset \mathcal{P}(\pounds)$$

The assignment of each label by the tonal function is given by the polynomial criterion. The polynomial criterion is the result of applying the Hungarian algorithm (Section 4.4.2) to the link matrix. There will be three cases for the labelling of polarized tonal functions depending on the number of voices that remain constant and those that change. If all voices remains the same except one or if a voice disappears (voicing dimensions are not equal), we will speak of tonic function T and will be denoted by (dashed arrows).

$$\not{\int}_G = T \iff (X \dashrightarrow Y) \text{ or } (Y \dashrightarrow X)$$

When two or more voices go down from X to Y, we will speak of dominant function. And finally, when two or more voices go up from X to Y, we will speak of subdominant function.

$$\not{\int}_G = D \iff X \to Y$$
$$\not{\int}_G = S \iff Y \to X$$

Since there are a total of seven labels in $\pounds$, then by elementary combinatorics 2^7 is the cardinal of the power set. That yields a total of 128 sets of labels assignable to each link (the empty label is included in the counting). The labels are classified into 7 sets in turn depending on their cardinal:

$$\pounds_1, ..., \pounds_7$$

Thus we call the **order of a tonal function** the number of labels that it contains out of the seven possible ones. We say that a function is of order one if only one label is assigned to the link we are studying, we say that it is of order 2 if two different labels are assigned to it, and so on.

5.3. The horizontal dimension of harmony

Once the problem of vertical harmony has been solved by using vertical dissonance graphs, that is, the problem of which notes can be played within a chord, we then set out to solve the problem for the **horizontal dimension of harmony**. This problem consists of finding out how chords relate to each other over time. Worded in a more philosophical way, the study of the horizontal dimension of harmony is itself the study of how chords relate one another as well as how the are linked one another. Again, we will assume the notation introduced in Chapter 2, especially in Section 2.2 (The musical universe). Recall that a class of progressions is notated by $[P] = \left(\Big|_{j=1}^{m} X_j\right)$, where the horizontal bar indicates the matrix extension, and where P is a progression. Brackets are used because $[P]$ are all the progressions (matrices) that have the same chords as P. The notation with the vertical bar only indicates the order of the chords, but they do not indicate the voice assignments in the columns of P. We call $[P]$ a class of P.

$$[P] = \left(\Big|_{j=1}^{m} X_j\right) = (X_1 \mid X_2 \mid \ldots \mid X_{m-1} \mid X_m) \tag{5.6}$$

We say that two progressions belong to the same class or that they are equivalent, if when taking classes of P and P* their chords coincide. Note that the chords are not in P; in P there are only columns with ordered elements of each chord X, or matrices of dimension $n \times 1$, but not chords. That is why the equivalence is made by taking classes:

$$P \simeq P^* \iff X_j = X_j^*, \forall X_j \subset [P], \forall X_j^* \subset [P^*]$$

It follows that the equivalence of progressions, on this level of abstraction, is independent of their arrangement. It is clear that P is a progression of m chords where the chords arrange their classes in the entries of each column of the progression matrix. Two progressions are equivalent if they have the same chords.

5.4. Links and tonal functions

The tonal function $\not\int_{(X|Y)}$ is defined for pairs of chords (antecedent and consequent chords). It will provide with a label for each pair of chords; typically, those labels will consist of **tonic**, **dominant**, and **subdominant** labels (as discussed in Section 5.1 above). Actually, since all equivalent links have the same tonal function, by the static function theorem (see Section 6.3), we can state that, given two links E, E^*, it holds

$$E \simeq E^* \Longrightarrow \not\int_{E_{(X|Y)}} = \not\int_{E^*_{(X^*|Y^*)}} \tag{5.7}$$

If the links are proportional in the sense they are the same up to a transposition, then their tonal functions are also the same.

Given a link and its tonal function, it is possible to compute the tonal function of a **link class**. The tonal function of a link class is the tonal function of any link of the class (this function is unique by the static function theorem). To denote the link class, we will use a bracket as in $\not\int_{[E]}$.

Let λ be a real number and $E = ([e_i]), i = 1, \ldots, n$, the rows of a link E. It then holds that

$$\lambda \cdot [E] = [\lambda \cdot E] = [([\lambda \cdot e_i])]$$

where the inner bracket indicates the octave class, the parenthesis the usual notation for matrices, and the outer bracket the link class. A consequence of the static function theorem is

$$\not\int_{[E]} = \not\int_{[\lambda \cdot E]} \Longleftrightarrow \lambda \neq 0$$

This expression means that the tonal function of a class of a link is the same in all its transpositions if λ is not null.

5.5. Voice leading

When trying to understand the relationship between two chords X, Y, we have to take into account their acoustic nature. These two chords X, Y sit in $\wp(LC_k/\sim)$. Let ψ be the map between the chords and their frequencies:

$$\psi : \wp(LC_k/\sim) \longrightarrow \Phi^n$$

$$\psi(X) = (\psi_1(X_1), \psi_2(X_2), \ldots, \psi_n(X_n))$$

The ψ **function** transforms an n-dimensional chord into a voicing of the same dimension in the n-dimensional frequency space; see Chapter 2. The only condition required of the application is that it should transform a chord into a voicing such that each frequency in the voicing belongs to the class of the original chord. In mathematical terms,

$$\psi_i(X_i) \subset [X_i], \forall i, 1 \leq i \leq n$$

In this way we can understand that each voicing is one of the possible physical manifestations that a chord can have. For a finite set of low cardinality classes, voicings can be varied and give different textures to the music. Our goal is still to study general properties of chords in order to understand how they can be arranged in time, but without losing sight of their physical manifestation.

Thus, we seek to establish a relationship between two chords X, Y in such a way that we can generate a progression P by relevant meeting musical criteria (smooth linking being one of them). Generating P is one of the most complicated objectives for a composer, who usually relies on learned patterns. The main contribution of this work is to provide a method to analyze pre-existing progressions and to generate new ones independently of the characteristics of each chord and its dimension.

Voice leading is an assignment of voicings between two consecutive chords. Traditionally this was done by following the minimum movement, i.e., by preserving the common notes and moving the rest as little as possible. In this work, we address two questions. The first one is to present a method to produce optimal voice leading even when the chord dimensions of the link are different. The second question is to prove that the minimization of the voice leading yields the minimization of the stimulus when the timbre is constant and when the amplitude variation at the moment of change is zero (this question is addressed in Chapter 4 and Section 6.3).

5.6. The universal link matrix

In this work, whose main context is Western music, we fixed a number of octave equivalence classes, which are well known theoretically speaking. Those classes they coexist with others, though, since there is always a certain degree of inflection and this generates more classes. Despite this fact, music is structured on a **written level** and in the field of **musical production**, around twelve octave classes usually tuned around 440 or 442 Hz. Thus, the number of classes can safely be assumed to be twelve. Since the WFC (Weber-Fechner connection, Section 4.4.3) proves that the minimization of stimulus is equivalent to the minimization of the global sum of the distances between the voices, then we can build a general matrix with the twelve classes, and then cancel those classes that are not involved in the harmonic link. That matrix is called the **universal link matrix** and is denoted by U; see Equation 5.8. This universal link matrix is chosen among many other possibilities for matrices describing the 12×12 classes. The more natural choice is the one presented here, which follows the circle of fifths.

When computing the tonal function of the global link $G = (X \mid Y)$, the rows corresponding to the notes of the tonal center X are selected and the rest are deleted. Similarly, the columns corresponding to the notes of the tonal center Y are selected and the rest deleted. The remaining submatrix is precisely the link matrix. These ideas will be developed in Section 7.1. Thus, given a particular link E, or a **global link** G between the open keys, we can calculate if the chords X and Y resolve or if the keys $\mathcal{O}_1$ and $\mathcal{O}_2$ modulate.

$$U = \begin{pmatrix}
\Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\
F & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- \\
Bb & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ \\
Eb & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- \\
Ab & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\
Db & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\
Gb & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\
B & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\
E & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\
A & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\
D & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- \\
G & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ \\
C & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0
\end{pmatrix} \tag{5.8}$$

The universal link matrix is then presented as a square matrix whose entries are the minimum distance in semitones between the class in the column and the class in the row. Each class in each row belongs to the antecedent tonal center and each class in each column belongs to the consequent tonal center. The superscripts indicate if the voices go up or down when we optimize the link. The antecedent goes to the consequent, which is the final tonal center. The distances are given by the minimum integral between two frequencies of each class. The

U matrix can be seen as the link matrix from which all the tonal functions in the tempered system can be extracted and it works as a tonal function calculator. Thus, only with U we can extract all the L matrices of links in this system and as well as all the L matrices between keys.

The operation of U consists first of selecting which are the tonal centers from among which we want to know the tonal function, that is, to know either $E = (X \mid Y)$ or $G = (\mathcal{O}_1 \mid \mathcal{O}_2)$. With this information we will replace the classes that are not present in those tonal centers with empty classes $[\emptyset]$. From here, we obtain that either $U_{(X|Y)}$ or $U_{(\mathcal{O}_1|\mathcal{O}_2)}$. At this point we only have to follow the steps of the Hungarian algorithm on the new matrix and complete the process until we obtain the polynomial we are looking for. However, the presence of empty classes will pose some issues. We need to extend the Hungarian algorithm so that it can handle that case. To that end, we introduce the $\infty - \infty$ method in the next section.

5.6.1. The $\infty - \infty$ method

The core of our method resides in minimizing distances between classes of notes. Such minimization occurs through the minimization of the integral found in the definition of $\Delta([\alpha],[\beta]) = \left|\int_{\alpha}^{\beta} \frac{\Omega}{\phi} d\phi\right|_{\Delta}$. However, a problem arises when two chords in a link have different dimensions (voices). In that case, one of the voices just disappears. That results in the need of measuring the distance between an octave class of an arbitrary frequency and an empty class. This section is dedicated to solving this part and giving meaning to the operation of the matrix U or any matrix L that comes from from a link where $dim(X) \neq dim(Y)$. In this situation, we have to worry about giving a definition for the minimum distance between a class in $(LC_k/\sim)$ and the empty class $[\emptyset]$. This is solved by the $\infty - \infty$ **method**.

Now we consider the metric between two classes where one class is an arbitrary class $[\beta]$ and the other is the empty class so that we are in the position of expressing the distance between a frequency of the class $[\beta]$ and the empty class (we could call it **mathematical silence**, if you will).

$$\Delta([\emptyset],[\beta]) = \left|\Omega \int_{0}^{\beta} \frac{d\phi}{\phi}\right|_{\Delta} \tag{5.9}$$

The subscript Δ is used here to mark the difference between the distance between classes and the distances between frequencies. When the Δ symbols appears, it means the class distance computed by taking the appropriate frequencies and then calculating the minimum distance between them.

In order to compute this metric, we will take limits of the distance when an arbitrary class $[\alpha]$ tends to $[\emptyset]$.

$$\Delta([\emptyset],[\beta]) = \lim_{[\alpha]\to[\emptyset]} \Delta([\alpha],[\beta])$$

With this clear notion we proceed to solve the limit to find the value of this distance. We also have the notation where the delta symbol appears as a subscript of a normal integral and therefore we can make the value of alpha tend to zero, then the value of the limit will be the minimum integral between classes when particularly alpha tends to zero. Thus, we have written the minimum distance between classes as a limit that we know how to solve by means of improper integration.

$$\lim_{[\alpha]\to[\emptyset]} \Delta([\alpha],[\beta]) = \lim_{\alpha\to 0} \left| \Omega \int_{\alpha}^{\beta} \frac{d\phi}{\phi} \right|_{\Delta}$$

In this way, the minimum distance between an arbitrary class and silence (disappearance of a class) is actually infinite. Therefore, we can write

$$\lim_{\alpha\to 0} \left| \Omega \int_{\alpha}^{\beta} \frac{d\phi}{\phi} \right|_{\Delta} = \infty$$

Since we have that by definition, the distance between classes is the minimum integral between a pair of frequencies, and specifically when a class is empty, we use improper integration to compute the distance:

$$\Delta([\emptyset],[\beta]) = \lim_{\alpha\to 0} \left| \Omega \int_{\alpha}^{\beta} \frac{d\phi}{\phi} \right|_{\Delta} = \infty$$

Finally, solving the improper integral and connecting it with the definition of distance between an arbitrary class and the silence, we see that the value of this metric is infinite.

$$\Delta([\emptyset],[\beta]) = \infty$$

The appearance of infinity causes many problems if it is not dealt with properly. In the next section, we introduce the so-called infinite arithmetic convention to solve this issue.

5.6.2. The infinite arithmetic convention

In the context of an L matrix where a pair of metrics $\Delta_1 = \infty$ and $\Delta_2 = \infty$ are involved into the operative of the Hungarian algorithm, we will follow the convention, taking k as number $k \in \mathbb{R}$.

$$\boxed{\infty - (\infty - k) = \infty - \infty + k = 0 + k = k} \tag{5.10}$$

This is known as the **infinite arithmetic convention** for the Hungarian algorithm, so the ∞ behaves "like a number" inside of an L matrix.

On the one hand, it has been seen that the area under the function that we use to measure between frequencies is worth infinity when we evaluate it between an arbitrary frequency and zero. This has been tested in the previous section.

The infinite arithmetic convention establishes how infinity has to be treated in this context and how this treatment is the one that completes the model of tonal function since it allows to get out of the limitation that the number of voices is the same for the two tonal centers involved and expands the calculation to cases where $\dim(X) \neq \dim(Y)$.

5.6.3. The Zero method

Given a matrix L to which the Hungarian algorithm is applied, several zeroes can appear on the same row or column. This corresponds to several solutions with the same nabla distance for different links. In order to find all the solutions, we will use the so-called **Zero method**. This method consists of forcing zeroes in matrix L_E^H, which is the matrix output by the Hungarian algorithm. The Zero method is described as follows.

(1) If a row contains a single zero, that zero is fixed. It will be called a **fixed zero**. Fixed zeroes will be boxed on the matrix;

(2) If a row contains more than one zero, select one of them and force it to be fixed. Box it, too. These zeroes will be called **moving zeroes**.

(3) Repeat the previous step until no more zeroes are available.

The Zero method finds all the solution by forcing zeroes. This method guarantees that no solution is missed.

To illustrate how the zero method works, we will next calculate the tonal function of the link $E = (X \mid Y)$ where $X = \{G, B, D\}$ and $Y = \{C, E, G, B\}$. Note that the first chord has fewer classes than the second chord.

$$E = \begin{pmatrix} [\varnothing] & B \\ D & G \\ B & E \\ G & C \end{pmatrix} \longrightarrow L_E = \begin{pmatrix} \infty & \infty & \infty & \infty \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 1}} \begin{pmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \\ 1 & 3 & \boxed{0} & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix}$$

$$\xrightarrow{\text{Step 2}} \begin{pmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \\ 1 & 3 & \boxed{0} & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix}$$

In this particular case, two solutions have arisen. Solution 1 (refer to the equations below) corresponds to the assignment $D \longrightarrow E, B \longrightarrow B, G \longrightarrow G$ and $[\o] \longrightarrow C$. With the voice-leading given by this solution, the chord $X = \{G, B, D\}$ is a subdominant chord with respect to the chord $Y = \{C, E, G, B\}$.

$$\xrightarrow{\text{Step 3}} \begin{pmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \\ 1 & 3 & \boxed{0} & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 4}} \begin{pmatrix} 0 & 0 & 0 & \boxed{0} \\ 1 & 3 & \boxed{0} & 0 \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 5}} \begin{pmatrix} - & - & - & \boxed{\infty} \\ - & - & \boxed{2} & - \\ \boxed{0} & - & - & - \\ - & \boxed{0} & - & - \end{pmatrix}$$

Solution 2 (refer to the equations below) works as follows. Now, the assignment is $D \longrightarrow C, B \longrightarrow B, G \longrightarrow G$ and $[\o] \longrightarrow E$. Analogously, with the voice leading provided by solution 2, the chord progression $X = \{G, B, D\} \longrightarrow Y = \{C, E, G, B\}$ is a non-polarized mixed semidefinite tonal function; see Section 5.2.

$$\xrightarrow{\text{Step 3}} \begin{pmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \\ 1 & 3 & \boxed{0} & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 4}} \begin{pmatrix} 0 & 0 & \boxed{0} & 0 \\ 1 & 3 & 0 & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 5}} \begin{pmatrix} - & - & \boxed{\infty} & - \\ - & - & - & \boxed{2} \\ \boxed{0} & - & - & - \\ - & \boxed{0} & - & - \end{pmatrix}$$

The importance of the convention of infinite arithmetic is enormous because it allows us to compute all cases within the tempered system, but also outside of it. Thus, in the context of tonal gravities, it also serves to characterize and distinguish them since we compute tonal functions in the set of real numbers endowed with the infinite arithmetic A^{∞}.The Zero method shows us all the solutions.

5.7. Algebraic model of tonal function

In Figure 4.1 in Chapter 4, we provided the reader with a mind map of the **hypervolumetric tonal function model** . In that **mind map** , one of the nodes consisted in the very algebraic model. In the mind map in Figure 5.1 on this page, we show the mind map corresponding to the algebraic model (the rest of the nodes has been summarized).

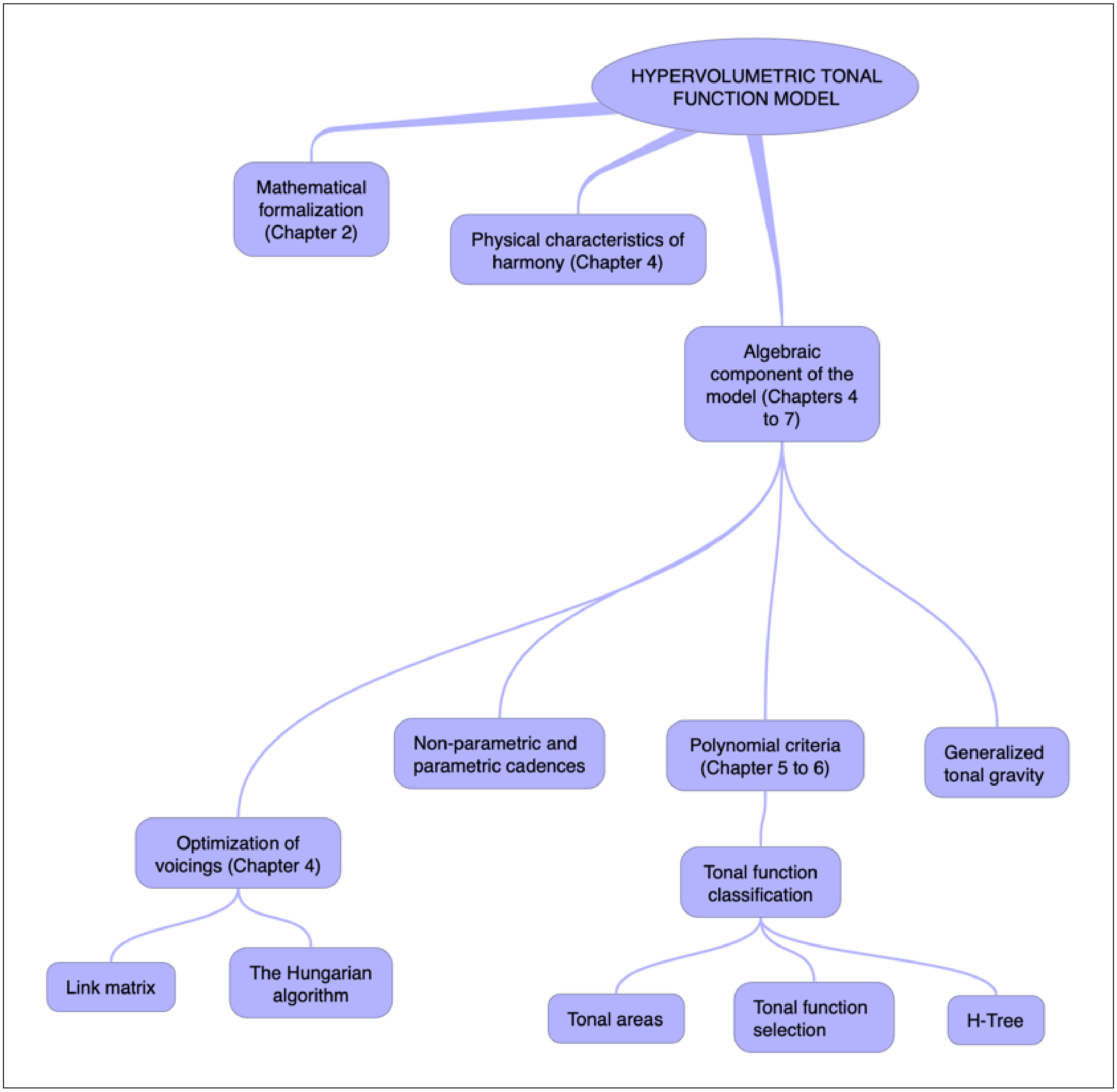


Figure 5.1: The hypervolumetric tonal function model (algebraic model)

The **formula** that represents the whole algebraic model of this work is

$$H : \wp(LC_k/\sim)^2 \longrightarrow \mathbb{C}[\lambda]$$
$$H_{(X|Y)} = \Delta\left(s^{\sigma(L_E^{BC}-W)} + P - \lambda \cdot Id_{\max\{\dim(X),\dim(Y)\}}\right)$$

In the next few sections, we will define each term of the formula. The formula is a function H defined on $\wp(LC_k/\sim)^2$, the set of pair of tonal centers, and it returns the tonal function when tonal center X lands on tonal center Y. In Figure 5.2, the components of the tonal function model are shown.

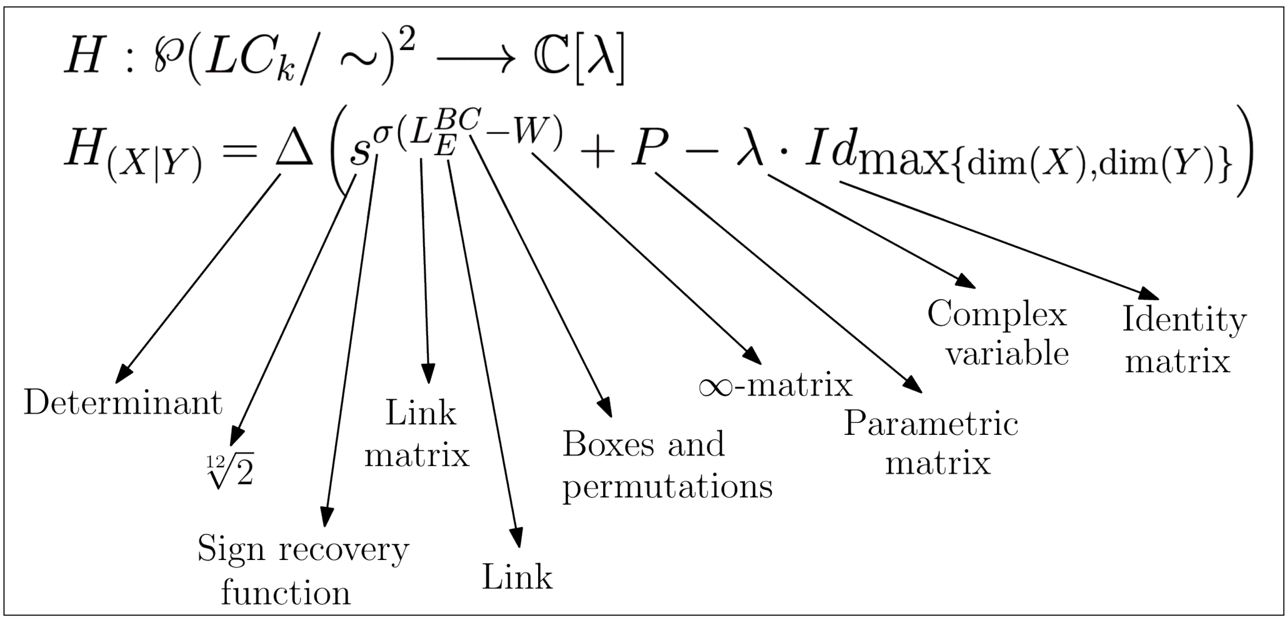


Figure 5.2: The components of the hypervolumetric tonal function model (algebraic model)

We start by classifying cadences into non-parametric and parametric cadences in the next section.

5.7.1. Parametric and non-parametric cadences

Giving a chord progression class $[P] = (X \mid Y)$, two important cases must be taken into account as tonal function classification is concerned, namely: (1) the dimensions (number of notes) of X and Y are the same or $\dim(X) \geq \dim(Y)$; (2) $\dim(X) < \dim(Y)$. In the former case, we speak of **non-parametric cadences** and in the latter of **parametric cadences**.

For a chord X, the mapping that takes any voicing of X through the optimal voice leading is an assignment of the voicing of X into a voicing of Y. We can define an endomorphism, to be called **cadence endomorphism** $C_{\mathbb{E}}$, $C_{\mathbb{E}} : \Phi^n \longrightarrow \Phi^n$ that transforms the original voicing into the another one through an optimal voice leading, say Y, that is, $C_{\mathbb{E}}(\psi(X)) = \psi(Y)$, where X is the given tonal center. Denote by s the quantity $2^{\frac{1}{12}}$, a semitone or half-tone in an equal division of the octave. Then, the endomorphism matrix for $C_{\mathbb{E}}$, $M_{C_{\mathbb{E}}}$, is given by

$$M_{C_{\mathbb{E}}} = \begin{pmatrix} s^{b_{11}} & 0 & \cdots & 0 \\ 0 & s^{b_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{b_{nn}} \end{pmatrix} = \begin{cases} s^{b_{ij}}, & \text{if } i = j \\ \\ 0, & \text{if } i \neq j \end{cases} \tag{5.11}$$

where the exponents b_{ij} are the motion of the voices; these exponents can take real values or $\pm\infty$.

We will abuse notation and will identify $M_{C_{\mathbb{E}}}$ and $C_{\mathbb{E}}$ and refer to the matrix by the endomorphism and vice versa. The context will clarify which sense is in use at the moment.

Let $P_{C_{\mathbb{E}}}(\lambda)$ be the **characteristic polynomial** associated to this endomorphism. The roots of $P_{C_{\mathbb{E}}}(\lambda)$ are precisely $s^{b_{11}}, \dots, s^{b_{nn}}$ and such polynomial can be written as

$$P_{C_{\mathbb{E}}}(\lambda) = \prod_{i=1}^{n} (s^{b_{ii}} - \lambda) \tag{5.12}$$

The exponents of those roots can be negative, positive, or zero. Below there is a classification of cadence endomorphisms according to the sign of the exponents. This classification serves as a means to define the tonal function in very general settings such as musical spaces with chords with a high number of voices.

Having taken the presence of infinite into account, some formal issues arise. In the previous example, Solution 1 (Section 5.6.3), one of the exponents is $+\infty$. The associated cadence endomorphism $C_{\mathbb{E}}$ will have an entry $s^{+\infty}$, which makes no sense. It would seem that there is no cadence endomorphism in this case, but that is not so. It is possible to construct a cadence endomorphism $M_{C_{\mathbb{E}}}$ that maps $\psi(X)$ onto $\psi(Y)$. In our example such endormorphism is

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4, \quad C_{\mathbb{E}} \cdot \psi(X) = \psi(Y)$$

$$C_{\mathbb{E}} = \begin{pmatrix} s^{\infty} & \psi(Y_1)/\psi(X_2) & 0 & 0 \\ w_2 & s^2 & 0 & 0 \\ w_3 & 0 & s^0 & 0 \\ w_4 & 0 & 0 & s^0 \end{pmatrix}$$

where $\psi(X)$ and $\psi(Y)$ are voicings of X and Y, respectively, and $w_1 \ldots, w_4$ are arbitrary parameters to be determined. The matrix expression of the cadence endomorphism is:

$$\begin{pmatrix} s^{\infty} & \psi(Y_1)/\psi(X_2) & 0 & 0 \\ w_2 & s^2 & 0 & 0 \\ w_3 & 0 & s^0 & 0 \\ w_4 & 0 & 0 & s^0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \psi(X_2) \\ \psi(X_3) \\ \psi(X_4) \end{pmatrix} = \begin{pmatrix} \psi(Y_1) \\ \psi(Y_2) \\ \psi(Y_3) \\ \psi(Y_4) \end{pmatrix}$$

Therefore, there are infinite endomorphisms that output the same optimal voice leading.

Let us examine what happens when the number of voices decreases. Take, for example, the chord $X = \{C, E, G, B\flat\}$ and set $Y = \{F, A, C\}$ as the second chord Below we have the execution of the Hungarian algorithm on this link.

$$E = \begin{pmatrix} B\flat & [\emptyset] \\ G & C \\ E & A \\ C & F \end{pmatrix} \longrightarrow L_E = \begin{pmatrix} \infty & 2 & 1 & 5 \\ \infty & 5 & 2 & 2 \\ \infty & 4 & 5 & 1 \\ \infty & 0 & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 1}} \begin{pmatrix} \infty - 1 & 1 & 0 & 4 \\ \infty - 2 & 3 & 0 & 0 \\ \infty - 1 & 3 & 4 & 0 \\ \infty & 0 & 3 & 5 \end{pmatrix}$$

$$\xrightarrow{\text{Step 2}} \begin{pmatrix} 1 & 1 & \boxed{0} & 4 \\ \boxed{0} & 3 & 0 & 0 \\ 1 & 3 & 4 & \boxed{0} \\ 2 & \boxed{0} & 3 & 5 \end{pmatrix} \xrightarrow{\text{Step 3}} \begin{pmatrix} - & - & \boxed{0} & - \\ \boxed{\infty} & - & - & - \\ - & - & - & \boxed{0} \\ - & \boxed{0} & - & - \end{pmatrix}$$

Reading off the voice leading from the last matrix above, it yields the voice leading

$$B\flat \longrightarrow A, E \longrightarrow F, C \longrightarrow C, G \longrightarrow [\emptyset]$$

This last voice leading makes the note G in the chord X disappear. Similarly to the previous case, the disappearance of a voice is considered as a voice moving down. As for the tonal function, two voices move down, two voices stay the same, and just one voice move up. Therefore, $X = \{C, E, G, B\flat\}$ has tonal function $\nexists_E = N_{msd}$ with respect to the chord $Y = \{F, A, C\}$.

