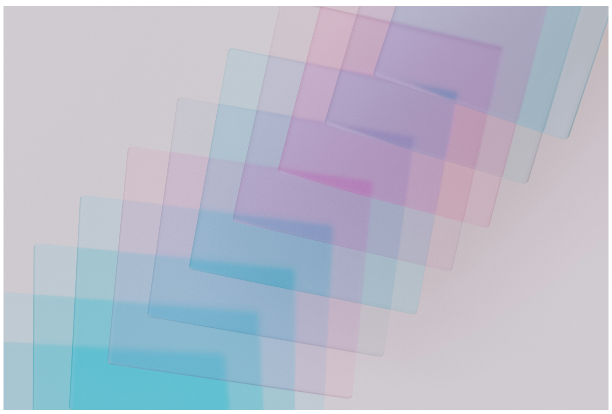
Appendix A

The Lydian Mode



Nick Fewings

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A.1. The Lydian Mode

A.1.1. II$\longrightarrow$ I Cadence

We focus on this first case and thus calculate the link between the second degree and the first degree in the Lydian mode. We will build a link with two triads at a distance of one tone and, by using the Hungarian algorithm, we will find the set of box distributions that indicate the optimal links between both structures.

$$E_{(VII^r_c|I^r_c)} = \begin{pmatrix} A & G \\ \#F & E \\ D & C \end{pmatrix} \tag{A.1}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(II^r|I^r)} = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 6 \\ 5 & 2 & 2 \end{pmatrix} \tag{A.2}$$

Once the matrix L is built, we develop the algorithm until we reach the matrix L^H, where we have not boxed any solution because it has multiple solutions. Therefore, we need to apply the zero method.

$$L_{(II^r|I^r)} = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 6 \\ 5 & 2 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r|I^r)} = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 5 \\ 3 & 0 & 0 \end{pmatrix} \longrightarrow L^H_{(II^r|I^r)} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 4 & 0 & 0 \end{pmatrix} \tag{A.3}$$

A.1.2. The Zero method over $L^H_{(II^r|I^r)}$

As we have seen that the matrix allows several solutions, then we select each zero from left to right and from top to bottom, going through the nine zeros. Every time we select a zero, a possible solution appears. The solutions need not be unique and at the end we will have to select those that are not repeated.

$$L^{Z_1}_{(II^r|I^r)} = \begin{pmatrix} \boxed{0}^* & 2 & 0 \\ 0 & \boxed{0} & 4 \\ 4 & 0 & \boxed{0} \end{pmatrix} \mid L^{Z_2}_{(II^r|I^r)} = \begin{pmatrix} 0 & 2 & \boxed{0}^* \\ \boxed{0} & 0 & 4 \\ 4 & \boxed{0} & 0 \end{pmatrix} \mid L^{Z_3}_{(II^r|I^r)} = \begin{pmatrix} 0 & 2 & \boxed{0} \\ \boxed{0}^* & 0 & 4 \\ 4 & \boxed{0} & 0 \end{pmatrix} \tag{A.4}$$

$$L^{Z_4}_{(II^r|I^r)} = \begin{pmatrix} \boxed{0} & 2 & 0 \\ 0 & \boxed{0}^* & 4 \\ 4 & 0 & \boxed{0} \end{pmatrix} \mid L^{Z_5}_{(II^r|I^r)} = \begin{pmatrix} 0 & 2 & \boxed{0} \\ \boxed{0} & 0 & 4 \\ 4 & \boxed{0}^* & 0 \end{pmatrix} \mid L^{Z_6}_{(II^r|I^r)} = \begin{pmatrix} \boxed{0} & 2 & 0 \\ 0 & \boxed{0} & 4 \\ 4 & 0 & \boxed{0}^* \end{pmatrix} \tag{A.5}$$

The solutions for $L^H_{(II^r|I^r)}$ when both triads are in root position become the following sets, which represents the minimum voice leading:

$$S^1(L^H_{(II^r|I^r)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

$$S^2(L^H_{(II^r|I^r)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

A.1.3. Tonal function for $S^1(L^H_{(II^r|I^r)})$

We take the first solution and expand it as a particular case to find the tonal function

$$\left[E^o_{1\,(II_c|I_c)}\right]_\nabla = \left[\begin{pmatrix} A & G \\ \#F & E \\ D & C \end{pmatrix}\right]_\nabla \tag{A.6}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{1\,(II_c|I_c)}) = 2 + 2 + 2 = 6$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(II|I)} = 6$.

Any optimal arrangement from an optimal progression $E^o_{1\,(II|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(II_c)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{-\Delta_{11}} & 0 & 0 \\ 0 & s^{-\Delta_{22}} & 0 \\ 0 & 0 & s^{-\Delta_{33}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ \#F_{z_2} \\ D_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \end{pmatrix} \tag{A.7}$$

Then the characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values $-\Delta_{11}$, $-\Delta_{22}$ and $-\Delta_{33}$:

$$\Delta(C_\mathbb{E} - \lambda Id) = \begin{pmatrix} s^{-2} - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{A.8}$$

We use the properties of the determinant to reach the hypervolumetric tonal function.

$$\Delta(C_\mathbb{E} - \lambda Id) = (s^{-2} - \lambda)(s^{-2} - \lambda)(s^{-2} - \lambda)$$

We see that the roots of the tonal function are classified according to their position with respect to the stabilizer of the Mersenne group.

$$\lambda^+ = \{\emptyset\}$$

$$\lambda^0 = \{\emptyset\}$$

$$\lambda^- = \{s^{-2}\}$$

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(II_c)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$.

It is a necessary condition for the minimization of the absolute perception that the subscripts of the octave classes appear correctly paired, this is equivalent to the fact that the voicing of each note does not jump octave.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II_c}(t) = \psi_{A_{z_1}}(t) + \psi_{\#F_{z_2}}(t) + \psi_{D_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

The physical phenomenon is the succession of two voicings with the same timbre distribution that link their voices, independently of their opening.

$$\psi_{II_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II_c)} - e^{-2\pi tki\psi_j(II_c)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree II related to the Lydian tonal center for n=3. In this case is unique and it can be represented only by one polynomial $\Phi(\lambda) \in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem and is written as a polynomial.

$$\boxed{\Phi[E^1_{(II|I)}] \in D^{\mathbb{R}[\lambda]}}$$

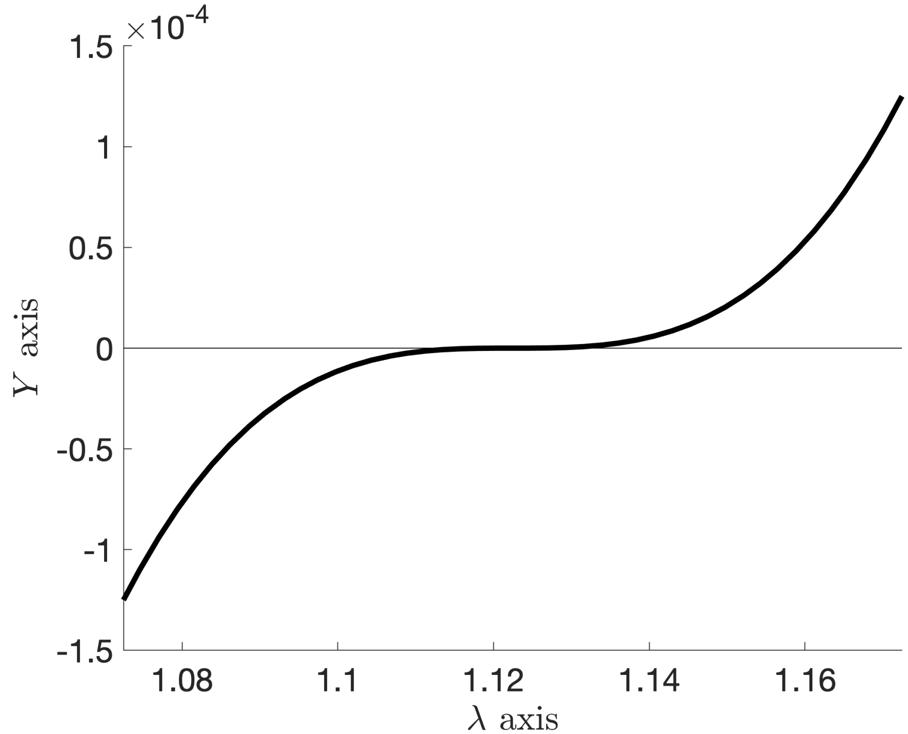


Figure A.1: Characteristic polynomial associated to the II-I cadence (first polynomial)

A.1.4. Tonal function for $S^2(L^H_{(II^r|I^r)})$

We take the second solution that we have obtained from the Zero method and calculate the nabla class of the optimal link:

$$\left[E^o_{2\,(II_c|I_c)}\right]_\nabla = \left[\begin{pmatrix} A & C \\ \#F & G \\ D & E \end{pmatrix}\right]_\nabla \tag{A.9}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{2\,(II_c|I_c)}) = 1+2+3 = 6$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(II|I)} = 6$

Any optimal arrangement from an optimal progression $E^o_{2\,(II|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(II_c)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 \\ 0 & 0 & s^{\Delta_{32}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ \#F_{z_2} \\ D_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} \tag{A.10}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values Δ_{12}, Δ_{23} and Δ_{31}:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \begin{pmatrix} s^3 - \lambda & 0 & 0 \\ 0 & s^1 - \lambda & 0 \\ 0 & 0 & s^2 - \lambda \end{pmatrix} \tag{A.11}$$

Using the properties of the determinant we calculate the second tonal function associated to the link between the pair of major triads.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^3 - \lambda)(s^1 - \lambda)(s^2 - \lambda)$$

The roots are polarized to the left side of $E(M)$.

$$\lambda^+ = \{s^1, s^2, s^3\}$$

$$\lambda^0 = \{\emptyset\}$$

$$\lambda^- = \{\emptyset\}$$

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(II_c)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. In the square brackets we write the two functions that represent the antecedent voicing and the consequent so that we have paired the subscripts in each octave following the indications of the second solution.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II_c}(t) = \psi_{A_{z_1}}(t) + \psi_{\#F_{z_2}}(t) + \psi_{D_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \end{cases}$$

For a certain timbral distribution we study how one function is transformed into another using the $\longrightarrow$ notation. Thus we see that when the voicings make up an optimal arrangement, then the absolute perception reaches the minimum for the set of options between the two tonal centers involved in the link.

$$\psi_{II_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II_c)} - e^{-2\pi tki\psi_j(II_c)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree II related to the Lydian tonal center for n=3. In this case we see that it is a dual function and that the two solutions are in different areas, so we will symbolize the relationship between degrees with a bidirectional arrow. The function for the second solution is generalized using the static tonal function theorem and is written as a polynomial.

$$\boxed{\Phi[E^2_{(II|I)}] \in S^{\mathbb{R}[\lambda]}}$$

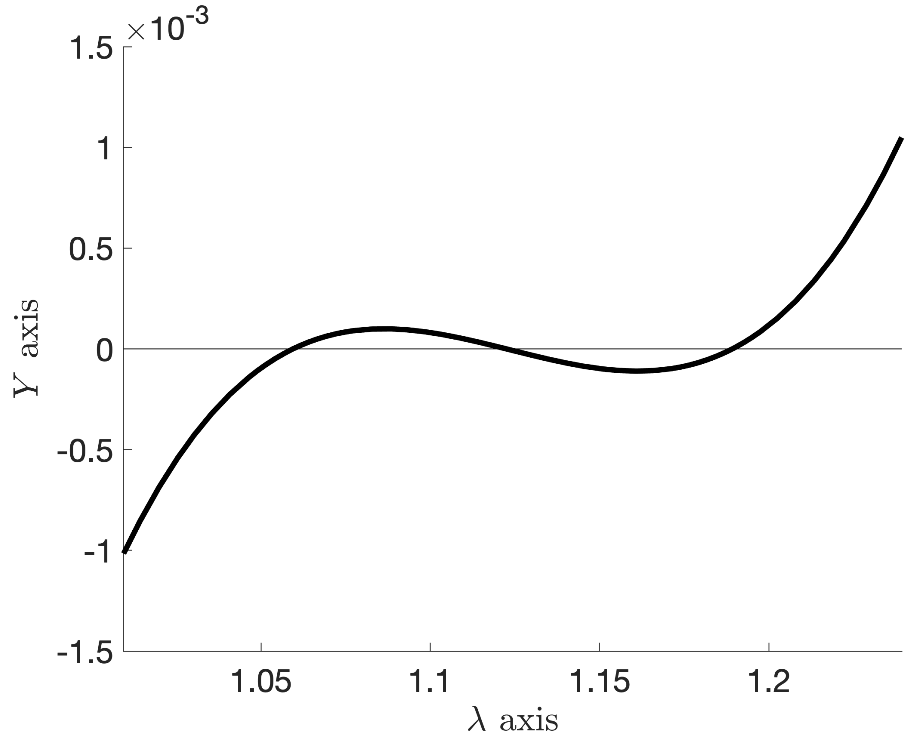


Figure A.2: Characteristic polynomial associated to the II-I cadence (second polynomial)

A.1.5. III-⟶ I Cadence

What we are particularly interested in is knowing how chords move within the diatonic context with three fixed voices. In this way we proceed to, fixing the first degree in the way that we are studying, understand the relationship between the remaining degrees. In this case, we continue studying the relationship between the minor third degree and the first degree. Based on W.F.C we build the L matrix and apply the Hungarian algorithm.

$$E_{(III^r_c-|I^r_c)} = \begin{pmatrix} B & G \\ G & E \\ E & C \end{pmatrix} \tag{A.12}$$

The link matrix will be constructed calculating every Δ_{ij}. The L matrix is the same for a link in every key so as usual, we omit the subindex.

$$L_{(III^r-|I^r)} = \begin{pmatrix} 4 & 5 & 1 \\ 0 & 3 & 5 \\ 3 & 0 & 4 \end{pmatrix} \tag{A.13}$$

Following the steps of the algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(III^r-|I^r)} = \begin{pmatrix} 4 & 5 & 1 \\ 0 & 3 & 5 \\ 3 & 0 & 4 \end{pmatrix} \longrightarrow L^F_{(III^r-|I^r)} = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 3 & 5 \\ 3 & 0 & 4 \end{pmatrix} \longrightarrow L^H_{(III^r-|I^r)} = \begin{pmatrix} 3 & 4 & \boxed{0} \\ \boxed{0} & 3 & 5 \\ 3 & \boxed{0} & 4 \end{pmatrix} \tag{A.14}$$

The solutions for $L^H_{(III^r|I^r)}$ when both triads are in root position becomes the following set wich represents the minimum voice leading $S(L^H_{(III^r-|I^r)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}, \}$.

Thus, we have managed to link the two chords and obtain the nabla class, which is made up of all the links whose nabla function is the minimum of all possible ones.

$$\left[E^o_{(III_c-|I_c)} \right]_\nabla = \left[\begin{pmatrix} B & C \\ G & G \\ E & E \end{pmatrix} \right]_\nabla \tag{A.15}$$

We calculate the optimal link class nabla value, $\nabla(E^o_{(III_c-|I_c)}) = 0 + 1 + 0 = 1$. We can generalize the value of nabla to the relationship between degrees regardless of the tonality, then we would reach the expression $\nabla^o_{(III-|I)} = 1$.

Now any optimal arrangement from an optimal progression $E^o_{(III-|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(III_c-)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} B_{z_1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} \tag{A.16}$$

Using the transformation T we have obtained the voicing transformation matrix in an optimal way, generalized for a selection of subscripts. Thus, we will assign values to each exponent using T and proceed to calculate the determinant. Then the characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with assigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^1 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{A.17}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = p_c(\lambda) = (s^0 - \lambda)^2 (s^1 - \lambda)$$

The roots have the following structure:

$$\lambda^+ = \{s^1\}$$
$$\lambda^0 = \{s^0\}$$
$$\lambda^- = \{\emptyset\}$$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(III_c-)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. In this way we obtain how the voicings are going to be linked when the arrangement is optimal. Then we have two functions $\psi(t)$ that will each represent a particular voicing for a particular gamma timbre.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{III_c-}(t) = \psi_{B_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \end{cases}$$

We represent the transition from one voicing to another using the arrow notation and expressing the functions in their analytical expression.

$$\psi_{III_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(III_c-)} - e^{-2\pi tki\psi_j(III_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

As we know the fundamental theorem and the timbres are the same and we also know W.F.C, we know that the value of absolute perception in this situation is going to be minimal, which is enough for, applying the polynomial criterion, to know the tonal function between both structures. We remember the polynomial criterion and we observe that the tonal function of the link is going to be unique and it is going to be classified as tonic.

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} < 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases} \text{or} \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} < 2 \end{cases}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$ then, following the polynomial criterion we obtain the function of the degree $III-$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(III-|I)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s-2)\ \lambda^2 + (2\,s+1)\ \lambda - s$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s}{3} - \frac{2}{3}\right)\ \lambda^3 + \left(s + \frac{1}{2}\right)\ \lambda^2 + (-s)\ \lambda$

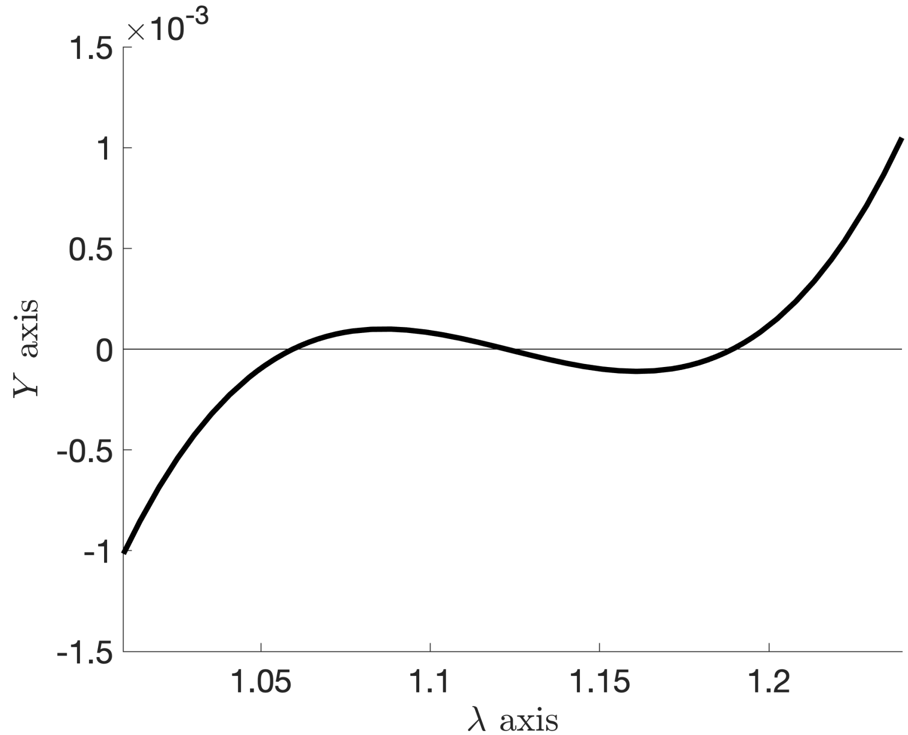


Figure A.3: Characteristic polynomial associated to the III-I cadence

A.1.6. #IVo⟶ I Cadence

We remember the polynomial criterion to study the placement of the roots of the characteristic polynomial. With this clear vision, we seek to optimize the link between both structures.This is a characteristic example where it is observed that the dimensional expansion of chords from three to four voices does not preserve the tonal function. The reader has to contrast this case with the extension to four voices and observe that the tonal function is inverted, changing area.

To classify a tonal function as tonic we use the following criteria:

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} < 2 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} = 0 \end{cases} \text{or} \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} = 0 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} < 2 \end{cases}$$

The tonal functions in the subdominant area must verify:

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} \geq 2 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} = 0 \end{cases}$$

The tonal functions in the dominant area are classified like this if in the optimal link the convergent algebraic multiplicity is greater than two, the convergent algebraic multiplicity is null and there are an arbitrary number of static voices.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} \geq 2 \\ M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} = 0 \end{cases}$$

We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(\#IV_c^r o|I_c^r)} = \begin{pmatrix} C & G \\ A & E \\ \#F & C \end{pmatrix} \tag{A.18}$$

Finding every Δ_{ij} distance and arranging them into the L matrix we calculate each minimum distance between classes starting with the first class of the first chord and going down the second column. Then we select the second class of the first chord and repeat the process. Finally we choose the third class and build the last row part of the L matrix .

$$L_{(\#IV^r o|I^r)} = \begin{pmatrix} 5 & 4 & 0 \\ 2 & 5 & 3 \\ 1 & 2 & 6 \end{pmatrix} \tag{A.19}$$

Then, by following the steps of the Hungarian algorithm we consider to apply the algorithm to the L matrix to find an optimum link. We transform the original matrix L using the steps of the algorithm until we find L^H.

$$L_{(\#IV^ro|I^r)} = \begin{pmatrix} 5 & 4 & 0 \\ 2 & 5 & 3 \\ 1 & 2 & 6 \end{pmatrix} \longrightarrow L^F_{\#IV^ro|I^r)} = \begin{pmatrix} 5 & 4 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix} \longrightarrow L^H_{(\#IV^ro|I^r)} = \begin{pmatrix} 5 & 3 & \boxed{0} \\ \boxed{0} & 2 & 1 \\ 0 & \boxed{0} & 5 \end{pmatrix} \tag{A.20}$$

Then the solutions for $L^H_{(\#IV^ro|I^r)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(\#IV^ro|I^r)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

We calculate the optimal link and incidentally its class, since the link admits the permutation of rows without losing the optimization conditions. Thus, we arrive at the set of links that share the smallest nabla function.

$$\left[E^o_{(\#IV_co|I_c)}\right] = \left[\begin{pmatrix} C & C \\ A & G \\ \#F & E \end{pmatrix}\right]_\nabla \tag{A.21}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(\#IV_co|I_c)}) = 2 + 2 + 0 = 4$$

We write the optimal nabla value as a generalization for every tonality:

$$\nabla^o_{(\#IVo|I)} = 4$$

Now any optimal arrangement from an optimal progression $E^o_{(\#IVo|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(\#IV_co)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ \#F_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} \tag{A.22}$$

Then the characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values l_1,l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{A.23}$$

Using the properties of the determinant the polynomial has the form. For the construction of the matrix, in this case, it is enough for us to calculate the product of the elements of the trace, obtaining the characteristic polynomial already factored.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^{-2} - \lambda)^2(s^0 - \lambda)$$

The structure of the roots is summarized in three sets that are $\lambda^- = \{s^{-2}\}$,$\lambda^0 = \{s^0\}$ and the set of divergent roots $\lambda^+ = \{\emptyset\}$.

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(\#IV_co)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We now calculate the pair of sinusoidal functions that represent the transition between two voicings of each tonal center for an arbitrary selection of zeta subscripts.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{\#IV_co}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{\#F_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \end{cases}$$

For a gamma distribution we will obtain the first voicing of the diminished chord that becomes the first degree of the Lydian tonal center. This is the transformation of sinusoidal functions where the absolute perception reaches the minimum value.

$$\psi_{\#IV_co}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(\#IV_co)} - e^{-2\pi tki\psi_j(\#IV_co)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$ then, following the polynomial criterion we obtain the function of the degree $\#IVo$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(\#IV|I)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{2}{s^2} - 1\right)\lambda^2 + \left(\frac{2}{s^2} + \frac{1}{s^4}\right)\lambda - \frac{1}{s^4}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^4 + 2\,s^2}{s^4\,3}\right)\lambda^3 + \frac{2\,s^2+1}{2\,s^4}\lambda^2 + \left(-\frac{1}{s^4}\right)\lambda$

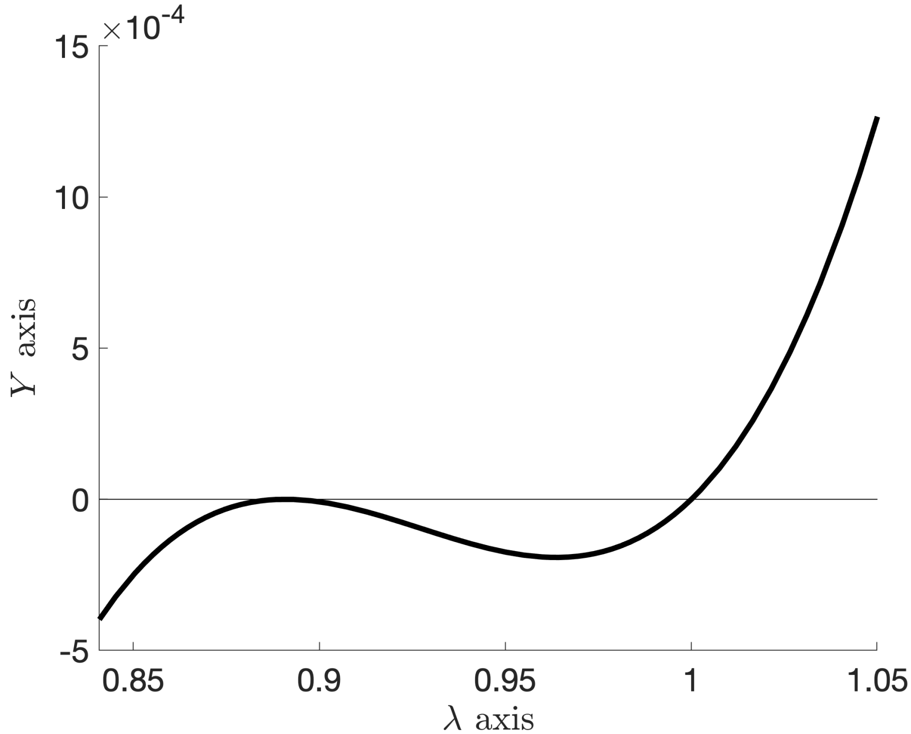


Figure A.4: Characteristic polynomial associated to the#IVo-I cadence

A.1.7. V⟶ I Cadence

This case is particularly important because it clearly explains how the results provided by the Hungarian algorithm are the key to a deep understanding of the cadential processes. We will see that since the dominant function depends on the presence of classes in the link and since $V7 \neq V$, we conclude that they do not share tonal function. This has been a reason for confrontation between musical styles, mainly between classical music and jazz since each cadence predominates in one of them. This problem also appears when, starting from the three-voice harmony, we try to extend the progression to four voices or when we carry out the opposite process, that is, starting with four voices and trying to reduce the P progression to three voices. In some links, depending on P, we will observe that the function is preserved, but not in all of them.

In order to classify this case and any other that may appear, we follow the steps for calculating the tonal function. Thus, we set up a link with the chords arranged in root state, where we will do this by pure order, since it is not relevant for the calculation of the tonal function.

$$E_{(V_c^r|I_c^r)} = \begin{pmatrix} D & G \\ B & E \\ G & C \end{pmatrix} \tag{A.24}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(V^r|I^r)} = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 5 & 1 \\ 0 & 3 & 5 \end{pmatrix} \tag{A.25}$$

Then, by following the steps of the Hungarian algorithm we consider to apply the algorithm to the L matrix to find an optimum link:

$$L_{(V^r|I^r)} = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 5 & 1 \\ 0 & 3 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r|I^r)} = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & 3 & 5 \end{pmatrix} \longrightarrow L^H_{(V^r|I^r)} = \begin{pmatrix} 3 & \boxed{0} & 0 \\ 3 & 4 & \boxed{0} \\ \boxed{0} & 3 & 5 \end{pmatrix} \tag{A.26}$$

Then the solutions for $L^H_{(V^r|I^r)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(V^r|I^r)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

We calculate the optimal link class:

$$\left[E^o_{(V_c|I_c)} \right]_\nabla = \left[\begin{pmatrix} D & E \\ B & C \\ G & G \end{pmatrix} \right]_\nabla \tag{A.27}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(V_c|I_c)}) = 2 + 1 + 0 = 3$$

We generalize the value of nabla for the link between the fifth degree and the first using the following equation:

$$\nabla^o_{(V|I)} = 3$$

Now any optimal arrangement from an optimal progresion $E^o_{(V|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(V_c)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ B_{z_2} \\ G_{z_3} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{A.28}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^1 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{A.29}$$

Thus, using the properties of the calculation of determinants, we will obtain the volume function that is canceled in the values that coincide with the proportion between voices.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2 - \lambda)(s^1 - \lambda)(s^0 - \lambda)$$

Thus, the points where the tonal function cancels out are precisely those that give us the proportions between voices in the same dimension.

$$\lambda^+ = \{s^2, s^1\}$$

$$\lambda^0 = \{s^0\}$$

$$\lambda^- = \{\emptyset\}$$

This is enough to be able to build the functions that represent each voicing and thus study how the energies behave in change. We know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(V_c)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$.

In this way we study the pair of sinusoidal sums and the transition between them. We now pay attention to the fact that the subscripts are analogous to those used in the midi notation, then we can choose them within a set of integers, this variable being the one that generates the set of optimal arrangements.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V_c}(t) = \psi_{D_{z_1}}(t) + \psi_{B_{z_2}}(t) + \psi_{G_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

Thus, for a particular set of integers, we will obtain the sinusoidal sums in the optimal arrangement.

$$\psi_{V_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(V_c)} - e^{-2\pi tki\psi_j(V_c)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

For all these selections where the arrangement is optimal regardless of the openings of the voicings, we will keep the value of absolute perception $\mid p \mid$ to a minimum.

To classify a tonal function as tonic we use the following criteria:

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} < 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases} \text{or} \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} < 2 \end{cases}$$

The tonal functions in the subdominant area must verify:

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} \geq 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases}$$

The tonal functions in the dominant area are classified like this if in the optimal link the convergent algebraic multiplicity is greater than two, the convergent algebraic multiplicity is null and there are an arbitrary number of static voices.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} \geq 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$ then, following the polynomial criteria we obtain the function of the degree V related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(V|I)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - s - 1)\ \lambda^2 + (s^3 + s^2 + s)\ \lambda - s^3$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2}{3} - \frac{s}{3} - \frac{1}{3}\right)\lambda^3 + \frac{s\ (s^2+s+1)}{2}\lambda^2 + \left(-s^3\right)\ \lambda$

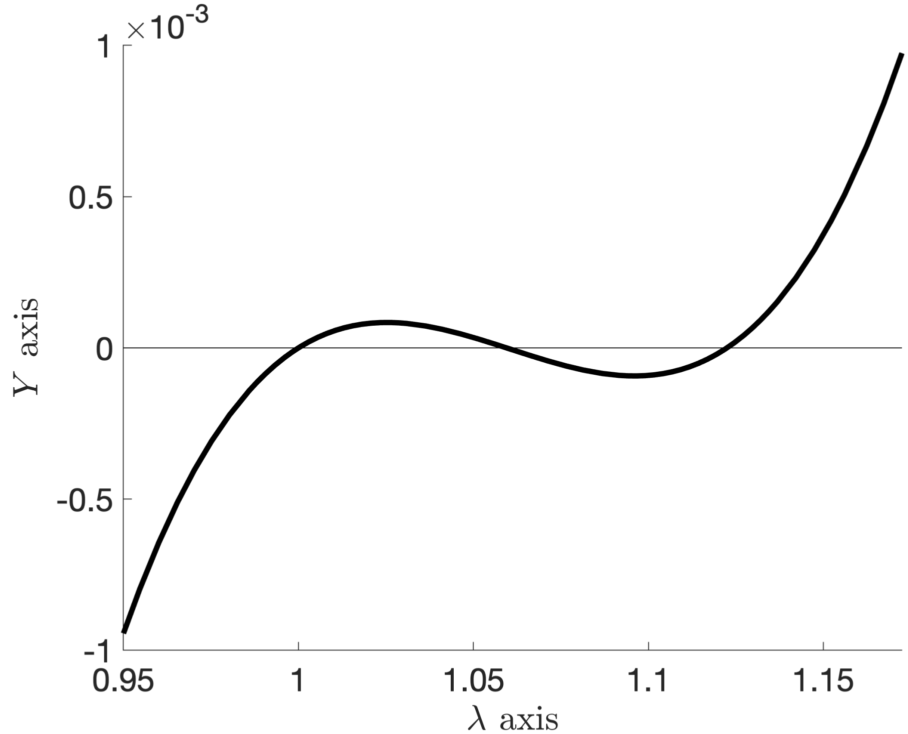


Figure A.5: Characteristic polynomial associated to the V-I cadence

A.1.8. VI-⟶ I Cadence

We now evaluate the relationship between the minor sixth degree and the first degree in the Lydian mode. We want to know what relationship exists between the two and if a possible progression involving both tonal centers is viable or not. We remember that convergence occurs when the voices drop at the optimum.

The link matrix will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(VI_c^r - |I_c^r)} = \begin{pmatrix} 3 & 0 & 4 \\ 5 & 4 & 0 \\ 2 & 5 & 3 \end{pmatrix} \tag{A.30}$$

Then following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(VI^r - |I^r)} = \begin{pmatrix} 3 & 0 & 4 \\ 5 & 4 & 0 \\ 2 & 5 & 3 \end{pmatrix} \longrightarrow L^F_{(VI^r - |I^r)} = \begin{pmatrix} 3 & 0 & 4 \\ 5 & 4 & 0 \\ 2 & 5 & 3 \end{pmatrix} \longrightarrow L^H_{(VI^r - |I^r)} = \begin{pmatrix} 3 & \boxed{0} & 4 \\ 5 & 4 & \boxed{0} \\ \boxed{0} & 3 & 1 \end{pmatrix} \tag{A.31}$$

Then, the solutions for $L^H_{(VI^r-|I^r)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(VI^r-|I^r)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

We calculate the optimal link class obtaining the set of optimal links for the progression between the mentioned chords.

$$\left[E^o_{(VI_c-|I_c)}\right]_\nabla = \left[\begin{pmatrix} E & E \\ C & C \\ A & G \end{pmatrix}\right]_\nabla \tag{A.32}$$

We calculate the optimal link class nabla value:

$$\nabla(E^o_{(VI_c-|I_c)}) = 0 + 0 + 2 = 2$$

We generalize the value of nabla for the progression formed by the cadence that we are studying, this value being the one given by the equation$\nabla^o_{(VI-|I)} = 2$.

Now any optimal arrangement from an optimal progression $E^o_{(VI-|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(VI_c-)) \longrightarrow \psi(I_c)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ A_{z_3} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{A.33}$$

Then the characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{A.34}$$

We calculate the exponents of the Mersenne number using the solutions that come from the operation of the Hungarian algorithm. Then each exponent l will correspond to one of the solutions of the S set.

$$\Delta(C_\mathbb{E} - \lambda Id) = (s^0 - \lambda)^2 (s^{-2} - \lambda)$$

Calculating the roots of the polynomial, we separate them according to their position with respect to the stabilizer of the Mersenne group. Thus, three sets appear that determine the movement of the voices in the optimal link.

$$\lambda^- = \{s^{-2}\}$$
$$\lambda^0 = \{s^0\}$$
$$\lambda^+ = \{\emptyset\}$$

By using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(VI_c-)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$.

Once we have carried out the optimization, we calculate the functions that represent the voicings in the time domain for a particular gamma harmonic distribution.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VI_c-}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{A_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

Regardless of the opening of the arrangement, the absolute perception will reach the minimum and it is at this point that we are going to study the placement of the voices.

$$\psi_{VI_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VI_c-)} - e^{-2\pi tki\psi_j(VI_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

To classify a tonal function as tonic we use the following criterion:

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} < 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases} \text{or} \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} < 2 \end{cases}$$

As $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$ then, following the polynomial criterion we obtain the function of the degree $VI-$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(VI-|I)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_\mathbb{E}}(\lambda) = \lambda^3 + \left(-\frac{1}{s^2} - 2\right) \lambda^2 + \left(\frac{2}{s^2} + 1\right) \lambda - \frac{1}{s^2}$

Integral of $p_{C_\mathbb{E}}(\lambda)$: $\int p_{C_\mathbb{E}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s^2+1}{3\,s^2}\right) \lambda^3 + \frac{s^2+2}{2\,s^2} \lambda^2 + \left(-\frac{1}{s^2}\right) \lambda$

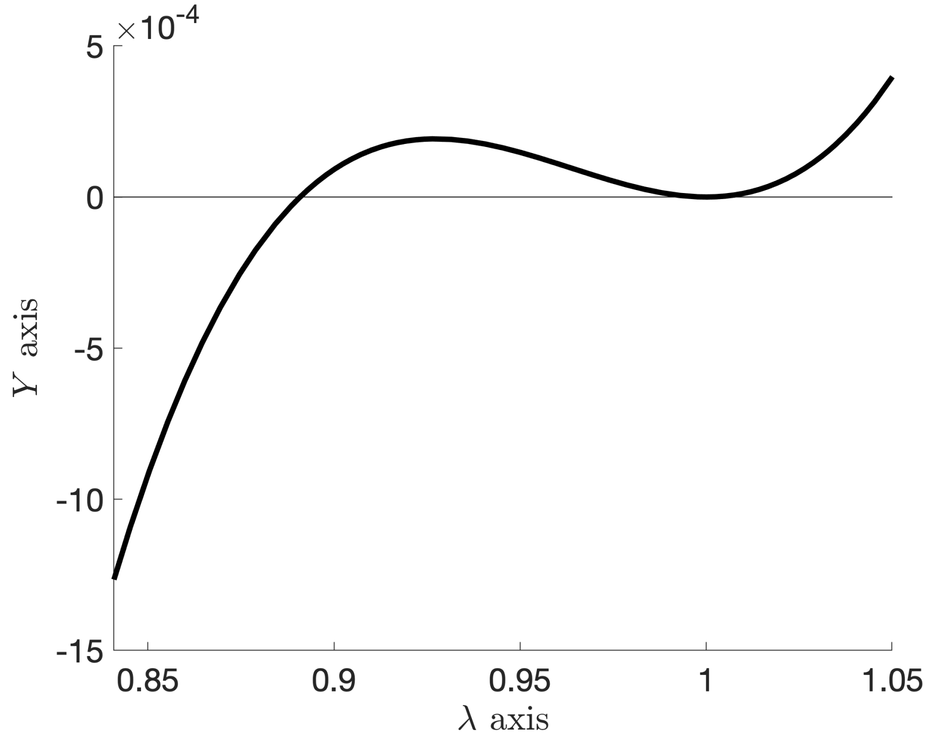


Figure A.6: Characteristic polynomial associated to the VI-I cadence

A.1.9. VII-⟶ I Cadence

In this section we want to know how the seventh degree behaves with respect to the first in the Lydian mode. To do this, we minimize the perception between both tonal centers and study how p behaves when we minimize $\mid p \mid$.

We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(VII_c^r - |I_c^r)} = \begin{pmatrix} \#F & G \\ D & E \\ B & C \end{pmatrix} \tag{A.35}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(VII^r - |I^r)} = \begin{pmatrix} 1 & 2 & 6 \\ 5 & 2 & 2 \\ 4 & 5 & 1 \end{pmatrix} \tag{A.36}$$

Then, following the steps of the Hungarian algorithm we consider to develope the algorithm through the L matrix to find an optimum link:

$$L_{(VII^r-|I^r)} = \begin{pmatrix} 1 & 2 & 6 \\ 5 & 2 & 2 \\ 4 & 5 & 1 \end{pmatrix} \longrightarrow L^F_{(VII^r-|I^r)} = \begin{pmatrix} 0 & 1 & 5 \\ 5 & 2 & 2 \\ 4 & 5 & 1 \end{pmatrix} \longrightarrow L^H_{(VII^r-|I^r)} = \begin{pmatrix} \boxed{0} & 1 & 5 \\ 3 & \boxed{0} & 0 \\ 3 & 4 & \boxed{0} \end{pmatrix} \tag{A.37}$$

Then the solutions for $L^H_{(VII^r-|I^r)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(VII^r-|I^r)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

We calculate the set of optimal links based on the distribution of boxes provided by the resolution of the L matrix.

$$\left[E^o_{(VII_c-|I_c)} \right]_\nabla = \left[\begin{pmatrix} \#F & G \\ D & E \\ B & C \end{pmatrix} \right]_\nabla \tag{A.38}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(VII_c-|I_c)}) = 1 + 2 + 1 = 4$$

We write the optimal nabla value as a generalization for every tonality:

$$\nabla^o_{(VII-|I)} = 4$$

Now any optimal arrangement from an optimal progresion $E^o_{(VII-|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(VII_c-)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} \#F_{z_1} \\ D_{z_2} \\ B_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \end{pmatrix} \tag{A.39}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^1 - \lambda & 0 & 0 \\ 0 & s^2 - \lambda & 0 \\ 0 & 0 & s^1 - \lambda \end{pmatrix} \tag{A.40}$$

Using the properties of the determinant the polynomial has the following form. We have already obtained the tonal function and we only need to classify it in the appropriate area by studying the position of its roots with respect to the stabilizer of the M group.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = p_c(\lambda) = (s^1 - \lambda)^2(s^2 - \lambda)$$

The roots are distributed in three sets according to the resolution of the Hungarian algorithm.

$$\lambda^+ = \{s^1, s^2\}$$

$$\lambda^0 = \{\emptyset\}$$

$$\lambda^- = \{\emptyset\}$$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(VII_c-)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VII_c-}(t) = \psi_{\#F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{B_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

Each of the voicings that arise from the progression will be a sinusoidal function for a particular choice of integers.

$$\psi_{VII_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VII_c-)} - e^{-2\pi tki\psi_j(VII_c-)}}{2i}$$

$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

The tonal functions in the subdominant area must verify:

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} \geq 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then, following the polynomial criteria we obtain the function of the degree $VII-$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(VII-|I)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - 2\,s)\ \lambda^2 + (2\,s^3 + s^2)\ \lambda - s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s\ (s+2)}{3}\right)\ \lambda^3 + \frac{s^2\ (2\,s+1)}{2}\ \lambda^2 + \left(-s^4\right)\ \lambda$

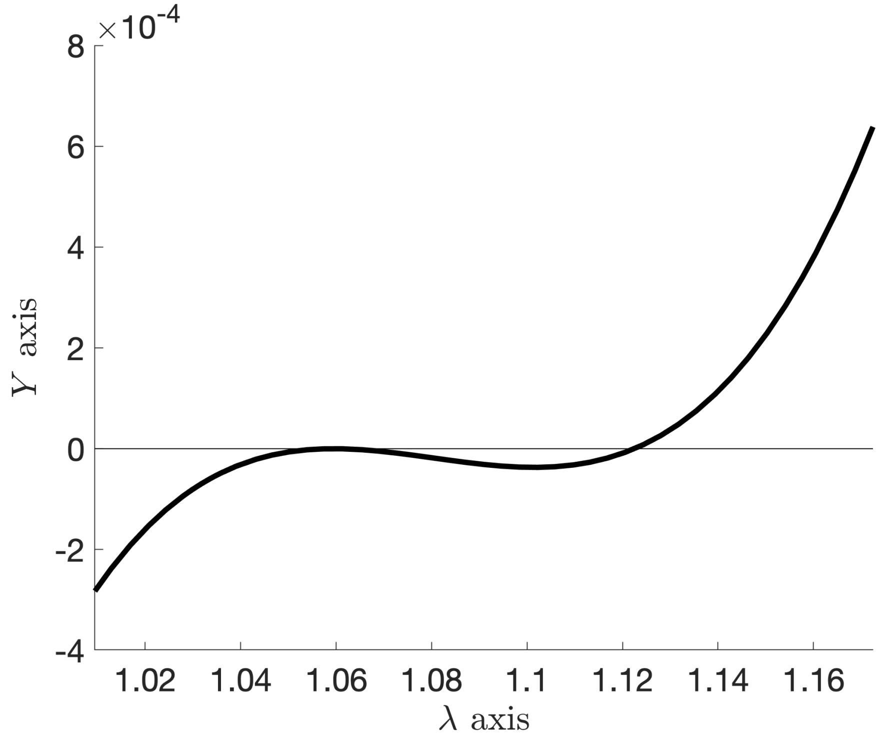


Figure A.7: Characteristic polynomial associated to the VII-I cadence

A.2. The Lydian Mode for n=4

A.2.1. II7⟶I△ Cadence

In the context of the four voices we are going to calculate the tonal function of each edge of the graph of tonal functions with the intention of having all the movements as the main compositional tool. At the end of the process, the polynomial criterion will be the same regardless of the number of voices.

To classify a tonal function as tonic we use the following criterion:

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} < 2 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} = 0 \end{cases} \text{or} \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} = 0 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} < 2 \end{cases}$$

The tonal functions in the subdominant area must verify:

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} \geq 2 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} = 0 \end{cases}$$

If the tonal function that we want to classify meets the following conditions, then we will classify it within the dominant area:

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} \geq 2 \\ M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} = 0 \end{cases}$$

We build the link by arranging each chord in root position and completing the E link.

$$E_{(II^r_c 7|I^r_c \triangle)} = \begin{pmatrix} C & E \\ A & C \\ \#F & B \\ D & G \end{pmatrix} \tag{A.41}$$

Then, we calculate the link matrix calculating all the distances Δ_{ij}:

$$L_{(II^r 7|I^r \triangle)} = \begin{pmatrix} 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \\ 5 & 1 & 2 & 6 \\ 3 & 5 & 2 & 2 \end{pmatrix} \tag{A.42}$$

Following the steps of the Hungarian algorithm we develope the L matrix:

$$L_{(II^r7|I^r\triangle)} = \begin{pmatrix} 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \\ 3 & 1 & 2 & 6 \\ 3 & 5 & 2 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r7|I^r\triangle)} = \begin{pmatrix} 1 & 5 & 4 & 0 \\ 0 & 0 & 3 & 1 \\ 5 & 1 & 2 & 6 \\ 3 & 5 & 2 & 2 \end{pmatrix} \longrightarrow L^H_{(II^r7|I^r\triangle)} = \begin{pmatrix} 1 & 5 & 4 & \boxed{0} \\ \boxed{0} & 0 & 3 & 1 \\ 4 & \boxed{0} & 1 & 5 \\ 1 & 3 & \boxed{0} & 0 \end{pmatrix} \tag{A.43}$$

The solutions provided by the algorithm will match the distribution of boxes over the matrix L^H.

$$S(L^H_{(II^r7|I^r\triangle)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(II_c7|I_c\triangle)}\right]_\nabla = \left[\begin{pmatrix} C & C \\ A & B \\ \#F & G \\ D & E \end{pmatrix}\right]_\nabla \tag{A.44}$$

Once the S set is calculated we can form a generalization of a dimensionally optimized cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(II_c7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ \#F_{z_3} \\ D_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ B_{z_2} \\ G_{z_3} \\ E_{z_4} \end{pmatrix} \tag{A.45}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id_4.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^2 - \lambda)(s^1 - \lambda)(s^2 - \lambda)$$

The algebraic multiplicity of each set of roots is: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(II7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We now suppose the pair of sums of sinusoidal functions that represent each voicing.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II7}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{\#F_{z_3}} + \psi_{D_{z_4}}(t) \\ \psi_{I\triangle}(t) = \psi_{C_{z_1}}(t) + \psi_{B_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{E_{z_4}}(t) \end{cases}$$

We represent the transition between the pair of voicings of each tonal center for a selection of integers representing the position in a particular octave of each class in each voicing.

$$\psi_{II7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(II7)} - e^{-2\pi tki\psi_j(II7)}}{2i}$$
$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then following the polynomial criteria we obtain the function of the degree $II7$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(II7|I\triangle)}] \in S^{\mathbb{R}[\lambda]}}$$

.

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-2\,s^2 - s - 1)\ \lambda^3 + (s^4 + 2\,s^3 + 2\,s^2 + s)\ \lambda^2 + (-s^5 - s^4 - 2\,s^3)\ \lambda + s^5$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{2} - \frac{s}{4} - \frac{1}{4}\right)\ \lambda^4 + \frac{s\ (s^3 + 2\,s^2 + 2\,s + 1)}{3}\ \lambda^3 + \left(-\frac{s^3\ (s^2 + s + 2)}{2}\right)\ \lambda^2 + s^5\ \lambda$

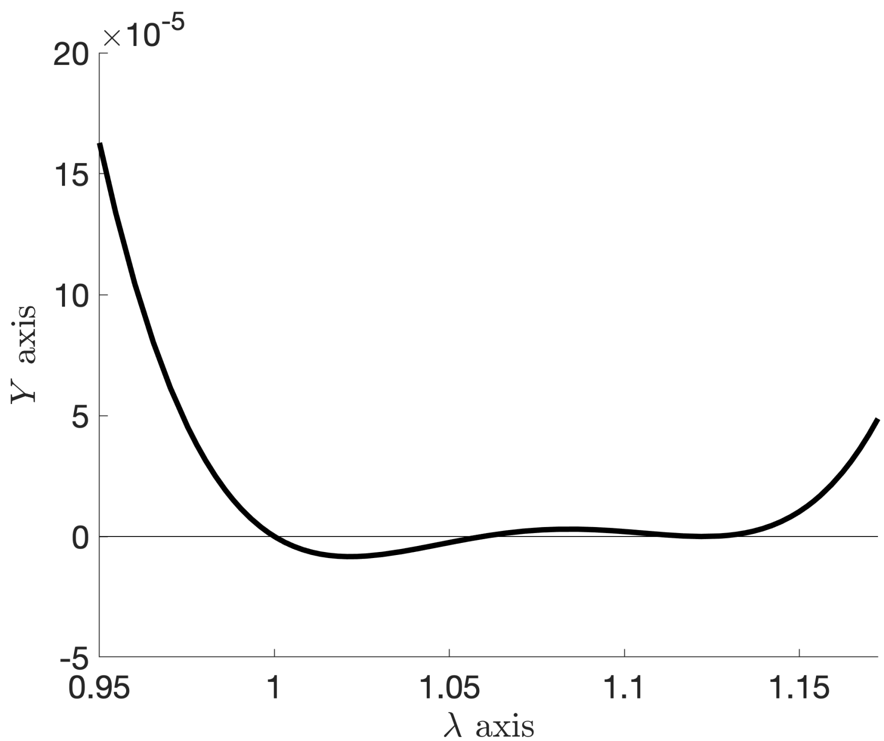


Figure A.8: Characteristic polynomial associated to the II7-I△ cadence

A.2.2. III-7⟶I△ Cadence

We study the cadence composed by the third degree and by the first in the context of the Lydian mode. We are going to study the behavior of the voices in the optimal link using the development of the algorithm through the L matrix. The first step that we have to carry out is to adequately prepare the link to later take the minimum distances between classes and form the L matrix.

$$E_{(III^r_c-7|I^r_c\triangle)} = \begin{pmatrix} D & B \\ B & G \\ G & E \\ E & C \end{pmatrix} \tag{A.46}$$

Then, we calculate the link matrix calculating all the distances Δ_{ij}. We build the matrix by taking the first voice of the first chord and noting the minimum distance between classes between said voice and each of the voices of the second chord in descending order. So we write the first row of the matrix L. We repeat the process with the second voice of the first chord and we obtain the second row of the link matrix. We do the same with the remaining voices until we get L.

$$L_{(III^r-7|I^r\triangle)} = \begin{pmatrix} 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \end{pmatrix} \tag{A.47}$$

Following the steps of the Hungarian algorithm we develope the L matrix:

$$L_{(III^r-7|I^r\triangle)} = \begin{pmatrix} 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \end{pmatrix} \longrightarrow L^F_{(III^r-7|I^r\triangle)} = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \end{pmatrix}$$

$$\longrightarrow L^H_{(III^r-7|I^r\triangle)} = \begin{pmatrix} 1 & 3 & 0 & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \\ 5 & 3 & \boxed{0} & 4 \end{pmatrix}$$

By reaching the matrix L^H we have obtained the solution, and therefore how both tonal centers are paired.

$$S(L^H_{(III^r-7|I^r\triangle)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

Thus, pairing both tonal centers, we obtain the set of links that share the nabla function and from which we will be able to deduce the tonal function.

$$\left[E^o_{(III_c - 7|I_c\triangle)} \right]_\nabla = \left[\begin{pmatrix} D & C \\ B & B \\ G & G \\ E & E \end{pmatrix} \right]_\nabla \tag{A.48}$$

Now we calculate the nabla value when the distance between class mappings is minimal. The minimum for all ∞ mappings is unique so the sum can be constructed:

$$\nabla(E^o(III_c - 7 \mid I_c\triangle)) = \sum_{j=1}^{n} \Omega \mid \int_{(E^o_{j1\,(III_c-7|I_c\triangle)})}^{(E^o_{j2\,(III_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\Delta = 5$$

Once the S set is calculated we can form a generalization of a dimensionally optimized cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(III_c - 7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ B_{z_2} \\ G_{z_3} \\ E_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ B_{z_2} \\ G_{z_3} \\ E_{z_4} \end{pmatrix} \tag{A.49}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-2} - \lambda)(s^0 - \lambda)(s^0 - \lambda)(s^0 - \lambda)$$

The multiplicities are respectively $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$.

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(III - 7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_I(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

We thus calculate the pair of functions that are the mathematical expression of any optimal arrangement between two voicings of each tonal center.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{III-7}(t) = \psi_{D_{z_1}}(t) + \psi_{B_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{E_{z_4}}(t) \\ \psi_{I\triangle}(t) = \psi_{C_{z_1}}(t) + \psi_{B_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{E_{z_4}}(t) \end{cases}$$

For the transition between both voicings we will use both functions separated by $\longrightarrow$, which in this case represents the temporal order of both voicings.

$$\psi_{III-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(III-7)} - e^{-2\pi tki\psi_j(III-7)}}{2i}$$
$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

To classify a tonal function as tonic we use the following criterion:

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} < 2 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} = 0 \end{cases} \text{or} \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} = 0 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} < 2 \end{cases}$$

As $\sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} = 1$ then following the polynomial criterion we obtain the function of the degree $III - 7$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(III-7|I\triangle)}] \in T^{\mathbb{R}[\lambda]}}$$

.

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s^2 - 3)\ \lambda^3 + (3\,s^2 + 3)\ \lambda^2 + (-3\,s^2 - 1)\ \lambda + s^2$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{4} - \frac{3}{4}\right)\ \lambda^4 + \left(s^2 + 1\right)\ \lambda^3 + \left(-\frac{3\,s^2}{2} - \frac{1}{2}\right)\ \lambda^2 + s^2\,\lambda$

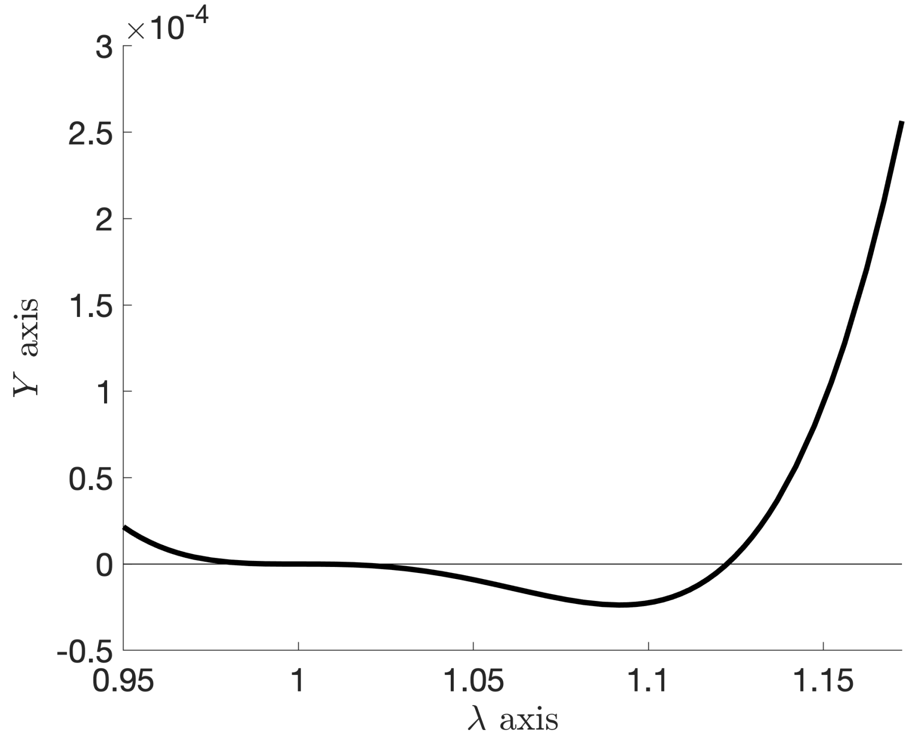


Figure A.9: Characteristic polynomial associated to the III-7-I△ cadence

A.2.3. #IVø7⟶I△ Cadence

In this case we are going to study the relationship between the fourth degree in the Lydian tonal center and the first degree. This relationship appears continuously in jazz music and is in itself very inspiring. The composer or the lecturer that has some experience playing the piano will be able to verify the beauty of the voice leading between both structures. It is not very common to find this cadence in traditional jazz music, and it appears more frequently in compositions and ballads by composers from the second half of the 20th century, although this appreciation is entirely personal. We are going to carry out the calculation of the function tonal to see how we can use it in our compositions.

$$E_{(\#IV_c^r ø7|I_c^r \triangle)} = \begin{pmatrix} E & B \\ C & G \\ A & E \\ \#F & C \end{pmatrix} \tag{A.50}$$

We calculate the link matrix calculating all the distances Δ_{ij}:

$$L_{(\#IV^r ø7|I^r\triangle)} = \begin{pmatrix} 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \\ 5 & 1 & 2 & 6 \end{pmatrix} \tag{A.51}$$

Following the steps of the Hungarian algorithm we develope the L matrix:

$$L_{(\#IV^r ø7|I^r\triangle)} = \begin{pmatrix} 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \\ 5 & 1 & 2 & 6 \end{pmatrix} \longrightarrow L^F_{(\#IVrø7|I^r\triangle)} = \begin{pmatrix} 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 0 & 0 & 3 & 1 \\ 4 & 0 & 1 & 5 \end{pmatrix} \tag{A.52}$$

$$\longrightarrow L^H_{(\#IV^r ø7|I^r\triangle)} = \begin{pmatrix} 5 & 3 & \boxed{0} & 4 \\ 1 & 5 & 4 & \boxed{0} \\ \boxed{0} & 0 & 3 & 1 \\ 4 & \boxed{0} & 1 & 5 \end{pmatrix}$$

Following the distribution of boxes of the Hungarian algorithm, we obtain those metrics that correspond to the minimization of the total sum of distances between the voices.

$$S(L^H_{(\#IV^r ø7|I^r\triangle)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(\#IV_c ø7|I_c\triangle)}\right]_\nabla = \left[\begin{pmatrix} E & E \\ C & C \\ A & B \\ \#F & G \end{pmatrix}\right]_\nabla \tag{A.53}$$

Now we calculate the Nabla value when the distance between class mappings is minimal. The minimum for all ∞ mappings is unique so the sum can be constructed:

$$\nabla(E^o(\sharp IV_c ø7 \mid I_c\triangle)) = \sum_{j=1}^{n} \Omega \mid \int_{(E^o_{j1\,(\sharp IV_c ø7|I_c\triangle)})}^{(E^o_{j2\,(\#IV_c ø7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\Delta = 3$$

Once the S set is calculated we can form a generalization of a dimensionally optimized cadence $C_{\mathbb{E}}$. Thus we have calculated the exponents of the Mersenne numbers that transform the voices from one voicing to another.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(\#IV_c ø7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ A_{z_3} \\ \#F_{z_4} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ B_{z_3} \\ G_{z_4} \end{pmatrix} \tag{A.54}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id_4.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^0 - \lambda)(s^2 - \lambda)(s^1 - \lambda)$$

We study the algebraic multiplicities of each of the roots and classify them based on the positions of said roots, obtaining $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$.

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(\#IVø7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_{I\triangle}(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

We calculate the pair of functions that represent the two voicings involved since we have obtained the proportions of transformation between them.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{\#IVø7}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{A_{z_3}} + \psi_{\#F_{z_4}}(t) \\ \psi_{I\triangle}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{B_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

We symbolize with $\longrightarrow$ the transition from the first voicing to the second when the link is optimal.

$$\psi_{\#IVø7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(\#IVø7)} - e^{-2\pi tki\psi_j(\#IVø7)}}{2i}$$
$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

The tonal functions in the subdominant area must verify:

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} \geq 2 \\ M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} = 0 \end{cases}$$

As the $\sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} = 2$ then following the polynomial criteria we obtain the function of the degree $\#IV\o7$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(\#IV\o7|I\triangle)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-2\,s^2 - s - 1)\,\lambda^3 + (s^4 + 2\,s^3 + 2\,s^2 + s)\,\lambda^2 + (-s^5 - s^4 - 2\,s^3)\,\lambda + s^5$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{2} - \frac{s}{4} - \frac{1}{4}\right)\lambda^4 + \frac{s\,(s^3 + 2\,s^2 + 2\,s + 1)}{3}\lambda^3 + \left(-\frac{s^3\,(s^2 + s + 2)}{2}\right)\lambda^2 + s^5\,\lambda$

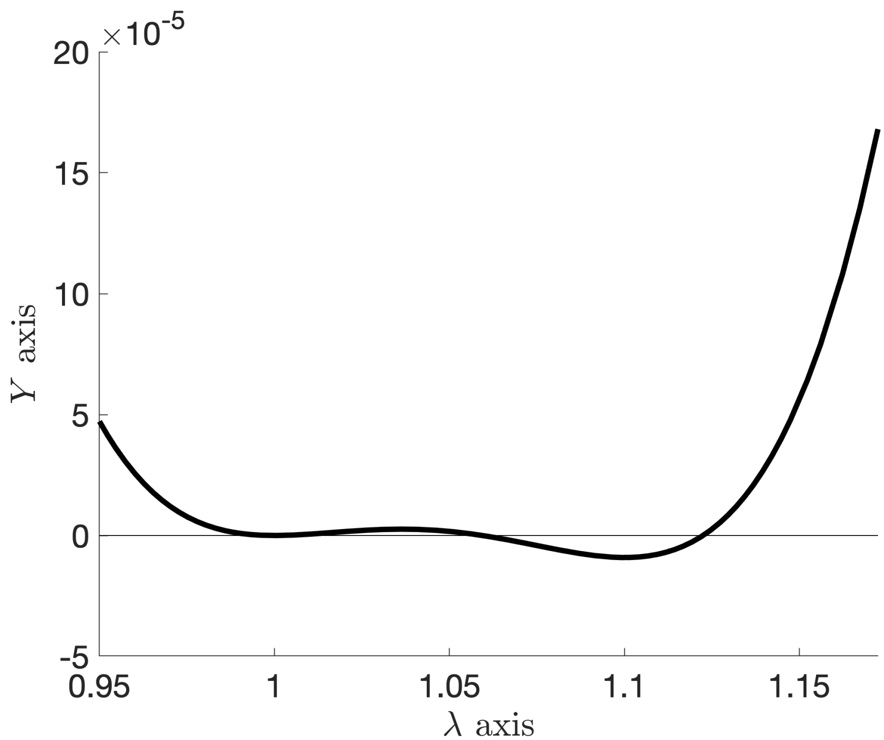


Figure A.10: Characteristic polynomial associated to the #IVø7-I△ cadence

A.2.4. V△ ⟶I△ Cadence

In many 21st century jazz traditions such as contemporary jazz, r&b, hip hop, or gospel, it is common to find chord progressions where there is no dominant chord and all chords have the same quality. This cadence appears frequently in this type of compositions and we are going to study why, when there is no supposed tritone resolution, as in classical music, the resolution between both tonal centers continues to exist.

We build the link between both structures looking for the explanation of the resolution between both tonal centers. Thus we write the link with both chords in root position.

$$E_{(V_c^r\triangle|I_c^r\triangle)} = \begin{pmatrix} \#F & B \\ D & G \\ B & E \\ G & C \end{pmatrix} \tag{A.55}$$

We calculate the link matrix calculating all the distances Δ_{ij}:

$$L_{(V^r\triangle|I^r\triangle)} = \begin{pmatrix} 5 & 1 & 2 & 6 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \end{pmatrix} \tag{A.56}$$

Following the steps of the Hungarian algorithm we develop the L matrix:

$$L_{(V^r\triangle|I^r\triangle)} = \begin{pmatrix} 5 & 1 & 2 & 6 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r\triangle|I^r\triangle)} = \begin{pmatrix} 4 & 0 & 1 & 5 \\ 1 & 3 & 0 & 0 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \end{pmatrix} \tag{A.57}$$

$$\longrightarrow L^H_{(V^r\triangle|I^r\triangle)} = \begin{pmatrix} 3 & 0 & \boxed{0} & 4 \\ 1 & 4 & 0 & \boxed{0} \\ \boxed{0} & 5 & 5 & 1 \\ 3 & \boxed{0} & 2 & 4 \end{pmatrix}$$

We calculate the S set where each box of the matrix L^H appears, which tells us how the voices should be linked to keep the nabla function at its minimum value.

$$S(L^H_{(V^r\triangle|I^r\triangle)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(V_c\triangle|I_c\triangle)}\right]_\nabla = \left[\begin{pmatrix} \#F & E \\ D & C \\ B & B \\ G & G \end{pmatrix}\right]_\nabla \tag{A.58}$$

Now we calculate the nabla value when the distance between class mappings is minimal. The minimum for al ∞ mappings is unique so the sum can be constructed:

$$\nabla(E^o(V_c\triangle \mid I_c\triangle)) = \sum_{j=1}^{n} \Omega \mid \int_{(E^o_{j1\,(V_c\triangle|I_c\triangle)})}^{(E^o_{j2\,(V_c\triangle|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\Delta = 4$$

Thus, all the links of the class of optimal links have the same value for their nabla function, which in this case is worth four. Now that we know how the chords are linked in the optimum we can calculate the matrix that transforms each voicing in an optimal way. We can form a generalization of a dimensinally optimized cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^4 \longrightarrow \Phi^4$$

$$C_\mathbb{E}(\psi(V_c\triangle)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{-\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} \#F_{z_1} \\ D_{z_2} \\ B_{z_3} \\ G_{z_4} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ B_{z_3} \\ G_{z_4} \end{pmatrix} \tag{A.59}$$

Then the polynomial is calculated as the determinant of the difference between $C_\mathbb{E}$ and Id_4.

$$p_{C_\mathbb{E}}(\lambda) = (s^{-2} - \lambda)(s^{-2} - \lambda)(s^0 - \lambda)(s^0 - \lambda)$$

The algebraic multiplicity of each root is $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(V\triangle)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_{I\triangle}(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

We now study the physical expression of the first and second voicing, where the absolute perception will take the minimum value.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V\triangle}(t) = \psi_{\#F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{B_{z_3}} + \psi_{G_{z_4}}(t) \\ \psi_{I\triangle}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{B_{z_3}} + \psi_{Gz_4}(t) \end{cases}$$

We remember that in the time domain the voicings behave like sums of trigonometric functions. We see the transition from one to the other and by W.F.C we can ensure that the absoute perception will reach the minimum.

$$\psi_{V\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(V\triangle)} - e^{-2\pi tki\psi_j(V\triangle)}}{2i}$$
$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

We classify the tonal function between both tonal centers studying the algebraic multiplicities. As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$ then following the polynomial criteria we obtain the function of the degree $V\triangle$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(V\triangle|I\triangle)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{2}{s^2} - 2\right) \lambda^3 + \left(\frac{4}{s^2} + \frac{1}{s^4} + 1\right) \lambda^2 + \left(-\frac{2}{s^2} - \frac{2}{s^4}\right) \lambda + \frac{1}{s^4}$

Integral of

$p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \dfrac{\lambda^5}{5} + \left(-\dfrac{s^2+1}{2\,s^2}\right) \lambda^4 + \dfrac{s^4 + 4\,s^2 + 1}{s^4\,3} \lambda^3 + \left(-\dfrac{2\,s^2+2}{2\,s^4}\right) \lambda^2 + \dfrac{\lambda}{s^4}$

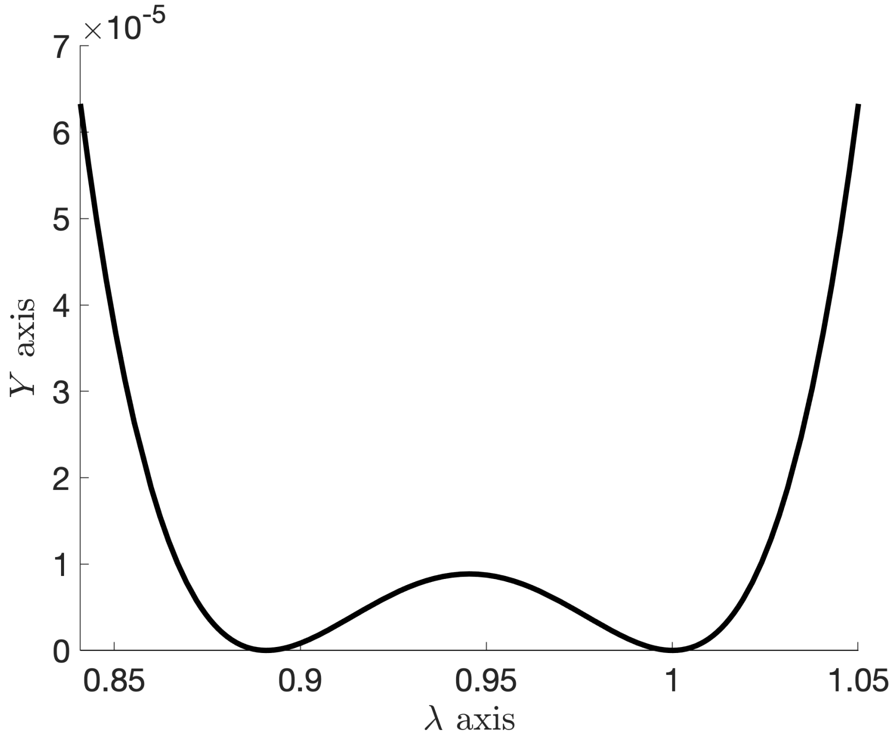


Figure A.11: Characteristic polynomial associated to the V△-I△ cadence

A.2.5. VI-7⟶I△ Cadence

In the context of classical music, this case is usually classified as having a tonic function, this classification being the one used by traditional books of tonal functions. What we intend to check is whether in this case the traditional tonal function of tonic agrees with the definition presented in this work. The answer is fortunately positive and coincides with the intuition of any music that both structures are substitutable in a reharmonization context, as long as the vertical dissonance graph allows it. We follow the usual process of calculating tonal functions, first assembling the link between both centers.

$$E_{(VI_c^r-7|I_c^r\triangle)} = \begin{pmatrix} G & B \\ E & G \\ C & E \\ A & C \end{pmatrix} \tag{A.60}$$

We calculate the link matrix calculating all the distances Δ_{ij}:

$$L_{(VI^r-7|I^r\triangle)} = \begin{pmatrix} 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \end{pmatrix} \tag{A.61}$$

Following the steps of the Hungarian algorithm we develope the L matrix. We are looking for the relationship between both tonal centers and we want to know how the voices behave.

$$L_{(VI^r-7|I^r\triangle)} = \begin{pmatrix} 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \end{pmatrix} \longrightarrow L^F_{(VI^r-7|I^r\triangle)} = \begin{pmatrix} 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \longrightarrow \tag{A.62}$$

$$L^H_{(VI^r-7|I^r\triangle)} = \begin{pmatrix} 4 & \boxed{0} & 3 & 5 \\ 5 & 3 & \boxed{0} & 4 \\ 1 & 5 & 4 & \boxed{0} \\ \boxed{0} & 0 & 3 & 1 \end{pmatrix}$$

The solutions given by the algorithm will be:

$$S(L^H_{(VI^r-7|I^r\triangle)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(VI_c-7|I_c\triangle)} \right]_\nabla = \left[\begin{pmatrix} G & G \\ E & E \\ C & C \\ A & B \end{pmatrix} \right]_\nabla \tag{A.63}$$

We calculate the nabla value when the distance between class mappings is minimal:

$$\nabla(E^o(VI_c - 7 \mid I_c\triangle)) = \sum_{j=1}^{n} \Omega \mid \int_{(E^o_{j1\,(VI_c-7|I_c\triangle)})}^{(E^o_{j2\,(VI_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\Delta = 2$$

Once the S set is calculated we can form a generalization of a dimensoinally optimized cadence $C_{\mathbb{E}}$. In the endomorphism in the frequency space, we have found the matrix that transforms each voicing of the first tonal center into another of the second so that the absolute perception is minimal.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(VI_c - 7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \\ A_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \\ B_{z_4} \end{pmatrix} \tag{A.64}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^0 - \lambda)(s^0 - \lambda)(s^2 - \lambda)$$

The multiplicities are $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 1$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(VI-7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_I(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We structure how the functions that symbolize each voicing and the transition between them will be.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VI-7}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{A_{z_4}}(t) \\ \psi_{I\triangle}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{B_{z_4}}(t) \end{cases}$$

The time domain function of each voicing will be a sum of trigonometric functions that follow the gamma distribution and where each fundamental extends proportionally along the harmonic distribution.

$$\psi_{VI-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VI-7)} - e^{-2\pi tki\psi_j(VI-7)}}{2i}$$

$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

To classify a tonal function as tonic we use the following criteria:

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} < 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases} \text{or} \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} < 2 \end{cases}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 1$ then following the polynomial criterion we obtain the function of the degree $VI-7$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(VI-7|I\triangle)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s^2 - 3)\ \lambda^3 + (3\,s^2 + 3)\ \lambda^2 + (-3\,s^2 - 1)\ \lambda + s^2$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{4} - \frac{3}{4}\right)\ \lambda^4 + \left(s^2 + 1\right)\ \lambda^3 + \left(-\frac{3\,s^2}{2} - \frac{1}{2}\right)\ \lambda^2 + s^2\,\lambda$

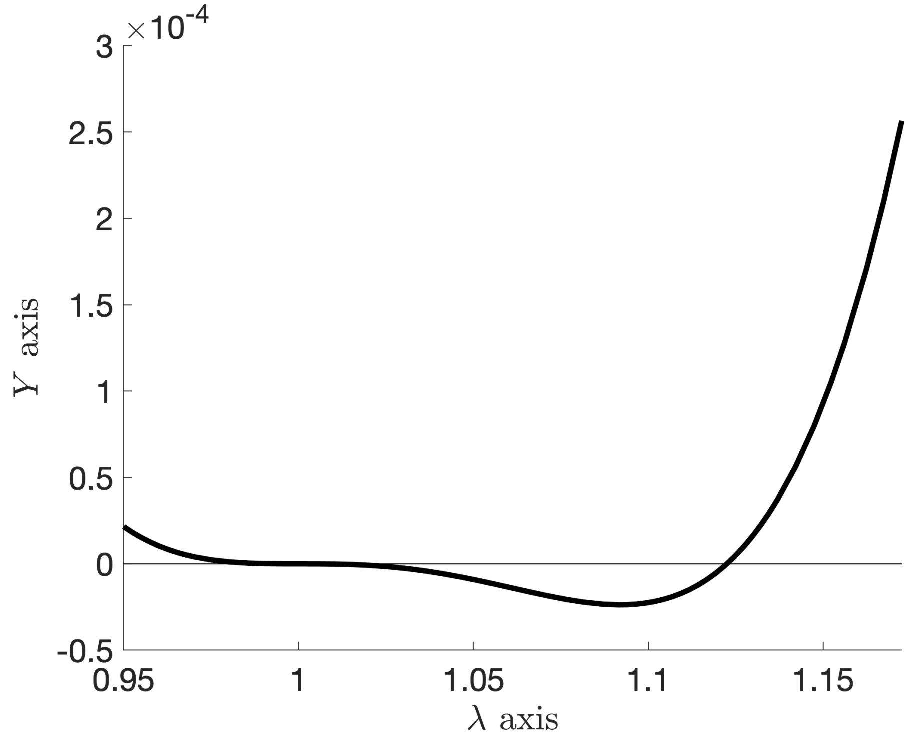


Figure A.12: Characteristic polynomial associated to the VI-7⟶I△ cadence

A.2.6. VII-7⟶I△ Cadence

We find in the calculation of this cadence one of the main reasons why the extensive calculation of tonal functions through the modes has been carried out. In this particular case we find the first example of a non-global dual tonal function, where we observe that for the same value of minimum nabla, there are several links with different tonal functions that are not in the same area. We will divide the results into three solutions based on the possible distributions of boxes over the solved matrix L^H. In this way, using the Zero method, we will have achieved those optimal links between both tonal centers and we will be able to deduce the tonal functions.