In the case of constant voices in the link, the parametric matrix is not necessary. The infinity in the last matrix in the example actually corresponds to $-\infty$(recovering the sign), since G disappears in the voice leading. Therefore, the entry $s^{-\infty}$, which is interpreted as $\lim_{K \to -\infty} s^K$, is zero. The characteristic polynomial of the above endomorphism is:

$$P_{C_{\mathbb{E}}}(\lambda) = -\lambda(s^{-1} - \lambda)(s^1 - \lambda)(s^0 - \lambda)$$

5.7.2. Change of space equations for non-parametric and parametric cadences

The link matrix L is next transformed into a **cadence matrix** $C_{\mathbb{E}}$ that holds

$$C_{\mathbb{E}}\psi(X) = \psi(Y) \tag{5.13}$$

The previous equation defines a space-changing transformation between an L matrix and $C_{\mathbb{E}}$, known as the T **transformation**.

$$T_i : \mathbb{L}_n(\mathbb{R} \cup \infty) \longrightarrow (C_{\mathbb{E}})_n(\mathbb{C})$$

$$T_i(L_E) = s^{\sigma(L_E^{B_iC_i} - W)} + P_i = (C_{\mathbb{E}})_i \tag{5.14}$$

The transformation T is carried out between the space of matrices $\mathbb{L}_n(\mathbb{R} \cup \infty)$ of dimension $n \times n$ and the space of matrices with complex coefficients $(C_{\mathbb{E}})_n(\mathbb{C})$ also of dimension $n \times n$. The subscript i is written since there are multiple box distributions depending on L. Given either a link or a global link the T transformation maps the optimization carried out on the L matrix onto the $C_{\mathbb{E}}$ matrix. Thus, the matrix $C_{\mathbb{E}}$ is the one that, as established in the WFC, transforms a voicing of the previous tonal center into a voicing that is contained in the following tonal center, keeping the measure of the perception of the sound stimulus between both to a minimum; in mathematical terms, we write $\mid p^{\Gamma} \mid ((\psi(X), \psi(Y)) = \min(\mid p^{\Gamma} \mid)$.We want to clarify that the perception of the stimulus $\mid p^{\Gamma} \mid = (\psi(X), \psi(Y))$ is different from $\mid p^{\Gamma} \mid (E(\psi(X)), E(\psi(Y))$. The correct measure for the perceived change of stimulus between two voicings is $\mid p^{\Gamma} \mid = (\psi(X), \psi(Y))$ where:

$$\mid p^{\Gamma} \mid (\psi(X), \psi(Y)) = \sum_{i=1}^{n} \mid p^{\Gamma} \mid ((\psi(X_i), \psi(Y_i))$$

Note that when $dim(X) = dim(Y)$, the absolute perception will take a value in $\mathbb{R}$, but when the number of voices is different, the perception of the stimulus will be measured in terms of infinities and they will be compared and computed by using infinite arithmetic over $\overset{\infty}{\underset{\infty}{\mathbb{R}}}$ (to be defined and examined in Section 5.9).

The reason why the perception of the link between two voicings is not calculated as $\mid p \mid (E(\psi(X), E(\psi(Y))$ is simply because the perception of the notes are perceived individually. Then when we calculate $C_{\mathbb{E}}$ in the equation $C_{\mathbb{E}}\psi(X) = \psi(Y)$, it is guaranteed that:

$$\mid p^{\Gamma} \mid (\psi(X), \psi(Y)) = \min \mid p^{\Gamma} \mid$$

In this way, by using the T transformation, we can transform the L matrix directly into the endomorphism matrix, guaranteeing that the absolute perception takes the minimum value in the stimulus change. We also realize at this point that there can be several solutions for the matrix L with the same value for perception. In these cases the tonal function will be dual and we will have a set of matrices $C_{\mathbb{E}}$ all with the same perception value.

5.7.3. Change of space for non-parametric cadences

Step 1 After the Hungarian algorithm is applied to a link matrix L, the final solution is given by a matrix with some entries boxed. This matrix will be notated as L^B. The matrix L^B is therefore the original matrix L formed from the tonal centers X and Y or from a pair of open keys $\mathcal{O}_1$ and $\mathcal{O}_2$, but with the boxes that make up the solution of the Hungarian algorithm drawn.

Step 2 Second, we have to find a C matrix that permutates the colums of L^B until all boxes are in the diagonal. The result will lead us to a new matrix named L^{BC} so;

$$L^{BC} = L^B C \tag{5.15}$$

Thus, when finding a matrix C that permutes the columns of L^B we will take the boxes that have been drawn when optimizing the voices using the Hungarian algorithm to the diagonal. The L^{BC} matrix is still in the space of matrices $\mathbb{L}_n$ of n dimension. One way to calculate the matrix C is to replace the boxes of L^B with ones and the rest of the entries with zeros. We will call this matrix the matrix Q, it is enough to transpose Q to find C having that $Q^t = C$. In this way we have already placed the solutions on the diagonal and it is observed that the rest of the information is outside our area of interest. Therefore, we use the matrix W to eliminate the rest of the entries.

Step 3 Once L^B has been rearranged, we use the W **matrix**. That matrix is defined by

$$W_{ij} = \begin{cases} W_{ij} = \infty & \text{if } i \neq j \\ W_{ij} = 0 & \text{if } i = j \end{cases} \tag{5.16}$$

Finally, we can write

$$L^B C - W = L^{BC} - W \tag{5.17}$$

Step 4 The step number four is to apply the change of space transformation once the optimization has been done in the logarithmic space. At this point we transform the space by calculating the matrix $s^{L^{BC}-W}$, which is the matrix whose entries are zero except for the diagonal elements; where each entry of said diagonal is equal to the result of raising s to the corresponding entry of $L^{BC} - W$. Since we are talking about non-parametric cadences where either the dimensions of both centers are equal or the dimension decreases in the link, then we safely assume that P is the zero matrix of dimension n, where $n = \max\{dim(X), dim(Y)\}$ or for the global link case $n = \max\{dim(\mathcal{O}_1), dim(\mathcal{O}_2)\}$.

Step 5 Once we have changed the space we have to recover the signs of each metric so that the voicing $\psi(X)$ is properly transformed into the voicing $\psi(Y)$. For this we use the function σ which retrieves the signs of the metrics on the diagonal.

$$\sigma(s^{L^{BC}-W}) = C_{\mathbb{E}}$$

Thus, if we start from the matrix U then the σ application will place the sign that originally appears in the superscript as the sign of the exponent of each element of the diagonal, obtaining the endomorphism matrix $C_{\mathbb{E}}$ with minimum absolute perception.

5.7.4. The complete process to find a tonal function

In this section, we will go through the complete process to find the tonal function of a link.

Step 1. We choose a pair of chords X and Y or open keys contained in $\wp(LC_k/\sim)$. At this point, it does not matter whether $dim(X) \neq dim(Y)$.

Step 2. Thus, the link class is formed, $[E] = (X \mid Y)$. The inversion of the chords do not matter at this point. For the purposes of the computation, we may assume that $E = (X^r \mid Y^r)$, that is, the chords are given in root position. If they are tonal centers, assume that they are arranged from bottom to top following the circle of fifths clockwise.

Step 3. Compute the L matrix where

$$L = (\Delta_{ij}) = \left(\left| \Omega \int_{X_i}^{Y_j} \frac{d\phi}{\phi} \right|_{\Delta} \right)$$

and $L_E = L_{(X|Y)}$. The L matrix is computed by calculating each minimal integral between the classes of each chord. Depending on the arrangement of the chords, the construction of the matrix L varies, but the result of the optimization will be the same. In order to simplify the process, we will arrange the tonal centers following the clockwise direction of $(LC_k/\sim)$.

Notation reminder: Sometimes the symbol Δ is used without the tilde if the context is clear enough to imply that it is measuring between octave classes and not between frequencies. Simultaneously, the symbol delta is used to indicate the determinant of a matrix as well as a subindex.

Step 4. Once L is made, then, if the dimensions of our tonal centers are identical, the entries of L will be real numbers. When the dimensions are different, some infinity symbols will appear on the matrix. Next, the Hungarian algorithm is applied to the matrix until one or more optimal solutions are attained. Use the extension of the algorithm and the $\infty - \infty$ **method** if $dim(X) \neq dim(Y)$; see Section 5.6.2. We represent the process starting from L_E, which is the matrix constructed for the link E, and we apply the first step of the Hungarian algorithm, subtracting the minimum of each row from its own row. We call this new matrix L_E^F. We then apply the second step by subtracting each column's minimum from its own column and reaching L_E^H.

$$L_E \longrightarrow L_E^F \longrightarrow L_E^H \tag{5.18}$$

Step 5. When several solutions with the same nabla for different links of the same class are present, then we will have to find them by using matrix L_E^H and forcing each zero to be part of the solution. Thus, for each zero that is found, we will obtain a solution to the link. On many occasions, forced zeros for the same solution will be found. This process was fully described in Section 5.6.3. It is necessary to force each zero in matrix L^H to find all the optimal solutions to the link. Thanks to this method, dual tonal functions can be computed.

Step 6. Once the box distributions are found as given by the Hungarian algorithm, then we will use transformation T defined by Equation 5.13 in Section 5.7.2 to compute as many tonal functions as there are different solutions generated by the link. If the zero method had to be used (due to the presence of many solutions), then the set of box distributions will be taken instead.

Step 7. Suppose now that we take one of the box distributions obtained by the Hungarian algorithm and we superimpose it over the original matrix L_E as in the example on page 81. We apply the transformation T to each of its solutions. Let $\mathcal{T} = \{T_1, ..., T_z\}$ be the set of the z distinct transformations that correspond to each box distribution over L_E. We can write

$$T_i : L_n(\mathbb{R} \cup \infty) \longrightarrow (C_{\mathbb{E}})_n(\mathbb{C})$$

$$T_i(L_E) = s^{\sigma(L_E^{B_i C_i} - W)} + P_i = (C_{\mathbb{E}})_i \tag{5.19}$$

Thus, set $\mathcal{T}$ is composed of all transformations that are solutions as given by the zero method and we will obtain as many matrices $(C_{\mathbb{E}})_i$ as transformations are involved. We call $\mathcal{C}$ the set of these matrices $\mathcal{C} = \{(C_{\mathbb{E}})_1, ..., (C_{\mathbb{E}})_z\}$ and we have that $\mathcal{T} = \mathcal{C}$. In this way we have calculated all the z matrices in $\mathcal{C}$, which correspond to all the transformations that take one voicing into another (while keeping the note stimulus perception to a minimum).

Step 8 Once $\mathcal{C}$ is computed, then the next step is to compute their **characteristic polynomials** of each of these matrices, that is, the hypervolume function as a function of λ that will take the value zero when λ takes the value of the ratio between two voices in the same dimension. We call this set of polynomials $\Delta(\mathcal{C} - \lambda Id)$ and it corresponds with all the tonal functions that exist in said link, unless the link is parametric, in which case there are infinitely many.

Step 9. At this point, we compute each characteristic polynomial of each matrix $(C_{\mathbb{E}})_i$ as follows.

$$\Phi[E_i] = \Delta((C_{\mathbb{E}})_i - \lambda Id)$$

We also write the tonal functions as polynomials for each distribution of boxes so that $\Phi[E_i] = H_{E_i}(\lambda)$. When we want to distinguish between different sets of tonal functions for different links we also use the notation $\Delta(\mathcal{C} - \lambda Id) = \mathbb{H}^E$ Thus, when calculating all the transformations and finding each characteristic polynomial, $\mathbb{H}^E$ is output, which is the set of tonal functions associated with a link. If the tonal functions we are studying are global, then, without loss of generality, the set of tonal functions will be $\mathbb{H}^G$ where G is the global link. Thus, what is sought is to know which tonal functions $\mathbb{H}^G$ are associated with the given link so that we determine how note perception behaves when changing from one tonal center to another.

Step 10. For each tonal function associated with a particular link and for each tonal function $H_{E_i}(\lambda)$ in $\mathbb{H}^E$ (as outuput by the zero method), the set of integrals of each tonal function $\mathbb{I}^E = \{\int H_{E_i}(\lambda)\, d\lambda, ..., \int H_{E_z}(\lambda)\, d\lambda\}$ is defined as follows.

$$\int H_{E_i}(\lambda)\, d\lambda = \int \Delta(s^{L_E^{B_i C_i} - W} + P_i - \lambda Id_n)\, d\lambda \tag{5.20}$$

If we are talking about a global tonal function, the set of integrals will be given by the symbol $\mathbb{I}^G$. As each tonal function is a complex variable polynomial we will carry out the integration in the usual way.

The case of parametric tonal functions is slightly more complicated and it will be addressed in Section 5.8. For that case, we will use one of the matrices of the radical class to calculate $\mathbb{H}^E$ and later we will calculate $\mathbb{I}^E$ from the inner set (to be explained below in that section).

5.7.5. Polynomial criterion $\mathcal{P}$

We call the **polynomial criterion** $\mathcal{P}$ the correspondence between the set of tonal functions of a normal or global link and a tonal area. Mathematically, a **tonal area** is a subset of the set of complex variable polynomials $\mathbb{C}[\lambda]$. Therefore, this criterion will classify any tonal function into a category belonging to the disjoint union of tonal areas is $\mathcal{A}$, where

$$\mathcal{A} = \{T^{\mathbb{C}[\lambda]}, D^{\mathbb{C}[\lambda]}, S^{\mathbb{C}[\lambda]}, I^{\mathbb{C}[\lambda]}, N^{\mathbb{C}[\lambda]}\} \tag{5.21}$$

The polynomial criterion classifies polarized tonal functions into different categories called **tonic**, **dominant**, **subdominant**, the **identity** one, and the **non-polarized** ones. A tonal function can belong to a single tonal area or if it turns out that a set of tonal functions appears in $\mathbb{H}^G$ or $\mathbb{H}^E$, then it can belong to several tonal areas simultaneously. The polynomial criterion is oriented to the classification of polarized tonal functions by combining the traditional knowledge of tonal functions with the new cases that appear in modern music. Each tonal area is a particular subset of $\mathbb{C}[\lambda]$ where each polynomial of said area has its roots clearly polarized to one side of the **stabilizer** of the **Mersenne group** $E(M) = 1$ (simply put, whether the roots are to the right or to the left of 1).

Both the cases of modulation and convergence between a three- and a four-voice chord are all polarized cases. However, sometimes there are musical situations where the tonal functions are non-polarized. In this work we provide a a classification of the tonal functions; it is resorts to a structure called the H **tree**, which will be described thoroughly in Section 5.13. Hence, the mathematical formalization of the polynomial criterion is the identification of the set $\mathbb{H}^G$ or the set $\mathbb{H}^E$ with one or more tonal areas.

5.7.6. Mathematical formalization of $\mathcal{P}$

We can see the polynomial criterion as the identification of the set of tonal functions $\mathbb{H}^E$ of a link obtained via the zero method with a subset of the power set of $\mathcal{A}$. Therefore, if the whole set of tonal functions is denoted by $\mathcal{H}$, in the system $\mathbb{S}$ (see Chapter 8.1.1), and $\mathbb{H}^E \subset \wp(\mathcal{H})$, then we will assign the union of certain tonal areas $\mathbb{A}$, where $\mathbb{A} \subset \wp(\mathcal{A})$, to each set of tonal functions of each link.

$$\begin{aligned} \mathcal{P} : \wp(\mathcal{H}) \longrightarrow \wp(\mathcal{A}) \\ \mathcal{P}(\mathbb{H}^E) = \mathbb{A} \subset \wp(\mathcal{A}) \end{aligned} \tag{5.22}$$

Another way of characterizing dual tonal functions is to establish that a link has a dual tonal function if $\mid \mathbb{A} \mid$ is greater or equal to 2, although this is not the only possible case. The zero method can provide more than one solution, in particular, more than 2. The fact that dual tonal functions exist is why we write $\wp(\mathcal{A})$ instead of $\mathcal{A}$, identifying thus each set of tonal functions of a link with a subset of the power set of tonal areas.

By considering this approach, we connect a link E or a global link G to a union of tonal areas $\mathbb{A}$, the tonal area being the one that delimits the direction of the arrows in a **graph of tonal functions**. These graphs model the relationship between the tonal centers (they will be describe in detail in Section 8.1.3). The polynomial criterion classifies all polarized tonal functions in tonal areas as a function of the stabilizer of the group $E(M)$ assigning each link a particular area $\mathbb{A}$. The following section establishes how these tonal areas are defined and bounded.

5.7.7. Tonal areas of $\mathcal{P}$

Given a link E or a global link G and a set of tonal functions $\mathbb{H}^E$ or $\mathbb{H}^G$, the goal is to classify those sets in a tonal area by using the polynomial criterion $\mathcal{P}$. In order to do so, we examine the behavior of the roots of each tonal function $H(\lambda)$ and determine in which area said tonal function lies. In this way, the tonal area of a set of tonal functions is the union of the areas of each of the tonal functions of the set.

$$\mathcal{P}(\mathbb{H}^E) = \mathbb{A}(H_i(\lambda)) \cup ... \cup \mathbb{A}(H_z(\lambda)) \tag{5.23}$$

where we have z distinct tonal functions that come from the **zero method**. To study the area of each tonal function we simply study its roots, and based on that, we determine how the voices behave when we take a voicing of an optimized link. There are two cases, namely, when the roots converge or when they diverge. Here it is the full description of these two cases.

Convergent roots. We call convergent roots of a tonal function those roots that are located to the left of the stabilizer of the **Mersenne group** $M =< s >= \{s^l : l \in \mathbb{Z}\}$; that stabilizer is just 1. Thus, the definition results in

$$\lambda^- = \{\lambda \in \mathbb{R}/H(\lambda) = 0 \wedge \lambda < E(M)\} \tag{5.24}$$

Divergent roots. Similarly, we define divergent roots of a tonal function as those that are to the right of the stabilizer of the Mersenne group.

$$\lambda^{+} = \{\lambda \in \mathbb{R}/H(\lambda) = 0 \wedge \lambda > E(M)\} \tag{5.25}$$

We now consider the classification of each tonal function within an area if it is true that said tonal function verifies the polynomial criterion depending on how their algebraic multiplicities are. We call **divergent algebraic multiplicity**

$$M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda^{+}_{i}} \tag{5.26}$$

which is itself the sum of each of the algebraic multiplicities of each divergent root.

Analogously, we call the sum of the **algebraic multiplicities of the convergent roots** the convergent algebraic multiplicity of a tonal function. It is given by

$$M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda^{-}_{i}} \tag{5.27}$$

where we have as many multiplicities as different convergent roots. The convergent algebraic multiplicity is equivalent to the number of voices that go down in the optimal link any of the links of the nabla class that are optimal.

When the roots are on the stabilizer of the Mersenne group, then its algebraic multiplicity is called **static** and is given by the formula

$$M^{o}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{o}|} m_{\lambda^{o}_{i}} \tag{5.28}$$

Dominant tonal area. The polynomial criterion $\mathcal{P}$ will take each tonal function to the dominant tonal area $D^{\mathbb{C}[\lambda]}$ when the convergent algebraic multiplicity of the tonal function is two or greater and the divergent one is zero; the values of the static algebraic multiplicity does not matter. Thus, given a pair of chords or open tonalities, we can write $X \longrightarrow Y$ or $\mathcal{O}_1 \longrightarrow \mathcal{O}_2$ to indicate that the pair of structures converges. Thus, studying the tonal function is equivalent to establishing how the pair of musical structures behave.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda^{-}_{i}} \geq 2 \\ M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda^{+}_{i}} = 0 \end{cases} \tag{5.29}$$

Subdominant tonal area. The polynomial criterion will classify a tonal function within the subdominant area $S^{\mathbb{C}[\lambda]}$ if, when studying the roots of the polynomial, it is fulfilled that there are two or more roots to the right of the stabilizer of the Mersenne group and there is no root to the left, being the number of roots above the stabilizer irrelevant.

The tonal function will then be classified as subdominant in this case. In a graph of both normal and global tonal functions we will represent this classification with an inverted arrow, where if X and Y are consecutive chords we will write $X \longleftarrow Y$. In the same way for the case of open tonalities $\mathcal{O}_1$ and $\mathcal{O}_2$ we will write without loss of generality that $\mathcal{O}_1 \longleftarrow \mathcal{O}_2$.

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} \geq 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases}$$

Tonic tonal area. When a single voice moves and the rest stay, this very small movement is what we classify as divergent tonic where we use the dashed arrow $X \dashleftarrow Y$, respectively $\mathcal{O}_1 \dashleftarrow \mathcal{O}_1$when the voice on the optimal link is going up. We will use the notation for chords or open keys $X \dashrightarrow Y$, respectively $\mathcal{O}_1 \dashrightarrow \mathcal{O}_2$ when we have a voice descending in the optimal link. The tonal function in this case is convergent tonic.

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} < 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases} \text{or} \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} < 2 \end{cases}$$

Identity tonal area. When the voices do not move in the optimal link, we speak of the identity tonal area. In this case, we have the following equality.

$$M^o_{H(\lambda)} = \sum_{i=1}^{|\lambda^o|} m_{\lambda_i^o} = n \tag{5.30}$$

Non-polarized tonal area. In this case, some voices go up whereas other voices go down; they can also be on the stabilizer. The sum of the algebraic multiplicities has to be n necessarily.

Thus, to determine the area $\mathbb{A}$ of a set of tonal functions $\mathbb{H}^E$ or indistinctly $\mathbb{H}^G$ we will calculate the tonal area of each tonal function and unite them obtaining the polynomial criterion for said normal link $\mathcal{P}(\mathbb{H}^E)$ or global $\mathcal{P}(\mathbb{H}^G)$. After the calculation, we will reduce the notation to arrows in a graph of tonal functions that will allow us to guide the composition of a harmonic progression, guaranteeing convergence between centers.

5.8. About parametric tonal functions

Consider the following **formula**:

$$
\boxed{
\begin{array}{l}
H : \wp(LC_k/\sim)^2 \longrightarrow \mathbb{C}[\lambda] \\
H_{(X|Y)} = \Delta\left(s^{\sigma(L_E^{BC} - W)} + P - \lambda \cdot Id_{\max\{\dim(X),\dim(Y)\}}\right)
\end{array}
}
$$

for the case where $\dim(X) < \dim(Y)$ and $P \neq 0$, that is, the **parametric case**.

These tonal functions are a bit more complicated because the cadence matrix is not unique in the endomorphism and the transformation can be carried out by infinitely many matrices, which also have characteristic polynomials that must be extended to the set of complex numbers. Let us illustrate the whole process for the parametric case through an example.

Assume we are given chords $X = \{C, E\}$ and $Y = \{F, A, C\}$. Let's compute the link by writing the chords in their root position[2]. Next, we apply the Hungarian algorithm.

$$
E = \begin{pmatrix} \emptyset & C \\ E & A \\ C & F \end{pmatrix}; \quad L_E = \begin{pmatrix} \infty^+ & \infty^+ & \infty^+ \\ 4^- & 5^+ & 1^+ \\ 0 & 3^- & 5^+ \end{pmatrix}
$$

As it can be seen, we indicate the sign of the metric beforehand using the superscript notation in order to recover the sign more easily.

We next calculate the boxes and rewrite them in the L matrix, thus obtaining L^B (box distribution). Hence, we have the original matrix L with the boxes marked and with the signs of the metrics.

$$
L_E^F = \begin{pmatrix} \not{0} & \boxed{0} & \not{0} \\ 3 & 4 & \boxed{0} \\ \boxed{0} & 3 & 5 \end{pmatrix} = L_E^H; \quad L_E^B = \begin{pmatrix} \infty^+ & \boxed{\infty}^+ & \infty^+ \\ 4^- & 5^+ & \boxed{1}^+ \\ \boxed{0} & 3^- & 5^+ \end{pmatrix}
$$

Following the process, we have to bring the boxes to the diagonal by using the equation below

$$
L_E^{BC} = L_E^B \cdot C \tag{5.31}
$$

where the matrix L_E^{BC} is the matrix with the boxed entries. To obtain C, we calculate the transpose of L_E^B and replace the boxes with ones and and the rest with zeros.

$$
C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
$$

[2]Letter E will be used to denote the musical note E as well as the link. They should not be confused as the context provides clear disambiguation.

In this way we arrive at L_E^{BC}:

$$L_E^{BC} = \begin{pmatrix} \boxed{\infty}^+ & \infty^+ & \infty^+ \\ 5^+ & \boxed{1}^+ & 4^- \\ 3^- & 5^+ & \boxed{0} \end{pmatrix}$$

We continue with the process of changing the space until we obtain the recovery of the sign where the signs of the superscripts go down. Next, we substract matrix W (Equation 5.16) with the goal of isolating the elements on the main diagonal. This allows to compute the endomorphism matrix.

$$L_E^{BC} - W = \begin{pmatrix} \boxed{\infty}^+ & \infty^+ & \infty^+ \\ 5^+ & \boxed{1}^+ & 4^- \\ 3^- & 5^+ & \boxed{0} \end{pmatrix} - \begin{pmatrix} 0 & \infty & \infty \\ \infty & 0 & \infty \\ \infty & \infty & 0 \end{pmatrix} = \begin{pmatrix} \infty^+ & -\infty & -\infty \\ -\infty & 1^+ & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}$$

In this way we are calculating the sign recovery, which will allow us to study the direction of the voices in the optimum link. When we retrieve the sign of a matrix, we only retrieve the metrics that are on the diagonal. Thus, by raising s to each cell we would obtain:

$$s^{\sigma(L_E^{BC} - W)} = \begin{pmatrix} s^\infty & 0 & 0 \\ 0 & s^1 & 0 \\ 0 & 0 & s^0 \end{pmatrix}$$

At this point P matrix steps in as the previous is not the matrix converting the given voicing into the optimum one.

$$\Delta\left(\begin{pmatrix} s^\infty & 0 & 0 \\ 0 & s^1 & 0 \\ 0 & 0 & s^0 \end{pmatrix} + P - \lambda \cdot Id_{\max\{\dim(X),\dim(Y)\}}\right)$$

Therefore, P is a **parametic matrix** where for each infinite root (s^∞) its row is filled with ratios and its column is filled with parameters as shown below.

$$P = \begin{pmatrix} 0 & n_{12} & n_{13} \\ p_{12} & 0 & 0 \\ p_{31} & 0 & 0 \end{pmatrix}$$

The particular combination of ratios on the matrix has to transform the given voicing into the optimal one. The **parameters** can take any value because they do not contribute to the final result as they will be multiplied by zero. Because the parameters can take any value, even complex numbers, we need to extend the model in the field of the complex numbers. Furthermore, the natural question that arises is, among all the matrices that transform one voicing into another, which one will have the precise characteristic polynomial associated with the corresponding tonal function?

The **radical form** or **radical class** of a parametric transformation matrix are all those matrices whose parameters are 0.

$$\mathcal{R}[A] = \left\{ A = \left(s^{\sigma(L_E^{BC} - W} + P \right) \mid p_{ij} = 0, 1 \leq i, j \leq \max(\dim(X), \dim(Y) \right\} \tag{5.32}$$

$$\mathcal{R}\left[\begin{pmatrix} s^{\infty} & n_{12} & n_{13} \\ p_{12} & s & 0 \\ p_{31} & 0 & 1 \end{pmatrix} \right] \tag{5.33}$$

Since the determinant of a matrix coincides with that of its transpose, then by using **Laplace's theorem** we will always be able to calculate the determinant by using the expansion by minors. In this way, we can ensure that any matrix of the radical class has as characteristic polynomial the product of the elements of its diagonal. Moreover, we identify a parametric link with the tonal function that results from calculating the characteristic polynomial of one of the matrices of its radical class.

In other words, given a link E and a matrix in its radical class, the tonal function is identified with the characteristic polynomial of any of the matrices of the radical class.

$$\begin{cases} E \subset \mathcal{M}_{(\max(\dim(X), \dim(Y)) \times 2}(LC_k / \sim) \\ A = \left(s^{\sigma(L_E^{BC} - W}) + P \right), A \subset \mathcal{R}[A] \end{cases} \Longrightarrow H_E(\lambda) = \Delta(A - \lambda Id_{\max(\dim(X), \dim(Y)}) \tag{5.34}$$

5.9. Generalized tonal gravity

One of the main research ideas in this work was the modeling of the attraction between tonal centers. This attraction was modelled through the Hungarian algorithm applied to the link matrix. Once we have taken the boxes to the diagonal, if we add their elements we obtain the value of nabla, which in turn determines an optimal link class in the link class. The nabla class is defined as follows.

$$[E]_{\nabla} = \{ E \subset [E] \mid \nabla(E) < \nabla(E^*)), \forall E^* \subset [E] \} \tag{5.35}$$

The **nabla class** is the set of links that have the minimum nabla distance; recall that the nabla distance is measured by taking into account all of the links of the class of E. Once we have computed the nabla of a link, we can retrieve how the voices move globally by retrieving the value of each metric individually. By doing this we arrive at the concept of **generalized tonal gravity**.

Let $E \subset [E]$ be a link where the class has the form $[E] = (X|Y)$ and the link is given as $E = (X^r|Y^r)$. Here we use the superscript r that arranges the chords in root position by distributing them in the entries of each column. If they are not conventional chords or are tonal centers what is being measured, then the r indicates that we will arrange the chords following the circle of fifths in a clockwise direction.

Given that the model consideres the cases where $\dim(X) \neq \dim(Y)$, it turns out that not all gravities fall on $\mathbb{R}$. Actually, the issue here is how to measure the gravities for the general case. That measurement involves the use of a set $\underset{\infty}{\overset{\infty}{\mathbb{R}}}$ of infinity arithmetic rules. The **symbol** $\underset{\infty}{\overset{\infty}{\mathbb{R}}}$ represents the union of two infinities. However, the usual infinity arithmetic (where $\infty + \infty = \infty$) is now replaced by an infinity arithmetic in which these infinities are treated like ordinary numbers; for example, $\infty + \infty = 2\infty$.

$$\underset{\infty}{\overset{\infty}{\mathbb{R}}} = (\mathbb{R} \cup \{+\infty, -\infty\}, A^{\infty}) \tag{5.36}$$

We then define the **generalized tonal gravity** as the recovery of the sign of the metrics of the nabla function:

$$\begin{aligned} G : \mathcal{M}_{\max\{\dim(X),\dim(Y)\}\times 2}(LC_k/\sim) &\longrightarrow \underset{\infty}{\overset{\infty}{\mathbb{R}}} \\ G(E) &= \sigma(\nabla(E)) \end{aligned} \tag{5.37}$$

Let $G(E)$ be the **generalized tonal gravity** of a link, then we carry out the following clarification:

$$\begin{aligned} \sigma : \underset{\infty}{\overset{\infty}{\mathbb{R}}} &\longrightarrow \underset{\infty}{\overset{\infty}{\mathbb{R}}} \\ \sigma(\nabla(E)) &= \sum_{i=1}^{n} \left(\Omega \int_{E_{i_1}}^{E_{i_2}} \frac{1}{\phi} d\phi \right)_{\Delta} \end{aligned} \tag{5.38}$$

where the expression $\left(\Omega \int_{E_{i_1}}^{E_{i_2}} \frac{1}{\phi} d\phi\right)_{\Delta}$ has the value of the minimum distance between classes but with the sign recovered. This means that for two classes E_{i_1} and E_{i_2}, the sign of the formula $\left(\Omega \int_{E_{i_1}}^{E_{i_2}} \frac{1}{\phi} d\phi\right)_{\Delta}$ is recovered as in matrix U.