We calculate the link for this cadence to optimize it:

$$E_{(VII_c^r-7|I_c^r\triangle)} = \begin{pmatrix} A & B \\ \#F & G \\ D & E \\ B & C \end{pmatrix} \tag{A.65}$$

We build the matrix L to understand how to calculate the possible tonal functions between both centers. We calculate the ordered collection of metrics Δ_{ij}:

$$L_{(VII^r-7|I^r\triangle)} = \begin{pmatrix} 2 & 2 & 5 & 3 \\ 5 & 1 & 2 & 6 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \end{pmatrix} \tag{A.66}$$

Following the steps of the Hungarian algorithm we develope the L matrix until we reach to the S set.We have used the notation of stars to symbolize that we are covering the zeros with lines as indicated by the algorithm.

$$L_{(VII^r-7|I^r\triangle)} = \begin{pmatrix} 2 & 2 & 5 & 3 \\ 5 & 1 & 2 & 6 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \end{pmatrix} \longrightarrow L^F_{(VII^r-7|I^r\triangle)} = \begin{pmatrix} 0 & 0 & 3 & 1 \\ 4 & 0 & 1 & 5 \\ 1 & 3 & 0 & 0 \\ 0 & 4 & 5 & 1 \end{pmatrix} \longrightarrow$$

$$L^H_{(VII^r-7|I^r\triangle)} = \begin{pmatrix} 0 & 0 & 3 & 1 \\ 4 & 0 & 1 & 5 \\ 1 & 3 & 0 & 0 \\ 0 & 4 & 5 & 1 \end{pmatrix}$$

After this process we calculate $L^{H^*} = L^{H^*}_{(VII^r-7|I^r\triangle)}$ and we asign a starting zero. That is, we have covered the zeros with the minimum number of lines and we have subtracted the minimum not covered with lines from the entries of the matrix that are not covered, we have added the same minimum to those entries that are doubly covered and we have left invariants those entries covered by a single line.

$$L^{H^*} = \begin{pmatrix} \boxed{0}^* & 0 & 2 & 0 \\ 4 & \boxed{0} & 0 & 4 \\ 2 & 4 & \boxed{0} & 0 \\ 0 & 4 & 4 & \boxed{0} \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} 0 & \boxed{0}^* & 2 & 0 \\ 4 & 0 & \boxed{0} & 4 \\ 2 & 4 & 0 & \boxed{0} \\ \boxed{0} & 4 & 4 & 0 \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} 0 & 0 & 2 & \boxed{0}^* \\ 4 & \boxed{0} & 0 & 4 \\ 2 & 4 & \boxed{0} & 0 \\ \boxed{0} & 4 & 4 & 0 \end{pmatrix}$$

$$L^{H^*} = \begin{pmatrix} \boxed{0} & 0 & 2 & 0 \\ 4 & \boxed{0}^* & 0 & 4 \\ 2 & 4 & \boxed{0} & 0 \\ 0 & 4 & 4 & \boxed{0} \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} 0 & \boxed{0} & 2 & 0 \\ 4 & 0 & \boxed{0}^* & 4 \\ 2 & 4 & 0 & \boxed{0} \\ \boxed{0} & 4 & 4 & 0 \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} \boxed{0} & 0 & 2 & 0 \\ 4 & \boxed{0} & 0 & 4 \\ 2 & 4 & \boxed{0}^* & 0 \\ 0 & 4 & 4 & \boxed{0} \end{pmatrix}$$

$$L^{H^*} = \begin{pmatrix} 0 & \boxed{0} & 2 & 0 \\ 4 & 0 & \boxed{0} & 4 \\ 2 & 4 & 0 & \boxed{0}^* \\ \boxed{0} & 4 & 4 & 0 \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} 0 & \boxed{0} & 2 & 0 \\ 4 & 0 & \boxed{0} & 4 \\ 2 & 4 & 0 & \boxed{0} \\ \boxed{0}^* & 4 & 4 & 0 \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} \boxed{0} & 0 & 2 & 0 \\ 4 & \boxed{0} & 0 & 4 \\ 2 & 4 & \boxed{0} & 0 \\ 0 & 4 & 4 & \boxed{0}^* \end{pmatrix}$$

For a fixed zero, sometimes there are more than one posible solutions. For fixed zero number four the solutions S_1 and S_2 are available. For fixed zero number six, both solutions S_1 and S_3 are options to reach the minimum. Also, zero number eight has two options; S_2 and S_3. Then the solutions for $L^{H^*}_{(VII^r-7|I^r\triangle)}$ when both chords are in root position are described by three sets, each one representing a link where $\nabla(E)$ is minimal.

$$S_1(L^{H^*}_{(VII^r-7|I^r\triangle)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}, \Delta_{44}\}$$

$$S_2(L^{H^*}_{(VII^r-7|I^r\triangle)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

$$S_3(L^{H^*}_{(VII^r-7|I^r\triangle)}) = \{\Delta_{14}, \Delta_{22}, \Delta_{33}, \Delta_{41}\}$$

A.2.7. $S_1(L^{H}_{(VII^r-7|I^r\triangle))})$

We consider the first solution within the solution set and study its characteristic polynomial.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(VII_c - 7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{\Delta_{11}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{22}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{33}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{44}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ \#F_{z_2} \\ D_{z_3} \\ B_{z_4} \end{pmatrix} = \begin{pmatrix} B_{z_1} \\ G_{z_2} \\ E_{z_3} \\ C_{z_4+1} \end{pmatrix} \tag{A.67}$$

The optimal link is the same as the one as it appear with both structures in root position. The cadence presented here is not going to change the result of the polynomial only when it is dimensionally optimized.

$$\left[E^1_{(VII_c-7|I_c\triangle)} \right] = \left[\begin{pmatrix} A & B \\ \#F & G \\ D & E \\ B & C \end{pmatrix} \right]_{\nabla} \tag{A.68}$$

The Nabla value is calculated for any pair of frequencies that $\Omega[\int_\alpha^\beta \phi^{-1} d\phi] \leq 6$ where $\alpha \in [\alpha]$ and $\beta \in [\beta]$. The delta subindex indicates that the integral is the minimum of the delta set Δ.

$$\nabla(E^1) = \sum_{j=1}^{n} \Omega \mid \int_{(E^1_{j1\,(VII_c-7|I_c\triangle)})}^{(E^1_{j2\,(VII_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_{\Delta} = 6$$

In this way, calculate one of the polynomials obtained from the minimization, then we have found the first of the tonal functions.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)^2 (s^1 - \lambda)^2$$

We study the algebraic multiplicities of each root for each type of root $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 4$ $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 0$.So all not null roots are divergent roots and the set describes the direction of the voices when the link is optimal.

$$\lambda^+ = \{s^1, s^2\}$$

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(VII - 7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_I(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ Adding each fundamental with all its harmonics and adding each function in turn, we obtain the pair of voicings as sums of trigonometric functions. Thus we have that the antecedent and consequent voicing appear as decomposed functions within the bracket.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VII-7}(t) = \psi_{A_{z_1}}(t) + \psi_{\#F_{z_2}}(t) + \psi_{D_{z_3}}(t) + \psi_{B_{z_3}}(t) \\ \psi_{I\triangle}(t) = \psi_{B_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

For a certain gamma harmonic distribution we have that in the time domain these functions are the sum of $n \times h$ sinusoidal functions. We symbolize the temporal order between these functions with an arrow. This arrow should not be confused with the one used in tonal function graphs that explicitly indicates convergence between structures.

$$\psi_{VII-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VII-7)} - e^{-2\pi tki\psi_j(VII-7)}}{2i}$$
$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

At this point we study the behavior of the voices at the optimum and use the polynomial criterion to see in which area the first solution of the link falls.

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} \geq 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 4$ then following the polynomial criterion we obtain the one function of the degree $VII-7$ related to the Lydian tonal center. This case is specially interesting because it has multiple functions that share nabla value although the polynomials are different.

$$\sum_{j=1}^{n} \Omega \mid \int_{(E^1_{j1\,(VII_c-7|I_c\triangle)})}^{(E^1_{j2\,(VII_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\Delta = \sum_{j=1}^{n} \Omega \mid \int_{(E^2_{j1\,(VII_c-7|I_c\triangle)})}^{(E^2_{j2\,(VII_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\Delta$$
$$= \sum_{j=1}^{n} \Omega \mid \int_{(E^3_{j1\,(VII_c-7|I_c\triangle)})}^{(E^3_{j2\,(VII_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\Delta$$

We see that all the functions share the value of nabla but that the tonal functions are different polynomials. As a matter of order, we limit ourselves to addressing the first solution and classifying it within the subdominant area.

$$\Phi[E^1_{(II-7|I\triangle)} \neq \Phi[E^2_{(II-7|I\triangle)} \neq \Phi[E^3_{(II-7|I\triangle)}]$$

We classify the first of the three tonal functions associated with the link within the subdominant area being a polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E^1_{(VII-7|I\triangle)}] \in S^{\mathbb{R}[\lambda]}}$$

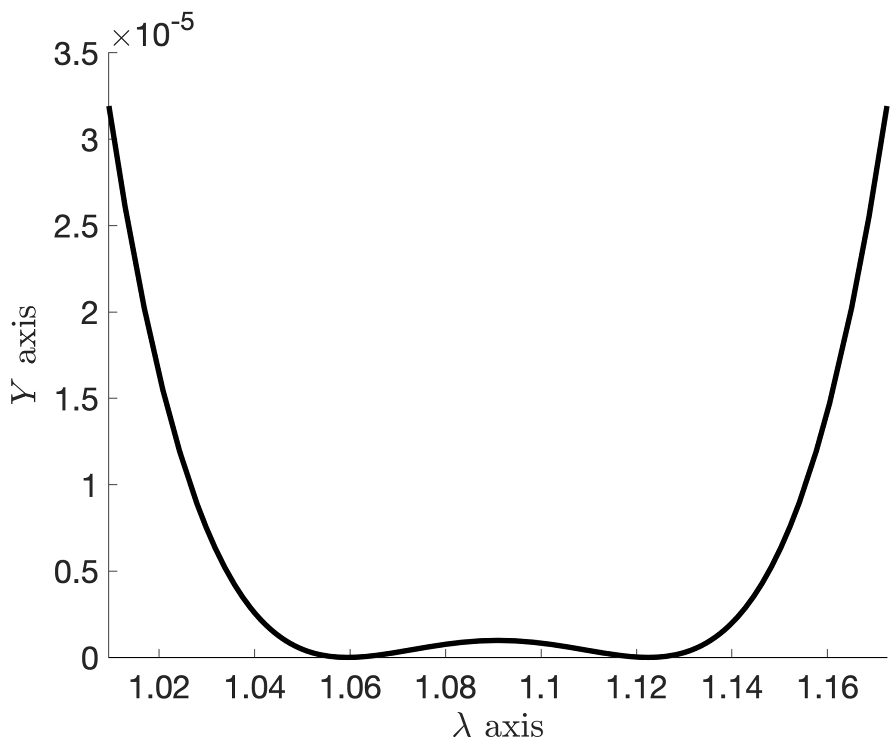


Figure A.13: Characteristic polynomial associated to the VII-7⟶I△ cadence

A.2.8. $S_2(L^H_{(VII^r-7|I^r\triangle))})$

Since the tonal function is dual, we develop the second solution in this section. We will carry out the same operation, but we will see that the endomorphism matrix changes and the tonal function changes as well.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(VII_c - 7)) \longrightarrow \psi(I_c\triangle)$$
$$\begin{pmatrix} s^{-\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{-\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ \#F_{z_2} \\ D_{z_3} \\ B_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \\ B_{z_4} \end{pmatrix} \tag{A.69}$$

The second optimized link will be a class matrix from the optimized class of links(nabla minimum class):

$$\left[E^2_{(VII_c-7|I_c\triangle)}\right]_\nabla = \left[\begin{pmatrix} A & G \\ \#F & E \\ D & C \\ B & B \end{pmatrix}\right]_\nabla \tag{A.70}$$

Measuring the link we obtain the same nabla value as $E^1_{(VII-7|I\triangle)}$in such a way that we add the minimum distances between voices $\nabla(E^2_{(VII_c-7|I_c\triangle)}) = \sum_{j=1}^n \Omega \mid \int_{(E^2_{j1\,(VII_c-7|I_c\triangle)})}^{(E^2_{j2\,(VII_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\triangle$

In this way, in the frequency space we have measured the distance between the voicings using the nabla function. Taking the characteristic polynomial of the endomorphism matrix, we arrive at the second tonal function having one root on the stabilizer of the group M and three on the left.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-2} - \lambda)^3(s^0 - \lambda)$$

The algebraic multiplicity of each roots is:

$$\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3, \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0, \sum_{i=1}^{|\lambda^o|} m_{\lambda_i^o} = 1$$

Thus we see that the sum of algebraic multiplicities coincides with the dimension of space and also describes the direction of the voices. We see that in this case the tonal function is polarized and is determined by a single type of root that indicates that there are three voices that go down one tone in the optimal link for solution number two. So all not null roots are divergent roots and the set describes the direction of the voices when the link is optimal.

$$\lambda^- = \{s^{-2}\}$$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi+$ as a component of $\psi(VII-7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector. written as $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$.

We visualize what the trigonometric functions that describe each of the voicings in the arrangement are like when we select particular octaves for the classes of both tonal centers.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VII-7}(t) = \psi_{A_{z_1}}(t) + \psi_{\#F_{z_2}}(t) + \psi_{D_{z_3}}(t) + \psi_{B_{z_3}}(t) \\ \psi_{I\triangle}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}}(t) + \psi_{B_{z_3}}(t) \end{cases}$$

The representation of the voicings depends on the gamma harmonic distribution, but we know that in the case of timbre equality between the antecedent voicing and the consequent voicing, said distribution does not affect the minimization of absolute perception, and therefore neither does it affect the tonal function.

$$\psi_{VII-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VII-7)} - e^{-2\pi tki\psi_j(VII-7)}}{2i}$$
$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

The tonal functions in the dominant area are classified like this if in the optimal link the convergent algebraic multiplicity is greater than two, the convergent algebraic multiplicity is null and there are an arbitrary number of static voices.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} \geq 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases}$$

As $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ then, following the polynomial criterion we obtain the function of the degree $VII-7$ related to the Lydian tonal center. In this case is not unique and can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E^2_{(VII-7|I\triangle)}] \in D^{\mathbb{R}[\lambda]}}$$

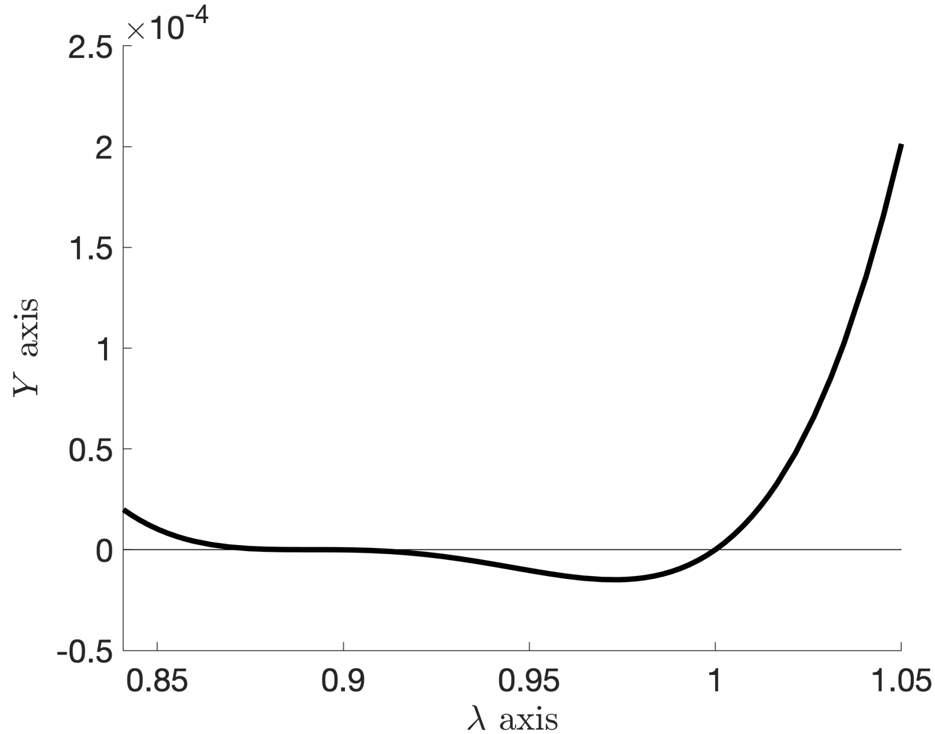


Figure A.14: Characteristic polynomial associated to the VII-7⟶I△ cadence

A.2.9. $S_3(L^H_{(VII^r-7|I^r\triangle)})$

Finally, we develop the third solution by the Zero method to complete the calculation of the tonal functions.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(VII-7)) \longrightarrow \psi(I\triangle)$$

$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{22}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{33}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ \#F_{z_2} \\ D_{z_3} \\ B_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ E_{z_2} \\ B_{z_3} \end{pmatrix} \tag{A.71}$$

We calculate the set of optimal links for the third solution:

$$\left[E^3_{(VII_c-7|I_c\triangle)} \right] = \left[\begin{pmatrix} A & C \\ \#F & G \\ D & E \\ B & B \end{pmatrix} \right]_{\nabla} \tag{A.72}$$

Now we calculate the nabla value when the distance between class mappings is minimal. The minimum for all ∞ mappings is unique so the sum can be constructed:

$$\nabla(E^3_{(VII_c-7|I_c\triangle)}) = \sum_{j=1}^{n} \Omega \mid \int_{(E^3_{j1\,(VII_c-7|I_c\triangle)})}^{(E^3_{j2\,(VII_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_{\Delta} = 6$$

In this way, the characteristic polynomial will have three roots to the right of the stabilizer of the group M and one on said stabilizer, then it is the third tonal function that the absolute perception is minimized.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^3 - \lambda)(s^1 - \lambda)(s^2 - \lambda)(s^0 - \lambda)$$

The algebraic multiplicity of each root is given by $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 0$. The set $\lambda^+ = \{s^1, s^2, s^3\}$ describes the direction of the voices when the link is optimal.

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(VII-7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_{I\triangle}(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. In this way we consider the pair of functions that represent the transition between voicings in the time domain.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VII-7}(t) = \psi_{A_{z_1}}(t) + \psi_{\#F_{z_2}}(t) + \psi_{D_{z_3}} + \psi_{B_{z_4}}(t) \\ \psi_{I\triangle}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}} + \psi_{B_{z_4}}(t) \end{cases}$$

This transition is the third that is possible minimizing the absolute perception and what we understand in this section is that the tonal function continues to be present regardless of which one is expressed in the music.

$$\psi_{VII-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VII-7)} - e^{-2\pi tki\psi_j(VII-7)}}{2i} \longrightarrow$$

$$\psi_{I\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

The tonal functions in the subdominant area must verify:

$$\mathbb{A}(H(\lambda)) = S^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} \geq 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then following the polynomial criterion we obtain the function of the degree $VII - 7$ related to the Lydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E^3(VII - 7 \mid I\triangle)] \in S^{\mathbb{R}[\lambda]}}$$

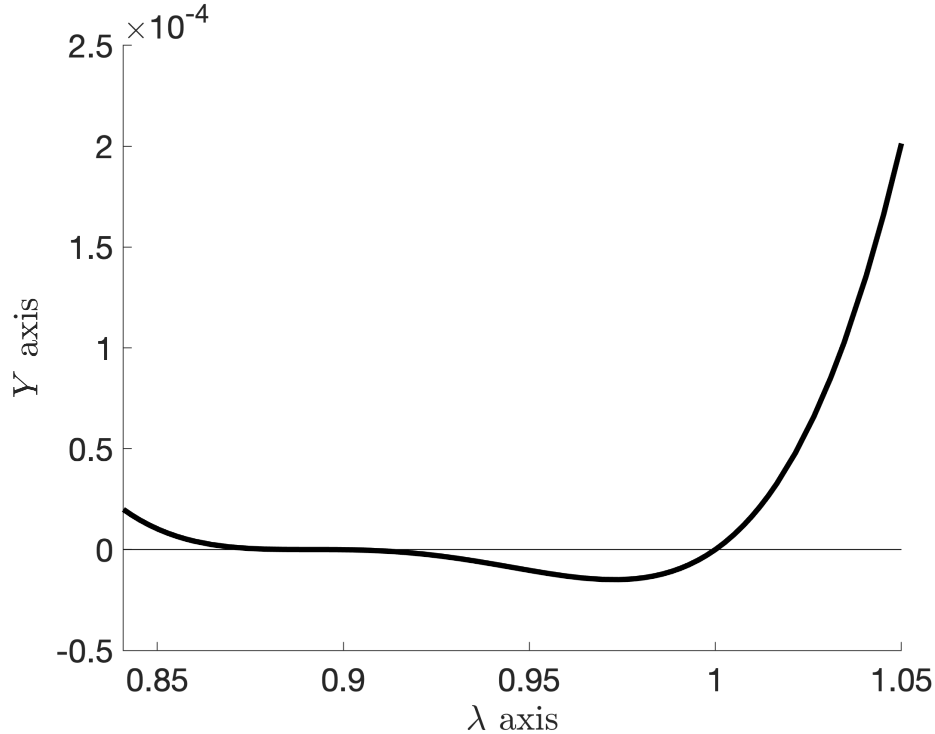


Figure A.15: Characteristic polynomial associated to the VII-7⟶I△ cadence

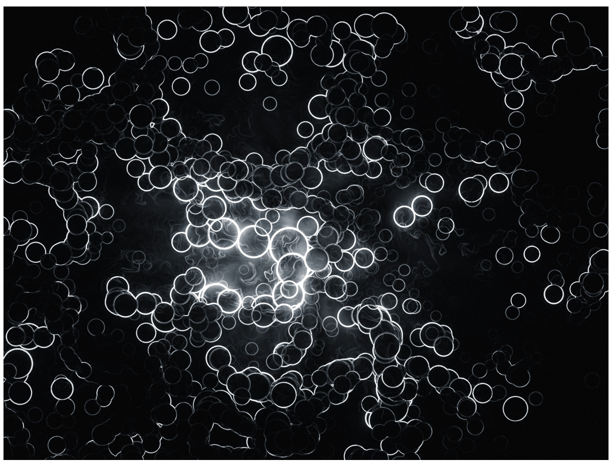
A.2.10. Lydian tonal functions

$II- \longrightarrow I$	$\Phi[E_{(II-\|I)}] \in S^{\mathbb{R}[\lambda]} \cup D^{\mathbb{R}[\lambda]}$
$III- \longrightarrow I$	$\Phi[E_{(III-\ \|I)}] \in T^{\mathbb{R}[\lambda]}$
$\#IVo \longrightarrow I$	$\Phi[E_{(\#IVo\|I)}] \in D^{\mathbb{R}[\lambda]}$
$V \longrightarrow I$	$\Phi[E_{(V\|I)}] \in S^{\mathbb{R}[\lambda]}$
$VI- \longrightarrow I$	$\Phi[E_{(VI-\|I)}] \in T^{\mathbb{R}[\lambda]}$
$VII- \longrightarrow I$	$\Phi[E_{(VII-\|I)}] \in S^{\mathbb{R}[\lambda]}$

$II7 \longrightarrow I\triangle$	$\Phi[E_{(II7\|I\triangle)}] \in S^{\mathbb{R}[\lambda]}$
$III-7 \longrightarrow I\triangle$	$\Phi[E_{(III-7\|I\triangle)}] \in T^{\mathbb{R}[\lambda]}$
$\#IV\o7\triangle \longrightarrow I\triangle$	$\Phi[E_{\#IV\o7\|I\triangle)}] \in S^{\mathbb{R}[\lambda]}$
$V\triangle \longrightarrow I\triangle$	$\Phi[E_{(V\triangle\|I\triangle)}] \in D^{\mathbb{R}[\lambda]}$
$VI-7 \longrightarrow I\triangle$	$\Phi[E_{(VI-7\|I\triangle)}] \in T^{\mathbb{R}[\lambda]}$
$VII-7 \longrightarrow I\triangle$	$\Phi[E_{(VII-7\|I\triangle)}] \in S^{\mathbb{R}[\lambda]} \cup D^{\mathbb{R}[\lambda]}$

Appendix B

The Ionian Mode



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B.1. The Ionian Mode for $n = 3$

B.1.1. II-$\longrightarrow$I Cadence

In this section we are going to dedicate ourselves to studying the tonal functions that correspond to each one of the edges of the graph of tonal functions that connects the first degree of the Ionian mode with the rest in the same mode. This is the most stable mode according to the number of prime harmonics of said mode and it is the one that throughout the history of music has been the center of harmonic progressions. Thus, in this chapter each of the tonal functions in three and four voices will be much more familiar to the reader than in the rest of the chapters for the simple reason that this mode is the one that appears most frequently in classical music.

We calculate the link matrix for this cadence to calculate the tonal function. The link will be a progression such as:

$$E_{(II_c^r-|I_c^r)} = \begin{pmatrix} A & G \\ F & E \\ D & C \end{pmatrix} \tag{B.1}$$

Once the link is written, the link cadence will be the same for all tonalities so the subindex c is omitted:

$$L_{(II^r-|I^r)} = \begin{pmatrix} 2 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 2 \end{pmatrix} \tag{B.2}$$

Then, following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix until we reach the solution set. The process is described in terms of matrices:

$$L_{(II^r-|I^r)} = \begin{pmatrix} 2 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r-|I^r)} = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 0 \end{pmatrix} \longrightarrow L^H_{(II^r-|I^r)} = \begin{pmatrix} \boxed{0} & 3 & 1 \\ 1 & \boxed{0} & 4 \\ 3 & 0 & \boxed{0} \end{pmatrix} \tag{B.3}$$

Then the solutions for $L^H_{(II^r-|I^r)}$ when both triads are in root position becomes the following set:

$$S(L^H_{(II^r-|I^r)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

These solutions give us a subset of cadences that are dimensionally optimized, so in the same dimension for an arbitrary set of integers the cadence will follow the solutions of the L^Hmatrix. A formal definition to dimensionally optimized voicings will be: Two voicings $\psi(X)$ and $\psi(Y)$ are **dimensionally optimized** if and only if every component of each vector $\psi(X)$ and $\psi(Y)$ is in the appropriate dimension given by the solutions of $L^H_{(X|Y)}$. We say that a cadence is dimensionally optimized if for any selection of integers the vector in the pre-image and the vector in the image are dimensionally optimized. If the cadence verifies that, we mark it as $C_{\mathbb{E}}$.

To complete the calculations, we write the equation for all dimensionally optimized cadences for arbitrary subindices in $\mathbb{Z}$:

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(II_c-)) = \psi(I_c)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ F_{z_2} \\ D_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \end{pmatrix} \tag{B.4}$$

Then, the characteristic polynomial will be calculated as:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^{-2} - \lambda & 0 & 0 \\ 0 & s^{-1} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{B.5}$$

$$p_{C_{\mathbb{E}}}(\lambda) = (\lambda - s^{-1})(\lambda - s^{-2})^2$$

So all not null roots are convergent roots and the set describes the direction of the voices when the link is optimal.

$$\lambda^{-} = \{s^{-1}, s^{-2}\}$$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^{+}$ as a component of $\psi(II-)$ or $\psi(I)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. So in case phase is 0 we can write each note of the voicing as a function $\psi_{X_j}(t)$. The subindex f is omitted due to the fact that in the sum, at least for musical analysis, it is not necessary to write the inversion.

$$\mathcal{B}_{\epsilon} = \begin{cases} \psi_{II_c-}(t) = \psi_{A_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

Thus, we write the sums for both voicing where Γ_k represents the amplitude of each k-harmonic when $\phi \in \Phi$ is a frecuency and Φ is a real vector space.

$$\psi_{II-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II-)} - e^{-2\pi tki\psi_j(II-)}}{2i}$$

$$\longrightarrow \psi_I(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I)} - e^{-2\pi tki\psi_j(I)}}{2i}$$

Following the polynomial criteria we obtain the function of the degree $II-$ related to the Ionian tonal center.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^{-}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{-}|} m_{\lambda_i^{-}} \geq 2 \\ M^{+}_{H(\lambda)} = \sum_{i=1}^{|\lambda^{+}|} m_{\lambda_i^{+}} = 0 \end{cases}$$

In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(II-|I)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s} - \frac{2}{s^2}\right)\,\lambda^2 + \left(\frac{2}{s^3} + \frac{1}{s^4}\right)\,\lambda - \frac{1}{s^5}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s+2}{3\,s^2}\right)\,\lambda^3 + \frac{2\,s+1}{2\,s^4}\,\lambda^2 + \left(-\frac{1}{s^5}\right)\,\lambda$

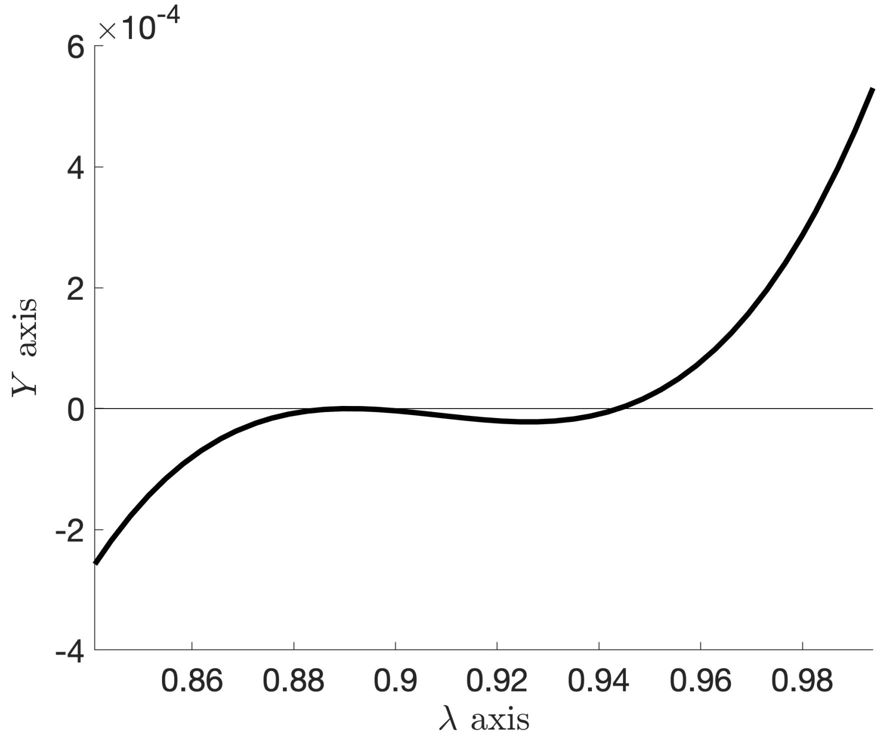


Figure B.1: Characteristic polynomial associated to the II-⟶I cadence

B.1.2. III-⟶I Cadence

Customarily, the third degree is a substitute for tonic in traditional harmony. What we want to understand is why this relationship occurs and not another. We are going to see that since most of the voices are common, the variation in perception is very little, so in subjective terms they are easily replaceable. We are going to develop the case in full to better understand the relationship between the two tonal centers.

We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as:

$$E_{(III^r_c - |I^r_c)} = \begin{pmatrix} B & G \\ G & E \\ E & C \end{pmatrix} \tag{B.6}$$

The link matrix is calculated using the minimum integral between two elements of the respective classes:

$$L_{(III^r-|I^r)} = \begin{pmatrix} 4 & 5 & 1 \\ 0 & 3 & 5 \\ 3 & 0 & 4 \end{pmatrix} \tag{B.7}$$

Then, following the steps of the Hungarian algorithm we want to develope the algorithm through the L matrix to reach $L^H_{(III-|I)}$.

$$L_{(III^r-|I^r)} = \begin{pmatrix} 4 & 5 & 1 \\ 0 & 3 & 5 \\ 3 & 0 & 4 \end{pmatrix} \longrightarrow L^F_{(III^r-|I^r)} = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 3 & 5 \\ 3 & 0 & 4 \end{pmatrix} \longrightarrow L^H_{(III^r-|I^r)} = \begin{pmatrix} 3 & 4 & \boxed{0} \\ \boxed{0} & 3 & 5 \\ 3 & \boxed{0} & 4 \end{pmatrix} \tag{B.8}$$

Then the solutions for $L^H_{(III-|I)}$ when both triads are in root position becomes the following set:

$$S(L^H_{(III^r-|I^r)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

Now we calculate an optimal link class following the solutions of the S set:

$$\left[E^o_{(III_c-|I_c)} \right]_\nabla = \left[\begin{pmatrix} B & C \\ G & G \\ E & E \end{pmatrix} \right]_\nabla \tag{B.9}$$

We calculate the optimal link class nabla value $\nabla(E^o_{(III_c-|I_c)}) = 1+0+0 = 1$ and we generalize the value to the relation purely between degrees using the static function theorem, then we can disregard all subscripts: $\nabla^o_{(III-|I)} = 1$.

Once we have found the solutions in S we can write a dimensionally optimized cadence for any set of integers.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(III_c-)) \longrightarrow \psi(I_c)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} B_{z_1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1+1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} \tag{B.10}$$

And we generalize the value purely to the relation purely between degrees using the static function theorem, then we can disregard all subscripts

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^1 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{B.11}$$

Thus, the factored polynomial is itself the tonal function and takes the following expression, such factorization being enough for us to see at first glance the algebraic multiplicities.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^1 - \lambda)(s^0 - \lambda)^2$$

All not null roots are divergent roots and the set is described by the following equation:

$$\lambda^+ = \{s^1\}$$

Since we have obtained the exponents of the Mersenne numbers that transform one voicing into another when the absolute perception is minimal, then we can calculate, for a certain gamma harmonic distribution, the equations of each voicing. Thus, in the bracket we present the note-by-note decomposition of both equations. Note the reader that the bracket is placed in this part of the calculation since for the optimization to make sense, there must be a relationship between the subscripts of the classes in midi notation.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{III-}(t) = \psi_{B_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \\ \psi_I(t) = \psi_{C_{z_1+1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \end{cases}$$

$$\psi_{III-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(III-)} - e^{-2\pi tki\psi_j(III-)}}{2i}$$
$$\longrightarrow \psi_I(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I)} - e^{-2\pi tki\psi_j(I)}}{2i}$$

Since only one voice is moving, there is not enough change considered to decide that the function is either subdominant or dominant. This is how we specify that the tonal area is tonic, following the polynomial criterion.

$$\boxed{\Phi[E_{(III-|I)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s-2)\ \lambda^2 + (2\,s+1)\ \lambda - s$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s}{3} - \frac{2}{3}\right)\ \lambda^3 + \left(s + \frac{1}{2}\right)\ \lambda^2 + (-s)\ \lambda$

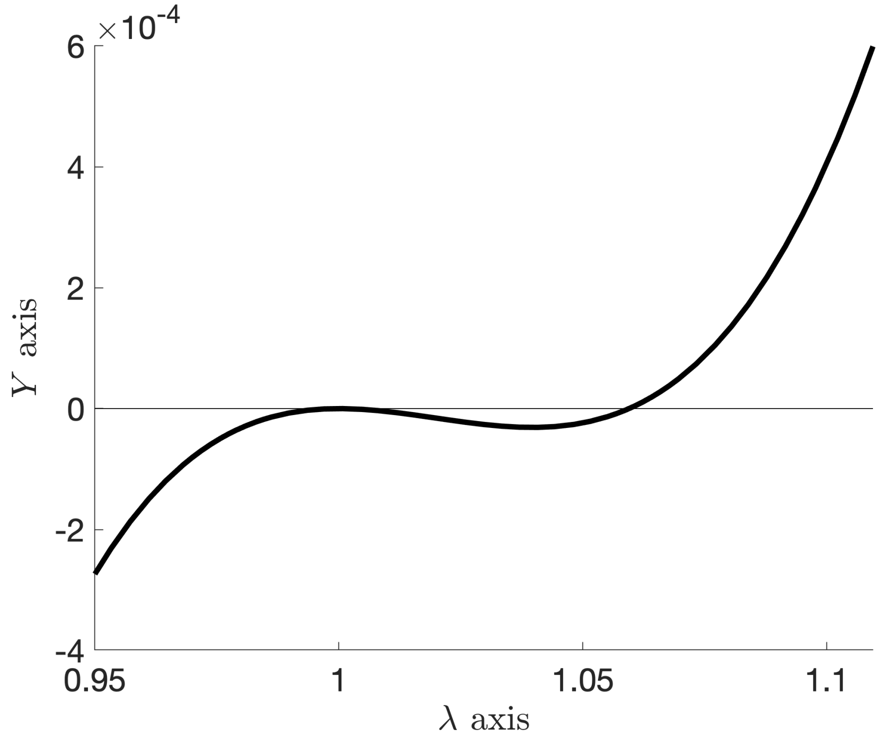


Figure B.2: Characteristic polynomial associated to the III-⟶I cadence

B.1.3. IV⟶ I Cadence

In this section we are going to study the plagal cadence from a new perspective. We are going to see how, according to the solutions of the Hungarian algorithm, the plagal cadence is actually resolving and the perfect cadence (without a seventh) is nothing more than, in reality, the cadence that arises from the retrograde link of a plagal cadence. As indicated, the perfect cadence in three voices is not solvable by the direction of the voices. This clearly goes against the confusion that exists between the perfect cadence with the seventh and the perfect cadence without it. As we have seen from the static tonal function theorem, the pitch function depends exclusively on the classes involved in the link, when the gamma distribution is constant for the two voicings of the arrangement.

The link will between both chords will be:

$$E_{(IV_c^r|I_c^r)} = \begin{pmatrix} C & G \\ A & E \\ F & C \end{pmatrix} \tag{B.12}$$

The link cadence will be constructed calculating every Δ_{ij} distance. The key subindex is omitted:

$$L_{(IV^r|I^r)} = \begin{pmatrix} 5 & 4 & 0 \\ 2 & 5 & 3 \\ 2 & 1 & 5 \end{pmatrix} \tag{B.13}$$

We develop the algorithm through the L matrix to find $L^H_{(IV^r|I^r)}$:

$$L_{(IV^r|I^r)} = \begin{pmatrix} 5 & 4 & 0 \\ 2 & 5 & 3 \\ 2 & 1 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r|I^r)} = \begin{pmatrix} 5 & 4 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{pmatrix} \longrightarrow L^H_{(IV^r|I^r)} = \begin{pmatrix} 5 & 4 & \boxed{0} \\ \boxed{0} & 3 & 1 \\ 1 & \boxed{0} & 4 \end{pmatrix} \tag{B.14}$$

Then, the solutions for $L^H_{(IV^r|I^r)}$ form a set that will be used as the reference to construct the link nabla class or optimal link class. $S(L^H_{(IV|I)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$ We calculate a dimensionally optimal link class following the results of $L^H_{(IV^r|I^r)}$. All permutations of the rows of an optimal link are in the same class.

$$\left[E^o_{(IV_c|I_c)}\right]_\nabla = \left[\begin{pmatrix} C & C \\ A & G \\ F & E \end{pmatrix}\right]_\nabla \tag{B.15}$$

We calculate an optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(IV_c|I_c)}) = 1 + 0 + 3 = 4$ and we write the optimal nabla value as $\nabla^o_{(IV|I)} = 4$. Notice that the c subindex has been omitted due to the colour theorem. The nabla value is an invariant for any tonality and any starting pair of chords, when we measure any of the optimal links.

We write the generalization of the dimensionally optimal cadences for an arbitrary selection of integers for every subindex of the components of the vectors in Φ^3:

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(IV)) \longrightarrow \psi(I)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ F_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} \tag{B.16}$$

Once we have found a dimensionally optimal cadence $C_{\mathbb{E}}$, for any selection of subindexes in $\mathbb{Z}$, the characterystic polynomial will be invariant. This polynomial will give us the direction of the voices when the link is optimal.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-1} - \lambda \end{pmatrix} \tag{B.17}$$

Thus, the characteristic polynomial arises from multiplying the entries of the diagonal of the previous matrix.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^{-2} - \lambda)(s^{-1} - \lambda)$$

So all roots are convergent roots and the set λ^- is described by:

$$\lambda^- = \{s^{-1}, s^{-2}\}$$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(IV)$ or $\psi(I)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. Both voicings are descomposed in single notes by the following equations:

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{F_{z_3}}(t) \\ \psi_I(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \end{cases}$$

Then, the voicings are expressed as sums of trigonometric functions for an arbitrary timbre Γ_k.

$$\psi_{IV}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(IV)} - e^{-2\pi tki\psi_j(IV)}}{2i}$$

$$\longrightarrow \psi_I(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I)} - e^{-2\pi tki\psi_j(I)}}{2i}$$

At this point, we apply the polynomial criterion to study the classification of the tonal function within the set of polynomials and determine the behavior of the progression.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} \geq 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases}$$

Following the polynomial criterion we obtain the function of the degree IV related to the Ionian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(IV|I)}] \in D^{\mathbb{R}[\lambda]}}$$

.

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s} - \frac{1}{s^2} - 1\right)\lambda^2 + \left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}\right)\lambda - \frac{1}{s^3}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2+s+1}{s^2\,3}\right)\lambda^3 + \frac{s^2+s+1}{s^3\,2}\lambda^2 + \left(-\frac{1}{s^3}\right)\lambda$

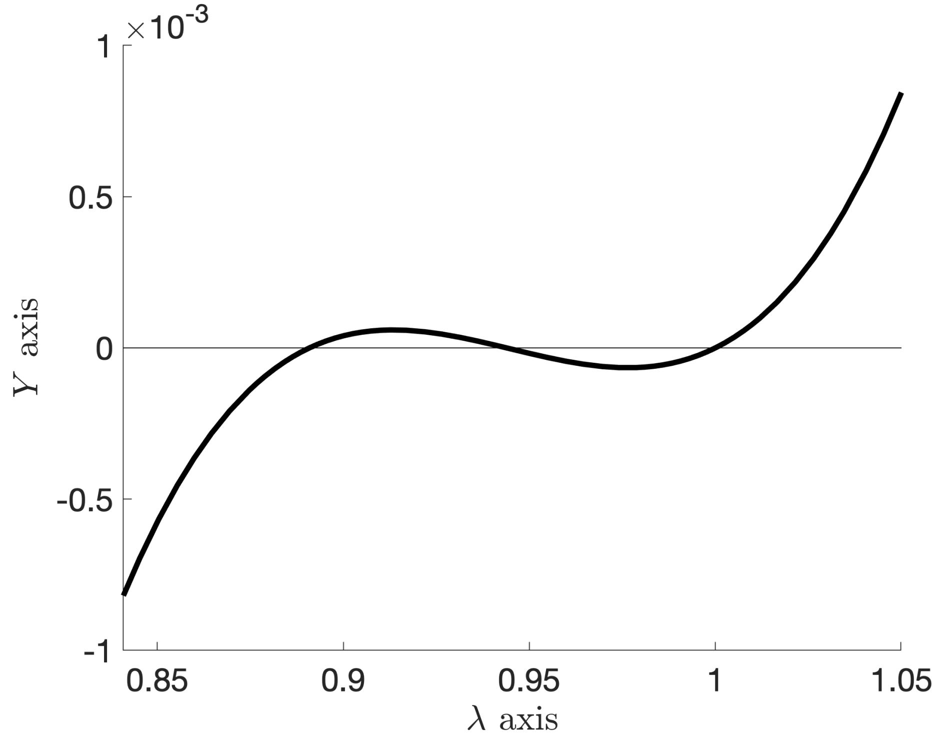


Figure B.3: Characteristic polynomial associated to the IV⟶ I cadence

B.1.4. V⟶ I Cadence

In the case of the cadence from V to I we are going to show that in the absence of more voices, this cadence is not resolving according to the polynomial criterion. That is to say, that its retrograde progression, which is the plagal cadence, is dominant, and therefore we have that the cadence itself does not have a convergent character. This idea will clash with the preconceived idea that the progression V to I resolves by eliminating a voice from the cadence $V7$ to I , but by having the W.F.C connection we see that the connection of perception with the voice leading allows us to examine this case precisely without leaving room for doubt.

We assemble the harmonic link to build the L matrix.

$$E_{(V_c^r|I_c^r)} = \begin{pmatrix} D & G \\ B & E \\ G & C \end{pmatrix} \tag{B.18}$$

Calculating every Δ_{ij} distance, we calculate the L matrix, which is the same for a relationship between degrees regardless of the key we are in.

$$L_{(V^r|I^r)} = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 5 & 1 \\ 0 & 3 & 5 \end{pmatrix} \tag{B.19}$$

Following the steps of the Hungarian algorithm we develope the L matrix into $L^H_{(V^r|I^r)}$:

$$L_{(V^r|I^r)} = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 5 & 1 \\ 0 & 3 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r|I^r)} = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & 3 & 5 \end{pmatrix} \longrightarrow L^H_{(V^r|I^r)} = \begin{pmatrix} 3 & \boxed{0} & 0 \\ 3 & 4 & \boxed{0} \\ \boxed{0} & 3 & 5 \end{pmatrix} \tag{B.20}$$

Then the solutions for $L^H_{(V^r|I^r)}$ when both triads are in root position becomes the following set:

$$S(L^H_{(V^r|I^r)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

Now we calculate the link nabla class:

$$\left[E^o_{(V_c|I_c)} \right]_\nabla = \left[\begin{pmatrix} D & E \\ B & C \\ G & G \end{pmatrix} \right]_\nabla \tag{B.21}$$

We calculate the optimal link class nabla value. This class contains all the posible link between a chord and the tonal center that share nabla value, where nabla is minimum $\nabla(E^o_{(V_c|I_c)}) = 2 + 1 + 0 = 3$ and we write the optimal nabla value as $\nabla^o_{(V|I)} = 3$

With the solutions of the L matrix found, we write the generalized equation of a dimensionally optimized cadence for an arbitrary set of integers.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(V_c)) = \psi(I_c)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ B_{z_2} \\ G_{z_3} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2+1} \\ G_{z_3} \end{pmatrix} \tag{B.22}$$

The polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^1 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{B.23}$$

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^2 - \lambda)(s^1 - \lambda)$$

So all not null roots are divergent roots and the set

$$\lambda^+ = \{s^1, s^2\}$$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(V)$ or $\psi(I)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$

$$\mathcal{B}_\epsilon = \begin{cases} \psi_V(t) = \psi_{D_{z_1}}(t) + \psi_{B_{z_2}}(t) + \psi_{G_{z_3}}(t) \\ \psi_I(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2+1}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

We calculate the trigonometric functions that are the temporal expression of each voicing for the two triads that we are studying.

$$\psi_V(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(V)} - e^{-2\pi tki\psi_j(V)}}{2i}$$
$$\longrightarrow \psi_I(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I)} - e^{-2\pi tki\psi_j(I)}}{2i}$$

The tonal functions in the dominant area are classified like this if in the optimal link the convergent algebraic multiplicity is greater than two, the convergent algebraic multiplicity is null and there are an arbitrary number of static voices.

$$\mathbb{A}(H(\lambda)) = D^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} \geq 2 \\ M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \end{cases}$$

Following the polynomial criterion we obtain the function of the degree V related to the Ionian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(V|I)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - s - 1)\ \lambda^2 + (s^3 + s^2 + s)\ \lambda - s^3$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2}{3} - \frac{s}{3} - \frac{1}{3}\right)\ \lambda^3 + \frac{s\ (s^2 + s + 1)}{2}\ \lambda^2 + \left(-s^3\right)\ \lambda$

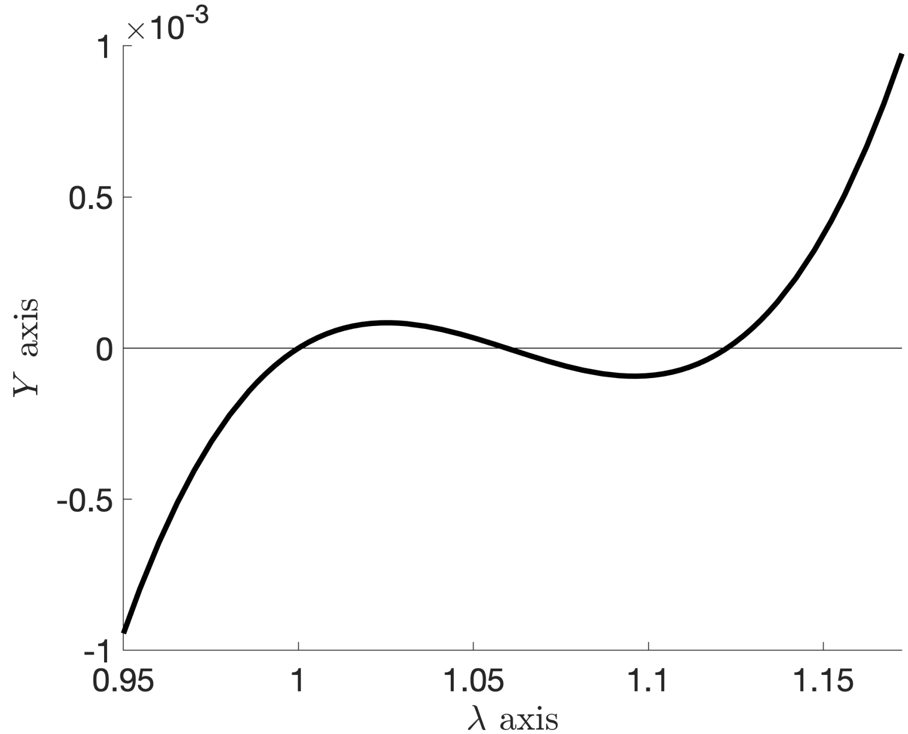


Figure B.4: Characteristic polynomial associated to the V⟶ I cadence

B.1.5. VI-⟶ I Cadence

We study the relationship between the relative minor and the first degree. We want to know how they are related to each other in order to generate the graph of tonal functions and use it with a view to reharmonizing harmonic progressions, composition and a deep understanding of how music works.

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them in the L matrix:

$$L_{(VI^r - |I^r)} = \begin{pmatrix} 3 & 0 & 4 \\ 5 & 4 & 0 \\ 2 & 5 & 3 \end{pmatrix} \tag{B.24}$$

Following the steps of the Hungarian algorithm, we develop the algorithm through the L matrix to find an optimum link:

$$L_{(VI^r - |I^r)} = \begin{pmatrix} 3 & 0 & 4 \\ 5 & 4 & 0 \\ 2 & 5 & 3 \end{pmatrix} \longrightarrow L^F_{(VI^r - |I^r)} = \begin{pmatrix} 3 & 0 & 4 \\ 5 & 4 & 0 \\ 0 & 3 & 1 \end{pmatrix} \longrightarrow L^H_{(VI^r - |I^r)} = \begin{pmatrix} 3 & \boxed{0} & 4 \\ 5 & 4 & \boxed{0} \\ \boxed{0} & 3 & 1 \end{pmatrix} \tag{B.25}$$

Then, the solutions for $L^H_{(VI^r-|I^r)}$ when both structures are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(VI^r-|I^r)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

We calculate an optimal link class using the solutions:

$$\left[E^o_{(VI_c-|I_c)}\right]_\nabla = \left[\begin{pmatrix} E & E \\ C & C \\ A & G \end{pmatrix}\right]_\nabla$$

We calculate the optimal link class nabla value:

$$\nabla(E^o_{(VI_c-|I_c)}) = 2+0+0 = 2$$

We generalize the value of nabla for the abstract relationship between degrees using the static function theorem:

$$\nabla^o_{(VI-|I)} = 2$$

Any optimal arrangement from an optimal progression $E^o_{(VI_c-|I_c)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence C.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}} : \psi(VI_c-) \longrightarrow \psi(I_c)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ A_{z_3} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{B.26}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{B.27}$$

Using the properties of the determinant, the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)^2 (s^{-2} - \lambda)$$

In this way we have found the roots of the characteristic polynomial, although we will expand it later to study its expression, the characteristics of the endomorphism matrix allow us to calculate the already factored polynomial.

$$\lambda^- = \{s^{-2}\}$$

$$\lambda^0 = \{s^0\}$$
$$\lambda^+ = \{\emptyset\}$$

Since we already know the roots, then for any optimal arrangement of the voicings of the chords in the progression we have calculated the transformation ratios between the voices. Thus we can obtain the pair of trigonometric functions that represent each voicing, in a generalized way based on a set of integers.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VI-}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{A_{z_3}}(t) \\ \psi_I(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

We represent the change from the antecedent voicing to the consequent voicing using the arrow notation, where, for a certain gamma distribution, we have the functions that represent each voicing.

$$\psi_{VI-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(VI-)} - e^{-2\pi tki\psi_j(VI-)}}{2i}$$
$$\longrightarrow \psi_I(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(I)} - e^{-2\pi tki\psi_j(I)}}{2i}$$

To classify a tonal function as tonic we use the following criterion:

$$\mathbb{A}(H(\lambda)) = T^{\mathbb{C}[\lambda]} \leftrightarrow \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} < 2 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0 \end{cases} \text{or} \begin{cases} M^+_{H(\lambda)} = \sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0 \\ M^-_{H(\lambda)} = \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} < 2 \end{cases}$$

Following the polynomial criterion we obtain the function of the degree $VI-$ related to the Ionian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. This case is very well known as it is the basic interchange for tonal function in Ionian context.

$$\boxed{\Phi[E_{(VI-|I)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_\mathbb{E}}(\lambda) = \lambda^3 + \left(-\frac{1}{s^2} - 2\right)\lambda^2 + \left(\frac{2}{s^2} + 1\right)\lambda - \frac{1}{s^2}$

Integral of $p_{C_\mathbb{E}}(\lambda) : \int p_{C_\mathbb{E}}(\lambda)d\lambda = \dfrac{\lambda^4}{4} + \left(-\dfrac{2\,s^2+1}{3\,s^2}\right)\lambda^3 + \dfrac{s^2+2}{2\,s^2}\lambda^2 + \left(-\dfrac{1}{s^2}\right)\lambda$

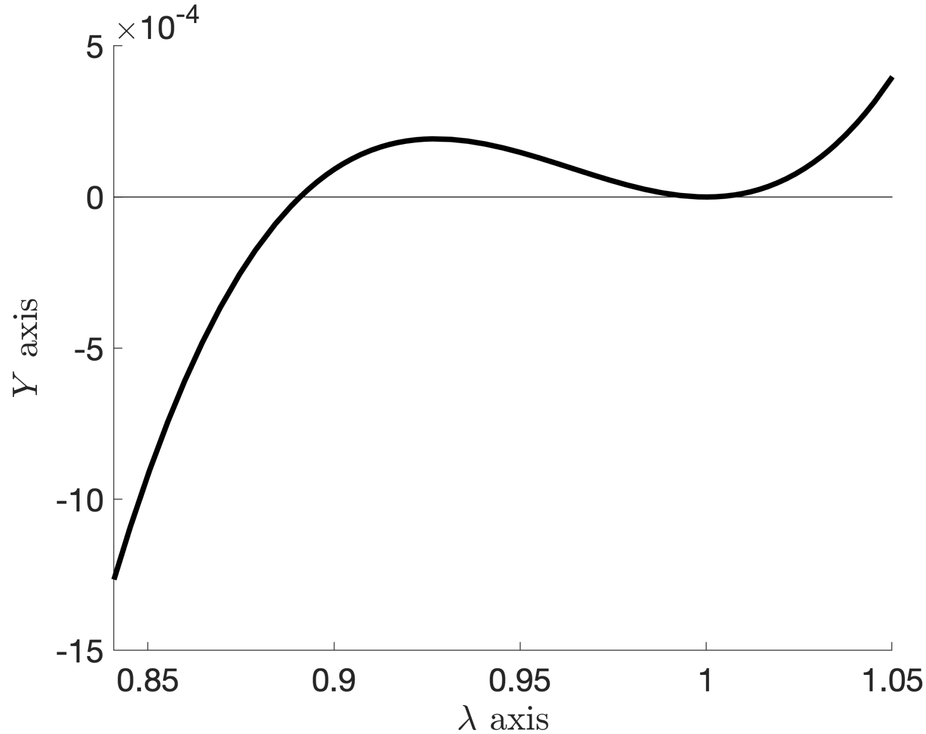


Figure B.5: Characteristic polynomial associated to the VI-$\longrightarrow$ I cadence

B.1.6. VII$\circ$ $\longrightarrow$ I Cadence

In this last case of the three-voice section we are going to understand the relationship between the seventh grade and the first one by following the optimization of the Hungarian algorithm. Since the dimension of both tonal centers is identical, it is not necessary to use infinite arithmetic and we can solve the matrix L using integer operations.

Then, we calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(VII^r_c o|I^r_c)} = \begin{pmatrix} F & G \\ D & E \\ B & C \end{pmatrix} \tag{B.28}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(VII^r o|I^r)} = \begin{pmatrix} 2 & 1 & 5 \\ 5 & 2 & 2 \\ 4 & 5 & 1 \end{pmatrix} \tag{B.29}$$

Then following the steps of the Hungarian algorithm we consider to develop it through the L matrix to find an optimum link:

$$L_{(VII^r o|I^r)} = \begin{pmatrix} 2 & 1 & 5 \\ 5 & 2 & 2 \\ 4 & 5 & 1 \end{pmatrix} \longrightarrow L^F_{(VII^r o|I^r)} = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 0 & 0 \\ 3 & 4 & 0 \end{pmatrix} \longrightarrow L^H_{(VII^r o|I^r)} = \begin{pmatrix} \boxed{0} & 0 & 4 \\ 2 & \boxed{0} & 0 \\ 2 & 4 & \boxed{0} \end{pmatrix} \tag{B.30}$$

Then the solutions for $L^H_{(VII^r o|I^r)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(VII^r o|I^r)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

By obtaining the link between voices, we immediately find the nabla class, which operationally allows us to build the optimal harmonic link for the progression.

$$\left[E^o_{(VII_c o|I_c)}\right]_\nabla = \left[\begin{pmatrix} F & G \\ D & E \\ B & C \end{pmatrix}\right]_\nabla \tag{B.31}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(VII_c o|I_c)}) = 2 + 2 + 1 = 5$$

We write the optimal nabla value as a generalization for every tonality:

$$\nabla^o_{(VIIo|I)} = 5$$

Now any optimal arrangement from an optimal progresion $E^o_{(VIIo|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(VII_c o)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ D_{z_2} \\ B_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \end{pmatrix} \tag{B.32}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^2 - \lambda & 0 \\ 0 & 0 & s^1 - \lambda \end{pmatrix} \tag{B.33}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2 - \lambda)^2 (s^1 - \lambda)$$

The roots have the following structure:

$$\lambda^- = \{\emptyset\}$$

$$\lambda^0 = \{\emptyset\}$$

$$\lambda^+ = \{s^2, s^1\}$$

Using the well known principle of sine waves superposition, it is clear that the function $\psi_I(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VIIo}(t) = \psi_{F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{B_{z_3}}(t) \\ \psi_I(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

The equations of the voicings are given in this case by:

$$\psi_{VII\circ}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VIIo)} - e^{-2\pi tki\psi_j(VIIo)}}{2i}$$

$$\longrightarrow \psi_I(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I)} - e^{-2\pi tki\psi_j(I)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $VIIo$ related to the Ionian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. This case is very well known as it is the basic interchange for tonal function in Ionian context. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(VIIo|I)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-2\,s^2 - s\right)\,\lambda^2 + \left(s^4 + 2\,s^3\right)\,\lambda - s^5$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s\,(2\,s+1)}{3}\right)\,\lambda^3 + \frac{s^3\,(s+2)}{2}\,\lambda^2 + \left(-s^5\right)\,\lambda$

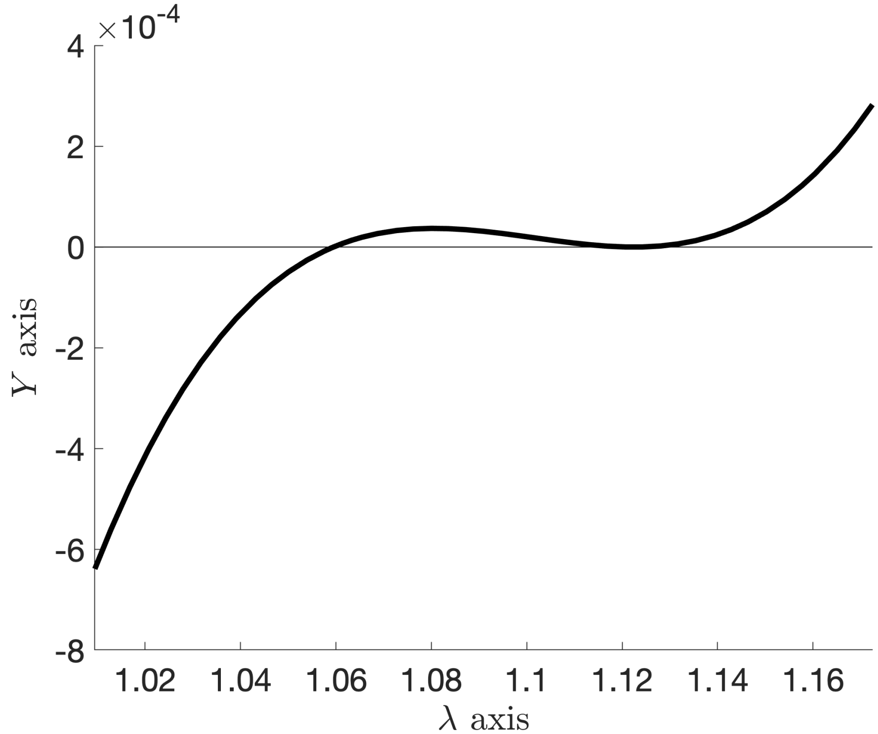


Figure B.6: Characteristic polynomial associated to the VII∘ ⟶ I cadence

B.2. The Ionian mode for $n = 4$

We enter this case with particular interest, for two important reasons for the analysis of jazz music. The first reason is the second grade's participation in the most commonly used progression in jazz music, the progression class $[P] = (II - 7 \mid V7 \mid I\triangle)$. The second reason is that the concept of dual function arises from the resolution by the Hungarian algorithm. It is very common that in musical practice, in some cases chords are cancelled, this is done to reduce the number of scales that the improvisor is thinking of when she improvises over a progression. Some modern contemporary jazz players cancel $V7$ chords in the progression class $[P] = (II - 7 \mid V7 \mid I\triangle)$. This leads us to wonder if this makes mathematical sense or is it just a whim that this practice is so common. At this point, the reader will appreciate that this cancellation operation of $V7$ preserves convergence, which is given by the first and third solutions of this case.

B.2.1. II-7⟶I△ Cadence

We calculate the link for this cadence to optimize it. The link will be a progression such as:

$$E_{(II_c^r-7|I_c^r\triangle)} = \begin{pmatrix} C & B \\ A & G \\ F & E \\ D & C \end{pmatrix} \tag{B.34}$$

The link cadence will be constructed using the ordered collection of metrics Δ_{ij}:

$$L_{(II^r-7|I^r\triangle)} = \begin{pmatrix} 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \\ 6 & 2 & 1 & 5 \\ 3 & 5 & 2 & 2 \end{pmatrix} \tag{B.35}$$

Following the steps of the Hungarian algorithm we develope the L matrix until we reach to the S set:

$$L_{(II^r-7|I^r\triangle)} = \begin{pmatrix} 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \\ 6 & 2 & 1 & 5 \\ 3 & 5 & 2 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r-7|I^r\triangle)} = \begin{pmatrix} 1 & 5 & 4 & 0 \\ 0 & 0 & 3 & 1 \\ 5 & 1 & 0 & 4 \\ 1 & 3 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow L^H_{(II^r-7|I^r\triangle)} = \begin{pmatrix} 1 & 5 & 4 & 0 \\ 0 & 0 & 3 & 1 \\ 5 & 1 & 0 & 4 \\ 1 & 3 & 0 & 0 \end{pmatrix}$$

Following the algorithm we solve the matrix L^H and apply the Zero Method, forcing each zero and constructing an optimal solution. We are going to find in this case three optimal solutions as a result of forcing each of the nine zeroes. After this process we calculate $L^{H^*} = L^{H^*}_{(II^r-7|I^r\triangle)}$ and we asign a starting zero.

$$L^{H^*} = \begin{pmatrix} \boxed{0} & 4 & 4 & 0 \\ 0 & \boxed{0} & 4 & 2 \\ 4 & 0 & \boxed{0} & 4 \\ 0 & 2 & 0 & \boxed{0}^* \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} \boxed{0}^* & 4 & 4 & 0 \\ 0 & \boxed{0} & 4 & 2 \\ 4 & 0 & \boxed{0} & 4 \\ 0 & 2 & 0 & \boxed{0} \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} \boxed{0} & 4 & 4 & 0 \\ 0 & \boxed{0} & 4 & 2 \\ 4 & 0 & \boxed{0}^* & 4 \\ 0 & 2 & 0 & \boxed{0} \end{pmatrix}$$

$$L^{H^*} = \begin{pmatrix} 0 & 4 & 4 & \boxed{0} \\ \boxed{0}^* & 0 & 4 & 2 \\ 4 & \boxed{0} & 0 & 4 \\ 0 & 2 & \boxed{0} & 0 \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} 0 & 4 & 4 & \boxed{0}^* \\ 0 & \boxed{0} & 4 & 2 \\ 4 & 0 & \boxed{0} & 4 \\ \boxed{0} & 2 & 0 & 0 \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} \boxed{0} & 4 & 4 & 0 \\ 0 & \boxed{0}^* & 4 & 2 \\ 4 & 0 & \boxed{0} & 4 \\ 0 & 2 & 0 & \boxed{0} \end{pmatrix}$$

$$L^{H^*} = \begin{pmatrix} 0 & 4 & 4 & \boxed{0} \\ 0 & \boxed{0} & 4 & 2 \\ 4 & 0 & \boxed{0} & 4 \\ \boxed{0}^* & 2 & 0 & 0 \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} 0 & 4 & 4 & \boxed{0} \\ \boxed{0} & 0 & 4 & 2 \\ 4 & \boxed{0} & 0 & 4 \\ 0 & 2 & \boxed{0}^* & 0 \end{pmatrix} \mid L^{H^*} = \begin{pmatrix} 0 & 4 & 4 & \boxed{0} \\ \boxed{0} & 0 & 4 & 2 \\ 4 & \boxed{0}^* & 0 & 4 \\ 0 & 2 & \boxed{0} & 0 \end{pmatrix}$$

Then the solutions for $L^{H^*}_{(II^r-7|I^r\triangle)}$ when both chords are in root position are described by three sets, each one representing a link where $\nabla(E)$ is minimal for al the permutations of the link:

$$S_1(L^{H^*}_{(II^r-7|I^r\triangle)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}, \Delta_{44}\}$$
$$S_2(L^{H^*}_{(II^r-7|I^r\triangle)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$
$$S_3(L^{H^*}_{(II^r-7|I^r\triangle)}) = \{\Delta_{14}, \Delta_{22}, \Delta_{33}, \Delta_{41}\}$$

B.2.2. $S_1(L^H_{(II^r-7|I^r\triangle))})$

In this way, we calculate from the set of solutions, the matrix that transforms one voicing into another in an optimal way.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(II_c - 7)) \longrightarrow \psi(I_c\triangle)$$
$$\begin{pmatrix} s^{-\Delta_{11}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{22}} & 0 & 0 \\ 0 & 0 & s^{-\Delta_{33}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{44}} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ F_{z_3} \\ D_{z_4} \end{pmatrix} = \begin{pmatrix} B_{z_1} \\ G_{z_2} \\ E_{z_3} \\ C_{z_4} \end{pmatrix} \tag{B.36}$$

The optimal link is the same as the one as it appear with both structures in root position. The cadence presented here is not going to change the result of the polynomial only when it is dimensionally optimized.

$$\left[E^1_{(II_c-7|I_c\triangle)} \right]_\nabla = \left[\begin{pmatrix} C & B \\ A & G \\ F & E \\ D & C \end{pmatrix} \right]_\nabla \tag{B.37}$$

The nabla value is calculated for any pair of frequencies that $\Omega[\int_\alpha^\beta \phi^{-1} d\phi] \leq 6$ where $\alpha \in [\alpha]$ and $\beta \in [\beta]$. The delta subindex indicates that the integral is the minimum of the delta set Δ.

$$\nabla(E^1) = \sum_{j=1}^{n} \Omega \mid \int_{(E^1_{j1\,(II_c-7|I_c\triangle)})}^{(E^1_{j2\,(II_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\Delta = 6$$

The characteristic polynomial of the endomorphism matrix is already factored:

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-2} - \lambda)^2 (s^{-1} - \lambda)^2$$

The multiplicities are $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 4$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 0$

So all not null roots are convergent roots and the set describes the direction of the voices when the link is optimal.

$$\lambda^- = \{s^{-1}, s^{-2}\}$$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(II-7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. For the first of the three optimal links we calculate the equations for the voicings:

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II-7}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{F_{z_3}}(t) + \psi_{D_{z_3}}(t) \\ \psi_{I\triangle}(t) = \psi_{B_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

$$\psi_{II-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II-7)} - e^{-2\pi tki\psi_j(II-7)}}{2i}$$
$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

Following the polynomial criteria we obtain the one function of the degree $II-7$ related to the Ionian tonal center. This case is specially interesting because it has multiple functions that share nabla value although the polynomials are different.

$$\sum_{j=1}^{n} \Omega \mid \int_{(E^1_{j1\,(II_c-7|I_c\triangle)})}^{\psi(E^1_{j2\,(II_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\triangle = \sum_{j=1}^{n} \Omega \mid \int_{(E^2_{j1\,(II_c-7|I_c\triangle)})}^{(E^2_{j2\,(II_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\triangle$$
$$= \sum_{j=1}^{n} \Omega \mid \int_{(E^3_{j1\,(II_c-7|I_c\triangle)})}^{(E^3_{j2\,(II_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\triangle$$

We can write that there are three distinct polynomials that are the result of optimization by the Hungarian algorithm.

$$\Phi[E^1_{(II-7|I\triangle)}] \neq \Phi[E^2_{(II-7|I\triangle)}] \neq \Phi[E^3_{(II-7|I\triangle)}]$$

The result for the first solution is concretely dominant, then we will use the appropriate superscript to distinguish the cases.

$$\boxed{\Phi[E^1_{(II-7|I\triangle)}] \in D^{\mathbb{R}[\lambda]}}$$

First characteristic polynomial: $p_{C_\mathbb{E}}(\lambda) = \lambda^4 + \left(-\frac{2}{s} - \frac{2}{s^2}\right) \lambda^3 + \left(\frac{1}{s^2} + \frac{4}{s^3} + \frac{1}{s^4}\right) \lambda^2 + \left(-\frac{2}{s^4} - \frac{2}{s^5}\right) \lambda + \frac{1}{s^6}$

Integral of the first polynomial $p_{C_\mathbb{E}}(\lambda)$: $\int p_{C_\mathbb{E}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s+1}{2\,s^2}\right) \lambda^4 + \frac{s^2+4\,s+1}{s^4\,3} \lambda^3 + \left(-\frac{s+1}{s^5}\right) \lambda^2 + \frac{\lambda}{s^6}$

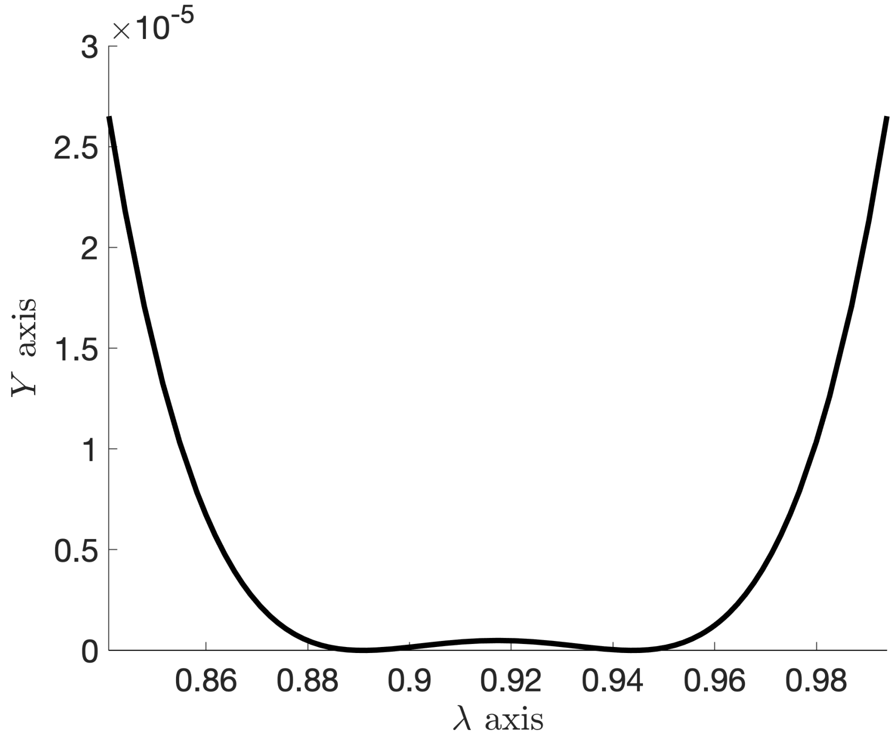


Figure B.7: Characteristic polynomial associated to the II-7⟶I△ I cadence (first polynomial)

B.2.3. $S_2(L^H_{(II^r-7|I^r\triangle))})$

Using the second set of solutions, we compute another matrix in the endomorphism that will optimally transform one voicing into another.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(II_c - 7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ F_{z_3} \\ D_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ B_{z_2} \\ G_{z_3} \\ E_{z_4} \end{pmatrix} \tag{B.38}$$

The second optimized link will be a class matrix from the optimized class of links (nabla minimum class):

$$\left[E^2_{(II_c-7|I_c\triangle)}\right]_{\nabla} = \left[\begin{pmatrix} C & C \\ A & B \\ F & G \\ D & E \end{pmatrix}\right]_{\nabla} \tag{B.39}$$

Measuring the link we obtain the same nabla value as $E^1_{(II-7|I\triangle)}$:

$$\nabla(E^2_{(II_c-7|I_c\triangle)}) = \sum_{j=1}^{n} \Omega \mid \int_{(E^2_{j1\,(II_c-7|I_c\triangle)})}^{(E^2_{j2\,(II_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_{\Delta} = 6$$

The characteristic polynomial that is already in itself the tonal function itself is given by the following expression

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2-\lambda)^3(s^0-\lambda)$$

Calculating each algebraic multiplicity we have that $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$, $\sum_{i=1}^{|\lambda_i^-|} m_{\lambda_i^-} = 0$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$.

So all not null roots are divergent roots and the set describes the direction of the voices when the link is optimal $\lambda^- = \{s^2\}$. Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(II-7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector written as $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$.

As we can see, the decomposition of the functions note by note can be done as a sum, which in the context of musical thought has the meaning of adding functions such as notes. So the first would be: $\psi_{II-7}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{F_{z_3}}(t) + \psi_{D_{z_3}}(t)$ and the second would be $\psi_{I\triangle}(t) = \psi_{C_{z_1}}(t) + \psi_{B_{z_2}}(t) + \psi_{G_{z_3}}(t) + \psi_{E_{z_3}}(t)$. We represent the change from one voicing to another as the transformation of functions in the time domain separated by the arrow$\longrightarrow$.

$$\psi_{II-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II-7)} - e^{-2\pi tki\psi_j(II-7)}}{2i}$$
$$\longrightarrow \psi_I(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\triangle)} - e^{-2\pi tki\psi_j(I\triangle)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $II-7$ related to the Ionian tonal center. In this case is not unique and can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E^2_{(II-7|I\triangle)}] \in S^{\mathbb{R}[\lambda]}}$$

Second characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-3\,s^2-1)\ \lambda^3 + (3\,s^4+3\,s^2)\ \lambda^2 + (-s^6-3\,s^4)\ \lambda + s^6$

Integral of the second polynomial $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{3\,s^2}{4} - \frac{1}{4}\right)\ \lambda^4 + \left(s^2\ \left(s^2+1\right)\right)\ \lambda^3 + \left(-\frac{s^6}{2} - \frac{3\,s^4}{2}\right)\ \lambda^2 + s^6\,\lambda$

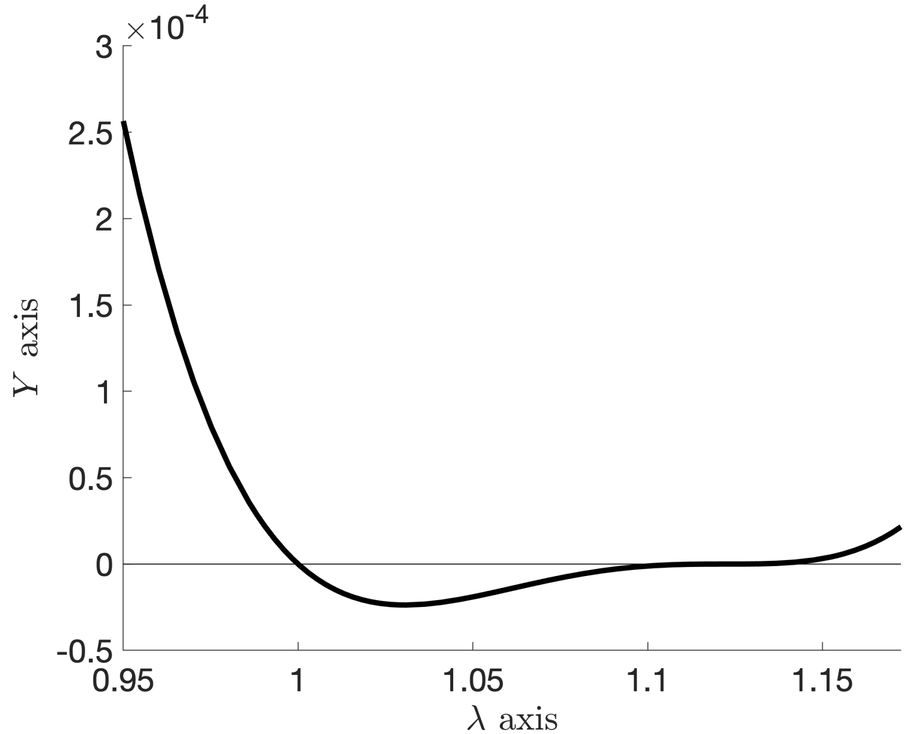


Figure B.8: Characteristic polynomial associated to the II-7⟶I△ I cadence (second polynomial)

B.2.4. $S_3(L^H_{(II^r-7|I^r\triangle))})$

Since we have already calculated the third set of solutions for the relationship between the second degree and the first in the context of the Ionian mode, we develop the section in the same way as the previous solutions.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(II_c - 7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{22}} & 0 & 0 \\ 0 & 0 & s^{-\Delta_{33}} & 0 \\ 0 & 0 & 0 & s^{-\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ F_{z_3} \\ D_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ E_{z_3} \\ B_{z_4} \end{pmatrix} \tag{B.40}$$

The last optimal link class will be:

$$\left[E^3_{(II_c-7|I_c\triangle)}\right]_{\nabla} = \left[\begin{pmatrix} C & C \\ A & G \\ F & E \\ D & B \end{pmatrix}\right]_{\nabla} \tag{B.41}$$

Now we calculate the nabla value when the distance between class mappings is minimal. The minimum for al ∞ mappings is unique so the sum can be constructed:

$$\nabla(E^3_{(II_c-7|I_c\triangle)}) = \sum_{j=1}^{n} \Omega \mid \int_{(E^3_{j1\,(II_c-7|I_c\triangle)})}^{(E^3_{j2\,(II_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_{\Delta} = 6$$

The characteristic polynomial of this third solution has all the roots either to the left of $E(M)$ or on $E(M)$ itself.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^{-2} - \lambda)(s^{-1} - \lambda)(s^{-3} - \lambda)$$

Thus, each one of the multiplicities of the roots of the tonal function would correspond respectively with $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$. The set that describes the direction of the voices when the link is optimal is $\lambda^- = \{s^{-1}, s^{-2}, s^{-3}\}$. Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(II-7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_I(t)$ can be created as the sum of every $\psi(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \le j \le 4, j \in \mathbb{N}$. Thus, the pair of functions that describe each voicing for an arbitrarily defined set of integers that place each class in a particular octave will be given by the following bracket

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II_c}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{F_{z_3}} + \psi_{D_{z_4}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}} + \psi_{B_{z_4}}(t) \end{cases}$$

We represent the moment of the change between the pair of voicings and we observe that the tonal function has changed area and is in the same area as the first solution.

$$\psi_{II_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II_c-7)} - e^{-2\pi tki\psi_j(II_c-7)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c\triangle)} - e^{-2\pi tki\psi_j(I_c\triangle)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $II-7$ related to the Ionian tonal center.

$$\boxed{\Phi[E^3_{(II-7|I\triangle)}] \in D^{\mathbb{R}[\lambda]}}$$

Third characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} - 1\right)\lambda^3 + \left(\frac{\frac{1}{s}+\frac{1}{s^2}}{s^3} + \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3}\right)\lambda^2 + \left(-\frac{\frac{1}{s}+\frac{1}{s^2}}{s^3} - \frac{1}{s^3} - \frac{1}{s^6}\right)\lambda + \frac{1}{s^6}$

Integral of the third polynomial $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^3+s^2+s+1}{s^3\,4}\right)\lambda^4 + \frac{s^4+s^3+2\,s^2+s+1}{s^5\,3}\lambda^3 + \left(-\frac{s^3+s^2+s+1}{s^6\,2}\right)\lambda^2 + \frac{\lambda}{s^6}$

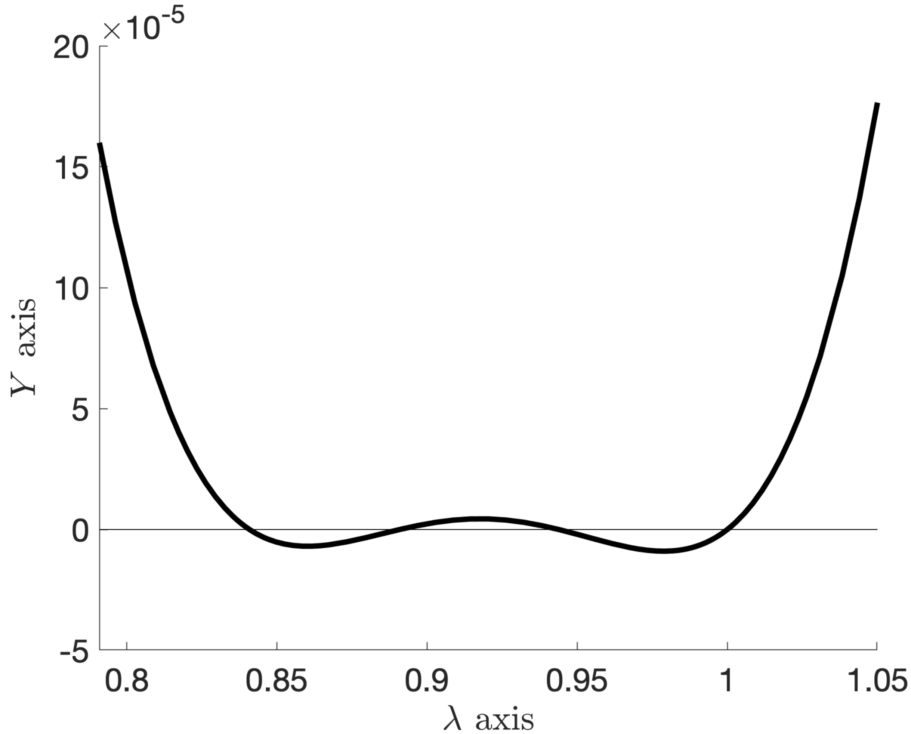


Figure B.9: Characteristic polynomial associated to the II-7⟶I△ I cadence (third polynomial)

B.2.5. III-7⟶I△ Cadence

We study the relationship of the third degree with the first in the next section. This case can be deduced by simple inspection but we develop it formally like the rest.