We define **tonal gravity** then as the retrieval sign of the smallest nabla values of the links that are optimal solutions provided by the Hungarian algorithm:

$$\begin{aligned} G^{\infty} : \mathcal{M}_{\max\{\dim(X),\dim(Y)\}\times 2}(LC_k/\sim) &\longrightarrow \underset{\infty}{\overset{\infty}{\mathbb{R}}} \times \overset{g}{\cdots} \times \underset{\infty}{\overset{\infty}{\mathbb{R}}} \\ G^{\infty}(E) &= \left(\mathrm{tr}(\sigma(L_E^{B_1 C_1} - W)), \ldots, \mathrm{tr}(\sigma(L_E^{B_g C_g} - W)) \right) \end{aligned} \tag{5.39}$$

where $\underset{\infty}{\overset{\infty}{\mathbb{R}}}$ is the real line extended with $\pm\infty$ and the ∞-arithmetic (said arithmetic was defined in Section 5.6.2) and g is the **dimension of the gravity vector**.

Since the calculation of infinities in this case comes from an **improper integration**, tonal gravity will be expressed with the term infinity as a number, and it will even be possible to order them and define an order on the gravity values. This can be done because infinity is represented by a metric that is equal to infinity while coexisting at the same time with finite metrics. Thus, by defining the infinity arithmetic, we can compute the gravity between tonal centers of different dimensions.

5.10. Tonal gravity

Since generalized tonal gravity can be associated to any link, then it is possible to associate the concept of generalized tonal gravity to a class so that the tonal gravity of a link can be fully specified. The equation below reflects this idea.

$$G^{\infty}([E]) = G([E]_{\nabla}) \tag{5.40}$$

The tonal gravity of a link class is computed by taking the links of the nabla class and obtaining their gravity values.

When discussing tonal gravity, we do refer to one of the optimal links in the class of a link. Gravity values are not unique, since the tonal function can be dual (due to the presence of many solutions output by the Hungarian algorithm). Thus, to delimit the concept, we define the tonal gravity of a link as a vector in $\begin{pmatrix}\infty\\ \mathbb{R}\\ \infty\end{pmatrix}^g$ corresponding to the values obtained when calculating the generalized gravity values of a link in the nabla class. Some of the properties of generalized gravity also holds for tonal gravity; for example:

$$G(E) = \sigma(\nabla(E)) \leq \nabla(E)$$

The only thing that is noted is that if the function is dual, then we will have the equality between vectors of the following form,

$$G^{\infty}(E) = (G^{\infty}(E_1^o), \ldots, G^{\infty}(E_g^o))$$

where the integer g is the number of distinct optimal solutions, which can arise either by simple inspection or by using the Zero method. It is posible to use other vectors that result from permuting the components of $G^{\infty}(E)$, since there is no order in the solutions of the Zero method, although we will use the gravity vector following the appearance of solutions of the zeros from left to right and from top to bottom in the L matrix. We will use that order by agreement. Furthermore, it can be proved that if $\lambda \neq 0$ and the tonal function is not dual, it holds

$$G^{\infty}([E]) = G^{\infty}([\lambda E]) \tag{5.41}$$

which is a consequence of the color theorem (see Section 2.3)

Let us clarify here that the scalar product of a class is the class with all its elements multiplied by said scalar:

$$\lambda E = \lambda(X^r|Y^r) = (\lambda X^r|\lambda Y^r) \tag{5.42}$$

Let us give an example. Starting from the link class $[E]$ we write a link with both chords in root position. We calculate the matrix of said link L_E:

$$[E] = (C \mid F) \implies E = \begin{pmatrix} G & C \\ E & A \\ C & F \end{pmatrix} \implies L_E = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 5 & 1 \\ 0 & 3 & 5 \end{pmatrix}$$

We follow the usual process using the Hungarian algorithm on L_E until we reach a distribution of boxes, described in L^B, where in this case said distribution is unique.

$$L_E^F = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & 3 & 5 \end{pmatrix} \Longrightarrow L^H = \begin{pmatrix} 3 & \boxed{0} & 0 \\ 3 & 4 & \boxed{0} \\ \boxed{0} & 3 & 5 \end{pmatrix} \Longrightarrow L_E^B = \begin{pmatrix} 5 & \boxed{2} & 2 \\ 4 & 5 & \boxed{1} \\ \boxed{0} & 3 & 5 \end{pmatrix}$$

Once we have reached L^B, we calculate C where $C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ using the usual calculation.

We recover the sign of the metric once we have subtracted W getting the matrix $\sigma(L_E^{BC} - W)$.

$$\sigma(L_E^{BC} - W) = \begin{pmatrix} \sigma(\Delta(G,A) & -\infty & -\infty \\ -\infty & \sigma(\Delta(E,F) & -\infty \\ -\infty & -\infty & \sigma(\Delta(C,C) \end{pmatrix}$$

Substituting the value of the metrics with its recovered sign, we find a completely numerical expression of the previous matrix, so it is enough to calculate the trace to reach the gravity value.

$$\sigma(L_E^{BC} - W) = \begin{pmatrix} 2 & -\infty & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}$$

Taking the trace on the previous matrix, we reach the value of gravity, where the value is positive and also coincides with the nabla function because it is a polarized link.

$$G^\infty(E) = \operatorname{tr}(\sigma(L_E^{BC} - W)) \Longrightarrow G^\infty(E) = 3$$

5.11. Expression of tonal gravity in $\overset{\infty}{\underset{\infty}{\mathbb{R}}}$

When the gravity of a link where $\dim(X) \neq \dim(Y)$ is measured, an infinite metric will appear at some point. The gravity of a link with infinity could take the value

$$G^\infty(E) = \infty - 2$$

Other examples would be:

$$G^\infty(E) = 3\infty - 1; \quad G^\infty(E) = -2\infty + 1; \quad G^\infty(E) = -6\infty$$

Thus, we can add gravities or based on these to define other operations in $\overset{\infty}{\underset{\infty}{\mathbb{R}}}$. For example, if we add $G^\infty(E) = 2\infty + 3$ with $G^\infty(E^*) = 2\infty + 2$, it yields $4\infty + 5$.

These examples are very simple, but they are written with the awareness that $\overset{\infty}{\underset{\infty}{\mathbb{R}}}$ ($\mathbb{R}$ double infinity) is an unusual set. This set fulfils its role in the understanding the movement between the voices and the convergences between the tonal centers of different dimension.

5.12. Gravity and polarization

Since $G^{\infty}(E) = \operatorname{tr}\left(\sigma(L_E^{BC} - W)\right)$, when the tonal function is **not dual**, the absolute value of $G^{\infty}(E)$ will coincide with $\nabla(E^o)$ if only if $H_E(\lambda)$ is polarized.

$$\mid G^{\infty}(E) \mid = \nabla(E^o) \Longleftrightarrow H_E(\lambda) \subset P^{\mathbb{C}[\lambda]} \tag{5.43}$$

where $P^{\mathbb{C}[\lambda]}$ are the polarized tonal areas, that is,

$$P^{\mathbb{C}[\lambda]} = \{T^{\mathbb{C}[\lambda]}, D^{\mathbb{C}[\lambda]}, S^{\mathbb{C}[\lambda]}, I^{\mathbb{C}[\lambda]}\}$$

Two cases may then arise:

(1) All the voices move in the same direction in an ascending direction or stay at $E(M)$.

This is equivalent to the fact that the sum of the divergent algebraic multiplicities is greater than zero and the sum of the convergent ones is zero.

(2) All the voices move in the same direction downwards, which is equivalent to the fact that the sum of the divergent algebraic multiplicities is greater than zero and the sum of the convergent ones is greater than zero.

$$|G^{\infty}(E)| = \nabla\left(E^o\right) \Leftrightarrow \left(\left\{\begin{array}{c} \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} \geq 0 \\ \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{array}\right\}\right) \text{or} \left(\left\{\begin{array}{c} \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} \geq 0 \\ \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{array}\right\}\right) \tag{5.44}$$

Regardless of whether the tonal function is polarized, we have that

$$|G^{\infty}(E)| \leq \nabla(E^o) \tag{5.45}$$

or if we unpack the absolute value

$$-\nabla(E^o) \leq G^{\infty}(E) \leq \nabla(E^o)$$

Then, the gravity values of a link is always in the closed interval in $\overset{\infty}{\underset{\infty}{\mathbb{R}}}$. In this way the value of gravity for a link $G^{\infty}(E)$ is bounded by the value of nabla $\nabla(E^o)$ and its opposite$-\nabla(E^o)$ in the set $\overset{\infty}{\underset{\infty}{\mathbb{R}}}$.

5.13. The H-tree

The number of solutions given by the Hungarian algorithm classifies the tonal functions into either dual or normal functions. Depending on the relation between the dimensions of the tonal centers, the tonal functions can be parametric, if $\dim(X) < \dim(Y)$, or non-parametric if $\dim(X) \geq \dim(Y)$. Once these two categories are determined, seven types of tonal functions arise depending on the position of the exponents of the roots in the characteristic polynomial.

Identity endomorphism (polarized)	All exponents are zero
Positive-definite endomorphism (polarized)	All exponents are positive
Negative-definite endomorphism (polarized)	All exponents are negative
Mixed-definite endomorphism	Some exponents are negative, and some are positive, there are no zero exponents
Positive-semidefinite endomorphism (polarized)	All exponents are either positive, or zero
Negative-semidefinite endomorphism (polarized)	All exponents are either negative, or zero
Mixed-semidefinite endomorphism	Some exponents are either positive, or negative, or zero, and the three cases occur

Table 5.2: Classification of tonal function

These types are further classified into polarized and non-polarized. The H-tree is a search tree that helps visualize all the possibilities for tonal function classification in the hypervolumetric tonal function model. The tree is highly symmetrical as the reader can notice. The left subtree takes care of the non-parametric case while the right subtree of the parametric case.

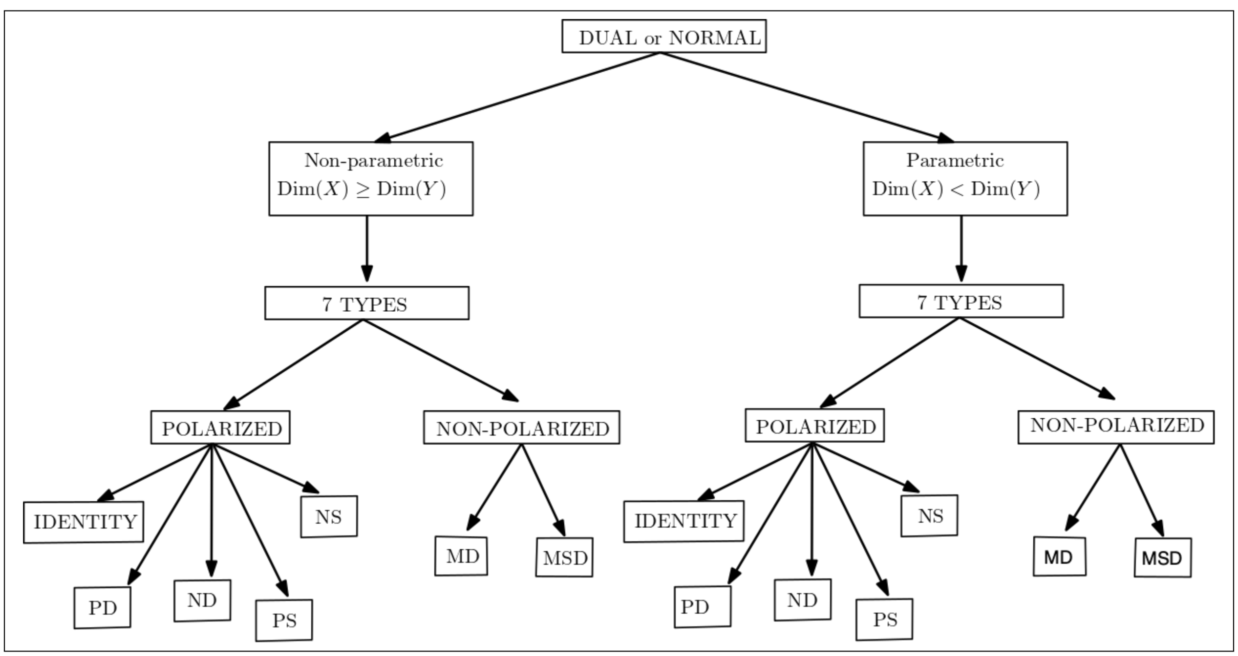


Figure 5.3: Classification tree for tonal functions

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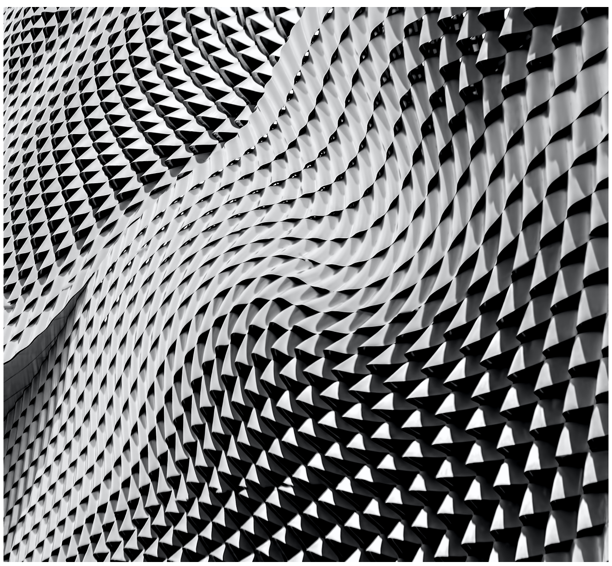
Chapter 6

A Definition of global tonal function



Ricardo Gómez

https://unsplash.com/photos/9AdeEdYB2yk

6.1. A definition of global tonal function

The musical discourse progresses through the variation of the musical material. A very important source of musical variation in the Western traditions and many others is harmonic variation. Tension in music has been elicited by changing tonal centers or by modulating between tonalities just to name but two important cases. In the case of Western tradition, modulating to farther and farther keys has been the thrive to make new music and construct new theory (even to build new instruments that could take on the new modulation necessities). Modulation and in general change of tonal centers have been studied since long ago by many authors. In the case of Western music, its harmony practice comprises from the early 1600 to present. It evolved from earlier musical practices such as polyphony of the late Middle Ages and the Renaissance. Later, **Jean-Philippe Rameau** [Ram22] wrote his *Traité de l'harmonie* (1722) where he lays the foundation of modern harmony in the commo period practice. Other authors wrote treatises of harmony since then such as **Charles Macpherson** [Mac20], **Arnold Schoenberg** [Sch48], **Walter Piston** [Pis50], or **Vicent Persichetti** [Per95].

In this chaper, we focus our attention on the problem of how to model the relationships between tonal centers and out of it how to define a **global tonal function**. This idea of global tonal function builds on previous ideas by a number of authors. For example, **Arnold Schoenberg** studied the **structural functions of harmony** [Sch69]. More recently, **William Caplin** [Cap83] addressed the harmony function in terms of metric accent while **Alexander Rehding**[Reh11] looked into the rule formation in harmony. **Topi Jarvinen** [Jar95] studied global tonalities in the context of jazz music. **Caroline Krumhansl** [KC10] examined this question from a psychological standpoint.

When we want to study how two musical sections written in different open keys are related, that is, we want to study the **modulation** between sections. To this end, we can also use the same concept of tonal function but generalized to sections. Indeed, once the key of each section is identified, it is enough to apply the formula of tonal functions on the open keys. We wonder, for example, if one key modulates to another by tone or if a symmetrical scale resolves or not to a particular key.These are typical questions within the framework of global tonal functions.The most interesting fact about the global tonal functions is that they solve many important questions about modulation. Using the universal link matrix U (refer to Section 5.6), it is possible to precisely know how the open tonalities modulate regardless of their dimension.

We first study how the class of a link involving two open tonalities is like. Let $[G] = ((\mathcal{O}_k/\sim)_1 \mid (\mathcal{O}_k/\sim)_2)$ be two open tonalities. The **static function theorem** (see Section 4.2.1 and also Section 6.3) states that regardless of the placement of the classes in a link, the tonal function remains the same. As a consequence, for our purposes, we take a link where the classes follow the circle of fifths, say, $G = (\mathcal{O}_1^r \mid \mathcal{O}_2^r)$. In order not to unnecessarily overcomplicate the notation and using the known results, we will omit the superscripts to indicate the order in which the classes appear in the open key and we will directly write $\mathcal{O}_1$ and $\mathcal{O}_2$ within the link, assuming that their classes follow the circle of fifths in clockwise order from bottom to top. Thus, we can write $G = (\mathcal{O}_1 \mid \mathcal{O}_2)$.

We then define the **hypervolumetric global tonal function** between two open tonalities for a global link G as the complex polynomial that arises from applying the general formula of non-dual tonal functions on open tonalities:

$$H_G(\lambda) = H_{(\mathcal{O}_1|\mathcal{O}_2)}(\lambda) = \Delta(s^{\sigma(L_G^{BC}-W)} + P - \lambda Id) \tag{6.1}$$

By using the U matrix, we can calculate any modulation relationship between tonal centers. Later in this work, the six fundamental cases of modulation for tonalities in the classical sense are studied and analyzed. These cases determine the modulation between Lydian tonal centers and in parallel, by reversing the order of the chords in each link, we can deduce the six remaining modulations (details will be provided in Section 7.1). This is a powerful enough tool for writing music at a professional level, although given the high level of abstraction of the method, the user will quickly appreciate that he can compute modulations of any open key himself by using the U matrix.

6.2. A definition of integral global function

Once we know the concept of tonal function, be it the normal one or the global one, we would like to consider operations on those tonal functions. It is not intended to generate a list of all the mathematical properties satisfied by each function $H_G(\lambda)$ because they are analogous to those of complex-variable polynomials.What is intended in this section is to highlight the idea of the integral of a global tonal function, given the connection of said integral with the roots of the tonal function. Another point of view to consider would be that of studying extrema. Instead of studying the intersection of the polynomials with the λ axis, we can study the maxima and minima of the integral global tonal function. We call the integral $I(H_G(\lambda))$ and we simply integrate as a function of λ.

$$I(H_G(\lambda)) = \int H_G(\lambda)\, d\lambda = \int H_{(\mathcal{O}_1|\mathcal{O}_2)}(\lambda)\, d\lambda \tag{6.2}$$

In this way, the integral of a tonal function has characterized the relative maxima and minima as the transformation values between voices when voice optimization has been carried out. In the case the tonal function is dual, there will be as many integrals as there are tonal functions.

At this point, the reader will wonder what is the reason for an integral formulation of the tonal function when polynomials are sufficient to characterize the tonal function. The reason resides in the fact that the transformation of the metrics into integrals eased the inclusion of infinity in the Hungarian algorithm. Although integrals are not used in the theoretical core of this work, it is intuitive to see that definite integrals can bear fruit in the future as to the classification of tonal functions, specifically for the case of global ones. The connection between computation of tonal functions through integrals with other fields of mathematics is left for the future, since the model has been proven functional and efficient for the generation of harmonic progressions and the analysis of pre-existing ones.Thus, for a parametric matrix P, two open keys $\mathcal{O}_1$, $\mathcal{O}_2$, and the well-known matrix W, the general formula is a complex variable polynomial given by $\Delta(s^{\sigma(L^{BC}_{(\mathcal{O}_1|\mathcal{O}_2)}-W)} + P - \lambda Id)$. Thus, its integral would remain as usual.

$$\int H_{(\mathcal{O}_1|\mathcal{O}_2)}(\lambda)\, d\lambda = \int \Delta(s^{\sigma(L^{BC}_{(\mathcal{O}_1|\mathcal{O}_2)}-W)} + P - \lambda Id)\, d\lambda \tag{6.3}$$

Thus, we have established not only the general formula for the calculation of tonal functions but also a connection with integrals that contains all the necessary information to represent them as other mathematical objects. In this way and without a more ambitious present purpose, the integral form of the global tonal functions remains determined.

6.3. Static function theorem

The **static function theorem** states that the hipervolumetric tonal function does not depend of the tone or the inversion of a pair of chords. Let i, k be two reference tones $i \neq k$, and let X_i, X_k, Y_i, Y_k be some chords defined on scale degrees (for example, $X_i = \flat \, \text{II}_i\Delta$). Moreover, let a, m, c, and b different inversion of the chords. In this section we will discuss and analyze this theorem in depth. This result allows to look at the tonal function as an algebraic invariant that transcends voice leading itself. In other words, even if the hypervolumetric tonal function is obtained by optimizing the corresponding voice leading, it is still there when voice leading obtained is not optimal.

Thus, we will prove that the tonal function is the same when the position of the chord classes varies. Put it in mathematical notation,

$$\boxed{\Phi[E_{(X_i^a|Y_i^m)}] = \Phi[E_{(X_k^c|Y_k^b)}]} \tag{6.4}$$

First, we will prove that the tonal function of a link is preserved by transposition, that is to say, the tonal function does not change depending on the tone. This result must be a requirement for the consistency of the model since without it the system of tonal functions could not be generalized to any tone, and that would pose a serious problem. Moreover, that would not be consistent with the musical phenomena. We want to prove the following equation

$$\boxed{\Phi[E_{(X_i|Y_i)}] = \Phi[E_{(X_k|Y_k)}]} \tag{6.5}$$

By using the **color theorem** (see Section 2.3), it is clear that if Δ_{ij} is each entry of $L_{E_{(X_i|Y_i)}}$ and Δ^*_{ij} is each entry of $L_{E_{(X_k|Y_k)}}$ then:

$$L_{E_{(X_i|Y_i)}} = L_{E_{(X_k|Y_k)}}$$

As we know from the color theorem that each input of the two L matrices is going to be identical, then we only need to build this first result following all the construction steps of the tonal function from two L matrices. In this way starting from the equality of matrices L we calculate the matrices L^H, which also happens to be equal.

$$L^H_{E_{(X_i|Y_i)}} = L^H_{E_{(X_k|Y_k)}}$$

Therefore, by using the general formula for tonal functions

$$H_E(\lambda) = H_{(\mathcal{O}_1|\mathcal{O}_2)}(\lambda) = \Delta(s^{\sigma(L_E^{BC} - W)} + P - \lambda Id)$$

we can find the matrices of the endomorphisms and check their equality, that is,

$$C_{\mathbb{E}}^{(X_i|Y_i)} = C_{\mathbb{E}}^{(X_k|Y_k)}$$

Again, by using the color theorem for each entry of the L matrices, we conclude that by means of the general formula of the tonal function, the resulting endomorphism matrix is identical when changing the tone. In this way, we will complete the process in the formal proof by computing the characteristic polynomial.

We understand that the notation of superscripts on the matrices of the endomorphism is practical to understand the rest of the process. Then, we subtract the identity matrices multiplied by λ from both sides.

$$C_{\mathbb{E}}^{(X_i|Y_i)} - \lambda Id = C_{\mathbb{E}}^{(X_k|Y_k)} - \lambda Id$$

Taking determinants on both sides of the equation, we are able to construct the tonal function for the link in tone i and tone k, in those cases where they are not the same.

$$\Delta(C_{\mathbb{E}}^{(X_i|Y_i)} - \lambda Id) = \Delta(C_{\mathbb{E}}^{(X_k|Y_k)} - \lambda Id)$$

Recovering the previous formula, we obtain the desired result:

$$\boxed{\Phi[E_{(X_i|Y_i)}] = \Delta(C_{\mathbb{E}}^{(X_i|Y_i)} - \lambda Id) = \Delta(C_{\mathbb{E}}^{(X_k|Y_k)} - \lambda Id) = \Phi[E_{(X_k|Y_k)}]}$$

This formula proves that the hypervolumetric tonal function is preserved by change of key in the link.

The second part proves that the change of the disposition of the chords does not affect the hipervolumetric tonal function. Let $a \neq c$ and $m \neq b$ be two different inversions for chords X and Y, respectively. We want to prove the following formula

$$\boxed{\Phi[E_{(X^a|Y^m)}] = \Phi[E_{(X^c|Y^b)}]}$$

A change in the inversion of a chord is the same as permuting the rows or columns of the L matrix. Therefore, if we want to compute the tonal function of the link $E_{(X^a|Y^m)}$, we can use the expression $\Sigma_r L \Sigma_c$, where matrices Σ_r, Σ_c are matrices that exchange rows and columns, respectively. We can see that although the calculation of the tonal function comes from the voice leading, it is, instead, the presence of octave classes that allows us to determine the tonal function for a link.

$$L_{E_{(X^c|Y^b)}} = (\Sigma_r L \Sigma_c)_{E_{(X^a|Y^m)}} \tag{6.6}$$

In this way, using the row and column transformation matrices for the L matrix, we study how the arrangement of the chords has changed. Since we have already established the connection between the change of the positions of the classes for the antecedent and the consequent chord and the transformation of a matrix L, then we proceed to solve said matrix by using the Hungarian algorithm.

Permutation of the rows and columns in L^H followed by the box distribution is commutative with the box distribution followed by the permutation of rows and columns. We can then write

$$(\Sigma_r L \Sigma_c)^H = (\Sigma_r L^H \Sigma_c)$$

Hence, we have the equivalence:

$$L^H_{E_{(X^c|Y^b)}} = (\Sigma_r L \Sigma_c)^H_{E_{(X^a|Y^m)}} \tag{6.7}$$

In this part it is shown that the order does not matter when solving a L matrix and swapping its rows or columns that is to say that the solution of the matrix by the Hungarian algorithm is maintained independently of the transformation of rows or columns carried out by the sigma matrices.

In this way, if we use the general formula for tonal functions $H_E(\lambda) = \Delta(s^{\sigma(L_E^{BC}-W)} + P - \lambda Id)$ and utilize the equality $L^H_{E_{(X^c|Y^b)}} = (\Sigma_r L \Sigma_c)^H_{E_{(X^a|Y^m)}}$, we arrive at the equivalence between tonal functions.

Since the solutions are preserved regardless of changing the order of rows or columns for matrices L, then we can establish the equality between tonal functions when assuming that L^B is unique .Thus, it is seen that the arrangement of the chords does not affect the existence of the tonal function. We can then substitute the equality between these matrices in the general formula for the tonal function and thus establish that it holds the following.

$$\begin{aligned} H_{E_{(X^c|Y^b)}}(\lambda) &= \Delta(s^{\sigma(L^{BC}_{E_{(X^c|Y^b)}}-W)} + P - \lambda Id) = \Delta(s^{\sigma((\Sigma_r L \Sigma_c)^{BC}_{E_{(X^a|Y^m)}}-W)} + P - \lambda Id) \\ &= H_{E_{(X^a|Y^m)}}(\lambda) \end{aligned}$$

As a matter of fact, if we remove the intermediate terms, equality is established between tonal functions of links that belong to the same link class but have their classes in a different way.

$$H_{E_{(X^c|Y^b)}}(\lambda) = H_{E_{(X^a|Y^m)}}(\lambda) \tag{6.8}$$

This situation holds for non-dual tonal functions. As for dual tonal functions, the zero method produces z solutions that are actually different distributions of boxes on the matrix L^H. This solutions are also preserved by changes of inversions in chords. Hence, we can extend the result to dual tonal functions, which makes our results quite general in scope.

Since the solutions of the different matrices are the same because an optimization problem is being solved, then it is necessary that in the case of several solutions that reach the minimum, then we will have different placements for the boxes on the original L matrix. Without loss of generality we can write the equality between the set of tonal functions of a link and another that underwent a permutation in the placement of the classes of one or both chords. The result is still valid for the case of open tonalities.

$$\boxed{\mathbb{H}^{E_{(X_i^c|Y_i^b)}} = \mathbb{H}^{E_{(X_i^a|Y_i^m)}}} \tag{6.9}$$

We saw above that the tonal function, even in the dual case, remains invariant when we transport the link. This established both results within the same set equivalence. Thus, we can change one of the subscripts i to a different subscript k indicating that a pitch transformation of the entire link does not affect the tonal function.

$$\boxed{\mathbb{H}^{E_{(X_k^c|Y_k^b)}} = \mathbb{H}^{E_{(X_i^a|Y_i^m)}}}$$

This result allows us to understand how the transposition of links or the change of the placement of the octave classes does not affect the calculation of the tonal function, regardless of duality. The result itself is known as the **static function theorem**, and it states that the tonal function is a general property of the link classes.

It is extremely important because in many cases the voice leading is not always optimal, but in this section we saw that the tonal function, even though it is being calculated from the optimization of the voices, ultimately depends on the presence of the classes in each chord.

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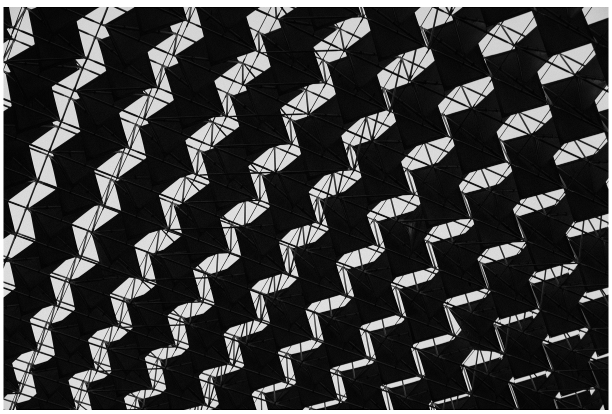
Chapter 7

Modulation



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7.1. Modulation

The classical definition of **modulation** in music refers to the process of changing from one key to another within a musical composition. This technique is fundamental to the structure and development of Western classical and popular music as well as other musical traditions. Modulation adds variety, tension, and interest to a piece, and it allows composers to explore different emotional and tonal landscapes.

Early Western music often revolved around modes rather than the major and minor scales we are familiar with today. Modulations during this period were less frequent and often involved a shift between related modes. The **Baroque** era saw the establishment of tonality, with major and minor scales becoming predominant. Composers like **Johann Sebastian Bach**, **George Philipp Telemann**, and **George Frideric Handel** experimented with modulation, developing more sophisticated harmonic progressions. Composers from the **Classical Era** (1750-1820) brought modulation to the forefront, using it to structure symphonies, sonatas, and other forms. **Mozart**'s operas and piano sonatas, **Haydn**'s symphonies, and **Beethoven**'s compositions are notable for their innovative use of modulation.