The link will be the following matrix:

$$E_{(III^r_c-7|I^r_c\triangle)} = \begin{pmatrix} D & B \\ B & G \\ G & E \\ E & C \end{pmatrix} \tag{B.42}$$

We calculate the link matrix calculating all the distances Δ_{ij}. As in other cases we have omitted the subscript that indicates the tone since the matrix L does not depend on the tone we are in.

$$L_{(III^r-7|I^r\triangle)} = \begin{pmatrix} 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \end{pmatrix} \tag{B.43}$$

Following the steps of the Hungarian algorithm we obtain a set of matrices.We are developing the steps through the matrix L until we reach a distribution of boxes over L^H.

$$L_{(III^r-7|I^r\triangle)} = \begin{pmatrix} 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \end{pmatrix} \longrightarrow L^F_{(III^r-7|I^r\triangle)} = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \end{pmatrix} \longrightarrow \tag{B.44}$$

$$L^H_{(III^r-7|I^r\triangle)} = \begin{pmatrix} 1 & 3 & 0 & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 1 \\ 5 & 3 & \boxed{0} & 4 \end{pmatrix}$$

The solutions given by the algorithm will be in the S set. In this way, the entries that appear inside a box must be rewritten on the original matrix L in order to obtain the transformation proportions when the link is optimal, recovering the sign.

$$S(L^H_{(III^r-7|I^r\triangle)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla class of E.

$$\left[E^o_{(III_c-7|I_c\triangle)}\right]_\nabla = \left[\begin{pmatrix} D & C \\ B & B \\ G & G \\ E & E \end{pmatrix}\right]_\nabla \tag{B.45}$$

Once the S set is calculated we can form a generalization of a dimensonally optimized cadence $C_{\mathbb{E}}$. The reader, at this point, has to realize that we are recovering the sign of the metrics in the endomorphism matrix. In case of doubt, you can see the cases of global tonal function where the transformation T is used explicitly.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(III_c - 7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{-\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ B_{z_2} \\ G_{z_3} \\ E_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ B_{z_2} \\ G_{z_3} \\ E_{z_4} \end{pmatrix} \tag{B.46}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id_4.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-2} - \lambda)(s^0 - \lambda)^3$$

We will now calculate the algebraic multiplicities of the tonal function which is analogous to studying the direction of the voices in the optimum $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$.

Following the polynomial criteria we obtain the function of the degree $III - 7$ related to the Ionian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(III-7|I\triangle)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{3}{s^2} - 1\right)\lambda^3 + \left(\frac{3}{s^2} + \frac{3}{s^4}\right)\lambda^2 + \left(-\frac{3}{s^4} - \frac{1}{s^6}\right)\lambda + \frac{1}{s^6}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \dfrac{\lambda^5}{5} + \left(-\dfrac{s^6 + 3\,s^4}{s^6\,4}\right)\lambda^4 + \dfrac{s^2+1}{s^4}\lambda^3 + \left(-\dfrac{3\,s^2+1}{2\,s^6}\right)\lambda^2 + \dfrac{\lambda}{s^6}$

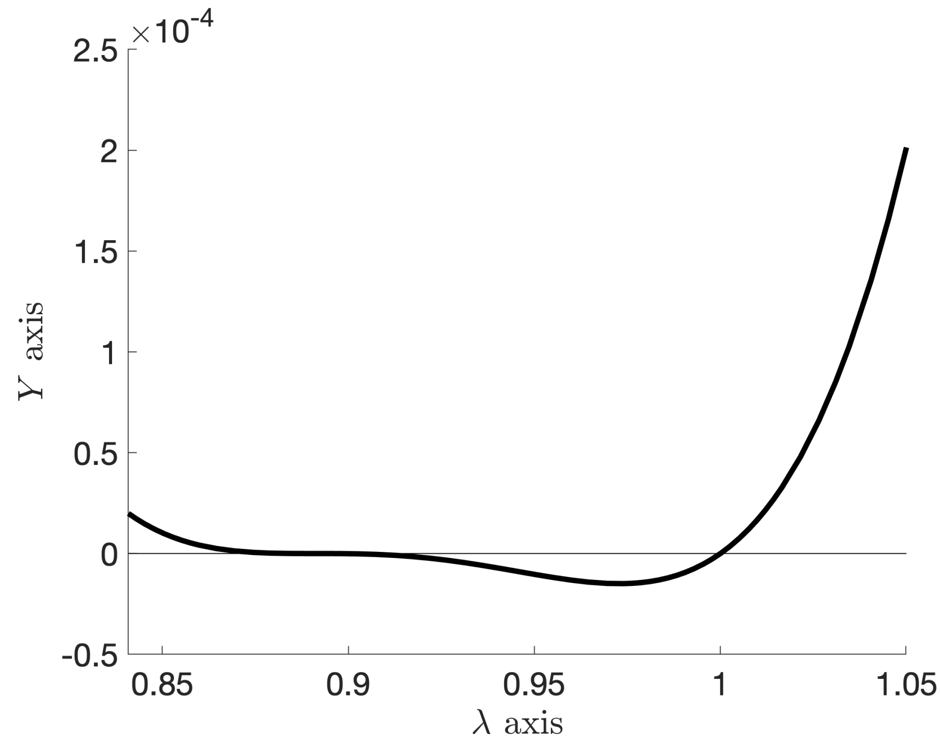


Figure B.10: Characteristic polynomial associated to the III-7⟶I△ cadence

B.2.6. IV△⟶I△ Cadence

In this case we analyze the convergence of major chords with a major seventh that are located a fifth away. This is a cadence, which, being classified within the subdominant area, as expected, appears in retrograde form, that is, it is common to see the link in its retrograde form as $\varrho(E)$, which is dominant. We pose the problem analogously to previous cases and without much difficulty, since for this example neither duality nor distinct dimensionality appears between the centers, so the resolution using the steps of the algorithm on the L matrix will end up in a single solution.

$$E_{(IV_c^r\triangle|I_c^r\triangle)} = \begin{pmatrix} E & B \\ C & G \\ A & E \\ F & C \end{pmatrix} \tag{B.47}$$

We take this opportunity to recall the color theorem applied in this section. We see that the value of the minimal nabla function is the same when the link is cast for a non-null λ scalar that casts the entire octave class. Here we see that the minimal nabla function of an optimal link, if the pitch function is unique, then it coincides with that of any optimal link carried by a certain non-zero λ.

This implies that we can minimize each integral with $\lambda \neq 0$:

$$\nabla(E) = \Omega \sum_{j=1}^{n} \mid \int_{E_{j1}^o}^{E_{j2}^o} \phi^{-1}\, d\phi \mid_\Delta = \nabla(\lambda E) = \Omega \sum_{j=1}^{n} \mid \int_{\lambda E_{j1}^o}^{\lambda E_{j2}^o} \phi^{-1}\, d\phi \mid_\Delta$$

To build the solution for this link we arrange the chords in state using the superscript r and place each class in an ordered fashion in the progression. So it would be written as follows:

$$E_{(IV_c^r\triangle|I_c^r\triangle)} = \begin{pmatrix} E & B \\ C & G \\ A & E \\ F & C \end{pmatrix} \tag{B.48}$$

We calculate the link matrix, calculating the minimum distances between each class by taking the first class of the first chord and annotating in the first row of L each one of the distances between the classes of the second chord. We repeat the process with the following classes of the first chord until writing L.

$$L_{(IV^r\triangle|I^r\triangle)} = \begin{pmatrix} 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \\ 6 & 2 & 1 & 5 \end{pmatrix} \tag{B.49}$$

Once we have built the matrix L we carry out the algorithm and use the final steps where we cover the zeroes with the minimum number of lines. In the end we found a solution that in this case is unique.

$$L_{(IV^r\triangle|I^r\triangle)} = \begin{pmatrix} 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \\ 6 & 2 & 1 & 5 \end{pmatrix} \longrightarrow L^{F*}_{(IV^r\triangle|I^r\triangle)} = \begin{pmatrix} 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 0 & 0 & 3 & 1 \\ 5 & 1 & 0 & 4 \end{pmatrix}$$

$$\longrightarrow L^{H*}_{(IV^r\triangle|I^r\triangle)} = \begin{pmatrix} 4 & 2 & \boxed{0} & 3 \\ 1 & 5 & 5 & \boxed{0} \\ \boxed{0} & 0 & 4 & 1 \\ 4 & \boxed{0} & 0 & 3 \end{pmatrix}$$

Then the solutions for $L^{H}_{(IV^r|I^r\triangle)}$ when both structures are in root position becomes the following set:

$$S(L_{(IV^r\triangle|I^r\triangle)} = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

$$\left[E^{o}_{(IV_c\triangle|I_c\triangle)}\right]_{\nabla} = \left[\begin{pmatrix} E & E \\ C & C \\ A & B \\ F & G \end{pmatrix}\right]_{\nabla} \tag{B.50}$$

Any optimal arrangement will be given by an dimensionally optimal cadence:

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(IV\triangle)) \longrightarrow \psi(I\triangle)$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ A_{z_3} \\ F_{z_4} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ B_{z_3} \\ G_{z_4} \end{pmatrix} \tag{B.51}$$

Then the polynomial comes in form by: $p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)^2(s^2 - \lambda)^2$ and ach of the algebraic multiplicities of each root will be concretely

$$\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2, \quad \sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2, \quad \sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 0$$

Following the polynomial criterion we obtain the function of the degree $IV\triangle$ related to the Ionian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(IV\triangle|I\triangle)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-2\,s^2 - 2)\,\lambda^3 + (s^4 + 4\,s^2 + 1)\,\lambda^2 + (-2\,s^4 - 2\,s^2)\,\lambda + s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{2} - \frac{1}{2}\right)\lambda^4 + \left(\frac{s^4}{3} + \frac{4\,s^2}{3} + \frac{1}{3}\right)\lambda^3 + \left(-s^2\,\left(s^2+1\right)\right)\lambda^2 + s^4\,\lambda$

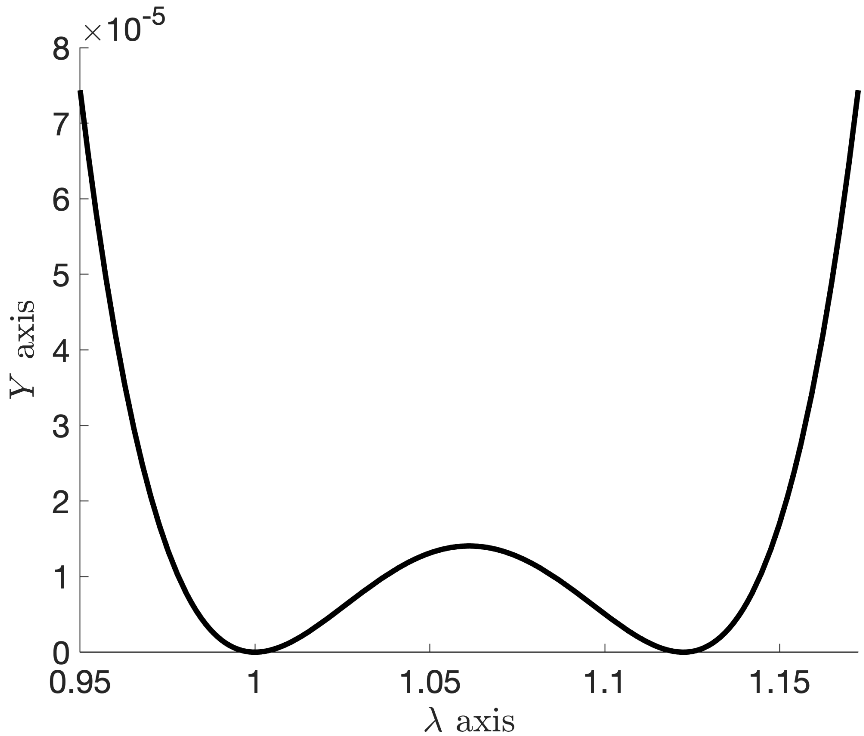


Figure B.11: Characteristic polynomial associated to the IV△⟶I△ cadence

B.2.7. V7⟶I△ Cadence

In this section we study the convergence of the cadence $[P] = (V7 \mid I\triangle)$. By putting the cadence under the vision of the Hungarian algorithm, we are going to confirm its convergence as expected, but we are also going to realize that there are several elements that have been the subject of debate and that are going to be resolved in this work. To begin with, the tritone determines a certain vertical sonority and, furthermore, since it is located on the axis of the circle of fifths, it determines a pair of tonalities which are facing halves in the circle, this occurs when the keys are compact. The presence of the tritone has many mathematical properties, since as we have already seen, the only dual tonal function that exists between traditional tonalities is the one that occurs by comparing tonalities a tritone away. This dual function between keys allows improvisers to carry out multiple substitutions (tritone substitutions) between dominant chords, but also allows superimposing structures that converge with each other in certain stylistic contexts. All this information can lead us to think that the tritone is responsible for the resolution of $V7$, but we are going to see that this is not the case, since if we focus on mathematics, all this range of possibilities that exist in the improvisation, where convergence predominates over vertical dissonance, are resources that coexist with the isolated case but do not influence its resolution.

Dominant chords are especially flexible since, by limiting the possibilities of belonging to two compact keys, they resolve in an especially strong way and also allow the superposition of a large number of scales, as seen in the tradition of blues and modern music. Here we note that the scales separately by having $dim(N) = 1$ where N is a note of a scale always has a tonic function in front of a tonal center, or when entering and leaving it. This lead us to think that a scale is momentarily generating a tonal function when it enters and leaves the matrix that we are studying. This fact, in conjunction with the limitation of possibilities presented by the presence of a tritone in a chord, is the explanation of why we can interchange as many scales in dominant chords as improvisers without the listener losing the sensation of tonal center at any time. Analyzing the case in a particular way and studying the voices of a dominant to dry, we are going to study the optimization problem as one more case:

$$E_{(V_c^r 7 \mid I_c^r \triangle)} = \begin{pmatrix} F & B \\ D & G \\ B & E \\ G & C \end{pmatrix} \tag{B.52}$$

Then we calculate the link matrix calculating all the distances Δ_{ij}.We build the array L as we already know and remove the subscripts on the way to abstraction.

$$L_{(V^r 7 \mid I^r \triangle)} = \begin{pmatrix} 6 & 2 & 1 & 5 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \end{pmatrix} \tag{B.53}$$

Following the steps of the Hungarian algorithm we obtain a set of matrices:

$$L_{(V^r7|I^r\triangle)} = \begin{pmatrix} 6 & 2 & 1 & 5 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r7|I^r\triangle)} = \begin{pmatrix} 5 & 1 & 0 & 4 \\ 1 & 3 & 0 & 0 \\ 0 & 4 & 5 & 1 \\ 4 & 0 & 3 & 5 \end{pmatrix}$$

$$\longrightarrow L^H_{(V7^r|I^r\triangle)} = \begin{pmatrix} 5 & 1 & \boxed{0} & 4 \\ 1 & 3 & 0 & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \\ 4 & \boxed{0} & 3 & 5 \end{pmatrix}$$

The solutions given by the algorithm will be those that correspond to the distribution of boxes provided by the Hungarian algorithm:

$$S(L^H_{(V^r7|I^r\triangle)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla class of E.

$$\left[E^o_{(V_c7|I_c\triangle)}\right]_\nabla = \left[\begin{pmatrix} F & E \\ D & C \\ B & B \\ G & G \end{pmatrix}\right]_\nabla \tag{B.54}$$

Once the S set is calculated we can form a generalization of a dimensoinally optimized cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(V_c7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{-\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ D_{z_2} \\ B_{z_3} \\ G_{z_4} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ B_{z_3} \\ G_{z_4} \end{pmatrix} \tag{B.55}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id_4.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-1} - \lambda)(s^{-2} - \lambda)(s^0 - \lambda)^2$$

Thus, analyzing the previous polynomial we observe that there are two static voices and two voices that descend in the optimum as they reflect the values of the algebraic multiplicities $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(V7)$ or $\psi(I\triangle)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

In this way we write the functions that represent the first and the second voicing in the optimal arrangement for an arbitrary selection of integers that fit each class in an octave respecting the minimum value of $\mid p \mid$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V_c7}(t) = \psi_{F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{B_{z_3}} + \psi_{G_{z_4}}(t) \\ \psi_{I_c\triangle}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{B_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

In an analytical way we represent the antecedent voicing and the consequent voicing using the arrow notation.

$$\psi_{V_c7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(V_c7)} - e^{-2\pi tki\psi_j(V_c7)}}{2i}$$
$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c\triangle)} - e^{-2\pi tki\psi_j(I_c\triangle)}}{2i}$$

Studying algebraic multiplicities and following the polynomial criterion we obtain the function of the degree $V7$ related to the Ionian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(V7|I\triangle)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s} - \frac{1}{s^2} - 2\right) \lambda^3 + \left(\frac{2}{s} + \frac{2}{s^2} + \frac{1}{s^3} + 1\right) \lambda^2 + \left(-\frac{1}{s} - \frac{1}{s^2} - \frac{2}{s^3}\right) \lambda + \frac{1}{s^3}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{2\,s^2 + s + 1}{s^2\,4}\right) \lambda^4 + \frac{s^3 + 2\,s^2 + 2\,s + 1}{s^3\,3} \lambda^3 + \left(-\frac{s^2 + s + 2}{s^3\,2}\right) \lambda^2 + \frac{\lambda}{s^3}$

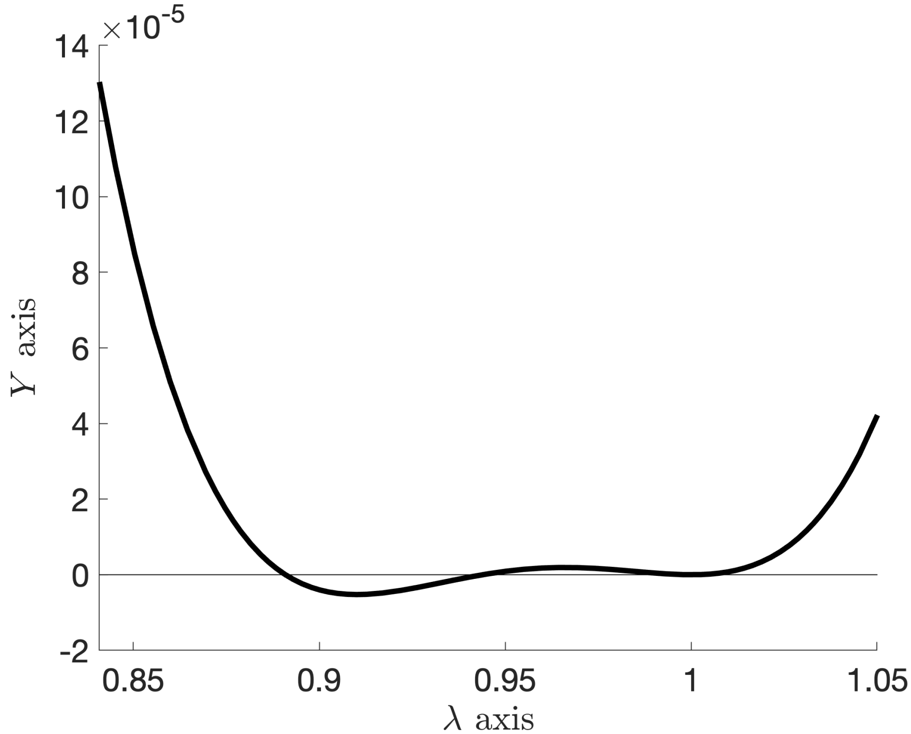


Figure B.12: Characteristic polynomial associated to the V7⟶I△ cadence

B.2.8. VI-7⟶I△ Cadence

Usually in jazz music the degree $VI-7$ is replaced with the degree $I\triangle$. It is very common for the order to be $I\triangle$ followed by $VI-7$ as in the turnaround. This leads us to think that although we distinguish the tonic function from the dominant and subdominant, within this function there is actually the convergent tonic and the divergent tonic. We do not make a specific classification of divergent and convergent tonic in the polynomial criterion since the movement of a single voice is so smooth that from our experience the two tonics can be used in both directions, although we comment in the following paragraph, and subjectively that there is a certain predominance of the convergent tonic over the divergent one, as it is natural. We take advantage of the case to comment that the tonal function of the tonic does not distinguish the direction of the voices since the variation is so small in perception that the direction of the voices does not matter when it is only one voice that moves. This is what usually happens due to our experience in music, although, since perception is conditioned by training, it is worth mentioning that a trained musician will hear the difference between the convergent tonic function and the divergent tonic function.

We have the link with both chords placed in root position. We do this to preserve some order in the calculation, since the results are the same.

$$E_{(VI^r_c - 7|I^r_c \triangle)} = \begin{pmatrix} G & B \\ E & G \\ C & E \\ A & C \end{pmatrix} \tag{B.56}$$

We calculate the link matrix calculating all the distances Δ_{ij}:

$$L_{(VI^r - 7|I^r \triangle)} = \begin{pmatrix} 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \end{pmatrix} \tag{B.57}$$

Following the steps of the Hungarian algorithm we obtain a set of matrices:

$$L_{(VI^r - 7|I^r \triangle)} = \begin{pmatrix} 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 2 & 2 & 5 & 3 \end{pmatrix} \longrightarrow L^F_{(VI^r - 7|I^r \triangle)} = \begin{pmatrix} 4 & 0 & 3 & 5 \\ 5 & 3 & 0 & 4 \\ 1 & 5 & 4 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

$$\longrightarrow L^H_{(VI^r - 7|I^r \triangle)} = \begin{pmatrix} 4 & \boxed{0} & 3 & 5 \\ 5 & 3 & \boxed{0} & 4 \\ 1 & 5 & 4 & \boxed{0} \\ \boxed{0} & 0 & 3 & 1 \end{pmatrix}$$

The solutions given by the algorithm will be:

$$S(L^H_{(VI^r - 7|I^r \triangle)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla class of E.

$$\left[E^o_{(VI_c - 7|I_c \triangle)} \right]_\nabla = \left[\begin{pmatrix} G & G \\ E & E \\ C & C \\ A & B \end{pmatrix} \right]_\nabla \tag{B.58}$$

In this case we build the nabla function by taking the sum of the minimum distances through the optimized link. Thus we calculate the minimum nabla function that is the common characteristic of the set of tonal functions associated with a link. In this case, the value of the minimal nabla function is very low because only one voice moves even though $n = 4$

$$\nabla(E^o_{(VI_c-7|I_c\triangle)}) = \sum_{j=1}^{n} \Omega \mid \int_{(E^o_{j1\,(VI_c-7|I_c\triangle)})}^{(E^o_{j2\,(VI_c-7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_{\Delta} = 2$$

In this way we have calculated the pairing between both tonal centers. Since we are covering cases of the same dimension, we do not have to use the infinite arithmetic criterion nor parametric cadences and in this way, and since the solution is also non-dual, we have that the endomorphism matrix $C_{\mathbb{E}}$ is unique and is enough for us, without the need to use the T transformation; use the solutions of the set S as exponents of each of the Mersenne numbers on the diagonal of the matrix. We have to remember that we have to retrieve the sign on each input of L.

The endomorphism is given in matrix form by the following map, where each z is an integer that changes the aperture of the optimal array. We are not going into calculating all the $\psi(X)$ voicings that are possible. We are interested in the transformation carried out by the matrix and its characteristic polynomial.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(VI_c - 7)) \longrightarrow \psi(I_c\triangle)$$

$$\begin{pmatrix} s^{\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \\ A_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \\ B_{z_4} \end{pmatrix} \tag{B.59}$$

Subtracting the identity from the previous matrix and taking the determinant, we calculate the characteristic polynomial, which, as a function of the variable λ gives us a hypervolume function that takes the value zero when λ matches the proportionality values between the voices by minimizing the absolute perception.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)(s^0 - \lambda)^3$$

Since we already have the characteristic polynomial, we study the algebraic multiplicities, which coincide, in our mathematical model, with the number of voices that move upwards (divergent algebraic multiplicity), downwards (convergent algebraic multiplicity) or that remain invariant (static algebraic multiplicity). In this way we have that $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 1$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$. Since we know the transformation coefficients between voicings for a finite set of integers, then, in order to study the phenomenon in its entirety, we propose the pair of functions in the temporal domain that are responsible for representing each voicing. The interested reader has here all the data, for from $k = 440$ or the tuning that he wants, to simulate the voicings in a graphing calculator. As an idea you can paint the first function using the harmonic series (for simplicity) in one color and overlay the second one in a different color. If the harmonic distribution does not converge to 0, you have to worry that the set of harmonics h is finite.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VI_c-7}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{A_{z_4}}(t) \\ \psi_{I_c\triangle}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{B_{z_4}}(t) \end{cases}$$

We represent the temporal transition between functions with the arrow $\longrightarrow$ where the antecedent and consequent functions are specified. Thus we have characterized two voicings for a finite set of integers that represent each of the chords studied at the beginning of the section.

$$\psi_{VI_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(VI_c-7)} - e^{-2\pi tki\psi_j(VI_c-7)}}{2i}$$
$$\longrightarrow \psi_{I_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_c\triangle)} - e^{-2\pi tki\psi_j(I_c\triangle)}}{2i}$$

Following the polynomial criteria we obtain the function of the degree $VI-7$ related to the Ionian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(VI-7|I\triangle)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s^2} - 3\right)\lambda^3 + \left(\frac{3}{s^2} + 3\right)\lambda^2 + \left(-\frac{3}{s^2} - 1\right)\lambda + \frac{1}{s^2}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{3\,s^2+1}{4\,s^2}\right)\lambda^4 + \frac{3\,s^2+3}{3\,s^2}\lambda^3 + \left(-\frac{s^2+3}{2\,s^2}\right)\lambda^2 + \frac{\lambda}{s^2}$

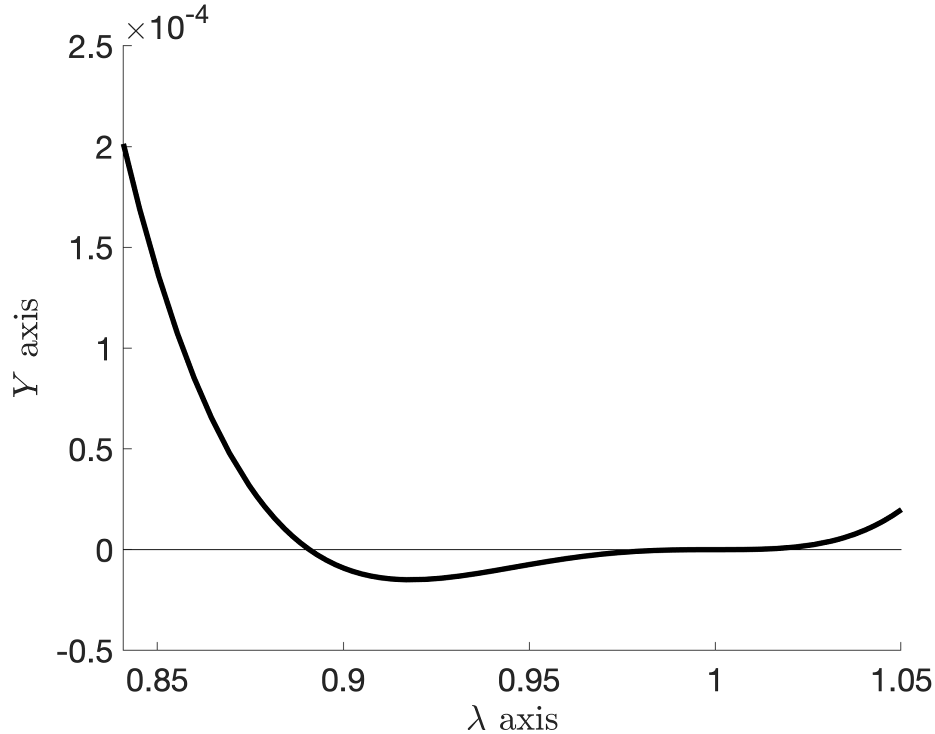


Figure B.13: Characteristic polynomial associated to the VI-7⟶I△ cadence

B.2.9. VIIø7⟶I△ Cadence

In this case, we study the relationship between the half-diminished seventh degree and the first degree. In the progressions, the first chord is usually used as a second degree that has a dominant function with respect to $V7(b9)$ where the sixth degree of the melodic minor scale is played, also called Locrian ♮2. Outside of the vertical contextualization of the chord, we want to determine its relation to the first degree. In this way we arrange the link placing both chords in root position to later build the distance matrix L.

$$E_{(VII^r_c\o7|I^r_c\triangle)} = \begin{pmatrix} A & B \\ F & G \\ D & E \\ B & C \end{pmatrix} \tag{B.60}$$

Then we calculate the link matrix calculating all the distances Δ_{ij}:

$$L_{(VII^r\o7|I^r\triangle)} = \begin{pmatrix} 2 & 2 & 5 & 3 \\ 6 & 2 & 1 & 5 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \end{pmatrix} \tag{B.61}$$

Following the steps of the Hungarian algorithm we obtain a set of matrices:

$$L_{(VII^r\o 7|I^r\triangle)} = \begin{pmatrix} 2 & 2 & 5 & 3 \\ 6 & 2 & 1 & 5 \\ 3 & 5 & 2 & 2 \\ 0 & 4 & 5 & 1 \end{pmatrix} \longrightarrow L^F_{(VII^r\o 7|I^r\triangle)} = \begin{pmatrix} 0 & 0 & 3 & 1 \\ 5 & 1 & 0 & 4 \\ 1 & 3 & 0 & 0 \\ 0 & 4 & 5 & 1 \end{pmatrix}$$

$$\longrightarrow L^H_{(VII^r\o 7|I^r\triangle)} = \begin{pmatrix} 0 & \boxed{0} & 3 & 1 \\ 5 & 1 & \boxed{0} & 4 \\ 1 & 3 & 0 & \boxed{0} \\ \boxed{0} & 4 & 5 & 1 \end{pmatrix}$$

In this way, the entries of the L matrix that arise from superimposing the distribution of boxes provided by the algorithm are the metrics in absolute value between the classes that are linked.

$$S(L^H_{(VII^r\o 7|I^r\triangle)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{32}, \Delta_{41}\}$$

With this information we are already able to calculate an optimal link and therefore, due to the link structure, we are immediately calculating all those optimal links that make up the nabla class. In this case, since it is a non-dual tonal function, the class nabla is unique, it can be generated by permutation of rows, a property that is not preserved for dual tonal functions.

$$\left[E^o_{(VII_c\o 7|I_c\triangle)}\right]_\nabla = \left[\begin{pmatrix} A & G \\ F & E \\ D & C \\ B & B \end{pmatrix}\right]_\nabla \tag{B.62}$$

Now we calculate the nabla value when the distance between class mappings is minimal. The minimum for al ∞ mappings is unique so the sum can be constructed. Calculating each metric, dimension by dimension in the optimal link, we obtain the minimum nabla function that characterizes the class of links that bears its name. In this case it is worth five, which is a relatively high value for $n = 4$ indicating that the voices move.

$$\nabla(E^o_{(VII_c\o 7|I_c\triangle)}) = \sum_{j=1}^{n} \Omega \mid \int_{(E^o_{j1\,(VII_c\o 7|I_c\triangle)})}^{(E^o_{j2\,(VII_c\o 7|I_c\triangle)})} \phi^{-1}\, d\phi \mid_\Delta = 5$$

With the information provided by the optimization techniques in conjunction with our previous knowledge, we can determine which is the endomorphism matrix between voicings, thus presenting the matrix equation.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(VII_c ø7)) \longrightarrow \psi(I_c\triangle)$$
$$\begin{pmatrix} s^{-\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{-\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ F_{z_2} \\ D_{z_3} \\ B_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \\ Bz_4 \end{pmatrix} \tag{B.63}$$

Since we have not chosen to use the method from U array but have instead built L array manually without annotating the sign exponents, we have retrieved the signs of the exponents by examining the classes instead of using the retrieval from the sign recovery function (σ). Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-2} - \lambda)^2(s^{-1} - \lambda)(s^0 - \lambda)$$

By simple inspection of the characteristic polynomial we calculate its algebraic multiplicities for the application of the polynomial criterion. Thus we have that the multiplicities are given by the equations: $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$. With this information obtained from solving the optimization problem we can deduce for a fixed tuning, the equations of the antecedent voicing and the consequent voicing

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VII_c ø7}(t) = \psi_{A_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}} + \psi_{B_{z_4}}(t) \\ \psi_{I_c\triangle}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{B_{z_4}}(t) \end{cases}$$

We have seen that the functions in the time domain can be separated note by note, but by formalism in the analysis we will write the functions of both voicings in the change.

$$\psi_{II_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VII_c ø7)} - e^{-2\pi tki\psi_j(VII_c ø7)}}{2i}$$
$$\longrightarrow \psi_{I_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c\triangle)} - e^{-2\pi tki\psi_j(I_c\triangle)}}{2i}$$

Following the polynomial criteria we obtain the function of the degree $VIIø7$ related to the Ionian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(VIIø7|I\triangle)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{3\,s^2+1}{s^2}\right)\lambda^3 + \frac{3\,s^2+3}{s^2}\lambda^2 + \left(-\frac{s^2+3}{s^2}\right)\lambda + \frac{1}{s^2}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \dfrac{\lambda^5}{5} + \left(-\dfrac{3\,s^2+1}{4\,s^2}\right)\lambda^4 + \dfrac{3\,s^2+3}{3\,s^2}\lambda^3 + \left(-\dfrac{s^2+3}{2\,s^2}\right)\lambda^2 + \dfrac{\lambda}{s^2}$

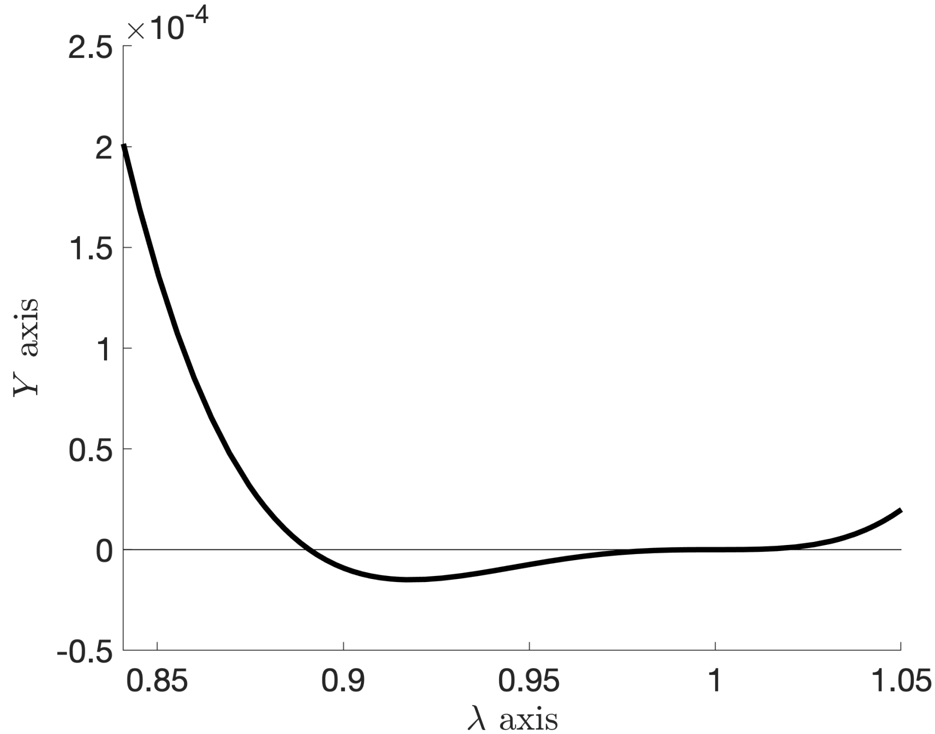


Figure B.14: Characteristic polynomial associated to the VII⟶I△ cadence

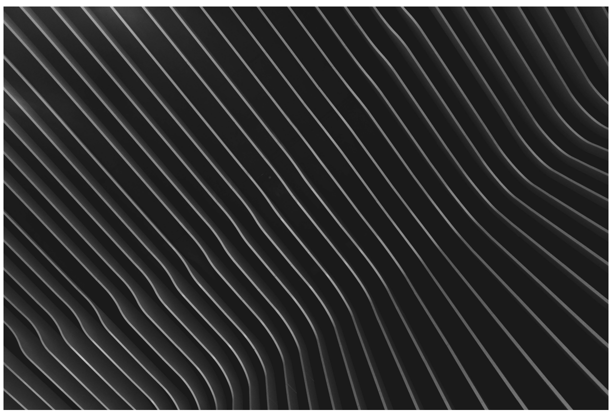
B.2.10. Ionian tonal functions

$II- \longrightarrow I$	$\Phi[E_{(II-\vert I)}] \in D^{\mathbb{R}[\lambda]}$
$III- \longrightarrow I$	$\Phi[E_{(III-\ \vert I)}] \in T^{\mathbb{R}[\lambda]}$
$IV \longrightarrow I$	$\Phi[E_{(IV\vert I)}] \in D^{\mathbb{R}[\lambda]}$
$V \longrightarrow I$	$\Phi[E_{(V\vert I)}] \in S^{\mathbb{R}[\lambda]}$
$VI- \longrightarrow I$	$\Phi[E_{(VI-\vert I)}] \in T^{\mathbb{R}[\lambda]}$
$VIIo \longrightarrow I$	$\Phi[E_{(VIIo\vert I)}] \in S^{\mathbb{R}[\lambda]}$

$II-7 \longrightarrow I\triangle$	$\Phi[E_{(II-7\vert I\triangle)}] \in S^{\mathbb{R}[\lambda]} \cup D^{\mathbb{R}[\lambda]}$
$III-7 \longrightarrow I\triangle$	$\Phi[E_{(III-7\vert I\triangle)}] \in T^{\mathbb{R}[\lambda]}$
$IV\triangle \longrightarrow I\triangle$	$\Phi[E_{(IV\triangle\vert I\triangle)}] \in S^{\mathbb{R}[\lambda]}$
$V7 \longrightarrow I\triangle$	$\Phi[E_{(V7\vert I\triangle)}] \in D^{\mathbb{R}[\lambda]}$
$VI-7 \longrightarrow I\triangle$	$\Phi[E_{(VI-7\vert I\triangle)}] \in T^{\mathbb{R}[\lambda]}$
$VII\o7 \longrightarrow I\triangle$	$\Phi[E_{(VII\o7\vert I\triangle)}] \in D^{\mathbb{R}[\lambda]}$

Appendix C

The Mixolydian Mode



Adrien Olichon

https://unsplash.com/photos/a-black-and-white-abstract-background-with-wavy-lines-gOdavfpH-3s

C.1. The Mixolydian mode for n =3

C.1.1. II-7⟶ I Cadence

As it happened in the Lydian mode, in the Mixolydian mode we find ourselves with the same situation in which the traditional theory does not cover the expectations regarding the resolution of the chords. Thus, there is no choice but to study each case in isolation to deduce the tonal function between the degrees of the Mixolydian mode when it is acting as a center. We started to develop the case in an analogous way, seeking to place the link and then solve the optimization problem. Ultimately, we will simplify the notation to graph theory to have a practical map when generating progressions. In this way we write the link E with the appropriate classes.

$$E_{(II_c^r - |I_c^r)} = \begin{pmatrix} A & G \\ F & E \\ D & C \end{pmatrix} \tag{C.1}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(II^r-|I^r)} = \begin{pmatrix} 2 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 2 \end{pmatrix} \tag{C.2}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link. As in previous cases, we note that we have dispensed with the subscript c, since by the static function theorem we know that the matrix L is independent of the key in which we are working.

$$L_{(II^r-|I^r)} = \begin{pmatrix} 2 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r-|I^r)} = \begin{pmatrix} 0 & 3 & 1 \\ 1 & 0 & 4 \\ 3 & 0 & 0 \end{pmatrix} \longrightarrow L^H_{(II^r-|I^r)} = \begin{pmatrix} \boxed{0} & 3 & 1 \\ 1 & \boxed{0} & 4 \\ 3 & 0 & \boxed{0} \end{pmatrix} \tag{C.3}$$

The solutions for $L^H_{(II^r-|I^r)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(II^r-|I^r)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

We calculate the optimal link class. In this case it matches the original link E.

$$\left[E^o_{(II_c-|I_c)} \right]_\nabla = \left[\begin{pmatrix} A & G \\ F & E \\ D & C \end{pmatrix} \right]_\nabla \tag{C.4}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(II_c-|I_c)}) = 2 + 1 + 2 = 5$$

We write the optimal nabla value as a generalization for every tonality:

$$\nabla^o_{(II-|I)} = 5$$

We know that any optimal arrangement from an optimal progression $E^o_{(II-|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$
$$C_{\mathbb{E}}(\psi(II_c-)) \longrightarrow \psi(I_c))$$
$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ F_{z_2} \\ D_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \end{pmatrix} \tag{C.5}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with assigned values l_1, l_2 and l_3. Each of these integer values are the metrics retrieved from the solution set S. So when substituting we have that the tonal function is given by the following expression.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^{-2} - \lambda & 0 & 0 \\ 0 & s^{-1} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{C.6}$$

From the algebraic properties of the determinant, we know that for this type of matrices, the calculation is simplified and it is enough to multiply the factors that appear on the diagonal.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^{-2} - \lambda)^2 (s^{-1} - \lambda)$$

The sets of roots that lie on the λ axis are classified according to their position relative to $E(M)$. Thus we have three perfectly determined sets.

$$\lambda^- = \{s^{-2}, s^{-1}\}$$
$$\lambda^0 = \{\emptyset\}$$
$$\lambda^+ = \{\emptyset\}$$

Since we already have the necessary information to know how the voicings are transformed in the optimization of voices, then we think about what physical expression the voicings will present, then, initially without specifically delimiting the octaves, we worry about giving an expression to these functions in the time domain. We use the parentheses since there must be a coordination between the classes of each voice so that the optimal voice leading is produced.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II_c-}(t) = \psi_{A_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

By using the arrow notation, we give an expression to each of the functions that each voicing represents, where each note of the voicing comes from the choice of a class of the optimal link, also respecting the coordination between octaves. Once we have optimized the link we will have:

$$\psi_{II_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II_c-)} - e^{-2\pi tki\psi_j(II_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we study how the voices behave in the optimal link and we determine the tonal function. We obtain the function of the degree $II-$ related to the Mixolydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(II-|I)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s} - \frac{2}{s^2}\right) \lambda^2 + \left(\frac{2}{s^3} + \frac{1}{s^4}\right) \lambda - \frac{1}{s^5}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \dfrac{\lambda^4}{4} + \left(-\dfrac{s+2}{3\,s^2}\right) \lambda^3 + \dfrac{2\,s+1}{2\,s^4}\,\lambda^2 + \left(-\dfrac{1}{s^5}\right) \lambda$

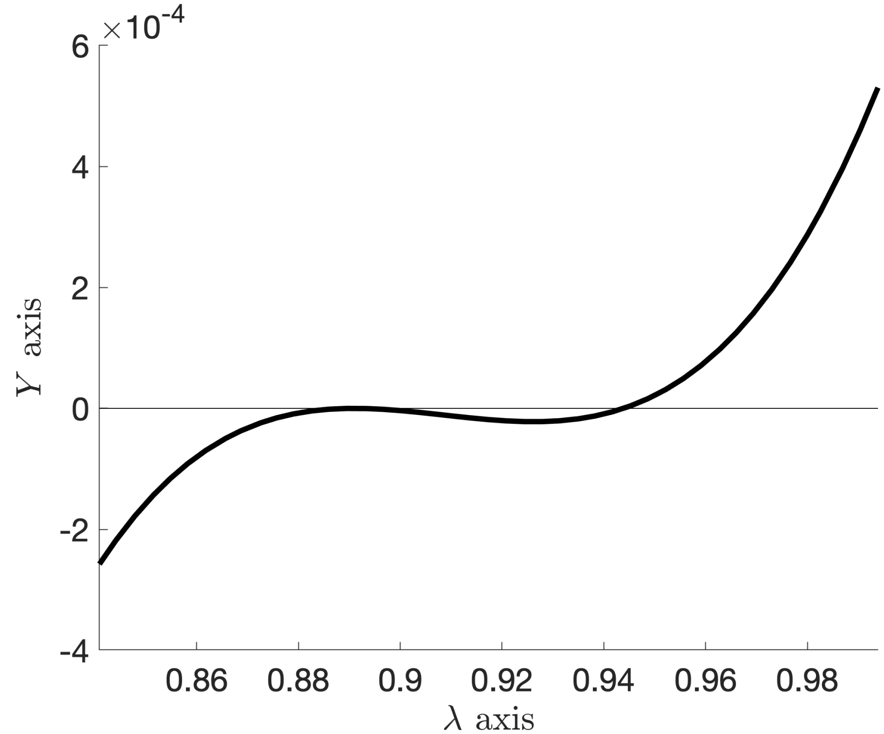


Figure C.1: Characteristic polynomial associated to the II-⟶ I cadence

C.1.2. III∘ ⟶ I Cadence

In our search for tonal functions, we continue advancing in the triads contained in the Mixolydian mode. Then, we calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(III^r_c o|I^r_c)} = \begin{pmatrix} Bb & G \\ G & E \\ E & C \end{pmatrix} \tag{C.7}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(III^r o|I^r)} = \begin{pmatrix} 3 & 6 & 2 \\ 0 & 3 & 5 \\ 3 & 0 & 4 \end{pmatrix} \tag{C.8}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(III^r o|I^r)} = \begin{pmatrix} 3 & 6 & 2 \\ 0 & 3 & 5 \\ 3 & 0 & 4 \end{pmatrix} \longrightarrow L^F_{(III^r o|I^r)} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 3 & 0 & 4 \end{pmatrix} \longrightarrow L^H_{(III^r o|I^r)} = \begin{pmatrix} 1 & 4 & \boxed{0} \\ \boxed{0} & 3 & 5 \\ 3 & \boxed{0} & 4 \end{pmatrix} \tag{C.9}$$

The solutions for $L^H_{(III^r o|I^r)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(III^r o|I^r)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

Now we calculate the optimal link class:

$$\left[E^o_{(III_c o|I_c)} \right]_\nabla = \left[\begin{pmatrix} Bb & C \\ G & G \\ E & E \end{pmatrix} \right]_\nabla \tag{C.10}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(III_c o|I_c)}) = 2 + 0 + 0 = 2$$

Any optimal arrangement from an optimal progression $E^o_{(IIIo|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$
$$C_{\mathbb{E}}(\psi(III_c o)) \longrightarrow \psi(I_c))$$
$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Bb_{z_1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} \tag{C.11}$$

Since we have the solutions of the matrix L^H , then we have found the exponents of the Mersenne numbers that appear in the transformation matrix from one voicing to another. Subtracting the identity matrix of appropriate dimensions and taking the determinant we find the tonal function. Sometimes we dispense with specifying the dimensions of the identity matrix because they are deduced from the context. Thus the volume function takes the following expression:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{C.12}$$

The expression for the tonal function is given by multiplying the inputs of the diagonal, being the volume as a function of λ the mathematical object in charge of compiling the relationship between the pair of tonal centers when perception is subject to the minimum.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2 - \lambda)(s^0 - \lambda)^2$$

The classification of the convergence of the roots facilitates three clearly differentiated sets $\lambda^- = \{\emptyset\}$, $\lambda^0 = \{s^0\}$ and $\lambda^+ = \{s^2\}$.

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(III_c\circ)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_j(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We generate the bracket that gives the frame to express the pair of functions for the optimal arrangement. This is how we study how the voicings have to be linked regardless of their opening.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{III_c\circ}(t) = \psi_{Bb_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \end{cases}$$

We have written the function of each voicing with each note separately where the choice of numbers z is arbitrary. In this way we see clearly how each function is.

As we have seen in previous cases, we write the functions of the voicings in the time domain and we symbolize the change using an arrow. By the Weber-Fechner law applied its energies in combination with the fundamental theorem, we know that for an arbitrary set of numbers z the absolute perception reaches the minimum.

$$\psi_{III_c\circ}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(III_co)} - e^{-2\pi tki\psi_j(III_co)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $IIIo$ related to the Mixolydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. Since only one voice moves, we classify the tonal function between degrees within the tonic area.

$$\boxed{\Phi[E_{(IIIo|I)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - 2)\ \lambda^2 + (2\, s^2 + 1)\ \lambda - s^2$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2}{3} - \frac{2}{3}\right)\ \lambda^3 + \left(s^2 + \frac{1}{2}\right)\ \lambda^2 + \left(-s^2\right)\ \lambda$

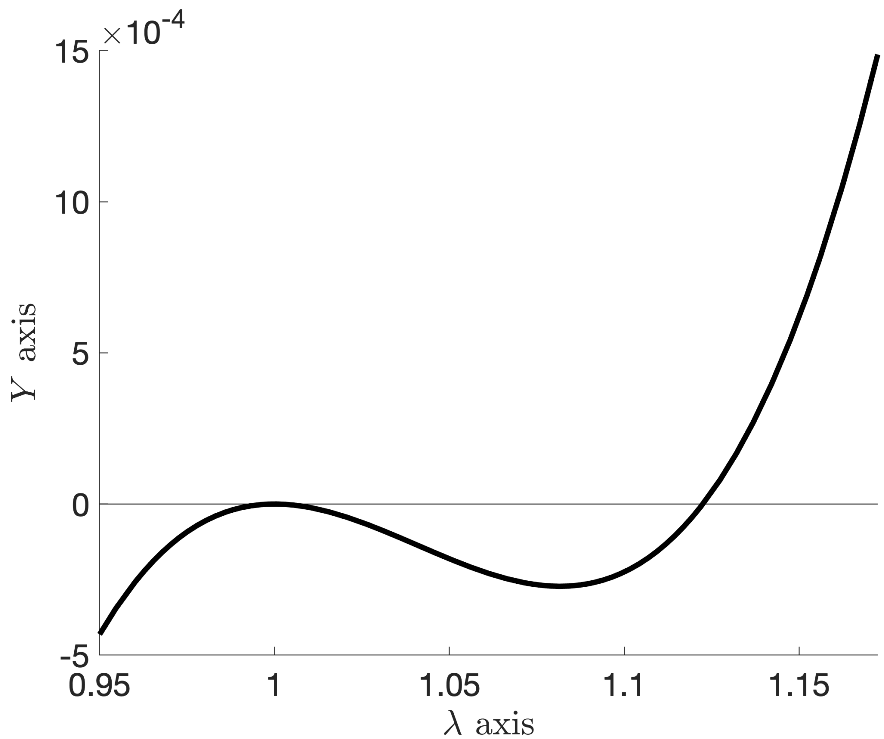


Figure C.2: Characteristic polynomial associated to the III∘ ⟶ I cadence

C.1.3. IV⟶ I Cadence

In this section we study the relationship between the fourth degree and the first when both are major triads. This progression, in its abstract expression, coincides with the retrograde progression class $\varrho([P]) = (V \mid I)$ where we point out that the convergence of a whole progression is only preserved if all the tonal functions are dual and with a solutions in both subdominant and dominant areas. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(IV_c^r|I_c^r)} = \begin{pmatrix} C & G \\ A & E \\ F & C \end{pmatrix} \tag{C.13}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix. We could deduce the tonal function of the retrograde progression class, but for a matter of formalism we are going to develop the calculations step by step.

$$L_{(IV^r|I^r)} = \begin{pmatrix} 5 & 4 & 0 \\ 2 & 5 & 3 \\ 2 & 1 & 5 \end{pmatrix} \tag{C.14}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(IV^r|I^r)} = \begin{pmatrix} 5 & 4 & 0 \\ 2 & 5 & 3 \\ 2 & 1 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r|I^r)} = \begin{pmatrix} 5 & 4 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{pmatrix} \longrightarrow L^H_{(IV^r|I^r)} = \begin{pmatrix} 5 & 4 & \boxed{0} \\ \boxed{0} & 3 & 1 \\ 1 & \boxed{0} & 4 \end{pmatrix} \tag{C.15}$$

Then the solutions for $L^H_{(IV^r|I^r)}$ when both triads are in root position becomes the set $S(L^H_{(IV^r|I^r)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$, wich represents the minimum voice leading. The distribution of boxes on L^H depends on the initial placement of the classes in the link, so for effectiveness and order we place the classes in the usual arrangement for chords and following the circle of fifths for both open and closed keys. This is not a necessary condition to reach the solution but it helps to visualize the whole process.

The position of each box on the L^H matrix indicates the pairing between the classes that minimizes perception. Thus we obtain an optimal link. We take classes since it is not the only optimal link.

$$\left[E^o_{(IV_c|I_c)}\right] = \left[\begin{pmatrix} C & C \\ A & G \\ F & E \end{pmatrix}\right]_{\triangledown} \tag{C.16}$$

Counting the total number of semitones that change in the link, we obtain the value of the minimum nabla function, which is the common characteristic of all optimal links. By the previously cited theorems, the value of this function is a feature that can be extended to a relationship between degrees.

$$\nabla^{o}_{(IV|I)} = 3$$

Any optimal arrangement from an optimal progression $E^{o}_{(IV|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(IV_c)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ F_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ E_{z_3} \end{pmatrix} \tag{C.17}$$

If we draw the distribution of boxes over the original matrix L and recover the signs appropriately then we will get the values for each exponent l. We subtract the identity matrix multiplied by λ and take the determinant.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-1} - \lambda \end{pmatrix} \tag{C.18}$$

Thus we arrive at the expression of the tonal function by multiplying the factors that appear on the diagonal.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)(s^{-2} - \lambda)(s^{-1} - \lambda)$$

When studying the roots we see that there are two to the left of the stabilizer of the group M, one on the stabilizer and none to its right:

$$\lambda^- = \{s^{-1}, s^{-2}\}$$

$$\lambda^0 = \{s^0\}$$

$$\lambda^+ = \{\emptyset\}$$

For every frecuency $\phi \in \Phi^+$ as a component of $\psi(IV_c)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. In the bracket we see which subscripts must be common for the arrangement to be optimal

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV_c}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{F_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{E_{z_3}}(t) \end{cases}$$

Using the arrow notation we indicate the transition between the first and the second voicing. The physical manifestation of the change would be the transition from one function to another.

$$\psi_{IV_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(IV_c)} - e^{-2\pi tki\psi_j(IV_c)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree IV related to the Mixolydian tonal center for n=3. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem and is written as a polynomial.

$$\boxed{\Phi[E_{(IV|I)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s} - \frac{1}{s^2} - 1\right)\lambda^2 + \left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}\right)\lambda - \frac{1}{s^3}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2+s+1}{s^2\,3}\right)\lambda^3 + \frac{s^2+s+1}{s^3\,2}\lambda^2 + \left(-\frac{1}{s^3}\right)\lambda$

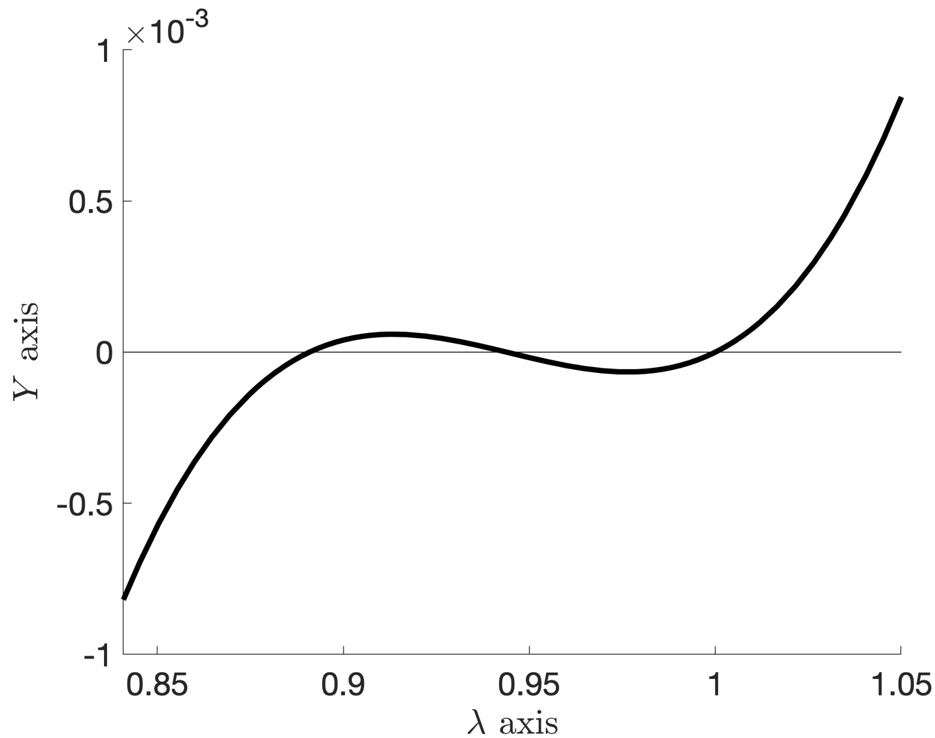


Figure C.3: Characteristic polynomial associated to the IV⟶ I cadence

C.1.4. V-$\longrightarrow$ I Cadence

We study in this section the cadence from the minor fifth degree to the first degree. This progression will appear retrograde, since the tonal function of the retrograde progression is dominant. It has a very particular sound and is used in many contexts such as film music, music for commercials or background music. We calculate in the link arranging the chords in root position:

$$E_{(V_c^r - | I_c^r)} = \begin{pmatrix} D & G \\ Bb & E \\ G & C \end{pmatrix} \tag{C.19}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix. In the calculation of the matrix L we have already omitted the subscripts c, since this matrix is independent of the key.

$$L_{(V^r - | I^r)} = \begin{pmatrix} 5 & 2 & 2 \\ 3 & 6 & 2 \\ 0 & 3 & 5 \end{pmatrix} \tag{C.20}$$

Following the steps of the Hungarian algorithm we consider to develope the algorithm through the L matrix to find an optimum link:

$$L_{(V^r - | I^r)} = \begin{pmatrix} 5 & 2 & 2 \\ 3 & 6 & 2 \\ 0 & 3 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r - | I^r)} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 3 & 5 \end{pmatrix} \longrightarrow L^H_{(V^r - | I^r)} = \begin{pmatrix} 3 & \boxed{0} & 0 \\ 1 & 4 & \boxed{0} \\ \boxed{0} & 3 & 5 \end{pmatrix} \tag{C.21}$$

Then the solutions for $L^H_{(V^r - | I^r)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(V^r - | I^r)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

Recovering the relationship between the voices that comes from the distribution of boxes over the L^H matrix, we obtain, an optimal link, and taking square brackets, its entire class.

$$\left[E^o_{(V_c - | I_c)} \right]_\nabla = \left[\begin{pmatrix} D & E \\ Bb & C \\ G & G \end{pmatrix} \right]_\nabla \tag{C.22}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value $\nabla(E^o_{(V_c - | I_c)}) = 2 + 2 + 0 = 4$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(V - | I)} = 4$.

Now any optimal arrangement from an optimal progression $E^o_{(V-|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(V_c-)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{\Delta_{12}} & 0 & 0 \\ 0 & s^{\Delta_{23}} & 0 \\ 0 & 0 & s^{\Delta_{31}} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ Bb_{z_2} \\ G_{z_3} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{C.23}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values Δ_{12}, Δ_{23} and Δ_{31}:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^2 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{C.24}$$

We have used the transformation T in an inexplicit way to obtain the endomorphism matrix. In this way we achieve, taking the determinant of the difference between said matrix and λId, the polynomial whose roots determine the direction of the voices in the optimum.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2 - \lambda)(s^2 - \lambda)(s^0 - \lambda)$$

We classify the roots of said polynomial based on how they intersect on the λ axis, distinguishing three sets based on their position with respect to the stability of the M group.

$$\lambda^- = \{\emptyset\}$$

$$\lambda^0 = \{s^0\}$$

$$\lambda^+ = \{s^2\}$$

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(V_c-)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We raise the pair of functions that appear in the change and place them in the bracket in such a way that we contrast the midi notation subscripts.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V_c-}(t) = \psi_{D_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2+1}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

Now for any pair of voicings in an optimal arrangement we will have two functions that are the sum of sinusoidal functions, where each function is a note that becomes the closest to the next chord following the solutions of the Hungarian algorithm. Then assuming that E is optimal and that the array is optimal as well.

$$\psi_{V_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(V_c-)} - e^{-2\pi tki\psi_j(V_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $V-$ related to the Mixolydian tonal center for n=3. In this case is unique and it can be represented only by one polynomial $\Phi(\lambda) \in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem and is written as a polynomial.

$$\boxed{\Phi[E_{(V-|I)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-2\,s^2 - 1)\ \lambda^2 + (s^4 + 2\,s^2)\ \lambda - s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s^2}{3} - \frac{1}{3}\right)\ \lambda^3 + \frac{s^2\ (s^2+2)}{2}\ \lambda^2 + \left(-s^4\right)\ \lambda$

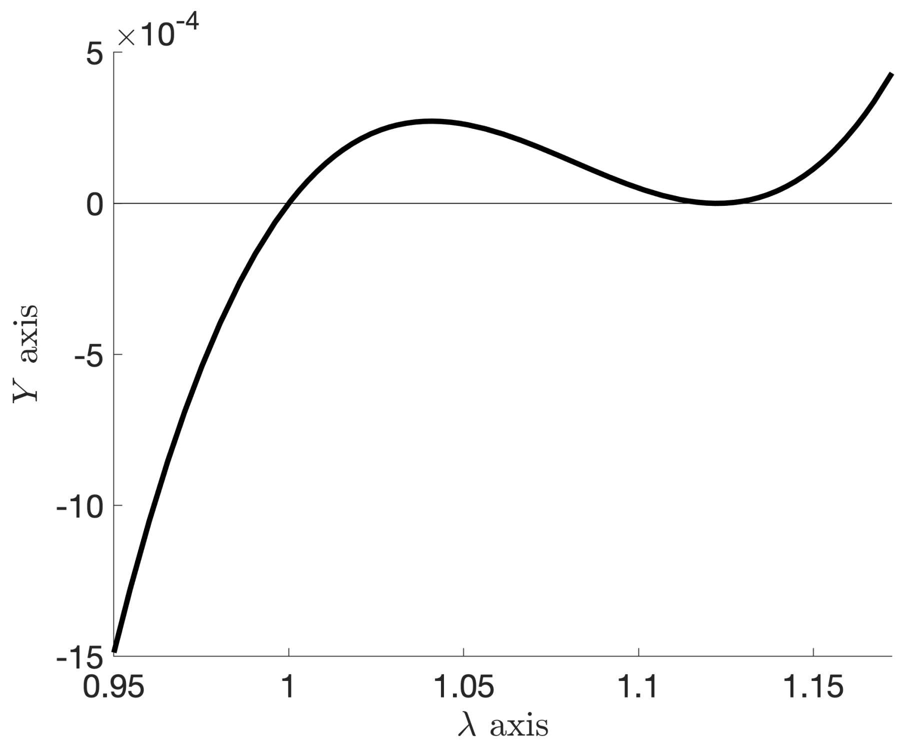


Figure C.4: Characteristic polynomial associated to the V-⟶ I cadence

C.1.5. VI-$\longrightarrow$ I Cadence

In this section we are going to study the relationship between the sixth degree and the first degree in the Mixolydian context. We will see that in this particular study we found the reason why these two chords usually appear together or sometimes substituted in the same section but a different structural repetition. On many occasions, a cadence is performed to the first degree and when this section is repeated, said first degree is repeated for the sixth. The explanation is that they have many notes in common, that is to say that this relation of intersection of octave equivalence classes allows us to use a substitution operation in certain progressions. As has been seen throughout the work, the substitution depends on the direction of the antecedent chord and the consequent chord that we are going to substitute. The link will be a progression such as the written below:

$$E_{(VI^r_c - |I^r_c)} = \begin{pmatrix} E & G \\ C & E \\ A & C \end{pmatrix} \tag{C.25}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(VI^r - |I^r)} = \begin{pmatrix} 3 & 0 & 4 \\ 5 & 4 & 0 \\ 2 & 5 & 3 \end{pmatrix} \tag{C.26}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(VI^r - |I^r)} = \begin{pmatrix} 3 & 0 & 4 \\ 5 & 4 & 0 \\ 2 & 5 & 3 \end{pmatrix} \longrightarrow L^F_{(VI^r - |I^r)} = \begin{pmatrix} 3 & 0 & 4 \\ 5 & 4 & 0 \\ 0 & 3 & 1 \end{pmatrix} \longrightarrow L^H_{(VI^r - |I^r)} = \begin{pmatrix} 3 & \boxed{0} & 4 \\ 5 & 4 & \boxed{0} \\ \boxed{0} & 3 & 1 \end{pmatrix} \tag{C.27}$$

The solutions for $L^H_{(VI^r - |I^r)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(VI^r - |I^r)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

As the solution provided by the algorithm has indicated, we compute an optimal link and take the nabla class from it.

$$\left[E^o_{(VI_c - |I_c)}\right]_\nabla = \left[\begin{pmatrix} E & E \\ C & C \\ A & G \end{pmatrix}\right]_\nabla \tag{C.28}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(V_c-|I_c)}) = 0 + 0 + 2 = 2$$

As we have already studied, we can generalize the value of nabla to a more abstract relationship exclusively between degrees, since by the static function theorem we know that we can carry out this generalization.

$$\nabla^o_{(VI-|I)} = 2$$

Any optimal arrangement from an optimal progression $E^o_{(VI-|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(VI_c-)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{\Delta_{12}} & 0 & 0 \\ 0 & s^{\Delta_{23}} & 0 \\ 0 & 0 & s^{-\Delta_{31}} \end{pmatrix} \cdot \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ A_{z_3} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{C.29}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values Δ_{12}, Δ_{23} and Δ_{31}:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{C.30}$$

Using the properties of the determinants we see that by multiplying each of the diagonal entries of the previous matrix, we obtain the characteristic polynomial of the endomorphism matrix. Thus we arrive at the tonal function.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)(s^0 - \lambda)(s^{-2} - \lambda)$$

Since we have already calculated the tonal function, we then study its roots to determine the movement of the voices and classify the tonal function within an area of the set of polynomials.

$$\lambda^- = \{s^{-2}\}$$

$$\lambda^0 = \{s^0\}$$

$$\lambda^+ = \{\emptyset\}$$

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(VI_c-)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. Using the brackets we study the functions available for each voicing when we carry out a selection of subscripts.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VI_c-}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{A_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

Considering that the arrangement is optimal, then for the same distribution of harmonics Γ we can ensure that the absolute perception is minimal and classify the tonal function of the behavior of the voices in the solution.

$$\psi_{VI_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VI_c-)} - e^{-2\pi tki\psi_j(VI_c-)}}{2i}$$

$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $VI-$ related to the Mixolydian tonal center for n=3. In this case is unique and it can be represented only by one polynomial $\Phi(\lambda) \in \mathbb{R}[\lambda]$.

The function is generalized using the static tonal function theorem and is written as a polynomial.

$$\boxed{\Phi[E_{(VI-|I)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{2}{s^2} - 1\right)\lambda^2 + \left(\frac{2}{s^2} + \frac{1}{s^4}\right)\lambda - \frac{1}{s^4}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^4 + 2\,s^2}{s^4\,3}\right)\lambda^3 + \frac{2\,s^2 + 1}{2\,s^4}\lambda^2 + \left(-\frac{1}{s^4}\right)\lambda$

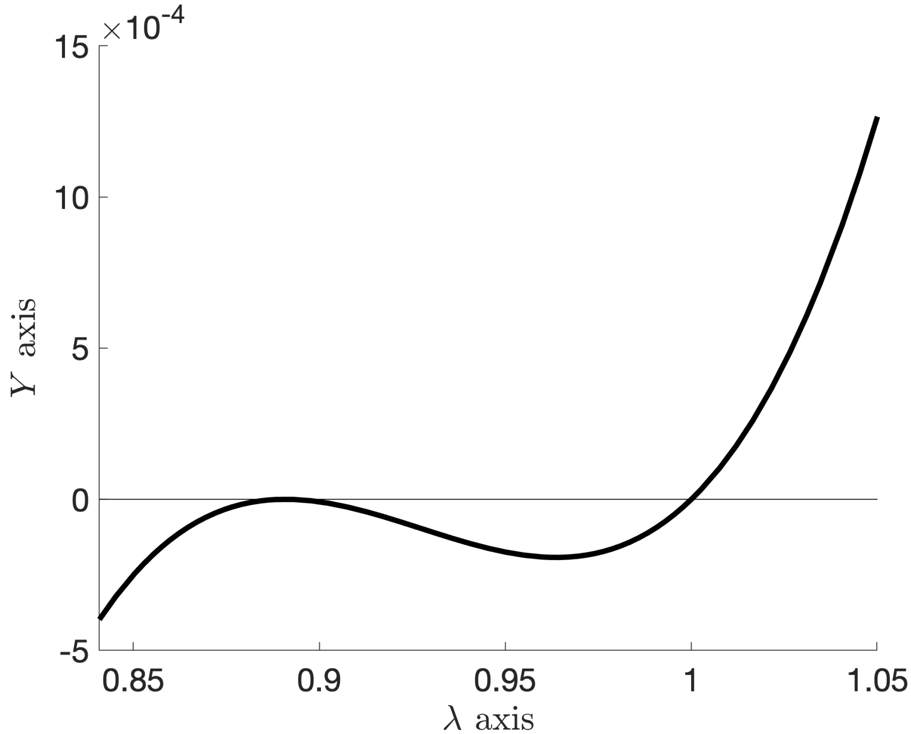


Figure C.5: Characteristic polynomial associated to the VI-⟶ I cadence

C.1.6. ♭VII⟶ I Cadence

In this case we are studying the relationship between two triads that are one tone away. In jazz music it is very common to superimpose upper structure triads (UST's) onto chords in the context of improvisation. When we are talking about a melodic line, the line itself always goes in and out of the chord of the moment, then increases and decreases its dimension if it has the same timbre. On the contrary, when the timbre is different, the line is perceived as a separate element that, as it always has dimension 1, does not have a tonal function other than tonic. This reflection is enough in itself, but we want to indicate that the perception of the melody is also affected by the reverberation conditions of a room, which increase the dimensionality of a melodic line since these conditions make the frequencies heard in time. For this second reflection regarding the reverb we have to take into account that the improvisation of melodic lines has to be approached, in relation to the tonal function, as a multidimensional phenomenon. Thus, there are pairs of triads that work particularly better than others like the ones we see below. The link will be a progression such as the written below:

$$E_{(bVII^r_c|I^r_c)} = \begin{pmatrix} F & G \\ D & E \\ Bb & C \end{pmatrix} \tag{C.31}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(VII^r|I^r)} = \begin{pmatrix} 2 & 1 & 5 \\ 5 & 2 & 2 \\ 3 & 6 & 2 \end{pmatrix} \tag{C.32}$$

With the matrix L built, we develop the algorithm until we reach the matrix L^H where we have not squared any solution because it has multiples, in such a way that we will have to apply the Zero Method.

$$L_{(bVII^r|I^r)} = \begin{pmatrix} 2 & 1 & 5 \\ 5 & 2 & 2 \\ 3 & 6 & 2 \end{pmatrix} \longrightarrow L^F_{(bVII^r|I^r)} = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 0 & 0 \\ 1 & 4 & 0 \end{pmatrix} \longrightarrow L^H_{(bVII^r|I^r)} = \begin{pmatrix} 0 & 0 & 4 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \tag{C.33}$$

C.1.7. The Zero method over $L^H_{(\flat VII^r|I^r)}$

As we have seen that the matrix allows several solutions, then we select each zero from left to right and from top to bottom, going through the nine zeros. Every time we select a zero, a possible solution appears. The solutions can appear several times later, finally we have to see how many different solutions we have.

$$L^{Z_1}_{(bVII^r|I^r)} = \begin{pmatrix} \boxed{0}^* & 0 & 4 \\ 2 & \boxed{0} & 0 \\ 0 & 4 & \boxed{0} \end{pmatrix} \mid L^{Z_2}_{(bVII^r|I^r)} = \begin{pmatrix} 0 & \boxed{0}^* & 4 \\ 2 & 0 & \boxed{0} \\ \boxed{0} & 4 & 0 \end{pmatrix} \mid L^{Z_3}_{(bVII^r|I^r)} = \begin{pmatrix} \boxed{0} & 0 & 4 \\ 2 & \boxed{0}^* & 0 \\ 0 & 4 & \boxed{0} \end{pmatrix} \tag{C.34}$$

$$L^{Z_4}_{(bVII^r|I^r)} = \begin{pmatrix} 0 & \boxed{0} & 4 \\ 2 & 0 & \boxed{0}^* \\ \boxed{0} & 4 & 0 \end{pmatrix} \mid L^{Z_5}_{(bVII^r|I^r)} = \begin{pmatrix} 0 & \boxed{0} & 4 \\ 2 & 0 & \boxed{0} \\ \boxed{0}^* & 4 & 0 \end{pmatrix} \mid L^{Z_6}_{(bVII^r|I^r)} = \begin{pmatrix} \boxed{0} & 0 & 4 \\ 2 & \boxed{0} & 0 \\ 0 & 4 & \boxed{0}^* \end{pmatrix} \tag{C.35}$$

The solutions for $L^H_{(VI^r|I^r)}$ when both triads are in root position becomes the following sets, wich represents the minimum voice leading:

$$S^1(L^H_{VI^r|I^r)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

$$S^2(L^H_{VI^r|I^r)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

C.1.8. Tonal function for $S^1(L^H_{(\flat VII^r|I^r)})$

We take the first solution and develop it as a particular case to find the tonal function

$$\left[E^o_{1\,(bVII_c|I_c)}\right]_\nabla = \left[\begin{pmatrix} F & G \\ D & E \\ Bb & C \end{pmatrix}\right]_\nabla \tag{C.36}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{1\,(bVII_c-|I_c)}) = 2+2+2 = 6$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(bVII|I)} = 6$.

Any optimal arrangement from an optimal progression $E^o_{1\,(bVII|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bVII_c)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{\Delta_{11}} & 0 & 0 \\ 0 & s^{\Delta_{22}} & 0 \\ 0 & 0 & s^{\Delta_{33}} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ D_{z_2} \\ Bb_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \end{pmatrix} \tag{C.37}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values Δ_{11},Δ_{22} and Δ_{33}:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2-\lambda & 0 & 0 \\ 0 & s^2-\lambda & 0 \\ 0 & 0 & s^2-\lambda \end{pmatrix} \tag{C.38}$$

We use the properties of the determinant to reach the tonal function.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2-\lambda)(s^2-\lambda)(s^2-\lambda)$$

We see that the roots of the tonal function are classified according to their position with respect to the stabilizer of the M group.

$$\lambda^- = \{\emptyset\}$$

$$\lambda^0 = \{\emptyset\}$$

$$\lambda^+ = \{s^2\}$$

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVII_c)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$.