Romantic composers like **Franz Schubert**, **Franz Liszt**, and **Richard Wagner** pushed the boundaries of modulation, exploring a wider emotional range. Wagner, in particular, used continuous modulation to blur the lines between keys, creating a sense of continuous tension and release. In the 20th century, **Debussy**'s impressionistic compositions and the expressionistic works of composers like **Arnold Schoenberg** introduced novel approaches to tonality and modulation. **Serialism**, a method of composition using a series of musical pitches, emerged as a way to organize pitch material without a tonal center.

Jazz embraced frequent and sometimes rapid modulations, with musicians like **Duke Ellington** and **John Coltrane** exploring new harmonic territories. In popular music, modulation became a common device for creating interest and excitement in songs, especially in genres like rock and pop. Moreover, contemporary composers often draw from a wide range of musical traditions, freely modulating between diverse tonalities. Electronic and experimental music have further expanded the possibilities of modulation, sometimes transcending traditional tonal structures.

Musical modulation has evolved over centuries, reflecting changes in musical tastes, styles, and technologies. From the rigid structures of the Baroque era to the experimental tonalities of the 20th century and beyond, modulation remains a powerful tool for composers to convey emotion, create tension, and shape the overall narrative of their musical works.

From a theoretical standpoint, modulation involves a change of key within a musical composition. It is a fundamental concept in music theory that affects the overall harmonic structure and tonal organization of a piece. Understanding modulation involves grasping key theoretical elements such as tonal centers, chord progressions, and harmonic relationships.

We briefly lay out a few key theoretical aspects of modulation:

(1) **Tonal centers**: In tonal music, a piece typically revolves around a central pitch or tonal center, often represented by a chord or by scale.

(2) Modulation involves a **shift** from one tonal center to another, achieved through a series of chord changes that establish a new key, or directly.

(3) **Pivot Chords**: Pivot chords are chords that exist in both the current and destination keys, facilitating a smooth transition. Composers often use pivot chords strategically to connect different keys and maintain coherence.

(4) **Modulation types**: There are many types of modulation. The most standard ones are direct modulation, common chord modulation, sequential modulation, modulation by interval, use of secondary dominants, and chromatic modulation.

Modulation is often linked to the larger structure of a musical composition, impacting the overall form. In the sonata form, for instance, the development section commonly involves modulation to explore new tonal territory.

In summary, modulation is a complex and versatile aspect of music theory, involving the manipulation of tonal centers, chord progressions, and harmonic relationships. It is a crucial tool for composers to create tension, release, and variety within a musical work. Theoretical analysis of modulation enhances our understanding of how composers achieve these effects and contributes to the broader appreciation of musical structure.

Modulation has always been of interest to musicians, composers, and music theorists. At each musical era, relevant authors wrote treatises and books about modulation. We mention the following among the most relevant both from a historical and conceptual standpoints: **Jean-Philippe Rameau** [Ram22], **Charles Macpherson** [Mac20], **Arnold Schoenberg** [Sch48, Sch69], **Walter Piston** [Pis50], **George Russell**[Rus01] **Vicent Persichetti** [Per95].

7.2. Case 1: $G_L \longrightarrow C_L$

In this chapter, we study how to compute global tonal functions between open keys. The definition and modeling of global tonal functions was addressed thoroughly in Chapter 6. To compute the global tonal functions between two open keys $\mathcal{O}_1$ and $\mathcal{O}_2$, we use the U matrix for the tempered system. Matrix U can have another form depending upon the tuning system under consideration, but in this context we will use the U matrix as it appears below.

$$U = \begin{pmatrix}
\Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\
F & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- \\
Bb & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ \\
Eb & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- \\
Ab & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\
Db & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\
Gb & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\
B & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\
E & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\
A & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\
D & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- \\
G & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ \\
C & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0
\end{pmatrix} \tag{7.1}$$

In U matrix, the sign recoveries of each metric have been annotated beforehand so that it is easier to see the whole process. We annotate the sign recovery as an exponent on each metric in such a way that it is visible to the naked eye. The U matrix helps us study the modulation between tonal centers and it is a fundamental tool to write musical sections once the mechanics of their operation is well acquired. Therefore, we will use the general formula of the tonal function on this matrix:

$$H_G(\lambda) = H_{(\mathcal{O}_1,\mathcal{O}_2)}(\lambda) = \Delta(s^{\sigma(U_G^{BC} - W)} + P - \lambda Id) \tag{7.2}$$

Only six cases arise for modulation instead of the expected twelve cases. Indeed, the tonal function of a link is the **symmetrical exponent tonal function** of its retrograde link; this tonal function is also called **retrograde tonal function**. This is true because the optimal voice leading between two chords X, Y is the same when X is the antecedent and Y the consequent than when Y is the antecedent and X is the consequent. In this way we inspect each case looking for whether the tonal centers modulate or not. If the readers are composer with some experience, they will immediately notice that each case of modulation coincides with some work that they themselves may have analyzed or with some typical modulation and they will find in this section the explanation of these modulations in music.

Given the extension of this work, no more cases of modulation are covered, but with this method, the interested reader may compute the global tonal function between melodic and harmonic minor scales and symmetrical scales, among others.

The cases have been written in a general way using open tonalities as if we were going to calculate open tonalities but in reality we are covering all the cases for compact tonalities, since a compact tonality is in turn an open tonality, although the name is not given. This facilitates the method using U to calculate the modulation in general and to be able to create sections.

For the first case we consider the global link $G = (\mathcal{O}_1 \mid \mathcal{O}_2)$ where $\mathcal{O}_1 = G_L$ and $\mathcal{O}_2 = C_L$. Classes that are outside of the open keys are empty classes. Consequently, the distance between an empty class and an arbitrary non-empty class is infinite and the distance between two empty classes is zero. We use the notation $U_G = U_{(\mathcal{O}_1|\mathcal{O}_2)}$ to name the transformation of the matrix U when we have delimited the open keys between which we want to calculate the tonal function.

$$U_{(\mathcal{O}_1|\mathcal{O}_2)} = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Bb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Eb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Ab & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Db & \infty & \infty & \infty & \infty & \infty & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\ Gb & \infty & \infty & \infty & \infty & \infty & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ B & \infty & \infty & \infty & \infty & \infty & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ E & \infty & \infty & \infty & \infty & \infty & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\ A & \infty & \infty & \infty & \infty & \infty & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\ D & \infty & \infty & \infty & \infty & \infty & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- \\ G & \infty & \infty & \infty & \infty & \infty & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ \\ C & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{pmatrix} \tag{7.3}$$

Once we have reached $U_{(\mathcal{O}_1|\mathcal{O}_2)}$ for two Lydian tonal centers, then we proceed to apply the Hungarian algorithm. We first compute $U^F_{(\mathcal{O}_1|\mathcal{O}_2)}$.

$$U^F_{(\mathcal{O}_1|\mathcal{O}_2)} = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Bb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Eb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Ab & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Db & (\infty-1) & (\infty-1) & (\infty-1) & (\infty-1) & (\infty-1) & 4^+ & 1^- & 2^+ & 3^- & 0^+ & 5^= & 0^- \\ Gb & \infty & \infty & \infty & \infty & \infty & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ B & \infty & \infty & \infty & \infty & \infty & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ E & \infty & \infty & \infty & \infty & \infty & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\ A & \infty & \infty & \infty & \infty & \infty & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\ D & \infty & \infty & \infty & \infty & \infty & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- \\ G & \infty & \infty & \infty & \infty & \infty & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ \\ C & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{pmatrix} \tag{7.4}$$

In this case the solution is unique and we see that $U^F_{(\mathcal{O}_1|\mathcal{O}_2)} = U^H_{(\mathcal{O}_1|\mathcal{O}_2)}$. Hence, we draw the solutions on U^F and we take the submatrix bounded by the tonal centers to apply the general formula for calculating tonal functions. This array is a global link array L.

$$U^F_{(\mathcal{O}_1|\mathcal{O}_2)} = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Bb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Eb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Ab & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Db & (\infty-1) & (\infty-1) & (\infty-1) & (\infty-1) & (\infty-1) & 4^+ & 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} \\ Gb & \infty & \infty & \infty & \infty & \infty & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ B & \infty & \infty & \infty & \infty & \infty & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ E & \infty & \infty & \infty & \infty & \infty & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- \\ A & \infty & \infty & \infty & \infty & \infty & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ \\ D & \infty & \infty & \infty & \infty & \infty & 4^+ & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- \\ G & \infty & \infty & \infty & \infty & \infty & 1^- & 4^+ & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ \\ C & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{pmatrix} \tag{7.5}$$

Thus, the matrix L on which we are going to compute the tonal function would be the one that matches the one with the square of the maximum dimension, which in this case would be:

$$L^F_{(\mathcal{O}_1|\mathcal{O}_2)} = \begin{pmatrix} 4^+ & 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- \\ 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ \\ 4^+ & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- \\ 1^- & 4^+ & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ \end{pmatrix} \tag{7.6}$$

With the matrix L calculated, it is enough to use the general formula $H_G(\lambda) = H_{(\mathcal{O}_1,\mathcal{O}_2)}(\lambda) = \Delta(s^{\sigma(L^{BC}_G - W)} + P - \lambda Id)$ to compute the polynomial. Then in this case the column transformation matrix C is given by:

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.7}$$

We can think of C as the matrix that takes the boxes to the diagonal. There are several methods to calculate it, but the simplest and most intuitive is the following. First, replace the boxes with ones and the rest of the entries with zeros in matrix L^B_G; lastly, transpose said matrix thus obtaining C.

$$L^{BC}_{(\mathcal{O}_1|\mathcal{O}_2)} = \begin{pmatrix} 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- \\ 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ \\ 4^+ & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- \\ 1^- & 4^+ & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this way we bring the boxes to the diagonal and place all the solutions of the Hungarian algorithm arranged in the diagonal of the matrix prepared to subtract W. So when we subtract the matrix W, we will have our solutions well placed and in the correct place to carry out the change of space.

$$L^{BC}_{(\mathcal{O}_1|\mathcal{O}_2)} - W = \begin{pmatrix} \boxed{1^-} & -\infty & -\infty & -\infty & -\infty & -\infty & -\infty \\ -\infty & \boxed{0} & -\infty & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & \boxed{0} & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & \boxed{0} & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty & \boxed{0} & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty & -\infty & \boxed{0} & -\infty \\ -\infty & -\infty & -\infty & -\infty & -\infty & -\infty & \boxed{0} \end{pmatrix}$$

Since we have already subtracted W then we recover the sign of each metric. Here we see that the negative infinities that come from W do not recover the sign because they are not a metric, since W functions as a transformation matrix between spaces. Thus, once we have recovered the metrics that are on the diagonal, we apply the exponentiation of base s, where s is the well-known Mersenne number or semitone ratio. In this way we obtain the transformation matrix between voicings.

$$s^{\sigma(L^{BC}_{(\mathcal{O}_1|\mathcal{O}_2)} - W)} = \begin{pmatrix} s^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^0 \end{pmatrix}$$

As we have already obtained the transformation matrix between voicings of both tonalities with the minimum level of note stimulus perception. We now convert the language of matrices to a polynomial language. Thus, first we calculate the difference between the matrix we have and the 7×7 identity matrix.

$$s^{\sigma(L^{BC}_{(\mathcal{O}_1|\mathcal{O}_2)} - W)} - \lambda Id_7 = \begin{pmatrix} s^{-1}-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s^0-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s^0-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^0-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s^0-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s^0-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^0-\lambda \end{pmatrix}$$

Once we have made the difference between matrices, since the tonal function is not parametric and $P = 0_7$ then we calculate the determinant Δ of said matrix. Thus we reach the tonal function by taking the determinant of the transformation. The tonal function $H(\lambda)$ will be the hypervolume of the difference between the transformation matrix with note stimulus perception at its minimum and the identity matrix as a function of lambda. We can see the tonal function as a hypervolume that will be canceled when λ is equal to 1 or when λ matches the ratio between one of the voices in the optimal link.

$$\Delta(s^{\sigma(L^{BC}_{(\mathcal{O}_1|\mathcal{O}_2)}-W)} - \lambda Id_7) = \Delta \begin{pmatrix} s^{-1}-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s^0-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s^0-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^0-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s^0-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s^0-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^0-\lambda \end{pmatrix}$$

Thus, by the properties of the determinant, in this case by the triangularity of the matrix, we can calculate the determinant as the product of the diagonal elements, then the characteristic polynomial of the endomorphism transformation matrix is itself the tonal function $H_{(G_L|C_L)}(\lambda)$. In this way we establish that $H(\lambda)$ is the tonal function between the two open keys G_L and C_L.

$$H_{(G_L|C_L)}(\lambda) = \Delta(s^{\sigma(L^{BC}_{(G_L|C_L)}-W)} - \lambda Id_7) = (s^{-1}-\lambda)(s^0-\lambda)^6$$

Once the calculation has been carried out to find the tonal function between the mentioned open tonalities, then we have to classify said polynomial in its corresponding area. We have seen along the way that the tonal function is polarized, so there are three areas of classification where the tonal function can be located. Since the tonal function is not dual since the solution by the Hungarian algorithm is unique, then the polynomial criterion will associate a single area to the link. We know that in order to classify a tonal function in the tonic area, the following proposition must be fulfilled:

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} < 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases} \text{or} \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} < 2 \end{cases}$$

We see that the tonal function that we have just calculated satisfies the left member of the logical proposition, the convergent algebraic multiplicity being less than 2 and the divergent one 0. This very small motion between the open tonalities makes the appropriate classification as tonic.

$$\mathbb{A}(H_{(G_L|C_L)}(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} < 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases}$$

We will use the notation for chords or open keys $\mathcal{O}_1 \dashrightarrow \mathcal{O}_2$ when we have a voice descending in the optimal link. In this way we conclude that $G_L \dashrightarrow C_L$ being the relationship between both open keys, a tonic relationship. Thus we see that the classical modulation by fifths is a tonic modulation for classical tonalities. We finally conclude that

$$\mathbb{A}(H_{(G_L|C_L)}(\lambda)) = T^{\mathbb{C}[\lambda]} \tag{7.8}$$

7.3. Case 2: $D_L \longrightarrow C_L$

Following the order of the circle of fifths, we want to see how two tonal centers that are one tone apart behave. By studying the global link G, we will be able to compute the tonal function of the retrograde link $\varrho(G)$ by changing the sign of the exponents of each root in the tonal function (see Equation 5.13).

$$U = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- \\ Bb & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ \\ Eb & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- \\ Ab & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\ Db & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\ Gb & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ B & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ E & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\ A & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\ D & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- \\ G & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ \\ C & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 \end{pmatrix} \tag{7.9}$$

Let's proceed to compute the tonal function between D_L and C_L, being these tonal centers, tonal centers a tone apart, the global link $G = (D_L \mid C_L)$, and the retrograde link $\varrho(G) = (C_L \mid D_L)$. With this information we assume the classes that do not belong to each tonal center are empty classes, taking into account that the distance between two empty classes is zero and the distance between an empty class and an arbitrary non-empty class is infinite. Thus U_G is the following matrix:

$$
U_G = \begin{pmatrix}
\Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\
F & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
Bb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
Eb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
Ab & \infty & \infty & \infty & \infty & \infty & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\
Db & \infty & \infty & \infty & \infty & \infty & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\
Gb & \infty & \infty & \infty & \infty & \infty & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\
B & \infty & \infty & \infty & \infty & \infty & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\
E & \infty & \infty & \infty & \infty & \infty & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\
A & \infty & \infty & \infty & \infty & \infty & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\
D & \infty & \infty & \infty & \infty & \infty & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- \\
G & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
C & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty
\end{pmatrix} \tag{7.10}
$$

Using the Hungarian algorithm on the matrix U_G, we obtain U_G^F where we subtract the minimum only in the first two rows of the submatrix L_G^F and therefore to the rows of infinities involved. In this case, the solution obtained is unique.

$$
U_G^F = \begin{pmatrix}
\Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\
F & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
Bb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
Eb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
Ab & \infty - 1 & \infty - 1 & \infty - 1 & \infty - 1 & \infty - 1 & 1^- & 2^+ & 3^- & 0^+ & 5^= & 0^- & 3^+ \\
Db & \infty - 1 & \infty - 1 & \infty - 1 & \infty - 1 & \infty - 1 & 4^+ & 1^- & 2^+ & 3^- & 0^+ & 5^= & 0^- \\
Gb & \infty & \infty & \infty & \infty & \infty & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\
B & \infty & \infty & \infty & \infty & \infty & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\
E & \infty & \infty & \infty & \infty & \infty & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\
A & \infty & \infty & \infty & \infty & \infty & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\
D & \infty & \infty & \infty & \infty & \infty & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- \\
G & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
C & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty
\end{pmatrix} \tag{7.11}
$$

With this operation we have already reached the solution by using the Hungarian algorithm. We box the zeros that are solutions. Later, these boxes will be taken to a submatrix L on which the space change will be applied as well as the computation of the determinant. Thus the matrix U_G^F with the boxes drawn would be given by:

$$U_G^F = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Bb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Eb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Ab & \infty-1 & \infty-1 & \infty-1 & \infty-1 & \infty-1 & 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} & 3^+ \\ Db & \infty-1 & \infty-1 & \infty-1 & \infty-1 & \infty-1 & 4^+ & 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} \\ Gb & \infty & \infty & \infty & \infty & \infty & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ B & \infty & \infty & \infty & \infty & \infty & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ E & \infty & \infty & \infty & \infty & \infty & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- \\ A & \infty & \infty & \infty & \infty & \infty & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ \\ D & \infty & \infty & \infty & \infty & \infty & 4^+ & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- \\ G & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ C & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{pmatrix} \tag{7.12}$$

By superimposing the boxes over the original L matrix, we clearly see the voice motion between the pair of tonal centers. Thus, in this case, we find out that there are five roots of the tonal function on the stabilizer and two that are to its left. Therefore, the voice motion in the optimum will be descending motion of two voices.

$$L_G^B = \begin{pmatrix} 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- \\ 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ \\ 4^+ & 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- \end{pmatrix} \tag{7.13}$$

With this placement of boxes we apply the general formula for calculating tonal functions between two tonalities. We assume that W is a square matrix of the same dimensions as L_G^B and whose diagonal entries are zero and the remainder is negative infinite. We also assume that we have computed the column permutation matrix C starting from L_G^B , substituting its boxes for ones, the rest of the entries for zeros, and transposing it. Thus, the transformation that we are going to apply is given by the formula:

$$H_G(\lambda) = H_{(\mathcal{O}_1,\mathcal{O}_2)}(\lambda) = \Delta(s^{\sigma(L_G^{BC}-W)} + P - \lambda Id)$$

In this case, we are dealing with a non-dual, non-parametric tonal function because the dimensions of both tonalities are the same. It is not necessary to use the auxiliary matrix P. Consequently, we have $P = 0_7$ and the tonal function is represented by a unique matrix in the endomorphism of frequency space Φ^7. All this yields

$$H_G(\lambda) = H_{(\mathcal{O}_1,\mathcal{O}_2)}(\lambda) = \Delta(s^{\sigma(L_G^{BC}-W)} - \lambda Id)$$

Since we have already obtained the tonal function, we proceed to classify the polynomial by using the polynomial criterion. At first glance, we see that the classification is going to be dominant, but we write a few inequalities of the algebraic multiplicities so that the process follows its formal structure.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} \geq 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases}$$

By using the general formula, then the tonal function between the mentioned tonal centers is given by the following polynomial.

$$H_G(\lambda) = (s^{-1} - \lambda)^2(1-\lambda)^5$$

Therefore, we calculate the tonal area of the tonal function and since there are two voices that move downwards and the remaining five are located on the stabilizer, then it is clear that the tonal function is in the dominant area and any optimal arrangement of both centers will lead to an decrease in energy, for all optimal voicings, which is situated against the divergence of centers.

$$\mathbb{A}(H_G(\lambda)) = D^{\mathbb{C}[\lambda]}$$

On the other hand, if we calculate the retrograde link $\varrho(G)$, then its tonal function will be given by

$$H_{\varrho(G)}(\lambda) = (s - \lambda)^2(1-\lambda)^5$$

Whenever we study a tonal function, we can calculate the tonal function of the retrograde link by changing the exponents of the Mersenne number in the tonal function. Regarding the classification by areas, for polarized tonal functions; this implies that for a tonal function in the dominant area, the tonal function of the retrograde link will be located in the subdominant area. The polynomial criterion for the subdominant cases outputs:

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} \geq 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases}$$

This is how with a single calculation of the tonal function we obtain the polynomial associated with the link and the polynomial associated with the retrograde link.

$$\mathbb{A}(H_{\varrho(G)}(\lambda)) = S^{\mathbb{C}[\lambda]}$$

7.4. Case 3: $A_L \longrightarrow C_L$

In the third case, we want to study the relationship between two Lydian tonal centers A_L and C_L, that is, $G = (A_L \mid C_L)$, and we want to see how both are related. This relationship between tonal centers that are a minor third apart from each other is very common in jazz music and for the composer with some experience, the results provided by the resolution of the Hungarian algorithm may come as a revelation since they will understand the behavior between these pairs of tonal centers.

This type of minor third motion is often at the limit of jazz music theory. By this, it is meant that musicians usually know how to use them but there are no handbooks to the best of our knowledge that provide composers and instrumentalists with a precise definition of these mathematical relationships. The U matrix and the Hungarian algorithm are sufficient tools to be able to determine the relationship between both centers. We start from U matrix to solve the problem:

$$U = \begin{pmatrix}
\Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\
F & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- \\
Bb & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ \\
Eb & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- \\
Ab & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\
Db & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\
Gb & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\
B & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\
E & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\
A & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\
D & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- \\
G & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ \\
C & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0
\end{pmatrix} \tag{7.14}$$

As we have seen in the previous cases, to compute the tonal function from a matrix U, what we must do is assume the classes not contained in the centers as empty classes and either calculate the tonal function on the transformation of U or take a L submatrix and perform the calculation on the L.

$$U_{(A_L|C_L)} = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Bb & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Eb & \infty & \infty & \infty & \infty & \infty & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- \\ Ab & \infty & \infty & \infty & \infty & \infty & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\ Db & \infty & \infty & \infty & \infty & \infty & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\ Gb & \infty & \infty & \infty & \infty & \infty & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ B & \infty & \infty & \infty & \infty & \infty & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ E & \infty & \infty & \infty & \infty & \infty & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\ A & \infty & \infty & \infty & \infty & \infty & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\ D & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ G & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ C & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{pmatrix} \tag{7.15}$$

We omit the submatrices that have all their entries with infinity and those that have all their entries zero. We write the L matrix to perform voice optimization by using the Hungarian algorithm.

$$L_{(A_L|C_L)} = \begin{pmatrix} 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- \\ 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\ 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\ 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \end{pmatrix} \tag{7.16}$$

We have managed to put the focus on the submatrix L and we already have the mathematical object to start the optimization process through the Hungarian algorithm. Next, by following the steps appropriately, it is enough to subtract the minimum of each row from its own row. In this case, since the minimum of the first three rows is one, it is enough to carry out this operation to obtain L^F.

$$L^F_{(A_L|C_L)} = \begin{pmatrix} 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} & 3^+ & 2^- \\ 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} & 3^+ \\ 4^+ & 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- \\ 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ \end{pmatrix} \tag{7.17}$$

With this in mind we move the boxes of the solved matrix L^F to the original submatrix L obtaining L^B. In this case, since there is no duality in the solutions, L^B is unique and represents the only possible distribution of boxes with a minimum value of the nabla function.

$$L^B_{(A_L|C_L)} = \begin{pmatrix} 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ & 3^- \\ 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- \\ 3^- & 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ \end{pmatrix} \tag{7.18}$$

Since we have already obtained the distribution of boxes on the L matrix, it is enough to apply the general formula for the tonal function, where P is the zero matrix of dimension seven, W is the matrix with all the entries with infinite value except its diagonal that is zero and C is the matrix that takes the boxes of L^B to the diagonal when we multiply it by C in the equation $L^BC = L^{BC}$. The general formula for this particular case would be:

$$H_G(\lambda) = H_{(A_L|C_L)}(\lambda) = \Delta(s^{\sigma(L_G^{BC} - W)} + P - \lambda Id)$$

If we apply the general formula to the matrix L^B that we have just calculated then we immediately arrive at the polynomial that contains all the information of the movement of voices between the two tonal centers. We remember here, that by the **static function theorem** (Section 6.3), this polynomial is the same if we transport the tonal centers. Thus the global tonal function in this case is given by the equality:

$$H_{(A_L|C_L)}(\lambda) = (1-\lambda)^4(s^{-1}-\lambda)^3$$

We see that in this case there are four voices that remain static and another three move downwards. Hence, in the polynomial the convergent algebraic multiplicity of the root s^{-1} will be three.This means that the first inequality of the dominant polynomial criterion will be fulfilled. It is also given, by simple inspection, that the divergent algebraic multiplicity is zero, therefore all the voices move in the same direction when the link is optimal.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} \geq 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases}$$

Applying the general conditions of the polynomial criterion, we classify this tonal function as a non-dual polarized tonal function that lies in the dominant area. So, A_L will be the center that gravitates to C_L and not the other way around. We state the classification of the tonal function as the equation:

$$\mathbb{A}(H_{(A_L|C_L)}(\lambda)) = D^{\mathbb{C}[\lambda]}$$

As we have seen in previous cases, the tonal function of the retrograde link is classified in the subdominant area since instead of lower three voices, the same three raise from C_L to A_L.

$$\mathbb{A}(H_{\varrho(A_L|C_L)}(\lambda)) = \mathbb{A}(H_{(C_L|A_L)}(\lambda)) = S^{\mathbb{C}[\lambda]}$$

7.5. Case 4: $E_L \longrightarrow C_L$

To compute the relationship between the next two tonal centers, we will apply the same method, using the universal matrix U as a general operator. This particular case is extremely important because it is implicit in the musical tradition of jazz music and is the backbone of an important period in the music of **John Coltrane**. Coltrane's changes of harmony were a revolution that has influenced many composers and has earned the attention of others to this day. It would be very strange if a composition technique like the one used by Coltrane and which has endured over time did not find its due mathematical foundation.

Thus the justification of the first section of **Giant Steps** finds its logic in the study of the following case.

$$U = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- \\ Bb & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ \\ Eb & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- \\ Ab & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\ Db & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\ Gb & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ B & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ E & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \\ A & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ \\ D & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ & 2^- \\ G & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 & 5^+ \\ C & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & 0 \end{pmatrix} \tag{7.19}$$

In the second section the cycle changes direction and we ask ourselves how it is possible if the result of the matrix U is a non-dual tonal function. Well, just look at Coltrane interpolating a $II-7$ before the $V7$ and this minor chord can be in three keys at the same time. A, following the movements that are exposed in these modulation cases, this $II-7$ must be momentarily Phrygian, and the antecedent momentarily Lydian to connect them with a tonal function in the tonic area. It is enough to analyze the ninth bar of Giant steps and conclude that $Eb\Delta$ is momentarily Lydian and $A-7$ momentarily Phrygian to see that the relation is actually the relation of tonic between Bb Ionian and F Ionian, which is a tonic relation already covered in case 1. As we can see, with the resolution of case four we were able to give a formal explanation to the Coltrane cycle inversions and understand why these mathematical relationships were chosen and not others.

$$U^{B}_{(E_L|C_L)} = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ Bb & \infty & \infty & \infty & \infty & \infty & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} \\ Eb & \infty & \infty & \infty & \infty & \infty & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} \\ Ab & \infty & \infty & \infty & \infty & \infty & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} \\ Db & \infty & \infty & \infty & \infty & \infty & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} \\ Gb & \infty & \infty & \infty & \infty & \infty & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} \\ B & \infty & \infty & \infty & \infty & \infty & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} \\ E & \infty & \infty & \infty & \infty & \infty & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} \\ A & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ D & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ G & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ C & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{pmatrix} \tag{7.20}$$

In this way we see that the first eight bars of Giant Steps are justified by Case 4. This justification will become very clear when the reader studies the resolution of this modulation case. Since each dominant and each resolution of it are in the same closed key, then in dimension seven, their tonal function is $H(\lambda) = (1-\lambda)^7$ since it has all the voices in common; and by using Case 4, we can move the blocks of two chords in a major third movement counterclockwise on the circle of fifths. Furthermore, the question that has intrigued performers and composers for years was how it is possible for the cycle to be reversed after bar eight. Since dual tonal functions were not study until recently (see [PG22a, PG22b]), many aspects of Coltrane's work remained unexplored. We will come to see that the tonal function in Case 4 is not dual. Thus, the only explanation to be provided is that Giant Steps is written in dimension 4, but when performed many more voices come into play due to saxophone improvisations or the instruments participating in the composition interpretation. So if we apply Case 1, we will find the justification to connect bars nine and ten, which are connected by tonic relationships.

$$L_{(E_L|C_L)} = \begin{pmatrix} 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ \\ 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- \\ 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\ 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ 2^+ & 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- \end{pmatrix} \tag{7.21}$$

Again, we take the L submatrix of the general operator and apply the Hungarian algorithm to it to study the convergence between both tonal centers. We reach the matrix L^F easily by subtracting the minimum of each row from the first 4 rows

$$L^F_{(E_L|C_L)} = \begin{pmatrix} 3^- & 0^+ & 5^= & \boxed{0^-} & 3^+ & 2^- & 1^+ \\ 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} & 3^+ & 2^- \\ 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} & 3^+ \\ 4^+ & 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- \end{pmatrix} \tag{7.22}$$

In this case we have already arrived at the solution and we write it as a distribution of boxes over L^F. At this point, we only have to move the boxes in the matrix with the written solution to the original submatrix L, then we add the minimum again to carry out the space change transformation and subsequently bring the boxes back to the diagonal. Thus, we rewrite L with the distribution of boxes provided by the Hungarian algorithm.

$$L^B_{(E_L|C_L)} = \begin{pmatrix} 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ & 3^- & 2^+ \\ 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ & 3^- \\ 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \\ 2^+ & 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- \end{pmatrix} \tag{7.23}$$

Once matrix L^B is calculated, we carry out the space change transformation for global tonal functions by applying the formula as in previous cases.

$$H_G(\lambda) = H_{(E_L|C_L)}(\lambda) = \Delta(s^{\sigma(L_G^{BC}-W)} + P - \lambda Id)$$

Since the tonal function is non-dual and non-parametric, then P is a null matrix and we only associate a polynomial to the global link G. With this transformation, we reach the tonal function immediately.

$$H_{(E_L|C_L)}(\lambda) = (s^{-1} - \lambda)^4(1 - \lambda)^3$$

At this point we are now concerned ourselves about applying the polynomial criterion to understand the relationship between the tonal centers. We see that the polynomial fulfils the conditions to be classified within the dominant area, conditions that are given by the inequality for the convergent algebraic multiplicity and the nullity of the divergent algebraic multiplicity.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} \geq 2 \\ M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} = 0 \end{cases}$$

Thus, we classify the movement by descending major thirds between Lydian tonal centers, as a convergent movement.

$$\mathbb{A}(H_{(E_L|C_L)}(\lambda)) = D^{\mathbb{C}[\lambda]}$$

Since we know that $H_G(\lambda) = \prod_{i=1}^{n}(s^{l_i} - \lambda)$. It implies that the tonal function of the retrograde link is given by the expression $H_{\varrho(G)}(\lambda) = \prod_{i=1}^{n}(s^{-l_i} - \lambda)$ so we conclude that when calculating the retrograde link we have classified the tonal function of both.

$$\mathbb{A}(H_{(C_L|E_L)}(\lambda)) = S^{\mathbb{C}[\lambda]}$$

7.6. Case 5: $B_L \longrightarrow C_L$

In this case we will cover modulation upwards by semitone. This case is specially clarifying since it is one of the most counterintuitive. Usually, in jazz music the motions between dominant chords are usually either by fifths or by descending semitones. When we are working with global links, the situation is different. Indeed, the result given by the matrix associated with the global link when performing the optimization with the Hungarian algorithm indicates that the upward movement between Lydian tonal centers a semitone away generates a tonal function within the dominant area. In this way, we start from the well-known U matrix to obtain a submatrix with the tonal centers that we are studying for this case.

$$U = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} \\ Bb & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} \\ Eb & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} \\ Ab & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} \\ Db & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} \\ Gb & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} \\ B & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} \\ E & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} \\ A & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} \\ D & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} \\ G & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} \\ C & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 \end{pmatrix} \tag{7.24}$$

Starting from the U matrix, we consider the classes that do not appear in B Lydian as empty classes. In the same way we consider the classes outside C Lydian to be empty. In this elegant way, the matrix L is drawn inside the universal matrix U.

$$U_{(B_L|C_L)} = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & \infty & \infty & \infty & \infty & \infty & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} \\ Bb & \infty & \infty & \infty & \infty & \infty & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} \\ Eb & \infty & \infty & \infty & \infty & \infty & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} \\ Ab & \infty & \infty & \infty & \infty & \infty & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} \\ Db & \infty & \infty & \infty & \infty & \infty & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} \\ Gb & \infty & \infty & \infty & \infty & \infty & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} \\ B & \infty & \infty & \infty & \infty & \infty & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} \\ E & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ A & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ D & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ G & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ C & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \end{pmatrix} \tag{7.25}$$

We start the process to find the tonal function using the L submatrix in such a way that said matrix would be as shown below. We remember at this point that we can solve the tonal function on the transformation of the U matrix, but for convenience, we solve it on the L submatrix.

$$L_{(B_L|C_L)} = \begin{pmatrix} 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- \\ 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- & 2^+ \\ 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ & 3^- \\ 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & 1^- \\ 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & 0 & 5^+ & 2^- & 3^+ & 4^- & 1^+ \end{pmatrix} \tag{7.26}$$

With the L matrix already computed, we apply the first step of the Hungarian algorithm and subtract the minimum of each row from its own row, obtaining L^F.The solutions for optimal voice leading are given by the distribution of boxes marked on the matrix L^F, as indicated.

$$L^F_{(B_L|C_L)} = \begin{pmatrix} 0^+ & 5^= & \boxed{0^-} & 3^+ & 2^- & 1^+ & 4^- \\ 3^- & 0^+ & 5^= & \boxed{0^-} & 3^+ & 2^- & 1^+ \\ 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} & 3^+ & 2^- \\ 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} & 3^+ \\ 4^+ & 1^- & 2^+ & 3^- & 0^+ & 5^= & \boxed{0^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \end{pmatrix} \tag{7.27}$$

Once we have obtained the solution, we rewrite the boxes on the original L matrix to study how the voices move. In this case, for any optimal link between the two tonal centers, there are five voices that move downwards and two that remain constant.

$$L^B_{(B_L|C_L)} = \begin{pmatrix} 1^+ & 6^= & \boxed{1^-} & 4^+ & 3^- & 2^+ & 5^- \\ 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ & 3^- & 2^+ \\ 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ & 3^- \\ 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 5^- & \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ \end{pmatrix} \tag{7.28}$$

At this point in the process, we have already carried out all the necessary steps to be able to apply the general formula. Since the tonal function is not dual since this is the only distribution of boxes that the Hungarian algorithm reaches, we only have to apply the general formula for such a distribution.