It is a necessary condition for the minimization of the absolute perception that the subscripts of the octave classes appear correctly paired, this is equivalent to the fact that the voicing of each note does not jump octave.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII_c}(t) = \psi_{F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

The physical phenomenon is the succession of two voicings with the same timbre distribution that link their voices, independently of their opening.

$$\psi_{bVII_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c)} - e^{-2\pi tki\psi_j(bVII_c)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bVII$ related to the Mixolydian tonal center for n=3. In this case is unique and it can be represented only by one polynomial $\Phi(\lambda) \in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem and is written as a polynomial.

$$\boxed{\Phi[E^1_{(bVII|I)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-3\,s^2)\ \lambda^2 + (3\,s^4)\ \lambda - s^6$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-s^2\right)\ \lambda^3 + \frac{3\,s^4}{2}\,\lambda^2 + \left(-s^6\right)\ \lambda + \frac{s^8}{4}$

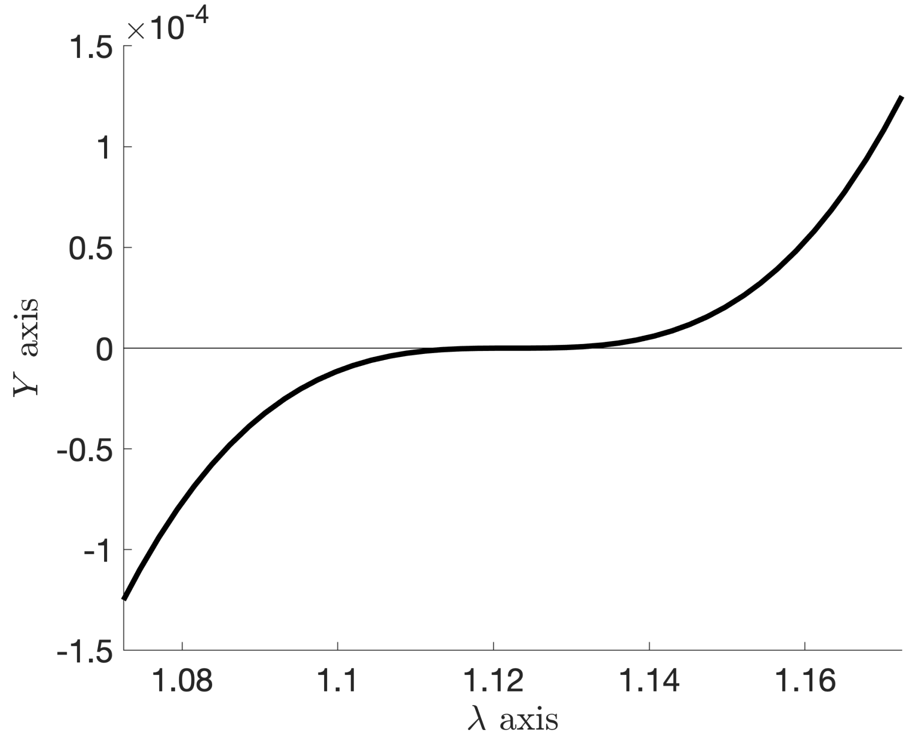


Figure C.6: Characteristic polynomial associated to the bVII⟶ I cadence (1)

C.1.9. Tonal function for $S^2(L^H_{(bVII^r|I^r)})$

We take the second solution that we have obtained from the Zero method and calculate the nabla class of the optimal link:

$$\left[E^o_{2\,(bVII_c|I_c)}\right]_\nabla = \left[\begin{pmatrix} F & E \\ D & C \\ Bb & G \end{pmatrix}\right]_\nabla \tag{C.39}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{2\,(bVII_c|I_c)}) = 1+2+3 = 6$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(bVII|I)} = 6$

Any optimal arrangement from an optimal progression $E^o_{2\,(bVII|I)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(bVII_c)) \longrightarrow \psi(I_c))$$

$$\begin{pmatrix} s^{-\Delta_{12}} & 0 & 0 \\ 0 & s^{-\Delta_{23}} & 0 \\ 0 & 0 & s^{-\Delta_{31}} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ D_{z_2} \\ Bb_{z_3} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{C.40}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values Δ_{12},Δ_{23} and Δ_{31}:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^{-1} - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-3} - \lambda \end{pmatrix} \tag{C.41}$$

Using the properties of the determinant we calculate the second tonal function associated to the link between the pair of major triads.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^{-1} - \lambda)(s^{-2} - \lambda)(s^{-3} - \lambda)$$

The roots are polarized to the left side of $E(M)$.

$$\lambda^{-} = \{s^{-1}, s^{-2}, s^{-3}\}$$

$$\lambda^{0} = \{\emptyset\}$$

$$\lambda^{+} = \{\emptyset\}$$

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^{+}$ as a component of $\psi(bVII_c)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. In the square brackets we write the two functions that represent the antecedent voicing and the consequent so that we have paired the subscripts in each octave following the indications of the second solution.

$$\mathcal{B}_{\epsilon} = \begin{cases} \psi_{bVII_c}(t) = \psi_{F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

For a certain timbral distribution we study how one function is transformed into another using the $\longrightarrow$ notation. Thus we see that when the voicings make up an optimal arrangement, then the absolute perception reaches the minimum for the set of options between the two tonal centers involved in the link.

$$\psi_{bVII_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c)} - e^{-2\pi tki\psi_j(bVII_c)}}{2i}$$

$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bVII$ related to the Mixolydian tonal center for n=3. In this case we see that it is a dual function and that the two solutions are in different areas, so we will symbolize the relationship between degrees with a bidirectional arrow. The function for the second solution is generalized using the static tonal function theorem and is written as a polynomial.

$$\boxed{\Phi[E^2_{(bVII|I)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3}\right)\lambda^2 + \left(\frac{\frac{1}{s}+\frac{1}{s^2}}{s^3} + \frac{1}{s^3}\right)\lambda - \frac{1}{s^6}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \dfrac{\lambda^4}{4} + \left(-\dfrac{s^2+s+1}{s^3\,3}\right)\lambda^3 + \dfrac{s^2+s+1}{s^5\,2}\lambda^2 + \left(-\dfrac{1}{s^6}\right)\lambda$

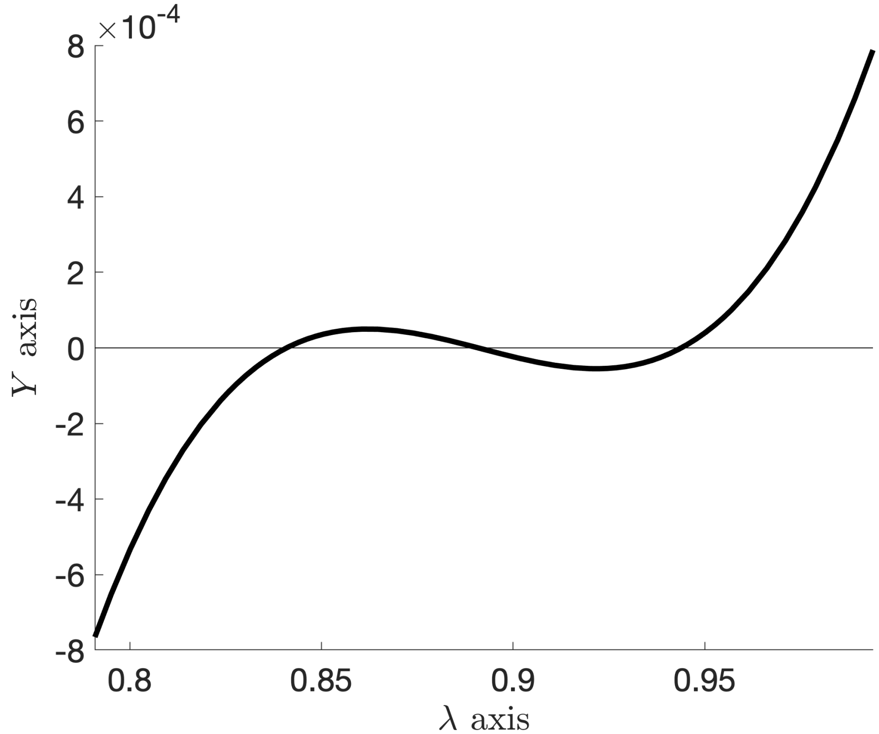


Figure C.7: Characteristic polynomial associated to the bVII⟶ I cadence (2)

C.2. The Mixolydian mode for n=4

C.2.1. II-7⟶I7 Cadence

Through the purpose of determining the tonal functions between triad chords and diatonic structures we have found some surprising results like the dual tonal function. In this particular case, we observe that our theoretical framework has to be coordinated with progressions established throughout history of the music. It would be very strange and would present a serious logical problem if, when determining the global pattern between the tonal centers that comes from connecting the Weber-Fechner law with the Hungarian algorithm, we find cases that, even though they are established as progression construction cells, contradicting the polynomial criterion. Fortunately this does not happen and until a paradigm shift, the cadence associated with the link class $[E] = (II - 7 \mid I7)$ continues to verify divergence within the polynomial criterion. Without giving this case special treatment, we are going to build the link with the chords placed in its fundamental expression. So the link E we build as:

$$E_{(II^r_c-7|I^r_c7)} = \begin{pmatrix} C & Bb \\ A & G \\ F & E \\ D & C \end{pmatrix} \tag{C.42}$$

We calculate the link matrix calculating all the distances Δ_{ij}. We build the L matrix analogously to previous cases.

$$L_{(II^r-7|I^r7)} = \begin{pmatrix} 2 & 5 & 4 & 0 \\ 1 & 2 & 5 & 3 \\ 5 & 2 & 1 & 5 \\ 4 & 5 & 2 & 2 \end{pmatrix} \tag{C.43}$$

With the matrix L built, we develop the whole process to find a distribution of boxes on the matrix L^H.

$$L_{(II^r-7|I^r7)} = \begin{pmatrix} 2 & 5 & 4 & 0 \\ 1 & 2 & 5 & 3 \\ 5 & 2 & 1 & 5 \\ 4 & 5 & 2 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r-7|I^r7)} = \begin{pmatrix} 2 & 5 & 4 & 0 \\ 0 & 1 & 4 & 2 \\ 4 & 1 & 0 & 4 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$
$$\longrightarrow L^H_{(II^r-7|I^r7)} = \begin{pmatrix} 2 & 5 & 4 & \boxed{0} \\ \boxed{0} & 0 & 4 & 2 \\ 4 & \boxed{0} & 0 & 4 \\ 2 & 2 & \boxed{0} & 0 \end{pmatrix}$$

Using the algorithm we mark the entries of the L^H matrix that, when rewritten over the L matrix, provide the metrics between the voices when the link is optimal.

$$S(L^H_{(II^r-7|I^r\triangle)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

Since we have found the pairing, which in this case is non-dual, we compute a link by arranging the first chord as it was and using the solutions to build the optimal link. We generalize the result calculating the nabla class. We use the sum of the minimum distances between classes to determine the value of the nabla function associated with an optimal link.

$$\left[E^o_{(II_c-7|I_c\triangle)}\right]_\nabla = \left[\begin{pmatrix} C & C \\ A & Bb \\ F & G \\ D & E \end{pmatrix}\right]_\nabla \tag{C.44}$$

We calculate the nabla value when the distance between class mappings is minimal. The minimum for al ∞ mappings is unique so the sum can be constructed:

$$\nabla(E^o(II_c - 7 \mid I_c 7)) = \sum_{j=1}^{n} \Omega \mid \int_{E^o_{j1\,(II_c-7|I_c7)}}^{E^o_{j2\,(II_c-7|I_c7)}} \phi^{-1}\, d\phi \mid_\Delta = 5$$

Once the S set is calculated we can form a generalization of a dimensoinally optimized cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^4 \longrightarrow \Phi^4$$

$$C_\mathbb{E}(\psi(II_c - 7)) \longrightarrow \psi(I_c 7)$$

$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ F_{z_3} \\ D_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ E_{z_4} \end{pmatrix} \tag{C.45}$$

Using the formula that we have already used in previous cases, we think about calculating the characteristic polynomial of the previous matrix that comes from the already mentioned transformation T. Thus we obtain the tonal function as the characteristic polynomial of the transformation matrix between voicings.

$$p_{C_\mathbb{E}}(\lambda) = (s^2 - \lambda)(s^2 - \lambda)(s^1 - \lambda)(s^0 - \lambda)$$

We leave annotated each one of the algebraic multiplicities to determine the area of classification of the tonal function. $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$,$\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(II - 7)$ or $\psi(I7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. As we have seen in previous cases, we give shape to the functions that in the temporal domain represent the pair of consecutive voicings for a particular selection of integers.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II-7}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{F_{z_3}} + \psi_{D_{z_4}}(t) \\ \psi_{I7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{E_{z_4}}(t) \end{cases}$$

Using the arrow notation, we represent the transition between the two voicings where the ψ functions of each component of the optimal link respect the optimization conditions.

$$\psi_{II-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II_c-7)} - e^{-2\pi tki\psi_j(II_c-7)}}{2i}$$
$$\longrightarrow \psi_{I_c7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c7)} - e^{-2\pi tki\psi_j(I_c7)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $II-7$ related to the Mixolydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(II-7|I7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-2\,s^2 - s - 1)\,\lambda^3 + (s^4 + 2\,s^3 + 2\,s^2 + s)\,\lambda^2 + (-s^5 - s^4 - 2\,s^3)\,\lambda + s^5$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{2} - \frac{s}{4} - \frac{1}{4}\right)\lambda^4 + \frac{s\,(s^3 + 2\,s^2 + 2\,s + 1)}{3}\lambda^3 + \left(-\frac{s^3\,(s^2 + s + 2)}{2}\right)\lambda^2 + s^5\,\lambda$

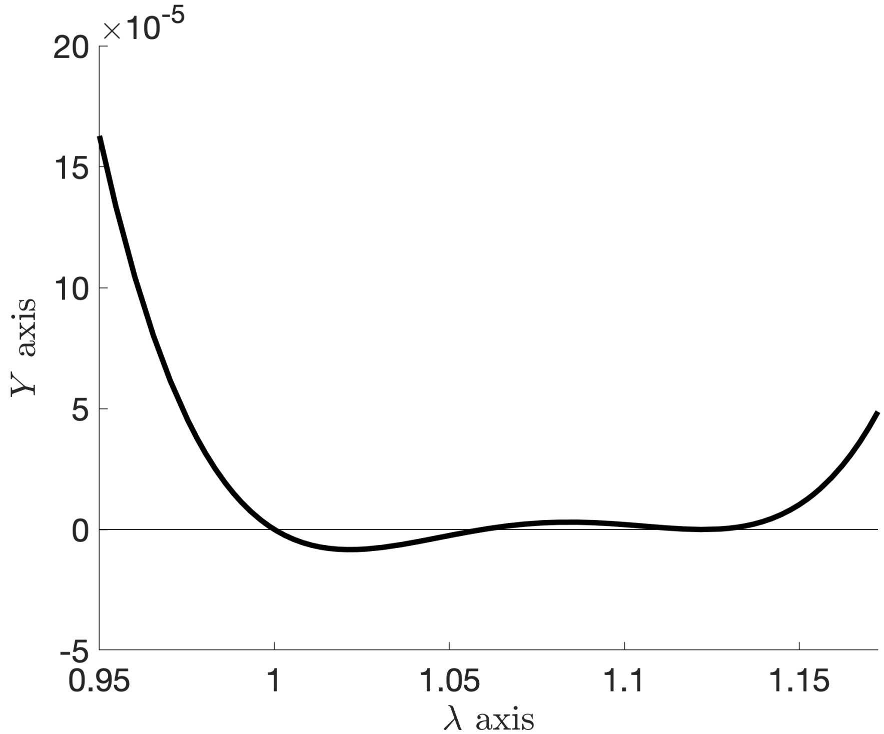


Figure C.8: Characteristic polynomial associated to the II-7⟶I7 cadence

C.2.2. III⌀7 ⟶I7 Cadence

It is very common for this pair of chords present in the link that we are going to develop below to appear substituted in the progressions of classical music and jazz music. We are still particularly interested in how they behave sequentially in time. Following W.F.C , we are going to develop the case using the general methodology. The link will be the following matrix:

$$E_{(III^r_c\o 7|I^r_c 7)} = \begin{pmatrix} D & Bb \\ Bb & G \\ G & E \\ E & C \end{pmatrix} \tag{C.46}$$

Taking the metrics appropriately based on the link we have written, we calculate the L matrix. We remember in this section that this matrix can be extracted from the U matrix, where we also have the exponents that indicate the sign of the distance for its recovery. In this case we build it manually.

$$L_{(III^r ø7|I^r 7)} = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 0 & 3 & 6 & 2 \\ 3 & 0 & 3 & 5 \\ 6 & 3 & 0 & 4 \end{pmatrix} \tag{C.47}$$

Starting from the matrix L we use the Hungarian algorithm until finding a solution on the matrix L^H.

$$L_{(III^r ø7|I^r 7)} = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 0 & 3 & 6 & 2 \\ 3 & 0 & 3 & 5 \\ 6 & 3 & 0 & 4 \end{pmatrix} \longrightarrow L^F_{(III^r ø7|I^r 7)} = \begin{pmatrix} 2 & 5 & 4 & 0 \\ 0 & 1 & 4 & 2 \\ 4 & 1 & 0 & 4 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow L^H_{(III^r ø7|I^r 7)} = \begin{pmatrix} 2 & 3 & 0 & \boxed{0} \\ \boxed{0} & 3 & 6 & 2 \\ 4 & \boxed{0} & 3 & 5 \\ 6 & 3 & \boxed{0} & 4 \end{pmatrix}$$

The solutions given by the algorithm will be:

$$S(L^H_{(III^r ø7|I^r 7)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(III_c ø7|I_c 7)} \right] = \left[\begin{pmatrix} D & C \\ Bb & Bb \\ G & G \\ E & E \end{pmatrix} \right]_\nabla \tag{C.48}$$

We see with the naked eye that the two tonal centers, which in this case are specifically chords, share three classes in common and only one differs. Thus, at first sight we specify that the value for the minimum nabla function is going to be 2. This value comes from the distance between the classes that D and C whose minimum distance between classes is 2. If we now focus on the physical manifestation of these two centers then, using the transformation T we reach the matrix that transforms a voicing in the four-dimensional frequency space into another voicing in the same space, respecting the optimization conditions that minimize the absolute perception.

$$
\begin{gathered}
C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4 \\
C_{\mathbb{E}}(\psi(III_c ø7)) \longrightarrow \psi(I_c 7) \\
\begin{pmatrix} s^{-\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ E_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ E_{z_4} \end{pmatrix}
\end{gathered} \tag{C.49}
$$

Calculating the characteristic polynomial using the formula for it, we arrive at the tonal function as such. Sometimes we make compatible different notations for the same mathematical object, in this case the tonal function. This is done since its study can be covered from different perspectives, that is to say that it is itself a characteristic polynomial of the endomorphism matrix that we are studying, but at the same time it is the function of a hypervolume.

$$
p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^0 - \lambda)(s^0 - \lambda)(s^{-2} - \lambda)
$$

As the algebraic multiplicities indicate, we see that there are three static voices and one moving downward in the optimal link. Thus, we see that in summary, the three multiplicities are given by the following three equations $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(III_c ø7)$ or $\psi(I_c 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

Now we see that in time we are going to have two signals that are decomposed note by note and we suppose the timbre of each note is fixed in both, where the timbre is given by the harmonic distribution Γ, which we remember can be visualized as a see in $\mathbb{R}^h$ which is the real vector space of h dimension.

$$
\mathcal{B}_\epsilon = \begin{cases} \psi_{IIIø7}(t) = \psi_{D_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{E_{z_4}}(t) \\ \psi_{I_c 7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{E_{z_4}}(t) \end{cases}
$$

Taking into account the conditions that the frequency assignment given by the ψ function that assigns a frequency to each class of the link must follow, we can represent the change from the first to the second voicing using the arrow notation:

$$
\begin{aligned}
\psi_{III_c ø7}(t) &= \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(III_c ø7)} - e^{-2\pi tki\psi_j(III_c ø7)}}{2i} \\
\longrightarrow \psi_{I_c 7}(t) &= \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c 7)} - e^{-2\pi tki\psi_j(I_c 7)}}{2i}
\end{aligned}
$$

We obtain the function of the degree $III\o7$ related to the Mixolydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(III\o7|I7)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s^2} - 3\right)\lambda^3 + \left(\frac{3}{s^2} + 3\right)\lambda^2 + \left(-\frac{3}{s^2} - 1\right)\lambda + \frac{1}{s^2}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{3\,s^2+1}{4\,s^2}\right)\lambda^4 + \frac{3\,s^2+3}{3\,s^2}\lambda^3 + \left(-\frac{s^2+3}{2\,s^2}\right)\lambda^2 + \frac{\lambda}{s^2}$

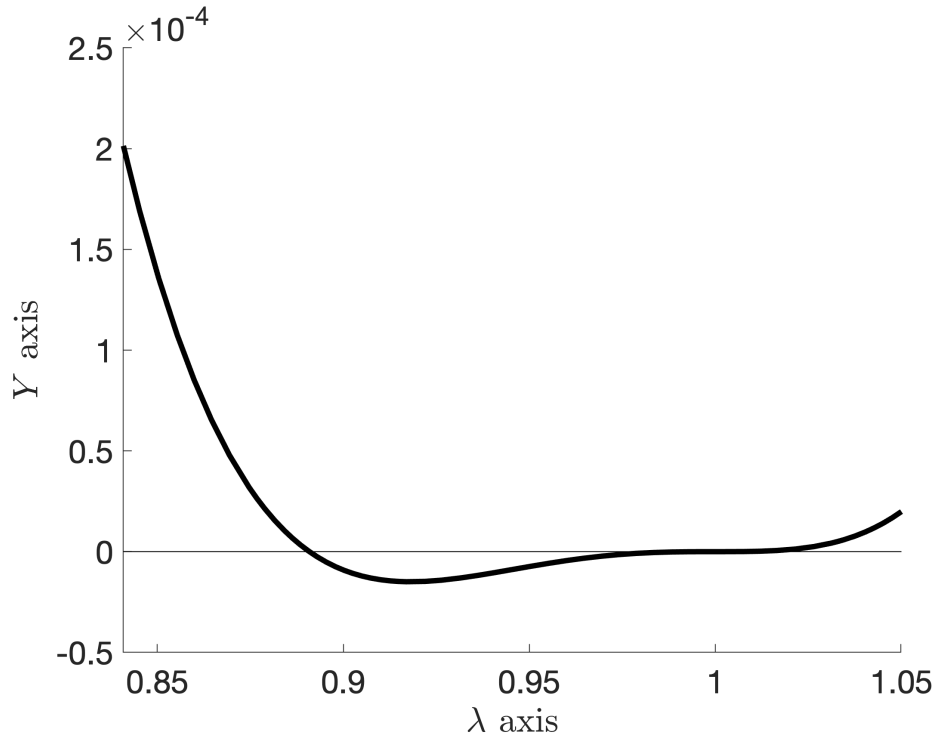


Figure C.9: Characteristic polynomial associated to the III∘7 ⟶I7 cadence

C.2.3. IV△ ⟶I7 Cadence

We continue working through each of the particular degrees of the Mixolydian mode with the purpose of generalizing the graph of tonal functions for $n = 4$ selecting the usual notes in four-voice chords. Given the large number of cases, in the work we cover a sufficient amount for common practice, leaving for the interested reader a large number of examples where he can consolidate the steps to generate himself other graphs with other scales that are of interest.

The link will be the following matrix:

$$E_{(IV_c^r \triangle | I_c^r 7)} = \begin{pmatrix} E & Bb \\ C & G \\ A & E \\ F & C \end{pmatrix} \tag{C.50}$$

Primarily, with the link ready, we are interested in generating the L matrix. In this case we do not have to use the infinite arithmetic criterion or other techniques since the dimensionality between both chords is the same.We calculate the link matrix organizing all the distances Δ_{ij}:

$$L_{(IV_c^r \triangle | I_c^r 7)} = \begin{pmatrix} 6 & 3 & 0 & 4 \\ 2 & 5 & 4 & 0 \\ 1 & 2 & 5 & 3 \\ 5 & 2 & 1 & 5 \end{pmatrix} \tag{C.51}$$

Following the steps of the Hungarian algorithm we develope the L matrix:

$$L_{(IV^r \triangle | I^r 7)} = \begin{pmatrix} 6 & 3 & 0 & 4 \\ 2 & 5 & 4 & 0 \\ 1 & 2 & 5 & 3 \\ 5 & 2 & 1 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r \triangle | I^r 7)} = \begin{pmatrix} 6 & 3 & 0 & 4 \\ 2 & 5 & 4 & 0 \\ 0 & 1 & 4 & 2 \\ 4 & 1 & 0 & 4 \end{pmatrix} \longrightarrow L^H_{(IV^r \triangle | I^r 7)} = \begin{pmatrix} 6 & 2 & \boxed{0} & 4 \\ 3 & 4 & 4 & \boxed{0} \\ \boxed{0} & 0 & 4 & 2 \\ 4 & \boxed{0} & 0 & 4 \end{pmatrix} \tag{C.52}$$

The solutions given by the algorithm will be:

$$S(L^H_{(IV^r \triangle | I^r 7)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(IV_c \triangle | I_c 7)} \right]_\nabla = \left[\begin{pmatrix} E & E \\ C & C \\ A & Bb \\ F & G \end{pmatrix} \right]_\nabla \tag{C.53}$$

Having calculated the nabla class, we are ready to calculate the value of the function nabla minimum. Thus we establish that the minimum nabla function is the sum of the minimum distances between classes when the link has been optimized by the Hungarian algorithm. This expression is consolidated in the following equation.

$$\nabla(E^o(IV_c\triangle \mid I_c\triangle)) = \sum_{j=1}^{n} \Omega \mid \int_{E^o_{j1\,(IV_c\triangle|I_c7)}}^{E^o_{j2\,(IV_c\triangle|I_c7)}} \phi^{-1}\, d\phi \mid_{\Delta} = 3$$

Immediately, after optimizing the relationship between the classes in the abstract, we ask ourselves what the physical manifestation of these chords is like and what matrix relates them. Ultimately we are going to group the information in the roots of a polynomial that has interesting properties, but this identification of the information with the polynomials is an extension of algebraic thinking, and could be expressed in another way. This has been chosen because it is considered the simplest. Once the S set is calculated we can form a generalization of a dimensionally optimized cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(IV_c\triangle)) \longrightarrow \psi(I_c7)$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ A_{z_3} \\ F_{z_4} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} \tag{C.54}$$

We take the characteristic polynomial to study where its roots lie with respect to the stabilizer of the Mersenne group $E(M)$.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^0 - \lambda)(s^0 - \lambda)(s^{-2} - \lambda)$$

We study the multiplicities of the polynomial obtaining three equations that musically represent the movement of the voices:$\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$,$\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(IV_c\triangle)$ or $\psi(I_c7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. Inside the bracket, we describe the note-by-note decomposition of the two functions involved in the change when we choose an arbitrary set of integers that determine a selection of octaves for both voicings.

$$\mathcal{B}_{\epsilon} = \begin{cases} \psi_{IV_c\triangle}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{A_{z_3}} + \psi_{F_{z_4}}(t) \\ \psi_{I_c7}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

By adjusting the function that chooses a particular frequency for each class and keeping the optimal arrangement then the physical expression of a voicing can be visualized with two functions with a fixed harmonic distribution that follow each other in time.

$$\psi_{IV_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(IV_c\triangle)} - e^{-2\pi tki\psi_j(IV_c\triangle)}}{2i} \longrightarrow \psi_{I_c7}(t)$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(I_c7)} - e^{-2\pi tki\psi_j(I_c7)}}{2i}$$

We obtain the function of the degree $IV\triangle$ related to the Mixoyidian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(IV\triangle|I7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s^2 - s - 2)\ \lambda^3 + (s^3 + 2\,s^2 + 2\,s + 1)\ \lambda^2 + (-2\,s^3 - s^2 - s)\ \lambda + s^3$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{4} - \frac{s}{4} - \frac{1}{2}\right) \lambda^4 + \left(\frac{s^3}{3} + \frac{2\,s^2}{3} + \frac{2\,s}{3} + \frac{1}{3}\right) \lambda^3 + \left(-\frac{s\ (2\,s^2 + s + 1)}{2}\right) \lambda^2 + s^3\,\lambda$

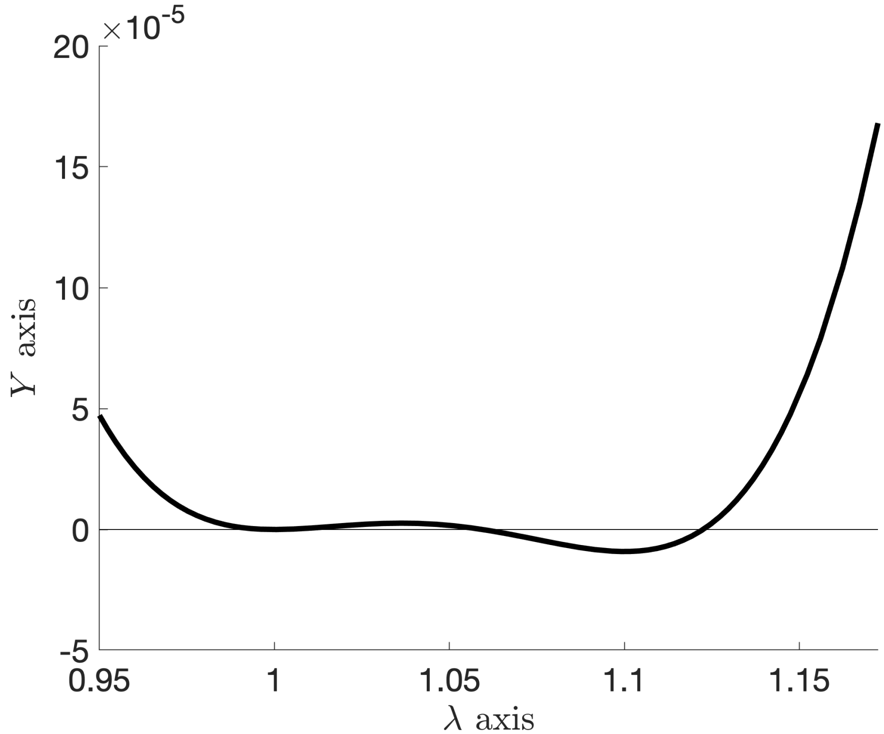


Figure C.10: Characteristic polynomial associated to the IV△ ⟶I7 cadence

C.2.4. V-7⟶I7 cadence

In this case, we study the convergence of the cadence $[E] = (V - 7 \mid I7)$ which, transporting the degrees appropriately, is equivalent to $[E] = (II - 7 \mid V7)$. Since $[E] = (II - 7 \mid V7)$ is an essential part of the progression $[E] = (II - 7 \mid V7 \mid I\triangle)$, we would enter into a serious logical contradiction if a progression so established in the history of jazz music did not have a convergent character, that is, that this case in isolation would be enough to re-evaluate the entire mathematical model that we have built in the work. The link will be the following matrix:

$$E_{(V_c^r - 7 | I_c^r 7)} = \begin{pmatrix} F & Bb \\ D & G \\ Bb & E \\ G & C \end{pmatrix} \tag{C.55}$$

Then, we calculate the link matrix organizing all the distances Δ_{ij}. We have described the relation of degrees with respect to one, because the graph of tonal functions of common practice is being constructed. The link is equivalent for other relationships between degrees as seen in the introduction to this section.

$$L_{(V_c^r-7|I_c^r7)} = \begin{pmatrix} 5 & 2 & 1 & 5 \\ 4 & 5 & 2 & 2 \\ 0 & 3 & 6 & 2 \\ 3 & 0 & 3 & 5 \end{pmatrix} \tag{C.56}$$

Once we have built the matrix L we develop each of the steps of the algorithm to find a distribution of boxes on the matrix L^H.

$$L_{(V^r-7|I^r7)} = \begin{pmatrix} 5 & 2 & 1 & 5 \\ 4 & 5 & 2 & 2 \\ 0 & 3 & 6 & 2 \\ 3 & 0 & 3 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r-7|I^r7)} = \begin{pmatrix} 4 & 1 & 0 & 4 \\ 2 & 3 & 0 & 0 \\ 0 & 3 & 6 & 2 \\ 3 & 0 & 3 & 5 \end{pmatrix}$$

$$\longrightarrow L^H_{(V^r-7|I^r7)} = \begin{pmatrix} 4 & 1 & \boxed{0} & 4 \\ 2 & 3 & 0 & \boxed{0} \\ \boxed{0} & 3 & 6 & 2 \\ 3 & \boxed{0} & 3 & 5 \end{pmatrix}$$

The solutions given by the algorithm will be:

$$S(L^H_{(V^r-7|I^r7)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(V_c-7|I_c7)}\right]_\nabla = \left[\begin{pmatrix} F & E \\ D & C \\ Bb & Bb \\ G & G \end{pmatrix}\right]_\nabla \tag{C.57}$$

Now we calculate the nabla value when the distance between class mappings is minimal. The minimum for all ∞ mappings is unique so the sum can be constructed:

$$\nabla(E^o(V_c-7 \mid I_c7)) = \sum_{j=1}^{n} \Omega \mid \int_{E^o_{j1\,(V_c-7|I_c7)}}^{E^o_{j2\,(V_c-7|I_c7)}} \phi^{-1}\, d\phi \mid_\Delta = 3$$

Once the S set is calculated we can form a generalization of a dimensionally optimized cadence $C_{\mathbb{E}}$. Since we have not used the T transformation explicitly, we have to pay attention to the recovery of the signs in the endomorphism matrix between the two consequent voicings.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(V_c - 7)) \longrightarrow \psi(I_c 7)$$

$$\begin{pmatrix} s^{-\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ D_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} = \begin{pmatrix} E_{z_1} \\ C_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} \tag{C.58}$$

Taking the matrix $C_{\mathbb{E}}$ and calculating its characteristic polynomial we find the tonal function; which in this case is unique for the pair of tonal centers under study. Thus, the correctly factored characteristic polynomial would be given by:

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-1} - \lambda)(s^{-2} - \lambda)(s^0 - \lambda)^2$$

Analyzing the roots and calculating the algebraic multiplicities we arrive at three equations that inform us of the movement of the voices when the link has been optimized. Thus we would have three equations whose sum must be the dimension of the frequency space: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$.

We know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(V_c - 7)$ or $\psi(I_c 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We thus have a pair of functions whose frequencies have to be linked in the optimal way for a particular harmonic distribution. We express in the bracket the need for the subscripts that indicate the position of each note in a particular octave to respect the optimization conditions.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V-7}(t) = \psi_{F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \\ \psi_{I_c 7}(t) = \psi_{E_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

We now represent analytically how the voicings will be for a particular harmonic distribution provided by a vector in $\mathbb{R}^h$. We point out as a reflection that there is some flexibility between the harmonic distributions without the tonal function varying. In the model we have assumed them to be equal since in this way we can apply the cancellation property in the fundamental theorem.

$$\psi_{V_c-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(V_c-7)} - e^{-2\pi tki\psi_j(V-7)}}{2i}$$

$$\longrightarrow \psi_{I_c 7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c 7)} - e^{-2\pi tki\psi_j(I_c 7)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $V-7$ related to the Mixolydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(V-7|I7)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s-3)\ \lambda^3 + (3\,s+3)\ \lambda^2 + (-3\,s-1)\ \lambda + s$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s}{4} - \frac{3}{4}\right)\ \lambda^4 + (s+1)\ \lambda^3 + \left(-\frac{3\,s}{2} - \frac{1}{2}\right)\ \lambda^2 + s\,\lambda$

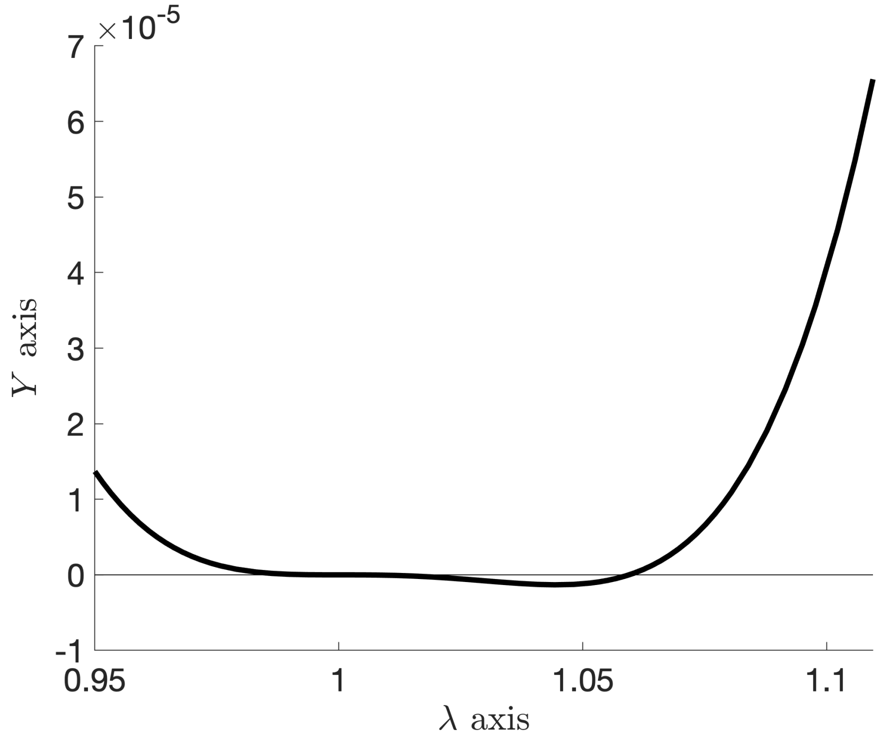


Figure C.11: Characteristic polynomial associated to the V-7⟶I7 cadence

C.2.5. VI-7⟶I7 cadence

The pair of chords present in the link is usually replaced on some occasions by the number of notes in common that appear between both structures. We are going to study the case without making any distinction by case and following the results at the demonstration level that we have previously exposed. That is, we are going to base ourselves on the connection of the Hungarian algorithm with the Weber-Fechner law of perception. This result is sufficient to safely generate convergent harmonic progressions for a fixed arbitrary timbre.

We build the link to know, ultimately, how the voices move when we minimize the value of absolute perception $\mid p \mid$:

$$E_{(VI^r_c-7|I^r_c7)} = \begin{pmatrix} G & Bb \\ E & G \\ C & E \\ A & C \end{pmatrix} \tag{C.59}$$

We place the classes in the way chords are usually placed, but as we have seen before, this is not an absolute requirement because the solutions by the algorithm do not change.

Then, we calculate the link matrix organizing all the distances Δ_{ij}. Building L we have carried out an abstraction process until collecting the metrics between classes to later apply the optimization method.

$$L_{(V^r_c I|I^r_c7)} = \begin{pmatrix} 3 & 0 & 3 & 5 \\ 6 & 3 & 0 & 4 \\ 2 & 5 & 4 & 0 \\ 1 & 2 & 5 & 3 \end{pmatrix} \tag{C.60}$$

Following the steps of the Hungarian algorithm we develop L until finding a distribution of boxes on said matrix forming L^H.

$$L_{(VI^r-7|I^r7)} = \begin{pmatrix} 3 & 0 & 3 & 5 \\ 6 & 3 & 0 & 4 \\ 2 & 5 & 4 & 0 \\ 1 & 2 & 5 & 3 \end{pmatrix} \longrightarrow L^F_{(VI^r-7|I^r7)} = \begin{pmatrix} 3 & 0 & 3 & 5 \\ 6 & 3 & 0 & 4 \\ 2 & 5 & 4 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix}$$
$$\longrightarrow L^H_{(VI^r-7|I^r7)} = \begin{pmatrix} 3 & \boxed{0} & 3 & 5 \\ 6 & 3 & \boxed{0} & 4 \\ 2 & 5 & 4 & \boxed{0} \\ \boxed{0} & 1 & 4 & 2 \end{pmatrix}$$

The solutions given by the algorithm will be:

$$S(L^H_{(VI^r-7|I^r7)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(VI_c-7|I_c7)}\right]_\nabla = \left[\begin{pmatrix} G & G \\ E & E \\ C & C \\ A & Bb \end{pmatrix}\right]_\nabla \tag{C.61}$$

By finding how the classes are paired, we have found all the optimal bindings that make up the nabla class. In this way we reach the set of links that represent the optimal voice leading between both tonal centers. This information is already enough to, with the naked eye, determine the tonal function between both structures if you have some experience. We are going to provide the rest of the process with a mathematical formalism using the transformation T to reach the endomorphism matrix in frequency space. In the practice of optimizing a P progression, it is usually enough to concatenate a series of optimal links like the previous one that make up the progression in such a way that we are optimizing link by link. Thus we build an optimal progression where P will converge if and only if each of the tonal functions associated with a given link is in either in the dominant or tonic area. This means that $\Phi[P]$ is a set of polynomials and $\mathbb{A}(\Phi[P])$ is the union of each of the areas.

In the space of frequencies from the transformation T we arrive at the matrix that optimally transforms two voicings of both tonal centers.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(VI_c - 7)) \longrightarrow \psi(I_c7)$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \\ A_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{C.62}$$

From this matrix we calculate the characteristic polynomial and substitute the delta values in the exponents, where in this case we have not had to recover any metric since they are all positive.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)^3 (s^1 - \lambda)$$

We study the multiplicities of the polynomial that are given by the equations: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 1$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi+$ as a component of $\psi(VI_c - 7)$ or $\psi(I_c 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We have two functions that are the representation of the physical manifestation of both voicings.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VI_c-7}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{A_{z_4}}(t) \\ \psi_{I_c 7}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Bb_{z_4}}(t) \end{cases}$$

Assuming that the transformation of each class at a particular frequency respects the optimization conditions, we represent the transition from one voicing to another for a particular selection of integers that determine the opening of the array.

$$\psi_{VI_c-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VI_c-7)} - e^{-2\pi tki\psi_j(VI_c-7)}}{2i}$$

$$\longrightarrow \psi_{I_c 7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c 7)} - e^{-2\pi tki\psi_j(I_c 7)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $VI - 7$ related to the Mixolydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(VI-7|I7)}] \in T^{\mathbb{R}[\lambda]}}$$

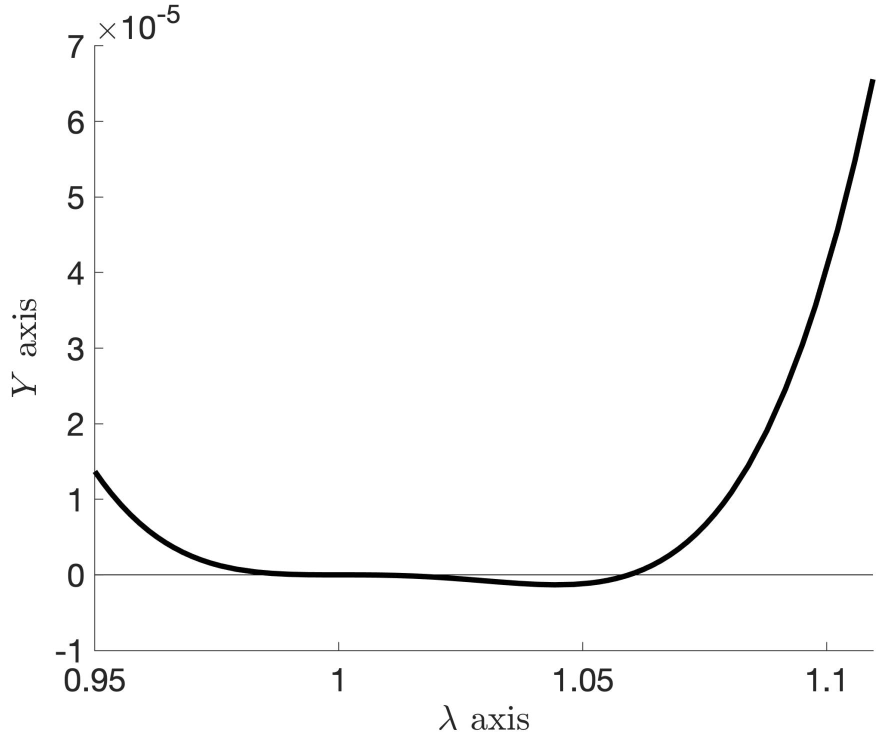


Figure C.12: Characteristic polynomial associated to the VI-7⟶ I7 cadence

C.2.6. ♭VII△ ⟶I-7 cadence

In our study on tonal functions we are going to evaluate the last case for four-voice chords in the Mixolydian mode. So we pose the problem in an analogous way.

$$E_{(bVII^r_c\triangle|I^r_c7)} = \begin{pmatrix} A & Bb \\ F & G \\ D & E \\ Bb & C \end{pmatrix} \tag{C.63}$$

The link cadence will be constructed using the ordered collection of metrics Δ_{ij}:

$$L_{(bVII^r\triangle|I^r7)} = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 5 & 2 & 1 & 5 \\ 4 & 5 & 2 & 2 \\ 0 & 3 & 6 & 2 \end{pmatrix} \tag{C.64}$$

Following the steps of the Hungarian algorithm we develope the L matrix until we reach to the S set:

$$L_{(bVII^r\triangle|I^r7)} = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 5 & 2 & 1 & 5 \\ 4 & 5 & 2 & 2 \\ 0 & 3 & 6 & 2 \end{pmatrix} \longrightarrow L^F_{(bVII^r\triangle|I^r7)} = \begin{pmatrix} 0 & 1 & 4 & 2 \\ 4 & 1 & 0 & 4 \\ 2 & 3 & 0 & 0 \\ 0 & 3 & 6 & 2 \end{pmatrix}$$
$$\longrightarrow L^H_{(bVII^r\triangle|I^r7)} = \begin{pmatrix} 0 & \boxed{0} & 4 & 2 \\ 4 & 0 & \boxed{0} & 4 \\ 2 & 2 & 0 & \boxed{0} \\ \boxed{0} & 2 & 6 & 2 \end{pmatrix}$$

In this way we have, first, subtracted the minimum of each row from its own row, second, subtracted the minimum of each column from its own column. This is how the zero of the matrix L^H of its last row is unique in its own row and forces a unique solution for the link, that is to say the distribution of boxes over L^H that gives us the optimal pairing is unique and therefore the tonal function is non-dual. With this information, it is enough for us to apply the transformation T on the distribution of boxes on the matrix L to reach the endomorphism matrix that transforms one voicing into another in an optimal way. We will see the development throughout the case.

We will use the transformation T for each distribution of boxes and we will study the position of the roots of the polynomial.

$$S(L^H_{(bVII^r\triangle|I^r7)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

Thus we reach the class nabla minimizing with respect to the solution and we study how the voices behave. Although the result is counterintuitive, we see that in the optimal link the voices go down and since it is the only solution, it is enough for us to apply the transformation T to reach the endomorphism matrix that transforms one voicing into another optimally in the four-dimension frequency space.

$$\left[E^o_{(VII_c\triangle|I_c7)} \right]_\nabla = \left[\begin{pmatrix} A & G \\ F & E \\ D & C \\ Bb & Bb \end{pmatrix} \right]_\nabla \tag{C.65}$$

In a formal way, we arrange the mapping in the frequency space that transforms the voicings of both tonal centers in an optimal way. We pay attention at this point, to the recovery of the sign of the metrics of the matrix L. Thus the matrix $C_{\mathbb{E}}$ will transform one voicing into another if and only if we recover the sign of the metrics of L.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bVII_c\triangle)) \longrightarrow \psi(I_c7)$$

$$\begin{pmatrix} s^{-\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{-\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ F_{z_2} \\ D_{z_3} \\ Bb_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ E_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{C.66}$$

As through the transformation we have reached the endomorphism matrix, it is enough for us to calculate its characteristic polynomial to arrive at the tonal function, which in this case is polarized, non-dual. Thus the characteristic polynomial takes the following expression.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-2} - \lambda)^2(s^{-1} - \lambda)(s^0 - \lambda)$$

We study the multiplicities of the polynomial that are given by the equations: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi+$ as a component of $\psi(bVII_c\triangle)$ or $\psi(I_c7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We have two functions that are the representation of the physical manifestation of both voicings.

We now consider the pair of functions that represent each voicing, decomposing each one note by note as individual functions. Thus, through the bracket, we know under what logical conditions the voicings have to be connected based on the solutions of the Hungarian algorithm.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII_c\triangle}(t) = \psi_{A_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}} + \psi_{Bb_{z_4}}(t) \\ \psi_{I_c7}(t) = \psi_{G_{z_1}}(t) + \psi_{E_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Bb_{z_4}}(t) \end{cases}$$

Assuming that the transformation of each class at a particular frequency respects the optimization conditions, we represent the transition from one voicing to another for a particular selection of integers that determine the opening of the array.

$$\psi_{bVII_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c\triangle)} - e^{-2\pi tki\psi_j(bVII_c\triangle)}}{2i}$$

$$\longrightarrow \psi_{I_c7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(I_c7)} - e^{-2\pi tki\psi_j(I_c7)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bVII\triangle$ related to the Mixolydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(bVII\triangle|I7)}] \in D^{\mathbb{R}[\lambda]}}$$

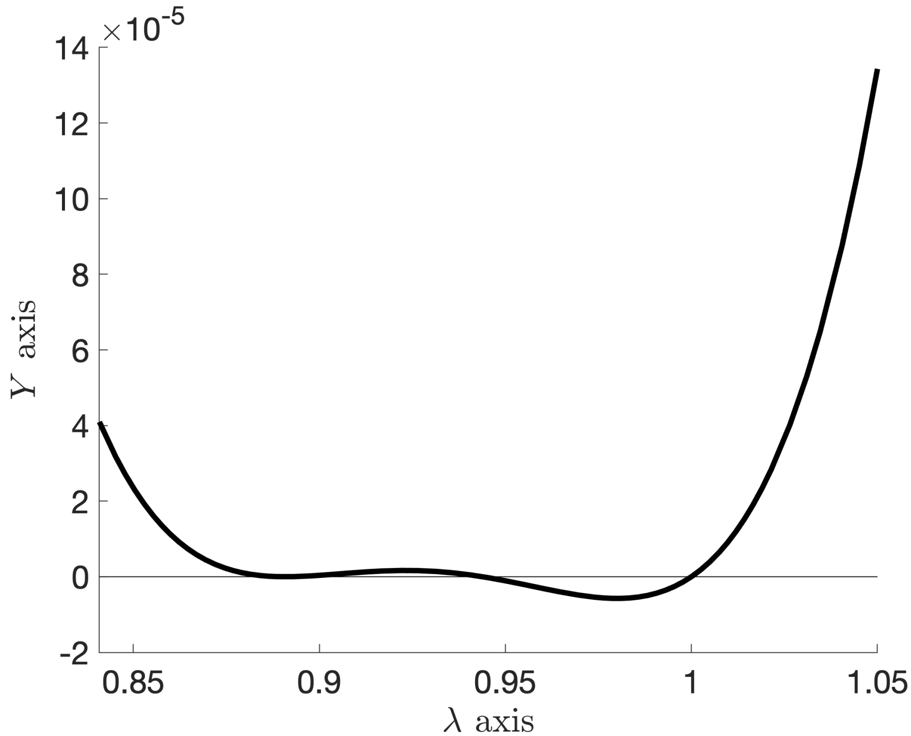


Figure C.13: Characteristic polynomial associated to the bVII△ | I7 cadence

C.2.7. Mixolydian tonal functions

$II- \longrightarrow I$	$\Phi[E_{(II-\|I)}] \in D^{\mathbb{R}[\lambda]}$
$IIIo \longrightarrow I$	$\Phi[E_{(IIIo\ \|I)}] \in T^{\mathbb{R}[\lambda]}$
$IV \longrightarrow I$	$\Phi[E_{(IV\|I)}] \in D^{\mathbb{R}[\lambda]}$
$V- \longrightarrow I$	$\Phi[E_{(V-\|I)}] \in S^{\mathbb{R}[\lambda]}$
$VI- \longrightarrow I$	$\Phi[E_{(VI-\|I)}] \in T^{\mathbb{R}[\lambda]}$
$bVII \longrightarrow I$	$\Phi[E_{(bVII\|I)}] \in S^{\mathbb{R}[\lambda]} \cup D^{\mathbb{R}[\lambda]}$

$II-7 \longrightarrow I7$	$\Phi[E_{(II-7\|I7)}] \in S^{\mathbb{R}[\lambda]}$
$III\text{ø}7 \longrightarrow I7$	$\Phi[E_{(III\text{ø}7\|I7)}] \in T^{\mathbb{R}[\lambda]}$
$IV\triangle \longrightarrow I7$	$\Phi[E_{IV\triangle\|I7)}] \in S^{\mathbb{R}[\lambda]}$
$V-7 \longrightarrow I7$	$\Phi[E_{(V-7\|I7)}] \in D^{\mathbb{R}[\lambda]}$
$VI-7 \longrightarrow I7$	$\Phi[E_{(VI-7\|I7)}] \in T^{\mathbb{R}[\lambda]}$
$bVII\triangle \longrightarrow I7$	$\Phi[E_{(bVII\triangle\|I7)}] \in D^{\mathbb{R}[\lambda]}$

Appendix D

The Dorian Mode

Taylor Leopold

https://unsplash.com/photos/a-blurry-photo-of-the-ocean-and-sky-fwsXlkNkwhI

D.1. The Dorian mode for n =3

D.1.1. II-⟶ I- Cadence

At this point, we are going to study the relationship between minor triads one tone away. These structures may come from other scales, but given the structure of the work, we arrive at them through the Dorian mode, where we have fixed the minor triad of the mode that works as the first degree. We are going to study how the second minor degree is related to the first in the tonality of C Dorian, where using the static tonal function theorem, we will generalize the result to an arbitrary key.

We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(II_c^r - | I_c^r)} = \begin{pmatrix} A & G \\ F & Eb \\ D & C \end{pmatrix} \tag{D.1}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(II^r-|I^r-)} = \begin{pmatrix} 2 & 6 & 3 \\ 2 & 2 & 5 \\ 5 & 1 & 2 \end{pmatrix} \tag{D.2}$$

Then following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(II^r-|I^r-)} = \begin{pmatrix} 2 & 6 & 3 \\ 2 & 2 & 5 \\ 5 & 1 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r-|I^r-)} = \begin{pmatrix} 0 & 4 & 1 \\ 0 & 0 & 3 \\ 4 & 0 & 1 \end{pmatrix} \longrightarrow L^H_{(II^r-|I^r-)} = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & 2 \\ 4 & 0 & 0 \end{pmatrix} \tag{D.3}$$

As we can see, there are multiple zeros in the matrix L^H , so we have to apply the Zero method to find the multiple solutions. If he gets some experience manipulating L matrices himself, there will come a point where he will be able to see the solutions directly. Even so, outside of this reflection, we have to follow the steps of the algorithm and evaluate the L^H matrix using the zero method to avoid losing solutions.

D.1.2. The zero method over $L^H_{(II^r-|I^r-)}$

From here we take the matrix L^H and select each one of the zeros, forcing said zero to be a solution. In this way we draw a distribution of boxes over the matrix L^H. We will obtain several distributions, of which several repeated equals can appear. We have to select those that are different from each other to know the different solutions. In this case the tonal function is dual and two solutions are presented, then there are two distributions of boxes that are related to two optimal ones that share the value of their nabla function.

$$L^{H_1}_{(II^r-|I^r)} = \begin{pmatrix} \boxed{0}^* & 4 & 0 \\ 1 & \boxed{0} & 2 \\ 4 & 0 & \boxed{0} \end{pmatrix} \mid L^{H_2}_{(II^r-|I^r-)} = \begin{pmatrix} 0 & 4 & \boxed{0}^* \\ \boxed{0} & 0 & 2 \\ 4 & \boxed{0} & 0 \end{pmatrix} \mid L^{H_3}_{(II^r-|I^r-)} = \begin{pmatrix} 0 & 4 & \boxed{0} \\ \boxed{0}^* & 0 & 2 \\ 4 & \boxed{0} & 0 \end{pmatrix} \tag{D.4}$$

$$L^{H_4}_{(II^r-|I^r-)} = \begin{pmatrix} \boxed{0} & 4 & 0 \\ 0 & \boxed{0}^* & 2 \\ 4 & 0 & \boxed{0} \end{pmatrix} \mid L^{H_5}_{(II^r-|I^r-)} = \begin{pmatrix} 0 & 4 & \boxed{0} \\ \boxed{0} & 0 & 2 \\ 4 & \boxed{0}^* & 0 \end{pmatrix} \mid L^{H_6}_{(II^r-|I^r-)} = \begin{pmatrix} \boxed{0} & 4 & 0 \\ 0 & \boxed{0} & 2 \\ 0 & 4 & \boxed{0}^* \end{pmatrix} \tag{D.5}$$

Then, the solutions for $L^H_{(II^r-|I^r-)}$ when both triads are in root position becomes the following sets, wich represents the minimum voice leading:

$$S^1(L^H_{(II^r-|I^r)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

$$S^2(L^H_{(II^r-|I^r)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

D.1.3. Tonal function for $S^1(L^H_{(II^r-|I^r-)})$

We evaluate the first solution, using the results of the Zero method. Thus, following the indications of pairing between voices of the first solution, we calculate an optimal link.

$$\left[E^o_{1\,(II_c-|I_c-)}\right]_\nabla = \left[\begin{pmatrix} A & G \\ F & Eb \\ D & C \end{pmatrix}\right]_\nabla \tag{D.6}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{1\,(II_c-|I_c-)}) = 2 + 2 + 2 = 6$$

We write the optimal nabla value as a generalization for every tonality:

$$\nabla^o_{(II-|I-)} = 6$$

Any optimal arrangement from an optimal progression $E^o_{1\,(II-|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(II_c-)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{-\Delta_{11}} & 0 & 0 \\ 0 & s^{-\Delta_{22}} & 0 \\ 0 & 0 & s^{-\Delta_{33}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ F_{z_2} \\ D_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \end{pmatrix} \tag{D.7}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values $-\Delta_{11}$, $-\Delta_{22}$ and $-\Delta_{33}$:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^{-2} - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{D.8}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^{-2} - \lambda)(s^{-2} - \lambda)(s^{-2} - \lambda)$$

The roots have the following structure:

$$\lambda^- = \{s^{-2}\}$$
$$\lambda^0 = \{\emptyset\}$$
$$\lambda^+ = \{\emptyset\}$$

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(II_c-)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. Using the bracket we determine that voice leading is optimal, that is, that when moving from voicing to voicing, the voices will move in an optimal way. Thus we have that the functions that transform a voicing into a specific frequency will do so optimally.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II_c-}(t) = \psi_{A_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

The interested reader will wonder how the optimization results are related to the physical expression of the voicings. Although we already know that the tonal function is a result that conceives the chords independently of the voicings that are in the arrangement, we have to provide the whole process with a mathematical formalism, so as in the rest of the cases we write the functions in the domain. For a fixed harmonic distribution, we present each voicing as a sum of trigonometric functions, which, for a set of integers that determines the opening of each voicing, generates the expression of said voicing as a sum of waves.

As can be seen, these functions depend on the harmonic distribution Γ and on the assignment of each function ψ that assigns to each input of the optimal link, a particular frequency.

$$\psi_{II_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II_c-)} - e^{-2\pi tki\psi_j(II_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As we have already described the whole process in different layers of abstraction, we now focus on studying the horizontal relationship between both tonal centers and, taking into account the distribution of the roots of the tonal function, using the polynomial criterion we determine the area of said function. Following the polynomial criterion we obtain a function of the degree $II-$ related to the Dorian tonal center for n=3. In this case is not unique, but the first solution is represented by the polynomial $\Phi(\lambda) \in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem and is written as a polynomial.

$$\boxed{\Phi[E^1_{(II-|I-)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-3\,s^2)\ \lambda^2 + (3\,s^4)\ \lambda - s^6$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-s^2\right)\ \lambda^3 + \frac{3\,s^4}{2}\ \lambda^2 + \left(-s^6\right)\ \lambda + \frac{s^8}{4}$

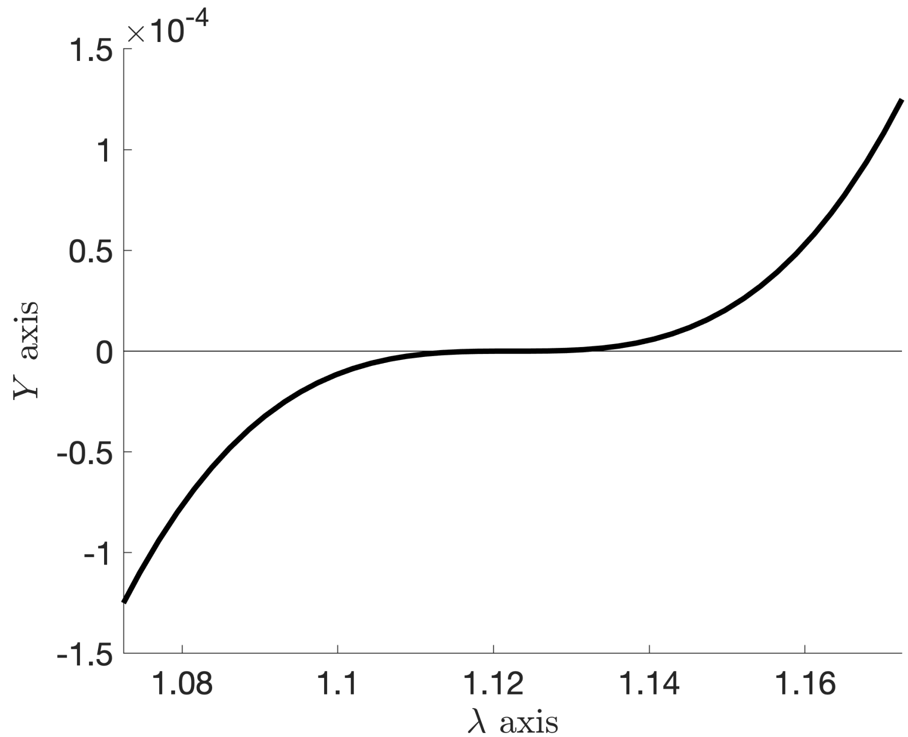


Figure D.1: Characteristic polynomial associated to the II-⟶ I- cadence (1)

D.1.4. Tonal function for $S^2(L^H_{(II^r-|I^r-)})$

Separately and with the aim of preserving the order, we analyze the second solution that appears from the application of the Zero method. Thus, we see that there is a second optimal link. If we consider the second distribution of boxes and apply the transformation T on the matrix L with the second distribution of boxes, we will observe that we will reach a new matrix $C_{\mathbb{E}}$ different from the one in the first case. We calculate the optimal link class:

$$\left[E^o_{2\,(II_c-|I_c-)}\right]_\nabla = \left[\begin{pmatrix} A & C \\ F & G \\ D & Eb \end{pmatrix}\right]_\nabla \tag{D.9}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{2\,(II_c-|I_c-)}) = 3+2+1 = 6$$

We write the optimal nabla value as a generalization for every tonality:

$$\nabla^o_{(II-|I-)} = 6$$

Any optimal arrangement from an optimal progression $E^o_{2\,(II_c-|I_c-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$. Thus, for the second solution of the Zero method, we obtain a second matrix that provides the transformation between voicings, the exponents being the metrics retrieved from the second set S^2.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(II_c-)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 \\ 0 & 0 & s^{\Delta_{32}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ F_{z_2} \\ D_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} \tag{D.10}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values Δ_{13}, Δ_{21} and Δ_{32}:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^3-\lambda & 0 & 0 \\ 0 & s^2-\lambda & 0 \\ 0 & 0 & s^1-\lambda \end{pmatrix} \tag{D.11}$$

Using the properties of the determinant the polynomial has the form. As we are seeing for the second solution, the roots have all polarized to the other side of the M stabilizer. Therefore the tonal function is dual, and polarized for all its functions.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^3 - \lambda)(s^2 - \lambda)(s^1 - \lambda)$$

The roots have the following structure:

$$\lambda^+ = \{s^3, s^2, s^1\}$$

$$\lambda^0 = \{\emptyset\}$$

$$\lambda^- = \{\emptyset\}$$

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(II_c-)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. Generalizing the functions of each voicing to a set of integers, we describe the physical manifestation of both voicings and how the antecedent and consequent voicing are connected, keeping the value of absolute perception $\mid p \mid$ at its minimum regardless of the opening of the arrangement.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II_c-}(t) = \psi_{A_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \end{cases}$$

Using the arrow notation $\longrightarrow$ and assuming that the functions ψ are consistent with the results of the Hungarian algorithm, then the following pair of trigonometric functions is sufficient, fixing a harmonic distribution Γ, to describe the change from one voicing to another and to model the physical phenomenon from voicing to voicing.

$$\psi_{II_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II_c-)} - e^{-2\pi tki\psi_j(II_c-)}}{2i}$$

$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c)} - e^{-2\pi tki\psi_j(I_c)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $II-$ related to the Dorian tonal center for n=3. In this case is not unique but one of the optimal link classes is represented by the tonal function as a polynomial $\Phi(\lambda) \in \mathbb{R}[\lambda]$.

The function is generalized using the static tonal function theorem and is written as a polynomial.

$$\boxed{\Phi[E^2_{(II-|I-)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-2\,s^3 - s^2)\ \lambda^2 + (s^6 + 2\,s^5)\ \lambda - s^8$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2\ (2\,s+1)}{3}\right) \lambda^3 + \frac{s^5\ (s+2)}{2}\,\lambda^2 + \left(-s^8\right)\ \lambda$

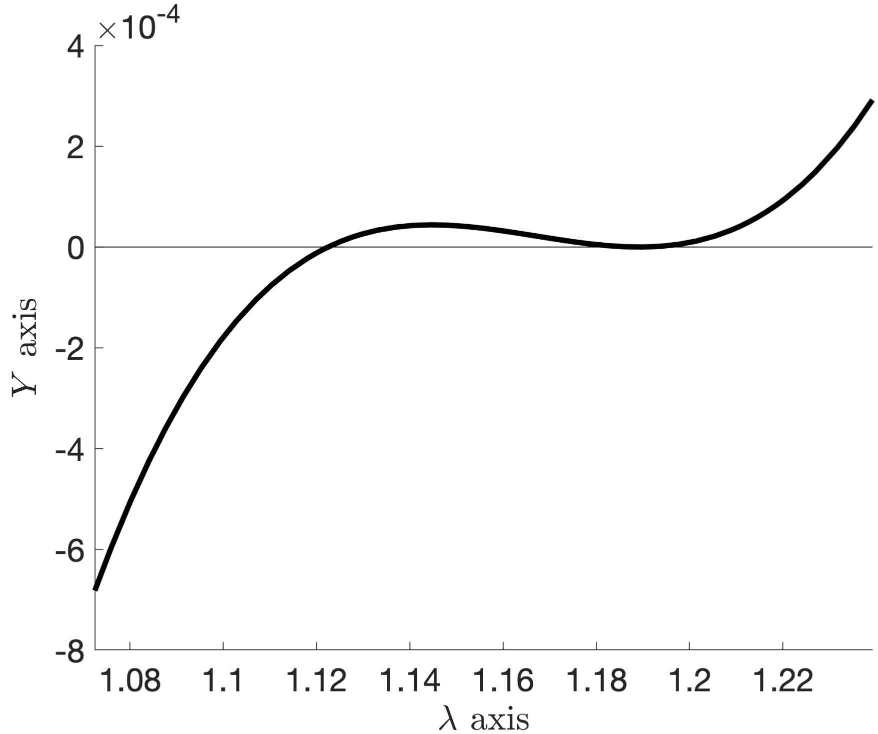


Figure D.2: Characteristic polynomial associated to the II-$\longrightarrow$ I- cadence (2)

D.1.5. bIII$\longrightarrow$ I- Cadence

In this case we are going to study the relationship between the third degree of the Dorian mode and its first degree. At first glance we already notice the tonal function since there are two voices in common. We are going to develop the case in a regular way to find the area where the tonal function is located. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(bIII_c^r|I_c^r-)} = \begin{pmatrix} Bb & G \\ G & Eb \\ Eb & C \end{pmatrix} \tag{D.12}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(bIII^r|I^r-)} = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 4 & 5 \\ 4 & 0 & 3 \end{pmatrix} \tag{D.13}$$

Note, the reader, that in the construction of the matrix L we have omitted the subscripts to indicate the tone because we know that the transport of a link E shares matrix L with the own E by the static function theorem.

Following the steps of the Hungarian algorithm we consider to develope the algorithm through the L matrix to find an optimum link. Starting from the original matrix L we apply the steps until we find the unique distribution of boxes on the matrix L^H.

$$L_{(bIII^r|I^r-)} = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 4 & 5 \\ 4 & 0 & 3 \end{pmatrix} \longrightarrow L^F_{(bIII^r|I^r-)} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 4 & 5 \\ 4 & 0 & 3 \end{pmatrix} \longrightarrow L^H_{(bIII^r|I^r-)} = \begin{pmatrix} 1 & 3 & \boxed{0} \\ \boxed{0} & 4 & 5 \\ 4 & \boxed{0} & 3 \end{pmatrix} \tag{D.14}$$

The solutions for $L^H_{(bIII^r|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(bIII^r|I^r-)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

We calculate the optimal link class which is made up of all optimal links that share its nabla function.

$$\left[E^o_{(bIII_c|I_c-)} \right]_\nabla = \left[\begin{pmatrix} Bb & C \\ G & G \\ Eb & Eb \end{pmatrix} \right]_\nabla \tag{D.15}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(bIII_c|I_c-)}) = 2 + 0 + 0 = 2$$

We write the optimal nabla value as a generalization for every tonality. This is the nabla value of an optimal link class.

$$\nabla^o_{(bIII|I)} = 2$$

Now any optimal arrangement from an optimal progresion $E^o_{(bIII|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(bIII_c)) \longrightarrow \psi(I_c-)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Bb_{(z_1-1)} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} \tag{D.16}$$

Then the characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{D.17}$$

Using the properties of the determinant the polynomial it has a root to the right of the stabilizer $E(M)$ and two roots above it that correspond to the two voices that do not change in the link. In this way, the tonal function is visualized as the polynomial of the variable λ that vanishes in the proportionality ratios between voices in the same dimension.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2 - \lambda)(s^0 - \lambda)^2$$

The group stabilizer $E(M)$ determines the collocation of the roots and allows us to classify the roots into three sets:

$$\lambda^- = \{\emptyset\}$$
$$\lambda^0 = \{s^0\}$$
$$\lambda^+ = \{s^2\}$$

We know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(bIII_c)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. In the physical manifestation of each voicing we will have two functions that are sums of trigonometric functions that describe the phenomenon in time.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bIII_c}(t) = \psi_{Bb_{(z_1-1)}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \end{cases}$$

Using the arrow notation we describe the antecedent voicing and the consequent voicing in the way we represent the phenomenon for a selection of integers that determine the opening of the arrangement. This is how we see music in terms of energy and we can understand from a new point, the tonal functions. The functions that the voicings represent are essential to understand the abstract relationship between the voices in conjunction with the physical manifestation of two specific voicings.

$$\psi_{bIII_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bIII_c)} - e^{-2\pi tki\psi_j(bIII_c)}}{2i}$$
$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bIII$ related to the Dorian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem to all keys.

$$\boxed{\Phi[E_{(bIII|I-)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - 2)\ \lambda^2 + (2\,s^2 + 1)\ \lambda - s^2$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2}{3} - \frac{2}{3}\right)\ \lambda^3 + \left(s^2 + \frac{1}{2}\right)\ \lambda^2 + \left(-s^2\right)\ \lambda$

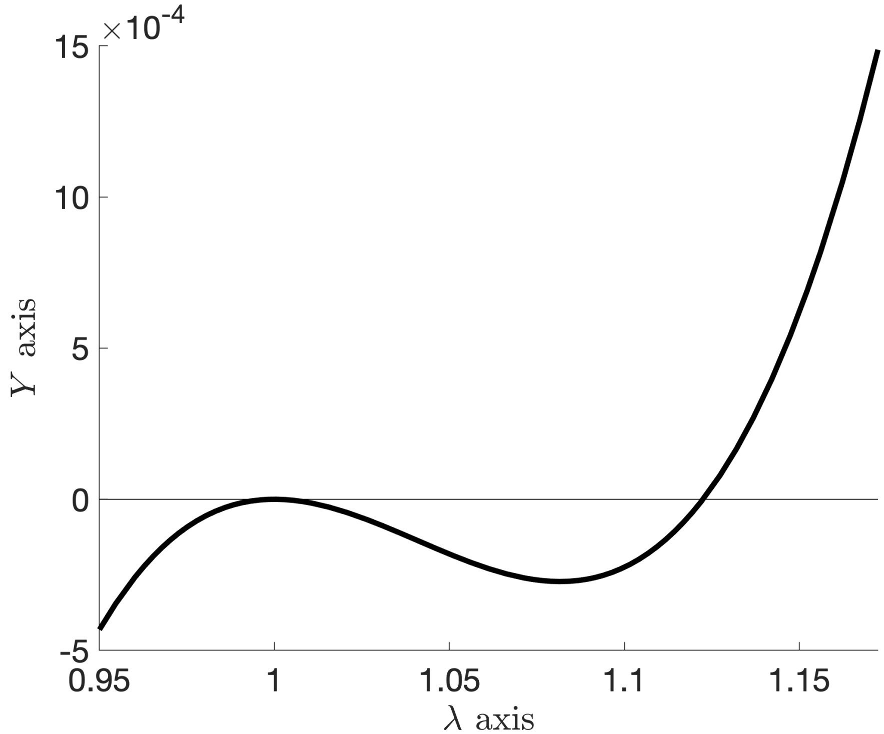


Figure D.3: Characteristic polynomial associated to the bIII-⟶ I- cadence

D.1.6. IV⟶ I- Cadence

We continue working on the identification of the edges of the graph of tonal functions, calculating case by case. Thus, we are going to study how the fourth degree of the Dorian mode is related to the tonic minor that behaves like the first degree in this mode. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(IV^r_c|I^r_c-)} = \begin{pmatrix} C & G \\ A & Eb \\ F & C \end{pmatrix} \tag{D.18}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(IV^r|I^r-)} = \begin{pmatrix} 5 & 3 & 0 \\ 2 & 6 & 3 \\ 2 & 2 & 5 \end{pmatrix} \tag{D.19}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(IV^r|I^r-)} = \begin{pmatrix} 5 & 3 & 0 \\ 2 & 6 & 3 \\ 2 & 2 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r|I^r-)} = \begin{pmatrix} 5 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\longrightarrow L^H_{(IV^r|I^r-)} = \begin{pmatrix} 5 & 3 & \boxed{0} \\ \boxed{0} & 4 & 1 \\ 0 & \boxed{0} & 3 \end{pmatrix}$$

The solutions for $L^H_{(IV^r|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(IV^r|I^r-)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

We calculate the optimal link class where we are already seeing that the voices clearly decrease, then at this point we can already know the value of the perception when the timbre behaves uniformly.

$$\left[E^o_{(IV_c|I_c-)} \right] = \left[\left(\begin{matrix} C & C \\ A & G \\ F & Eb \end{matrix} \right) \right]_\nabla \tag{D.20}$$

We calculate the optimal link class nabla value $\nabla(E^o_{(IV_c|I_c-)}) = 0 + 2 + 2 = 4$ and we write the optimal nabla value as a generalization for every tonality. This is the nabla value of an optimal link class. $\nabla^o_{(IV-|I)} = 4$.

Any optimal arrangement from an optimal progression $E^o_{(IV|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(IV_c)) \longrightarrow \psi(I_c-)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ F_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} \tag{D.21}$$

Then the characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{D.22}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = p_c(\lambda) = (s^0 - \lambda)(s^{-2} - \lambda)^2$$

We now study how the roots of the tonal function are classified and we observe the polarization. As usual, we classify them into three sets in relation to $E(M)$. The sets are respectively $\lambda^+ = \{\emptyset\}$, $\lambda^0 = \{s^0\}$ and $\lambda^- = \{s^{-2}\}$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(IV_c)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. After class optimization we consider the physical expression of the voicings where the function ψ plays a fundamental role.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV_c}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{F_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \end{cases}$$

We symbolize the physical manifestation of the optimal arrangement for an arbitrary harmonic distribution and taking into account that the functions ψ respect the optimization conditions.

$$\psi_{IV_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(IV_c)} - e^{-2\pi tki\psi_j(IV_c)}}{2i}$$

$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

Since the voices descend, are polarized and are placed to the left of $E(M)$, we know how the perception of energy will behave in the optimum. Using the polynomial criterion we classify the tonal function within the dominant area. The function is generalized using the static tonal function theorem to all keys.

$$\boxed{\Phi[E_{(IV|I-)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s^2} - 2\right)\lambda^2 + \left(\frac{2}{s^2} + 1\right)\lambda - \frac{1}{s^2}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s^2+1}{3\,s^2}\right)\lambda^3 + \frac{s^2+2}{2\,s^2}\lambda^2 + \left(-\frac{1}{s^2}\right)\lambda$

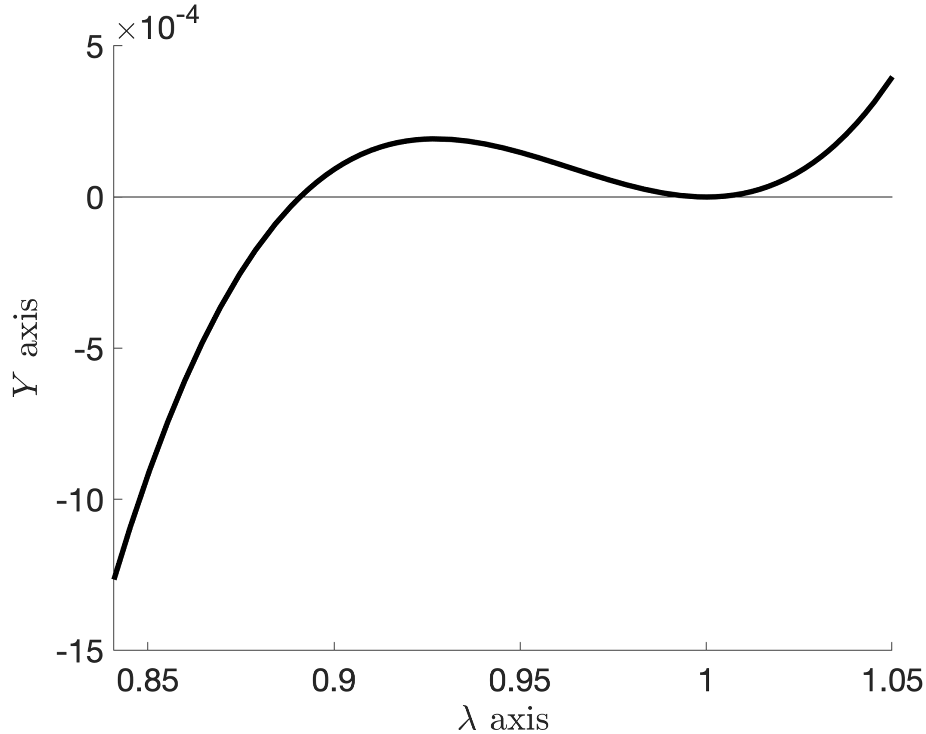


Figure D.4: Characteristic polynomial associated to the IV⟶ I- cadence

D.1.7. V-⟶ I- Cadence

We continue advancing and in this case we calculate the relationship between the fifth grade and the first, both minors. As in the rest of the cases, we want to know the polynomial that relates both tonal centers to know if we can place them on the line, guaranteeing convergence. Thus, we continue the process in an orderly manner. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(V_c^r-|I_c^r-)} = \begin{pmatrix} D & G \\ Bb & Eb \\ G & C \end{pmatrix} \tag{D.23}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix. We build the matrix with each metric, which is itself the minimum distance between classes. For this case we write L as:

$$L_{(V^r-|I^r-)} = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 5 & 2 \\ 0 & 4 & 5 \end{pmatrix} \tag{D.24}$$

Then following the steps of the Hungarian algorithm we consider to develope the algorithm through the L matrix to find an optimum link:

$$L_{(V^r-|I^r-)} = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 5 & 2 \\ 0 & 4 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r-|I^r-)} = \begin{pmatrix} 4 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix} \longrightarrow L^H_{(V^r-|I^r-)} = \begin{pmatrix} 4 & \boxed{0} & 1 \\ 1 & 3 & \boxed{0} \\ \boxed{0} & 4 & 5 \end{pmatrix} \tag{D.25}$$

The solutions for $L^H_{(V-^r|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(V^r-|I^r-)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

By establishing the pairing between the voices we have found an optimum link and immediately, the rest that share the minimum value for the nabla function.

$$\left[E^o_{(V_c-|I_c-)} \right]_\nabla = \left[\begin{pmatrix} D & Eb \\ Bb & C \\ G & G \end{pmatrix} \right]_\nabla \tag{D.26}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(V_c-|I_c-)}) = 1 + 2 + 0 = 3$ and we write the optimal nabla value as a generalization for every tonality. This is the nabla value of an optimal link class. $\nabla^o_{(V-|I)} = 3$

Any optimal arrangement from an optimal progression $E^o_{(V-|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(V_c-)) \longrightarrow \psi(I_c-)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ Bb_{z_2} \\ G_{z_3} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{D.27}$$

Then the characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values l_1, l_2 and l_3. We see that by substituting the values l for each of the metrics with the retrieved sign, we obtain the transformation matrix in the three-dimensional frequency space.