By using the general formula for pairs of tonal centers we assume P to be a null matrix and we assume that C is the appropriate matrix to bring the boxes to the diagonal.

$$H_G(\lambda) = H_{(B_L|C_L)}(\lambda) = \Delta(s^{\sigma(L_G^{BC}-W)} + P - \lambda Id)$$

Applying the transformation directly to L^B we arrive at the global tonal function between both centers

$$H_{(B_L|C_L)}(\lambda) = (s^{-1} - \lambda)^5(1-\lambda)^2$$

Now applying the polynomial criterion we see that the tonal function will be classified within the dominant area.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} \geq 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases}$$

By using the tonal area calculation, we classify the relationship between both centers as a dominant relationship when the antecedent tonal center is B_L and the consequent is C_L. Thus the equation for the tonal area would be determined as:

$$\mathbb{A}(H_{(B_L|C_L)}(\lambda)) = D^{\mathbb{C}[\lambda]}$$

Since we know that $H_G(\lambda) = \prod_{i=1}^n (s^{l_i} - \lambda)$ implies that the tonal function of the retrograde link is given by the expression $H_{\varrho(G)}(\lambda) = \prod_{i=1}^n (s^{-l_i} - \lambda)$, we conclude that when calculating the retrograde link, we have classified the tonal function of both. Changing the sign of each exponent of each root, we obtain the tonal function of the retrograde link, which, contrary to intuition, is classified within the subdominant area.

$$\mathbb{A}(H_{(C_L|B_L)}(\lambda)) = S^{\mathbb{C}[\lambda]}$$

7.7. Case 6: $Gb_L \longrightarrow C_L$

In this section, we will study the relationship between two tonal centers that are a tritone apart. In this way, the relationship between **opposing halves** in the circle of fifths will be clear. Thus, as usual for global tonal functions, we will use the matrix U as the general operator.

$$U = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} \\ Bb & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} \\ Eb & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} \\ Ab & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} \\ Db & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} \\ Gb & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} \\ B & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} \\ E & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} \\ A & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} & 3^{+} \\ D & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} & 2^{-} \\ G & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 & 5^{+} \\ C & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 \end{pmatrix} \tag{7.29}$$

We select both tonal centers in the U matrix and obtain the L submatrix.

$$U_{(Gb_L|C_L)} = \begin{pmatrix} \Delta & F & Bb & Eb & Ab & Db & Gb & B & E & A & D & G & C \\ F & \infty & \infty & \infty & \infty & \infty & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} \\ Bb & \infty & \infty & \infty & \infty & \infty & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} \\ Eb & \infty & \infty & \infty & \infty & \infty & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} \\ Ab & \infty & \infty & \infty & \infty & \infty & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} \\ Db & \infty & \infty & \infty & \infty & \infty & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} \\ Gb & \infty & \infty & \infty & \infty & \infty & 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} \\ B & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ E & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ A & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ D & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ G & 0 & 0 & 0 & 0 & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ C & \infty & \infty & \infty & \infty & \infty & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 \end{pmatrix} \tag{7.30}$$

In this way, we already have focused the L matrix coming from U with each one of the metrics between the tonal centers separated by the tritone ratio. A composer with some experience will understand at this point why tritone modulation is of utmost importance as we develop the case.

$$L_{(Gb_L|C_L)} = \begin{pmatrix} 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} \\ 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} \\ 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} & 3^{-} \\ 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} & 4^{+} \\ 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} \\ 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} \\ 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 \end{pmatrix} \tag{7.31}$$

We apply the first step of the Hungarian algorithm and obtain the L^F matrix in such a way that we try to see if we have reached the solution. As we can see, the matrix has no solution and we have to apply the zero-minimum coverage where the number of lines that cover the zeros must be the square root of the dimension of L.

$$L^F_{(Gb_L|C_L)} = \begin{pmatrix} 0^{+} & 5^{=} & 0^{-} & 3^{+} & 2^{-} & 1^{+} & 4^{-} \\ 3^{-} & 0^{+} & 5^{=} & 0^{-} & 3^{+} & 2^{-} & 1^{+} \\ 2^{+} & 3^{-} & 0^{+} & 5^{=} & 0^{-} & 3^{+} & 2^{-} \\ 1^{-} & 2^{+} & 3^{-} & 0^{+} & 5^{=} & 0^{-} & 3^{+} \\ 4^{+} & 1^{-} & 2^{+} & 3^{-} & 0^{+} & 5^{=} & 0^{-} \\ 0 & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} \\ 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & 0 \end{pmatrix} \tag{7.32}$$

Covering columns $1, 3, 5, 7$ of L^F and rows $2, 4$ and subtracting the uncoated minimum which is min $= 1$ from the uncoated entries, adding it to the double coated ones and leaving the simply covered ones invariant ; we arrive at the matrix L^H that provides us with several solutions for the convergence between tonal centers at a tritone distance.

$$L^H_{(Gb_L|C_L)} = \begin{pmatrix} 0^{+} & 4^{=} & 0^{-} & 2^{+} & 2^{-} & 0^{+} & 4^{-} \\ 4^{-} & 0^{+} & 6^{=} & 0^{-} & 4^{+} & 2^{-} & 2^{+} \\ 2^{+} & 2^{-} & 0^{+} & 4^{=} & 0^{-} & 2^{+} & 2^{-} \\ 2^{-} & 2^{+} & 4^{-} & 0^{+} & 6^{=} & 0^{-} & 4^{+} \\ 4^{+} & 0^{-} & 2^{+} & 2^{-} & 0^{+} & 4^{=} & 0^{-} \\ 0 & 4^{+} & 2^{-} & 2^{+} & 4^{-} & 0^{+} & 5^{=} \\ 6^{=} & 0^{-} & 4^{+} & 2^{-} & 2^{+} & 4^{-} & 0 \end{pmatrix} \tag{7.33}$$

Since the matrix L^H has different solutions, to find them we have to apply the **zero method**. Then we will select each zero one of the sixteen zeros and see if holding it fixed generates a solution or not; that is to say, if when choosing it nullifies another zero in the same column that forces a solution within the matrix.

At this point we would look for each solution over L^H and migrate box distribution over the original L matrix. In this case we are going to directly write the box migrations for each zero that generates a solution. Thus we have that by forcing the first zero (the order of the zeros is established from left to right, from top to bottom in the matrix), we have that the first optimal solution is the diagonal itself. The value of the nabla function for this optimal link is 6.

$$L^{B_1}_{(Gb_L|C_L)} = \begin{pmatrix} \boxed{1^+} & 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- \\ 4^- & \boxed{1^+} & 6^= & 1^- & 4^+ & 3^- & 2^+ \\ 3^+ & 4^- & \boxed{1^+} & 6^= & 1^- & 4^+ & 3^- \\ 2^- & 3^+ & 4^- & \boxed{1^+} & 6^= & 1^- & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & \boxed{1^+} & 6^= & 1^- \\ 0 & 5^+ & 2^- & 3^+ & 4^- & \boxed{1^+} & 6^= \\ 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & \boxed{0} \end{pmatrix} \quad (7.34)$$

We apply the general formula on the matrix L^{B_1} and obtain one of the tonal functions associated with the link

$$(H_G(\lambda))_1 = (H_{(Gb_L|C_L)}(\lambda))_1 = \Delta(s^{\sigma(L_G^{B_1C_1} - W)} + P - \lambda Id)$$

Applying the formula, we have:

$$(H_{(Gb_L|C_L)}(\lambda))_1 = (s-\lambda)^6(1-\lambda)$$

Then, we have to apply the polynomial criterion on the first solution. That would determine that the polynomial verifies the two conditions to be within the subdominant area.

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} \geq 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases}$$

In this way we classify the first of the four solutions within the subdominant area.

$$\mathbb{A}((H_{(Gb_L|C_L)}(\lambda))_1) = S^{\mathbb{C}[\lambda]}$$

If we use the zero method and select the second zero of the matrix L in the established order of zeros, we obtain the second solution by drawing the second distribution on the matrix L and forming L^{B_2}.

$$L^{B_2}_{(Gb_L|C_L)} = \begin{pmatrix} 1^+ & 6^= & \boxed{1^-} & 4^+ & 3^- & 2^+ & 5^- \\ 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ & 3^- & 2^+ \\ 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ & 3^- \\ 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= & \boxed{1^-} \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 6^= & \boxed{1^-} & 4^+ & 3^- & 2^+ & 5^- & 0 \end{pmatrix} \quad (7.35)$$

When visualizing this second distribution of boxes, we observe that we only have to apply the general formula on the second optimal solution to reach the tonal function.

$$(H_G(\lambda))_2 = (H_{(Gb_L|C_L)}(\lambda))_2 = \Delta(s^{\sigma(L_G^{B_2C_2}-W)} + P - \lambda Id)$$

In this way we reach the tonal function as:

$$(H_{(Gb_L|C_L)}(\lambda))_2 = (s^{-1} - \lambda)^6(1 - \lambda)$$

Applying the formula we have:

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} \geq 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases}$$

So for the second zero, applying the polynomial criterion we have that the tonal function is dominant.

$$\mathbb{A}((H_{(Gb_L|C_L)}(\lambda))_2) = D^{\mathbb{C}[\lambda]}$$

Using the zero method we are going to study what happens now if we force the third zero to be a solution for the tonal function. So when applying this restriction we see that a new solution appears.

$$L^{B_3}_{(Gb_L|C_L)} = \begin{pmatrix} 1^+ & 6^= & 1^- & 4^+ & 3^- & \boxed{2^+} & 5^- \\ 4^- & \boxed{1^+} & 6^= & 1^- & 4^+ & 3^- & 2^+ \\ 3^+ & 4^- & \boxed{1^+} & 6^= & 1^- & 4^+ & 3^- \\ 2^- & 3^+ & 4^- & \boxed{1^+} & 6^= & 1^- & 4^+ \\ 5^+ & 2^- & 3^+ & 4^- & \boxed{1^+} & 6^= & 1^- \\ \boxed{0} & 5^+ & 2^- & 3^+ & 4^- & 1^+ & 6^= \\ 6^= & 1^- & 4^+ & 3^- & 2^+ & 5^- & \boxed{0} \end{pmatrix} \tag{7.36}$$

As we have found L^{B_3}, it is enough to apply the general formula again to find one of the tonal functions associated with the link

$$(H_G(\lambda))_3 = (H_{(Gb_L|C_L)}(\lambda))_3 = \Delta(s^{\sigma(L_G^{B_3C_3}-W)} + P - \lambda Id)$$

In this way we calculate the third solution for the modulation by tritone and we reach a third polynomial that is located in the subdominant area. So it would be written as:

$$(H_{(Gb_L|C_L)}(\lambda))_1 = (s - \lambda)^4(s^2 - \lambda)(1 - \lambda)^2$$

We state the polynomial criterion for polynomials in the subdominant area and we verify that indeed the divergent algebraic multiplicity of said polynomial is greater than or equal to two.

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} \geq 2 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} = 0 \end{cases}$$

Since we are talking about a dual tonal function, we have to classify each of the tonal functions that are a solution by the Hungarian algorithm and in this case, the third solution is classified as subdominant.

$$\mathbb{A}((H_{(Gb_L|C_L)}(\lambda))_3) = S^{\mathbb{C}[\lambda]}$$

By using the zero method, we continue trying to find one that generates a solution. We do not describe the process here, but when we reach the tenth zero, solution 4 is formed. This solution is a new distribution of boxes over the matrix L. Thus, by forcing the tenth zero we would obtain said distribution as illustrated in the following matrix:

$$L^{B_4}_{(Gb_L|C_L)} = \begin{pmatrix} 1^{+} & 6^{=} & \boxed{1^{-}} & 4^{+} & 3^{-} & 2^{+} & 5^{-} \\ 4^{-} & 1^{+} & 6^{=} & \boxed{1^{-}} & 4^{+} & 3^{-} & 2^{+} \\ 3^{+} & 4^{-} & 1^{+} & 6^{=} & \boxed{1^{-}} & 4^{+} & 3^{-} \\ 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & \boxed{1^{-}} & 4^{+} \\ 5^{+} & \boxed{2^{-}} & 3^{+} & 4^{-} & 1^{+} & 6^{=} & 1^{-} \\ \boxed{0} & 5^{+} & 2^{-} & 3^{+} & 4^{-} & 1^{+} & 6^{=} \\ 6^{=} & 1^{-} & 4^{+} & 3^{-} & 2^{+} & 5^{-} & \boxed{0} \end{pmatrix} \tag{7.37}$$

Using the tenth zero we obtain a distribution of boxes from which, applying the general formula, we can obtain the fourth tonal function that is a solution by the Hungarian algorithm of the voice optimization problem.

$$(H_G(\lambda))_4 = (H_{(Gb_L|C_L)}(\lambda))_4 = \Delta(s^{\sigma(L^{B_4 C_4}_{G} - W)} + P - \lambda Id)$$

Applying the formula with P being the null matrix of appropriate dimensions we arrive at the fourth polynomial.

$$(H_{(Gb_L|C_L)}(\lambda))_1 = (s^{-1} - \lambda)^4 (s^{-2} - \lambda)(1 - \lambda)^2$$

With the polynomial calculated, then we apply the polynomial criterion for its correct classification:

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} \geq 2 \\ M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} = 0 \end{cases}$$

Thus, the fourth tonal function of the set of solutions is classified within the dominant area.

$$\mathbb{A}((H_{(Gb_L|C_L)}(\lambda))_4) = D^{\mathbb{C}[\lambda]}$$

Applying the polynomial criterion and its formalization, we determine that the area associated with a dual tonal function is the conjunctistic union of the areas of each of the optimal solutions by the Hungarian algorithm.

$$\mathcal{P}(\mathbb{H}^G) = \mathbb{A}(H_i(\lambda)) \cup ... \cup \mathbb{A}(H_z(\lambda))$$

Just as we have four solutions, which are located in the dominant area and in the subdominant area, the dual tonal function will be a function that is both subdominant and dominant simultaneously. This tonal function will be represented by a double arrow in a graph of global tonal functions.

$$\mathcal{P}(\mathbb{H}^G) = D^{\mathbb{C}[\lambda]} \cup S^{\mathbb{C}}[\lambda]$$

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Chapter 8

Methods of Progression Construction

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https://unsplash.com/photos/WmnsGyaFnCQ

8.1. Global tonal functions

In this chapter, we study **global tonal functions** in great detail from a mathematical standpoint. A first encounter with global tonal functions was took place in Chapter 6 and there the musical meaning was elaborated upon.

Modulation is often linked to the larger structure of a musical composition, impacting the overall form. In the sonata form, for instance, the development section commonly involves modulation to explore new tonal territory.

In summary, modulation is a complex and versatile aspect of music theory, involving the manipulation of tonal centers, chord progressions, and harmonic relationships. It is a crucial tool for composers to create tension, release, and variety within a musical work. Theoretical analysis of modulation enhances our understanding of how composers achieve these effects and contributes to the broader appreciation of musical structure.

Modulation has always been of interest to musicians, composers, and music theorists. At each musical era, relevant authors wrote treatises and books about modulation. We mention the following among the most relevant both from a historical and conceptual standpoints: **Jean-Philippe Rameau** [Ram22], **Charles Macpherson** [Mac20], **Arnold Schoenberg** [Sch48, Sch69], **Walter Piston** [Pis50], **George Russell**[Rus01] **Vicent Persichetti** [Per95].

8.1.1. The set of links $\mathcal{E}_{\mathbb{S}}$

Usually, we work with the 12-class equal temperament as established by the quotient set $LC_k/\sim$. The **number of links** is related to a tuning system $\mathbb{S}$ under use in the given musical practice. In this work we will use $\mathbb{S} = LC_k/\sim$, the usual equal temperament. Let $\mathbb{U}_{\mathbb{S}}$ be the set of matrices whose entries are elements taken from $\mathbb{S}$ and are given by the metric; formally, its definition is

$$\mathbb{U}_{\mathbb{S}} = \{U = (\Delta_{ij}),\ \Delta_{ij} = \Delta([\theta_i], [\theta_j]), [\theta_i], [\theta_j] \in \mathbb{S}, i, j = 1, \ldots, \dim(\mathbb{S})\} \tag{8.1}$$

Furthermore, $\mathbb{L}_{\mathbb{S}}$ will be the set of sub-matrices of $\mathbb{U}_{\mathbb{S}}$ obtained by removing either some rows or some columns. The set of links $\mathcal{E}_{\mathbb{S}}$ is the set of links produced by the system $\mathbb{S}$.

From the above, reasoning on the cardinality of $\mathbb{S}$, we deduce that the number of k-permutations is $P_i^{|\mathbb{S}|}P_j^{|\mathbb{S}|}$, where $dim(X) = i$ and a consequent such that $dim(Y) = j$. Since the links are mathematical objects where the voices are distributed in dimensions then, for the cases where $dim(X) = i \neq dim(Y) = j$ we have to multiply the product of k-permutations by the factor $\left(\frac{\left(\frac{max\{i,j\}!}{(max\{i,j\}-|i-j|)!}\right)}{|i-j|!}\right)$.

$$| \mathcal{E}_{\mathbb{S}} | = \sum_{i=1}^{|\mathbb{S}|}\sum_{j=1}^{|\mathbb{S}|} P_i^{|\mathbb{S}|}P_j^{|\mathbb{S}|} \left(\frac{\left(\frac{max\{i,j\}!}{(max\{i,j\}-|i-j|)!}\right)}{| i-j |!}\right)$$

The reason behind this multiplication is that when taking the maximum dimension for any of the k-permutations we have to divide by the factorial of the **number of filled cells** $(max\{i,j\}- | i-j |)!$ for do not count the rearrangements of the variations in duplicate, and immediately afterwards divide by the factorial of the **number of empty cells** $| i-j |!$ since the empty class symbol [ ] is equivalent when the number of empty cells is two or greater.

8.1.2. Mathematical properties of tonal functions

Let $\forall\!\!\!/_E$ be the tonal function associated to a link E or $\forall\!\!\!/_G$ the tonal function associated to a global link G. By the definition of $\forall\!\!\!/_E$:

$$\forall\!\!\!/ : \mathcal{E} \longrightarrow \wp(\pounds)$$

$$\forall\!\!\!/_{E\in\mathcal{E}} = \mathcal{L} \subset \wp(\pounds) \tag{8.2}$$

Since $\pounds$ is a family of labels into seven types and $\wp(\pounds)$ is the set of subsets of labels that can receive a link, then we write some of the mathematical properties that help establish a context for the tonal function and thus apply the method of construction of graphs of tonal functions (upcoming in Section 8.1.3).

Property 1: Taking into account the above, we can write $\not\int_{E \in \mathcal{E}} = \not\int_{\lambda E \in \lambda \mathcal{E}}$ given the fact that $E \in \mathcal{E}$ and $\lambda E \in \lambda \mathcal{E}$. By applying the **static tonal function theorem** (Section 4.2.1), we conclude that $\mathbb{H}^E = \mathbb{H}^{\lambda E}$. Therefore,

$$\mathbb{H}^E = \mathbb{H}^{\lambda E} \iff \not\int_E = \not\int_{\lambda E} \tag{8.3}$$

Property 2: Given a link E contained in the set of links $\mathcal{E}$ generated by the system $\mathbb{S}$ under consideration, in our case $\mathbb{S} = LC_k / \sim$, the tonal function of a link contained in the link set arises from the last construction of $\mathcal{E}$. This is independent of $\mathbb{S}$ when $\mathbb{S}$ is finite. Therefore,

$$\not\int_E = \not\int_{E_1^o} \cup \cdots \cup \not\int_{E_q^o} \tag{8.4}$$

where the set $\mathcal{Z}_{(X|Y)} = \{E_1^o, \ldots, E_q^o\}$ are the links that arise from the application of the **zero method** in the process of obtaining the tonal function by using the Hungarian algorithm.

Property 3: Given a link E in $\mathcal{E}$ and a class contained in $(\wp(\mathbb{S}))^2$, it is true that the class of the link shares a tonal function with the link; that is to say, we can define the tonal function on the entire class also as a mapping from a set of links to a set of labels. By using the definition of a system as a set of finite octave classes, we can define an equivalence relation in the set of classes. Indeed, two links are equivalent if their classes are equal. This relation has a quotient set $\mathcal{E}/\sim = (\wp(\mathbb{S}))^2$, for which we can write the following expression.

$$\not\int : (\wp(\mathbb{S}))^2 \longrightarrow \wp(\pounds)$$

$$\not\int_{[E]} = \mathcal{L} \subset \wp(\pounds) \tag{8.5}$$

By the static tonal function theorem and by the fact that all the links of the same class share a tonal function and by the properties of the matrix L and its resolution by means of the Hungarian algorithm, then we conclude know that the tonal function of a link is the same as its whole class:

$$\not\int_E = \not\int_{[E]} \tag{8.6}$$

Property 4: Taking into account the definition, then we have that if $\not\int_E$ is polarized, that is, $\not\int_E = P$, we can conclude that

$$\not\int_E = P \iff \not\int_{\varrho(E)} = P \tag{8.7}$$

where $\varrho(E)$ is the retrograde link. To understand this property, we resort to the definition of hypervolumetric tonal function and express it in its general form with $H_E(\lambda) = \prod_{i=1}^n (s^{l_i} - \lambda)$. This definition implies that in the event that the hypervolumetric tonal function is polarized, then we will have:

$$H_E(\lambda) = \prod_{i=1}^n (s^{l_i} - \lambda) \to H_{\varrho(E)}(\lambda) = \prod_{i=1}^n (s^{-l_i} - \lambda)$$

Since this property holds, we immediately deduce that

$$H_E(\lambda) = \prod_{i=1}^{n}(s^{l_i} - \lambda) \iff H_{\varrho(E)}(\lambda) = \prod_{i=1}^{n}(s^{-l_i} - \lambda) \tag{8.8}$$

The above reasoning implies that polarization is preserved under the retrograde operation on a link E, which is expressed in terms of tonal functions as

$$\not{\oint}_E = P \iff \not{\oint}_{\varrho(E)} = P \tag{8.9}$$

Alternatively, we can carry out the same reasoning but in terms of L matrices, where we will refer to the set of matrices as $\mathbb{L}$ when the context is the usual, where $\mathbb{S} = LC_k/\sim$, and where we will refer to said set as $\mathbb{L}_{\mathbb{S}}$ when the set of matrices is built based on another tuning system different from the one used that we will specify in advance.

Since for each link E we construct a matrix $L \subset \mathbb{L}$ with signs as in the case of the matrix $U \subset \mathbb{U}$ which is the universal matrix of our system, then we can define the tonal function alternatively, as a label that is assigned to each matrix $L \subset \mathbb{L}$. In this way for matrices, the tonal functions would be:

$$\not{\oint} : \mathbb{L}^{\sigma}_{\mathbb{S}} \longrightarrow \wp(\pounds) \tag{8.10}$$

$$\not{\oint}_{L^{\sigma}_E} = \mathcal{L} \subset \wp(\pounds) \tag{8.11}$$

where $\sigma = (\sigma_{ij})$ is the **sign matrix** that is assigned to the matrix L based on the sign recovery of each one of its metrics.

By considering this definition of tonal function of a matrix, we can deduce how the polarization will be preserved when retrograding said link just by observing a property of L matrices with respect to retrogradation of the links. Let be a matrix L^{σ}_E where $\sigma = (\sigma_{ij})$ is the **sign matrix** that is assigned to the matrix L based on the sign recovery of each one of its metrics. It holds that

$$L^{\sigma}_E = (L^t_{\varrho(E)})^{-\sigma^t} \tag{8.12}$$

This is revealing the fact that the matrices L of a link E and its retrograde are related by transposing the original matrix and multiplying the transposed sign matrix $\sigma^t = (\sigma_{ij})^t$ by -1. Here we see that if the tonal function is polarized, then in the retrograde link, all its signs will change, and therefore their polarization will be preserved if all the signs are equal or zero in the first function, that is, if the tonal function is polarized. From the previous equation, we take tonal functions on both sides, which yields

$$\not{\oint}_{L^{\sigma}_E} = P \iff \not{\oint}_{L^{-\sigma^t}_{\varrho(E)}} = P \tag{8.13}$$

8.1.3. Tonal functions graphs

The algebraic model presented here to determine the tonal function of pairs of chords can be generalized to **global tonal functions**. We first need the definition of a **tonal center**. A tonal center is a subset of $(LC_k/\sim)$.

Let $[P] = \left(\Big|_{j=1}^{m} X_j\right)$ be the class of a progression P and T the tonal center in which this progression is located. Such tonal center is

$$T = \bigcup_{j=1}^{m} X_j,\ T \subset \wp(LC_k/\sim)$$

which is nothing but the union of all its chords.

Let P^* be another progression, to be called the **consequent** of P; assume it contains q chords and its class is $[P^*] = \left(\Big|_{j=1}^{q} X_j^*\right)$. Analogously, let T^* be its tonal center

$$T^* = \bigcup_{j=1}^{q} X_j^*,\ T^* \subset \wp(LC_k/\sim)$$

The **global tonal function** H_G of two progressions is the tonal function obtained by computing the tonal function of the two tonal centers. Note that the term tonality is sometimes used to indicate the progression in T, indistinctly. Here we refer to T as an element of the power set of the set of octave classes, and we refer to P as a progression indicating that the chords are expressed in a certain order; where the vertical bar indicates **antecedent** and **consequent**, and serves to organize the classes in time.

The global tonal function is computed on a link, $[E] = ([T]|[T^*])$. When writing $(T)^r$ and $(T^*)^r$, the order of the circle of fifths will be followed so that the notation is not abused unnecessarily. For this purpose, we assume that a tonal center is following the direction of the circle of fifths clockwise.We calculate the tonal function between two progressions as the function between the tonal centers to which they belong.

$$H : \wp(LC_k/\sim)^2 \longrightarrow \mathbb{C}[\lambda]$$

$$H_{G=(P|P^*)}(\lambda) = H_{(T^r|(T^*)^r)}(\lambda) = \Delta\left(s^{\sigma(L^{BC}_{(T^r|(T^*)^r)}-W)} + P - \lambda \cdot Id_{\max\{\dim(T^r),\dim((T^*)^r)\}}\right) \tag{8.14}$$

Here are a couple of examples of the discussion above. Given $C\Delta \in C_L$ and $D7 \in C_L$, then

$$H_G(C_L, C_L) = (1-\lambda)^7$$

which leads to $H_G \in I^{\mathbb{C}[\lambda]}$

In another example, if $C\Delta \in C_L$ and E♭Δ ∈E♭$_L$, then the global tonal function is $H_G(C_L, E\flat_L) \in D^{\mathbb{C}[\lambda]}$.

Thus, if we take $(T)^r$ and $(T^*)^r$, then by following the notation defined above of arrows for tonal functions, we produce a graph to visualize those tonal functions, the **graph of tonal**

functions. The **nodes** of the graph are the tonal centers and the **edges** are the relations between said polarized tonal functions (denoted with our special set of **arrows**). To represent the **dominant modulation** relationship between sections we will use a graph like the one in the following figure:

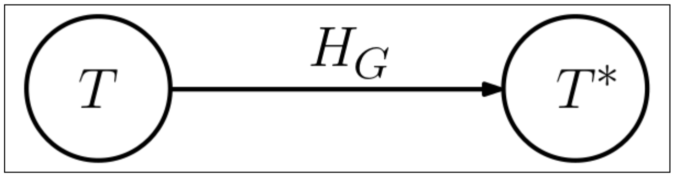

Figure 8.1: Dominant modulation

Every progression P has a tonal center such that for every chord of its class $[P] = \left(\Big|_{j=1}^{m} X_j\right)$, we have that the union of all of them is in $T = \bigcup_{j=1}^{m} X_j$, $T \subset \wp(LC_k/\sim)$. Therefore, when we draw the graphs of global tonal functions, we are actually drawing the graph of modulation between **sections** given because of the relationship between each progression and its tonal center.

A graph of tonal functions with k nodes and a finite number of tonal functions between consequent tonalities is written, for $1 \leq d \leq k, 1 \leq e \leq k$, as

$$G(N, A) = (\{T_1, \ldots, T_k\}, \{H_{G=(T^d \bigotimes T^e)}(\lambda)\})$$

where N is the set of k nodes and A are the edges (polynomials) between consequent progressions. $G(N, A)$ need not be unique, though. The symbol $\otimes$ is used to indicate that one center is the antecedent and the other is the consequent, where the symbol distinguishes the order between the left center and the right center, the left center being the antecedent, and the right center, the consequent. This symbol is used exclusively for a graph of tonal functions since it must show the order of the chords in the total progression. Thus, if we write $T^d \otimes T^e$, it means that the first tonal center is antecedent to the second. The set of edges A is therefore the subset of edges of the complete graph where the edges connect tonal centers that are consecutive in time.

$$A = \left\{H_{G=(T^d \bigotimes T^e)}(\lambda)\right\}$$

The symbol $\otimes$ is included here since a graph of tonal functions is not necessarily a complete graph as only some edges between consequent tonal centers appear on it. In the following example we represent a section where there are three tonal centers; see Figure 8.2.

$$G(N, A) = (\{T^1, T^2, T^3\}, \{H_{G=(T^1 \bigotimes T^2)}(\lambda), H_{G=(T^2 \bigotimes T^3)}(\lambda)\}, H_{G=(T^3 \bigotimes T^1)}(\lambda))$$

Thus, $H_{G=(T^1 \bigotimes T^2)}(\lambda)$ has a dominant tonal function; moreover, $H_{G=(T^1 \bigotimes T^2)}(\lambda)$ is one of the polynomials between tonal centers T_2 and T_3 because the tonal function between them is dual. For another, $H_{G=(T^3 \bigotimes T^1)}(\lambda)$ is dominant, but it is not dual. Within each node are written the sub-nodes that are the chords contained in the tonal center with their respective non-global tonal functions.

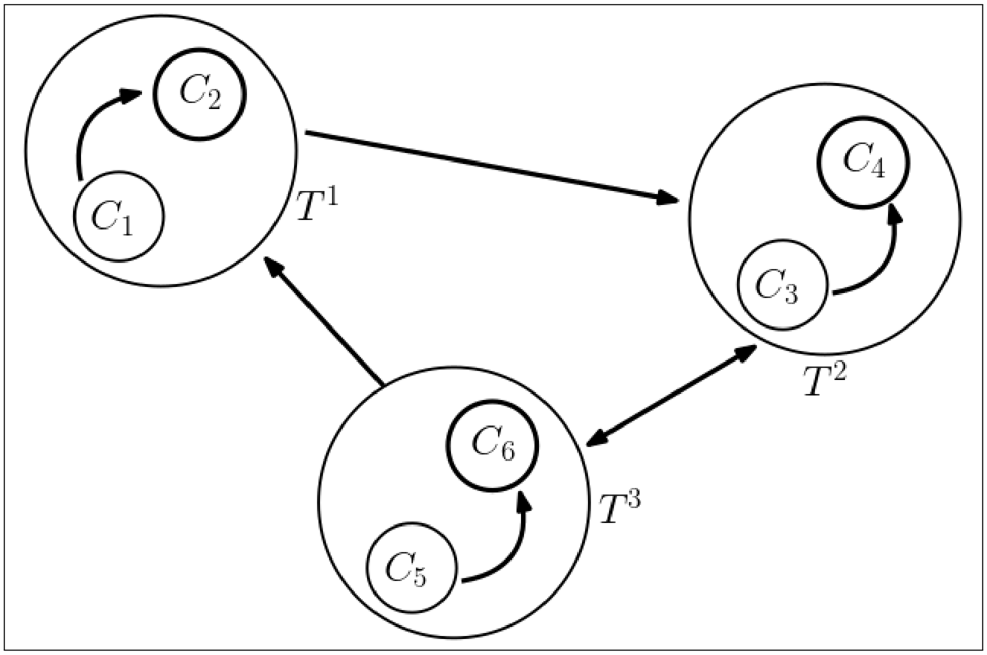

Figure 8.2: A section with three tonal centers

Each **node** can host an arbitrary number of **sub-nodes**, which can be chords or poly-chords. We call the **ratio** the number of sub-nodes in a node. The ratio is important because it substantially contributes to the perception of progress in the **harmonic rhythm**. The harmonic rhythm is the number of chords along with their durations per measure and the ratio is the number of chords per tonal center. Thus, depending on the style and our compositional needs, we will choose one ratio or another. It is convenient to think that an excess of change in the tonal centers will force the musicians to be changing scale too fast, that is, they will be forced to adapt to change scale and not to develop expression at the time of interpreting the music. Therefore, if we want the compositions to be expressive, it is convenient to find high ratios or look for tonal centers with intersections of high cardinality in order to be able to find common scales to the tonal centers. It is important to realize that these scales must have good mathematical properties.

The minor pentatonics are clear candidates for this job, since, as we saw earlier, their vertical dissonance graph is stable and they also appear as a series of fifths in the circle; see Figure 8.3.

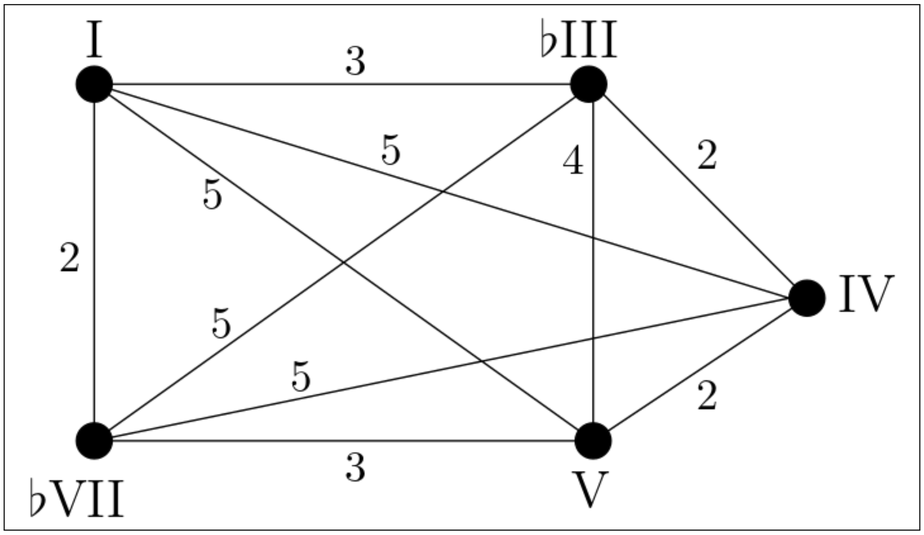

Figure 8.3: VDG for the minor pentatonic scale

8.1.4. General graphs

When working with tonal functions, we face a combinatorial explosion in the number of tonal functions because its number is the cardinal of the power set squared,$|H| = |\wp(LC_k/\sim)|^2$. Thus, the number of tonal functions for $LC_k/\sim$ is 16,777,216; some of them are dual, where some of them may be dual or not. As the number is so large, at first one considers calculating the most common tonal functions that arise from studying pre-existing material to confirm that the results are consistent with the usual changes in classical music and jazz, so as a result of the calculation we obtain the graph of non-global tonal functions for the Greek modes; see Figure 8.4.

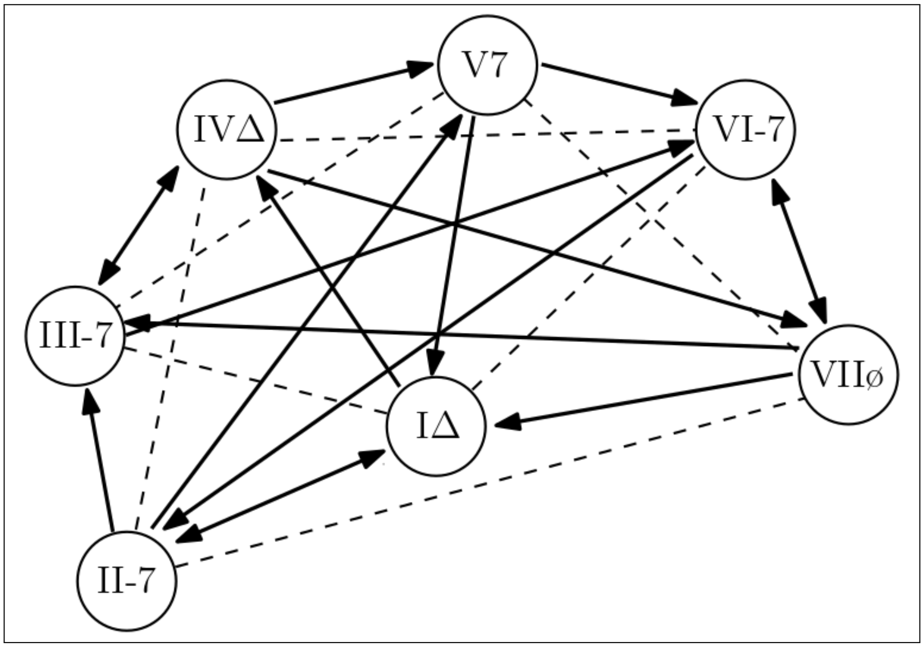

Figure 8.4: Non-global tonal functions for diatonic seventh chords

Of course, here we have calculated a few where we will be able to verify most of the usual tonal relationships. We examined some of those relationships to seek for patterns that can be properly generalized; see Figure 8.5. In the case of $[P] = (V_c7|I_c\Delta)$, we see that the relationship is clear. If this progression did not follow the arrows, the model would not be coherent with the whole harmonic tradition of the 20th century and earlier.

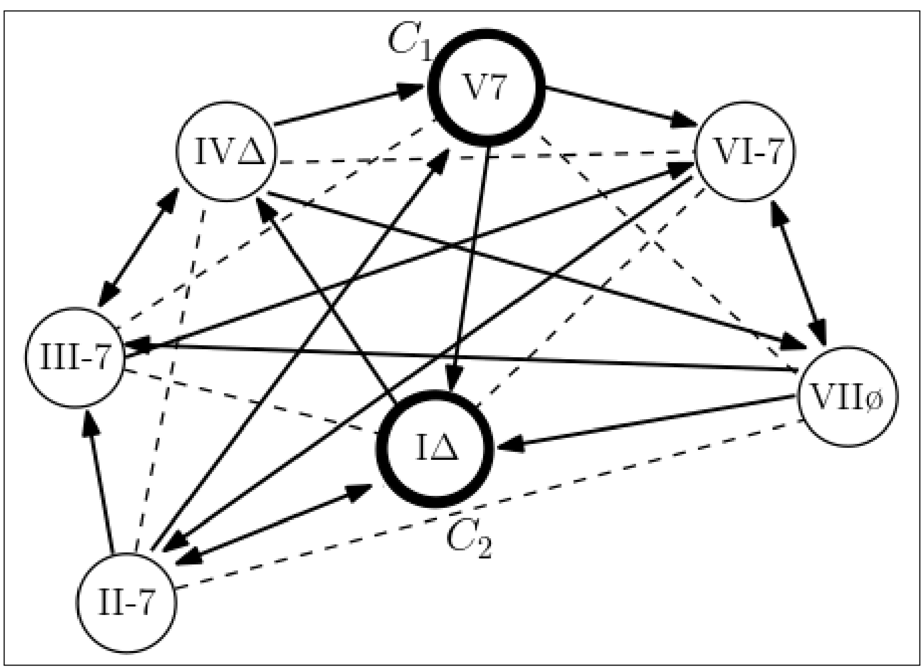

Figure 8.5: Tonal functions for the progression $[P] = (V7|I\Delta)$

In the case of $[P] = (II_c - 7|V_c 7|I_c \Delta)$, we see that the relationship is also clear. In this case, this progression is the most common one in jazz music; see Figure 8.6 on the next page.

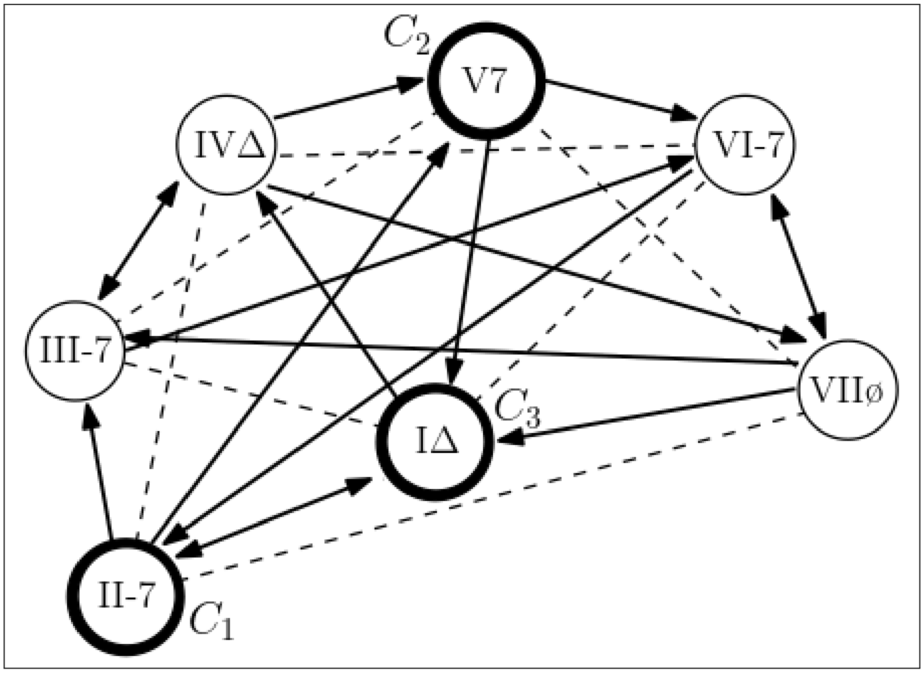

Figure 8.6: Tonal functions for progression$[P] = (II - 7|V7|I\Delta)$

We wonder whether in the case of $[P] = (VI_c - 7|II_c - 7|V_c 7|I_c \Delta)$ it woud turn out that convergence in the graph of tonal functions still holds. This means that the tonal functions are polarized and that the convergent algebraic multiplicity is greater than or equal to 2. Indeed, after carrying out the proper computations, we can verify that these traditional progressions comply with the pattern proposed by the model. By using the digraph we can trace paths to generate other compositions and always ensure that the changes are working; see Figure 8.7 on next page.

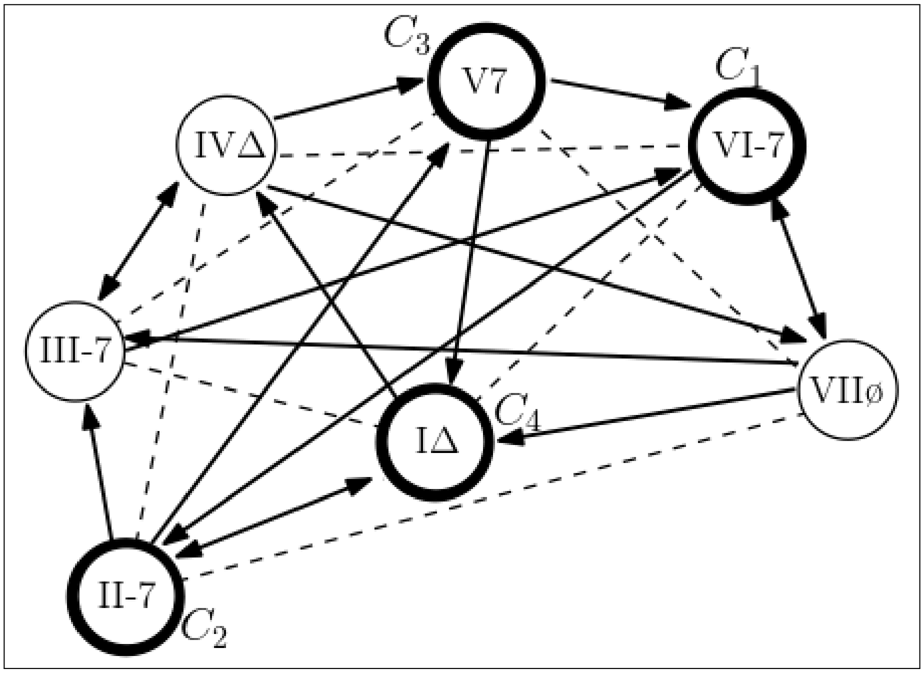

Figure 8.7: Tonal functions for progression $[P] = (VI - 7|II - 7|V7|I\Delta)$

Another case of frequent reasoning is when $[P]$ is compared with $[\lambda P]$. We have that $[P] = (I_c\Delta|II_c - 7|III_c - 7|VI_c - 7|II_c - 7|V_c7)$ and we can generalize the result to any tone.

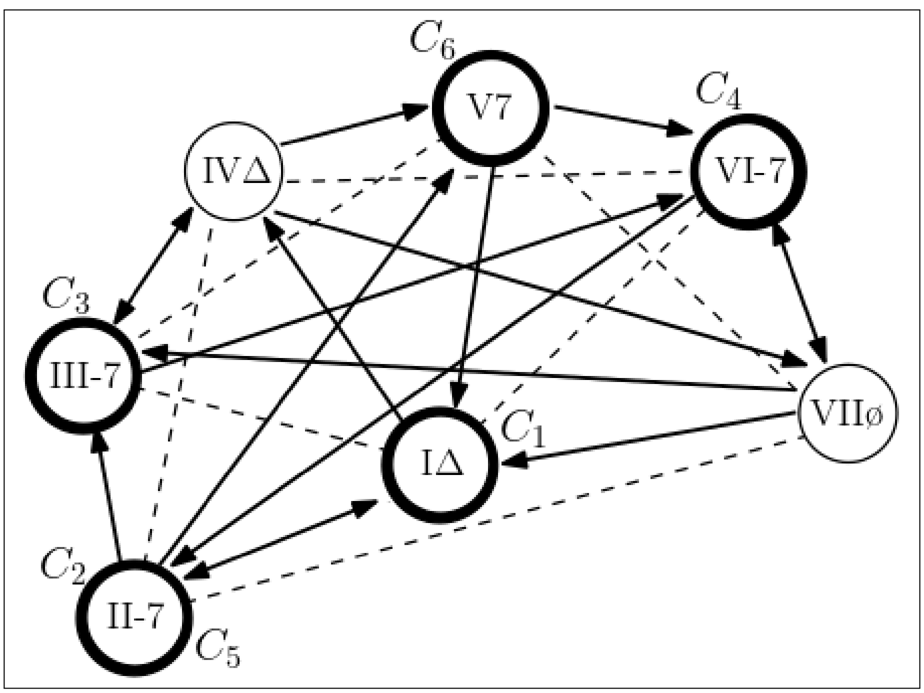

Figure 8.8: Tonal functions for progression $[P] = (I\Delta|II - 7|III - 7|VI - 7|II - 7|V7)$

If $\lambda \neq 0$, the progression would be as shown above (the tonal functions are preserved).

$$[\lambda P] = (\lambda I_c\Delta|\lambda II_c - 7|\lambda III_c - 7|\lambda VI_c - 7|\lambda II_c - 7|\lambda V_c7)$$

The **general graph** is a graph whose nodes are the set of tonal centers of arbitrary dimension and whose edges are all possible tonal functions among pairs of tonal centers. This graph is a complete graph and will be denoted by G^{ϕ}:

$$G^{\phi} = (N, A) = (N^{G^{\phi}}, A^{G^{\phi}})$$

Then the number of edges for $N^{G^{\phi}}$ will be, using combinatorics:

$$|A^{G^{\phi}}| = C(|N^{G^{\phi}}|, 2) = \frac{|N^{G^{\phi}}|!}{2!(|N^{G^{\phi}}| - 2)!}$$

Then specifying cardinals in the general graph formula would leave:

$$G^{\phi} = (N, A) = \left(\left\{ (N^{G^{\phi}})^1, \ldots, (N^{G^{\phi}})^{|\wp(LC_k/\sim)|} \right\}, \left\{ (A^{G^{\phi}})^1, \ldots, (A^{G^{\phi}})^{\frac{|N^{G^{\phi}}|!}{2!(|N^{G^{\phi}}| - 2)!}} \right\} \right)$$

Any proper subset of N will constitute a possible set of tonal centers. By elementary combinatorics, there are 4,096. It is clear that the construction of the graph is uniquely related to the number of classes of $LC_k/\sim$. Then, it is enough to change the initial system to generate another one. It is clear that if a quotient set is in another set, then the graph generated by both preserves the membership relations between those sets. Since the model of tonal functions can be adapted to other systems, it would suffice to take another quotient set to construct the general graph associated with the new set and thus establish the total set of tonal centers that determines the quotient set in question.

In this work the tempered system is used because it is the established system in western music and it has good mathematical properties allowing in particular to modulate to other keys, although the system is not the only one available.

Let us now consider the **power graph** $\wp(G^{\phi})$, where S_i are subgraphs of the general one. Thus, the power of the general graph would be given by the set of subgraphs that are in the general graph, each with the respective edges linking the subset of nodes in question.

$$\wp(G^{\phi}) = \bigcup_{i=1}^{|\wp(N^{G^{\phi}})|} S_i^{G^{\phi}} \text{ such as } S_i^{G^{\phi}} = \left(N_i \subset \wp(N^{G^{\phi}}), A_i \subset A^{\wp(G^{\phi})} \right)$$

Now we ask ourselves how many graphs there are in $\wp(G^{\phi})$, which is not difficult since it is enough to calculate the number of subgraphs of the general graph,

$$|\wp(N^{G^{\phi}})| = 2^{|\wp(LC_k/\sim)|} = 2^{4096}$$

Thus, this amount is the total number of subgraphs of the general graph.

1044388 881413 152 506 691752 710 716 624 382 579 964 249 047383 780 384233
483 283 953 907 971557456 848 826 811 934 997558 340 890 106 714439 262
837 987573 438 185 793 607 263 236 087 851 365 277 945 956 976 543 709 998
340361 590134383 718 314428 070 011855 946 226 376 318 839 397 712 745 672
334684 3445866174968079087058 03 704071 284048740118609 114467977783 598
029 006 686 938 976 881787 785 946 905 630190 260940 599 579453 432 823 469
303 026 696 443 059025 015 972 399 867 714215 541693 835 559 885 291486318
237914434496734087811872639496475 100189041349 008417061675 093 668 333
850 551032 972 088 269 550 769 983 616 369 411933 015 213 796 825 837 188
091833 656 751221318 492 846 368 125 550 225 998300 412 344 784 862 595
674492194 617023 806505913 245 610 825 731835 380087 608622 102834270 197
698 202 313 169017 678 006 675 195 485 079 921636419 370285 375 124 784
014907159 135459982790513399611 551794271106831 134090584272884 279 791554
849 782 954 323 534 517 065 223 269 061394 905 987 693 002122 963 395 687 782
878 948 440 616 007412 945 674 919 823 050 571 642 377154 816 321 380 631045
902 916136 926 708 342 856 440 730447 899 971 901781465 763 473 223 850 267
253 059 899 795 996 090 799 469 201774 624 817 718 449 867455 659 250178 329
070473 119 433 165 550 807 568 221846 571746 373 296 884 912 819 520 317457
002440 926 616 910 874148 385 078 411929 804 522 981857338 977 648103126 085
903 001302413 467189 726673 216491511 131602920 781738 033 436 090 243 804
708 340 403 154190 336

Note that this is not the number of possible progressions since the graph does not allow for repetition. It is the number of sets containing non-repeating chords that are represented in the general graph.

8.1.5. Method for increasing dimension or the D-method

In this section we will present a method to compose sections of music given a set of tonal centers without computing the tonal functions of each progression. This method, called the **D-method**, takes advantage of a certain property of modulation. That property states that when the dimension of a chord is incrementally increased, its tonal function either converges to the **identity** or the **tonic global function**. The algorithms terminates when the algorithm outputs a tonal function that satisfies the criteria of the composer. We next present the description of such algorithm.

ALGORITHM

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(1) We compute a graph of global tonal functions with an arbitrary number of nodes k and verify that every pair of consequent tonal centers has a dominant or tonic function.

$$G(N, A) = \Big(\{T_1, \ldots, T_k\}, \{H_{G=(T^d \bigotimes T^e)}(\lambda)\}\Big)$$

Thus, we draw a number of tonal centers and form a section.

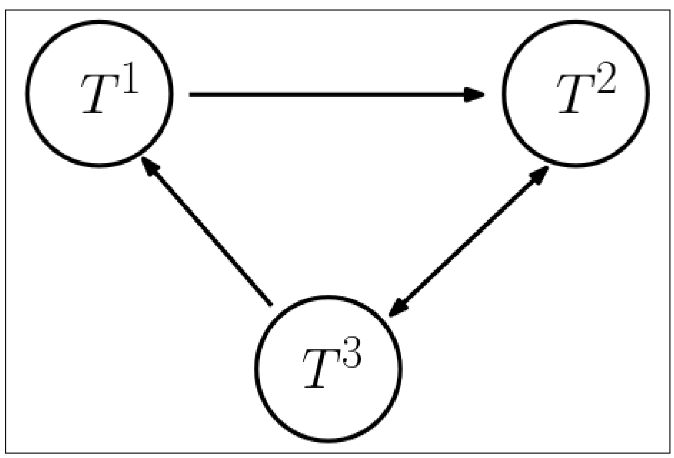

Figure 8.9: Section formed by some tonal centers

The section does not yet have chords, but the assignment will not be a problem. The relations between the tonal centers have been calculated previously and for the Greek modes there are only 6, since the other 6 come from inverting the arrow of the previous relation. These relations in C_L are summarized in a simple way; let us use the notation $[X]_L$ to indicate that it is a Lydian mode. It would be:

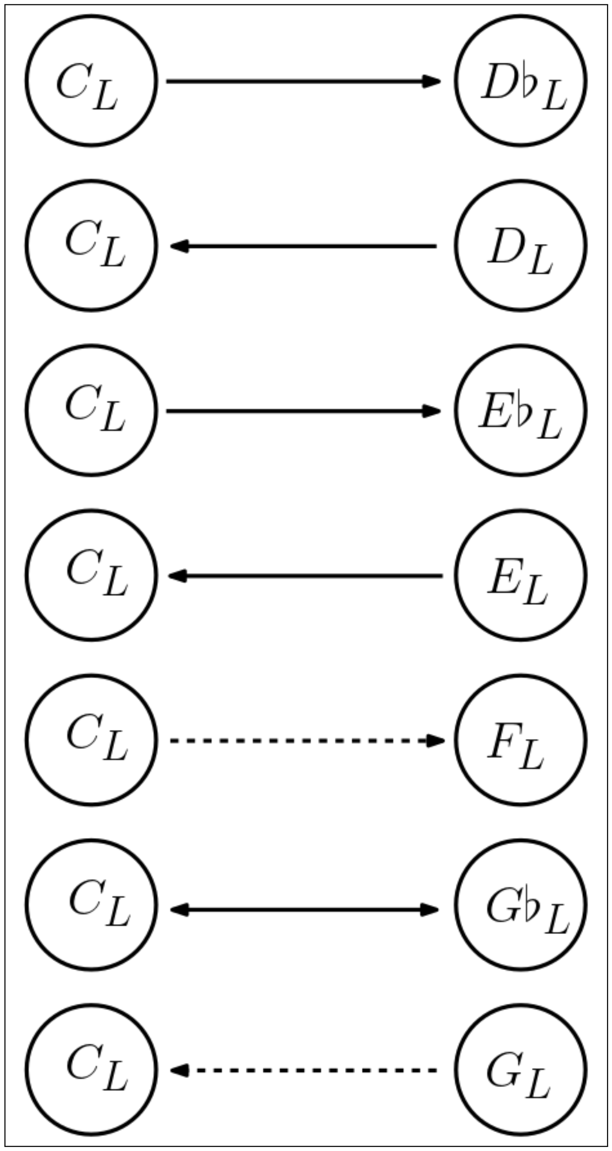

Figure 8.10: Relations between tonal centers

Under these conditions we can now generate a section. It must be taken into account that for each pair of consequent tonal centers the function must be tonic or dominant.

(2) We now choose a **ratio** for each tonal center; in principle, we can choose the same for all the tonal centers of the section, but that is not necessary. We refer to the ratio of the tonal center i as the number of **sub-nodes** in T^i, where $i = 1, \ldots, k$. Depending on the style and how we want the music to work, we will set the ratios to a lower or higher number. The ratio of a section is given by the sum of the ratios divided by the **number of tonal centers**, and gives an idea of what the **global tonal function graph** will be like.

(3) Once we have drawn the section, we choose chords from the row where the fourth degree (Lydian) is located and we draw them as sub-nodes of each tonal center; see Table 8.1.