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^1 - \lambda & 0 & 0 \\ 0 & s^2 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{D.28}$$

Due to the properties of the determinant, the polynomial appears already factored, then it remains directly as:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^1 - \lambda)(s^2 - \lambda)(s^0 - \lambda)$$

We consider the placement of the roots based on the stabilizer of the M group. In this case the roots are polarized.

$$\lambda^+ = \{s^1, s^2\}$$
$$\lambda^0 = \{s^0\}$$
$$\lambda^- = \{\emptyset\}$$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(V_c-)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_i}(t)$. In the bracket we have the two functions with the decomposition note by note, in such a way that we see how the frequencies are connected in the optimal arrangement.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V_c-}(t) = \psi_{D_{z_1}}(t) + \psi_{Bb_{z_2-1}}(t) + \psi_{G_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

Using the arrow notation, we arrange the functions that represent each voicing as a sum of trigonometric functions. Thus we have that:

$$\psi_{V_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(V_c-)} - e^{-2\pi tki\psi_j(V_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $V-$ related to the Dorian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem to all keys.

$$\boxed{\Phi[E_{(V-|I-)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - s - 1)\ \lambda^2 + (s^3 + s^2 + s)\ \lambda - s^3$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2}{3} - \frac{s}{3} - \frac{1}{3}\right) \lambda^3 + \frac{s\ (s^2 + s + 1)}{2} \lambda^2 + \left(-s^3\right)\ \lambda$

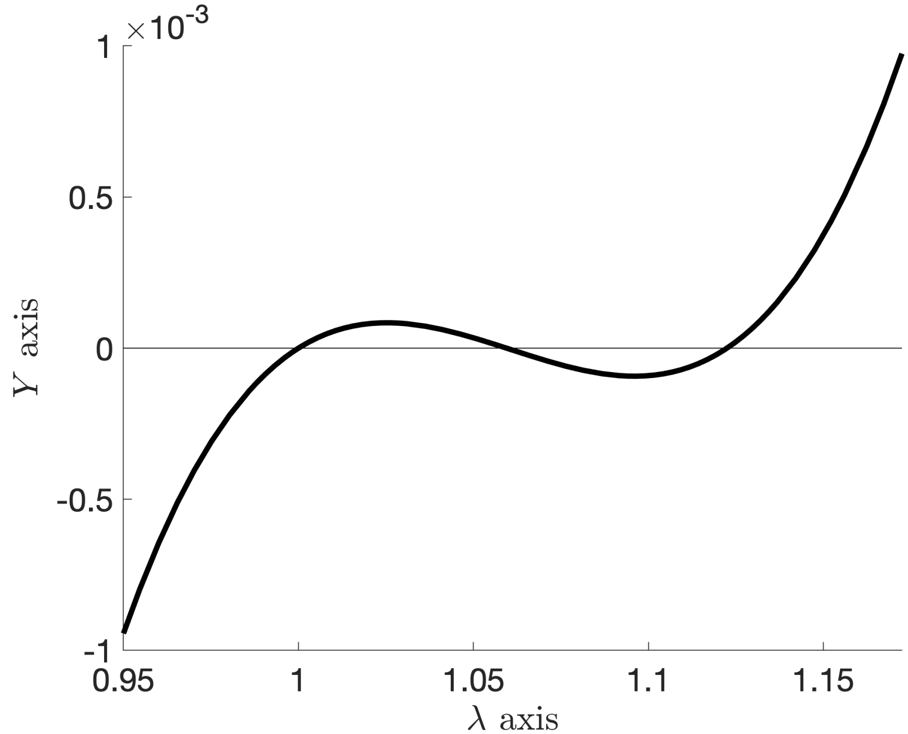


Figure D.5: Characteristic polynomial associated to the V-$\longrightarrow$ I- cadence

D.1.8. VI$\circ$ $\longrightarrow$ I- Cadence

In this case we are going to study the relationship between the first degree of the Dorian mode and the diminished sixth degree. We already see that there are two classes in common between both tonal centers, therefore, according to the polynomial criterion, the only tonal function that the link can have is in the tonic area. However, for prudence and order, we are going to develop the case in an orderly way as we have been doing throughout the previous cases and, also with the aim of providing enough examples to the interested person so that, in his own search, he finds other tonal functions that are of interest. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(VI^r_c o|I^r_c-)} = \begin{pmatrix} Eb & G \\ C & Eb \\ A & C \end{pmatrix} \tag{D.29}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(VI^r o|I^r-)} = \begin{pmatrix} 4 & 0 & 3 \\ 5 & 3 & 0 \\ 2 & 6 & 3 \end{pmatrix} \tag{D.30}$$

Then following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(VI^r o|I^r -)} = \begin{pmatrix} 4 & 0 & 3 \\ 5 & 3 & 0 \\ 2 & 6 & 3 \end{pmatrix} \longrightarrow L^F_{(VI^r o|I^r -)} = \begin{pmatrix} 4 & 0 & 3 \\ 5 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix} \longrightarrow L^H_{(VI^r o|I^r -)} = \begin{pmatrix} 4 & \boxed{0} & 3 \\ 5 & 3 & \boxed{0} \\ \boxed{0} & 4 & 1 \end{pmatrix} \tag{D.31}$$

Then the solutions for $L^H_{(VI^r \circ|I^r -)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(VI^r o|I^r -)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

This is how we have built the set of solutions S where we find the metrics with the sign recovery of the matrix L that correspond to the optimization problem solved. Using these solutions in S we find an optimal binding that we generalize to the nabla class.

$$\left[E^o_{(VI_c o|I_c -)} \right]_\nabla = \left[\begin{pmatrix} Eb & Eb \\ C & C \\ A & G \end{pmatrix} \right]_\nabla \tag{D.32}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(VI_c \circ|I_c -)}) = 0 + 0 + 2 = 2$ and we write the optimal nabla value as a generalization for every tonality. This is the nabla value of an optimal link class. $\nabla^o_{(VI\circ|I-)} = 2$.

Any optimal arrangement from an optimal progression $E^o_{(VIo|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(VI_c o)) \longrightarrow \psi(I_c -)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ A_{z_3} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{D.33}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{D.34}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)(s^0 - \lambda)(s^{-2} - \lambda)$$

As we have been doing, we classify the roots of the tonal function according to their distribution with respect to the stabilizer of the M group. In this way we find three sets that place two voices on the stabilizer and one on its left. In set terminology we describe these three sets as: $\lambda^+ = \{\emptyset\}$, $\lambda^0 = \{s^0\}$ and $\lambda^- = \{s^{-2}\}$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(VI_c o)$ or $\psi(I_c -)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_i}(t)$. We visualize the function that combines each voicing for a chosen set of integers that determine its opening, in this way we would have a bracket that pairs the notes of both voicings following the results of the optimization algorithm.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VI_c o}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{A_{z_3}}(t) \\ \psi_{I_c -}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

We write both functions separated by an arrow that indicates which is the antecedent function and which is the consequent. You can graph both functions separately if you want to study them on a graphing calculator by setting a h for the harmonic distribution of your choice.

$$\psi_{VI_c \circ}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VI_c \circ)} - e^{-2\pi tki\psi_j(VI_c \circ)}}{2i}$$
$$\longrightarrow \psi_{I_c -}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c -)} - e^{-2\pi tki\psi_j(I_c -)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree VIo related to the Dorian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem to all keys.

$$\boxed{\Phi[E_{(VI\circ|I-)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s^2} - 2\right) \lambda^2 + \left(\frac{2}{s^2} + 1\right) \lambda - \frac{1}{s^2}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s^2 + 1}{3\,s^2}\right) \lambda^3 + \frac{s^2 + 2}{2\,s^2} \lambda^2 + \left(-\frac{1}{s^2}\right) \lambda$

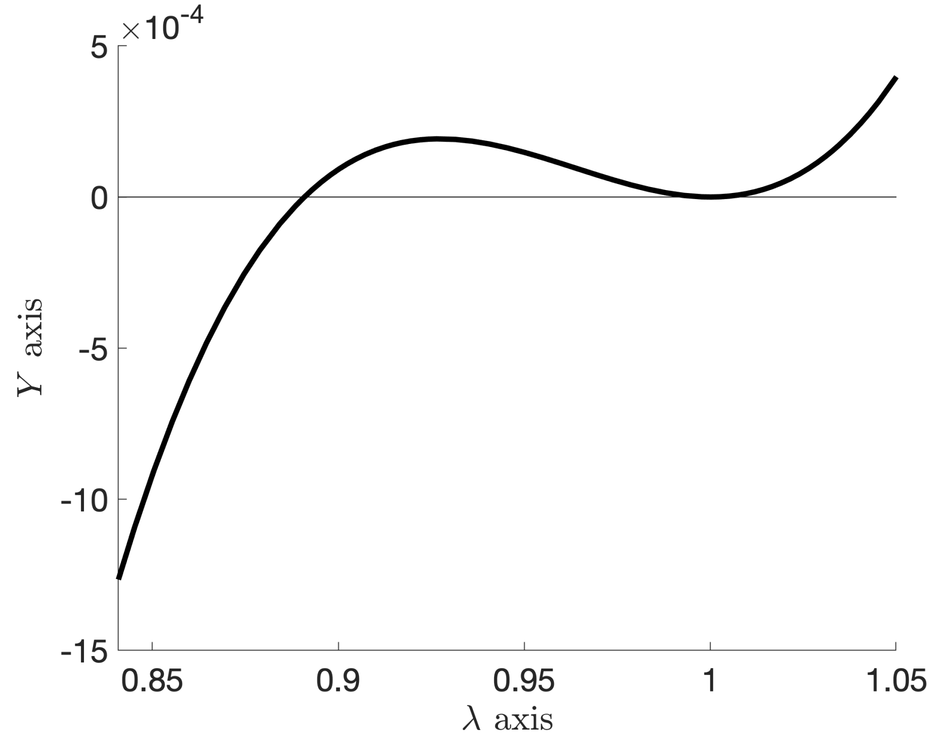


Figure D.6: Characteristic polynomial associated to the VI∘ ⟶ I- cadence

D.1.9. bVII⟶ I- Cadence

We build the link E of the last case for three voices of the Dorian mode. Usually, when we are going to calculate a tonal function, we follow the same process, initially we calculate the link E, then, based on said link, we build L and find the solutions using said matrix. The link will be a progression such as the written below:

$$E_{(bVII^r_c|I^r_c-)} = \begin{pmatrix} F & G \\ D & Eb \\ Bb & C \end{pmatrix} \tag{D.35}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(bVII^r|I^r-)} = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 1 & 2 \\ 3 & 5 & 2 \end{pmatrix} \tag{D.36}$$

Following the steps of the Hungarian algorithm we consider to develope the algorithm through the L matrix to find an optimum link.In this case the solution is unique and the original link E that we had built coincides with the optimal link.

$$L_{(bVII^r|I^r-)} = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 1 & 2 \\ 3 & 5 & 2 \end{pmatrix} \longrightarrow L^F_{(bVII^r|I^r-)} = \begin{pmatrix} 0 & 0 & 3 \\ 4 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix} \longrightarrow L^H_{(bVII^r|I^r-)} = \begin{pmatrix} \boxed{0} & 0 & 3 \\ 4 & \boxed{0} & 1 \\ 1 & 3 & \boxed{0} \end{pmatrix} \tag{D.37}$$

The solutions for $L^H_{(bVII^r|I^r-)}$ when both triads are in root position becomes the set S, that represents the minimum voice leading:

$$S(L^H_{(bVII^r|I^r-)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

Since the link itself with the two structures arranged in the ground state corresponds to the optimal link according to the algorithm, then we have found the pairing between the voices, directly writing the nabla class.

$$\left[E^o_{(bVII_c|I_c-)} \right]_\nabla = \left[\begin{pmatrix} F & G \\ D & Eb \\ Bb & C \end{pmatrix} \right]_\nabla \tag{D.38}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(bVII_c|I_c-)}) = 2 + 1 + 2 = 5$$

We write the optimal nabla value as a generalization for every tonality. This is the nabla value of an optimal link class.

$$\nabla^o_{(bVII|I-)} = 5$$

Any optimal arrangement from an optimal progression $E^o_{(bVII|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$. Thus we propose the endomorphism from the solutions of the matrix L where, implicitly, we are using the transformation T for non-parametric tonal functions, therefore P is the null matrix in the T transformation.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bVII_c)) \longrightarrow \psi(I_c-)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ D_{z_2} \\ Bb_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \end{pmatrix} \tag{D.39}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^1 - \lambda & 0 \\ 0 & 0 & s^2 - \lambda \end{pmatrix} \tag{D.40}$$

Using the properties of the determinant we understand that by simply multiplying the entries of the diagonal of the previous matrix we are able to reach the tonal function, where the only remaining process that we need for its classification is the study of the positioning of the roots with respect to the stabilizer of the M group.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2 - \lambda)(s^1 - \lambda)(s^2 - \lambda)$$

The roots of the previous polynomial are classified into three sets depending on the stabilizer of the cyclic group M. In this case there are no convergent roots nor on the stabilizer.

$$\lambda^- = \{\emptyset\}$$
$$\lambda^0 = \{\emptyset\}$$
$$\lambda^+ = \{s^2, s^1\}$$

If every frequency $\phi \in \Phi^+$ is a component of $\psi(bVII_c)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. Each particular note is represented by a sum of h harmonics, in such a way that the sum of the n notes makes up the representation of a voicing as a wave. The wave functions of each voicing must be coordinated with the result of the optimization of L, then we propose in the bracket that said coordination is given by the equality of subscripts, once we have fixed those that interest us, which in turn time will determine the opening of both voicings.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII_c}(t) = \psi_{F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

The sum of each note is summarized in the wave function of each voicing, where using the arrow notation we indicate the change from one to another. It is clear that despite the variety of forms of representation of the sound phenomenon, the tonal function continues to be a mathematical object that relates both tonal centers transversally, following the results that have been demonstrated in detail in the first chapters of this work.

$$\psi_{bVII_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c)} - e^{-2\pi tki\psi_j(bVII_c)}}{2i}$$
$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bVII$ related to the Dorian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem to all keys.

$$\boxed{\Phi[E_{(bVII|I-)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - 2\,s)\ \lambda^2 + (2\,s^3 + s^2)\ \lambda - s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s\ (s+2)}{3}\right)\ \lambda^3 + \frac{s^2\ (2\,s+1)}{2}\ \lambda^2 + \left(-s^4\right)\ \lambda$

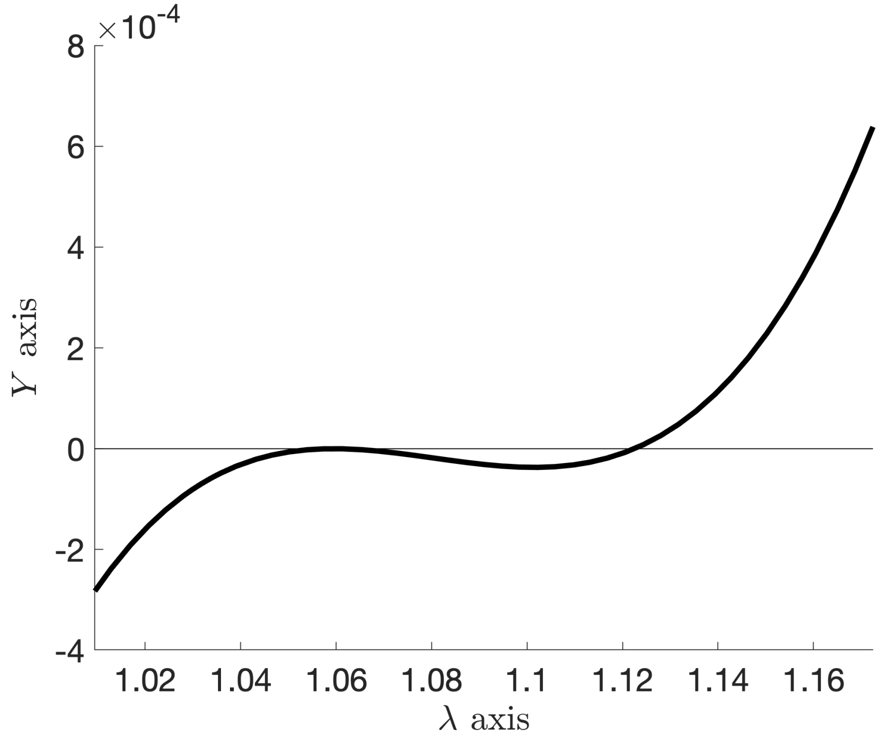


Figure D.7: Characteristic polynomial associated to the bVII⟶ I- cadence

D.2. The Dorian mode for n=4

D.2.1. II-7⟶I-7 Cadence

We now structure the four-voice chord part and focus on studying each case separately using the method. In this way we now study the relationship between two minor chords that are one tone apart. Thus, through the study of the Dorian mode we understand the relationship between these structures. The link will be the following matrix:

$$E_{(II^r_c-7|I^r_c-7)} = \begin{pmatrix} C & Bb \\ A & G \\ F & Eb \\ D & C \end{pmatrix} \tag{D.41}$$

We write the link matrix calculating all the distances Δ_{ij}:

$$L_{(II^r-7|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 1 & 2 & 6 & 3 \\ 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \end{pmatrix} \tag{D.42}$$

Following the steps of the Hungarian algorithm we develop the L matrix. As usual we start from the matrix L. then we write L^F and finally arrive at L^H.

$$L_{(II^r-7|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 1 & 2 & 6 & 3 \\ 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r-7|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 0 & 1 & 5 & 2 \\ 3 & 0 & 0 & 3 \\ 3 & 4 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow L^H_{(II^r-7|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & \boxed{0} \\ \boxed{0} & 1 & 5 & 2 \\ 3 & \boxed{0} & 0 & 3 \\ 3 & 4 & \boxed{0} & 1 \end{pmatrix}$$

The solutions given by the algorithm will be:

$$S(L^H_{(II^r-7|I^r-7)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^{o}_{(II_c-7|I_c-7)}\right]_{\nabla} = \left[\begin{pmatrix} C & C \\ A & Bb \\ F & G \\ D & Eb \end{pmatrix}\right]_{\nabla} \tag{D.43}$$

Using the transformation T that changes from one space to another, we obtain the endomorphism matrix in the frequency space that transforms one vector of frequencies into another, in such a way that we know that the absolute perception takes the minimum when we fix the subscripts that they determine the opening of the voicing itself.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(II_c - 7)) \longrightarrow \psi(I_c - 7)$$
$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ A_{z_2} \\ F_{z_3} \\ D_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ Eb_{z_4} \end{pmatrix} \tag{D.44}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)(s^1 - \lambda)(s^1 - \lambda)(s^0 - \lambda)$$

In this way we visualize the three algebraic multiplicities that serve us, using the polynomial criterion, to understand how the roots of the tonal function are distributed along the λ axis. Thus, we will have three equations that are: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$

We know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(II - 7)$ or $\psi(I - 7)$ for a given tonal center, its clear that the function $\psi_{X_j}(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. Since we have already located how the fundamental frequencies of one voicing are transformed to another when we set an opening for the optimal arrangement, then, using the bracket we have two functions, which following the optimization conditions, represent each of each voicing for a specific selection of subscripts.

$$\mathcal{B}_{\epsilon} = \begin{cases} \psi_{II-7}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{F_{z_3}} + \psi_{D_{z_4}}(t) \\ \psi_{I-7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{Eb_{z_4}}(t) \end{cases}$$

As in other cases, the wave function of each voicing is provided, separated by an arrow that indicates their relationship in time as isolated phenomena.

$$\psi_{II-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(II-7)} - e^{-2\pi tki\psi_j(II-7)}}{2i}$$
$$\longrightarrow \psi_{I7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then following the polynomial criterion we obtain the function of the degree $II-7$ related to the mixolydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(II-7|I-7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s^2 - 2\,s - 1)\ \lambda^3 + (2\,s^3 + 2\,s^2 + 2\,s)\ \lambda^2 + (-s^4 - 2\,s^3 - s^2)\ \lambda + s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{(s+1)^2}{4}\right)\ \lambda^4 + \frac{s\ (s^2+s+1)\ 2}{3}\ \lambda^3 + \left(-\frac{s^2\ (s+1)^2}{2}\right)\ \lambda^2 + s^4\ \lambda$

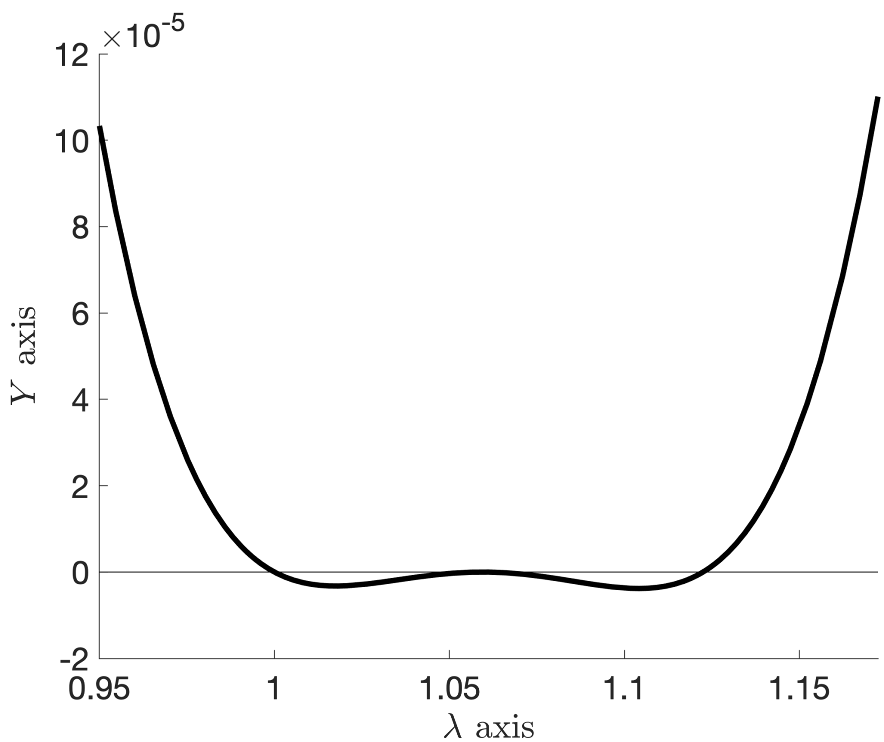


Figure D.8: Characteristic polynomial associated to the II-7⟶I-7 cadence

D.2.2. bIII△ ⟶I-7 Cadence

Throughout this section we are going to study the relationship between the third degree of the mixolydian mode and its first degree. As usual we will generate a link E and taking metrics we will build the distance matrix L. It is clear that L is enough for us to apply the transformation T and reach the endomorphism matrix. We follow the process in an orderly fashion as usual and start with the E link. The link will be the following matrix:

$$E_{(bIII^r_c\triangle|I^r_c-7)} = \begin{pmatrix} D & Bb \\ Bb & G \\ G & Eb \\ Eb & C \end{pmatrix} \tag{D.45}$$

We calculate the link matrix calculating all the distances Δ_{ij}. Note that depending on the placement of the classes in E we are going to obtain a matrix L or another that is related by permutation of rows and columns. To maintain consistency in the method we chose to place the chords by ascending thirds, getting the following matrix.

$$L_{(bIII^r\triangle|I^r7)} = \begin{pmatrix} 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix} \tag{D.46}$$

Following the steps of the Hungarian algorithm we develope the L matrix:

$$L_{(bIII^r\triangle|I^r-7)} = \begin{pmatrix} 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix} \longrightarrow L^F_{(bIII^r\triangle|I^r-7)} = \begin{pmatrix} 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix}$$

$$\longrightarrow L^H_{(bIII^r\triangle|I^r7)} = \begin{pmatrix} 3 & 4 & 0 & \boxed{0} \\ \boxed{0} & 3 & 5 & 1 \\ 3 & \boxed{0} & 4 & 4 \\ 5 & 4 & \boxed{0} & 2 \end{pmatrix}$$

With the previous distribution of boxes, we have found the optimal pairing of the voices and what metrics are associated with the voices. Thus, the set S will be the set of entries of the matrix L that corresponds to the metrics of an optimal link. We write S without recovering the sign, since we will do this later.

$$S(L^H_{(bIII^r\triangle|I^r-7)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^{o}_{(bIII_c\triangle|I_c-7)}\right]_{\nabla} = \left[\begin{pmatrix} D & C \\ Bb & Bb \\ G & G \\ Eb & Eb \end{pmatrix}\right]_{\nabla} \tag{D.47}$$

We can use the transformation T implicitly to reach the transformation matrix between frequency vectors, which themselves are an abstraction of the physical manifestation of a voicing. We also call these vectors voicings as described above. Thus, said matrix allows us to study the movement of the voices when $\mid p \mid$ reaches the minimum. Formally, we write the endomorphism as:

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(II_c - 7)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{-\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ Eb_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ Eb_{z_4} \end{pmatrix} \tag{D.48}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id. We see that the metric between the voices of the first dimension has recovered the sign , so we are using T implicitly. On the other hand we see that we have not encrypted the dimension of the identity matrix since it is clearly seen by the context, so we will not always specify the dimension of said matrix.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)^3(s^{-2} - \lambda)$$

Studying the tonal function we see that the distribution of the roots is determined by the values of the algebraic multiplicities. Thus we would have the convergent algebraic multiplicity, the static and the divergent, where, for this case, the last one is null $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$. On the other hand, we also want to know how the wave functions of the antecedent and consequent voicing behave, then it is enough for us to know the frequencies of the first voicing, which we will assign by deciding on a particular opening, for, later, using the solutions algorithm, calculate those of the second voicing under the optimization conditions. It is clear that the function $\psi_{X_j}(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

For an assumed distribution of harmonics Γ. We have in each equation the decomposition note by note of the wave function of each voicing. Thus, each note will be the sum of trigonometric functions of a fundamental (depending on the opening of the voicing), and of all its harmonics, depending on Γ.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{III_c\triangle}(t) = \psi_{C_{z_1}}(t) + \psi_{A_{z_2}}(t) + \psi_{F_{z_3}} + \psi_{D_{z_4}}(t) \\ \psi_{I_c-7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{Eb_{z_4}}(t) \end{cases}$$

Understanding that the function ψ_j assigns the indicated frequency in each case following the indications of the brackets, and respects the optimization conditions for a set of integers, we have the representation as wave functions of the antecedent voicing and the consequent voicing.

$$\psi_{bIII_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k\frac{e^{2\pi tki\psi_j(bIII_c\triangle)} - e^{-2\pi tki\psi_j(bIII_c\triangle)}}{2i}$$
$$\longrightarrow \psi_{I_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k\frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$ then following the polynomial criterion we obtain the function of the degree $bIII\triangle$ related to the mixolydian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(bIII\triangle|I-7)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s^2} - 3\right)\lambda^3 + \left(\frac{3}{s^2} + 3\right)\lambda^2 + \left(-\frac{3}{s^2} - 1\right)\lambda + \frac{1}{s^2}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{3\,s^2+1}{4\,s^2}\right)\lambda^4 + \frac{3\,s^2+3}{3\,s^2}\lambda^3 + \left(-\frac{s^2+3}{2\,s^2}\right)\lambda^2 + \frac{\lambda}{s^2}$

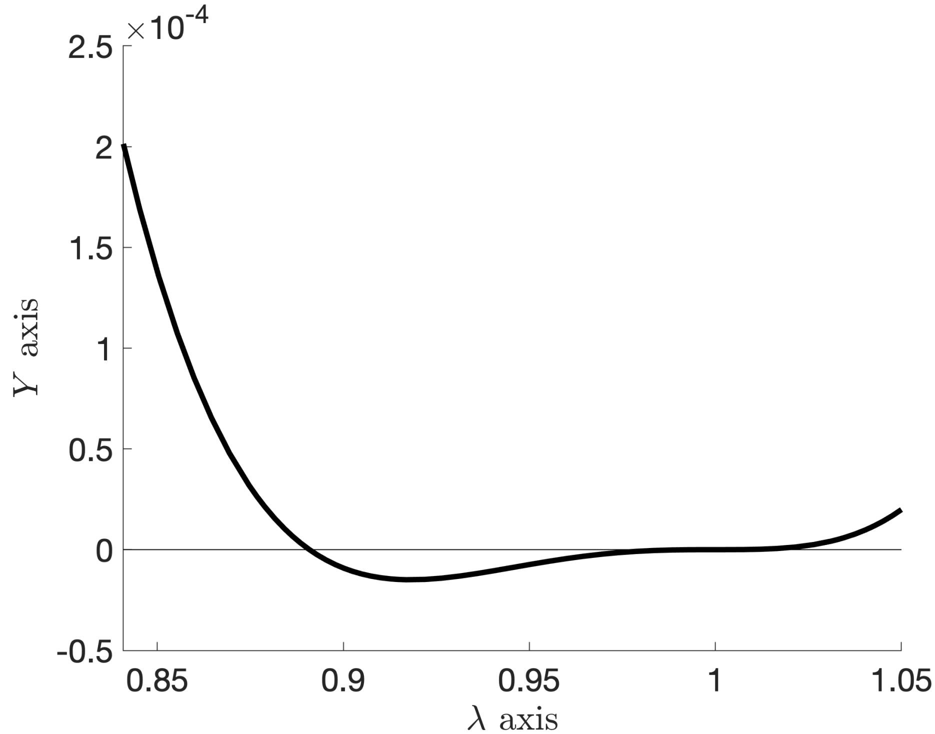


Figure D.9: Characteristic polynomial associated to the bIII-7⟶I-7 cadence

D.2.3. IV7⟶I-7 Cadence

We study the following cadential process building the link between both structures to later build a matrix L on which to apply the optimization algorithm. An E link would be given by the classes matrix that in each column contains the classes of the antecedent chord and the consequent chord, that is, it can be seen as a progression P with two columns where P does not have to be optimal. Recall that P is optimal if and only if every link is optimal in P. The link will be the following matrix:

$$E_{(IV_c^r 7|I_c^r-7)} = \begin{pmatrix} Eb & Bb \\ C & G \\ A & Eb \\ F & C \end{pmatrix} \tag{D.49}$$

Then, we calculate the link matrix calculating all the distances Δ_{ij}. Once we have chosen a link E, we take the metrics in order to generate a matrix L.

$$L_{(IV^r 7|I^r-7)} = \begin{pmatrix} 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 1 & 2 & 6 & 3 \\ 5 & 2 & 2 & 5 \end{pmatrix} \tag{D.50}$$

Following the steps of the Hungarian algorithm we develop the L matrix. In this case we only find a distribution of boxes on the matrix L^H, therefore the solution is unique.

$$L_{(IV^r7|I^r-7)} = \begin{pmatrix} 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 1 & 2 & 6 & 3 \\ 5 & 2 & 2 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r7|I^r-7)} = \begin{pmatrix} 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 0 & 1 & 5 & 2 \\ 3 & 0 & 0 & 3 \end{pmatrix}$$
$$\longrightarrow L^H_{(IV^r7|I^r-7)} = \begin{pmatrix} 5 & 4 & \boxed{0} & 3 \\ 2 & 5 & 3 & \boxed{0} \\ \boxed{0} & 1 & 5 & 2 \\ 3 & \boxed{0} & 0 & 3 \end{pmatrix}$$

The positions of the boxes are collected in the set S where we see that each class is optimally assigned another.

$$S(L^H_{(IV^r7|I^r-7)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E. In this way we fix the first chord and using the results of the algorithm we place the classes of the second forming an optimal link. Since the solution is unique, it is enough for us to take square brackets with the nabla subscript to obtain the nabla class of all the optimal links between the pair of chords that we are studying.

$$\left[E^o_{(IV_c7|I_c-7)} \right]_\nabla = \left[\begin{pmatrix} Eb & Eb \\ C & C \\ A & Bb \\ F & G \end{pmatrix} \right]_\nabla \tag{D.51}$$

Using the transformation T we reach the matrix $C_{\mathbb{E}}$ that transforms one voicing into another in the four-dimensional frequency space.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(IV_c7)) \longrightarrow \psi(I_c-7)$$
$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ A_{z_3} \\ F_{z_4} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} \tag{D.52}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id. In this way we have two roots on the stabilizer $E(M)$ and another two to its right, indicating the ascending movement of the voices in the optimum. We translate this into the language of algebraic multiplicities to study the equations.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)^2 (s^1 - \lambda)(s^2 - \lambda)$$

The algebraic multiplicities are summarized in three formulas whose sum coincides with the dimension of the space in which we are working, so we have that $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(IV7)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We describe in the bracket the two wave functions of each voicing, distinguishing each fundamental frequency.

For the sake of completeness and to provide the reader with several points of view, we write the wave functions of the voicings when we have minimized the absolute perception where the functions of the voicings follow the optimization conditions for a set of subscripts and the distribution Γ is an arbitrary harmonic distribution identical for each function.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{A_{z_3}} + \psi_{F_{z_4}}(t) \\ \psi_{I-7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

$$\psi_{IV7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(IV7)} - e^{-2\pi tki\psi_j(IV7)}}{2i}$$
$$\longrightarrow \psi_{I-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$ then following the polynomial criterion we obtain the function of the degree $IV7$ related to the Dorian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(IV7|I-7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s^2 - s - 2)\ \lambda^3 + (s^3 + 2\,s^2 + 2\,s + 1)\ \lambda^2 + (-2\,s^3 - s^2 - s)\ \lambda + s^3$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{4} - \frac{s}{4} - \frac{1}{2}\right) \lambda^4 + \left(\frac{s^3}{3} + \frac{2\,s^2}{3} + \frac{2\,s}{3} + \frac{1}{3}\right) \lambda^3 + \left(-\frac{s\ (2\,s^2 + s + 1)}{2}\right) \lambda^2 + s^3\,\lambda$

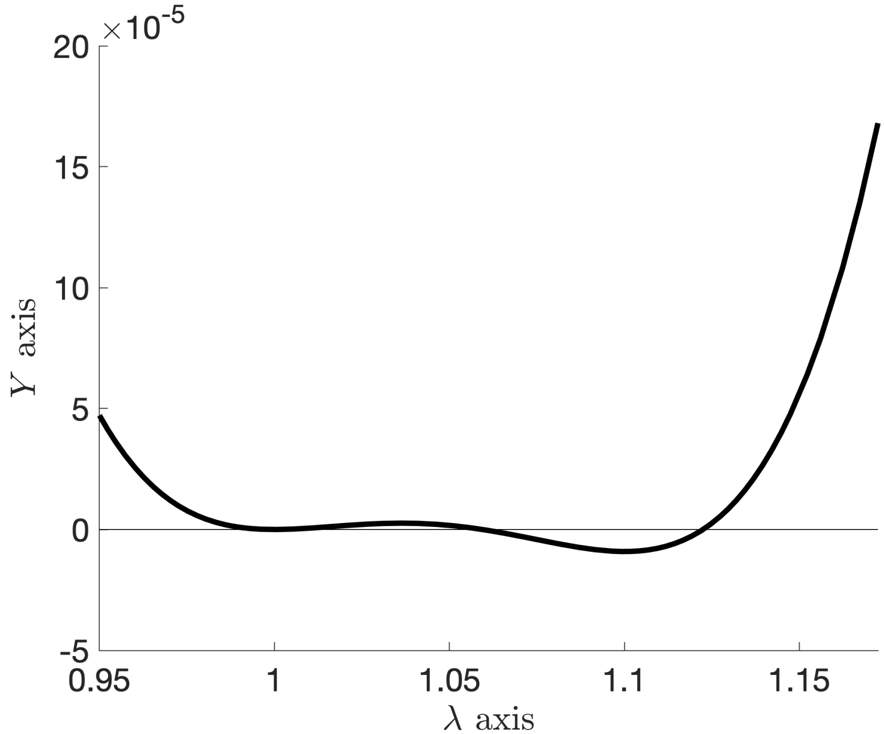


Figure D.10: Characteristic polynomial associated to the IV7⟶I-7 cadence

D.2.4. V-7⟶I-7 Cadence

The relationship between minor chords a fifth apart is very common. We are going to formalize how both tonal centers are related through the Hungarian algorithm. The link will be represented by the following matrix:

$$E_{(V_c^r-7|I_c^r-7)} = \begin{pmatrix} F & Bb \\ D & G \\ Bb & Eb \\ G & C \end{pmatrix} \tag{D.53}$$

We calculate the link matrix calculating all the distances Δ_{ij}. We have built L in the usual way, taking each class of the first chord and relating it to each class of the second, going down each column. This is the way we have to build L, although there are others that solve the problem in an equivalent way.

$$L_{(V^r-7|I^r-7)} = \begin{pmatrix} 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \end{pmatrix} \tag{D.54}$$

We develop the steps of the algorithm through the matrices L, L^F and L^H. We reach the solution, which in this case is a single distribution of boxes. We note the positions of the boxes to know the value of the sum of metrics between voices of the same dimension when the link is optimal.

$$L_{(V^r-7|I^r-7)} = \begin{pmatrix} 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r-7|I^r-7)} = \begin{pmatrix} 3 & 0 & 0 & 3 \\ 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \end{pmatrix}$$

$$\longrightarrow L^H_{(V^r-7|I^r7)} = \begin{pmatrix} 3 & 0 & \boxed{0} & 2 \\ 3 & 4 & 0 & \boxed{0} \\ \boxed{0} & 3 & 5 & 1 \\ 3 & \boxed{0} & 4 & 4 \end{pmatrix}$$

The solutions given by the algorithm will be:

$$S(L^H_{(V^r-7|I^r-7)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(V_c-7|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} F & Eb \\ D & C \\ Bb & Bb \\ G & G \end{pmatrix}\right]_\nabla \tag{D.55}$$

Then, selecting one of the cadences of the dimensionally optimal class we have a cadence wich is related to the hipervolume that has the information related to the cadential process:

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(V_c - 7)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ D_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Bb_{z_2} \\ G_{z_3} \end{pmatrix} \tag{D.56}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id_4.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-2} - \lambda)^2 (s^0 - \lambda)^2$$

We see that in this case we have two voices that are going down in the optimal link, and two voices that remain invariant. The polynomial provides us with this information. We will now study algebraic multiplicities. Thus we can apply the polynomial criterion and clarify the cadential process.

Each of the multiplicities of the above polynomial is specified by the following formulas: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(V-7)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We write the wave functions of both voicings in such a way that they are specified for an arbitrary set of subscripts that determine the opening of the voicings. For the arrangement to be optimal, the frequencies of the antecedent and consequent voicing must be linked, which motivates the use of the bracket in this section.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V-7}(t) = \psi_{F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \\ \psi_{I-7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

We write the formula as usual for the wave functions of each of the voicings.

$$\psi_{V-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(V-7)} - e^{-2\pi tki\psi_j(V-7)}}{2i}$$

$$\longrightarrow \psi_{I-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$ then following the polynomial criterion we obtain the function of the degree $V-7$ related to the Dorian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(V-7|I-7)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{2}{s^2} - 2\right) \lambda^3 + \left(\frac{4}{s^2} + \frac{1}{s^4} + 1\right) \lambda^2 + \left(-\frac{2}{s^2} - \frac{2}{s^4}\right) \lambda + \frac{1}{s^4}$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2+1}{2\,s^2}\right) \lambda^4 + \frac{s^4 + 4\,s^2 + 1}{s^4\,3} \lambda^3 + \left(-\frac{2\,s^2+2}{2\,s^4}\right) \lambda^2 + \frac{\lambda}{s^4}$

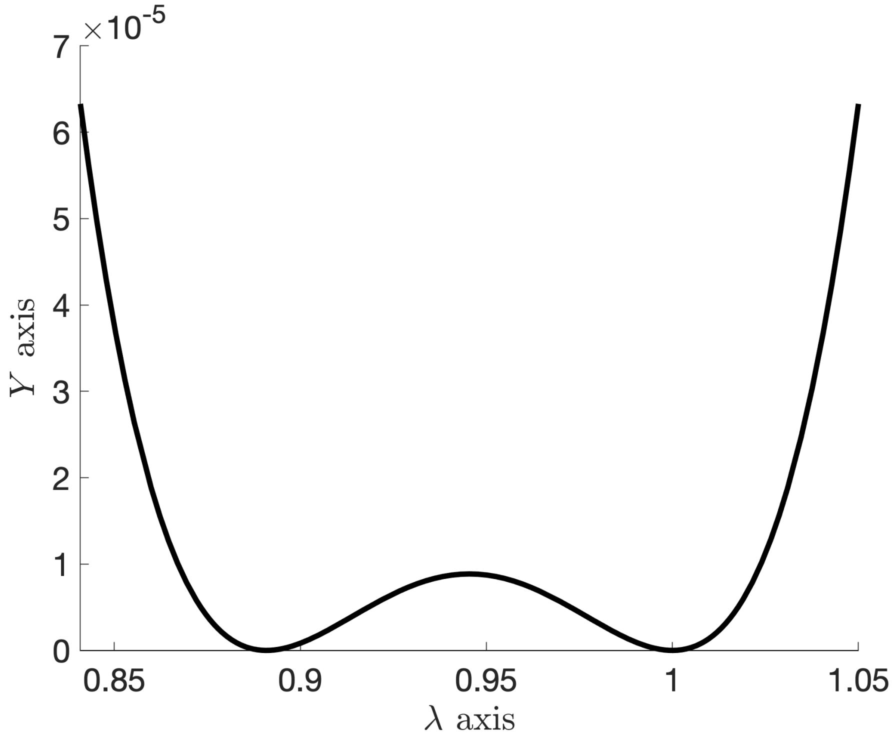


Figure D.11: Characteristic polynomial associated to the V-7⟶I-7 cadence

D.2.5. VIø7⟶I-7 Cadence

Without exception for the half-diminished chord, we carry out the same procedure to study the convergence of said tonal center on the following one. The link will be represented by the following matrix:

$$E_{(VI^r_c\o 7|I^r_c-7)} = \begin{pmatrix} G & Bb \\ Eb & G \\ C & Eb \\ A & C \end{pmatrix} \tag{D.57}$$

We calculate the link matrix calculating all the distances Δ_{ij}. We construct the L matrix in the usual way.

$$L_{(VI^r\o 7|I^r 7)} = \begin{pmatrix} 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & 6 & 3 \end{pmatrix} \tag{D.58}$$

From the matrix L we build the solutions as we have been doing throughout the work. In this case, the solution is unique and we have not had to use infinite arithmetic or any special procedure, we simply apply the steps of the algorithm on the matrix that we have built and continue until we find a distribution of boxes.

$$L_{(VI^r\o 7|I^r-7)} = \begin{pmatrix} 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 1 & 2 & 6 & 3 \end{pmatrix} \longrightarrow L^F_{(VI^r\o 7|I^r-7)} = \begin{pmatrix} 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 0 & 1 & 5 & 2 \end{pmatrix}$$
$$\longrightarrow L^H_{(VI^r\o 7|I^r-7)} = \begin{pmatrix} 3 & \boxed{0} & 4 & 5 \\ 5 & 4 & \boxed{0} & 3 \\ 2 & 5 & 3 & \boxed{0} \\ \boxed{0} & 1 & 5 & 2 \end{pmatrix}$$

The solutions given by the algorithm will be contained in the S set.

$$S(L^H_{(V^r\o 7|I^r-7)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(VI_c\o 7|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} G & G \\ Eb & Eb \\ C & C \\ A & Bb \end{pmatrix}\right]_\nabla \tag{D.59}$$

Once we have found the optimal link, we implicitly use the T transformation and reach the endomorphism matrix. By taking the polynomial we specify the movement of the voices in the optimum using the language of the polynomials.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(VI_c\o 7)) \longrightarrow \psi(I_c - 7)$$
$$\begin{pmatrix} s^{\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ A_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{D.60}$$

The polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id_4.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)^3 (s^1 - \lambda)$$

It is important that once we have calculated the characteristic polynomial, we worry about writing the equations of the algebraic multiplicities that we have defined at the beginning. So these would look like: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 1$.

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(VI\o7)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_{X_j}(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We now assume that the physical expression of each voicing is going to be each of the wave functions that we write inside the brackets. We have used the midi notation in order to indicate which particular frequency is written when we choose a particular subscript.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{VI\o7}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{A_{z_4}}(t) \\ \psi_{I-7}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Bb_{z_4}}(t) \end{cases}$$

As usual, we symbolize the change from the first wave function to the second in such a way that we have the equations of both voicings separated by an arrow.

$$\psi_{VI\o7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(VI\o7)} - e^{-2\pi tki\psi_j(VI\o7)}}{2i}$$
$$\longrightarrow \psi_{I-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $VI\o7$ related to the Dorian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E_{(VI\o7|I-7)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s-3)\ \lambda^3 + (3\,s+3)\ \lambda^2 + (-3\,s-1)\ \lambda + s$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s}{4} - \frac{3}{4}\right)\ \lambda^4 + (s+1)\ \lambda^3 + \left(-\frac{3\,s}{2} - \frac{1}{2}\right)\ \lambda^2 + s\,\lambda$

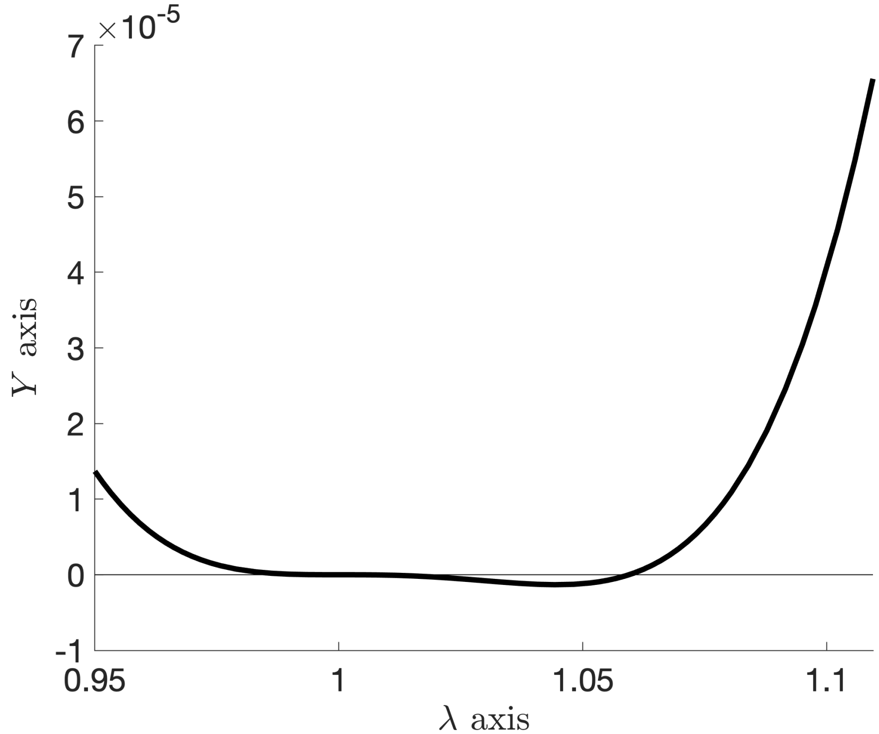


Figure D.12: Characteristic polynomial associated to the VI∅7⟶I-7 cadence

D.2.6. bVII△ ⟶I-7 Cadence

The cadence that we are going to develop now is very common in jazz music and other genres. There are even entire themes whose main section is structured by these two chords or around these two. We will understand when developing the case mathematically what is the justification for the use of this progression and why its popularity is due. As the tonal function is dual and the solutions are in complementary areas then we can move from one chord to another guaranteeing convergence in both the bond E and the retrograde link.

We calculate the link for this cadence to optimize it. The link will be a progression such as:

$$E_{(bVII_c^r\triangle|I_c^r-7)} = \begin{pmatrix} A & Bb \\ F & G \\ D & Eb \\ Bb & C \end{pmatrix} \tag{D.61}$$

The link cadence will be constructed using the ordered collection of metrics Δ_{ij}:

$$L_{(bVII^r\triangle|I^r-7)} = \begin{pmatrix} 1 & 2 & 6 & 3 \\ 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \end{pmatrix} \tag{D.62}$$

Following the steps of the Hungarian algorithm we develope the L matrix until we reach to the S set:

$$L_{(bVII^r\triangle|I^r-7)} = \begin{pmatrix} 1 & 2 & 6 & 3 \\ 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \end{pmatrix} \longrightarrow L^F_{(bVII^r\triangle|I^r-7)} = \begin{pmatrix} 0 & 1 & 5 & 2 \\ 3 & 0 & 0 & 3 \\ 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 2 \end{pmatrix}$$

$$\longrightarrow L^H_{(bVII^r\triangle|I^r-7)} = \begin{pmatrix} 0 & 1 & 5 & 1 \\ 3 & 0 & 0 & 2 \\ 3 & 4 & 0 & 0 \\ 0 & 3 & 5 & 1 \end{pmatrix}$$

We carry out all the operations and since L^H has no solution, we follow the steps of the Hungarian algorithm, covering the zeros with the minimum number of lines, in this case represented by cancellations on the inputs. After having covered the zeros with the minimum number of lines, we look for the minimum uncovered and we are to the uncovered entries, we add it to the doubly covered ones and leave invariant those entries that are only covered by one line. Once we obtain the L^H matrix, we find the non canceled minimum and we operate until we reach L^{H^*}:

$$L^H_{(bVII^r\triangle|I^r-7)} = \begin{pmatrix} 0 & 1 & 5 & 1 \\ 3 & 0 & 0 & 2 \\ 3 & 4 & 0 & 0 \\ 0 & 3 & 5 & 1 \end{pmatrix} \longrightarrow L^{H^*}_{(bVII^r\triangle|I^r-7)} = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 4 & 0 & 0 & 2 \\ 4 & 4 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{pmatrix} \tag{D.63}$$

After this process we calculate $L^{H^*} = L^{H^*}_{(bVII^r\triangle|I^r-7)}$ and we assign a starting zero. This is how, using the Zero method, we are going to find all those distributions of boxes on the new matrix that make up a solution. Forcing each zero we will find, in this case, three solutions.

$$L^{H^*}_1 = \begin{pmatrix} \boxed{0}^* & 0 & 4 & 0 \\ 4 & \boxed{0} & 0 & 2 \\ 4 & 4 & \boxed{0} & 0 \\ 0 & 2 & 4 & \boxed{0} \end{pmatrix} \mid L^{H^*}_2 = \begin{pmatrix} 0 & \boxed{0}^* & 4 & 0 \\ 4 & 0 & \boxed{0} & 2 \\ 4 & 4 & 0 & \boxed{0} \\ \boxed{0} & 2 & 4 & 0 \end{pmatrix} \mid L^{H^*}_3 = \begin{pmatrix} 0 & 0 & 4 & \boxed{0}^* \\ 4 & \boxed{0} & 0 & 2 \\ 4 & 4 & \boxed{0} & 0 \\ \boxed{0} & 2 & 4 & 0 \end{pmatrix} \tag{D.64}$$

$$L^{H^*}_4 = \begin{pmatrix} 0 & 0 & 4 & \boxed{0} \\ 4 & \boxed{0}^* & 0 & 2 \\ 4 & 4 & \boxed{0} & 0 \\ \boxed{0} & 2 & 4 & 0 \end{pmatrix} \mid L^{H^*}_5 = \begin{pmatrix} 0 & \boxed{0} & 4 & 0 \\ 4 & 0 & \boxed{0}^* & 2 \\ 4 & 4 & 0 & \boxed{0} \\ \boxed{0} & 2 & 4 & 0 \end{pmatrix} \mid L^{H^*}_6 = \begin{pmatrix} 0 & 0 & 4 & \boxed{0} \\ 4 & \boxed{0} & 0 & 2 \\ 4 & 4 & \boxed{0}^* & 0 \\ \boxed{0} & 2 & 4 & 0 \end{pmatrix} \tag{D.65}$$

$$L_7^{H^*} = \begin{pmatrix} 0 & \boxed{0} & 4 & 0 \\ 4 & 0 & \boxed{0} & 2 \\ 4 & 4 & 0 & \boxed{0}^* \\ \boxed{0} & 2 & 4 & 0 \end{pmatrix} \mid L_8^{H^*} = \begin{pmatrix} 0 & \boxed{0} & 4 & 0 \\ 4 & 0 & \boxed{0} & 2 \\ 4 & 4 & 0 & \boxed{0} \\ \boxed{0}^* & 2 & 4 & 0 \end{pmatrix} \mid L_9^{H^*} = \begin{pmatrix} \boxed{0} & 0 & 4 & 0 \\ 4 & \boxed{0} & 0 & 2 \\ 4 & 4 & \boxed{0} & 0 \\ 0 & 2 & 4 & \boxed{0}^* \end{pmatrix} \tag{D.66}$$

Then the solutions for $L^{H^*}_{(VII^r-7|I^r\triangle)}$ when both chords are in root position are described by three sets, each one representing a link where $\nabla(E)$ is minimal for al the permutations of the link:

$$S_1(L^{H^*}_{(bVII^r\triangle|I^r-7)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}, \Delta_{44}\}$$
$$S_2(L^{H^*}_{(bVII^r\triangle|I^r-7)}) = \{\Delta_{14}, \Delta_{22}, \Delta_{33}, \Delta_{41}\}$$
$$S_3(L^{H^*}_{(bVII^r\triangle|I^r-7)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

Since we have found three sets S of solutions, then, by conservation of order, we are going to develop each case separately. So, we'll start from the same set and try to find the tonal function using the transformation T implicitly. Once we have finished we will continue with the next set S to finally find the union of areas where the multiple tonal functions appear.

D.2.7. $S_1(L^{H}_{(bVII^r\triangle|I^r-7))})$

Using the solutions of the first set we match the classes of each of the tonal centers and find an optimal link E that immediately connects with its nabla class. Thus we visualize how the voices behave for this first solution in the optimal link.

$$\left[E^1_{(bVII_c\triangle|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} A & Bb \\ F & G \\ D & Eb \\ Bb & C \end{pmatrix}\right]_\nabla \tag{D.67}$$

Using the transformation T we connect the first solution of the matrix L with the matrix of endomorphism, with the certainty that the metric minimization that we have carried out is minimizing the absolute perception between structures with the same timbre, under the conditions established.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(bVII_c\triangle)) \longrightarrow \psi(I_c - 7)$$
$$\begin{pmatrix} s^{\Delta_{11}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{22}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{33}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{44}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ F_{z_2} \\ D_{z_3} \\ Bb_{z_4} \end{pmatrix} = \begin{pmatrix} Bb_{z_1} \\ G_{z_2} \\ Eb_{z_3} \\ C_{z_4} \end{pmatrix} \tag{D.68}$$

The nabla value is calculated for any pair of frequencies that $\Omega[\int_{\alpha}^{\beta} \phi^{-1} d\phi] \leq 6$ where $\alpha \in [\alpha]$ and $\beta \in [\beta]$. The delta subindex indicates that the integral is the minimum of the delta set Δ.

$$\nabla(E^1) = \sum_{j=1}^{n} \Omega \mid \int_{E^1_{j1\,(bVII_c\triangle|I_c-7)}}^{E^1_{j2\,(bVII_c\triangle|I_c-7)}} \phi^{-1}\, d\phi \mid_{\Delta} = 6$$

We calculate the function nabla of the first optimized link, where the subscript Δ indicates the minimum distance between classes, in such a way that this function is the sum of the minimum distances between classes when the link is optimal. For the first solution provided by the Zero method, we find a first tonal function whose roots are all placed to the right of the M group stabilizer. In this way we have found the first one that would already be factored appropriately.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)^2 (s^1 - \lambda)^2$$

We now calculate the algebraic multiplicities of the first polynomial in order to apply the polynomial criterion and determine in which area the tonal function is located: $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 4$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 0$

Since we already know how the voices are transformed in the optimum, then knowing a fixed tuning, we can determine the wave functions of the voicings, where depending on the desired opening, to obtain an optimal arrangement, we will choose one set of subscripts or another.

$$\mathcal{B}_{\epsilon} = \begin{cases} \psi_{bVII\triangle}(t) = \psi_{A_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I-7}(t) = \psi_{Bb_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

We write the wave functions of each voicing where each voicing depends on the set of subscripts.

$$\psi_{bVII\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII\triangle)} - e^{-2\pi tki\psi_j(bVII\triangle)}}{2i}$$
$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 4$ then following the polynomial criterion we obtain the one function of the degree $bVII\triangle$ related to the Dorian tonal center. In this case it can be represented by the polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E^1_{(bVII\triangle|I-7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-2\,s^2 - 2\,s\right)\,\lambda^3 + \left(s^4 + 4\,s^3 + s^2\right)\,\lambda^2 + \left(-2\,s^5 - 2\,s^4\right)\,\lambda + s^6$

Integral of $p_{C_{\mathbb{E}}}(\lambda)$: $\int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s\,(s+1)}{2}\right)\,\lambda^4 + \frac{s^2\,(s^2+4\,s+1)}{3}\,\lambda^3 + \left(-s^4\,(s+1)\right)\,\lambda^2 + s^6\,\lambda$

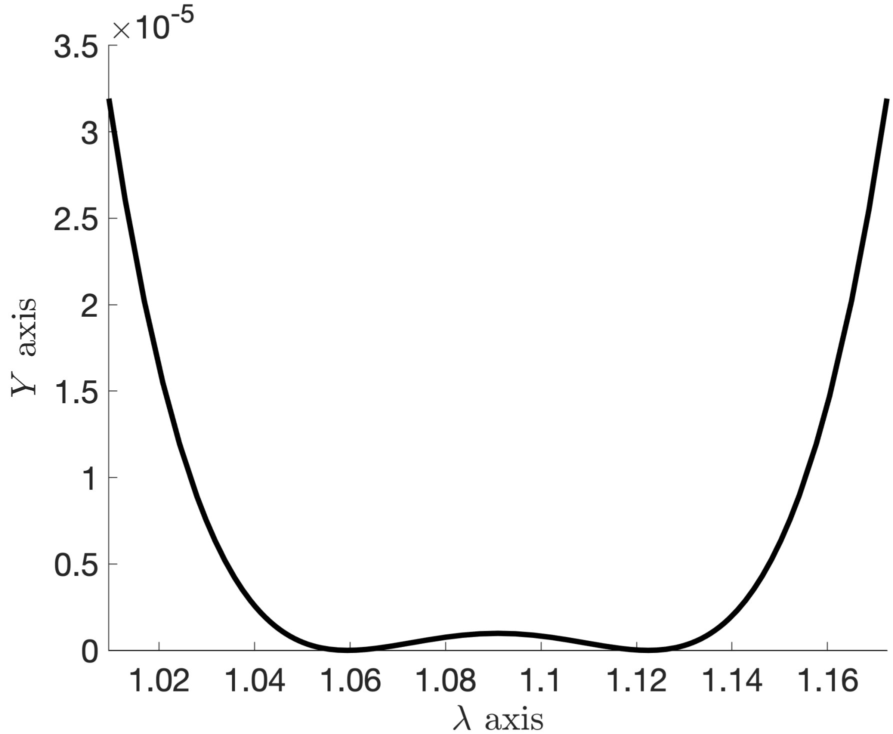


Figure D.13: Characteristic polynomial associated to the VII△ ⟶I-7 cadence (1)

D.2.8. $S_2(L^H_{(bVII^r\triangle|I^r-7))})$

Using a second box distribution we pair the classes of both chords differently but keeping the value of the nabla function to a minimum.

$$\left[E^2_{(bVII_c\triangle|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} A & C \\ F & G \\ D & Eb \\ Bb & Bb \end{pmatrix}\right]_\nabla \tag{D.69}$$

Using the transformation T we compute the endomorphism matrix, where sign recovery is not retrieving any negative value for any exponent.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bVII_c\triangle)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{22}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{33}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ F_{z_2} \\ D_{z_3} \\ Bb_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ Eb_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{D.70}$$

Measuring the link we obtain the same nabla value as $E^1_{(bVII\triangle|I-7)}$.In this way, if we measure the nabla function of the second optimal link, its value must coincide with the value of the nabla function of the first solution.

$$\nabla(E^2_{(bVII\triangle|I-7)}) = \sum_{j=1}^{n} \Omega \mid \int_{E^2_{j1\,(bVII\triangle|I-7)}}^{E^2_{j2\,(bVII\triangle|I-7)}} \phi^{-1}\, d\phi \mid_\Delta = 6$$

The tonal function is nothing more than the characteristic polynomial of the endomorphism matrix associated to the second solution by the transformation T. Then, as we have already described, in this case we find several tonal functions that are the result of applying the T transformation to each of the box distributions that are a solution by the Zero method.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^3 - \lambda)(s^2 - \lambda)(s^1 - \lambda)(s^0 - \lambda)$$

We write the algebraic multiplicities in such a way that we are left with three equations that describe the movement of the voices in the second optimal link. This is how we came to determine the area where the second tonal function is located: $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$.

As usual, we write the decomposed wave functions note by note to study each voicing in a particular way. Each wave function of each note is composed of the sum of the sinusoidal functions from the function to the harmonic number h where the distribution of the amplitudes of each harmonic is specified by the distribution Γ.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII\triangle}(t) = \psi_{A_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I-7}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) + \psi_{Bb_{z_3}}(t) \end{cases}$$

Once the subscripts that will determine the opening of the array have been fixed, we will then obtain a pair of wave functions that go one behind the other in time. This is how we represent, in another layer of abstraction more linked to analysis, the transition of the physical manifestation of one voicing to another.

$$\psi_{bVII\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(bVII\triangle)} - e^{-2\pi tki\psi_j(bVII\triangle)}}{2i}$$
$$\longrightarrow \psi_{X_j}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then, following the polynomial criterion we obtain the function of the degree $II-7$ related to the Dorian tonal center. In this case is not unique and can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E^2_{(bVII\triangle|I-7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s^3 - s^2 - s - 1)\ \lambda^3 + (s + s^2 + s^3 + s^5 + s\,(s^3 + s^2))\ \lambda^2 + (-s^5 - s^6 - s\ (s^3 + s^2))\ \lambda + s^6$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \dfrac{\lambda^5}{5} + \left(-\dfrac{s^3}{4} - \dfrac{s^2}{4} - \dfrac{s}{4} - \dfrac{1}{4}\right)\ \lambda^4 + \dfrac{s\ (s^4 + s^3 + 2\,s^2 + s + 1)}{3}\ \lambda^3 + \left(-\dfrac{s^3\ (s^3 + s^2 + s + 1)}{2}\right)\ \lambda^2 + s^6\ \lambda$

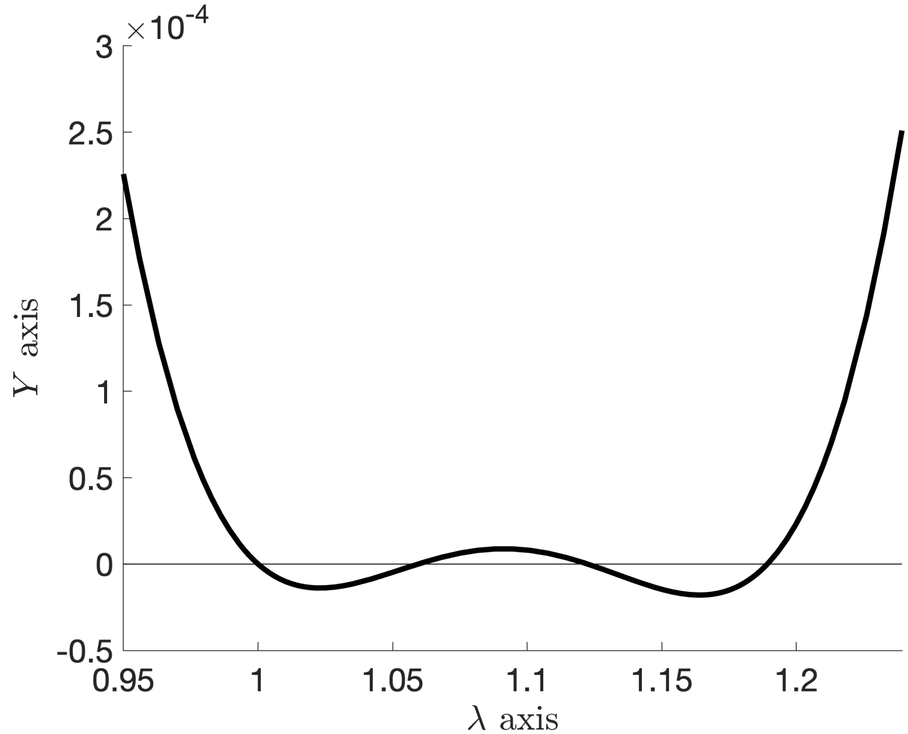


Figure D.14: Characteristic polynomial associated to the VII△ ⟶I-7 cadence (2)

D.2.9. $S_3(L^H_{(bVII^r\triangle|I^r-7))})$

Using the third distinct solution that we haven't used yet, we find a third optimal link. We write such a link and use the brackets to generalize the result. Thus we have used the third pending solution that we found by the Zero method.

$$\left[E^3_{(bVII_c\triangle|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} A & G \\ F & Eb \\ D & C \\ Bb & Bb \end{pmatrix}\right]_\nabla \tag{D.71}$$

Using the transformation T we arrive at a third matrix that transforms a voicing of the first tonal center into a voicing of the second in an optimal way. We use the signed metrics retrieved from the set S in such a way that they are the exponents of each Mersenne number s. Thus we write the matrix equation between voicings in frequency space.

$$C_\mathbb{E} : \Phi^4 \longrightarrow \Phi^4$$

$$C_\mathbb{E}(\psi(bVII\triangle)) \longrightarrow \psi(I\triangle)$$

$$\begin{pmatrix} s^{-\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{-\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} A_{z_1} \\ F_{z_2} \\ D_{z_3} \\ Bb_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{D.72}$$

Now we calculate the Nabla value when the distance between class mappings is minimal. The minimum for al ∞ mappings is unique so the sum can be constructed:

$$\nabla(E^3_{(bVII_c\triangle|I_c-7)}) = \sum_{j=1}^{n} \Omega \mid \int_{E^3_{j1\,(bVII_c\triangle|I_c-7)}}^{E^3_{j2\,(bVII_c\triangle|I_c-7)}} \phi^{-1}\, d\phi \mid_{\Delta} = 6$$

This is how we have calculated the nabla function of the optimal third link, which in this case is worth 6. On the other hand, since our objective is to calculate the tonal function and we already have the matrix $C_{\mathbb{E}}$, calculating the characteristic polynomial we reach said third function immediately. Thus, we observe that this third tonal function has its roots polarized with respect to the stabilizer of M, but it is also classified in the dominant area, in a different way than the previous solutions did for this case.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-2} - \lambda)^3 (s^0 - \lambda)$$

Observing the structure of the factored polynomial, we determine the equations for each of the multiplicities, where we observe that there are three voices that are descending in the optimum and one remains invariant. Thus we write these three equations for the formal classification of the tonal function: $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$. In this case there is only one convergent root that is described by the appropriate set, said set being:

$$\lambda^- = \{s^{-2}\}$$

Regarding the physical manifestation of the vocings we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVII_c\triangle)$ or $\psi(I_c - 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum\limits_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We write the decomposition of the wave functions of each voice to visualize the placement of the notes along the spectrum. This function, as we have already seen, depends on our decision when choosing a set of subscripts that will determine the opening of the arrangement.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII_c\triangle}(t) = \psi_{A_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}} + \psi_{Bb_{z_4}}(t) \\ \psi_{I_c-7}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Bb_{z_4}}(t) \end{cases}$$

We write the wave functions of both voicings separated by an arrow to indicate the antecedent and consequent voicing. This is how we study these functions.

$$\psi_{bVII_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c\triangle)} - e^{-2\pi tki\psi_j(bVII_c\triangle)}}{2i} \longrightarrow \psi_{I_c-7}(t) =$$

$$\sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

As $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$, following the polynomial criterion we obtain the function of the degree $bVII\triangle$ related to the Dorian tonal center. It can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi^3[E_{(bVII\triangle|I-7)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{3}{s^2} - 1\right)\lambda^3 + \left(\frac{3}{s^2} + \frac{3}{s^4}\right)\lambda^2 + \left(-\frac{3}{s^4} - \frac{1}{s^6}\right)\lambda + \frac{1}{s^6}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \dfrac{\lambda^5}{5} + \left(-\dfrac{s^6 + 3\,s^4}{s^6\,4}\right)\lambda^4 + \dfrac{s^2+1}{s^4}\lambda^3 + \left(-\dfrac{3\,s^2+1}{2\,s^6}\right)\lambda^2 + \dfrac{\lambda}{s^6}$

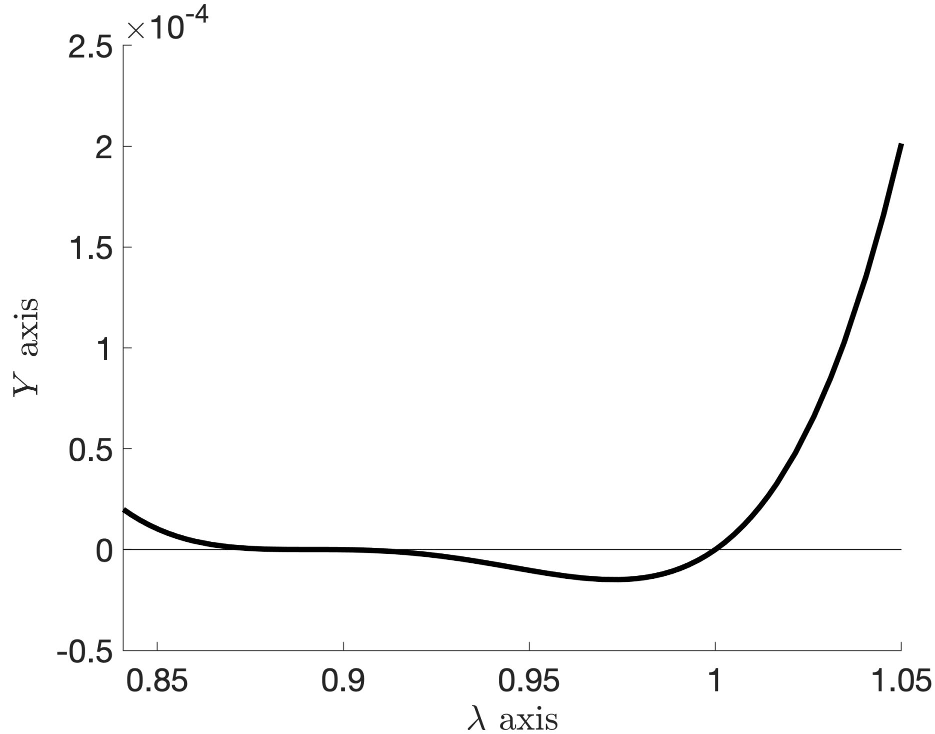


Figure D.15: Characteristic polynomial associated to the VII△ ⟶I-7 cadence (3)

D.2.10. Dorian tonal functions

$II- \longrightarrow I-$	$\Phi[E_{(II-\vert I-)}] \in S^{\mathbb{R}[\lambda]} \cup D^{\mathbb{R}[\lambda]}$
$bIII- \longrightarrow I-$	$\Phi[E_{(bIII-\ \vert I-)}] \in T^{\mathbb{R}[\lambda]}$
$IV \longrightarrow I-$	$\Phi[E_{(IV\vert I-)}] \in D^{\mathbb{R}[\lambda]}$
$V- \longrightarrow I-$	$\Phi[E_{(V-\vert I-)}] \in S^{\mathbb{R}[\lambda]}$
$VIo \longrightarrow I-$	$\Phi[E_{(VIo\vert I-)}] \in T^{\mathbb{R}[\lambda]}$
$bVII \longrightarrow I-$	$\Phi[E_{(bVII\vert I-)}] \in S^{\mathbb{R}[\lambda]}$

$II-7 \longrightarrow I-7$	$\Phi[E_{(II-7\vert I-7)}] \in S^{\mathbb{R}[\lambda]}$
$bIII-7 \longrightarrow I-7$	$\Phi[E_{(bIII-7\vert I-7)}] \in T^{\mathbb{R}[\lambda]}$
$IV7 \longrightarrow I-7$	$\Phi[E_{(IV7\vert I-7)}] \in S^{\mathbb{R}[\lambda]}$
$V-7 \longrightarrow I-7$	$\Phi[E_{(V-7\vert I-7)}] \in D^{\mathbb{R}[\lambda]}$
$VIø7 \longrightarrow I-7$	$\Phi[E_{(VIø7\vert I-7)}] \in T^{\mathbb{R}[\lambda]}$
$bVII \longrightarrow I-7$	$\Phi[E_{(bVII7\vert I-7)}] \in S^{\mathbb{R}[\lambda]} \cup D^{\mathbb{R}[\lambda]}$

Appendix E

The Aeolian Mode

Tao Yuan

https://unsplash.com/photos/a-black-and-white-photo-of-a-feather-DvMRYghfArY

E.1. The Aeolian mode for n =3

E.1.1. IIo⟶ I- Cadence

In this section, we concern ourselves with calculating the tonal functions, taking the consequent tonal center as the first degree of the Aeolian mode. We start by performing the process at $n = 3$ with the usual chord constructions. Later we will do it in $n = 4$ leaving enough examples so that the interested composer can calculate relationships between structures based on his interests. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(II^r_c o|I^r_c -)} = \begin{pmatrix} Ab & G \\ F & Eb \\ D & C \end{pmatrix} \tag{E.1}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(II^r o|I^r -)} = \begin{pmatrix} 1 & 5 & 4 \\ 2 & 2 & 5 \\ 5 & 1 & 2 \end{pmatrix} \tag{E.2}$$

Then following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(II^r o|I^r -)} = \begin{pmatrix} 1 & 5 & 4 \\ 2 & 2 & 5 \\ 5 & 1 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r o|I^r -)} = \begin{pmatrix} 0 & 4 & 3 \\ 0 & 0 & 3 \\ 4 & 0 & 1 \end{pmatrix} \longrightarrow L^H_{(II^r o|I^r -)} = \begin{pmatrix} \boxed{0} & 4 & 2 \\ 0 & \boxed{0} & 2 \\ 4 & 0 & \boxed{0} \end{pmatrix} \tag{E.3}$$

Then the solutions for $L^H_{(II^r o|I^r -)}$ when both triads are in root position becomes the following set, wich represents the minimum voice-leading:

$$S(L^H_{(II^r o|I^r -)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

Reading the subscripts of each entry Δ_{ij}, then we can build an optimal link and take the nabla class of it.

$$\left[E^o_{(II_c o|I_c -)} \right]_\nabla = \left[\begin{pmatrix} Ab & G \\ F & Eb \\ D & C \end{pmatrix} \right]_\nabla \tag{E.4}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(II_c o|I_c -)}) = 2 + 1 + 2 = 5$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(IIo-|I-)} = 5$

Any optimal arrangement from an optimal progression $E^o_{(IIo|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(II_c o)) \longrightarrow \psi(I_c -))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Ab_{z_1} \\ F_{z_2} \\ D_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \end{pmatrix} \tag{E.5}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3. Substituting each value l for the appropriate exponent, we recover the metrics of the matrix L that are solved by the Hungarian algorithm. So the expression would look like:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^{-2} - \lambda & 0 & 0 \\ 0 & s^{-1} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{E.6}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^{-2} - \lambda)^2 (s^{-1} - \lambda)$$

As we can see with the naked eye, the roots are to the left of $E(M)$, in this way three sets are formed depending on the polarization of the roots where two of them are empty and the only non-empty one is the convergent roots.

$$\lambda^{-} = \{s^{-2}, s^{-1}\}$$

$$\lambda^{0} = \{\emptyset\}$$

$$\lambda^{+} = \{\emptyset\}$$

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^{+}$ as a component of $\psi(II_c o)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the wave functions of each voicing as their note-by-note decomposition.

$$\mathcal{B}_{\epsilon} = \begin{cases} \psi_{II_c-}(t) = \psi_{Ab_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

Taking into account the optimization conditions and the relationship between the functions for a particular harmonic distribution and a selection of subscripts, we can then write the functions of the antecedent voicing and the consequent voicing as two functions separated by an arrow.

$$\psi_{II_c o}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II_c o)} - e^{-2\pi tki\psi_j(II_c o)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ then, following the polynomial criterion we obtain the function of the degree IIo related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(IIo|I-)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s} - \frac{2}{s^2}\right)\lambda^2 + \left(\frac{2}{s^3} + \frac{1}{s^4}\right)\lambda - \frac{1}{s^5}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s+2}{3\,s^2}\right)\lambda^3 + \frac{2\,s+1}{2\,s^4}\lambda^2 + \left(-\frac{1}{s^5}\right)\lambda$

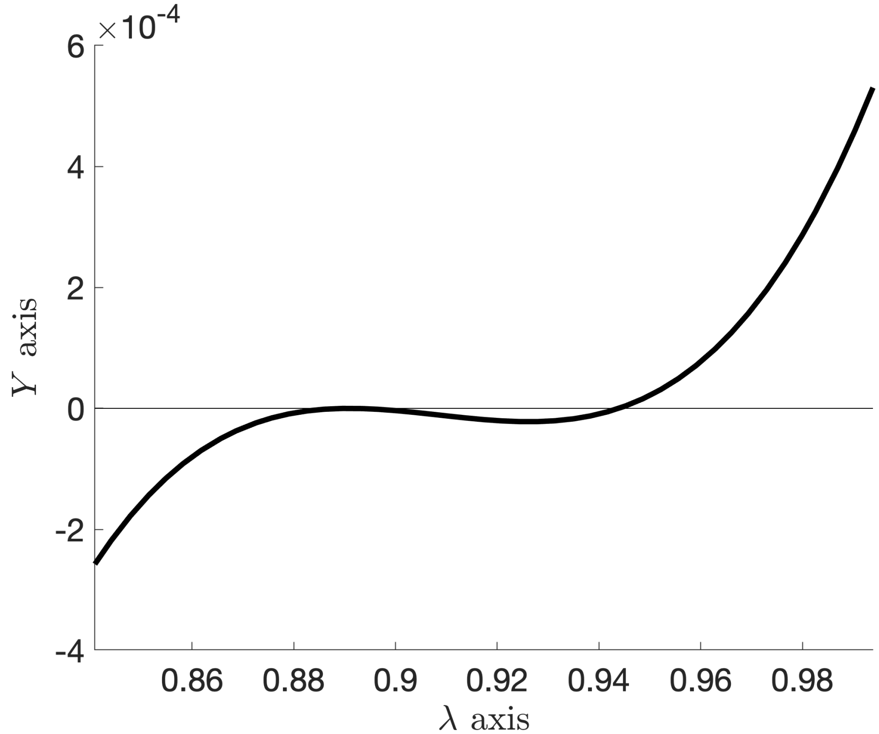


Figure E.1: Characteristic polynomial associated to the IIo⟶ I- cadence

E.1.2. bIII⟶ I- Cadence

We continue advancing through the mode with the aim of calculating each of the tonal functions associated with each edge that connects the chords of the mode. We have chosen this type of organization for the calculation, but it could be another. In this way we are calculating each tonal function between each pair of chords, taking the first degree of the Aoelian mode as the fixed center with the appropriate dimension for this section. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(bIII^r_c|I^r_c-)} = \begin{pmatrix} Bb & G \\ G & Eb \\ Eb & C \end{pmatrix} \tag{E.7}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix. We calculate the matrix L in the usual way, but this does not exclude that there are other ways to solve the same optimization problem.

$$L_{(bIII^r|I^r-)} = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 4 & 5 \\ 4 & 0 & 3 \end{pmatrix} \tag{E.8}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bIII^r|I^r-)} = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 4 & 5 \\ 4 & 0 & 3 \end{pmatrix} \longrightarrow L^F_{(bIII^r|I^r-)} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 4 & 5 \\ 4 & 0 & 3 \end{pmatrix} \longrightarrow L^H_{(bIII^r|I^r-)} = \begin{pmatrix} 1 & 3 & \boxed{0} \\ \boxed{0} & 4 & 5 \\ 4 & \boxed{0} & 3 \end{pmatrix} \tag{E.9}$$

Then the solutions for $L^H_{(bIII^r|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice-leading:

$$S(L^H_{(bIII^r|I^r-)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

Using the placement of each box in the distribution of boxes found, we know how the voices are linked, where the subscript of each solution indicates the pairing between the class i of the first chord and the class j of the second, where the classes are numbered from top to bottom, taking as reference the initial link E.

$$\left[E^o_{(bIII_c|I_c-)}\right]_\triangledown = \left[\begin{pmatrix} Bb & C \\ G & G \\ Eb & Eb \end{pmatrix}\right]_\triangledown \tag{E.10}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^{o}_{(bIII_{c}o|I_{c}-)}) = 2+0+0 = 2$ and we write the optimal nabla value as a generalization for every tonality:

$$\nabla^{o}_{(bIII|I-)} = 2$$

Any optimal arrangement from an optimal progression $E^{o}_{(bIII|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bIII_c)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Bb_{z_1} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} \tag{E.11}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with assigned values l_1, l_2 and l_3. We have replaced each of the metrics with the sign recovered by each of the exponents in the endomorphism matrix. This matrix would be as follows:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{E.12}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2 - \lambda)(s^0 - \lambda)^2$$

We write the sets of roots based on their placement with respect to the M stabilizer. Thus we would have: $\lambda^- = \{s^2\}$, $\lambda^0 = \{1\}$ and $\lambda^+ = \{\emptyset\}$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bIII_c)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the decomposed wave functions note by note:

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{\flat III_c}(t) = \psi_{Bb_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \\ \psi_{I_c}(t) = \psi_{C_{z_1+1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \end{cases}$$

We use the usual notation to represent the transition from the first voicing to the second, leaving both formulas written as:

$$\psi_{bIII_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k\frac{e^{2\pi tki\psi_j(bIII_c)} - e^{-2\pi tki\psi_j(bIII_c)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k\frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bIII$ related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bIII|I-)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-2\,s^2 - 1)\,\lambda^2 + (s^4 + 2\,s^2)\,\lambda - s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s^2}{3} - \frac{1}{3}\right)\lambda^3 + \frac{s^2\,(s^2+2)}{2}\lambda^2 + (-s^4)\,\lambda$

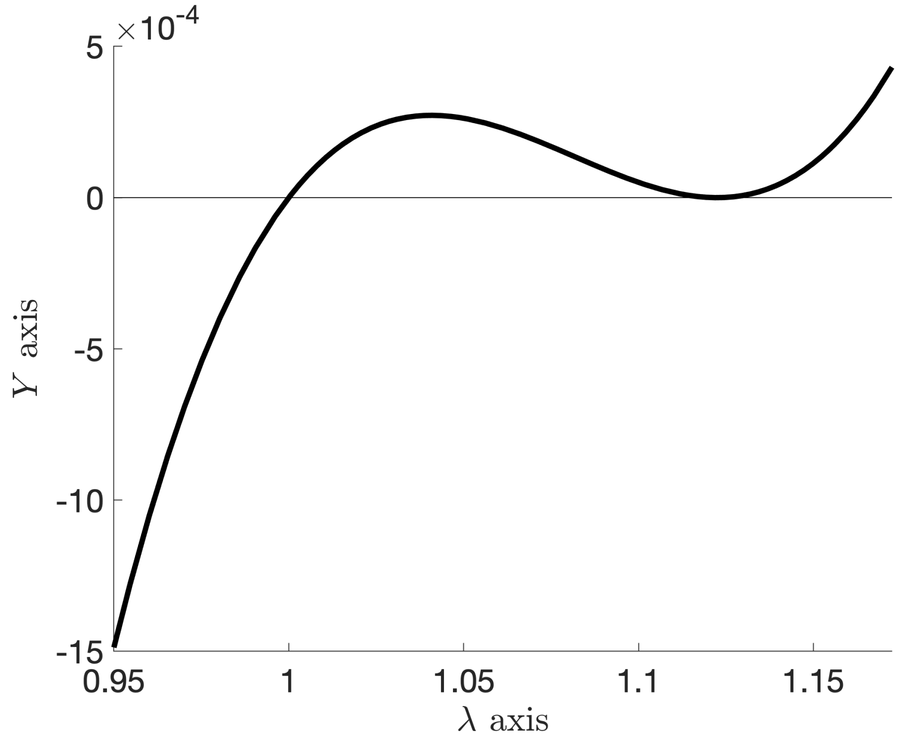


Figure E.2: Characteristic polynomial associated to the bIII⟶ I- cadence

E.1.3. IV-$\longrightarrow$ I- Cadence

In this section we are going to study the relationship between the minor fourth degree of the Aoelian mode and its first degree, thus determining its tonal function. We are going to follow the usual procedure building a matrix of metrics and using the Hungarian algorithm to find the pairing between the voices. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(IV_c^r-|I_c^r-)} = \begin{pmatrix} C & G \\ Ab & Eb \\ F & C \end{pmatrix} \tag{E.13}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix. We have removed the subscripts from the key since we know the property of the matrix L regarding the change of key and therefore we can work with them already directly in the next layer of abstraction in order to study a tonal function between degrees independently of the given the tonality.

$$L_{(IV^r-|I^r-)} = \begin{pmatrix} 5 & 3 & 0 \\ 1 & 5 & 0 \\ 2 & 2 & 5 \end{pmatrix} \tag{E.14}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(IV^r-|I^r-)} = \begin{pmatrix} 5 & 3 & 0 \\ 1 & 5 & 0 \\ 2 & 2 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r-|I^r-)} = \begin{pmatrix} 5 & 3 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \longrightarrow L^H_{(IV^r-|I^r-)} = \begin{pmatrix} 4 & 2 & \boxed{0} \\ \boxed{0} & 4 & 0 \\ 0 & \boxed{0} & 4 \end{pmatrix} \tag{E.15}$$

Then, the solutions for $L^H_{(IV^r-|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(IV^r-|I^r-)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

Following the indications provided by the subscripts that we find in each solution of the set S, we build an optimal link immediately. We take square brackets to generalize the result.

$$\left[E^o_{(IV_c-|I_c-)}\right]_\nabla = \left[\begin{pmatrix} C & C \\ Ab & G \\ F & Eb \end{pmatrix}\right]_\nabla \tag{E.16}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(IV_c-|I_c-)}) = 2 + 1 + 0 = 3$ and we write the optimal nabla value as a generalization for every tonality: $\nabla^o_{(IV-|I-)} = 3$.