	IΔ	II-7	III-7	IVΔ	V7	VI-7	VII∅
	CΔ	D-7	E-7	FΔ	G7	A-7	B∅
♭	FΔ	G-7	A-7	B♭Δ	C7	D-7	E∅
♭♭	B♭Δ	C-7	D-7	E♭Δ	F7	G-7	A∅
♭♭♭	E♭Δ	F-7	G-7	A♭Δ	B♭7	C-7	D∅
♭♭♭♭	A♭Δ	B♭-7	C-7	D♭Δ	E♭7	F-7	G∅
♭♭♭♭♭	D♭Δ	E♭-7	F-7	G♭Δ	A♭7	B♭-7	C∅
♭♭♭♭♭♭	G♭Δ	A♭-7	B♭-7	BΔ	D♭7	E♭-7	F∅
♯♯♯♯♯	BΔ	C♯-7	D♯-7	EΔ	F♯7	G♯-7	A♯∅
♯♯♯♯	EΔ	F♯-7	G♯-7	AΔ	B7	C♯-7	D♯∅
♯♯♯	AΔ	B-7	C♯-7	DΔ	E7	F♯-7	G♯∅
♯♯	DΔ	E-7	F♯-7	GΔ	A7	B-7	F♯∅
♯	GΔ	A-7	B-7	CΔ	D7	E-7	F♯∅

Table 8.1: Master table

We next take chords from each Lydian tonal center and choose, for the chosen ratio, as many chords as the ratio per tonal center sets. In Figure 8.11, we drew the section without sub-nodes and placed the corresponding row of the master table (Table 8.1) to later assign the chords to each center.

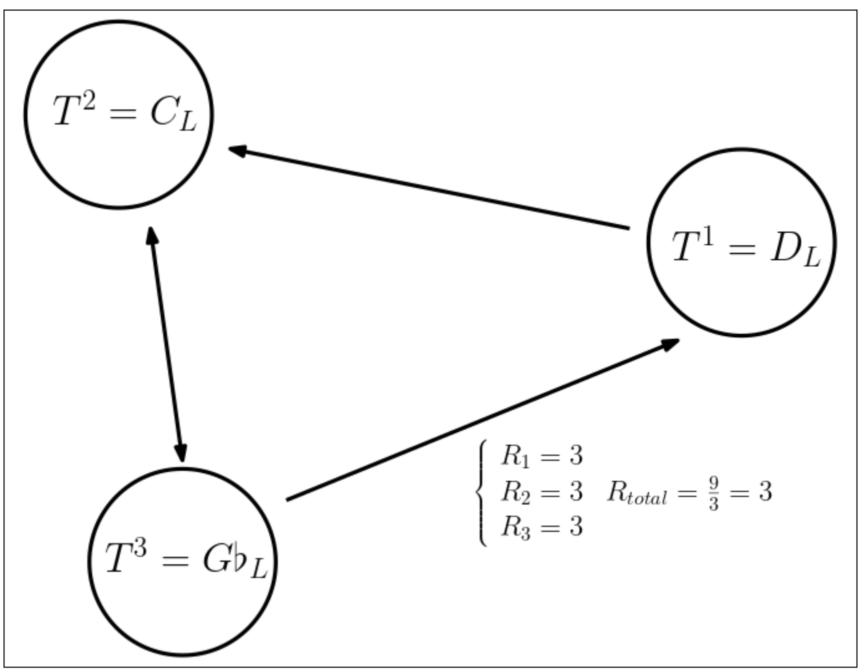

Figure 8.11: Tonal centers and ratios

In order to generate a progression class $[P]$, we now randomly assign chords that are in the row. We could do this by rolling a seven-sided dice or by making cards with the chords and drawing them from each deck of 7 cards at random. Once the assignment is done, then the section is equipped with sub-nodes. This would be as follows:

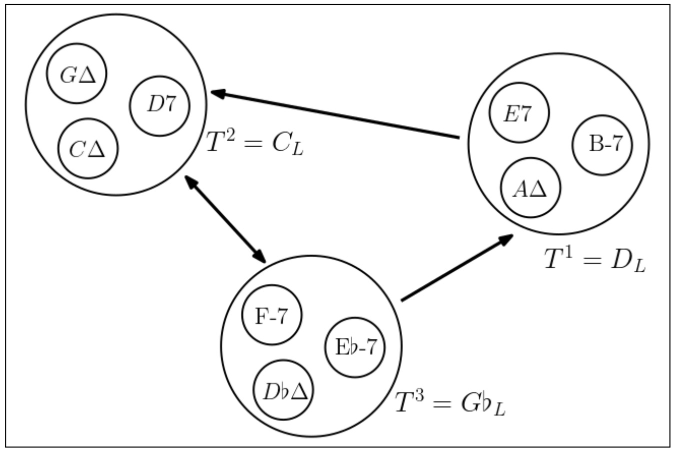

Figure 8.12: Sub-nodes in the global tonal function graph

Therefore, we have assured the convergence for the consequent tonal centers, that is to say, that every tonal center converges to the following one. The problem is that we do not yet know what the relationships between the sub-nodes are like at the transition between two tonal centers. We can get to know the relationship between symbols in the same center, but at modulation time, a decision must be made.

$$[P] = \overbrace{X_1 \underbrace{\otimes}_{H_{(X_1|X_2)(\lambda)}} X_2 \underbrace{\otimes}_{H_{(X_2|X_3)(\lambda)}} X_3}^{T^1} \underbrace{\otimes}_{?} \overbrace{X_4 \underbrace{\otimes}_{H_{(X_4|X_5)(\lambda)}} X_5 \underbrace{\otimes}_{H_{(X_5|X_6)(\lambda)}} X_6}^{T^2} \underbrace{\otimes}_{?}$$

$$\overbrace{X_7 \underbrace{\otimes}_{H_{(X_7|X_8)(\lambda)}} X_8 \underbrace{\otimes}_{H_{(X_8|X_9)(\lambda)}} X_9}^{T^3}$$

Let us say we calculate all the pairs of tonal centers between T^1 and T^2 , i.e. ; for the transition between T^1 and T^2 we do not find a way to find a path in the digraph, i.e. once the chords are chosen, there is no way for the arrows to reach the fourth chord; see Figure 8.13.

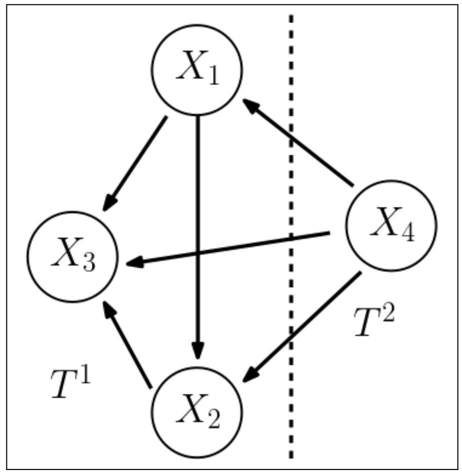

Figure 8.13: Potential tonal functions for T^1 and T^2

For the second figure, the tonal functions between the first 3 chords do not allow to permute them, i.e. we have to end on X_3. The problem is that then the arrow is still inverted between X_3 and a hypothetical X_4 in T^2

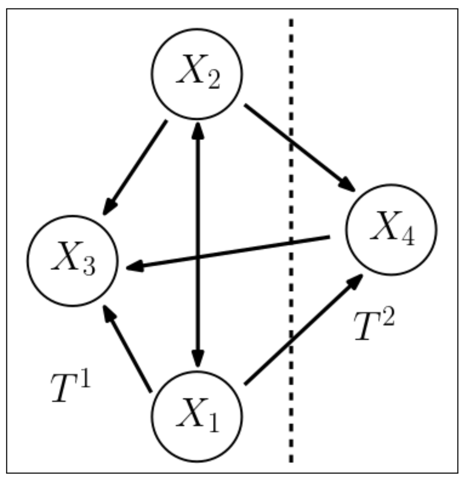

Figure 8.14: Another potential tonal functions for T^1 and T^2

Since this problem would have no solution for a fixed dimension, we considered extending the dimension. We can use **slash chords** to increase the dimension. The notation followed here is the slash indistinctly for **polychords** or **slash chords**, although when writing chords in a score the bar is used for polychords and the slash for slash chords.

(4) Since the tonal functions are unknown between the last chord of T^1 and the first chord of T^2 and it may happen that there is no way to rearrange the chords so that there is a path in the digraph of tonal functions, then we consider increasing the dimension of the last chord. We use the symbol $\oplus$ to indicate that two tonal centers are **consecutive** and also **converge**. That is, the tonal function between them is dominant or tonic.

Suppose we have a progression class $[Q]$ where consecutive chords do not necessarily converge:

$$[Q] = \left(\overbrace{X_a \underbrace{\bigotimes}_{H_{(X_a|X_b)}(\lambda)} X_b \underbrace{\bigotimes}_{H_{(X_b|X_c)}(\lambda)} X_c}^{T^e} \right)$$

Given that we assume that $T^e \bigoplus T^d$, then we can form **poly-chords** so that by increasing the dimension, sooner or later the convergence between the chords and between the tonal centers will occur simultaneously.

$$[Q \bigoplus Q^*] = \left(\overbrace{(X_a/Y_a) \underbrace{\bigoplus}_{H_{((X_a/Y_a)|(X_b/Y_b))}(\lambda)} (X_b/Y_b) \underbrace{\bigoplus}_{H_{((X_b/Y_b)|(X_c/Y_c))}(\lambda)} (X_c/Y_c)}^{T^e} \right)$$

$$\overbrace{\underbrace{\bigoplus}_{H_{((X_c/Y_c)|(X_d/Y_d))}(\lambda)}}^{H_{G(T^e|T^u)}(\lambda)} \left(\overbrace{(X_d/Y_d) \underbrace{\bigoplus}_{H_{((X_d/Y_d)|(X_{d+1}/Y_{d+1})}(\lambda)} \ldots\ldots}^{T^u} \right)$$

This happens because we know the convergence between the centers $T^e \bigoplus T^d$ and we know that by increasing the number of voices of a chord, it will surely converge when it is equal to its center.

Thus, we can write a progression with the convergence between centers assured and then write a few chords randomly and expand them one by one until we find the convergence between them. This is the incremental dimension method or D-method and it is a compositional tool that maintains the logic between structures, but adds a random component that stylistically refreshes the most common functional progressions. The musical results are always coherent and allows us to generate a lot of musical material to later apply arranging and production techniques to the progressions.

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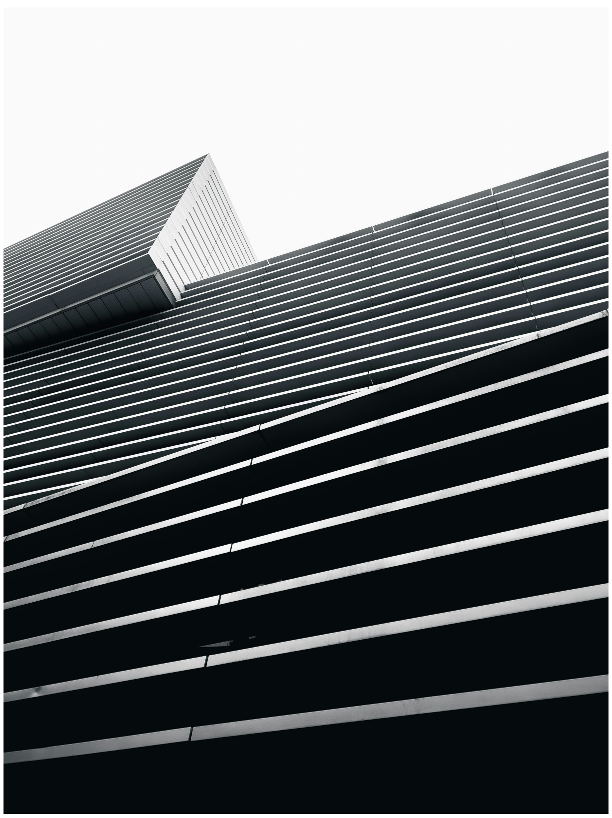
Chapter 9

Applications



Shubham Dhage

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9.1. Applications

In this chapter, we lay out applications of the theoretical work done so far to musical composition. These applications are present in the compositional activity of every composer, arranger or improviser.

9.2. Vertical applications

9.2.1. Vertical dissonance graphs

1 Use of **VDG** graphs as an analytical tool in the analysis of compositions and pre-existing works.

2 Use of **VDG** graphs for the computation of a clean form contained in an open form $\mathcal{C}(\mathcal{F}^{\mathcal{O}}) \subset \mathcal{F}^{\mathcal{O}}$.

3 Use of the **VDG** graphs for the analysis of the scales that can be used on each chord without establishing b9 relationships or, as is sometimes interesting, establishing them on purpose to generate dissonance.

9.2.2. Poly-chords

1 Use of poly-chords for the composition of material and in improvisation using each of the forms that arise from combining basic structures.

2 Use of open forms for the enrichment of the harmonic language for the jazz pianist or the amateur.

3 Expansion of the harmonic language creating combinations of shapes and generating poly-chord exercises in all keys to integrate them into the language of improvisation.

4 Use of polychords as an object of study for melodic instruments and understanding of their operation through the knowledge of VDG in combination with the static function theorem.

9.3. Horizontal applications

9.3.1. Voice leading

With the use of the Hungarian algorithm in combination with infinite arithmetic (extended Hungarian algorithm) we can solve any voice leading problem, which immediately precipitates us to write either music, or exercises for harmonic instruments. As we have seen we can always link two chords X and Y using some matrix L.

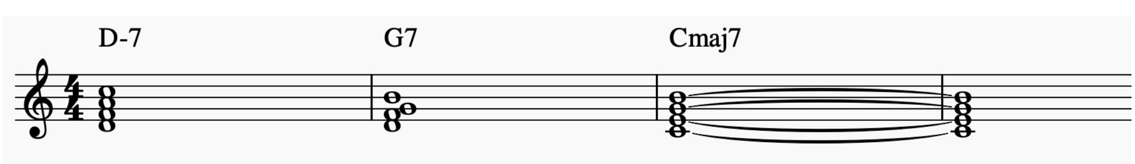


Figure 9.1: II-7-V7-I△ voice leading

9.3.2. Progression generation

If $\mid \mathbb{S} \mid$ is the cardinal of the system and $\mid \mathcal{E}_{\mathbb{S}} \mid$ is the number of links, then if the horizontal dimension of $[P]$ is m, we can compute any P by concatenating links of $\mid \mathcal{E}_{\mathbb{S}} \mid$. If P is in $\mathbb{P}_{\mathbb{S}}$, using the properties of the cartesian product we have $(\mid \mathcal{E}_{\mathbb{S}} \mid)^m = \mid P \mid$

Thus we build progressions where we can use the D-method or graphs of tonal functions or other techniques. In the next pages, some graphs of constant structures in all keys are shown with the notation of tonal functions presented previously.

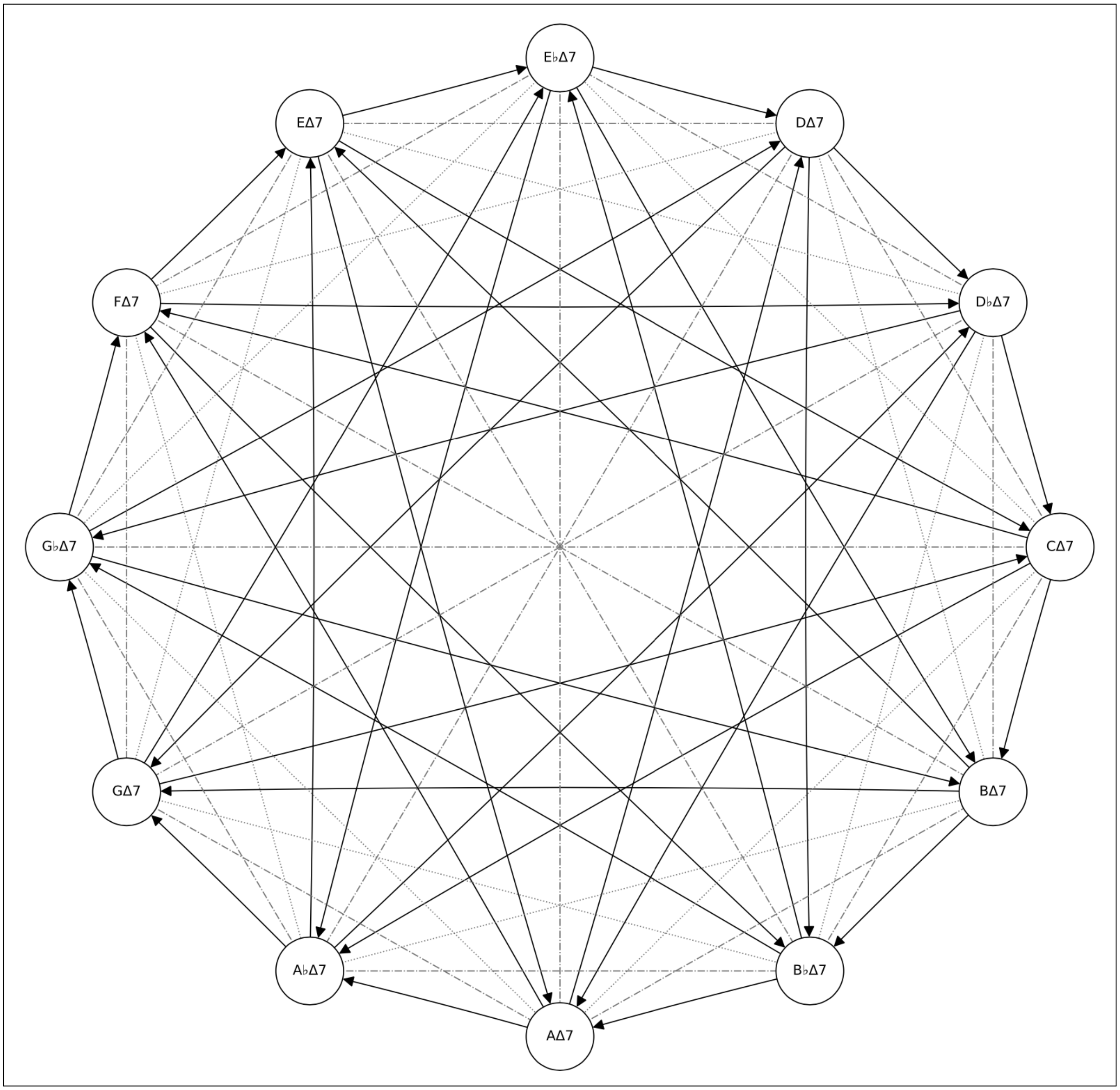


Figure 9.2: Tonal functions for constant major seventh chords

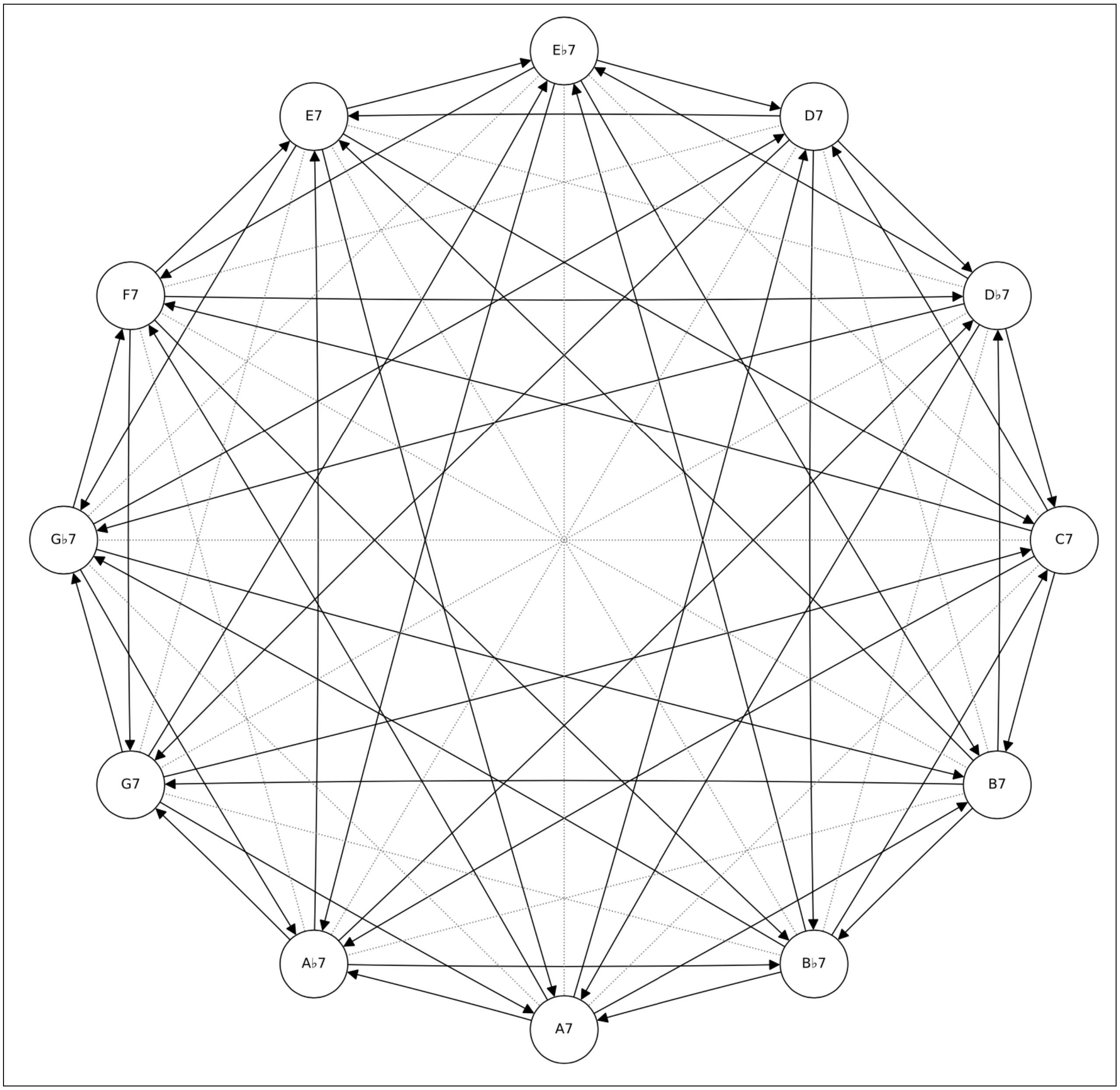


Figure 9.3: Tonal functions for constant dominant chords

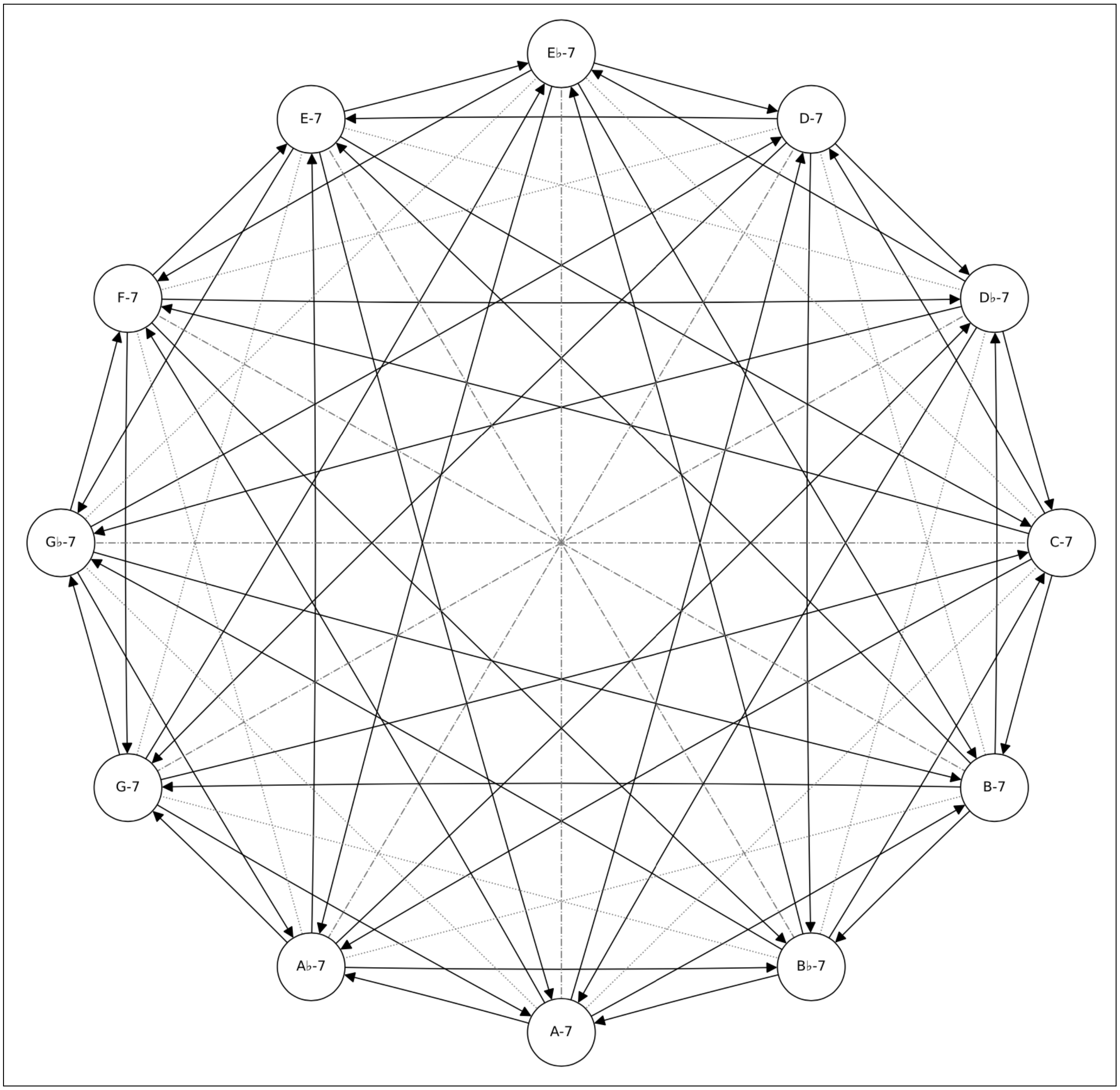


Figure 9.4: Tonal functions for constant minor seventh chords

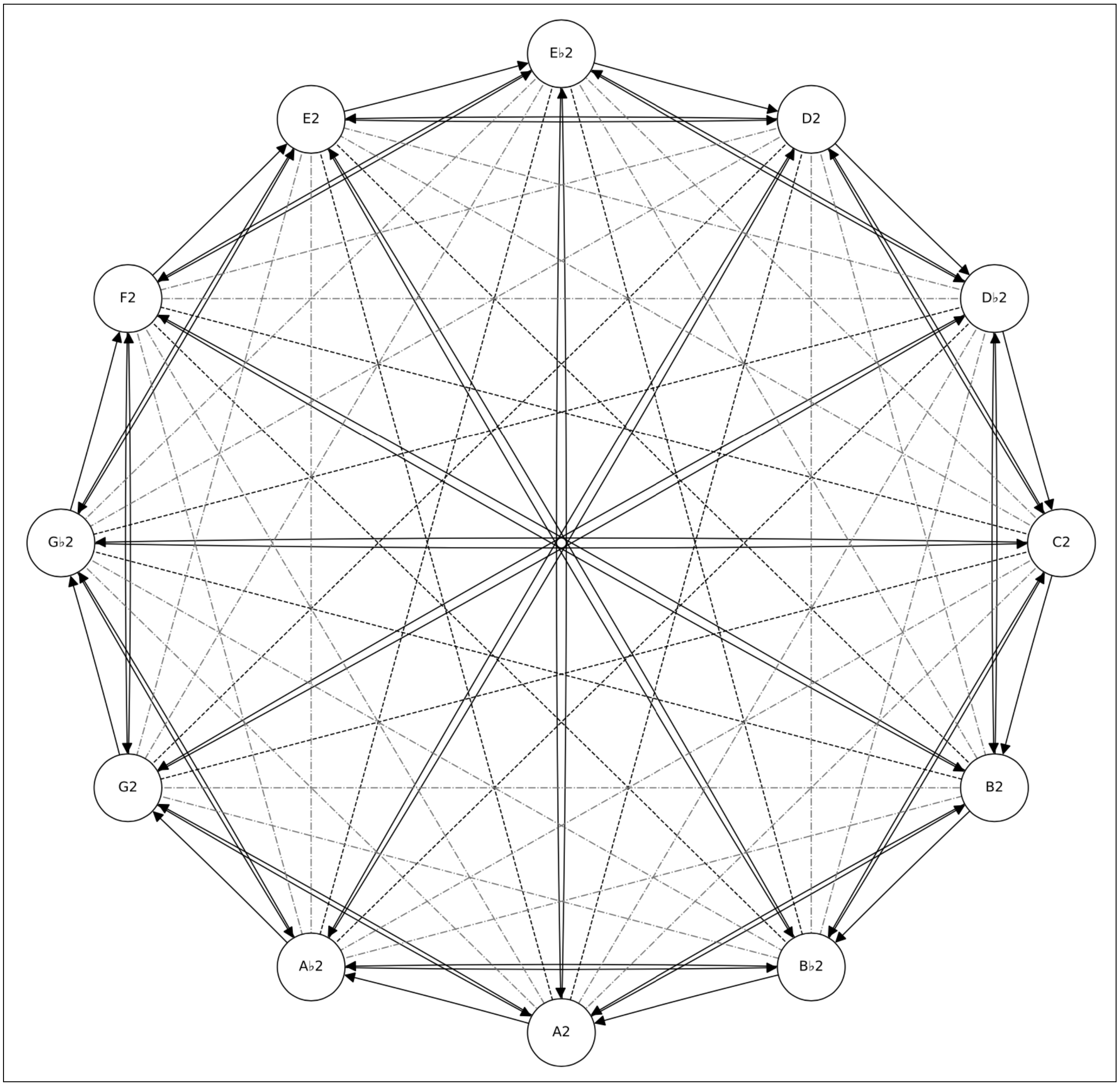


Figure 9.5: Tonal functions for constant sus 2 chords

9.3.3. Tune generation

Since we have delimited the concept of global tonal functions $\forall_G$, then the problem of linking multiple harmonic progressions is solved since we have a method to understand modulation, even for **open keys**, calculations for traditional tonalities have been covered in the work, but the method is global and can be applied to any structure. Next, we show some graphs calculated for open keys.