Any optimal arrangement from an optimal progression $E^{o}_{(IV|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bIII_c)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ Ab_{z_2} \\ F_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} \tag{E.17}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^{-1} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{E.18}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^{-2} - \lambda)(s^0 - \lambda)(s^{-1} - \lambda)$$

According to the distribution of the roots of the characteristic polynomial, we classify these roots into three sets. So for this case we have:

$$\lambda^- = \{s^{-2}, s^{-1}\}$$

$$\lambda^0 = \{s^0\}$$

$$\lambda^+ = \{\emptyset\}$$

We see that the roots have been polarized to the left $E(M)$ and that we have one root (the remaining one) on the M group stabilizer. We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(IV_c-)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the voicing functions inside the bracket leaving the subscripts open, since the opening of the array does not affect the minimum value of absolute perception.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV_c-}(t) = \psi_{C_{z_1}}(t) + \psi_{Ab_{z_2}}(t) + \psi_{F_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \end{cases}$$

We represent the transition between voicings in the usual way. We consider that the optimization conditions under which these functions are written are known, then:

$$\psi_{IV_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(IV_c-)} - e^{-2\pi tki\psi_j(IV_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$ then, following the polynomial criterion we obtain the function of the degree IV related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(IV-|I-)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s} - \frac{1}{s^2} - 1\right)\lambda^2 + \left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}\right)\lambda - \frac{1}{s^3}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2+s+1}{s^2\,3}\right)\lambda^3 + \frac{s^2+s+1}{s^3\,2}\lambda^2 + \left(-\frac{1}{s^3}\right)\lambda$

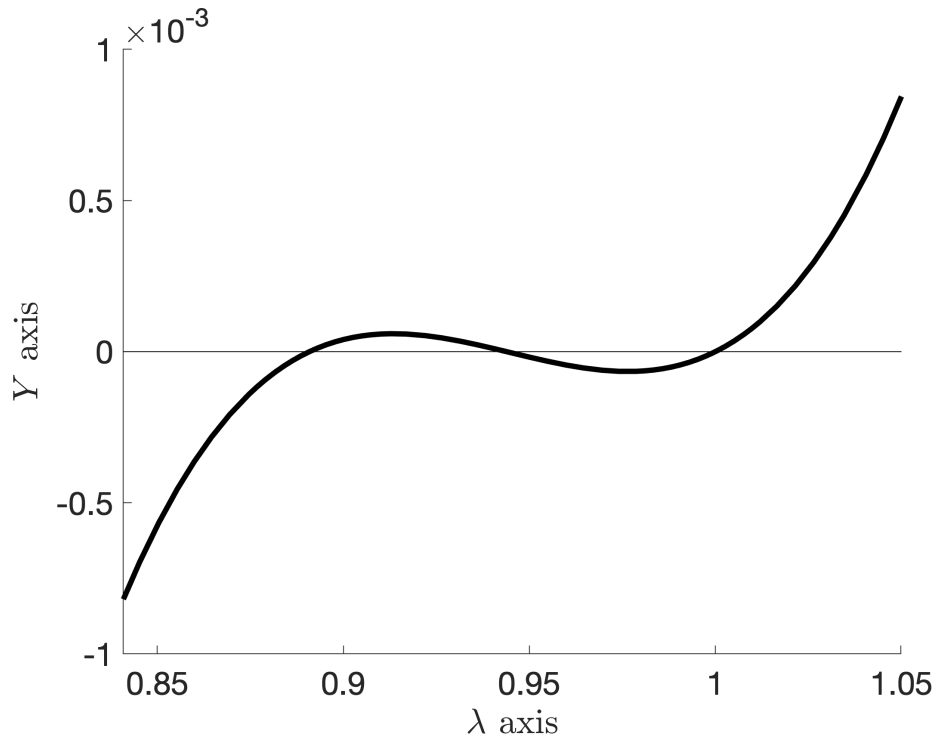


Figure E.3: Characteristic polynomial associated to the IV-⟶ I- cadence

E.1.4. V-⟶ I- Cadence

From the Aoelian mode, we are going to study the relationship between its minor fifth degree and the first degree of said mode. So we are going to write a link E with both chords in the root position, and from there, we are going to use the calculation method to reach the tonal function. Since the chords have the same dimension, we are not going to use the infinite arithmetic criterion nor are we going to need anything more than what is strictly usual in this work. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(V_c^r-|I_c^r-)} = \begin{pmatrix} D & G \\ Bb & Eb \\ G & C \end{pmatrix} \tag{E.19}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(V^r-|I^r-)} = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 5 & 2 \\ 0 & 4 & 5 \end{pmatrix} \tag{E.20}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(V^r-|I^r-)} = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 5 & 2 \\ 0 & 4 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r-|I^r-)} = \begin{pmatrix} 4 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix} \longrightarrow L^H_{(V^r-|I^r-)} = \begin{pmatrix} 4 & \boxed{0} & 1 \\ 1 & 3 & \boxed{0} \\ \boxed{0} & 4 & 5 \end{pmatrix} \tag{E.21}$$

The solutions for $L^H_{(V^r-|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(V^r-|I^r-)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

Once we have found the set S, it is enough to follow the row and column indications of the positions of each box in the distribution of boxes found to find an optimal link and generalize the result to the nabla class.

$$\left[E^o_{(V_c-|I_c-)}\right]_\nabla = \left[\begin{pmatrix} D & Eb \\ Bb & C \\ G & G \end{pmatrix}\right]_\nabla \tag{E.22}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(V_c-|I_c-)}) = 2+1+0 = 3$ and we write the optimal Nabla value as a generalization for every tonality: $\nabla^o_{(V-|I-)} = 3$.

Any optimal arrangement from an optimal progression $E^o_{(IV|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(V_c-)) \longrightarrow \psi(I_c-)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ Bb_{z_2} \\ G_{z_3} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{E.23}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with assigned values l_1, l_2 and l_3. In this case, all the exponents are positive and by substituting the values l for the values of the metrics found by the Hungarian algorithm, then we have that the endomorphism matrix has the following determinant as its characteristic polynomial:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^1-\lambda & 0 & 0 \\ 0 & s^2-\lambda & 0 \\ 0 & 0 & s^0-\lambda \end{pmatrix} \tag{E.24}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^1-\lambda)(s^2-\lambda)(s^0-\lambda)$$

The roots of the previous polynomial are organized in three sets, where we observe one root on the stabilizer and two to its right, which is enough to apply the polynomial criterion.

$$\lambda^+ = \{s^2, s^1\}$$

$$\lambda^0 = \{s^0\}$$

$$\lambda^- = \{\emptyset\}$$

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(V_c-)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. Both voicing wave functions have a general form regardless of the chosen aperture, satisfying the equations inside the bracket when we choose a set of subscripts

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V_c-}(t) = \psi_{D_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

The representation of the physical phenomenon consists of the transition from one wave function to another, where the absolute perception between one and the other reaches the minimum.

$$\psi_{V_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(V_c-)} - e^{-2\pi tki\psi_j(V_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$ then, following the polynomial criterion we obtain the function of the degree IV related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(V-|I-)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - s - 1)\ \lambda^2 + (s^3 + s^2 + s)\ \lambda - s^3$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2}{3} - \frac{s}{3} - \frac{1}{3}\right)\ \lambda^3 + \frac{s\left(s^2+s+1\right)}{2}\ \lambda^2 + \left(-s^3\right)\ \lambda$

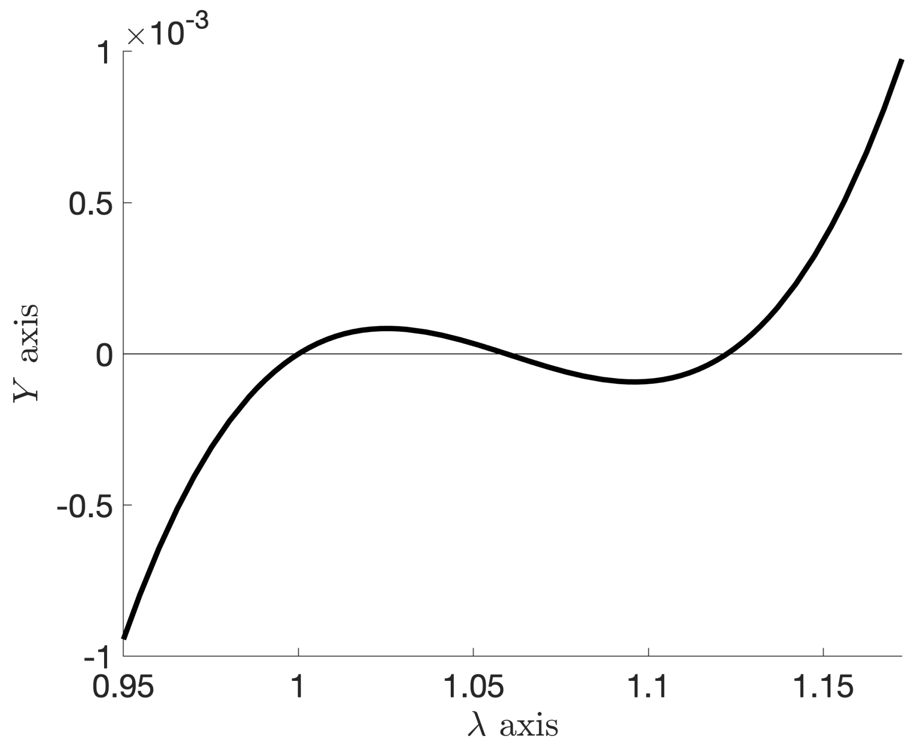


Figure E.4: Characteristic polynomial associated to the V-⟶ I- cadence

E.1.5. bVI⟶ I- Cadence

In the calculation process that is being carried out, we are going to study the relationship between the sixth degree and the first within the mode. As usual, we start by writing the link E. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(bVI_c^r|I_c^r-)} = \begin{pmatrix} Eb & G \\ C & Eb \\ Ab & C \end{pmatrix} \tag{E.25}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(VI^r-|I^r-)} = \begin{pmatrix} 4 & 0 & 3 \\ 5 & 3 & 0 \\ 1 & 5 & 4 \end{pmatrix} \tag{E.26}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bVI^r|I^r-)} = \begin{pmatrix} 4 & 0 & 3 \\ 5 & 3 & 0 \\ 1 & 5 & 4 \end{pmatrix} \longrightarrow L^F_{(bVI^r|I^r)} = \begin{pmatrix} 4 & 0 & 3 \\ 5 & 3 & 0 \\ 0 & 4 & 3 \end{pmatrix} \longrightarrow L^H_{(bVI^r|I^r-)} = \begin{pmatrix} 4 & \boxed{0} & 3 \\ 5 & 3 & \boxed{0} \\ \boxed{0} & 4 & 3 \end{pmatrix} \tag{E.27}$$

Then the solutions for $L^H_{(bVI^r|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice-leading:

$$S(L^H_{(VI^r-|I^r-)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

We calculate the optimal link following the solution of the set S. So we write:

$$\left[E^o_{(bVI_c|I_c-)}\right]_\nabla = \left[\begin{pmatrix} Eb & Eb \\ C & C \\ Ab & G \end{pmatrix}\right]_\nabla \tag{E.28}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bVI_c|I_c-)}) = 0 + 0 + 1 = 1$ We write the optimal nabla value as a generalization for every tonality: $\nabla^o_{(bVI|I-)} = 1$.

Any optimal arrangement from an optimal progression $E^o_{(bVI_c|I_c-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bVI_c)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Ab_{z_3} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{E.29}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^{-1} - \lambda \end{pmatrix} \tag{E.30}$$

Using the properties of the determinant the polynomial has the following form. Despite the fact that later we noted the expansion of the polynomial for the mere interest that its study entails, here we see that it is already factored and in it we can observe the placement of the roots with respect to $E(M)$.

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)^2(s^{-1} - \lambda)$$

The roots are grouped into three sets based on $E(M)$. Thus they remain for this case as indicated below:

$$\lambda^- = \{s^{-1}\}$$

$$\lambda^0 = \{s^0\}$$

$$\lambda^+ = \{\emptyset\}$$

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVI_c)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. As usual, we study wave functions when we select a particular set of subscripts that fix each frequency for each wave function of each particular note. Thus, in the bracket we have that both functions have to maintain the connection between their notes by the solutions of the Hungarian algorithm.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVI_c}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Ab_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

The transition between wave functions is symbolized by $\longrightarrow$ between functions. The interested reader has the tools to draw the functions using a graphing calculator if it is of interest to carry out this activity.

$$\psi_{bVI_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(bVI_c-)} - e^{-2\pi tki\psi_j(bVI_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$ then, following the polynomial criterion we obtain the function of the degree bVI related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bVI|I-)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - s - 1)\ \lambda^2 + (s^3 + s^2 + s)\ \lambda - s^3\lambda^3$
$+ \left(-\frac{2}{s} - 1\right)\ \lambda^2 + \left(\frac{2}{s} + \frac{1}{s^2}\right)\ \lambda - \frac{1}{s^2}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s+2}{3\,s}\right)\ \lambda^3 + \frac{2\,s+1}{2\,s^2}\ \lambda^2 + \left(-\frac{1}{s^2}\right)\ \lambda$

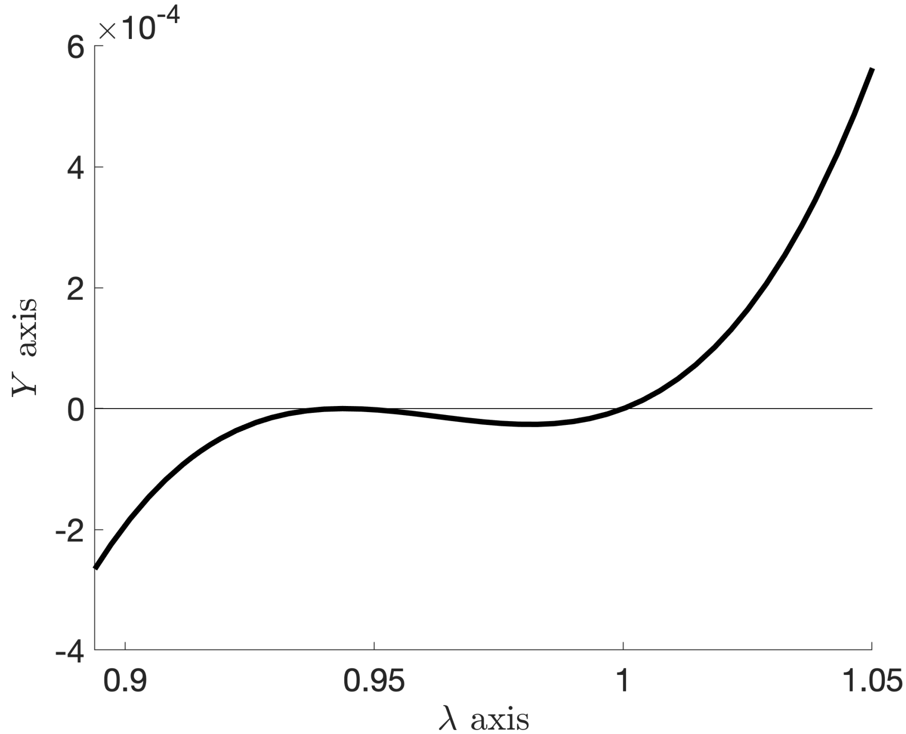


Figure E.5: Characteristic polynomial associated to the bVI⟶ I- cadence

E.1.6. bVII⟶ I- Cadence

In this section we study the relationship between the seventh degree and the first in the context of the aAoelian mode. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(bVII_c^r|I_c^r-)} = \begin{pmatrix} F & G \\ D & Eb \\ Bb & C \end{pmatrix} \quad \text{(E.31)}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(\flat VII^r|I^r-)} = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 1 & 2 \\ 3 & 5 & 2 \end{pmatrix} \quad \text{(E.32)}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bVII^r|I^r-)} = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 1 & 2 \\ 3 & 5 & 2 \end{pmatrix} \longrightarrow L^F_{(bVII^r|I^r-)} = \begin{pmatrix} 0 & 0 & 3 \\ 4 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix} \longrightarrow L^H_{(bVII^r|I^r-)} = \begin{pmatrix} \boxed{0} & 0 & 3 \\ 4 & \boxed{0} & 1 \\ 1 & 3 & \boxed{0} \end{pmatrix} \tag{E.33}$$

Then the solutions for $L^H_{(bVII^r|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice leading:

$$S(L^H_{(\flat VII^r|I^r-)}) = \{\Delta_{11}, \Delta_{21}, \Delta_{33}\}$$

Following the subscript notation of the solutions of the set S, we build the optimal link, which in this case matches the original link E.

$$\left[E^o_{(bVII_c|I_c-)} \right]_\nabla = \left[\begin{pmatrix} F & G \\ D & Eb \\ Bb & C \end{pmatrix} \right]_\nabla \tag{E.34}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bVII_c|I_c-)}) = 2 + 1 + 2 = 5$ and we write the optimal abla value as a generalization for every tonality: $\nabla^o_{(bVII|I-)} = 5$

Any optimal arrangement from an optimal progression $E^o_{(bVII|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bVII_c)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ D_{z_2} \\ Bb_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \end{pmatrix} \tag{E.35}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^1 - \lambda & 0 \\ 0 & 0 & s^2 - \lambda \end{pmatrix} \tag{E.36}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)^2(s^1 - \lambda)$$

We organize the roots based on their placement with respect to the stabilizer $E(M)$, forming three sets, two of them empty in this case.

$$\lambda^+ = \{s^1, s^2\}$$

$$\lambda^0 = \{\emptyset\}$$

$$\lambda^- = \{\emptyset\}$$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(bVII_c)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the wave functions of each voicing following the decomposition note by note.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII_c}(t) = \psi_{F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

Using the arrow notation we indicate the equations of the antecedent voicing and the consequent voicing.

$$\psi_{bVII_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c)} - e^{-2\pi tki\psi_j(bVII_c)}}{2i}$$

$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$ then, following the polynomial criterion we obtain the function of the degree $bVII$ related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bVII|I-)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-2\,s^2 - s)\ \lambda^2 + (s^4 + 2\,s^3)\ \lambda - s^5$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s\,(2\,s+1)}{3}\right)\ \lambda^3 + \frac{s^3\,(s+2)}{2}\ \lambda^2 + (-s^5)\ \lambda$

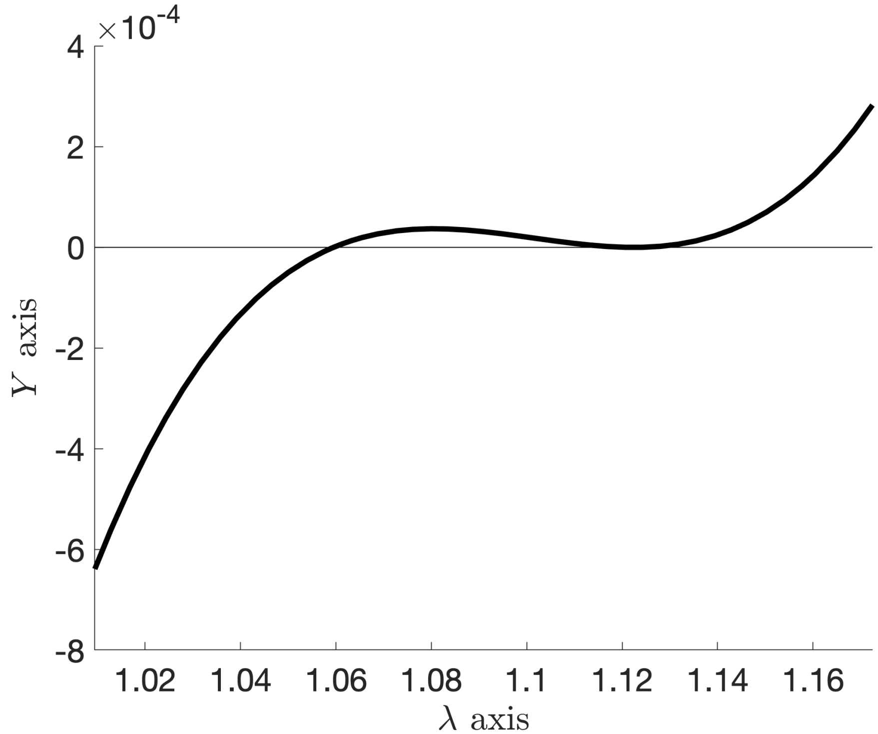


Figure E.6: Characteristic polynomial associated to the bVII⟶ I- cadence

E.2. The Aeolian mode for n=4

E.2.1. IIø7⟶I-7 cadence

From here, we are going to develop the cases of the Aeolian mode in four voices for the chords commonly used in jazz music. The number of chords in four voices in each mode depends on how we set the number of voices and its calculation is simple. In the work a large number of cases are exposed, but the interested composer has a method for the study of the sets of chords that are of interest to him in particular. The link will be the following matrix:

$$E_{(II^r_c\text{ø}7|I^r_c-7)} = \begin{pmatrix} C & Bb \\ Ab & G \\ F & Eb \\ D & C \end{pmatrix} \tag{E.37}$$

Then, we calculate the link matrix calculating all the distances Δ_{ij}.We assemble the L matrix in the usual way, thus calculating each of the metrics between the chord classes:

$$L_{(II^r\o7|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \end{pmatrix} \tag{E.38}$$

Following the steps of the Hungarian algorithm we develop the L matrix:

$$L_{(II^r\o7|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \end{pmatrix} \longrightarrow L^F_{(II^r\o7|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 1 & 0 & 4 & 3 \\ 3 & 0 & 0 & 3 \\ 3 & 4 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow L^H_{(II^r\o7|I^r-7)} = \begin{pmatrix} 1 & 5 & 3 & \boxed{0} \\ \boxed{0} & 0 & 4 & 3 \\ 2 & \boxed{0} & 0 & 3 \\ 2 & 4 & \boxed{0} & 1 \end{pmatrix}$$

We construct the set S of solutions for this link, where S is unique since the pitch function is non-dual.

$$S(L^H_{(II^r\o7|I^r-7)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

With the solutions of the S set, it is enough for us to find how the two consecutive chords are linked and to build the optimal link between them.

$$\left[E^o_{(II_c\o7|I_c-7)} \right]_\nabla = \left[\begin{pmatrix} C & C \\ Ab & Bb \\ F & G \\ D & Eb \end{pmatrix} \right]_\nabla \tag{E.39}$$

Using the transformation T we transform the solutions of the matrix L into the matrix $C_{\mathbb{E}}$, from which we calculate the characteristic polynomial we obtain the tonal function.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(II_c\o7)) \longrightarrow \psi(I_c - 7)$$
$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ Ab_{z_2} \\ F_{z_3} \\ D_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ Eb_{z_4} \end{pmatrix} \tag{E.40}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id_4.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^2 - \lambda)^2(s^1 - \lambda)$$

Observing the characteristic polynomial , we calculate the algebraic multiplicities in order to apply the polynomial criterion: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(II\o 7)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We write the succession of voicings breaking down each wave function note by note to study how they are linked in the physical manifestation.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{II\o 7}(t) = \psi_{C_{z_1}}(t) + \psi_{Ab_{z_2}}(t) + \psi_{F_{z_3}} + \psi_{D_{z_4}}(t) \\ \psi_{I7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{Eb_{z_4}}(t) \end{cases}$$

The functions would be written as indicated, separated by an arrow indicating the temporal order.

$$\psi_{II\o 7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(II\o 7)} - e^{-2\pi tki\psi_j(II\o 7)}}{2i}$$
$$\longrightarrow \psi_{I-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then following the polynomial criterion we obtain the function of the degree $II\o 7$ related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(II\o 7|I-7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-2\,s^2 - s - 1)\ \lambda^3 + (s^4 + 2\,s^3 + 2\,s^2 + s)\ \lambda^2 + (-s^5 - s^4 - 2\,s^3)\ \lambda + s^5$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{2} - \frac{s}{4} - \frac{1}{4}\right)\ \lambda^4 + \frac{s\,\left(s^3+2\,s^2+2\,s+1\right)}{3}\ \lambda^3 + \left(-\frac{s^3\,\left(s^2+s+2\right)}{2}\right)\ \lambda^2 + s^5\,\lambda$

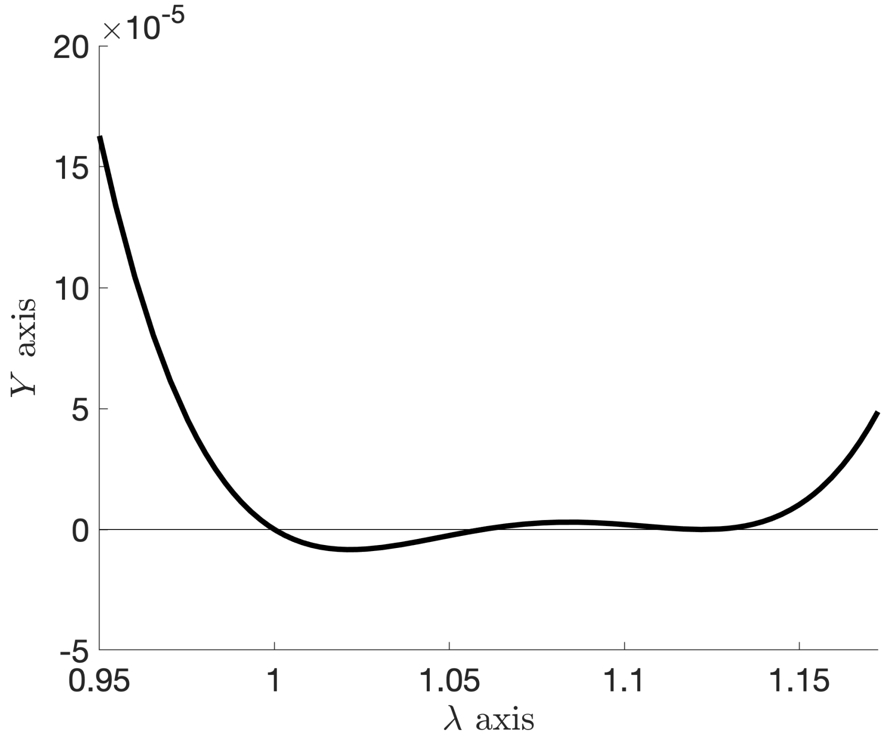


Figure E.7: Characteristic polynomial associated to the IIø7⟶I-7 cadence

E.2.2. bIII△ ⟶I-7 Cadence

Progressing in the study of the relationship between degrees of the mode, we are going to study the relationship between the first degree and the third degree. in this way we build an E link to place both chords in root state and build an L matrix. The link will be the following matrix:

$$E_{(bIII^r\triangle|I^r-7)} = \begin{pmatrix} D & Bb \\ Bb & G \\ G & Eb \\ Eb & C \end{pmatrix} \tag{E.41}$$

Then ,we calculate the link matrix calculating all the distances Δ_{ij}:

$$L_{(bIII^r\triangle|I^r-7)} = \begin{pmatrix} 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix} \tag{E.42}$$

Following the steps of the Hungarian algorithm we develop the L matrix:

$$L_{(bIII^r\triangle|I^r-7)} = \begin{pmatrix} 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix} \longrightarrow L^F_{(bIII\triangle|I^r7)} = \begin{pmatrix} 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix}$$

$$\longrightarrow L^H_{(bIII^r\triangle|I^r7)} = \begin{pmatrix} 3 & 4 & 0 & \boxed{0} \\ \boxed{0} & 3 & 5 & 1 \\ 3 & \boxed{0} & 4 & 4 \\ 5 & 4 & \boxed{0} & 2 \end{pmatrix}$$

We look at the positions of the boxes on the matrix L^H to build the S set. Thus, when building S we realize how the chords link and thus we can build the optimal link.

$$S(L^H_{(bIII^r\triangle|I^r-7)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

We build an optimal link from the solutions of S and in the same process we generalize the result to the nabla class.

$$\left[E^o_{(bIII_c\triangle|I_c-7)} \right]_\nabla = \left[\begin{pmatrix} D & C \\ Bb & Bb \\ G & G \\ Eb & Eb \end{pmatrix} \right]_\nabla \tag{E.43}$$

Knowing the transformation T we reach the endomorphism matrix in the four-dimensional frequency space, where the metrics in S appear as exponents of the Mersenne numbers with the appropriate sign recovery.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bIII_c\triangle)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{-\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} D_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ Eb_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ Eb_{z_4} \end{pmatrix} \tag{E.44}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)^3 (s^{-2} - \lambda)$$

The algebraic multiplicities are determined by the roots of the polynomial, and are summarized as the three formulas that we written: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bIII\triangle)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_I(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We write the wave functions for a set of subscripts that determine its opening.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bIII\triangle}(t) = \psi_{D_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{Eb_{z_4}}(t) \\ \psi_{I-7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{Eb_{z_4}}(t) \end{cases}$$

Assuming that the assignment functions of each class respect the optimization conditions, we write the functions of each voicing for a particular opening, separated by the arrow indicating their temporal order.

$$\psi_{bIII\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bIII\triangle)} - e^{-2\pi tki\psi_j(bIII\triangle)}}{2i}$$

$$\longrightarrow \psi_{I-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bIII\triangle$ related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bIII\triangle|I-7)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{3}{s^2} - 1\right) \lambda^3 + \left(\frac{3}{s^2} + \frac{3}{s^4}\right) \lambda^2 + \left(-\frac{3}{s^4} - \frac{1}{s^6}\right) \lambda + \frac{1}{s^6}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^6+3\,s^4}{s^6\,4}\right) \lambda^4 + \frac{s^2+1}{s^4} \lambda^3 + \left(-\frac{3\,s^2+1}{2\,s^6}\right) \lambda^2 + \frac{\lambda}{s^6}$

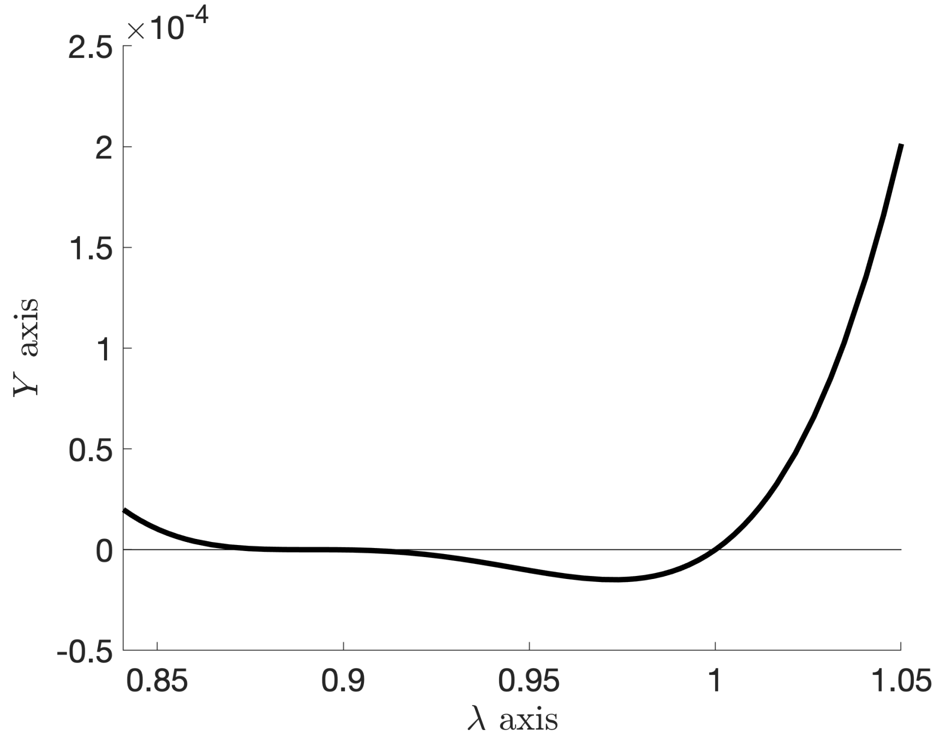


Figure E.8: Characteristic polynomial associated to the bIII△ ⟶I-7 cadence

E.2.3. IV-7⟶I-7 Cadence

We continue advancing in the study of the relationships between the degrees of the Aoelian mode, with the aim of understanding the behavior of chords within the traditional tonality. We mount the link E between the two chords involved in the calculation. The link will be the following matrix:

$$E_{(IV_c^r-7|I_c^r-7)} = \begin{pmatrix} Eb & Bb \\ C & G \\ Ab & Eb \\ F & C \end{pmatrix} \tag{E.45}$$

Then, we calculate the link matrix calculating all the distances Δ_{ij}. The calculation of the L matrix is done as we have been doing, calculating the minimum distance between classes in an ordered way.

$$L_{(IV^r-7|I^r-7)} = \begin{pmatrix} 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \end{pmatrix} \tag{E.46}$$

Following the steps of the Hungarian algorithm we develop the L matrix. In this case the solution is unique and is represented by a unique distribution of boxes over L^H.

$$L_{(IV^r-7|I^r-7)} = \begin{pmatrix} 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix} \longrightarrow L^F_{(IV^r-7|I^r-7)} = \begin{pmatrix} 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 1 & 0 & 4 & 3 \\ 3 & 0 & 0 & 3 \end{pmatrix}$$

$$\longrightarrow L^H_{(IV^r-7|I^r-7)} = \begin{pmatrix} 4 & 4 & \boxed{0} & 3 \\ 1 & 5 & 3 & \boxed{0} \\ \boxed{0} & 0 & 4 & 3 \\ 2 & \boxed{0} & 0 & 3 \end{pmatrix}$$

The solutions given by the algorithm will be in S:

$$S(L^H_{(IV^r-7|I^r-7)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

Following the solutions in S we build an optimal binding that generalizes to its nabla class immediately.

$$\left[E^o_{(IV_c-7|I_c-7)}\right] = \left[\begin{pmatrix} Eb & Eb \\ C & C \\ Ab & Bb \\ F & G \end{pmatrix}\right]_\nabla \tag{E.47}$$

We only have to use the transformation T to connect the matrix L with the matrix $C_{\mathbb{E}}$ in such a way that we can study how the frequency vectors behave in the corresponding space. Recovering the solutions obtained by the Hungarian algorithm, we are able to determine the exponents of each number s.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(IV_c - 7)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Ab_{z_3} \\ F_{z_4} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} \tag{E.48}$$

The polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)^2(s^2 - \lambda)^2$$

Inspecting the characteristic polynomial we find each one of the equations for the multiplicities, where they appear: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$,$\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(IV-7)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

We write the wave functions of each voicing in its decomposition note by note to understand how the notes are connected when a certain opening is chosen, understanding that the opening depends exclusively on the selection of subscripts chosen.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV-7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Ab_{z_3}} + \psi_{F_{z_4}}(t) \\ \psi_{I-7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

For the understanding of the musical phenomenon that we are studying, we give the corresponding wave functions for each voicing, with the usual notation.

$$\psi_{IV-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(IV-7)} - e^{-2\pi tki\psi_j(IV-7)}}{2i}$$

$$\longrightarrow \psi_{I-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $IV-7$ related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(IV-7|I-7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-2\,s^2 - 2)\ \lambda^3 + (s^4 + 4\,s^2 + 1)\ \lambda^2 + (-2\,s^4 - 2\,s^2)\ \lambda + s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{2} - \frac{1}{2}\right)\ \lambda^4 + \left(\frac{s^4}{3} + \frac{4\,s^2}{3} + \frac{1}{3}\right)\ \lambda^3 + (-s^2\ (s^2+1))\ \lambda^2 + s^4\ \lambda$

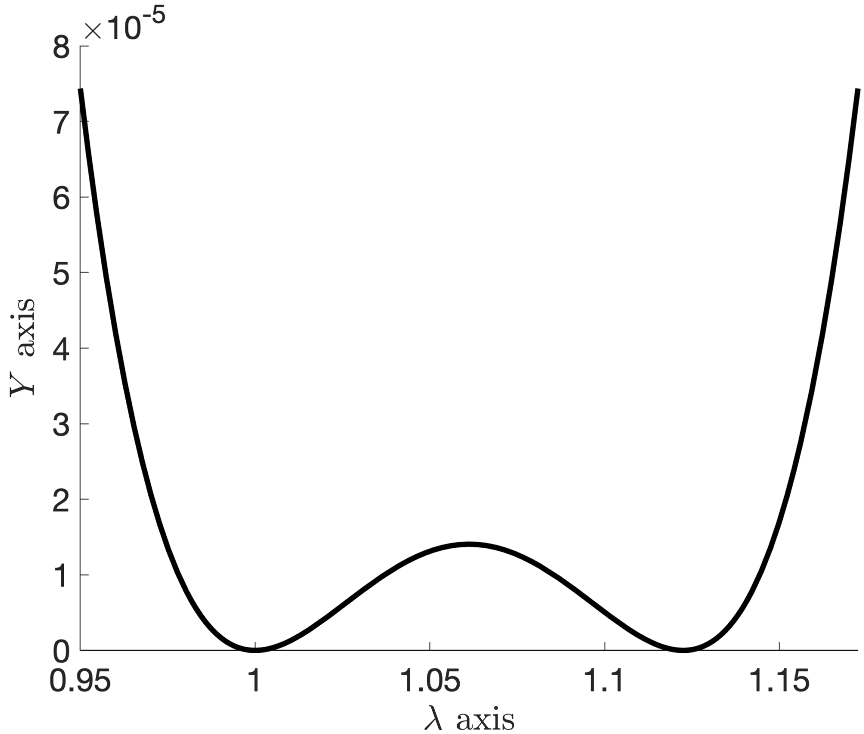


Figure E.9: Characteristic polynomial associated to the IV-7⟶I-7 cadence

E.2.4. V-7⟶I-7 Cadence

We study the relationship between $V-7$ and $I-7$ in the usual way. This progression is common in modern music and usually breaks the intuition of composers because it is a resolution whose first chord does not have a tritone, so it could not be justified that the existence of the tritone is the element that determines the resolution from one chord to another. Using the techniques that have been exposed, the Hungarian algorithm, the transformation T and in general the construction of L matrices, we are able to understand what is the justification of this resolution, a mechanism that resides exclusively in the movement of the voices in the optimum link.The link will be the following matrix:

$$E_{(V_c^r-7|I_c^r-7)} = \begin{pmatrix} F & Bb \\ D & G \\ Bb & Eb \\ G & C \end{pmatrix} \tag{E.49}$$

We calculate the link matrix calculating all the distances Δ_{ij}.We construct L in the usual way:

$$L_{(V^r-7|I^r-7)} = \begin{pmatrix} 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \end{pmatrix} \tag{E.50}$$

Following the steps of the Hungarian algorithm we develop the L matrix. We start at L until we reach L^H following the process. We end up finding a distribution of boxes over L^H:

$$L_{(V^r-7|I^r-7)} = \begin{pmatrix} 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r-7|I^r-7)} = \begin{pmatrix} 3 & 0 & 0 & 3 \\ 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \end{pmatrix}$$

$$\longrightarrow L^H_{(V^r-7|I^r-7)} = \begin{pmatrix} 3 & 0 & \boxed{0} & 2 \\ 3 & 4 & 0 & \boxed{0} \\ \boxed{0} & 3 & 5 & 1 \\ 3 & \boxed{0} & 4 & 4 \end{pmatrix}$$

Studying the position of the boxes on the last matrix, we find the metrics that make up the solution that optimally links the voices. In this way, the set S is determined by the following equation, which is enough to build an optimal link.

$$S(L^H_{(V^r-7|I^r-7)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

Following the results of the previous process we end up finding the optimal binding, which is immediately generalizable to all of its nabla class.

$$\left[E^o_{(V_c-7|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} F & Eb \\ D & C \\ Bb & Bb \\ G & G \end{pmatrix}\right]_\nabla \tag{E.51}$$

Using the transformation T we are going to reach the matrix of the endomorphism. In this case we have that there are voices that descend and therefore the recovery of the sign plays a crucial role. Thus, we observe that the first two voices descend when applying the matrix and the last two remain invariant.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(V_c - 7)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{-\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ D_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} \tag{E.52}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)^2 (s^{-2} - \lambda)^2$$

We compute the algebraic multiplicities by inspecting the characteristic polynomial: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(V-7)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. Next we write the wave functions of each voice without specifying its opening and decomposing them note by note:

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V_c-7}(t) = \psi_{F_{z_1}}(t) + \psi_{D_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \\ \psi_{I_c-7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

The wave functions for a distribution of harmonics Γ will be specific depending on the subscripts chosen.

$$\psi_{V_c-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(V_c-7)} - e^{-2\pi tki\psi_j(V_c-7)}}{2i}$$
$$\longrightarrow \psi_{I_c-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $V-7$ related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(V-7|I-7)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{2}{s^2} - 2\right) \lambda^3 + \left(\frac{4}{s^2} + \frac{1}{s^4} + 1\right) \lambda^2 + \left(-\frac{2}{s^2} - \frac{2}{s^4}\right) \lambda + \frac{1}{s^4}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2+1}{2\,s^2}\right) \lambda^4 + \frac{s^4+4\,s^2+1}{s^4\,3} \lambda^3 + \left(-\frac{2\,s^2+2}{2\,s^4}\right) \lambda^2 + \frac{\lambda}{s^4}$

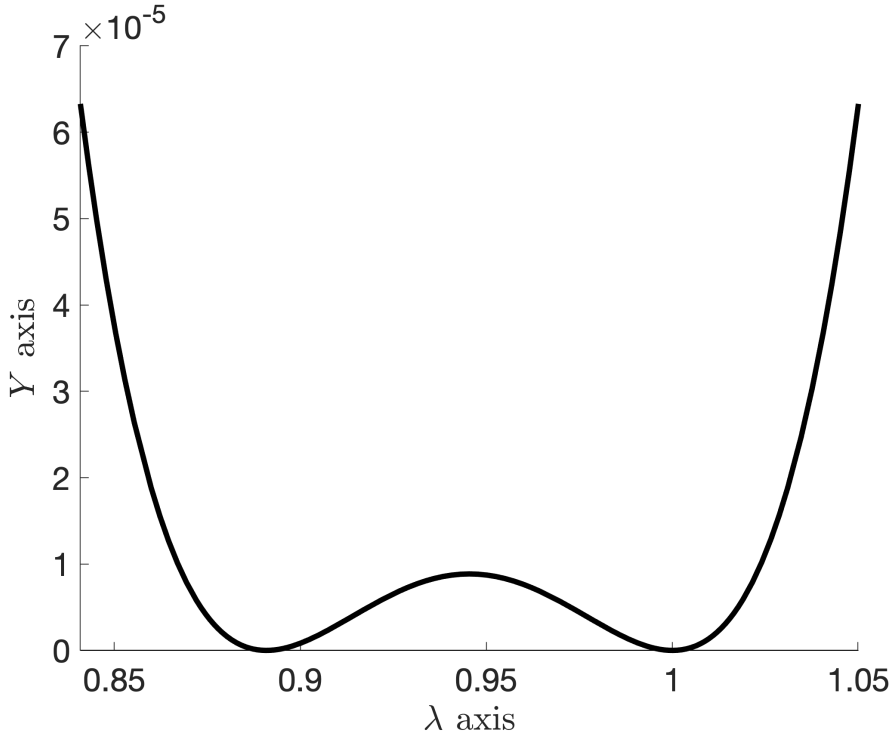


Figure E.10: Characteristic polynomial associated to the V-7⟶I-7 cadence

E.2.5. bVI△ ⟶I-7 Cadence

We are completing the calculations corresponding to the relationship between degrees of the Aeloian mode. In this way we structure the edges of the graph of tonal functions. We begin by assembling a link with the chords of this section:

$$E_{(bVI^r_c\triangle|I^r_c-7)} = \begin{pmatrix} G & Bb \\ Eb & G \\ C & Eb \\ Ab & C \end{pmatrix} \tag{E.53}$$

We calculate the link matrix calculating all the distances Δ_{ij}. We build the matrix L in the usual way where each row collects the metrics between the first class of the first chord and the metrics of the second in an ordered manner.

$$L_{(bVI^r\triangle|I^r-7)} = \begin{pmatrix} 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 0 & 0 & 4 & 3 \end{pmatrix} \tag{E.54}$$

Following the steps of the Hungarian algorithm we develope the L matrix:

$$L_{(bVI^r\triangle|I^r-7)} = \begin{pmatrix} 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \end{pmatrix} \longrightarrow L^F_{(bVI^r\triangle|I^r-7)} = \begin{pmatrix} 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 1 & 0 & 4 & 3 \end{pmatrix}$$

$$\longrightarrow L^H_{(bVI^r\triangle|I^r-7)} = \begin{pmatrix} 2 & \boxed{0} & 4 & 5 \\ 4 & 4 & \boxed{0} & 3 \\ 1 & 5 & 3 & \boxed{0} \\ \boxed{0} & 0 & 4 & 3 \end{pmatrix}$$

From the distribution of boxes that we have found on L^H, we obtain the solutions contained in the set S. Thus, when constructing S, we have all the information to obtain an optimal link later.

$$S(L^H_{(bVI^r\triangle|I^r-7)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

Using S we find the optimal pairing between voices between both chords, then, fixing the first chord, we pair the second one, being the function nabla, minimal.

$$\left[E^o_{(bVI_c\triangle|I_c-7)} \right]_\nabla = \left[\begin{pmatrix} G & G \\ Eb & Eb \\ C & C \\ Ab & Bb \end{pmatrix} \right]_\nabla \tag{E.55}$$

Using the transformation T we implicitly connect the matrix L with the matrix of the endomorphism, where in this case it is not necessary to recover the sign of any metric.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bVI_c\triangle)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Ab_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{E.56}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)(s^0 - \lambda)^3$$

Inspecting the characteristic polynomial we study the algebraic multiplicities, in such a way that three formulas remain: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 1$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bIV\triangle)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We break down the wave functions of each voicing into separate notes to visualize how they connect when we choose an opening for each voicing.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVI\triangle}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Ab_{z_4}}(t) \\ \psi_{I-7}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Bb_{z_4}}(t) \end{cases}$$

The wave functions are specified by the harmonic distribution the choice of subscripts:

$$\psi_{bVI_c\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVI_c\triangle)} - e^{-2\pi tki\psi_j(bVI\triangle)}}{2i}$$

$$\longrightarrow \psi_{I_c-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bVI\triangle$ related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bVI\triangle|I-7)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s^2 - 3)\ \lambda^3 + (3\,s^2 + 3)\ \lambda^2 + (-3\,s^2 - 1)\ \lambda + s^2$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{4} - \frac{3}{4}\right)\ \lambda^4 + (s^2 + 1)\ \lambda^3 + \left(-\frac{3\,s^2}{2} - \frac{1}{2}\right)\ \lambda^2 + s^2\ \lambda$

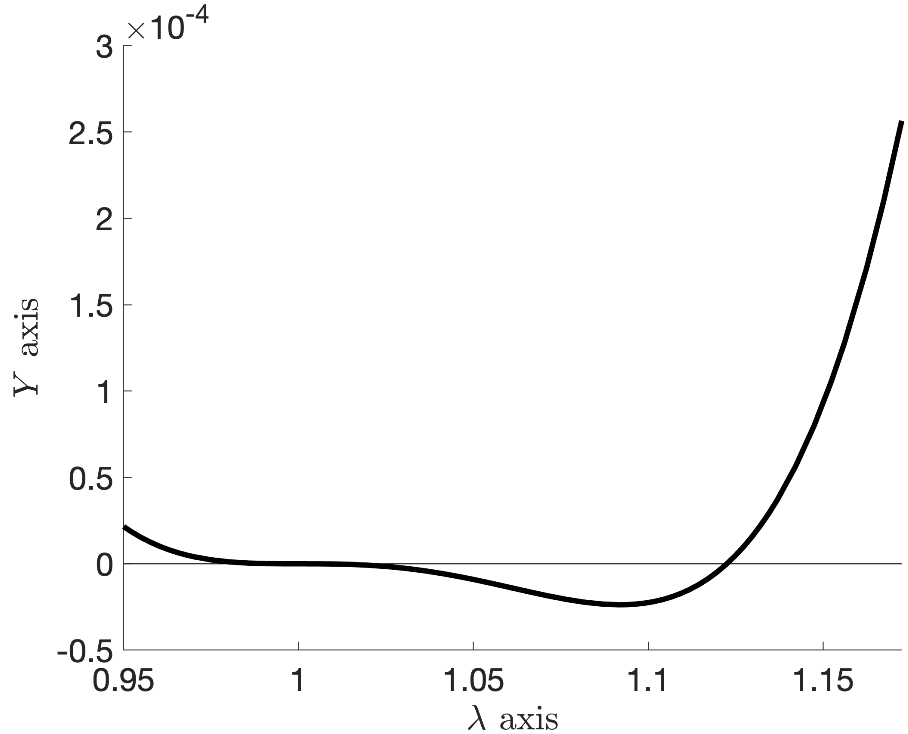


Figure E.11: Characteristic polynomial associated to the bVI△ ⟶I-7 cadence

E.2.6. bVII 7⟶I-7 Cadence

The link will be the following matrix:

$$E_{(bVII_c^r7|I_c^r-7)} = \begin{pmatrix} Ab & Bb \\ F & G \\ D & Eb \\ Bb & C \end{pmatrix} \tag{E.57}$$

Then, we calculate the link matrix calculating all the distances Δ_{ij}:

$$L_{(bVII^r7|I^r-7)} = \begin{pmatrix} 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \end{pmatrix} \tag{E.58}$$

Following the steps of the hungarian algorithm we develope the L matrix:

$$L_{(bVII^r\triangle|I^r-7)} = \begin{pmatrix} 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \\ 4 & 5 & 1 & 2 \\ 0 & 3 & 5 & 2 \end{pmatrix} \longrightarrow L^F_{(bVII^r7|I^r-7)} = \begin{pmatrix} 1 & 0 & 4 & 3 \\ 3 & 0 & 0 & 3 \\ 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 2 \end{pmatrix}$$

$$\longrightarrow L^H_{(bVII^r7|I^r-7)} = \begin{pmatrix} 1 & \boxed{0} & 4 & 2 \\ 3 & 0 & \boxed{0} & 2 \\ 3 & 4 & 0 & \boxed{0} \\ \boxed{0} & 3 & 5 & 1 \end{pmatrix}$$

The solutions given by the algorithm will be:

$$S(L^H_{(bVII^r7|I^r-7)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(bVII_c7|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} Ab & G \\ F & Eb \\ D & C \\ Bb & Bb \end{pmatrix}\right]_\nabla \tag{E.59}$$

Now we calculate the Nabla value when the distance between class mappings is minimal. The minimum for al ∞ mappings is unique so the sum can be constructed:

$$\nabla(E^o(bVII_c\triangle \mid I_c-7)) = \sum_{j=1}^{n} \Omega \mid \int_{(E^o_{j1\,(bVII_c7|I_c-7)})}^{(E^o_{j2\,(bVII_c7|I_c-7)})} \phi^{-1}\, d\phi \mid_\Delta = 5$$

Once the S set is calculated we can form a generalization of a dimensionally optimized cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bVII_c\triangle)) \longrightarrow \psi(I_c-7)$$

$$\begin{pmatrix} s^{-\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{-\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} Ab_{z_1} \\ F_{z_2} \\ D_{z_3} \\ Bb_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{E.60}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^{-1} - \lambda)(s^{-2} - \lambda)^2$$

The multiplicity of each set of roots is: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(bVII7)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_I(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII7}(t) = \psi_{Ab_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{D_{z_3}} + \psi_{Bb_{z_4}}(t) \\ \psi_{I-7}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Bb_{z_4}}(t) \end{cases}$$

$$\psi_{bVII7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII7)} - e^{-2\pi tki\psi_j(bVII7)}}{2i} \longrightarrow$$
$$\psi_{I-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ then following the polynomial criterion we obtain the function of the degree $bVII7$ related to the Aoelian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bVII7|I-7)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s} - \frac{2}{s^2} - 1\right)\lambda^3 + \left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} + \frac{1}{s^4}\right)\lambda^2 + \left(-\frac{2}{s^3} - \frac{1}{s^4} - \frac{1}{s^5}\right)\lambda + \frac{1}{s^5}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2+s+2}{s^2\,4}\right)\lambda^4 + \frac{s^3+2\,s^2+2\,s+1}{s^4\,3}\lambda^3 + \left(-\frac{2\,s^2+s+1}{s^5\,2}\right)\lambda^2 + \frac{\lambda}{s^5}$

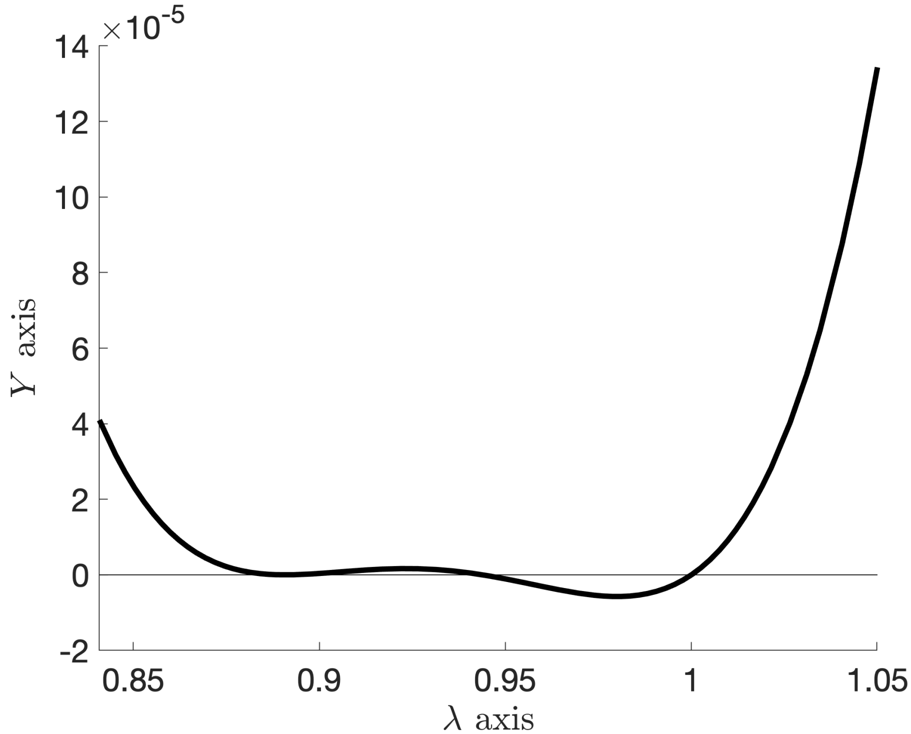


Figure E.12: Characteristic polynomial associated to the bVII 7⟶I-7 cadence

E.2.7. Aeolian tonal functions

$IIo \longrightarrow I-$	$\Phi[E_{(IIo\|I-)}] \in D^{\mathbb{R}[\lambda]}$
$bIII \longrightarrow I-$	$\Phi[E_{(bIII\ \|I-)}] \in T^{\mathbb{R}[\lambda]}$
$IV- \longrightarrow I-$	$\Phi[E_{(IV-\|I-)}] \in D^{\mathbb{R}[\lambda]}$
$V- \longrightarrow I-$	$\Phi[E_{(V-\|I-)}] \in S^{\mathbb{R}[\lambda]}$
$bVI \longrightarrow I-$	$\Phi[E_{(bVI\|I-)}] \in T^{\mathbb{R}[\lambda]}$
$bVII \longrightarrow I-$	$\Phi[E_{(bVII\|I-)}] \in S^{\mathbb{R}[\lambda]}$

$II\o 7 \longrightarrow I-7$	$\Phi[E_{(II\o 7\|I-7)}] \in S^{\mathbb{R}[\lambda]}$
$bIII\triangle \longrightarrow I-7$	$\Phi[E_{(bIII\triangle\|I-7)}] \in T^{\mathbb{R}[\lambda]}$
$IV-7 \longrightarrow I-7$	$\Phi[E_{(IV-7\|I-7)}] \in S^{\mathbb{R}[\lambda]}$
$V-7 \longrightarrow I-7$	$\Phi[E_{(V-7\|I-7)}] \in D^{\mathbb{R}[\lambda]}$
$bVI\triangle \longrightarrow I-7$	$\Phi[E_{(bVI\triangle\|I-7)}] \in T^{\mathbb{R}[\lambda]}$
$bVII7 \longrightarrow I-7$	$\Phi[E_{(bVII7\|I-7)}] \in D^{\mathbb{R}[\lambda]}$

Appendix F

The Phrygian Mode

Christian Perner

https://unsplash.com/photos/white-concrete-building-wall-fYO1T495QCM

F.1. The Phrygian mode for n =3

F.1.1. bII$\longrightarrow$ I- Cadence

We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(bII^r_c|I^r_c-)} = \begin{pmatrix} Ab & G \\ F & Eb \\ Db & C \end{pmatrix} \tag{F.1}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(bII^r|I^r-)} = \begin{pmatrix} 1 & 5 & 4 \\ 2 & 2 & 5 \\ 6 & 2 & 1 \end{pmatrix} \tag{F.2}$$

Then, following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bII^r|I^r-)} = \begin{pmatrix} 1 & 5 & 4 \\ 2 & 2 & 5 \\ 6 & 2 & 1 \end{pmatrix} \longrightarrow L^F_{(bII^r|I^r-)} = \begin{pmatrix} 0 & 4 & 3 \\ 0 & 0 & 3 \\ 5 & 1 & 0 \end{pmatrix} \longrightarrow L^H_{(bII^r|I^r-)} = \begin{pmatrix} \boxed{0} & 4 & 3 \\ 0 & \boxed{0} & 3 \\ 5 & 1 & \boxed{0} \end{pmatrix} \quad \text{(F.3)}$$

The solutions for $L^H_{(bII^r|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice- leading $S(L^H_{(bII^r-|I^r-)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$ and we calculate the optimal link class:

$$\left[E^o_{(bII_c|I_c-)}\right]_\nabla = \left[\begin{pmatrix} Ab & G \\ F & Eb \\ Db & C \end{pmatrix}\right]_\nabla \quad \text{(F.4)}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bII_c|I_c)}) = 2 + 1 + 1 = 4$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(bII-|I)} = 4$.

Any optimal arrangement from an optimal progresion $E^o_{(bII|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bII_c-)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Ab_{z_1} \\ F_{z_2} \\ Db_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \end{pmatrix} \quad \text{(F.5)}$$

Then the characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^{-1} - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-1} - \lambda \end{pmatrix} \quad \text{(F.6)}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^{-1} - \lambda)^2 (s^{-2} - \lambda)$$

The roots of the polynomial make up three clearly differentiated sets based on their position with respect to the stabilizer of the M group. According to its relationship with $E(M)$, the convergence of one chord on another is determined, where we have that these three sets would be: $\lambda^- = \{s^{-2}, s^{-1}\}$, $\lambda^0 = \{\emptyset\}$ and $\lambda^+ = \{\emptyset\}$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bII_c)$ or $\psi(I_c)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We describe the wave functions of each voicing where the subscripts determine the opening of both the antecedent and the consequent voicing. Thus, following the optimization conditions that maintain the value of absolute perception at its minimum, we have:

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bII_c}(t) = \psi_{Ab_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{Db_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

The wave functions can also be written as summation, where for an arbitrary harmonic distribution Γ and a set of functions ψ that respect the optimization conditions, they will be written as:

$$\psi_{bII_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bII_c)} - e^{-2\pi tki\psi_j(bII_c)}}{2i}$$

$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ then, following the polynomial criterion we obtain the function of the degree bII related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bII|I-)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{2}{s} - \frac{1}{s^2}\right) \lambda^2 + \left(\frac{1}{s^2} + \frac{2}{s^3}\right) \lambda - \frac{1}{s^4}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s+1}{3\,s^2}\right) \lambda^3 + \frac{s+2}{2\,s^3} \lambda^2 + \left(-\frac{1}{s^4}\right) \lambda$

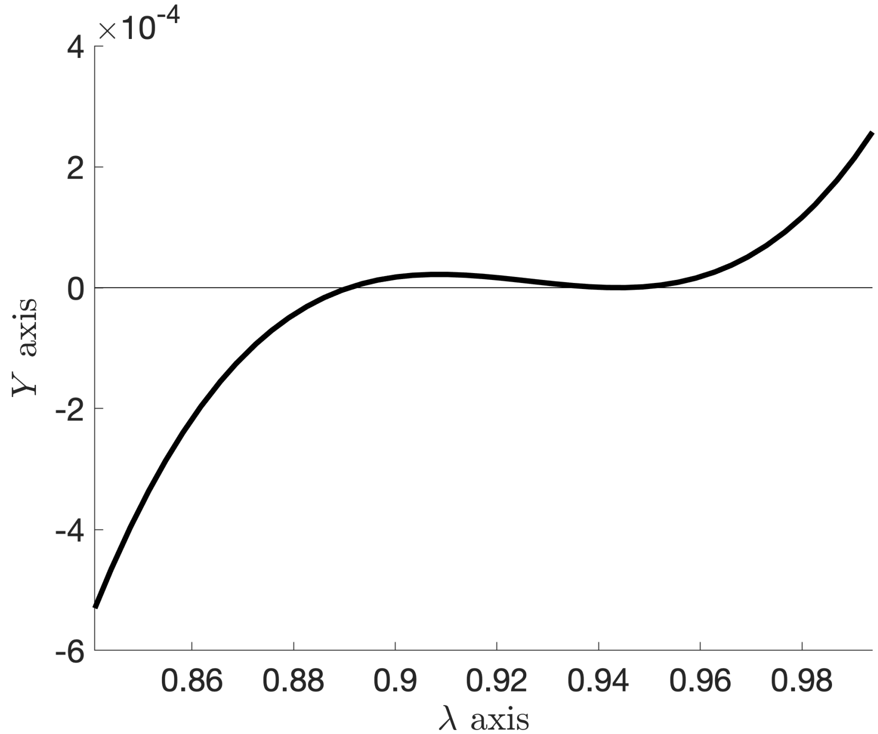


Figure F.1: Characteristic polynomial associated to the bII⟶ I- cadence

F.1.2. bIII⟶ I- Cadence

We continue calculating the tonal functions between degrees in the Phrygian mode with the aim of having the directed edges in the graph of tonal functions. Thus, we perform the calculations using the construction of L and its resolution using the Hungarian algorithm to find each tonal function. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(bIII_c^r|I_c^r-)} = \begin{pmatrix} Bb & G \\ G & Eb \\ Eb & C \end{pmatrix} \tag{F.7}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(bIII^r|I^r-)} = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 4 & 5 \\ 4 & 0 & 3 \end{pmatrix} \tag{F.8}$$

Then following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bIII^r|I^r-)} = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 4 & 5 \\ 4 & 0 & 3 \end{pmatrix} \longrightarrow L^F_{(bIII^r|I^r-)} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 4 & 5 \\ 4 & 0 & 3 \end{pmatrix} \longrightarrow L^H_{(bIII^r|I^r-)} = \begin{pmatrix} 1 & 3 & \boxed{0} \\ \boxed{0} & 4 & 5 \\ 4 & \boxed{0} & 3 \end{pmatrix} \tag{F.9}$$

Then the solutions for $L^H_{(bIII^r|I^r-)}$ when both triads are in root position becomes the following set, wich represents the minimum voice-leading:

$$S(L^H_{(bIII^r|I^r-)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

We calculate the optimal link class:

$$\left[E^o_{(bIII_c|I_c-)} \right]_\nabla = \left[\begin{pmatrix} Bb & C \\ G & G \\ Eb & Eb \end{pmatrix} \right]_\nabla \tag{F.10}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bIII_c|I_c)}) = 2 + 0 + 0 = 2$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(bIII|I)} = 2$.

Any optimal arrangement from an optimal progresion $E^o_{(bIII-|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(bIII_c)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Bb_{z_1} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1+1} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} \tag{F.11}$$

The characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with assigned values l_1, l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{F.12}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)^2 (s^2 - \lambda)$$

The roots of the characteristic polynomial are separated into three sets based on their placement with respect to the stabilizer $E(M)$ in such a way that we have $\lambda^+ = \{s^2\}$, $\lambda^0 = \{\emptyset\}$ and $\lambda^- = \{\emptyset\}$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bIII_c)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the wave functions inside the bracket for each voicing, where using the typical midi notation to indicate the specific octave of each class, we have two wave functions that have two common classes and one that varies.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bIII_c}(t) = \psi_{Bb_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{C_{z_1+1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \end{cases}$$

Wave functions in compact sum form would be written using the conventional notation as:

$$\psi_{bIII_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bIII_c)} - e^{-2\pi tki\psi_j(bIII_c)}}{2i}$$
$$\longrightarrow \psi_{I_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bIII$ related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bIII|I-)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - 2)\ \lambda^2 + (2\, s^2 + 1)\ \lambda - s^2$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2}{3} - \frac{2}{3}\right)\ \lambda^3 + \left(s^2 + \frac{1}{2}\right)\ \lambda^2 + (-s^2)\ \lambda$

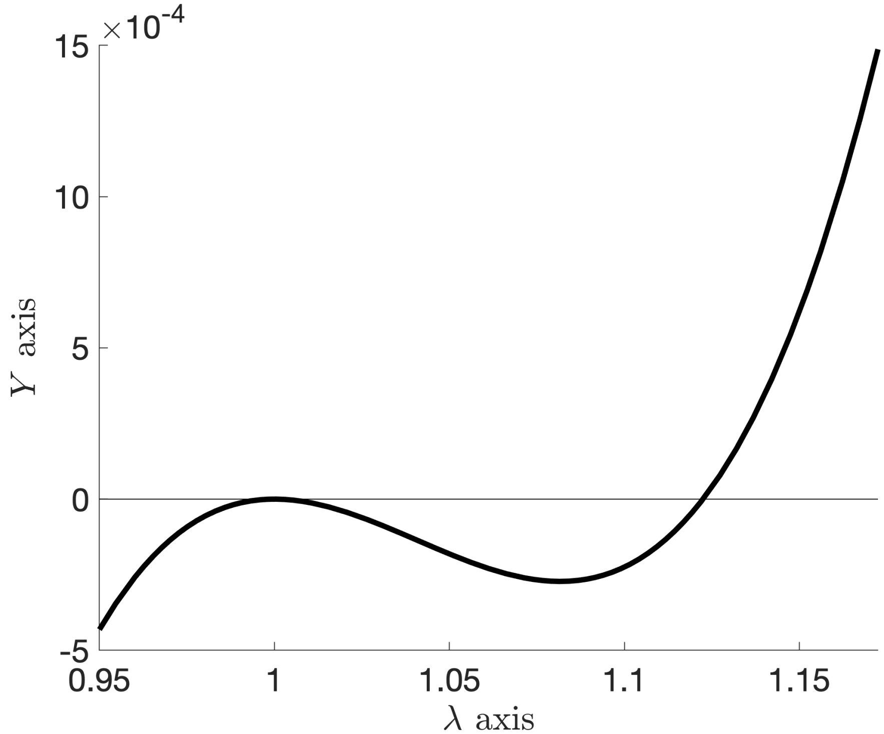


Figure F.2: Characteristic polynomial associated to the bIII⟶ I- cadence

F.1.3. IV-⟶ I- Cadence

Continuing with our task of obtaining the tonal functions in each mode when we fix the dimension of the chords, we are going to study how the roots of the tonal function behave between two minor triads that are a fourth apart. In this way, we set up the link between the fourth and the first degree of the Phrygian mode. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(IV_c^r - | I_c^r -)} = \begin{pmatrix} C & G \\ Ab & Eb \\ F & C \end{pmatrix} \tag{F.13}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(IV^r - | I^r -)} = \begin{pmatrix} 5 & 3 & 0 \\ 1 & 5 & 4 \\ 2 & 2 & 5 \end{pmatrix} \tag{F.14}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(IV^r-|I^r-)} = \begin{pmatrix} 5 & 3 & 0 \\ 1 & 5 & 4 \\ 2 & 2 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r-|I^r-)} = \begin{pmatrix} 5 & 3 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 3 \end{pmatrix} \longrightarrow L^H_{(IV^r-|I^r-)} = \begin{pmatrix} 5 & 3 & \boxed{0} \\ \boxed{0} & 4 & 3 \\ 0 & \boxed{0} & 3 \end{pmatrix} \tag{F.15}$$

The solutions for $L^H_{(IV^r|I^r-)}$ when both triads are in root position becomes the following set, that represents the minimum voice leading:

$$S(L^H_{(IV^r|I^r-)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

Following the positions of the boxes in the matrix that ends the algorithm, we find how the classes of the antecedent and consequent chords are paired. In this way we write the optimal link and immediately generalize it to the nabla class.

$$\left[E^o_{(IV_c-|I_c-)} \right]_\nabla = \left[\begin{pmatrix} C & C \\ Ab & G \\ F & Eb \end{pmatrix} \right]_\nabla \tag{F.16}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(IV_c-|I_c)}) = 0 + 1 + 2 = 3$ and we write the optimal nabla value as a generalization for every tonality: $\nabla^o_{(IV-|I)} = 3$

Any optimal arrangement from an optimal progression $E^o_{(IV-|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(IV_c-)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ Ab_{z_2} \\ F_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ Eb_{z_3} \end{pmatrix} \tag{F.17}$$

The characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^{-1} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{F.18}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)(s^{-1} - \lambda)(s^{-2} - \lambda)$$

Studying the roots of the tonal function, we distinguish the three sets that arise from the contrast of the position of each root on the λ axis with the stabilizer of the group $E(M)$.Thus we obtain these sets that are written as: $\lambda^- = \{s^{-2}, s^{-1}\}$, $\lambda^0 = \{s^0\}$ and $\lambda^+ = \{\emptyset\}$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(IV_c-)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. Note by note, we specify, in the bracket, the wave functions of each voicing. We rely on the midi notation, since it reflects the equivalence between octaves. Thus, said functions would be written as follows, where the subscripts determine the opening of each one of the voicings.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV_c-}(t) = \psi_{C_{z_1}}(t) + \psi_{Ab_{z_2}}(t) + \psi_{F_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \end{cases}$$

Wave functions can be written in terms of sines, where each sine is a harmonic of the voicing in question. We clearly see that each voicing can be understood as a sum of $n \times h$ harmonics that follow the Γ distribution. In this way, the general formulas of the antecedent and consequent voicing will be written as:

$$\psi_{IV_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(IV_c-)} - e^{-2\pi tki\psi_j(IV_c-)}}{2i}$$

$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$ then, following the polynomial criterion we obtain the function of the degree $IV-$ related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(IV-|I-)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s} - \frac{1}{s^2} - 1\right) \lambda^2 + \left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}\right) \lambda - \frac{1}{s^3}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2+s+1}{s^2\,3}\right) \lambda^3 + \frac{s^2+s+1}{s^3\,2} \lambda^2 + \left(-\frac{1}{s^3}\right) \lambda$

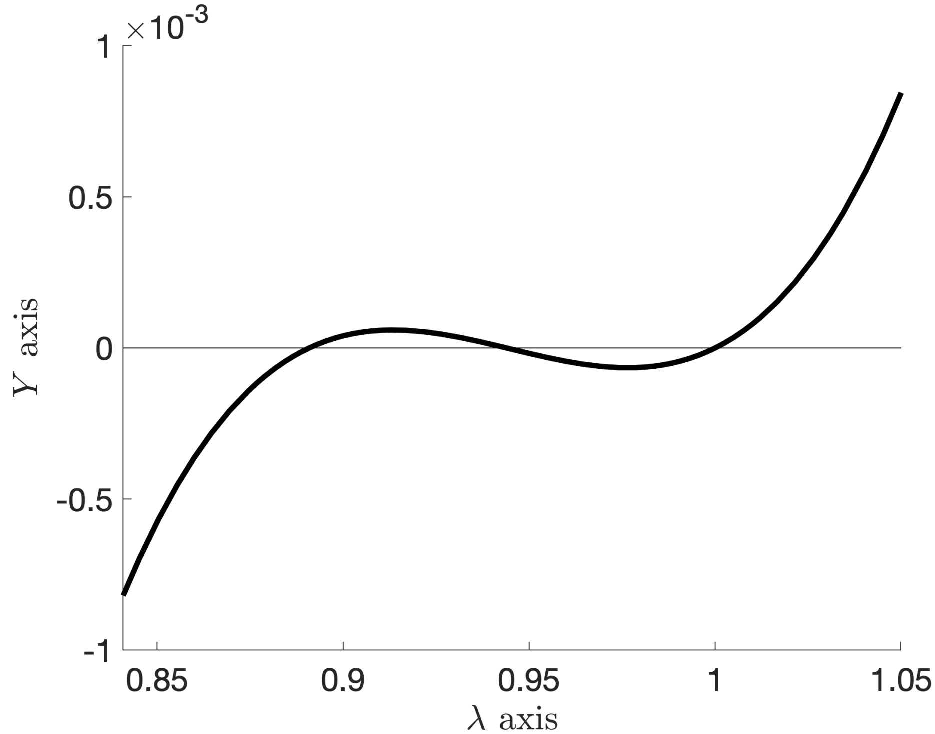


Figure F.3: Characteristic polynomial associated to the IV-⟶ I- cadence

F.1.4. Vo⟶ I- Cadence

We are still working on establishing the tonal functions between the degrees of the Phrygian mode. Through this process, in some moments, we will reach unusual cadences that will be of special interest for the construction of P progressions that stand out from the common progressions in modern music. Thus, we are going to study the relationship between the diminished fifth degree and the minor first degree. We use the usual problem construction, starting with the E link and ending at the tonal function.

$$E_{(V_c^r o|I_c^r -)} = \begin{pmatrix} Db & G \\ Bb & Eb \\ G & C \end{pmatrix} \tag{F.19}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(V^r o|I^r -)} = \begin{pmatrix} 6 & 2 & 1 \\ 3 & 5 & 2 \\ 0 & 4 & 5 \end{pmatrix} \tag{F.20}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(V^r o|I^r -)} = \begin{pmatrix} 6 & 2 & 1 \\ 3 & 5 & 2 \\ 0 & 4 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r o|I^r -)} = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix} \longrightarrow L^H_{(V^r o|I^r -)} = \begin{pmatrix} 5 & \boxed{0} & 0 \\ 1 & 2 & \boxed{0} \\ \boxed{0} & 3 & 5 \end{pmatrix} \tag{F.21}$$

Then the solutions for $L^H_{(V^r o|I^r -)}$ when both triads are in root position becomes the following set $S(L^H_{(V^r o|I^r -)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$.

With the information that appears in the distribution of boxes drawn by the algorithm, we are able to understand how the classes are matched between both tonal centers, then we can calculate the tonal function of the antecedent and consequent chord.

$$\left[E^o_{(V_c o|I_c -)} \right]_\nabla = \left[\begin{pmatrix} Db & Eb \\ Bb & C \\ G & G \end{pmatrix} \right]_\nabla \tag{F.22}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(V_c o|I_c -)}) = 2 + 2 + 0 = 4$$

We write the optimal nabla value as a generalization for every tonality:

$$\nabla^o_{(V o|I -)} = 4$$

We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(V_c o)) \longrightarrow \psi(I_c -))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Db_{z_1} \\ Bb_{z_2} \\ G_{z_3} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{F.23}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^2 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{F.24}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2 - \lambda)(s^2 - \lambda)(s^0 - \lambda)$$

Using the implicit transformation T, we reach the roots of the tonal function, then we think about its classification using the polynomial criterion. In the set of divergent roots we only find one root:

$$\lambda^+ = \{s^2\}$$

On the stabilizer $E(M)$ we find a root that has the same value as the stabilizer and is synonymous with the fact that there is at least one voice that moves us in the optimal link. We note that using the conjunctistic notation does not provide how many roots appear on the stabilizer, this being the main reason why we have used the language of multiplicities, since we do care about the number of roots that appear on $E(M)$.

$$\lambda^0 = \{s^0\}$$

To the left of $E(M)$ we find an empty set of roots, since none exist, which means that there are no voices moving down the optimal link.

$$\lambda^- = \{\emptyset\}$$

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(V_c o)$ or $\psi(I_c -)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. With the help of midi notation, we write the wave functions of the optimal voicings of the antecedent and consequent tonal centers. In this way we establish that for a set of integers that determine the opening of the array, we will be able to have the equations for that opening just by choosing said set.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{V_c o}(t) = \psi_{Db_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}}(t) \\ \psi_{I_c -}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

For completeness and insight, we also write the same wave functions in their general form as we have already done in previous cases. We use the arrow notation $\longrightarrow$ to indicate the temporal order in perception.

$$\psi_{V_c o}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(V_c o)} - e^{-2\pi tki\psi_j(V_c o)}}{2i}$$

$$\longrightarrow \psi_{I_c -}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c -)} - e^{-2\pi tki\psi_j(I_c -)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$ then, following the polynomial criteria we obtain the function of the degree Vo related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(Vo|I-)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-2\,s^2 - 1)\ \lambda^2 + (s^4 + 2\,s^2)\ \lambda - s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s^2}{3} - \frac{1}{3}\right)\ \lambda^3 + \frac{s^2\,\left(s^2+2\right)}{2}\,\lambda^2 + (-s^4)\ \lambda$

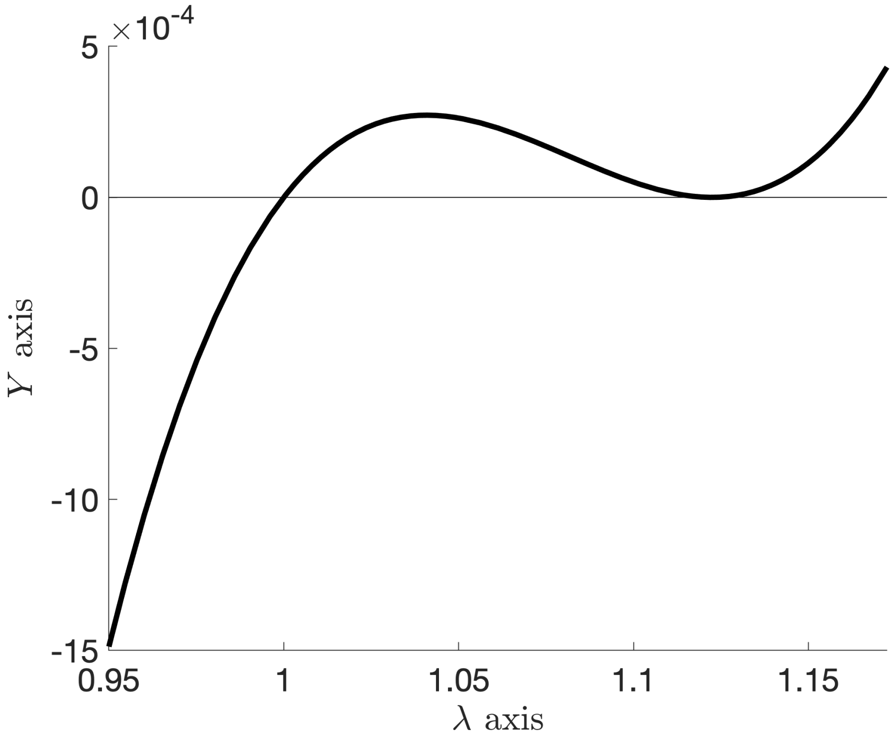


Figure F.4: Characteristic polynomial associated to the Vo⟶ I- cadence

F.1.5. bVI⟶ I- Cadence

We continue with the process of calculating the tonal functions. At this point we are going to study how the sixth degree and the first degree of the Phrygian mode are related when we are considering their triads. Thus we follow the procedure, taking advantage of the connection W.F.C. We will follow the usual procedure, generating a link E calculating its matrix L and using the transformation T to at some point reach the solution.

$$E_{(bVI_c^r|I_c^r-)} = \begin{pmatrix} Eb & G \\ C & Eb \\ Ab & C \end{pmatrix} \tag{F.25}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix. At this point we can remove the subscripts indicating the key since L is preserved:

$$L_{(bVI^r|I^r-)} = \begin{pmatrix} 6 & 2 & 1 \\ 3 & 5 & 2 \\ 0 & 4 & 5 \end{pmatrix} \tag{F.26}$$

We point out that the L matrices are the same between degrees, that is to say that, regardless of the tone in which we are speaking, the L matrix between two degrees is the same. This occurs since the color theorem equates each entry of the matrix by transporting the classes of each metric involved in each entry. For this reason we omit the subscripts.

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bVI^r|I^r-)} = \begin{pmatrix} 4 & 0 & 3 \\ 5 & 1 & 0 \\ 1 & 5 & 4 \end{pmatrix} \longrightarrow L^F_{(bVI^r|I^r-)} = \begin{pmatrix} 4 & 0 & 3 \\ 5 & 1 & 0 \\ 0 & 4 & 3 \end{pmatrix} \longrightarrow L^H_{(bVI^r|I^r-)} = \begin{pmatrix} 4 & \boxed{0} & 3 \\ 5 & 1 & \boxed{0} \\ \boxed{0} & 4 & 3 \end{pmatrix} \tag{F.27}$$

The solutions for $L^H_{(bVI^r|I^r-)}$ when both triads are in root position becomes the following set: $S(L^H_{(bVI^r|I^r-)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$:

$$\left[E^o_{(bVI_c|I_c-)}\right]_\nabla = \left[\begin{pmatrix} Eb & Eb \\ C & C \\ Ab & G \end{pmatrix}\right]_\nabla \tag{F.28}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share abla value: $\nabla(E^o_{(bVI_c|I_c-)}) = 0 + 0 + 1 = 1$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(bVI|I-)} = 1$.

Any optimal arrangement from an optimal progresion $E^{o}_{(bVI|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bVI_c)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Ab_{z_3} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{F.29}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^{-1} - \lambda \end{pmatrix} \tag{F.30}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)(s^0 - \lambda)(s^{-1} - \lambda)$$

By using the transformation T and reaching the tonal function we see that a root appears to the left of $E(M)$. Thus, in this set the said root appears, which is representing that at least one voice moves downwards.

$$\lambda^- = \{s^{-1}\}$$

On the stabilizer $E(M)$ there are two roots that take the value $E(M) = 1$ on the λ axis. Thus, on said axis there are two roots on the stabilizer, although the set language at this point only informs us that there is at least one root on the stabilizer. This is the reason why the polynomial criterion is related to the algebraic multiplicities described above, since set theory modeling is not enough to accurately characterize the movement of voices. This is a specific question of voices that move with the same proportion and in the same direction, where the language of the sets we are talking about makes them indistinguishable. Although the cardinality of the set is one, there are two roots over the stabilizer for this case.

$$\lambda^0 = \{s^0\}$$

On the other hand, to the right of the stabilizer. We see that no roots appear.

$$\lambda^+ = \{\emptyset\}$$

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVI_c)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the wave functions of the pair of voicings breaking them down into the wave functions of each note. Using the bracket and focusing on the subscripts we see how the voices are linked in the optimal link.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVI_c}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Ab_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

Following the optimization conditions and given a particular aperture, we write the wave function of the voicings for a Γ distribution.

$$\psi_{bVI_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVI_c)} - e^{-2\pi tki\psi_j(bVI_c)}}{2i}$$

$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$ then, following the polynomial criterion we obtain the function of the degree bVI related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bVI|I-)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s-2)\ \lambda^2 + (2\,s+1)\ \lambda - s$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s}{3} - \frac{2}{3}\right)\ \lambda^3 + \left(s + \frac{1}{2}\right)\ \lambda^2 + (-s)\ \lambda$

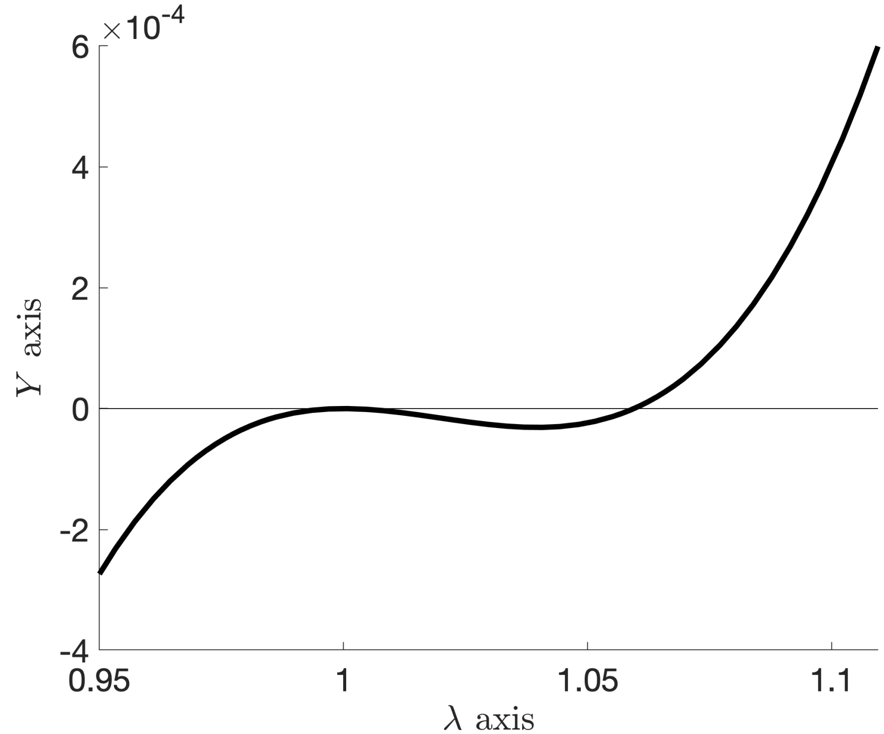


Figure F.5: Characteristic polynomial associated to the bVI⟶ I- cadence

F.1.6. bVII-⟶ I- Cadence

We continue to progress in the calculation of the tonal functions, calculating the function between the degrees of the Phrygian mode. Thus, we establish the following tonal function using the usual method. We calculate the link E and build the system of matrices L that will end up finding one or several distributions of boxes. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(bVII_c^r-|I_c^r-)} = \begin{pmatrix} F & G \\ Db & Eb \\ Bb & C \end{pmatrix} \tag{F.31}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(bVII^r-|I^r-)} = \begin{pmatrix} 2 & 2 & 5 \\ 6 & 2 & 1 \\ 3 & 5 & 2 \end{pmatrix} \tag{F.32}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bVII^r-|I^r-)} = \begin{pmatrix} 2 & 2 & 5 \\ 6 & 2 & 1 \\ 3 & 5 & 2 \end{pmatrix} \longrightarrow L^F_{(bVII^r-|I^r-)} = \begin{pmatrix} 0 & 0 & 3 \\ 5 & 1 & 0 \\ 1 & 3 & 0 \end{pmatrix} \longrightarrow L^H_{(bVII^r-|I^r-)} = \begin{pmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{F.33}$$

We have omitted the writing of the L^{FC} matrix but in fact we have applied a zero covering of two lines to said matrix to reach the L^H matrix that does not appear with a distribution of boxes since it has multiple solutions. Thus $L^{FC} = L^F$ and using zero covering across the first row and third column we would reach

$$L^H = \begin{pmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

This is why we pass from the matrix L^F to the matrix L^H. Since the matrix L^H has multiple solutions, then we have to apply the Zero method to find them, then we will set each zero in an ordered way until we find all the solutions.

F.1.7. The zero method over $L^H_{(bVII^r-|I^r-)}$

As we have seen that the matrix allows several solutions, then we select each zero from left to right and from top to bottom, going through the nine zeros. Every time we select a zero, a possible solution appears. The solutions can appear several times later, finally we have to see how many different solutions we have.

$$L^{Z_1}_{(bVII^r-|I^r-)} = \begin{pmatrix} \boxed{0}^* & 0 & 4 \\ 4 & \boxed{0} & 0 \\ 0 & 2 & \boxed{0} \end{pmatrix} \mid L^{Z_2}_{(bVII^r-|I^r-)} = \begin{pmatrix} 0 & \boxed{0}^* & 4 \\ 4 & 0 & \boxed{0} \\ \boxed{0} & 2 & 0 \end{pmatrix} \mid L^{Z_3}_{(bVII^r-|I^r-)}$$
$$= \begin{pmatrix} \boxed{0} & 0 & 4 \\ 4 & \boxed{0}^* & 0 \\ 0 & 2 & \boxed{0} \end{pmatrix}$$

$$L^{Z_4}_{(bVII^r-|I^r-)} = \begin{pmatrix} 0 & \boxed{0} & 4 \\ 4 & 0 & \boxed{0}^* \\ \boxed{0} & 2 & 0 \end{pmatrix} \mid L^{Z_5}_{(bVII^r-|I^r-)} = \begin{pmatrix} 0 & \boxed{0} & 4 \\ 4 & 0 & \boxed{0} \\ \boxed{0}^* & 2 & 0 \end{pmatrix} \mid L^{Z_6}_{(bVII^r-|I^r-)}$$
$$= \begin{pmatrix} \boxed{0} & 0 & 4 \\ 4 & \boxed{0} & 0 \\ 0 & 2 & \boxed{0}^* \end{pmatrix}$$

The solutions for $L^H_{(bVII^r-|I^r)}$ when both triads are in root position becomes the following sets, wich represents the minimum voice-leading:

$$S^1(L^H_{(bVII^r-|I^r-)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$$

$$S^2(L^H_{(bVII^r-|I^r-)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$$

F.1.8. Tonal function for $S^1(L^H_{(bVII_c-|I_c-)})$

Using the first solution, we can now compute an optimal link, where the classes are not going to move from the original arrangement since the boxes appear along the diagonal of the L^H matrix. We calculate the optimal link class:

$$\left[E^o_{(bVII_c-|I_c-)}\right]_\nabla = \left[\begin{pmatrix} F & G \\ Db & Eb \\ Bb & C \end{pmatrix}\right]_\nabla \tag{F.34}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bVII_c-|I_c-)}) = 2+2+2 = 6$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(bVII-|I-)} = 6$.

Any optimal arrangement from an optimal progression $E^o_{(bVII-|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(bVII_c-)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ Db_{z_2} \\ Bb_{z_3} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3+1} \end{pmatrix} \tag{F.35}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with assigned values l_1, l_2 and l_3. We see that by substituting the values of each number l we find the number of semitones in the voices. In this case it is not necessary to recover the sign since all the voices rise in the first solution of the optimum. Thus, when substituting, the exponents are positive. We take the characteristic polynomial following the expression:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^2-\lambda & 0 & 0 \\ 0 & s^2-\lambda & 0 \\ 0 & 0 & s^2-\lambda \end{pmatrix} \tag{F.36}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2-\lambda)(s^2-\lambda)(s^2-\lambda)$$

Knowing the characteristic polynomial we distinguish three sets based on the placement of the roots based on the stabilizer of the M group. Then, based on $E(M)$, these three sets are drawn: $\lambda^+ = \{s^2\}$, $\lambda^0 = \{\emptyset\}$ and $\lambda^- = \{\emptyset\}$.

Using the well known principle of sine waves superposition, we know that for every frecuency $\phi \in \Phi^+$ as a component of $\psi(bVII_c-)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frecuency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the wave functions of each voicing leaving open the choice of the opening of each voicing, which depends, in the last instance; of the selection of subscripts of each wave function of each note.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII_c-}(t) = \psi_{F_{z_1}}(t) + \psi_{Db_{z_2}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3+1}}(t) \end{cases}$$

We write the general wave functions for each voicing, following the optimization conditions and respecting the voice leading.

$$\psi_{bVII_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c-)} - e^{-2\pi tki\psi_j(bVII_c-)}}{2i}$$

$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then, following the polynomial criteria we obtain the function of the degree $bVII-$ related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bVII-|I-)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-3\,s^2)\;\lambda^2 + (3\,s^4)\;\lambda - s^6$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + (-s^2)\;\lambda^3 + \frac{3\,s^4}{2}\,\lambda^2 + (-s^6)\;\lambda + \frac{s^8}{4}$

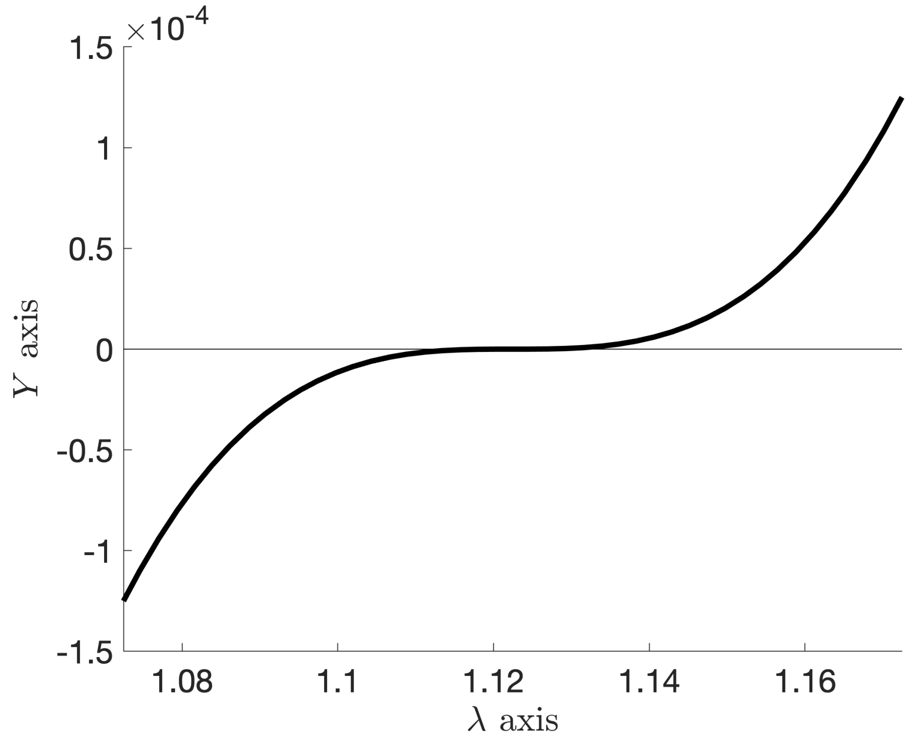


Figure F.6: Characteristic polynomial associated to the bVII-⟶ I- cadence

F.1.9. Tonal function for $S^2(L^H_{(bVII_c-|I_c-)})$

Using the set $S^2(L^H_{(bVII^r-|I^r-)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$ that comes from the Zero method we can study the second solution. Thus, we see that the optimal link is not unique and that the tonal function is dual. We calculate the optimal link class:

$$\left[E^o_{(bVII_c-|I_c-)}\right]_\nabla = \left[\begin{pmatrix} F & Eb \\ Db & C \\ Bb & G \end{pmatrix}\right]_\nabla \tag{F.37}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bVII_c-|I_c-)}) = 2+2+2 = 6$ and we write the optimal nabla value as a generalization for every tonality: $\nabla^o_{(bVII-|I-)} = 6$

Any optimal arrangement from an optimal progression $E^o_{(bVII-|I-)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(bVII_c-)) \longrightarrow \psi(I_c-))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ Db_{z_2} \\ Ab_{z_3} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ G_{z_3} \end{pmatrix} \tag{F.38}$$

The characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values l_1, l_2 and l_3. We see that the second tonal function is polarized, but on the other side of $E(M)$. That is to say that although both solutions are governed by the minimum value of the nabla function, the placement of the roots is different.

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^{-2} - \lambda & 0 & 0 \\ 0 & s^{-1} - \lambda & 0 \\ 0 & 0 & s^{-3} - \lambda \end{pmatrix} \tag{F.39}$$

Using the properties of the determinant , we arrive at the characteristic polynomial, and therefore, the second tonal function. Thus, for this link, we have two polarized tonal functions, each on one side of $E(M)$.

$$\Delta(C_\mathbb{E} - \lambda Id) = (s^{-2} - \lambda)(s^{-1} - \lambda)(s^{-3} - \lambda)$$

In this case the sets of roots change y and remain polarized to the left of $E(M)$. So these sets are written as: $\lambda^+ = \{\emptyset\}$, $\lambda^0 = \{\emptyset\}$ and $\lambda^- = \{s^{-2}, s^{-1}, s^{-3}\}$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVII_c-)$ or $\psi(I_c-)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the voicing wave functions, decomposing, as we have been doing, the wave function into an assumption of wave functions for each note, where; by choosing a particular set of subscripts we will determine an opening for each voicing.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII_c-}(t) = \psi_{F_{z_1}}(t) + \psi_{Db_{z_2}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{G_{z_3}}(t) \end{cases}$$

On the other hand, and to describe each case with some completeness, we also write the wave function of the voicing itself. So we have that, as in previous cases, we have two wave functions that follow one another in time.

$$\psi_{bVII_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c-)} - e^{-2\pi tki\psi_j(bVII_c-)}}{2i}$$
$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_c-)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ then, following the polynomial criterion we obtain the function of the degree $bVII-$ related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E^2_{(bVII|I-)}] \in D^{\mathbb{R}[\lambda]}}$$

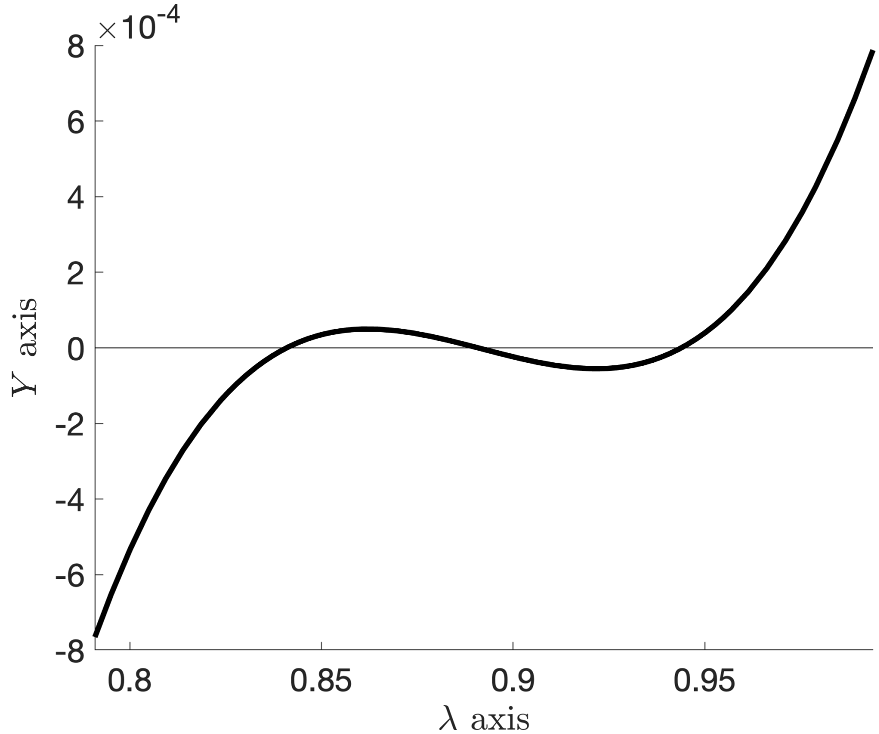


Figure F.7: Characteristic polynomial associated to the bVII-⟶ I- cadence

F.2. The Phrygian mode for n=4

F.2.1. bII△ ⟶I-7 Cadence

We continue advancing in the calculations and now we study the tonal functions of the Phrygian mode for four-voice chords. The interested reader can study the relationships between any chord regardless of its dimensions using the method provided in this work. In this case we are going to see that the tonal function is dual and that we have to use the Zero method to reach the solutions. In the first place, as usual, we calculate the link E, which will allow us the immediate construction of the matrix L.

$$E_{(bII_c^r\triangle|I_c^r-7)} = \begin{pmatrix} C & Bb \\ Ab & G \\ F & Eb \\ Db & C \end{pmatrix} \tag{F.40}$$

The link cadence will be constructed using the ordered collection of metrics Δ_{ij}. Thus, we calculate each metric by taking the pairs of classes in an ordered manner. We have already warned that we could build the L matrices in another way, but that, to preserve the order, we choose to always place the chords in the root position and then calculate the collection of meters to write L in an ordered way and be able to apply the steps to find the solution.

$$L_{(bII^r\triangle|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \\ 3 & 6 & 2 & 1 \end{pmatrix} \tag{F.41}$$

Following the steps of the Hungarian algorithm we develop the L matrix until we reach to the S set:

$$L_{(bII^r\triangle|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \\ 3 & 6 & 2 & 1 \end{pmatrix} \longrightarrow L^F_{(bII^r\triangle|I^r-7)} = \begin{pmatrix} 2 & 5 & 3 & 0 \\ 1 & 0 & 4 & 3 \\ 3 & 0 & 0 & 3 \\ 2 & 5 & 1 & 0 \end{pmatrix}$$

$$\longrightarrow L^H_{(bII^r\triangle|I^r-7)} = \begin{pmatrix} 1 & 5 & 3 & 0 \\ 0 & 0 & 4 & 3 \\ 2 & 0 & 0 & 3 \\ 1 & 5 & 1 & 0 \end{pmatrix}$$

Once we obtain the L^H matrix, we find the non canceled minimum and we operate until we reach L^{H^*}. We see that when reaching L^H this matrix has no solution, and we have to follow another step in the algorithm using zero covering. Thus we arrive at the matrix L^{H^*} where we have to apply the Zero method to separate the solutions:

$$L^{H}_{(bII^r\triangle|I^r-7)} = \begin{pmatrix} 1 & 5 & 3 & 0 \\ 0 & 0 & 4 & 3 \\ 2 & 0 & 0 & 3 \\ 1 & 5 & 1 & 0 \end{pmatrix} \longrightarrow L^{H^*}_{(bII^r\triangle|I^r-7)} = \begin{pmatrix} 0 & 4 & 2 & 0 \\ 0 & 0 & 4 & 4 \\ 2 & 0 & 0 & 4 \\ 0 & 4 & 0 & 0 \end{pmatrix} \tag{F.42}$$

F.2.2. The zero method over $L^{H^*}_{(bII^r\triangle|I^r-7)}$

We have taken the matrix L^H and we have applied the algorithm since it had no solution, covering the zeros with the minimum number of lines and applying the addition and subtraction steps of the minimum not covered. Thus we have reached a new matrix L^{H^*}. We assign, in an orderly manner, a fixed zero to each matrix and look for the solution. We will do this process as many times as there are zeros, in this case nine.

$$L^{H^*}_1 = \begin{pmatrix} \boxed{0}^* & 4 & 2 & 0 \\ 0 & \boxed{0} & 4 & 4 \\ 2 & 0 & \boxed{0} & 4 \\ 0 & 4 & 0 & \boxed{0} \end{pmatrix} \mid L^{H^*}_2 = \begin{pmatrix} 0 & 4 & 2 & \boxed{0}^* \\ 0 & \boxed{0} & 4 & 4 \\ 2 & 0 & \boxed{0} & 4 \\ \boxed{0} & 4 & 0 & 0 \end{pmatrix} \mid L^{H^*}_3 = \begin{pmatrix} 0 & 4 & 2 & \boxed{0} \\ \boxed{0}^* & 0 & 4 & 4 \\ 2 & \boxed{0} & 0 & 4 \\ 0 & 4 & \boxed{0} & 0 \end{pmatrix} \tag{F.43}$$

$$L^{H^*}_4 = \begin{pmatrix} \boxed{0} & 4 & 2 & 0 \\ 0 & \boxed{0}^* & 4 & 4 \\ 2 & 0 & \boxed{0} & 4 \\ 0 & 4 & 0 & \boxed{0} \end{pmatrix} \mid L^{H^*}_5 = \begin{pmatrix} 0 & 4 & 2 & \boxed{0} \\ \boxed{0} & 0 & 4 & 4 \\ 2 & \boxed{0}^* & 0 & 4 \\ 0 & 4 & \boxed{0} & 0 \end{pmatrix} \mid L^{H^*}_6 = \begin{pmatrix} 0 & 4 & 2 & \boxed{0} \\ 0 & \boxed{0} & 4 & 4 \\ 2 & 0 & \boxed{0}^* & 4 \\ \boxed{0} & 4 & 0 & 0 \end{pmatrix} \tag{F.44}$$

$$L^{H^*}_7 = \begin{pmatrix} 0 & 4 & 2 & \boxed{0} \\ 0 & \boxed{0} & 4 & 4 \\ 2 & 0 & \boxed{0} & 4 \\ \boxed{0}^* & 4 & 0 & 0 \end{pmatrix} \mid L^{H^*}_8 = \begin{pmatrix} 0 & 4 & 2 & \boxed{0} \\ \boxed{0} & 0 & 4 & 4 \\ 2 & \boxed{0} & 0 & 4 \\ 0 & 4 & \boxed{0}^* & 0 \end{pmatrix} \mid L^{H^*}_9 = \begin{pmatrix} \boxed{0} & 4 & 2 & 0 \\ 0 & \boxed{0} & 4 & 4 \\ 2 & 0 & \boxed{0} & 4 \\ 0 & 4 & 0 & \boxed{0}^* \end{pmatrix} \tag{F.45}$$

Then the solutions for L^{H^*} when both chords are in root position are described by three sets, each one representing a link where $\nabla(E)$ is minimal for al the permutations of the link:

$$S_1(L^{H^*}_{(bII^r\triangle|I^r-7)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}, \Delta_{44}\}$$
$$S_2(L^{H^*}_{(bII^r\triangle|I^r-7)}) = \{\Delta_{14}, \Delta_{22}, \Delta_{33}, \Delta_{41}\}$$
$$S_3(L^{H^*}_{(bII^r\triangle|I^r-7)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

When is time to choose specific notes we use the representation of a cadence as $C \in \Phi^4 \times \Phi^4$. In this case we have three optimal cadences for any z octave. We separate each solution in a particular case of study, then for each set S we will calculate an optimal link that comes from linking the voices following the distribution of boxes associated with said set S. Thus we distinguish three cases for this link, being the dual tonal function.

F.2.3. $S_1(L^H_{(bII^r\triangle|I^r-7)})$

Following the box distribution of the first solution, we find one of the optimal links. At this point it is enough for us to apply the transformation T implicitly on the distribution of boxes mentioned to find the matrix $C_{\mathbb{E}}$.

$$\left[E^1_{(bII_c\triangle|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} C & Bb \\ Ab & G \\ F & Eb \\ Db & C \end{pmatrix}\right]_\nabla \tag{F.46}$$

We now study the first ad of the three solutions for this cadence in the Phrygian mode. Thus, since the first solution is a distribution of boxes on the diagonal of the matrix L^{H^*} then we will arrange the frequency vectors as chords in their root position. Thus we will write the matrix equation of the endomorphism as follows:

$$\begin{gathered} C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4 \\ C_{\mathbb{E}}(\psi(bII_c\triangle)) \longrightarrow \psi(I_c - 7) \\ \begin{pmatrix} s^1 & 0 & 0 & 0 \\ 0 & s^2 & 0 & 0 \\ 0 & 0 & s^1 & 0 \\ 0 & 0 & 0 & s^2 \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ Ab_{z_2} \\ F_{z_3} \\ Db_{z_4} \end{pmatrix} = \begin{pmatrix} Bb_{z_1-1} \\ G_{z_2} \\ Eb_{z_3} \\ C_{z_4} \end{pmatrix} \end{gathered} \tag{F.47}$$

Taking the matrix $C_{\mathbb{E}}$ and calculating its characteristic polynomial we arrive at the tonal function, which we have to study for its classification. In this way, the polynomial would remain as:

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)^2 (s^1 - \lambda)^2$$

By observing said polynomial we have three equations for each of the algebraic multiplicities, which are nothing more than a classification of the algebraic multiplicity based on the position of each root with respect to $E(M)$. So we have these three equations: $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 4$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 0$. We clearly see that the divergent algebraic multiplicity is the only non-zero. Thus we observe the behavior of the voices in the first optimal link, where all the voices rise at the rate of a tone or semitone. In this way, the rest of the multiplicities are worth zero, since there is no voice that is repeated between the antecedent and the consequent tonal centers for this solution. There is also no voice that descends. With the use of multiplicities we completely characterize the movement of the voices and we can classify the tonal functions within the areas.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bII\triangle)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the decomposed functions note by note. In other words, we write them as the sum of each wave function of each note. We carry out this very explicit process for a matter of depth of concept. We see here, not differently from the rest of the cases, that we leave each function expressed while waiting for a selection of subscripts, which will not alter the minimum value of perception if voice-leading is optimal.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bII\triangle}(t) = \psi_{C_{z_1}}(t) + \psi_{Ab_{z_2}}(t) + \psi_{F_{z_3}}(t) + \psi_{Db_{z_3}}(t) \\ \psi_{I-7}(t) = \psi_{Bb_{z_1-1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

For completeness and order we write the general wave functions for each voicing where each function is composed of $n \times h$ harmonics, each harmonic has an amplitude determined by the distribution Γ which is a vector in $\mathbb{R}^h$ and the arrow $\longrightarrow$ indicates the temporal order of each of the functions. It is understood that the openness of these functions depends on the selection of subscripts in a permissive way and that the selection of fundamental frequencies respects the optimization conditions.

$$\psi_{bII\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bII\triangle)} - e^{-2\pi tki\psi_j(bII\triangle)}}{2i}$$

$$\longrightarrow \psi_{I\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I-7)} - e^{-2\pi tki\psi_j(I-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 4$ then following the polynomial criteria we obtain the one function of the degree $bII\triangle$ related to the dorian tonal center. This case is specially interesting because it has multiple functions that share nabla value although the polynomials are different.

$$\Phi[E^1_{(bII\triangle|I-7)}] \neq \Phi[E^2_{(bII\triangle|I-7)}] \neq \Phi[E^3_{(bII\triangle|I-7)}]$$

For the first solution, we have that the tonal function falls within the subdominant area, so we finally state it. In this case it can be represented by a polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E^1_{(bII\triangle|I-7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{2}{s} - \frac{2}{s^2}\right)\lambda^3 + \left(\frac{1}{s^2} + \frac{4}{s^3} + \frac{1}{s^4}\right)\lambda^2 + \left(-\frac{2}{s^4} - \frac{2}{s^5}\right)\lambda + \frac{1}{s^6}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s+1}{2\,s^2}\right)\lambda^4 + \frac{s^2+4\,s+1}{s^4\,3}\lambda^3 + \left(-\frac{s+1}{s^5}\right)\lambda^2 + \frac{\lambda}{s^6}$

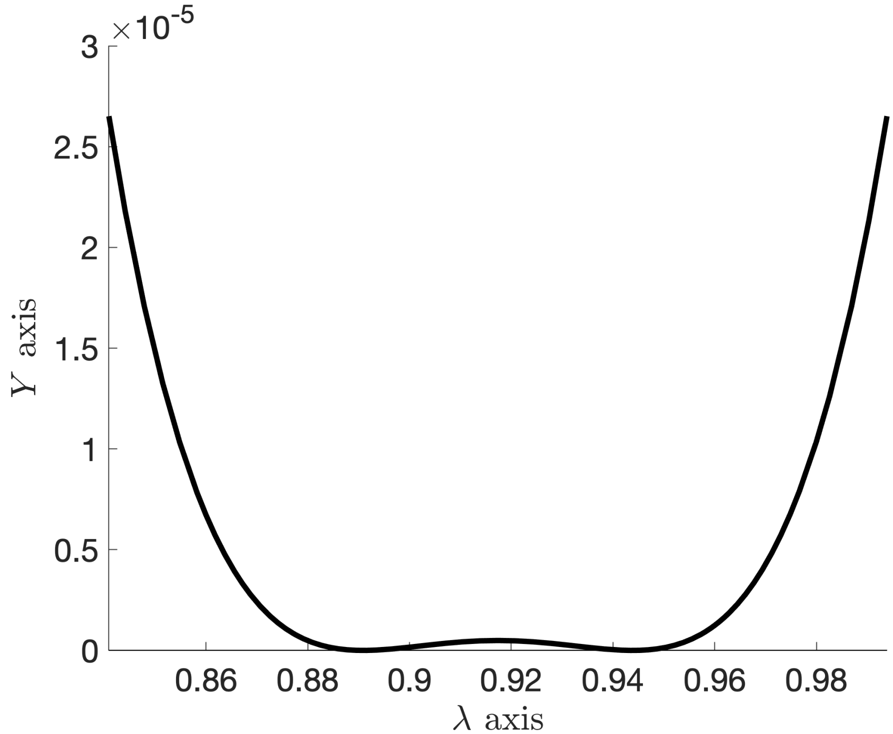


Figure F.8: Characteristic polynomial associated to the bII△ ⟶I-7 cadence (I)

F.2.4. $S_2(L^H_{(bII^r\triangle|I^r-7))})$

Using the second solution, we calculate an optimal link to use the T transformation to find a second tonal function. We shouldn't worry too much about dual tonal functions, just be aware that they represent a major perspective shift in musical thinking. In fact, without the participation of mathematics, we would continue to think that the tonal functions, in their first approach, are unidirectional. As we can see, the mathematical model provides us with an alternative to this thinking when we break a few layers in the abstraction and solve the voice-leading using the Zero method.

$$\left[E^2_{(bII_c\triangle|I_c-7)}\right] = \left[\begin{pmatrix} C & C \\ Ab & G \\ F & Eb \\ Db & Bb \end{pmatrix}\right]_{\triangledown} \tag{F.48}$$

Once we have found a second optimal link, we only have to use the transformation T to reach the endomorphism matrix, and from there we already know the route to calculate the tonal function. Although there are other ways, it is recommended to think about this transformation as suggested in the work, for a simple mechanics of simplicity and economy at the time of calculation.

Thus, if we are thinking of using the transformation T implicitly, we will quickly find the matrix $C_{\mathbb{E}}$. In this case, for each Mersenne number, we see that a different exponent appears, either on $E(M)$ or to its left. Due to the descending character of the voices, we have to use the recovery of the signs in the appropriate way.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(bII_c\triangle)) \longrightarrow \psi(I_c - 7)$$
$$\begin{pmatrix} s^0 & 0 & 0 & 0 \\ 0 & s^{-1} & 0 & 0 \\ 0 & 0 & s^{-2} & 0 \\ 0 & 0 & 0 & s^{-3} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ Ab_{z_2} \\ F_{z_3} \\ Db_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ G_{z_2} \\ Eb_{z_3} \\ Bb_{z_4 - 1} \end{pmatrix} \tag{F.49}$$

If we calculate the characteristic polynomial of $C_{\mathbb{E}}$ we reach the second tonal function for this link. In this case, the tonal function is polarized and reports the movement of the voices in one of the optimal links. The value of the nabla function in this optimal link is six, which helps us, among other things, to verify that we have carried out all the calculations correctly.

The divergent algebraic multiplicity is zero since, then we have that in the optimum there is no voice that ascends, then we have that $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$. The convergent algebraic multiplicity is worth three, since we have three voices that descend in the optimal link independent of the ratio between each one of them in the same dimension, then we have $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$. Finally, we study the static algebraic multiplicity, which is worth one because there is a voice that is mandated in the change between centers.$\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bII\triangle)$ or $\psi(I - 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector written as $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the wave functions of each voicing, decomposed note by note:

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bII_c\triangle}(t) = \psi_{C_{z_1}}(t) + \psi_{Ab_{z_2}}(t) + \psi_{F_{z_3}}(t) + \psi_{Db_{z_3}}(t) \\ \psi_{I_c-7}(t) = \psi_{C_{z_1}}(t) + \psi_{G_{z_2}}(t) + \psi_{Eb_{z_3}}(t) + \psi_{Bb_{z_3}-1}(t) \end{cases}$$

On the other hand, the general functions of each voicing would be written as usual:

$$\psi_{bII_c\triangle}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bII_c\triangle)} - e^{-2\pi tki\psi_j(bII_c\triangle)}}{2i}$$
$$\longrightarrow \psi_{I_c-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

As $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ then, following the polynomial criterion we obtain the function of the degree $bII\triangle$ related to the Phrygian tonal center. In this case is not unique and can be represented only by one polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E^2_{(bII\triangle|I-7)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s} - \frac{1}{s^3} - 2\right)\lambda^3 + \left(\frac{2}{s} + \frac{2}{s^3} + \frac{1}{s^4} + 1\right)\lambda^2 + \left(-\frac{1}{s} - \frac{1}{s^3} - \frac{2}{s^4}\right)\lambda + \frac{1}{s^4}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{2\,s^4+s^3+s}{s^4\,4}\right)\lambda^4 + \frac{s^4+2\,s^3+2\,s+1}{s^4\,3}\lambda^3 + \left(-\frac{s^3+s+2}{s^4\,2}\right)\lambda^2 + \frac{\lambda}{s^4}$

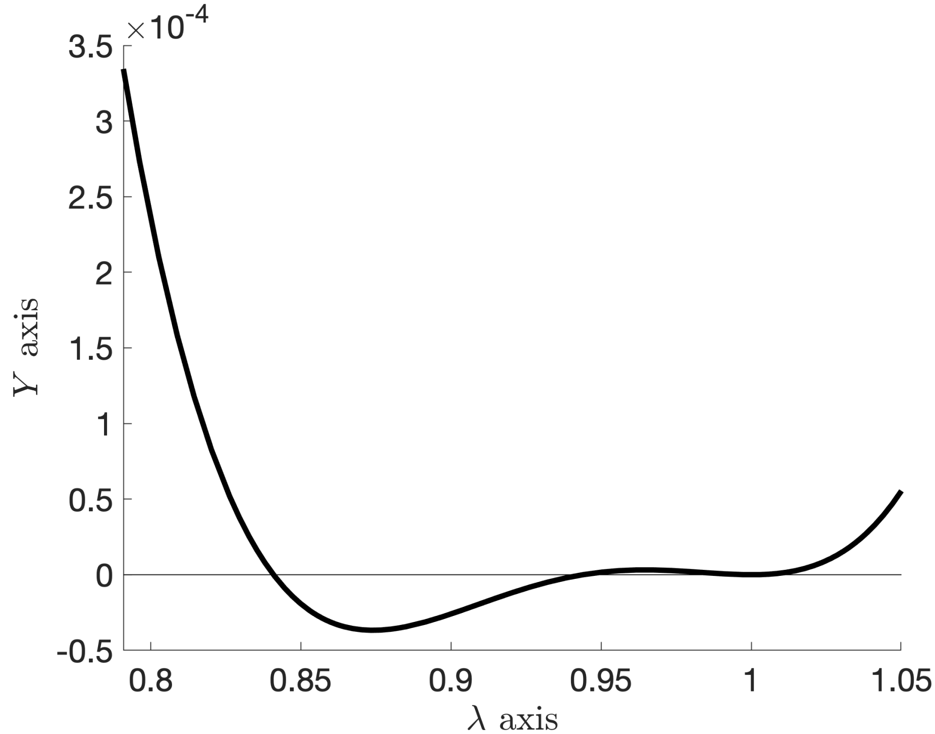


Figure F.9: Characteristic polynomial associated to the bII△ ⟶I-7 cadence (II)

F.2.5. $S_3(L^H_{(bII^r\triangle|I^r-7))})$

Following the solutions of the third S set we find a third way to pair both chords keeping the value of the nabla function to a minimum. We see that immediately, we take the square brackets and reach the class nabla, this being the set of links whose nabla function reaches the minimum.

$$\left[E^3_{(bII_c\triangle|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} C & C \\ Ab & Bb \\ F & G \\ Db & Eb \end{pmatrix}\right]_\nabla \tag{F.50}$$

Using the T transformation we reach the endomorphism matrix for the third distribution of boxes. We are not talking about the permutation matrix C implicit in the process since it is assumed to be known. Thus we arrive at the matrix that transforms a vector of frequencies from the first tonal center into another from the second center, keeping the value of absolute perception at a minimum when timbre is constant and there are no amplitude variations between voices as explained in W.F.C. At this point, the reader can take the opportunity to look at regardless of how we collate the dimensions, in case we had started with a state other than the fundamental state, even though the matrix $C_{\mathbb{E}}$ is different, by taking the tonal function, those tonal functions will coincide. This occurs because the factors are going to be the same.

The matrix equation in frequency space Φ^4 has to be verified for any selection of subscripts. Although we haven't particularly modeled it, an aperture in frequency space can be seen as an element $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bII_c\triangle)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^0 & 0 & 0 & 0 \\ 0 & s^2 & 0 & 0 \\ 0 & 0 & s^2 & 0 \\ 0 & 0 & 0 & s^2 \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ Ab_{z_2} \\ F_{z_3} \\ Db_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ Eb_{z_4} \end{pmatrix} \tag{F.51}$$

Regardless of the chosen aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$, the characteristic polynomial of the previous matrix will be the same, that is, the tonal function will be preserved.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)^3(s^0 - \lambda)$$

By simple inspection, since the polynomial already appears factored, we calculate the algebraic multiplicities, where: $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bII\triangle)$ or $\psi(I - 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We study the wave functions where the value of $\mid p \mid$ is going to be the minimum regardless of the aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bII_c\triangle}(t) = \psi_{C_{z_1}}(t) + \psi_{Ab_{z_2}}(t) + \psi_{F_{z_3}} + \psi_{Db_{z_4}}(t) \\ \psi_{I_c-7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{Eb_{z_4}}(t) \end{cases}$$

The wave functions of the antecedent and consequent voicing are still subject to the distribution of harmonics and the opening, but they are stated as we have been doing in the usual way.

Thus we write the general formulas that depend on the aforementioned parameters.

$$\psi_{bII_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(bII_c\triangle)} - e^{-2\pi tki\psi_j(bII_c\triangle)}}{2i}$$
$$\longrightarrow \psi_X(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then following the polynomial criteria we obtain the function of the degree $bII\triangle$ related to the dorian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E^3_{(bII\triangle|I-7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-3\,s^2 - 1)\ \lambda^3 + (3\,s^4 + 3\,s^2)\ \lambda^2 + (-s^6 - 3\,s^4)\ \lambda + s^6$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{3\,s^2}{4} - \frac{1}{4}\right)\ \lambda^4 + (s^2\ (s^2+1))\ \lambda^3 + \left(-\frac{s^6}{2} - \frac{3\,s^4}{2}\right)\ \lambda^2 + s^6\,\lambda$

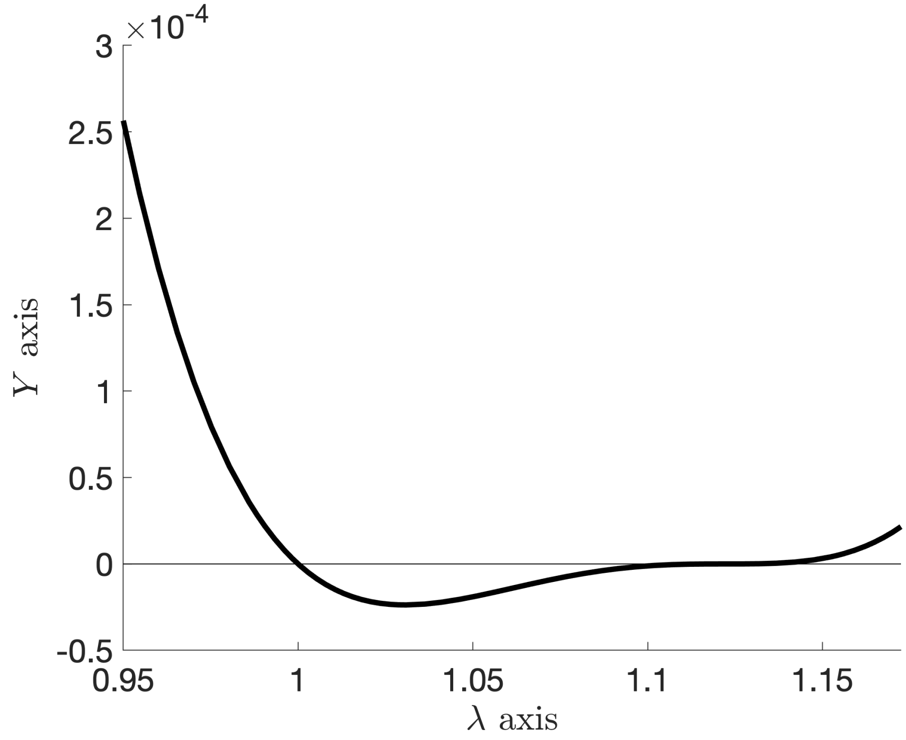


Figure F.10: Characteristic polynomial associated to the bII△ ⟶I-7 cadence (III)

F.2.6. bIII7⟶I-7 Cadence

We continue making the appropriate calculations to determine the tonal functions between the degrees of the Phrygian mode. In this way, just by calculating it in one key, as we know the static tonal function theorem, we can immediately generalize the tonal function to the rest of the tonalities or to others outside the temperate system. We continue to use the same system, ensuring order and focusing on respecting each step. Thus, we start by calculating the link E to assemble the matrix L taking the metrics.

$$E_{(bIII^r_c7|I^r_c-7)} = \begin{pmatrix} Db & Bb \\ Bb & G \\ G & Eb \\ Eb & C \end{pmatrix} \tag{F.52}$$

We calculate the link matrix calculating all the distances Δ_{ij}. We carry out this process on a regular basis, as we have done before. Here our only objective is to construct L to apply the algorithm.

$$L_{(bIII^r7|I^r-7)} = \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix} \tag{F.53}$$

Following the steps of the Hungarian algorithm we develop the L matrix. We always follow the process of transforming L into L^F and trying to reach L^H or a variant of it as soon as possible. If we don't have to use the Zero method we will arrive sooner but sometimes it is unavoidable. In this case, we reach the solution with ease.

$$L_{(bIII^r7|I^r-7)} = \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix} \longrightarrow L^F_{(bIII^r7|I^r-7)} = \begin{pmatrix} 2 & 5 & 1 & 0 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \end{pmatrix}$$

$$\longrightarrow L^H_{(bIII^r7|I^r-7)} = \begin{pmatrix} 2 & 5 & 1 & \boxed{0} \\ \boxed{0} & 3 & 5 & 2 \\ 3 & \boxed{0} & 4 & 5 \\ 5 & 4 & \boxed{0} & 3 \end{pmatrix}$$

The set S for this matrix is the set of metrics that are indicated by the distribution of boxes:

$$S(L^H_{(bIII^r7|I^r-7\triangle)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

Reading the placement of said boxes, we then compute an optimal link and generalize, immediately, to the nabla class. This is how we visualize, in an abstract way, how the chords are linked, although in their physical manifestation what is linked are voicings. This is a more abstract way of thinking about music that allows for fluent composing and improvisation.

$$\left[E^{o}_{(bIII_c7|I_c-7)}\right]_{\nabla} = \left[\begin{pmatrix} Db & C \\ Bb & Bb \\ G & G \\ Eb & Eb \end{pmatrix}\right]_{\nabla} \tag{F.54}$$

Once an optimal link is reached we are ready to use the transformation T and arrive at the matrix $C_{\mathbb{E}}$. We see that according to the notation that we are using in this case we have to recover the sign of one of the metrics. For all other, this is unnecessary because its value is cero.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bIII_c7)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{-\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} Db_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ Eb_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ G_{z_3} \\ Eb_{z_4} \end{pmatrix} \tag{F.55}$$

Using $C_{\mathbb{E}}$ and taking its characteristic polynomial, we reach the tonal function. Thus this function has three roots on $E(M)$ and one to the left of said stabilizer.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-1} - \lambda)(s^0 - \lambda)(s^0 - \lambda)(s^0 - \lambda)$$

Inspecting the previous polynomial, we study the algebraic multiplicities, where three formulas appear whose sum coincides with the dimension of the frequency space in this case. Thus the convergent algebraic multiplicity $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$. The static algebraic multiplicity has value three being $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$. Finally, the divergent algebraic multiplicity has a null value, since there are no voices that rise in the optimum $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bIII7)$ or $\psi(I - 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

$$\mathcal{B}_{\epsilon} = \begin{cases} \psi_{bIII7}(t) = \psi_{Db_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{Eb_{z_4}}(t) \\ \psi_{I7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{G_{z_3}} + \psi_{Eb_{z_4}}(t) \end{cases}$$

We write below, using the notation of parentheses, the wave functions of each voicing. We see that as a function of the aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$ where $\epsilon = (z_1, \ldots, z_4)$ each of the functions that assign each class to a particular frequency will assign said fundamental frequency as a function of ϵ, keeping, for the optimization conditions, the value of absolute perception to a minimum.

$$\psi_{bIII_c7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bIII_c7)} - e^{-2\pi tki\psi_j(bIII_c7)}}{2i}$$
$$\longrightarrow \psi_{I_c-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$ then following the polynomial criterion we obtain the function of the degree $bIII7$ related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bIII7|I-7)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s} - 3\right) \lambda^3 + \left(\frac{3}{s} + 3\right) \lambda^2 + \left(-\frac{3}{s} - 1\right) \lambda + \frac{1}{s}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{3\,s+1}{4\,s}\right) \lambda^4 + \frac{3\,s+3}{3\,s} \lambda^3 + \left(-\frac{s+3}{2\,s}\right) \lambda^2 + \frac{\lambda}{s}$

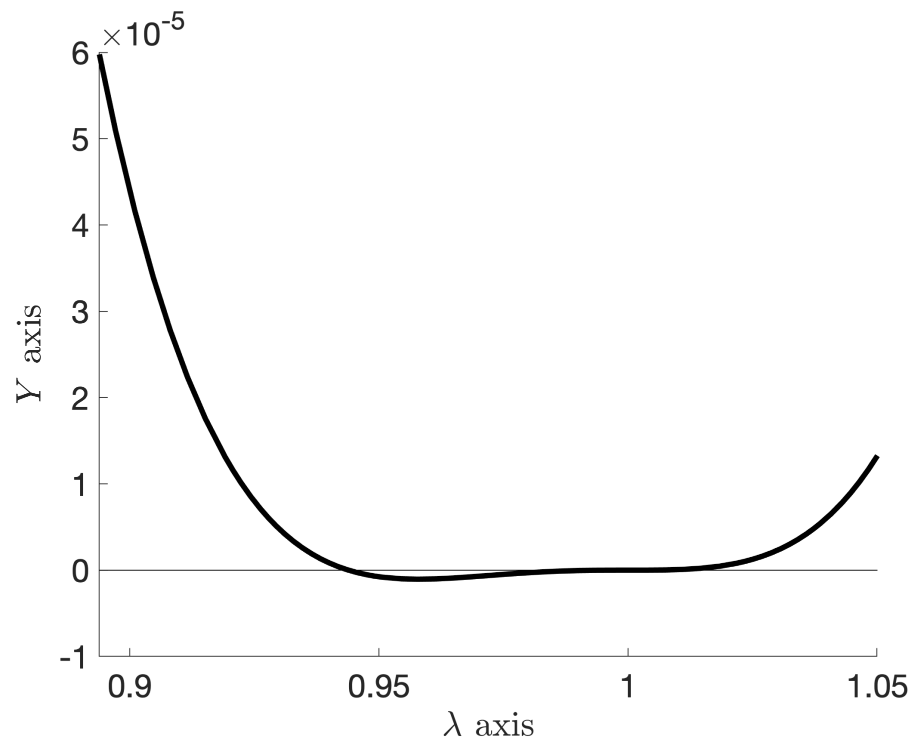


Figure F.11: Characteristic polynomial associated to the bIII7⟶I7 cadence

F.2.7. IV-7⟶I-7 Cadence

We are going to study the relationship between the minor fourth degree and the first. This relationship arises naturally from setting the first degree of the Phrygian mode to one and studying the degrees of the mode that are generated by taking subsets in the mode. Thus, through this vision, we study some of the tonal functions that appear most frequently in habitual compositional practice.

$$E_{(IV_c^r-7|I_c^r-7)} = \begin{pmatrix} Eb & Bb \\ C & G \\ Ab & Eb \\ F & C \end{pmatrix} \tag{F.56}$$

Then, we calculate the link matrix calculating all the distances Δ_{ij}. We calculate the matrix L in the usual way, taking the metrics in an ordered way.

$$L_{(IV^r-7|I^r-7)} = \begin{pmatrix} 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \end{pmatrix} \tag{F.57}$$

Following the steps of the Hungarian algorithm we develope the L matrix:

$$L_{(IV^r-7|I^r-7)} = \begin{pmatrix} 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r-7|I^r-7)} = \begin{pmatrix} 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 1 & 0 & 4 & 3 \\ 3 & 0 & 0 & 3 \end{pmatrix}$$

$$\longrightarrow L^H_{(IV^r-7|I^r-7)} = \begin{pmatrix} 4 & 4 & \boxed{0} & 3 \\ 1 & 5 & 3 & \boxed{0} \\ \boxed{0} & 0 & 4 & 3 \\ 2 & \boxed{0} & 0 & 3 \end{pmatrix}$$

Reading the box distribution of the matrix L^H we obtain the set S. At this point we are ready to calculate an optimal link.

$$S(L^H_{(IV^r-7|I^r-7)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^{o}_{(IV_c-7|I_c-7)}\right]_{\nabla} = \left[\begin{pmatrix} Eb & Eb \\ C & C \\ Ab & Bb \\ F & G \end{pmatrix}\right]_{\nabla} \tag{F.58}$$

Using the transformation T implicitly we reach the transformation matrix of the endomorphism where the exponents of each Mersenne number coincide with the recovery of the sign of each metric indicated in the matrix L by the Hungarian algorithm.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(IV_c - 7)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Ab_{z_3} \\ F_{z_4} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} \tag{F.59}$$

Then the polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id_4.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)(s^2 - \lambda)(s^0 - \lambda)(s^0 - \lambda)$$

Inspecting the characteristic polynomial we observe the algebraic multiplicities in order to classify the tonal function using the polynomial criterion. We have two voices that remain invariant and another two that ascend. This occurs in the optimal link, which in this case is unique because the pitch function is non-dual: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(IV - 7)$ or $\psi(I - 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We write the wave functions leaving the subscripts open, in such a way that when choosing an opening we will specify the fundamental frequencies of each of the wave functions of each note.

$$\mathcal{B}_{\epsilon} = \begin{cases} \psi_{IV_c-7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Ab_{z_3}} + \psi_{F_{z_4}}(t) \\ \psi_{I_c-7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

The general wave functions for each voicing are specified by the following formulas, where they act as a conceptual summary of the physical phenomenon.

$$\psi_{IV_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(IV_c-7)} - e^{-2\pi tki\psi_j(IV_c-7)}}{2i}$$
$$\longrightarrow \psi_{I_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$ then following the polynomial criterion we obtain the function of the degree $IV-7$ related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(IV-7|I-7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-2\,s^2 - 2)\ \lambda^3 + (s^4 + 4\,s^2 + 1)\ \lambda^2 + (-2\,s^4 - 2\,s^2)\ \lambda + s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{2} - \frac{1}{2}\right)\ \lambda^4 + \left(\frac{s^4}{3} + \frac{4\,s^2}{3} + \frac{1}{3}\right)\ \lambda^3 + (-s^2\ (s^2+1))\ \lambda^2 + s^4\,\lambda$

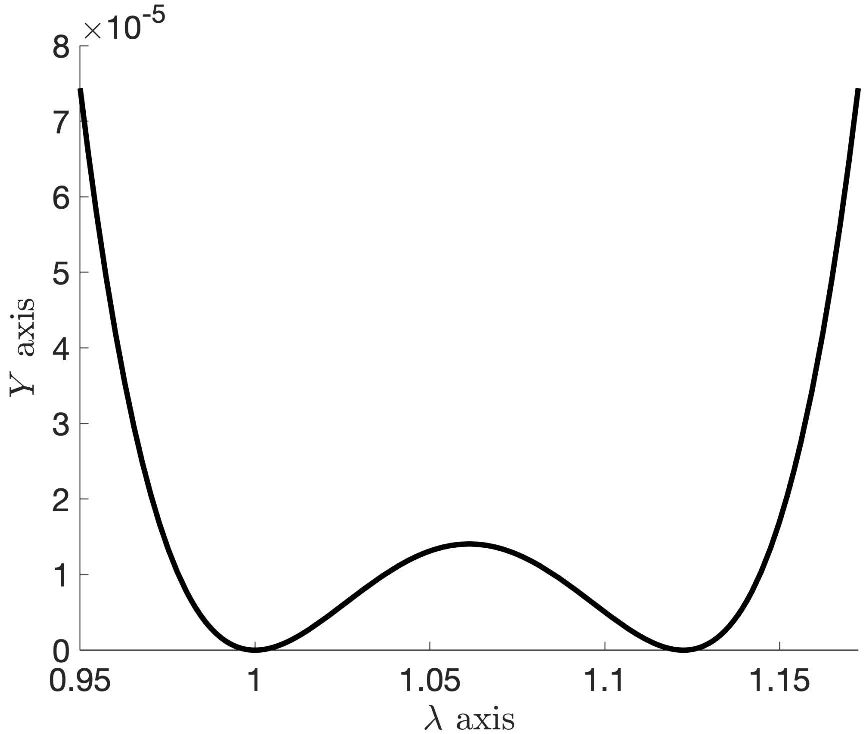


Figure F.12: Characteristic polynomial associated to the IV-7⟶I-7 cadence

F.2.8. Vø7 ⟶I-7 Cadence

In our study of tonal functions, we continue to advance in the relationships between degrees through the Phrygian mode. Thus we are calculating each tonal function, which ultimately will be drawn as an edge in the graph of tonal functions. In the first instance, we calculate the link between the fifth half-diminished degree and the first minor degree:

$$E_{(V_c^r \o 7 | I_c^r - 7)} = \begin{pmatrix} F & Bb \\ Db & G \\ Bb & Eb \\ G & C \end{pmatrix} \tag{F.60}$$

Then, we calculate the link matrix calculating all the distances Δ_{ij}. We calculate the matrix L in the usual way, taking the metrics in an ordered way.

$$L_{(V^r \o 7 | I^r - 7)} = \begin{pmatrix} 5 & 2 & 2 & 5 \\ 3 & 6 & 2 & 1 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \end{pmatrix} \tag{F.61}$$

Following the steps of the Hungarian algorithm we develop the L matrix:

$$L_{(V^r\o7|I^r-7)} = \begin{pmatrix} 5 & 2 & 2 & 5 \\ 3 & 6 & 2 & 1 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \end{pmatrix} \longrightarrow L^F_{(V^r\o7|I^r-7)} = \begin{pmatrix} 3 & 0 & 0 & 3 \\ 2 & 5 & 1 & 0 \\ 0 & 3 & 5 & 2 \\ 3 & 0 & 4 & 5 \end{pmatrix}$$

$$\longrightarrow L^H_{(V^r\o7|I^r-7)} = \begin{pmatrix} 3 & 0 & \boxed{0} & 3 \\ 2 & 5 & 1 & \boxed{0} \\ \boxed{0} & 3 & 5 & 2 \\ 3 & \boxed{0} & 4 & 5 \end{pmatrix}$$

We use the distribution of boxes to find the solutions in S. With this information we are ready to calculate the optimal link.

$$S(L^H_{(V^r\o7|I^r-7)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(V_c\o7|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} F & Eb \\ Db & C \\ Bb & Bb \\ G & G \end{pmatrix}\right]_\nabla \tag{F.62}$$

Since we have already found the optimal link E^o from the original E, then using the transformation T we immediately reach the endomorphism matrix. Thus, we write the matrix equation, where the subscripts remain open to the choice of an opening $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$ that does not alter the minimization of absolute perception.

$$C_\mathbb{E} : \Phi^4 \longrightarrow \Phi^4$$

$$C_\mathbb{E}(\psi(IV_c - 7)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ Db_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Bb_{z_3} \\ G_{z_4} \end{pmatrix} \tag{F.63}$$

From the previous matrix, we calculate the characteristic polynomial reaching the tonal function. We will do this process as many times as necessary, since despite the high number of calculations, every time we carry out this process, we are discovering a new way to generate chord progressions, guaranteeing convergence.

$$p_{C_\mathbb{E}}(\lambda) = (s^{-2} - \lambda)(s^{-1} - \lambda)(s^0 - \lambda)(s^0 - \lambda)$$

We have three formulas for the algebraic multiplicities that will reflect the movement of the voices in the optimum. These formulas will be written as: $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$, for convergent algebraic multiplicity, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ for static algebraic multiplicity and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$ for divergent algebraic multiplicity.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(V\o7)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We write the wave functions leaving the subscripts open, in such a way that when choosing an opening we will specify the fundamental frequencies of each of the wave functions of each note.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV_c-7}(t) = \psi_{F_{z_1}}(t) + \psi_{Db_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \\ \psi_{I_c-7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{G_{z_4}}(t) \end{cases}$$

We write the general formulas that are still connected with the idea of opening. Then these formulas will depend, in addition to the harmonic distribution that we choose, on the opening under consideration $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$.

$$\psi_{V_c\o7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(V_c\o7)} - e^{-2\pi tki\psi_j(V_c\o7)}}{2i}$$
$$\longrightarrow \psi_{I_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$ then following the polynomial criterion we obtain the function of the degree $V\o7$ related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(V\o7|I-7)}] \in D^{\mathbb{R}[\lambda]}}$$

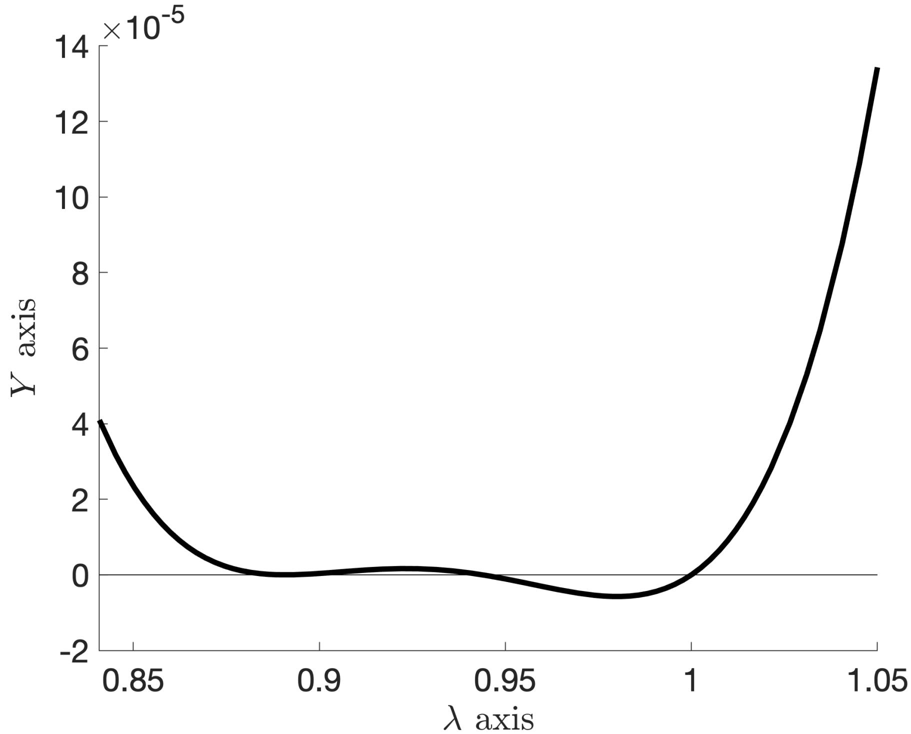


Figure F.13: Characteristic polynomial associated to the Vø7⟶I-7 cadence

F.2.9. ♭VI△ ⟶I-7 Cadence

In this study, we continue advancing in the calculation of tonal functions. In this case, we study the relationship between the sixth degree and the first degree in the context of the Phrygian mode. Calculating these relationships within the framework of tonal functions allows us to fully understand the relationship between two tonal centers in the abstract.

$$E_{(bVI^r_c\triangle|I^r_c-7)} = \begin{pmatrix} G & Bb \\ Eb & G \\ C & Eb \\ Ab & C \end{pmatrix} \tag{F.64}$$

We build the matrix L from the link E.

$$L_{(bVI^r\triangle|I^r-7)} = \begin{pmatrix} 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \end{pmatrix} \tag{F.65}$$

We develop the entire calculation process through the Hungarian algorithm until we find the solution. We are transforming the matrix L until we reach the distribution of boxes that reflects the optimal pairing between the voices. In this way we write using arrows, the sequence of transformation between matrices:

$$L_{(bVI^r\triangle|I^r-7)} = \begin{pmatrix} 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 2 & 1 & 5 & 4 \end{pmatrix} \longrightarrow L^F_{(bVI^r\triangle|I^r-7)} = \begin{pmatrix} 3 & 0 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 5 & 3 & 0 \\ 1 & 0 & 4 & 3 \end{pmatrix}$$

$$\longrightarrow L^H_{(bVI^r\triangle|I^r-7)} = \begin{pmatrix} 2 & \boxed{0} & 4 & 5 \\ 4 & 4 & \boxed{0} & 3 \\ 1 & 5 & 3 & \boxed{0} \\ \boxed{0} & 0 & 4 & 3 \end{pmatrix}$$

Interpreting the distribution of boxes, which in this case is unique, we write the set S. With the solution in mind we now know how the classes in each dimension are matched.

$$S(L^H_{(bVI^r\triangle|I^r-7)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

This set allows us to know the shape of an optimized link, calculating the nabla minimum class of E.

$$\left[E^o_{(bVI^r\triangle|I_c-7)} \right]_\nabla = \left[\begin{pmatrix} G & G \\ Eb & Eb \\ C & C \\ Ab & Bb \end{pmatrix} \right]_\nabla \tag{F.66}$$

Using the transformation T we reach the endomorphism matrix $C_{\mathbb{E}}$ that carries a voicing from the first tonal center to another for any aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$. In this way, we visualize the transformation in the frequency space that is generalized for any opening, keeping the value of absolute perception to a minimum.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bVI_c\triangle)) \longrightarrow \psi(I_c - 7)$$

$$\begin{pmatrix} s^{\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Ab_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{F.67}$$

The polynomial is calculated as the determinant of the difference between $C_{\mathbb{E}}$ and Id_4.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)(s^0 - \lambda)(s^0 - \lambda)(s^0 - \lambda)$$

If we look at how the voices move in the optimal link, we observe that there is only one voice that moves at the rate of pitch. Then we already see that the tonal function will belong to the tonic area according to the polinomial criterion, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$, $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ and $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 1$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVI\triangle)$ or $\psi(I-7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We write the wave functions leaving the subscripts open, in such a way that when choosing an opening we will specify the fundamental frequencies of each of the wave functions of each note.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVI\triangle_c}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Ab_{z_4}}(t) \\ \psi_{I_c-7}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Bb_{z_4}}(t) \end{cases}$$

The general wave functions for each voicing are specified by the following formulas, where they act as a conceptual summary of the physical phenomenon.

$$\psi_{bVI_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVI_c\triangle)} - e^{-2\pi tki\psi_j(bVI_c\triangle)}}{2i}$$

$$\longrightarrow \psi_{I_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 1$ then following the polynomial criterion we obtain the function of the degree $bVI\triangle$ related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bVI\triangle|I-7)}] \in T^{\mathbb{R}[\lambda]}}$$

F.2.10. bVII-7⟶I-7 Cadence

We finish the chapter studying the last case in four voices in the Phrygian mode where we see the relationship between the seventh degree and the first, both of minor quality. Although we have already covered this relationship in at least one other way, we rewrite the case for compactness and for elegance. Thus, we focus on applying the entire process from the formation of the E link.

$$E_{(bVII_c^r-7|I_c^r-7)} = \begin{pmatrix} Ab & Bb \\ F & G \\ Db & Eb \\ Bb & C \end{pmatrix} \tag{F.68}$$

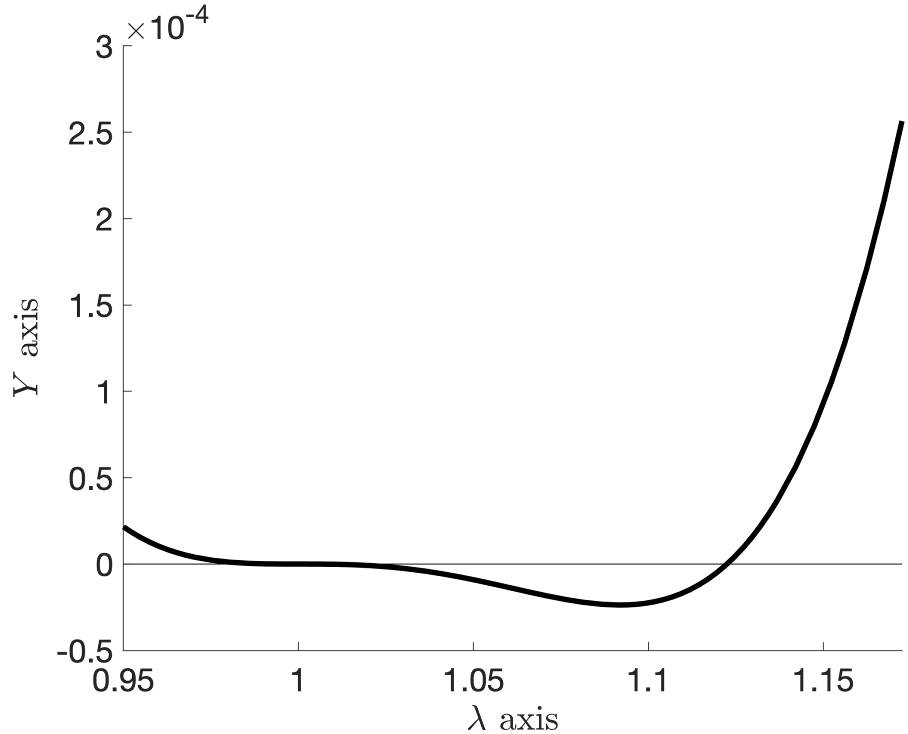


Figure F.14: Characteristic polynomial associated to the bVI△ ⟶I-7 cadence

Then, we calculate the link matrix calculating all the distances Δ_{ij}. Having formed the link arranging the chords in root position, we are now ready for the construction of the L matrix.

$$L_{(bVII^r-7|I^r-7)} = \begin{pmatrix} 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \\ 3 & 6 & 2 & 1 \\ 0 & 3 & 5 & 2 \end{pmatrix} \tag{F.69}$$

Following the steps of the Hungarian algorithm we develop the L matrix:

$$L_{(bVII^r-7|I^r-7)} = \begin{pmatrix} 2 & 1 & 5 & 4 \\ 5 & 2 & 2 & 5 \\ 3 & 6 & 2 & 1 \\ 0 & 3 & 5 & 2 \end{pmatrix} \longrightarrow L^F_{(bVII^r-7|I^r-7)} = \begin{pmatrix} 1 & 0 & 4 & 3 \\ 3 & 0 & 0 & 3 \\ 2 & 5 & 1 & 0 \\ 0 & 3 & 5 & 2 \end{pmatrix}$$

$$\longrightarrow L^H_{(bVII^r-7|I^r-7)} = \begin{pmatrix} 1 & \boxed{0} & 4 & 3 \\ 3 & 0 & \boxed{0} & 3 \\ 2 & 5 & 1 & \boxed{0} \\ \boxed{0} & 3 & 5 & 2 \end{pmatrix}$$

Reading the box distribution of the matrix L^H we obtain the set S. At this point we are ready to calculate an optimal link.

$$S(L^H_{(bVII^r-7|I^r-7)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

Using the distribution of boxes that appears in the L^H matrix, we construct S , so we are ready to find an optimal link, which we will immediately generalize to its entire nabla class.

$$\left[E^o_{(bVII_c-7|I_c-7)}\right]_\nabla = \left[\begin{pmatrix} Ab & G \\ F & Eb \\ Db & C \\ Bb & Bb \end{pmatrix}\right]_\nabla \tag{F.70}$$

From the previous solution, we can calculate, using the transformation T, the transformation matrix between voicings in the frequency space. We will leave the subscripts open since the same matrix is valid regardless of the opening $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(bVII_c - 7)) \longrightarrow \psi(I_c - 7)$$
$$\begin{pmatrix} s^{-\Delta_{12}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{23}} & 0 & 0 \\ 0 & 0 & s^{-\Delta_{34}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{41}} \end{pmatrix} \cdot \begin{pmatrix} Ab_{z_1} \\ F_{z_2} \\ Db_{z_3} \\ Bb_{z_4} \end{pmatrix} = \begin{pmatrix} G_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{F.71}$$

We see that we have recovered the sign of the metrics of the matrix L and that with this we can calculate the characteristic polynomial of the matrix $C_{\mathbb{E}}$, thus reaching the tonal function.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-2} - \lambda)(s^{-2} - \lambda)(s^{0} - \lambda)(s^{-1} - \lambda)$$

Observing said polynomial by simple inspection, we realize which are the equations associated with the algebraic multiplicities that we have been writing throughout this work.