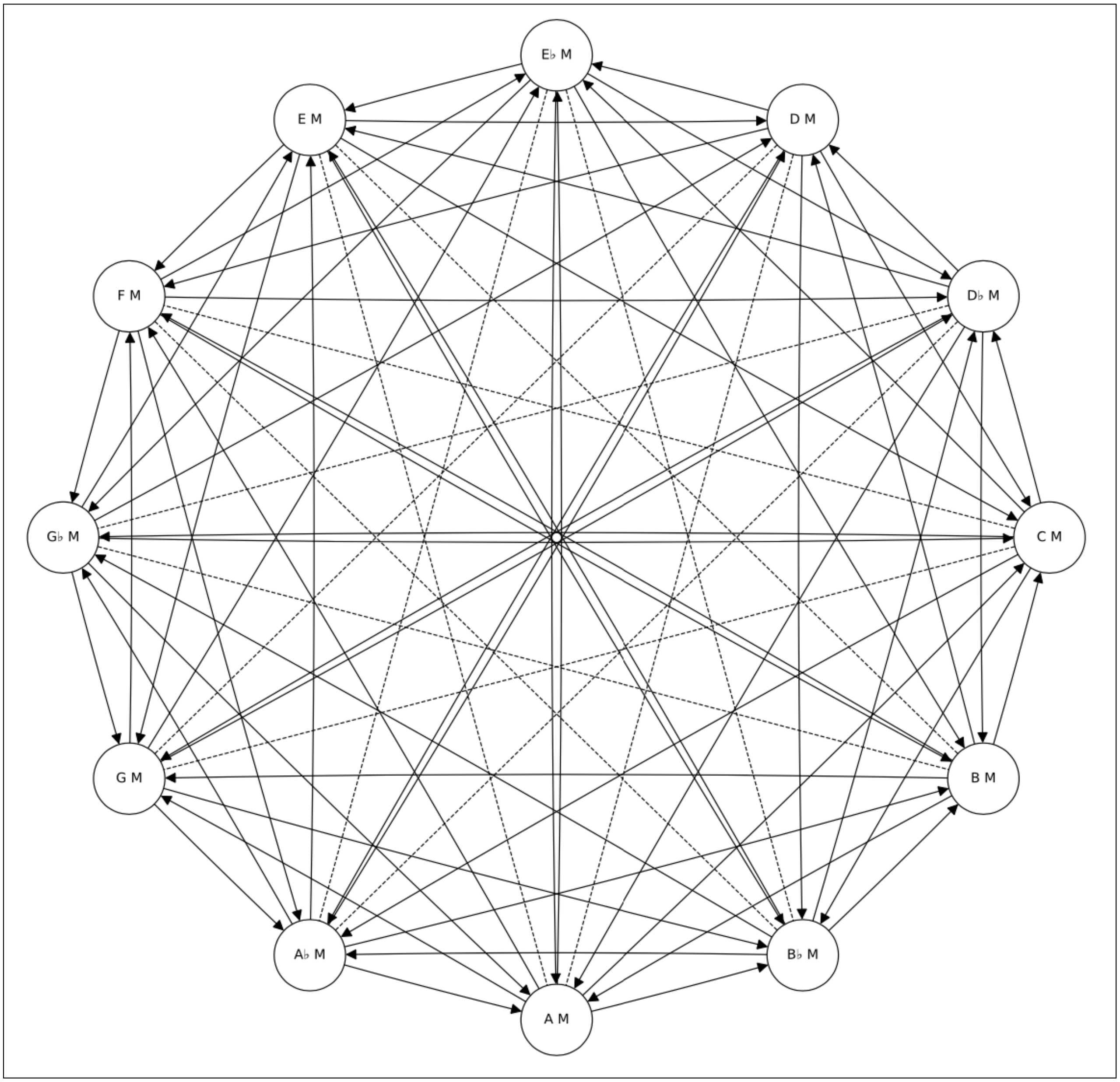


Figure 9.6: Major scales graph

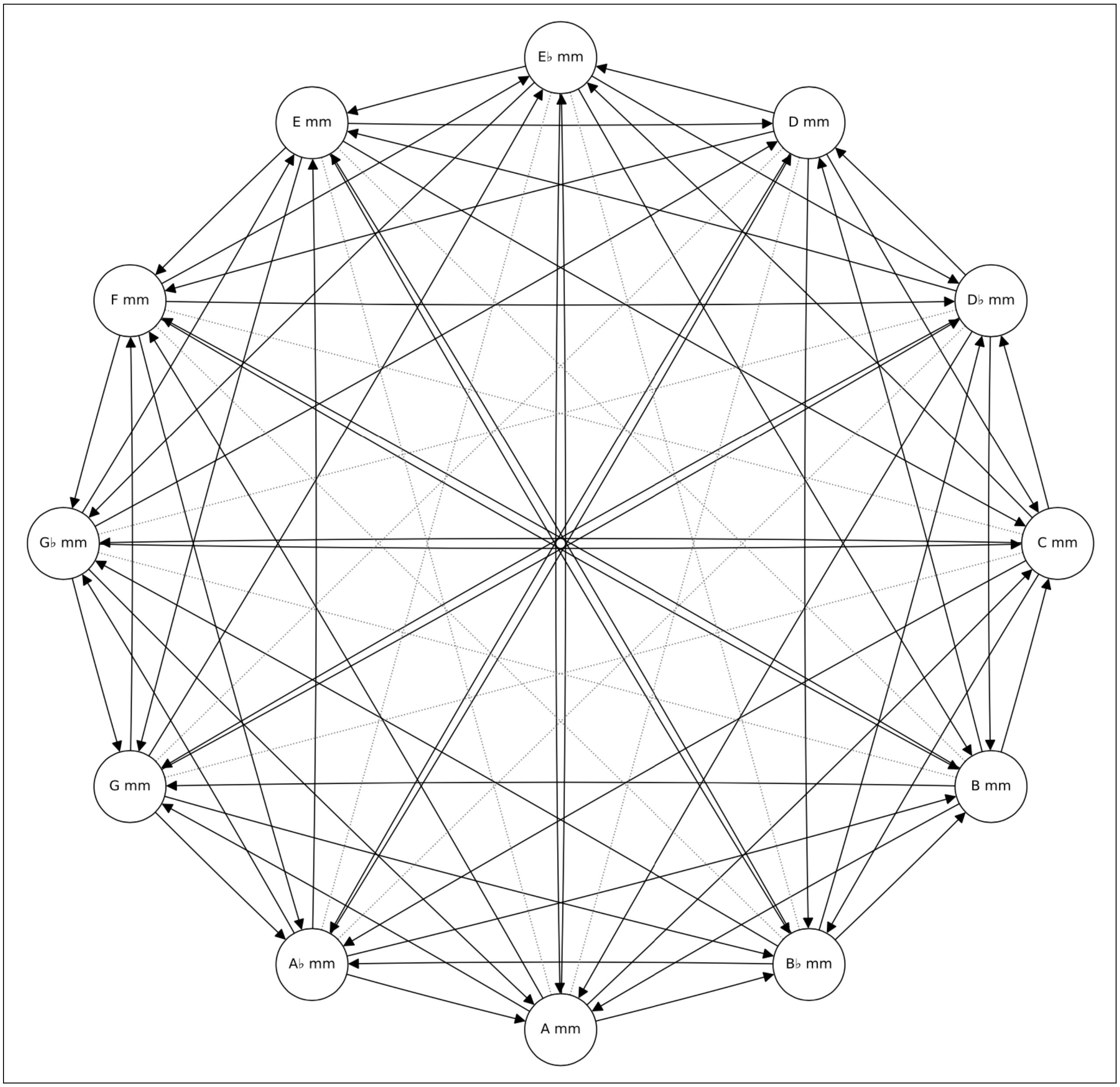


Figure 9.7: Melodic minor scales graph

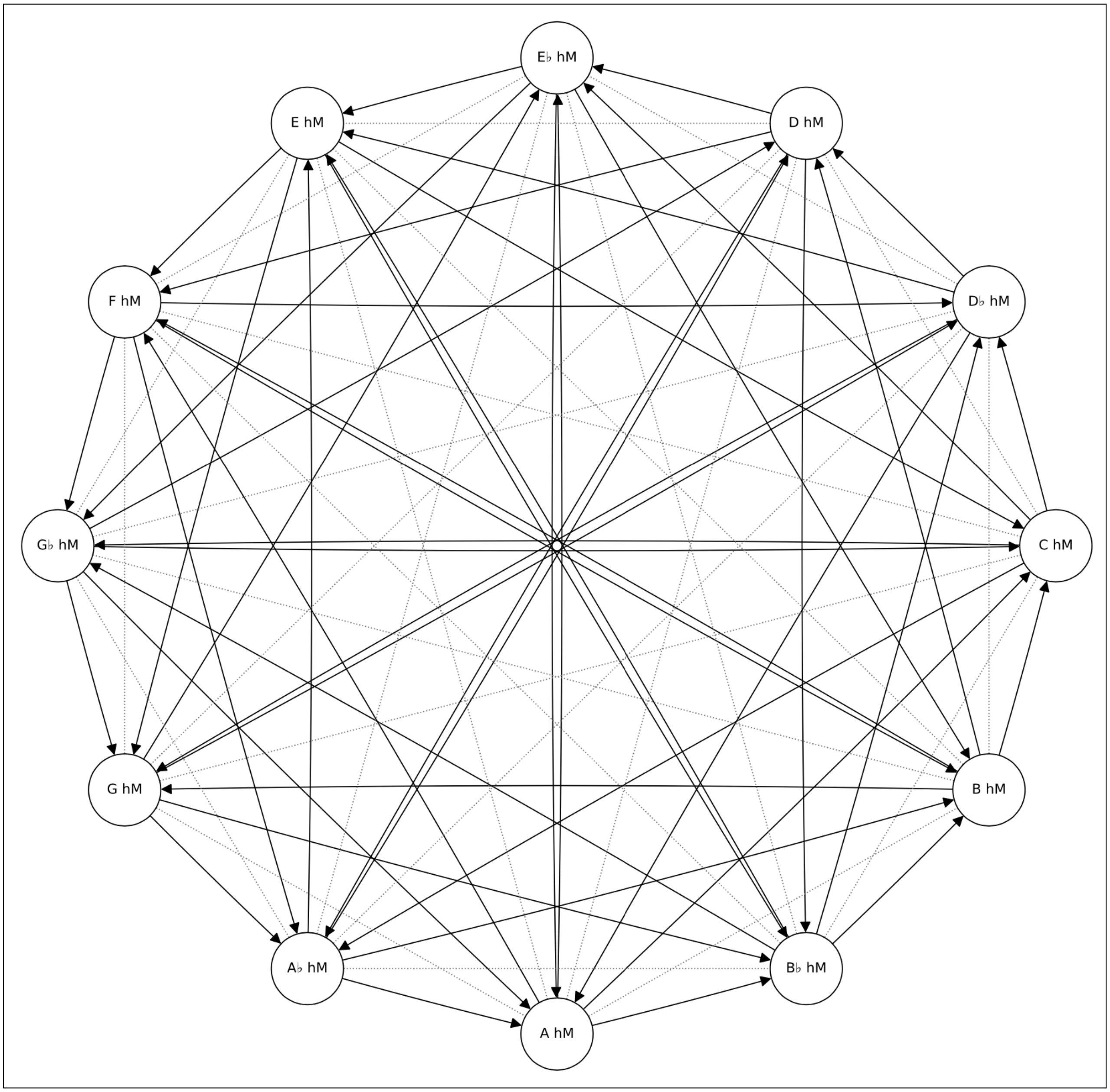


Figure 9.8: Harmonic major scales graph

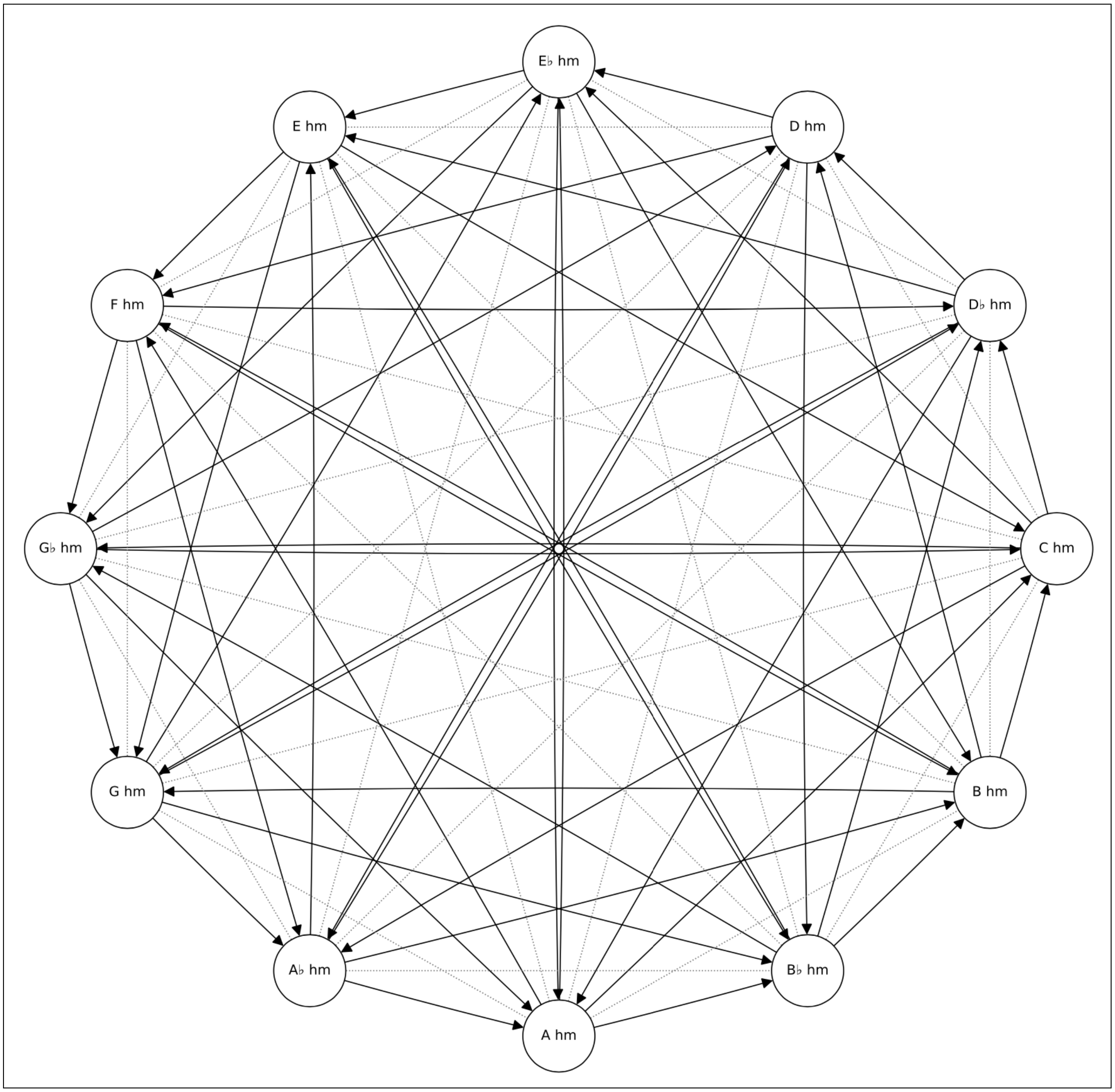


Figure 9.9: Harmonic minor scales graph

9.3.4. Walking bass line writing

The line writing problem is also solved since we added a build factor based on voice-leading. The lines, being one-dimensional, have a tonal function of tonic since the dimension of the line is one, so in reality they can do anything. We choose the notes that intervene for its construction. If $[P] = (C-7 \mid F7 \mid Bb\triangle \mid Eb\triangle)$ and we restrict the voices to only tonic and fifth with $n = 2$, we create P.

$$P = \begin{pmatrix} G & F & F & Eb \\ C & C & Bb & Bb \end{pmatrix} \tag{9.1}$$

Starting from restricting the bass voices, it is enough to draw a **contour** on the optimized matrix P to write the bass line:

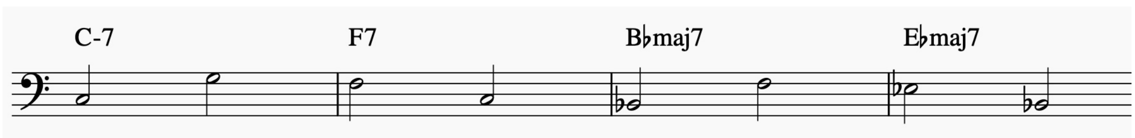


Figure 9.10: Two-voiced bass line

If we want to write a walking bass line, then instead of restricting to two voices, we will restrict to three, being the matrix W the progression where we only admit the tonic, the fifth and the third of each chord. This decision to eliminate tensions in the bass is made for known reasons that have to do with the harmonic series. We optimize W using the Hungarian algorithm and draw a contour C_1, repeating the third voice.

$$W = \begin{pmatrix} G & A & Bb & Bb \\ Eb & F & F & G \\ C & C & D & Eb \end{pmatrix} \tag{9.2}$$

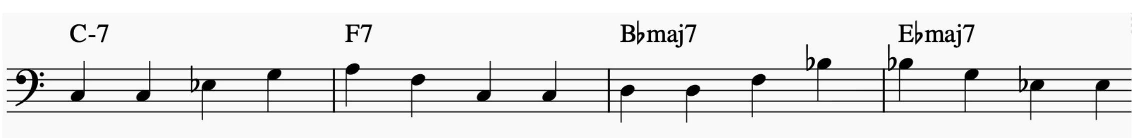


Figure 9.11: A first contour of a three-voiced bass line

Tracing another different contour we write another different bass line from said contour C_2.

Figure 9.12: A second contour of a three-voiced bass line

9.3.5. Melodic line writing

For the writing of melodic lines for the study of improvisation or for composition, we can use the method of **tracing** on matrices of four voices $n = 4$. In this way if we take a progression M and optimize said matrix:

$$M = \begin{pmatrix} C & B & B & A \\ A & G & G & F \\ F & E & E & E \\ D & D & C & C \end{pmatrix} \tag{9.3}$$

All we have to do is trace over M to write a melodic line.

Figure 9.13: Four-voiced melodic contour

It can also be double traced on each column of M in such a way that we express the progression M in two octaves, obtaining the following melodic line.

Figure 9.14: A second four-voiced melodic contour

Eliminating the first and fourth voices of M, we obtain a **sub-progression** of M called S (removing rows from a progression) that follows the conductions of voices of M, that is to say, that S is not optimal in itself but it is on M. This S subprogression is traceable to obtaining melodic lines called **subvoices**.

$$S = \begin{pmatrix} A & G & G & F \\ F & E & E & E \end{pmatrix} \tag{9.4}$$

9.3.6. Reharmonization table

Given a melody seen as a vector of notes of dimension c, that is, we see that $\mathbb{M} = (\mathbb{M}_1, \ldots, \mathbb{M}_c)$, then we suppose that said frequencies sound over a progression P where $\mathbb{M}$ is distributed in chunks where the first chunk is $f_1 = (\mathbb{M}_1, \ldots, \mathbb{M}_a)$, the second is $f_2 = (\mathbb{M}_{a+1}, \ldots, \mathbb{M}_b)$ and the third is $f_{b+1} = (\mathbb{M}_{b+1}, \ldots, \mathbb{M}_c)$. Thus, we split the frequency vector into chunks (also called fragments), where each fragment is on each of the chords of P; that is, f_1 is sounding on X_1, f_2 is sounding on X_2 and f_3 is sounding on X_3.

$\mathbb{M}$	$(\mathbb{M}_1, \ldots, \mathbb{M}_a)$	$(\mathbb{M}_{a+1}, \ldots, \mathbb{M}_b)$	$(\mathbb{M}_{b+1}, \ldots, \mathbb{M}_c)$
P	X_1	X_2	X_3
R_1	X_1^*	X_2^*	X_3^*
R_2	X_1^{**}	X_2^{**}	X_3^{**}
R_3	X_1^{***}	X_2^{***}	X_3^{***}

Table 9.1: Reharmonization table

This is the so-called reharmonization table, where we see which fragment is placed on which chord of the P progression. Thus, our objective when reharmonizing is that, starting from a melody $\mathbb{M}$ and a pre-existing P, find other progressions R that work with $\mathbb{M}$. So the first thing we are going to do is write a progression R within the key that is convergent. If $\mathbb{M}$ is in several keys, then using the graph of global tonal functions we will try to see which pair of available keys it is in and which of these pairs have a convergent modulation relationship. With the above in mind we wrote a reharmonization R with the methods explained in the thesis. We will say that R is diatonic if the chords of R are contained in the same power set of a compact tonality as $\mathbb{M}$,($\mathbb{M}$ can be in several at the same time). We will say that R is not diatonic if instead, the chords of R are contained in the power set of an open key to which $\mathbb{M}$ does not belong.

9.3.7. Reharmonization conditions

With the reharmonization table present and a graph of tonal functions, we understand that there are some conditions for R to work, these conditions can be altered for artistic reasons but in general it is desirable that they be met for our R reharmonization since that we want R to work by itself and that by overlaying $\mathbb{M}$ (which has its own tonal function because it is one-dimensional, and generally has a different timbre) we do not incur vertical dissonance relationships. For this we define the tonal function of a progression $\not\forall_R$ as the union of the tonal functions of the $m-1$ links of a class progression $[R]$ with m chords. $\mathbb{M}_k$ is defined as each of the components of $\mathbb{M}$ that sounds on the chord X_j, where $1 \leq k \leq c$

$$\begin{cases} \not\forall_R = \bigcup_{i=1}^{m-1} \not\forall_{E_i} \subseteq \{D, T\} \\ \\ G_{(X_j) \cup [\mathbb{M}_k]} \text{ is stable for all } \mathbb{M} \end{cases}$$

9.3.8. Enclosures

For the study of the enclosures or approximation notes we consider the problem as that each approximation note is at most one tone from the target note so there are only four available $A = X - 2, B = X - 1, C = X + 1$ and $D = X + 2$. Thus we write the equations of these notes in ratios of semitones for convenience, although in reality these equations are only the exponents.

Taking all the approximation notes we would have all the permutations, being a total of 24 for four approximation notes.

If we consider 3 notes, then we have three sets from which to take permutations that would be

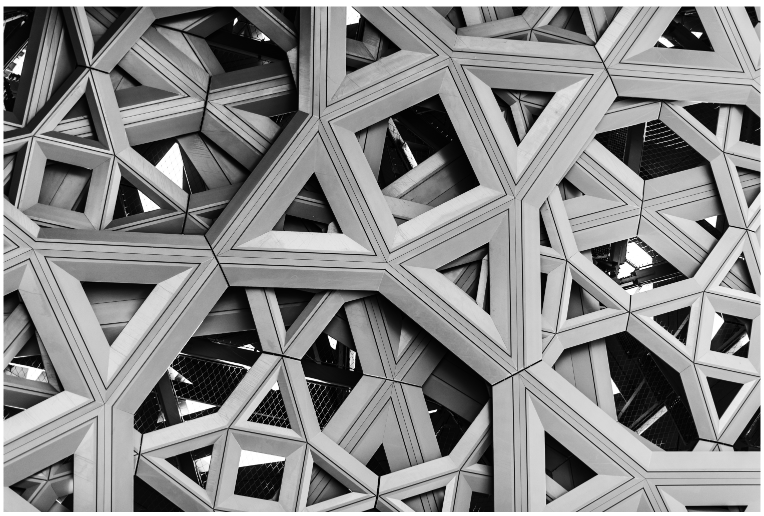
$\cancel{A}BCD$	$A\cancel{B}CD$	$AB\cancel{C}D$	$ABC\cancel{D}$

Table 9.2: Table of permutations

From the above we consider four sets of 6 permutations each, so we have a total of 24 possibilities for three-note enclosures. For the two-note enclosures we can visualize them as being all the ordered pairs of the set of approximation notes. Finally, we have to add 4 one-note enclosures. In total we have 24 of four notes, 24 of 3 notes, 12 of 2 notes and 4 of one note, so we have a total number of 64 enclosures.

Chapter 10

Conclusions and Future Work



Thomas Drouault

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10.1. Conclusions

In this thesis, a mathematical model for the tonal function has been presented. This model is termed the hypervolumetric tonal function model (HTFM). It combines physical, mathematical, and musical elements to explain and classify tonal function. This model has been designed with several features of interest in mind. First, the model should possess a great explanatory power and as we can see in the chapter on applications, the model is able to explain music in very general contexts, ranging from classical music to the common practice and including avant-garde jazz, just to name a few.

Second, the model should be of computational nature so that classification of the tonal function can be carried out automatically. For the computation of the nabla distances, an application was designed and programmed as described in Section 2.4; this application can be found on the usual software platforms. Furthermore, the whole HTFM was programmed on a computer and it is now possible to obtain a striking panoply of tonal function graphs. In

Section 9.1 and following these applications of tonal function are laid out (VDG, poly-chords, voice leading, progression generation, or tune generation, just to name but a few).

Third, the HTFM has another good feature and that is its agreement with human perception. As established in Chapter 4, the model has to be consistent with human perception and cognition of music and in that chapter all the physical, cognitive, and perceptual foundations of the HTFM model were comprehensively described and expatiated upon. The main result here was to prove how the minimization of voices in the model matched minimization of voices in the human perception (that was done in Section 4.3, by the fundamental theorem in Section 4.3.1 and in Section 4.4.3).

Fourth, the HTFM presents a much more nuanced and broader classification of tonal function classification. It goes beyond the standard classification into three functions —tonic, dominant, and subdominant—. The model provides other categories such as convergent tonic or other tonal function categories based on unusual voice movements (see 5.13, for example). Another outstanding quality is that the tonal function of chords with different number of voices are considered and computed in the HTFM. This part of the model was indeed quite delicate. The Hungarian algorithm had to be extended with an infinity arithmetic so that it could handle the appearance and disappearance of voices during the execution of the algorithm.

Fifth, the model should be musically meaningful, that is, not only does the model have to explain music in the most general context possible, but also it has to show great generative capabilities. As for its explanatory power, the HTFM agrees with the standard classification of tonal function for all the musical traditions we examined (which as a matter of fact includes the most relevant one in the Western traditions). As for the generative capabilities, the HTFM should be a useful tool for analysis and composition. In the chapter on applications (Chapter 9), many applications to music composition have been shown and we consider they constitute an excellent proof of this fact.

In summary, the HTFM provides a mathematical approach to defining and classifying tonal functions based on the physical and perceptual characteristics of optimal voice leading.

We next address the limits and the context of this thesis. The HTFM is suitable for music based on voice leading and chords, as is the case of Western music. However, there exist other kinds of music not based on vertical harmony as is the Western music tradition; think of the gamelan music, the traditional ensemble music of the Javanese, Sundanese, and Balinese peoples of Indonesia, just to name an example. In gamelan music, there exist the concept of simultaneous melodic lines occurring at the same time —which provides the sense of harmony —, but the musical goal is related to targets in the form of clusters and interlockingness between the melody lines.

Secondly, in some parts of the HTFM, we assumed that the timbre was constant (Sections 4.2.1 and section-4-3-1). This case is one of the most frequent and important found in the Western music tradition but also in many others. In that respect, the HTFM covers a great deal of practical cases. However, in other cases, even in the Western tradition, the constant timbre is not the case and then the model fails to fully explain and predict certain music phenomena. Notice that the fact the timbre is not constant only affects to some perceptual aspects to the model, especially those describe in Chapter 4. The classification of tonal function through the Hungarian algorithm and the polynomial criterion remain intact.

We are aware that the level of abstraction of this work is very high. Results were obtain and presented with the immediate goal to obtain a model for the problem of tonal function classification. Layers of abstraction and mathematical tools were wielded in order to subdue this problem. We are also aware that some parts of this thesis would benefit from a synthesis process leading to a more concise research (this is set as future work in the next section).

Lastly, we would like to have conducted some empirical work, but that was not possible given the magnitude and reach of the model itself. In this thesis, we laid out a theoretical work. The perceptual part of it is based on the Weber-Fechner and although that roots the model into actual perceptual facts, the study on actual subjects is of great interest. Informally, the author has shown to fellow musicians the model and how to use it; they all were delighted by the power, versatility and usefulness of the model, especially when it came down to music composition.

10.2. Future Work

Many venues for further research can be derived from this thesis. Some have been mentioned throughout the thesis. We summarize below the most important ones.

(1) Apply the HTFM to other music styles. As we mentioned earlier, the HTFM model is a general model, but it has been tested mainly on Western music or at least music based on voice leading and chord progressions.

(2) One piece of research that we were not able to carry out (due to time constraints) was empirically check how much time it would take to learn how to improvise with our model of tonal function. We would have liked to conduct experiments to answer that question.

(3) Generalize the HTFM so that non-constant timbre and amplitude are taken into account in the model. This seems to be a hard problem as the equations explode in complexity.

(4) We keep assuming that the timbre is kept constant as we did in Chapter 4. From Section 4.2.4, we consider Equation 4.16

$$\gamma = \Delta(\alpha, \beta) - \frac{1}{2}\Delta^{\mathbb{E}}(E_\alpha, E_\beta) \tag{10.1}$$

If we fix the two sounds α, β, we can ignore the arguments and write the previous equation just as mere functions.

$$\Delta^{\mathbb{E}} = 2\Delta - 2\gamma$$

Let us consider this relationship in the context of a link matrix L associated to a chord progression $(X \mid Y)$. As matrices, we can write, for $i = 1, \ldots, \max\{\dim(X), \dim(Y)\}$,

$$\left(\Delta^{\mathbb{E}}_{ij}\right) = 2(\Delta_{ij}) - 2(\gamma_{ij})$$

Considering a global link G, we substitute the terms in the previous equation for L matrices, which gives place to a new matrix $L_G^{\mathbb{E}}$ contained in $\mathbb{L}^{\mathbb{E}}$. The set $\mathbb{L}^{\mathbb{E}}$ is the set of matrices whose entries are the energy metrics between pairs of tonal centers. Therefore, the above yields

$$L_G^{\mathbb{E}} = 2L_G - 2(\gamma_{ij}) \tag{10.2}$$

This equation establishes the matrix the relationship of the L with the actual production of sound. Finally, we take tonal functions on the previous equation for the L matrices.

$$\not\oint_{L_G^{\mathbb{E}}} = \not\oint_{(2L_G - 2(\gamma_{ij}))} \tag{10.3}$$

The previous equations are telling us that within certain margin of tolerance for the amplitudes, the HTFM works.

(5) As we pointed out in the previous section, the HTFM could be streamlined and made more concise. When results are first obtained, they always come somewhat raw. We are aware that this our case and in the future these results should be refined in its presentation.

(6) As a future project, we plan on programming the whole model so that the tonal graphs and the tonal functions can be obtained automatically. Some advances were presented in this thesis with the Nabla application. The programming of the model is in progress at the time of the writing of this thesis.

(7) The pedagogical aspects to this work is a topic that was barely addressed here (due to time constraints and also for the reach of the thesis). That topic would give place to another thesis on its own.