Being the convergent algebraic multiplicity $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$, the static algebraic multiplicity $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$ and the divergent algebraic multiplicity $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+}$ =0.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVII - 7)$ or $\psi(I - 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$. We write the wave functions of each voicing respecting the existence of each aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$. Thus, inside the bracket, either both voicings coincide in the opening, or the components of said opening partially coincide, since there is some class that is in the octave change of the midi notation. In this case, the opening of both voicings will coincide.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII_c-7}(t) = \psi_{Ab_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{Db_{z_3}} + \psi_{Bb_{z_4}}(t) \\ \psi_{I_c-7}(t) = \psi_{G_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Bb_{z_4}}(t) \end{cases}$$

The general formula that is associated with each voicing and that reflects the physical phenomenon is written below in such a way that we write the sums of $n \times h$ harmonics for a harmonic distribution Γ

$$\psi_{bVII_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c-7)} - e^{-2\pi tki\psi_j(bVII_c-7)}}{2i}$$

$$\longrightarrow \psi_{I_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(I_c-7)} - e^{-2\pi tki\psi_j(I_c-7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ then following the polynomial criterion we obtain the function of the degree $IV - 7$ related to the Phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bVII-7|I-7)}] \in D^{\mathbb{R}[\lambda]}}$$

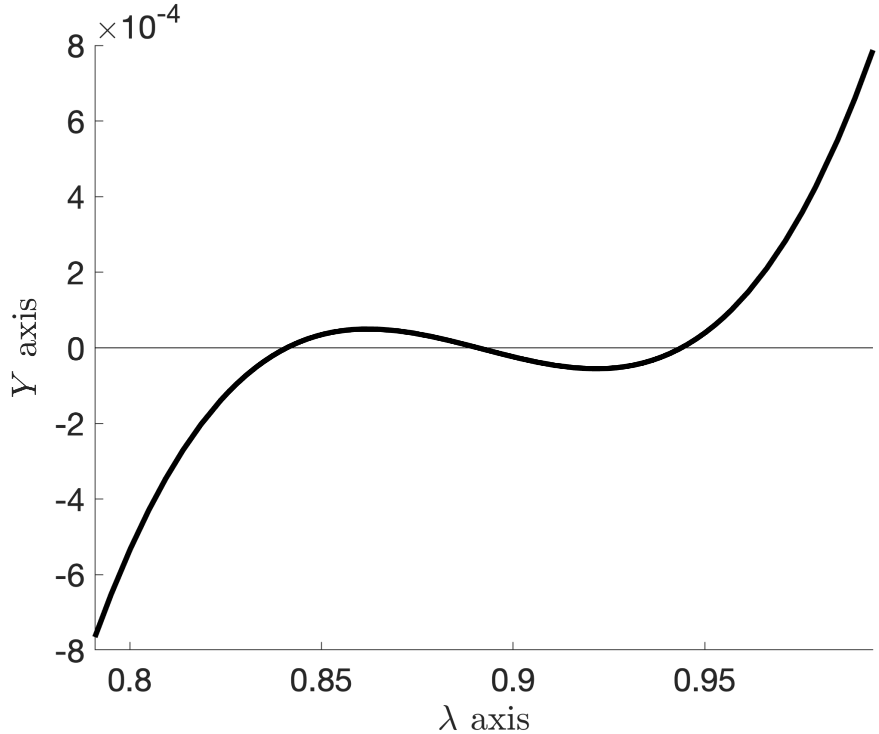


Figure F.15: Characteristic polynomial associated to the bVII-7⟶I-7 cadence

F.2.11. Phrygian tonal functions

$bII \longrightarrow I-$	$\Phi[E_{(bII\|I-)}] \in D^{\mathbb{R}[\lambda]}$
$bIII \longrightarrow I-$	$\Phi[E_{(bIII\ \|I-)}] \in T^{\mathbb{R}[\lambda]}$
$IV- \longrightarrow I-$	$\Phi[E_{(IV-\|I-)}] \in D^{\mathbb{R}[\lambda]}$
$Vo \longrightarrow I-$	$\Phi[E_{(Vo\|I-)}] \in S^{\mathbb{R}[\lambda]}$
$bVI \longrightarrow I-$	$\Phi[E_{(bVI\|I-)}] \in T^{\mathbb{R}[\lambda]}$
$bVII- \longrightarrow I-$	$\Phi[E_{(VII-\|I-)}] \in S^{\mathbb{R}[\lambda]} \cup D^{\mathbb{R}[\lambda]}$

$bII\triangle \longrightarrow I-7$	$\Phi[E_{(bII\triangle\|I-7)}] \in S^{\mathbb{R}[\lambda]} \cup D^{\mathbb{R}[\lambda]}$
$bIII7 \longrightarrow I-7$	$\Phi[E_{(bIII7\|I-7)}] \in T^{\mathbb{R}[\lambda]}$
$IV-7 \longrightarrow I-7$	$\Phi[E_{(IV-7\|I-7)}] \in S^{\mathbb{R}[\lambda]}$
$V\o 7 \longrightarrow I-7$	$\Phi[E_{(V\o 7\|I-7)}] \in D^{\mathbb{R}[\lambda]}$
$bVI\triangle \longrightarrow I-7$	$\Phi[E_{(bVI\triangle\|I-7)}] \in T^{\mathbb{R}[\lambda]}$
$bVII-7 \longrightarrow I-7$	$\Phi[E_{(bVII-7\|I-7)}] \in D^{\mathbb{R}[\lambda]}$

Appendix G

The Locrian Mode

G.1. The Locrian mode

Pawel Czerwinski

https://unsplash.com/photos/a-black-and-white-photo-of-a-bunch-of-feathers-aZc1SQ5WXAs

G.2. The Locrian mode for n =3

G.2.1. bII⟶ Io Cadence

We structure the calculations in the Locrian mode by carrying out the same procedure. In this case, as it is a mode that does not predominate because it is obscure, the methodology we have developed is especially useful since it allows us to know for sure how chords behave in this mode. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(bII_c^r|I_c^r o)} = \begin{pmatrix} Ab & Gb \\ F & Eb \\ Db & C \end{pmatrix} \tag{G.1}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(bII^r|I^ro)} = \begin{pmatrix} 2 & 5 & 4 \\ 1 & 2 & 5 \\ 5 & 2 & 1 \end{pmatrix} \tag{G.2}$$

Then, following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bII^r|I^ro)} = \begin{pmatrix} 2 & 5 & 4 \\ 1 & 2 & 5 \\ 5 & 2 & 1 \end{pmatrix} \longrightarrow L^F_{(bII^r|I^ro)} = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 1 & 4 \\ 4 & 1 & 0 \end{pmatrix} \longrightarrow L^H_{(bII^r|I^ro)} = \begin{pmatrix} \boxed{0} & 2 & 2 \\ 0 & \boxed{0} & 4 \\ 4 & 0 & \boxed{0} \end{pmatrix} \tag{G.3}$$

The solutions for $L^H_{(bII^r|I^ro)}$ when both triads are in root position becomes the following set, wich represents the minimum voice- leading $S(L^H_{(bII^r|I^ro)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$ and we calculate the optimal link class:

$$\left[E^o_{(bII_c|I_co)} \right]_\nabla = \left[\begin{pmatrix} Ab & Gb \\ F & Eb \\ Db & C \end{pmatrix} \right]_\nabla \tag{G.4}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bII_c|I_co)}) = 2 + 1 + 1 = 4$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(bII|Io)} = 4$.

Any optimal arrangement from an optimal progresion $E^o_{(bII|Io)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(bII_c-)) \longrightarrow \psi(I_co)$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Ab_{z_1} \\ F_{z_2} \\ Db_{z_3} \end{pmatrix} = \begin{pmatrix} Gb_{z_1} \\ Eb_{z_2} \\ C_{z_3} \end{pmatrix} \tag{G.5}$$

Then the characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^{-2} - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-1} - \lambda \end{pmatrix} \tag{G.6}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^{-2} - \lambda)^2(s^{-1} - \lambda)$$

The roots of the polynomial make up three clearly differentiated sets based on their position with respect to the stabilizer of the M group. According to its relationship with $E(M)$, the convergence of one chord on another is determined, where we have that these three sets would be: $\lambda^- = \{s^{-2}, s^{-1}\}$, $\lambda^0 = \{\emptyset\}$ and $\lambda^+ = \{\emptyset\}$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bII_c)$ or $\psi(I_co)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We describe the wave functions of each voicing where the subscripts determine the opening of both the antecedent and the consequent voicing. Thus, following the optimization conditions that maintain the value of absolute perception at its minimum, we have:

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bII_c}(t) = \psi_{Ab_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{Db_{z_3}}(t) \\ \psi_{I_co}(t) = \psi_{Gb_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

The wave functions can also be written as summation, where for an arbitrary harmonic distribution Γ and a set of functions ψ that respect the optimization conditions, they will be written as:

$$\psi_{bII_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(bII_c)} - e^{-2\pi tki\psi_j(bII_c)}}{2i}$$

$$\longrightarrow \psi_{I_co}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_c-)} - e^{-2\pi tki\psi_j(I_co)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ then, following the polynomial criterion we obtain the function of the degree bII related to the phrygian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bII|Io)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s} - \frac{2}{s^2}\right)\lambda^2 + \left(\frac{2}{s^3} + \frac{1}{s^4}\right)\lambda - \frac{1}{s^5}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda)d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s+2}{3\,s^2}\right)\lambda^3 + \frac{2\,s+1}{2\,s^4}\lambda^2 + \left(-\frac{1}{s^5}\right)\lambda$

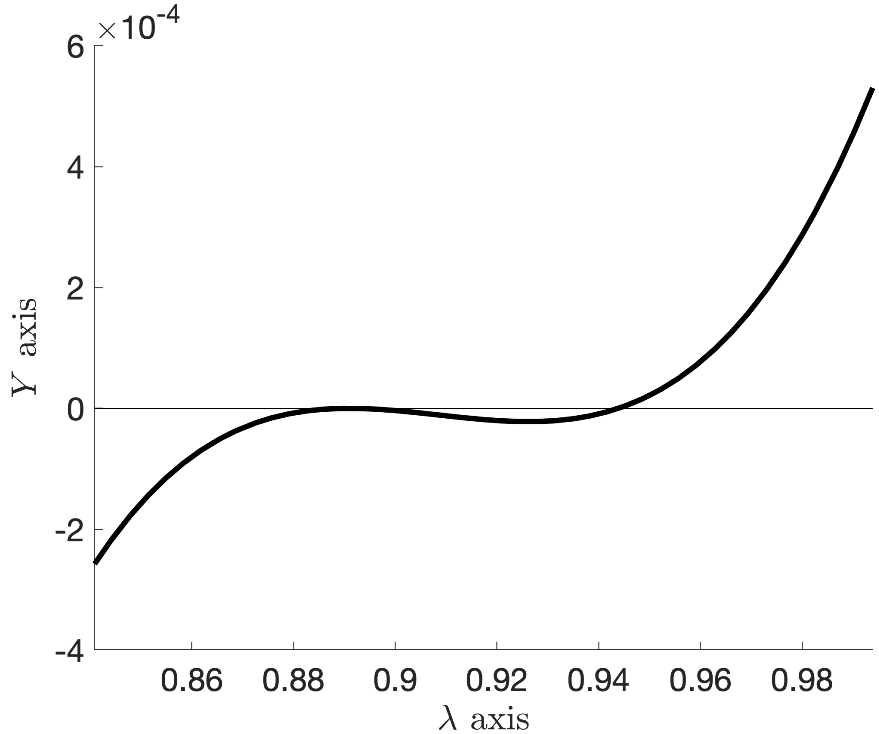


Figure G.1: Characteristic polynomial associated to the bII-⟶Io cadence (I)

G.2.2. bIII-⟶Io Cadence

We continue calculating the tonal functions between degrees in the Locrian mode with the aim of having the directed edges in the graph of tonal functions. Thus, we perform the calculations using the construction of L and its resolution using the Hungarian algorithm to find each tonal function. We calculate the link matrix for this cadence to optimize the previous cadence. We continue structuring the calculations with the objective of calculating each tonal function.

$$E_{(bIII^r_c - |I^r_c o)} = \begin{pmatrix} Bb & G \\ Gb & Eb \\ Eb & C \end{pmatrix} \tag{G.7}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(bIII^r - |I^r o)} = \begin{pmatrix} 4 & 5 & 2 \\ 0 & 3 & 6 \\ 3 & 0 & 3 \end{pmatrix} \tag{G.8}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link. We carry out the process like this, first calculating L and later developing the calculations through L^F until reaching a distribution of boxes in L^H.

$$L_{(bIII^r-|I^r o)} = \begin{pmatrix} 4 & 5 & 2 \\ 0 & 3 & 6 \\ 3 & 0 & 3 \end{pmatrix} \longrightarrow L^F_{(bIII^r-|I^r o)} = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 3 & 0 & 3 \end{pmatrix} \longrightarrow L^H_{(bIII^r-|I^r o)} = \begin{pmatrix} 2 & 3 & \boxed{0} \\ \boxed{0} & 3 & 6 \\ 3 & \boxed{0} & 3 \end{pmatrix} \tag{G.9}$$

Then the solutions for $L^H_{(bIII^r-|I^r o)}$ when both triads are in root position becomes the following set, wich represents the minimum voice-leading:

$$S(L^H_{(bIII^r-|I^r o)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

From the results of the set S, we can immediately establish the optimal link. So, setting the first chord, we build the second in the link, and generalize the result to the nabla class.

$$\left[E^o_{(bIII_c-|I_c o)}\right]_\nabla = \left[\begin{pmatrix} Bb & C \\ Gb & Gb \\ Eb & Eb \end{pmatrix}\right]_\nabla \tag{G.10}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bIII_c-|I_c o)}) = 2+0+0 = 2$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(bIII-|Io)} = 2$.

Any optimal arrangement from an optimal progression $E^o_{(bIII-|Io)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(bIII_c-)) \longrightarrow \psi(I_c o))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Bb_{z_1} \\ Gb_{z_2} \\ Eb_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1+1} \\ Gb_{z_2} \\ Eb_{z_3} \end{pmatrix} \tag{G.11}$$

The characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with assigned values l_1, l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \tag{G.12}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)^2(s^2 - \lambda)$$

The roots of the characteristic polynomial are separated into three sets based on their placement with respect to the stabilizer $E(M)$ in such a way that we have $\lambda^+ = \{s^2\}$, $\lambda^0 = \{\emptyset\}$ and $\lambda^- = \{\emptyset\}$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bIII_c-)$ or $\psi(I_co)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the wave functions inside the bracket for each voicing, where using the typical midi notation to indicate the specific octave of each class, we have two wave functions that have two common classes and one that varies.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bIII_c-}(t) = \psi_{Bb_{z_1}}(t) + \psi_{Gb_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \\ \psi_{I_c-}(t) = \psi_{C_{z_1+1}}(t) + \psi_{Gb_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \end{cases}$$

Wave functions in compact sum form would be written using the conventional notation.

$$\psi_{bIII_c-}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bIII_c-)} - e^{-2\pi tki\psi_j(bIII_c-)}}{2i}$$

$$\longrightarrow \psi_{I_co}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_co)} - e^{-2\pi tki\psi_j(I_co)}}{2i}$$

Following the polynomial criterion we obtain the function of the degree $bIII-$ related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bIII-|Io)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-s^2 - 2)\ \lambda^2 + (2\,s^2 + 1)\ \lambda - s^2$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^2}{3} - \frac{2}{3}\right)\ \lambda^3 + \left(s^2 + \frac{1}{2}\right)\ \lambda^2 + (-s^2)\ \lambda$

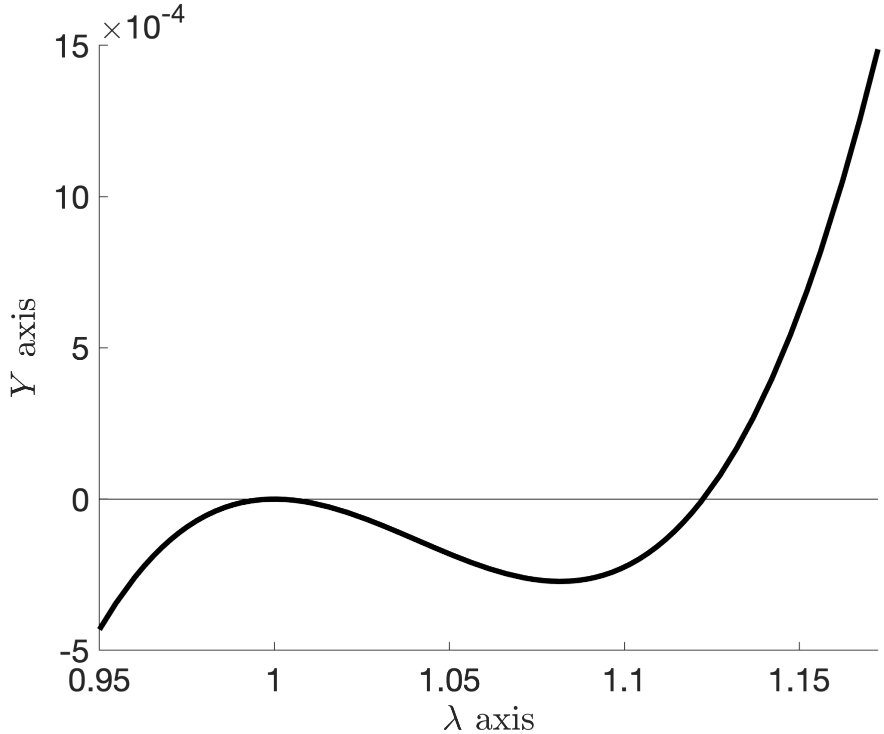


Figure G.2: Characteristic polynomial associated to the bIII-⟶Io cadence

G.2.3. IV-⟶Io Cadence

Continuing with our task of obtaining the tonal functions in each mode when we fix the dimension of the chords, we are going to study how the roots of the tonal function behave between two minor triads that are a fourth apart. In this way, we set up the link between the fourth and the first degree of the Locrian mode. We calculate the link matrix for this cadence to optimize the previous cadence. The link will be a progression such as the written below:

$$E_{(IV^r_c - |I^r_c o)} = \begin{pmatrix} C & Gb \\ Ab & Eb \\ F & C \end{pmatrix} \tag{G.13}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(IV^r - |I^r o)} = \begin{pmatrix} 6 & 3 & 0 \\ 2 & 5 & 4 \\ 1 & 2 & 5 \end{pmatrix} \tag{G.14}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(IV^r-|I^ro)} = \begin{pmatrix} 6 & 3 & 0 \\ 2 & 5 & 4 \\ 1 & 2 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r-|I^ro)} = \begin{pmatrix} 6 & 3 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{pmatrix} \longrightarrow L^H_{(IV^r-|I^ro)} = \begin{pmatrix} 6 & 2 & \boxed{0} \\ \boxed{0} & 2 & 2 \\ 0 & \boxed{0} & 4 \end{pmatrix} \tag{G.15}$$

The solutions for $L^H_{(IV^r-|I^ro)}$ when both triads are in root position becomes the following set, that represents the minimum voice leading:

$$S(L^H_{(IV^r|I^ro)}) = \{\Delta_{13}, \Delta_{21}, \Delta_{32}\}$$

Following the positions of the boxes in the matrix that ends the algorithm, we find how the classes of the antecedent and consequent chords are paired. In this way we write the optimal link and immediately generalize it to the nabla class.

$$\left[E^o_{(IV_c-|I_co)} \right]_\nabla = \left[\begin{pmatrix} C & C \\ Ab & Gb \\ F & Eb \end{pmatrix} \right]_\nabla \tag{G.16}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(IV_c-|I_co)}) = 0 + 1 + 2 = 4$ and we write the optimal nabla value as a generalization for every tonality: $\nabla^o_{(IV-|Io)} = 4$

Any optimal arrangement from an optimal progression $E^o_{(IV-|Io)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi((IV_c-)) \longrightarrow \psi(I_co))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ Ab_{z_2} \\ F_{z_3} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Gb_{z_2} \\ Eb_{z_3} \end{pmatrix} \tag{G.17}$$

The characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with assigned values l_1, l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^{-2} - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{G.18}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)(s^{-2} - \lambda)(s^{-2} - \lambda)$$

Studying the roots of the tonal function, we distinguish the three sets that arise from the contrast of the position of each root on the λ axis with the stabilizer of the group $E(M)$.Thus we obtain these sets that are written as: $\lambda^- = \{s^{-2}\}$, $\lambda^0 = \{s^0\}$ and $\lambda^+ = \{\emptyset\}$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(IV_c-)$ or $\psi(I_c o)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. Note by note, we specify, in the bracket, the wave functions of each voicing. We rely on the midi notation, since it reflects the equivalence between octaves. Thus, said functions would be written as follows, where the subscripts determine the opening of each one of the voicings.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV_c-}(t) = \psi_{C_{z_1}}(t) + \psi_{Ab_{z_2}}(t) + \psi_{F_{z_3}}(t) \\ \psi_{I_c o}(t) = \psi_{C_{z_1}}(t) + \psi_{Gb_{z_2}}(t) + \psi_{Eb_{z_3}}(t) \end{cases}$$

Wave functions can be written in terms of sines, where each sine is a harmonic of the voicing in question. We clearly see that each voicing can be understood as a sum of $n \times h$ harmonics that follow the Γ distribution. In this way, the general formulas of the antecedent and consequent voicing will be written as:

$$\psi_{IV_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(IV_c-)} - e^{-2\pi tki\psi_j(IV_c-)}}{2i}$$

$$\longrightarrow \psi_{I_c-}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c o)} - e^{-2\pi tki\psi_j(I_c o)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$ then, following the polynomial criterion we obtain the function of the degree $IV-$ related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(IV-|Io)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{2}{s^2} - 1\right) \lambda^2 + \left(\frac{2}{s^2} + \frac{1}{s^4}\right) \lambda - \frac{1}{s^4}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{s^4+2\,s^2}{s^4\,3}\right) \lambda^3 + \frac{2\,s^2+1}{2\,s^4} \lambda^2 + \left(-\frac{1}{s^4}\right) \lambda$

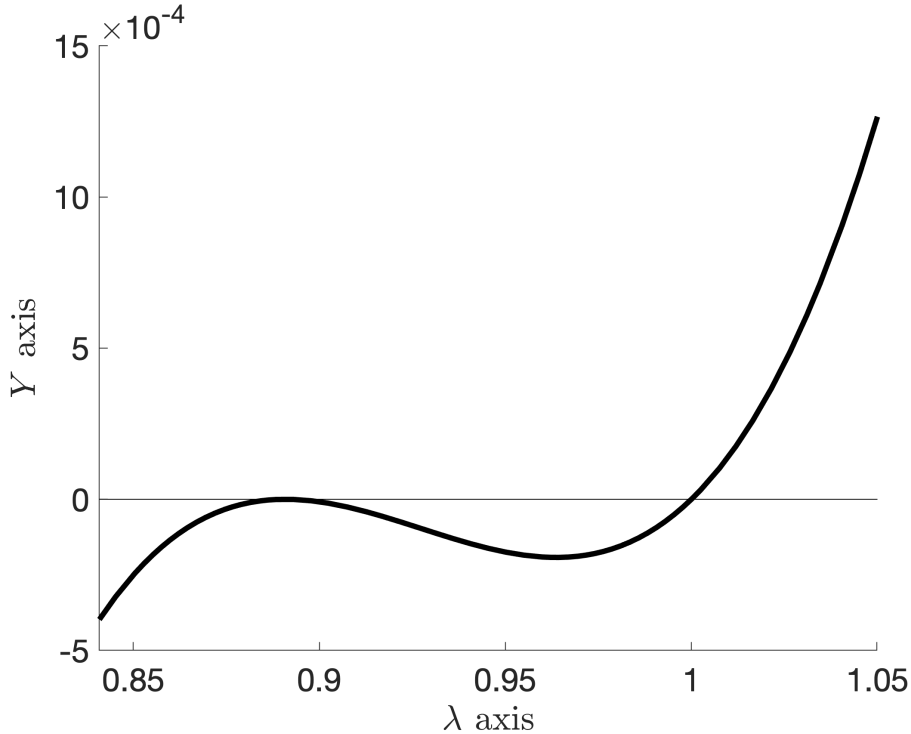


Figure G.3: Characteristic polynomial associated to the IV-⟶Io cadence

G.2.4. bV⟶Io Cadence

We are still working on establishing the tonal functions between the degrees of the Locrian mode. Through this process, in some moments, we will reach unusual cadences that will be of special interest for the construction of P progressions that stand out from the common progressions in modern music. We use the usual problem construction, starting with the E link and ending at the tonal function.

$$E_{(bV_c^r|I_c^r o)} = \begin{pmatrix} Db & Gb \\ Bb & Eb \\ Gb & C \end{pmatrix} \tag{G.19}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix:

$$L_{(bV^r|I^r o)} = \begin{pmatrix} 5 & 2 & 1 \\ 4 & 5 & 2 \\ 0 & 3 & 6 \end{pmatrix} \tag{G.20}$$

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bV^r|I^r o)} = \begin{pmatrix} 5 & 2 & 1 \\ 4 & 5 & 2 \\ 0 & 3 & 6 \end{pmatrix} \longrightarrow L^F_{(bV^r|I^r o)} = \begin{pmatrix} 4 & 1 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & 6 \end{pmatrix} \longrightarrow L^H_{(bV^r|I^r o)} = \begin{pmatrix} 4 & \boxed{0} & 0 \\ 2 & 2 & \boxed{0} \\ \boxed{0} & 2 & 6 \end{pmatrix} \quad (G.21)$$

Then the solutions for $L^H_{(bV^r|I^r o)}$ when both triads are in root position becomes the following set $S(L^H_{(bV^r|I^r o)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$.

With the information that appears in the distribution of boxes drawn by the algorithm, we are able to understand how the classes are matched between both tonal centers, then we can calculate the tonal function of the antecedent and consequent chord.

$$\left[E^o_{(bV_c|I_c o)} \right]_\nabla = \left[\begin{pmatrix} Db & Eb \\ Bb & C \\ Gb & Gb \end{pmatrix} \right]_\nabla \quad (G.22)$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value:

$$\nabla(E^o_{(bV_c|I_c o)}) = 2 + 2 + 0 = 4$$

We write the optimal nabla value as a generalization for every tonality:

$$\nabla^o_{(bV|Io)} = 4$$

We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_\mathbb{E}$.

$$C_\mathbb{E} : \Phi^3 \longrightarrow \Phi^3$$

$$C_\mathbb{E}(\psi(bV_c)) \longrightarrow \psi(I_c o))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Db_{z_1} \\ Bb_{z_2} \\ Gb_{z_3} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Gb_{z_3} \end{pmatrix} \quad (G.23)$$

The characteristic polynomial is calculated using $\Delta(C_\mathbb{E} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_\mathbb{E} - \lambda Id) = \Delta \begin{pmatrix} s^2 - \lambda & 0 & 0 \\ 0 & s^2 - \lambda & 0 \\ 0 & 0 & s^0 - \lambda \end{pmatrix} \quad (G.24)$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^2 - \lambda)(s^2 - \lambda)(s^0 - \lambda)$$

Using the implicit T transformation, we reach the roots of the tonal function, then we think about its classification using the polynomial criterion. In the set of divergent roots we only find one root:

$$\lambda^+ = \{s^2\}$$

On the stabilizer $E(M)$ we find a root that has the same value as the stabilizer and is synonymous with the fact that there is at least one voice that moves us in the optimal link. We note that using the conjunctistic notation does not provide how many roots appear on the stabilizer, this being the main reason why we have used the language of multiplicities, since we do care about the number of roots that appear on $E(M)$.

$$\lambda^0 = \{s^0\}$$

To the left of $E(M)$ we find an empty set of roots, since none exist, which means that there are no voices moving down the optimal link.

$$\lambda^- = \{\emptyset\}$$

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bV_c)$ or $\psi(I_co)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. With the help of midi notation, we write the wave functions of the optimal voicings of the antecedent and consequent tonal centers. In this way we establish that for a set of integers that determine the opening of the array, we will be able to have the equations for that opening just by choosing said set.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bV_c}(t) = \psi_{Db_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{Gb_{z_3}}(t) \\ \psi_{I_co}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Gb_{z_3}}(t) \end{cases}$$

For completeness and insight, we also write the same wave functions in their general form as we have already done in previous cases. We use the arrow notation $\longrightarrow$ to indicate the temporal order in perception.

$$\psi_{bV_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\Gamma_k \frac{e^{2\pi tki\psi_j(bV_c)} - e^{-2\pi tki\psi_j(bV_c)}}{2i}$$

$$\longrightarrow \psi_{I_co}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h}\ \Gamma_k \frac{e^{2\pi tki\psi_j(I_co)} - e^{-2\pi tki\psi_j(I_co)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$ then, following the polynomial criterion we obtain the function of the degree bV related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bV|Io)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-2\,s^2 - 1\right)\,\lambda^2 + \left(s^4 + 2\,s^2\right)\,\lambda - s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s^2}{3} - \frac{1}{3}\right)\,\lambda^3 + \frac{s^2\,\left(s^2+2\right)}{2}\,\lambda^2 + \left(-s^4\right)\,\lambda$

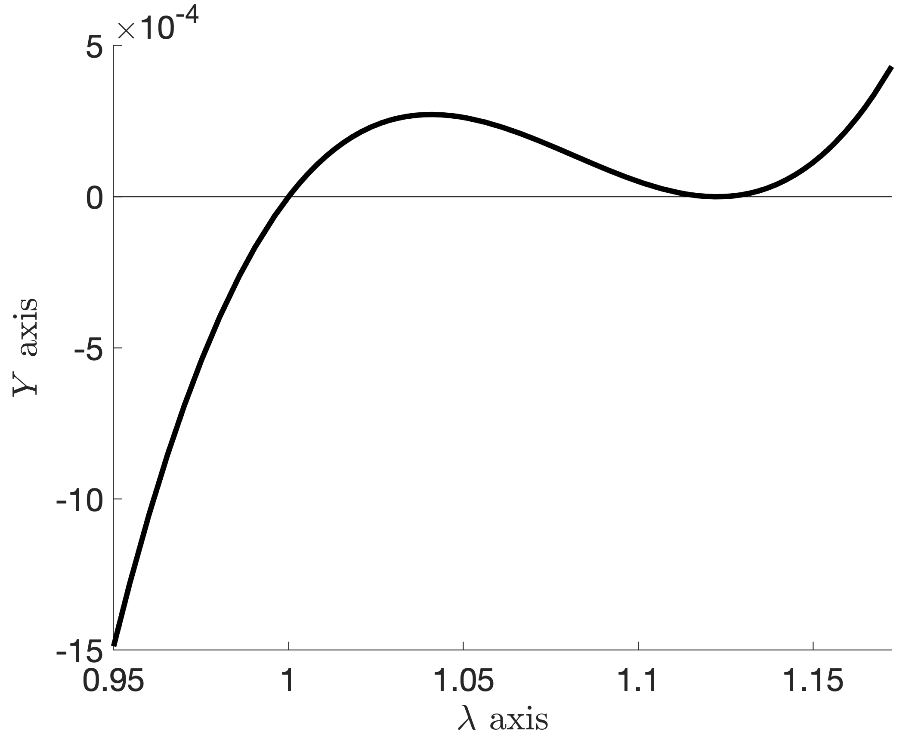


Figure G.4: Characteristic polynomial associated to the bV⟶Io cadence

G.2.5. bVI⟶Io Cadence

We continue with the process of calculating the tonal functions. At this point we are going to study how the sixth degree and the first degree of the Phrygian mode are related when we are considering their triads. Thus we follow the procedure, taking advantage of the connection W.F.C. We will follow the usual procedure, generating a link E calculating its matrix L and using the transformation T to at some point reach the solution.

$$E_{(bVI^r_c|I^r_c o)} = \begin{pmatrix} Eb & Gb \\ C & Eb \\ Ab & C \end{pmatrix} \tag{G.25}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix. At this point we can remove the subscripts indicating the key since L is preserved:

$$L_{(bVI^r|I^r o)} = \begin{pmatrix} 3 & 0 & 3 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{pmatrix} \tag{G.26}$$

We point out that the L matrices are the same between degrees, that is to say that, regardless of the tone in which we are speaking, the L matrix between two degrees is the same. This occurs since the color theorem equates each entry of the matrix by transporting the classes of each metric involved in each entry. For this reason we omit the subscripts.

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bVI^r|I^r o)} = \begin{pmatrix} 3 & 0 & 3 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{pmatrix} \longrightarrow L^F_{(bVI^r|I^r o)} = \begin{pmatrix} 3 & 0 & 3 \\ 6 & 3 & 0 \\ 0 & 3 & 2 \end{pmatrix} \longrightarrow L^H_{(bVI^r|I^r o)} = \begin{pmatrix} 3 & \boxed{0} & 3 \\ 6 & 3 & \boxed{0} \\ \boxed{0} & 3 & 2 \end{pmatrix} \tag{G.27}$$

The solutions for $L^H_{(bVI^r|I^r o)}$ when both triads are in root position becomes the following set: $S(L^H_{(bVI^r|I^r o)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$:

$$\left[E^o_{(bVI_c|I_c o)} \right]_\nabla = \left[\begin{pmatrix} Eb & Eb \\ C & C \\ Ab & Gb \end{pmatrix} \right]_\nabla \tag{G.28}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bVI_c|I_c o)}) = 0 + 0 + 2 = 2$ and we write the optimal nabla value as a generalization for every tonality$\nabla^o_{(bVI|Io)} = 2$.

Any optimal arrangement from an optimal progression $E^o_{(bVI|Io)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bVI_c)) \longrightarrow \psi(I_c o))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Ab_{z_3} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Gb_{z_3} \end{pmatrix} \tag{G.29}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with assigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^0 - \lambda & 0 & 0 \\ 0 & s^0 - \lambda & 0 \\ 0 & 0 & s^{-2} - \lambda \end{pmatrix} \tag{G.30}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^0 - \lambda)(s^0 - \lambda)(s^{-2} - \lambda)$$

By using the transformation T and reaching the tonal function we see that a root appears to the left of $E(M)$. Thus, in this set the said root appears, which is representing that at least one voice moves downwards.

$$\lambda^- = \{s^{-2}\}$$

On the stabilizer $E(M)$ there are two roots that take the value $E(M) = 1$ on the λ axis. Thus, on said axis there are two roots on the stabilizer, although the set theory at this point only informs us that there is at least one root on the stabilizer. This is the reason why the polynomial criterion is related to the algebraic multiplicities described above, since set theory modeling is not enough to accurately characterize the movement of voices. This is a specific question of voices that move with the same proportion and in the same direction, where the language of the sets we are talking about makes them indistinguishable. Although the cardinality of the set is one, there are two roots over the stabilizer for this case.

$$\lambda^0 = \{s^0\}$$

On the other hand, to the right of the stabilizer. We see that no roots appear.

$$\lambda^+ = \{\emptyset\}$$

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVI_c)$ or $\psi(I_co)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the wave functions of the pair of voicings breaking them down into the wave functions of each note. Using the bracket and focusing on the subscripts we see how the voices are linked in the optimal link.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVI_c}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Ab_{z_3}}(t) \\ \psi_{I_co}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Gb_{z_3}}(t) \end{cases}$$

Following the optimization conditions and given a particular aperture, we write the wave function of the voicings for a Γ distribution.

$$\psi_{bVI_c}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVI_c)} - e^{-2\pi tki\psi_j(bVI_c)}}{2i}$$

$$\longrightarrow \psi_{I_co}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \ \Gamma_k \frac{e^{2\pi tki\psi_j(I_co)} - e^{-2\pi tki\psi_j(I_co)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$ then, following the polynomial criterion we obtain the function of the degree bVI related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bVI|Io)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + \left(-\frac{1}{s^2} - 2\right) \lambda^2 + \left(\frac{2}{s^2} + 1\right) \lambda - \frac{1}{s^2}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s^2+1}{3\,s^2}\right) \lambda^3 + \frac{s^2+2}{2\,s^2} \lambda^2 + \left(-\frac{1}{s^2}\right) \lambda$

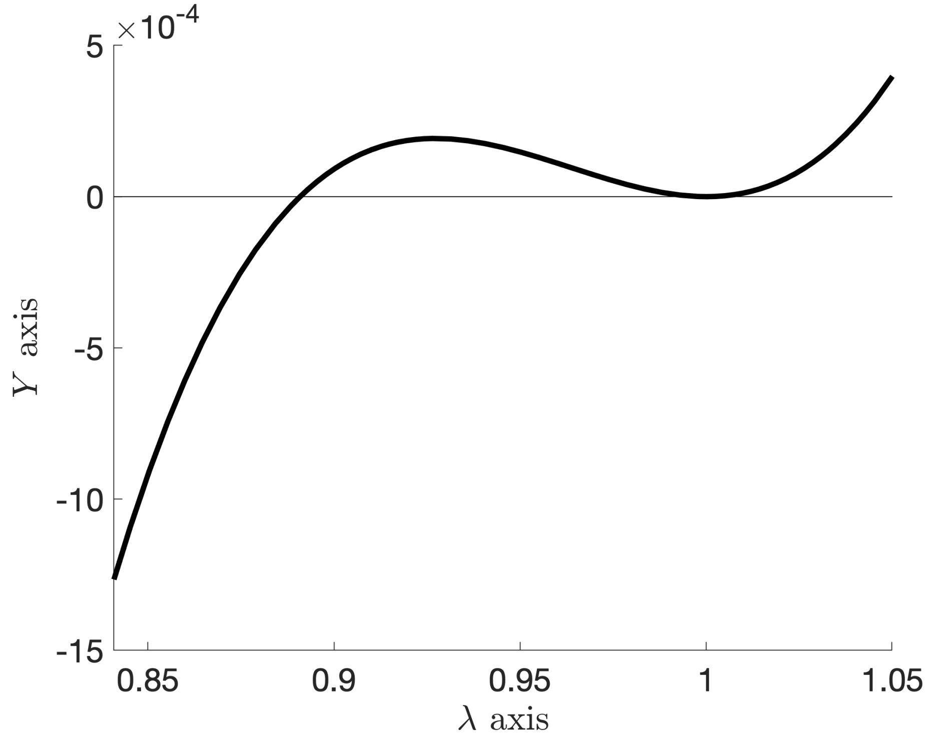


Figure G.5: Characteristic polynomial associated to the bVI⟶Io cadence

G.2.6. bVII⟶Io Cadence

We continue with the process of calculating the tonal functions. At this point we are going to study how the sixth degree and the first degree of the phrygian mode are related when we are considering their triads. Thus we follow the procedure, taking advantage of the connection W.F.C. We will follow the usual procedure, generating a link E calculating its matrix L and using the transformation T to at some point reach the solution.

$$E_{(bVII^r_c|I^r_c o)} = \begin{pmatrix} F & Gb \\ Db & Eb \\ Bb & C \end{pmatrix} \tag{G.31}$$

The link cadence will be constructed calculating every Δ_{ij} distance and arranging them into the L matrix. At this point we can remove the subscripts indicating the key since L is preserved:

$$L_{(bVII^r|I^r o)} = \begin{pmatrix} 1 & 2 & 5 \\ 5 & 2 & 1 \\ 4 & 5 & 2 \end{pmatrix} \tag{G.32}$$

We point out that the L matrices are the same between degrees, that is to say that, regardless of the tone in which we are speaking, the L matrix between two degrees is the same. This occurs since the color theorem equates each entry of the matrix by transporting the classes of each metric involved in each entry. For this reason we omit the subscripts.

Following the steps of the Hungarian algorithm we consider to develop the algorithm through the L matrix to find an optimum link:

$$L_{(bVI^r|I^r o)} = \begin{pmatrix} 1 & 2 & 5 \\ 5 & 2 & 1 \\ 4 & 5 & 2 \end{pmatrix} \longrightarrow L^F_{(bVI^r|I^r o)} = \begin{pmatrix} 0 & 1 & 4 \\ 4 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix} \longrightarrow L^H_{(bVI^r|I^r o)} = \begin{pmatrix} \boxed{0} & 0 & 4 \\ 4 & \boxed{0} & 0 \\ 2 & 3 & \boxed{0} \end{pmatrix} \tag{G.33}$$

The solutions for $L^H_{(bVII^r|I^r o)}$ when both triads are in root position becomes the following set: $S(L^H_{(bVII^r|I^r o)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}\}$. We build the optimal link with the previous solutions:

$$\left[E^o_{(bVII_c|I_c o)}\right]_\nabla = \left[\begin{pmatrix} F & Gb \\ Db & Eb \\ Bb & C \end{pmatrix}\right]_\nabla \tag{G.34}$$

We calculate the optimal link class nabla value, the class all the posible link between a chord and the tonal center that share nabla value: $\nabla(E^o_{(bVII_c|I_c o)}) = 1 + 2 + 2 = 5$ and we write the optimal nabla value as a generalization for every tonality $\nabla^o_{(bVII|Io)} = 5$.

Any optimal arrangement from an optimal progresion $E^o_{(bVII|Io)}$ has a cadence with the same characteristic polynomial. We calculate a cadence of the optimal link located in $\Phi^3 \times \Phi^3$. Notice that for every value of the subindex $z \in \mathbb{Z}$, we can use the same cadence $C_{\mathbb{E}}$.

$$C_{\mathbb{E}} : \Phi^3 \longrightarrow \Phi^3$$

$$C_{\mathbb{E}}(\psi(bVII_c)) \longrightarrow \psi(I_c o))$$

$$\begin{pmatrix} s^{l_1} & 0 & 0 \\ 0 & s^{l_2} & 0 \\ 0 & 0 & s^{l_3} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ Db_{z_2} \\ Bb_{z_3} \end{pmatrix} = \begin{pmatrix} Gb_{z_1} \\ Eb_{z_2} \\ C_{z_3} \end{pmatrix} \tag{G.35}$$

The characteristic polynomial is calculated using $\Delta(C_{\mathbb{E}} - \lambda Id)$ with asigned values l_1, l_2 and l_3:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = \Delta \begin{pmatrix} s^1 - \lambda & 0 & 0 \\ 0 & s^2 - \lambda & 0 \\ 0 & 0 & s^2 - \lambda \end{pmatrix} \tag{G.36}$$

Using the properties of the determinant the polynomial has the form:

$$\Delta(C_{\mathbb{E}} - \lambda Id) = (s^1 - \lambda)(s^2 - \lambda)(s^2 - \lambda)$$

Using the transformation T we see that when reaching the tonal function, it has two roots to the right of the stabilizer of M, one of them being double. Thus, the rest of the sets are empty. We write the three sets in the usual way: $\lambda^+ = \{s^1, s^2\}$, $\lambda^0 = \{\emptyset\}$ and $\lambda^- = \{\emptyset\}$.

We know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVII_c)$ or $\psi(I_c o)$ for a given tonal center, it is clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the wave functions of the pair of voicings breaking them down into the wave functions of each note. Using the bracket and focusing on the subscripts we see how the voices are linked in the optimal link.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII_c}(t) = \psi_{F_{z_1}}(t) + \psi_{Db_{z_2}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I_c o}(t) = \psi_{Gb_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

Following the optimization conditions and given a particular aperture, we write the wave function of the voicings for a Γ distribution.

$$\psi_{bVII_c}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII_c)} - e^{-2\pi tki\psi_j(bVII_c)}}{2i}$$

$$\longrightarrow \psi_{I_c o}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c o)} - e^{-2\pi tki\psi_j(I_c o)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then, following the polynomial criterion we obtain the function of the degree $bVII$ related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$. The function is generalized using the static tonal function theorem.

$$\boxed{\Phi[E_{(bVII|Io)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^3 + (-2\,s^2 - 1)\ \lambda^2 + (s^4 + 2\,s^2)\ \lambda - s^4$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^4}{4} + \left(-\frac{2\,s^2}{3} - \frac{1}{3}\right)\ \lambda^3 + \frac{s^2\,\left(s^2+2\right)}{2}\ \lambda^2 + (-s^4)\ \lambda$

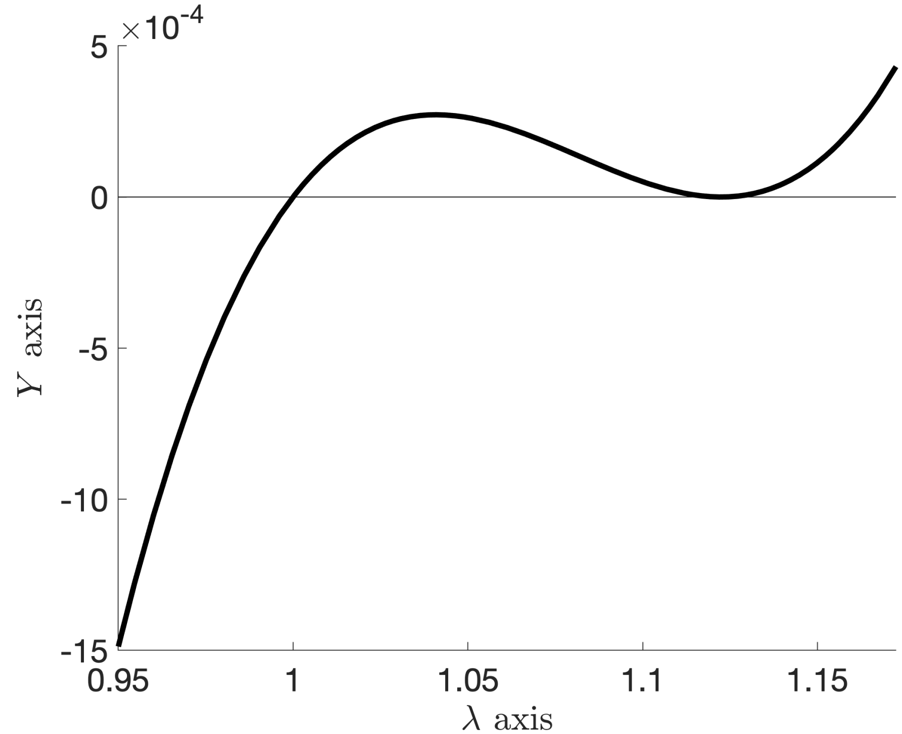


Figure G.6: Characteristic polynomial associated to the bVII⟶Io cadence

G.3. The Locrian mode for n=4

G.3.1. bII△ ⟶Iø7 Cadence

In the process of calculating all the tonal functions in chords within the framework of common practice, we continue to calculate the functions for the Locrian mode in $n = 4$ with the usual class choices.

$$E_{(bII_c^r\triangle|I_c^r\o7)} = \begin{pmatrix} C & Bb \\ Ab & Gb \\ F & Eb \\ Db & C \end{pmatrix} \tag{G.37}$$

Once we have written the link in the usual way, we have to calculate the matrix L in order to develop the algorithm through said matrix and reach a distribution of boxes over L^H.

$$L_{(bII^r\triangle|I^r\o7)} = \begin{pmatrix} 2 & 6 & 3 & 0 \\ 2 & 2 & 5 & 4 \\ 5 & 1 & 2 & 5 \\ 3 & 5 & 2 & 1 \end{pmatrix} \tag{G.38}$$

Following the steps of the Hungarian algorithm we develop the L matrix. We always follow the process of transforming L into L^F and trying to reach L^H or a variant of it as soon as possible. In this case, we reach the solution with ease. We carry out the calculations calculating the matrices in sequence as we have been doing:

$$L_{(bII^r\triangle|I^r\text{ø}7)} = \begin{pmatrix} 2 & 6 & 3 & 0 \\ 2 & 2 & 5 & 4 \\ 5 & 1 & 2 & 5 \\ 3 & 5 & 2 & 1 \end{pmatrix} \longrightarrow L^F_{(bII^r\triangle|I^r\text{ø}7)} = \begin{pmatrix} 2 & 6 & 3 & 0 \\ 0 & 0 & 3 & 2 \\ 4 & 0 & 1 & 4 \\ 2 & 4 & 1 & 0 \end{pmatrix}$$

$$\longrightarrow L^H_{(bII^r\triangle|I^r\text{ø}7)} = \begin{pmatrix} 2 & 6 & 2 & \boxed{0} \\ \boxed{0} & 0 & 2 & 2 \\ 4 & \boxed{0} & 0 & 4 \\ 2 & 4 & \boxed{0} & 0 \end{pmatrix}$$

The set S for this matrix is the set of metrics that are indicated by the distribution of boxes:

$$S(L^H_{(bII^r\triangle|I^r\text{ø}7\triangle)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

Using the calculated box distribution we find the pairing between classes that corresponds to the optimal link E^o.

$$\left[E^o_{(bII_c\triangle|I_c\text{ø}7)}\right]_\nabla = \left[\begin{pmatrix} C & C \\ Ab & Bb \\ F & Gb \\ Db & Eb \end{pmatrix}\right]_\nabla \tag{G.39}$$

Once an optimal link is reached we are ready to use the transformation T and arrive at the matrix $C_{\mathbb{E}}$. We see that according to the notation that we are using in this case we have to recover the sign of one of the metrics. For all other, this is unnecessary because its value is cero.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bII_c\triangle)) \longrightarrow \psi(I_c\text{ø}7)$$

$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} C_{z_1} \\ Ab_{z_2} \\ F_{z_3} \\ Db_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ Gb_{z_3} \\ Eb_{z_4} \end{pmatrix} \tag{G.40}$$

Using $C_{\mathbb{E}}$ and taking its characteristic polynomial, we reach the tonal function. Thus this function has three roots on $E(M)$ and one to the left of said stabilizer.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^2 - \lambda)(s^1 - \lambda)(s^2 - \lambda)$$

Inspecting the previous polynomial, we study the algebraic multiplicities, where three formulas appear whose sum coincides with the dimension of the frequency space in this case. Thus the convergent algebraic multiplicity is $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$. The static algebraic multiplicity has value $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$. Finally, the divergent algebraic multiplicity has value $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bII\triangle)$ or $\psi(I\o 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bII\triangle}(t) = \psi_{C_{z_1}}(t) + \psi_{Ab_{z_2}}(t) + \psi_{F_{z_3}} + \psi_{Db_{z_4}}(t) \\ \psi_{I\o 7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{Gb_{z_3}} + \psi_{Eb_{z_4}}(t) \end{cases}$$

We write below, using the notation of parentheses, the wave functions of each voicing. We see that as a function of the aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$ where $\epsilon = (z_1, ..., z_4)$ each of the functions that assign each class to a particular frequency will assign said fundamental frequency as a function of ϵ, keeping, for the optimization conditions, the value of absolute perception to a minimum.

$$\psi_{bII_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bII_c\triangle)} - e^{-2\pi tki\psi_j(bII_c\triangle)}}{2i}$$

$$\longrightarrow \psi_{I_c\o 7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c\o 7)} - e^{-2\pi tki\psi_j(I_c\o 7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 3$ then following the polynomial criterion we obtain the function of the degree $bII\triangle$ related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bII\triangle|I\o 7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-2\,s^2 - s - 1\right)\,\lambda^3$
$+ \left(s^4 + 2\,s^3 + 2\,s^2 + s\right)\,\lambda^2 + \left(-s^5 - s^4 - 2\,s^3\right)\,\lambda + s^5$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{2} - \frac{s}{4} - \frac{1}{4}\right)\,\lambda^4 + \frac{s\,\left(s^3+2\,s^2+2\,s+1\right)}{3}\,\lambda^3$
$+ \left(-\frac{s^3\,\left(s^2+s+2\right)}{2}\right)\,\lambda^2 + s^5\,\lambda$

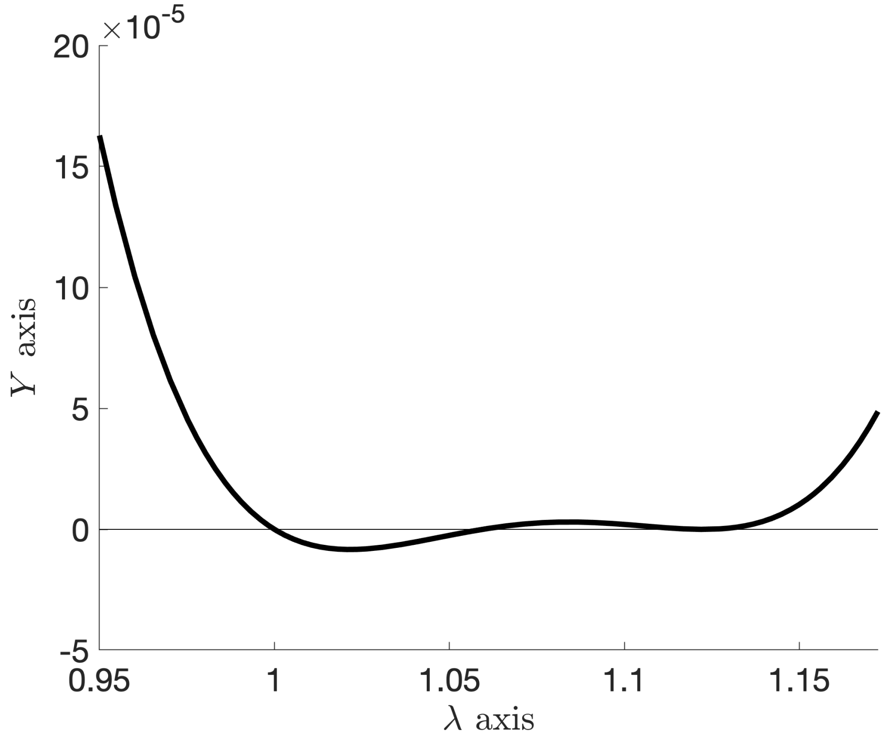


Figure G.7: Characteristic polynomial associated to the bII△ ⟶Iø7 cadence

G.3.2. bIII-7⟶Iø7 Cadence

In order to study the relationship between the third Locrian degree and the first, we first calculate the bond E, which will be enough to, from this information, develop the calculations until reaching the tonal function.

$$E_{(bIII^r_c-7|I^r_c\o 7)} = \begin{pmatrix} C & Bb \\ Bb & Gb \\ Gb & Eb \\ Eb & C \end{pmatrix} \tag{G.41}$$

From the previous link we calculate the matrix L. Thus, it is enough for us to find one or several distributions of boxes on said matrix L and later use the transformation T.

$$L_{(bIII^r-7|I^r\o 7)} = \begin{pmatrix} 3 & 5 & 2 & 1 \\ 0 & 4 & 5 & 2 \\ 4 & 0 & 3 & 6 \\ 5 & 3 & 0 & 3 \end{pmatrix} \tag{G.42}$$

Starting from the well-known L matrix, we develop the calculations using the steps of the Hungarian algorithm, where in this case we reach a unique distribution of boxes over L^H. In this way, we collect the solutions in the set S, which gives us information on both the metrics involved in the optimal link and the optimal pairing.

$$L_{(bIII^r-7|I^r\o 7)} = \begin{pmatrix} 3 & 5 & 2 & 1 \\ 0 & 4 & 5 & 2 \\ 4 & 0 & 3 & 6 \\ 5 & 3 & 0 & 3 \end{pmatrix} \longrightarrow L^F_{(bIII^r-7|I^r\o 7)} = \begin{pmatrix} 2 & 4 & 1 & 0 \\ 0 & 4 & 5 & 2 \\ 4 & 0 & 3 & 6 \\ 5 & 3 & 0 & 3 \end{pmatrix}$$

$$\longrightarrow L^H_{(bIII^r-7|I^r\o 7)} = \begin{pmatrix} 2 & 4 & 1 & \boxed{0} \\ \boxed{0} & 4 & 5 & 2 \\ 4 & \boxed{0} & 3 & 6 \\ 5 & 3 & \boxed{0} & 3 \end{pmatrix}$$

The set S for this matrix is the set of metrics that are indicated by the distribution of boxes:

$$S(L^H_{(bIII^r-7|I^r\o 7\triangle)}) = \{\Delta_{14}, \Delta_{21}, \Delta_{32}, \Delta_{43}\}$$

Using the calculated box distribution we find the pairing between classes that corresponds to the optimal link E^o.

$$\left[E^o_{(bIII_c-7|I_c\o 7)} \right]_\nabla = \left[\begin{pmatrix} Db & C \\ Bb & Bb \\ Gb & Gb \\ Eb & Eb \end{pmatrix} \right]_\nabla \tag{G.43}$$

Inspecting the optimal link, we realize that only one voice is moving in the optimum. As we already have some experience in the process of calculating tonal functions, we understand that the tonal function of this link E is going to be classified within the tonic area, following the polynomial criterion, but even so we develop the whole process formally.

Using the transformation T where $T(L_E) = C_{\mathbb{E}}$ we reach the matrix $C_{\mathbb{E}}$ where in this case we do not require multiple transformations since the distribution of boxes that make up the solution is unique. This is how we write the matrix equation for this case:

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(bIII_c - 7)) \longrightarrow \psi(I_c\o 7)$$
$$\begin{pmatrix} s^{-\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} Db_{z_1} \\ Bb_{z_2} \\ Gb_{z_3} \\ Eb_{z_4} \end{pmatrix} = \begin{pmatrix} C_{z_1} \\ Bb_{z_2} \\ Gb_{z_3} \\ Eb_{z_4} \end{pmatrix} \tag{G.44}$$

Taking the characteristic polynomial of the previous matrix where there is recovery of the sign of the first metric, we reach the characteristic polynomial which is itself the tonal function.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^{-1} - \lambda)(s^0 - \lambda)(s^0 - \lambda)(s^0 - \lambda)$$

Inspecting the previous polynomial, we study the algebraic multiplicities, where three formulas appear whose sum coincides with the dimension of the frequency space in this case. Thus the convergent algebraic multiplicity is $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 1$. The static algebraic multiplicity has value $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$. Finally, the divergent algebraic multiplicity has value $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$. With this information, we can now apply the polynomial criterion and establish in which area the tonal function is. In this case, applying this criterion we see that it will be the tonic area. We continue studying the case in the formal context to classify this function and be able to assign a direction to the edge in the graph of tonal functions.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bIII - 7)$ or $\psi(I\o 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bIII-7}(t) = \psi_{Db_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{Gb_{z_3}} + \psi_{Eb_{z_4}}(t) \\ \psi_{I\o 7}(t) = \psi_{C_{z_1}}(t) + \psi_{Bb_{z_2}}(t) + \psi_{Gb_{z_3}} + \psi_{Eb_{z_4}}(t) \end{cases}$$

We write below, using the notation of parentheses, the wave functions of each voicing. We see that as a function of the aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$ where $\epsilon = (z_1, ..., z_4)$ each of the functions that assign each class to a particular frequency will assign said fundamental frequency as a function of ϵ, keeping, for the optimization conditions, the value of absolute perception to a minimum.

$$\psi_{bIII_c-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bIII_c-7)} - e^{-2\pi tki\psi_j(bIII_c-7)}}{2i}$$

$$\longrightarrow \psi_{I_c\o 7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c\o 7)} - e^{-2\pi tki\psi_j(I_c\o 7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ then following the polynomial criterion we obtain the function of the degree $bIII - 7$ related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bIII-7|I\o 7)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s} - 3\right) \lambda^3 + \left(\frac{3}{s} + 3\right) \lambda^2 + \left(-\frac{3}{s} - 1\right) \lambda + \frac{1}{s}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{3s+1}{4s}\right) \lambda^4 + \frac{3s+3}{3s} \lambda^3 + \left(-\frac{s+3}{2s}\right) \lambda^2 + \frac{\lambda}{s}$

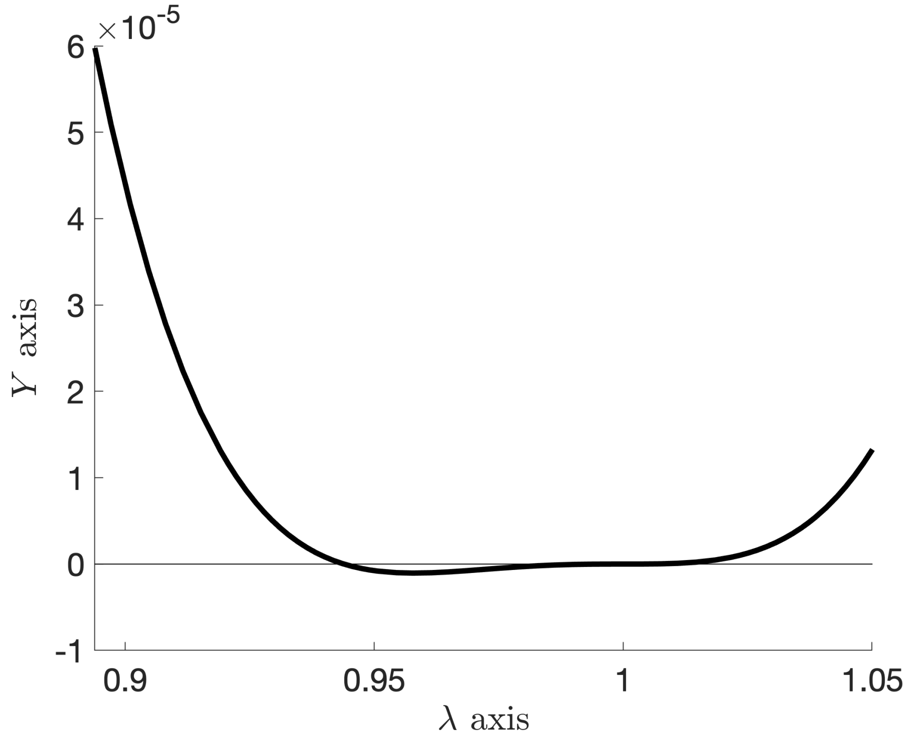


Figure G.8: Characteristic polynomial associated to the bIII-7⟶Iø7 cadence

G.3.3. IV-7⟶Iø7 Cadence

We continue studying the cases until obtaining each tonal function of each edge of the graph for $n = 4$. We set up the link E in such a way that we can build a matrix L on which to apply the Hungarian algorithm.

$$E_{(IV_c^r-7|I_c^r\o 7)} = \begin{pmatrix} Eb & Bb \\ C & Gb \\ Ab & Eb \\ F & C \end{pmatrix} \tag{G.45}$$

From the above link, we build an L matrix, then compute each metric and sort them into that matrix:

$$L_{(IV^r-7|I^r\o 7)} = \begin{pmatrix} 5 & 3 & 0 & 3 \\ 2 & 6 & 3 & 0 \\ 2 & 2 & 5 & 4 \\ 5 & 1 & 2 & 5 \end{pmatrix} \tag{G.46}$$

With the construction of the previous matrix, we pose the optimization problem that we are going to solve with the Hungarian algorithm until we reach a distribution of boxes.

$$L_{(IV^r-7|I^r\o7)} = \begin{pmatrix} 5 & 3 & 0 & 3 \\ 2 & 6 & 3 & 0 \\ 2 & 2 & 5 & 4 \\ 5 & 1 & 2 & 5 \end{pmatrix} \longrightarrow L^F_{(IV^r-7|I^r\o7)} = \begin{pmatrix} 5 & 3 & 0 & 3 \\ 2 & 6 & 3 & 0 \\ 0 & 0 & 3 & 2 \\ 4 & 0 & 1 & 4 \end{pmatrix} \longrightarrow$$
$$L^H_{(IV^r-7|I^r\o7)} = \begin{pmatrix} 5 & 3 & \boxed{0} & 3 \\ 2 & 6 & 3 & \boxed{0} \\ \boxed{0} & 0 & 3 & 2 \\ 4 & \boxed{0} & 1 & 4 \end{pmatrix} \tag{G.47}$$

The set S for this matrix is the set of metrics that are indicated by the distribution of boxes:

$$S(L^H_{(IV^r-7|I^r\o7)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

From the S array above, we compute an optimal link, and immediately generalize it to its nabla class.

$$\left[E^o_{(IV^r-7|I^r\o7)} \right] = \left[\begin{pmatrix} Eb & Eb \\ C & C \\ Ab & Bb \\ F & Gb \end{pmatrix} \right]_{\nabla} \tag{G.48}$$

By looking at the optimal link we already have a good idea of how voices behave on the optimal link. We are going to complete the formal process to the end to classify the tonal function within an area and thus know the direction of the arrow in the graph of tonal functions and be able to use this link as material for the composition.

Using the transformation T where $T(L_E) = C_{\mathbb{E}}$ we reach the matrix $C_{\mathbb{E}}$ where in this case we do not require multiple transformations since the distribution of boxes that make up the solution is unique. This is how we write the matrix equation for this case:

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(IV_c - 7)) \longrightarrow \psi(I_c\o7)$$
$$\begin{pmatrix} s^{\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Ab_{z_3} \\ F_{z_4} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Bb_{z_3} \\ Gb_{z_4} \end{pmatrix} \tag{G.49}$$

Taking the characteristic polynomial of the previous matrix as it has been done, we calculate it in a factorized way, obtaining the expression where we can see the algebraic multiplicities that are of our interest.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^0 - \lambda)(s^0 - \lambda)(s^2 - \lambda)(s^1 - \lambda)$$

Inspecting the previous polynomial, we study the algebraic multiplicities, where three formulas appear whose sum coincides with the dimension of the frequency space in this case. Thus the convergent algebraic multiplicity is $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$. The static algebraic multiplicity has value $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$. Finally, the divergent algebraic multiplicity has value $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 2$.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(IV - 7)$ or $\psi(I\o 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{IV-7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Ab_{z_3}} + \psi_{F_{z_4}}(t) \\ \psi_{I\o 7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{Gb_{z_4}}(t) \end{cases}$$

We write below, using the notation of parentheses, the wave functions of each voicing. We see that as a function of the aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$ where $\epsilon = (z_1, \ldots, z_4)$ each of the functions that assign each class to a particular frequency will assign said fundamental frequency as a function of ϵ, keeping, for the optimization conditions, the value of absolute perception to a minimum.

$$\psi_{IV_c-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(IV_c-7)} - e^{-2\pi tki\psi_j(IV_c-7)}}{2i}$$
$$\longrightarrow \psi_{I_c\o 7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c\o 7)} - e^{-2\pi tki\psi_j(I_c\o 7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ then following the polynomial criterion we obtain the function of the degree $IV - 7$ related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(IV-7|I\o 7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s^2 - s - 2)\ \lambda^3 + (s^3 + 2\,s^2 + 2\,s + 1)\ \lambda^2 + (-2\,s^3 - s^2 - s)\ \lambda + s^3$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{4} - \frac{s}{4} - \frac{1}{2}\right)\ \lambda^4 + \left(\frac{s^3}{3} + \frac{2\,s^2}{3} + \frac{2\,s}{3} + \frac{1}{3}\right)\ \lambda^3 + \left(-\frac{s\left(2\,s^2+s+1\right)}{2}\right)\ \lambda^2 + s^3\,\lambda$

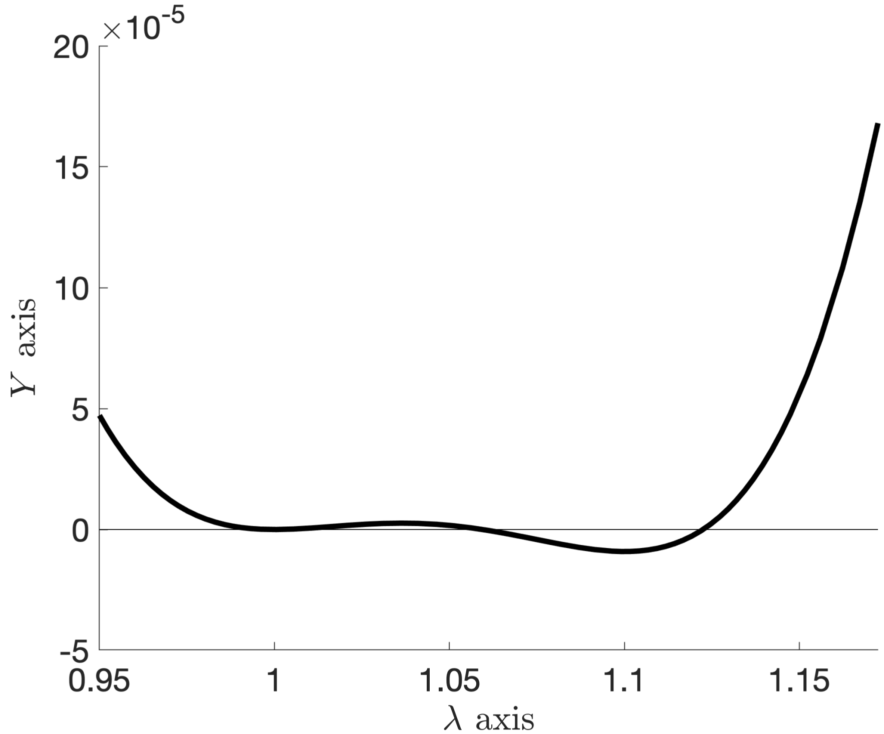


Figure G.9: Characteristic polynomial associated to the P IV-7⟶Iø7 cadence

G.3.4. bV△ ⟶Iø7 Cadence

We continue in the calculation of tonal functions in such a way that we are covering each case in each hand making the appropriate calculations. So we are going to write the link for the fifth degree of the Locrian mode to carry out the whole process.

$$E_{(bV_c^r\triangle|I_c^r\o 7)} = \begin{pmatrix} F & Bb \\ Db & Gb \\ Bb & Eb \\ Gb & C \end{pmatrix} \tag{G.50}$$

From the previous link we calculate the L matrix. Thus, it is enough for us to find one or several distributions of boxes on said matrix L and later use the transformation T.

$$L_{(bV^r\triangle|I^r\o 7)} = \begin{pmatrix} 5 & 1 & 2 & 5 \\ 3 & 5 & 2 & 1 \\ 0 & 4 & 5 & 2 \\ 4 & 0 & 3 & 6 \end{pmatrix} \tag{G.51}$$

We carry out the entire optimization process, in such a way that we end up reaching a distribution of boxes over the matrix L^H, which in this case is unique.

$$L_{(bV^r\triangle|I^r\o 7)} = \begin{pmatrix} 5 & 1 & 2 & 5 \\ 3 & 5 & 2 & 1 \\ 0 & 4 & 5 & 2 \\ 4 & 0 & 3 & 6 \end{pmatrix} \longrightarrow L^F_{(bV^r\triangle|I^r\o 7)} = \begin{pmatrix} 4 & 0 & 1 & 4 \\ 2 & 4 & 1 & 0 \\ 0 & 4 & 5 & 2 \\ 4 & 0 & 3 & 6 \end{pmatrix}$$

$$\longrightarrow L^H_{(bV^r\triangle|I^r\o 7)} = \begin{pmatrix} 4 & 0 & \boxed{0} & 4 \\ 2 & 4 & 0 & \boxed{0} \\ \boxed{0} & 4 & 4 & 2 \\ 4 & \boxed{0} & 2 & 6 \end{pmatrix}$$

The set S for this matrix is the set of metrics that are indicated by the distribution of boxes:

$$S(L^H_{(bV^r\triangle|I^r\o 7\triangle)}) = \{\Delta_{13}, \Delta_{24}, \Delta_{31}, \Delta_{42}\}$$

Using the calculated box distribution we find the pairing between classes that corresponds to the optimal link E^o.

$$\left[E^o_{(bV_c\triangle|I_c\o 7)}\right]_\nabla = \left[\begin{pmatrix} F & Eb \\ Db & C \\ Bb & Bb \\ Gb & Gb \end{pmatrix}\right]_\nabla \tag{G.52}$$

Using the T transformation where $T(L_E) = C_\mathbb{E}$ we reach the matrix $C_\mathbb{E}$ where in this case we do not require multiple transformations since the distribution of boxes that make up the solution is unique. This is how we write the matrix equation for this case:

$$C_\mathbb{E} : \Phi^4 \longrightarrow \Phi^4$$

$$C_\mathbb{E}(\psi(bV_c\triangle)) \longrightarrow \psi(I_c\o 7)$$

$$\begin{pmatrix} s^{-\Delta_{13}} & 0 & 0 & 0 \\ 0 & s^{-\Delta_{24}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{31}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{42}} \end{pmatrix} \cdot \begin{pmatrix} F_{z_1} \\ Db_{z_2} \\ Bb_{z_3} \\ Gb_{z_4} \end{pmatrix} = \begin{pmatrix} Eb_{z_1} \\ C_{z_2} \\ Bb_{z_3} \\ Gb_{z_4} \end{pmatrix} \tag{G.53}$$

Taking the characteristic polynomial of the previous matrix where there is recovery of the sign of the first two metrics, we reach the characteristic polynomial which is itself the tonal function.

$$p_{C_\mathbb{E}}(\lambda) = (s^{-2} - \lambda)(s^{-1} - \lambda)(s^0 - \lambda)(s^0 - \lambda)$$

Inspecting the previous polynomial, we study the algebraic multiplicities, where three formulas appear whose sum coincides with the dimension of the frequency space in this case. Thus the convergent algebraic multiplicity is $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 2$. The static algebraic multiplicity has value $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$. Finally, the divergent algebraic multiplicity has value $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$. With this information, we can now apply the polynomial criterion and establish in which area the tonal function is. In this case, applying this criterion we see that it will be the tonic area. We continue studying the case in the formal context to classify this function and be able to assign a direction to the edge in the graph of tonal functions.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bV\triangle)$ or $\psi(I\o 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bV\triangle}(t) = \psi_{F_{z_1}}(t) + \psi_{Db_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{Gb_{z_4}}(t) \\ \psi_{I\o 7}(t) = \psi_{Eb_{z_1}}(t) + \psi_{C_{z_2}}(t) + \psi_{Bb_{z_3}} + \psi_{Gb_{z_4}}(t) \end{cases}$$

We write below, using the notation of parentheses, the wave functions of each voicing. We see that as a function of the aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$ where $\epsilon = (z_1, ..., z_4)$ each of the functions that assign each class to a particular frequency will assign said fundamental frequency as a function of ϵ, keeping, for the optimization conditions, the value of absolute perception to a minimum.

$$\psi_{bV_c\triangle}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bV_c\triangle)} - e^{-2\pi tki\psi_j(bV_c\triangle)}}{2i}$$

$$\longrightarrow \psi_{I_c\o 7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c\o 7)} - e^{-2\pi tki\psi_j(I_c\o 7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 2$ then following the polynomial criterion we obtain the function of the degree $bV\triangle$ related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bV\triangle|I\o 7)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s} - \frac{1}{s^2} - 2\right)\lambda^3 + \left(\frac{2}{s} + \frac{2}{s^2} + \frac{1}{s^3} + 1\right)\lambda^2 + \left(-\frac{1}{s} - \frac{1}{s^2} - \frac{2}{s^3}\right)\lambda + \frac{1}{s^3}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{2\,s^2+s+1}{s^2\,4}\right)\lambda^4 + \frac{s^3+2\,s^2+2\,s+1}{s^3\,3}\lambda^3 + \left(-\frac{s^2+s+2}{s^3\,2}\right)\lambda^2 + \frac{\lambda}{s^3}$

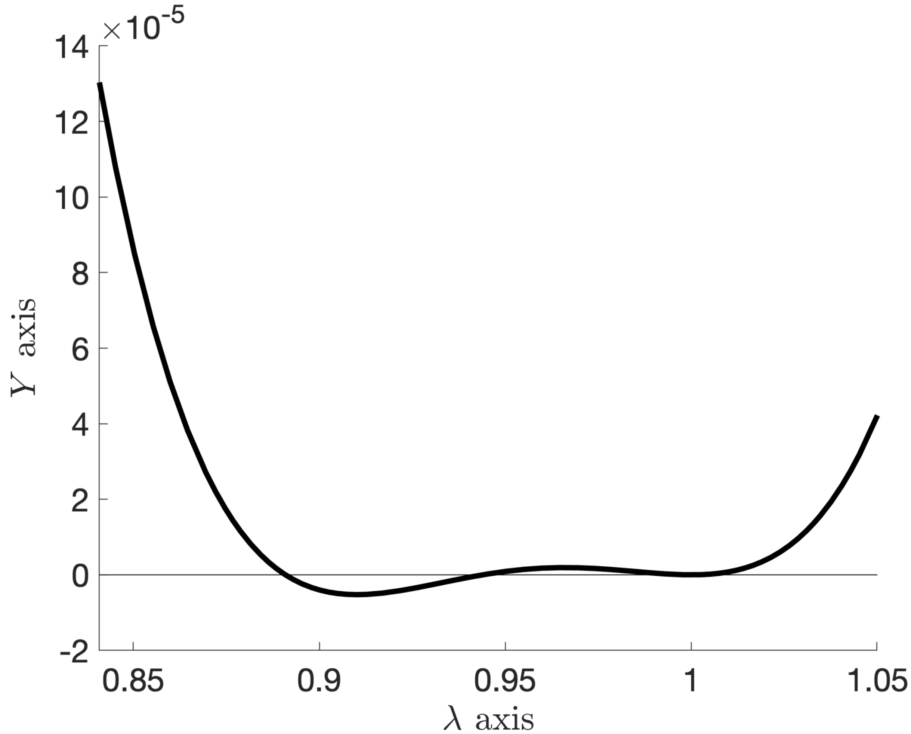


Figure G.10: Characteristic polynomial associated to the bV△ ⟶Iø7 cadence

G.3.5. P bVI7⟶Iø7 cadence

In our calculation process, we continue to refine the method by relying on W.F.C. With this in mind we calculate the link for the sixth degree of the Locrian mode. Thus we have that the link E takes the following form:

$$E_{(bVI^r_c7|I^r_c\text{ø}7)} = \begin{pmatrix} Gb & Bb \\ Eb & Gb \\ C & Eb \\ Ab & C \end{pmatrix} \tag{G.54}$$

From the previous link we calculate the matrix L. Thus, it is enough for us to find one or several distributions of boxes on said matrix L and later use the transformation T.

$$L_{(bVI^r7|I^r\text{ø}7)} = \begin{pmatrix} 4 & 0 & 3 & 6 \\ 5 & 3 & 0 & 3 \\ 2 & 6 & 3 & 0 \\ 2 & 2 & 5 & 4 \end{pmatrix} \tag{G.55}$$

Using the matrix L, we carry out the entire optimization process until we find a distribution of boxes over the matrix L^H. In this case, the box distribution is unique since the tonal function is not dual.

$$L_{(bVI^r7|I^r\o 7)} = \begin{pmatrix} 4 & 0 & 3 & 6 \\ 5 & 3 & 0 & 3 \\ 2 & 6 & 3 & 0 \\ 2 & 2 & 5 & 4 \end{pmatrix} \longrightarrow L^F_{(bVI^r7|I^r\o 7)} = \begin{pmatrix} 4 & 0 & 3 & 6 \\ 5 & 3 & 0 & 3 \\ 2 & 6 & 3 & 0 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

$$\longrightarrow L^H_{(bVI^r7|I^r\o 7)} = \begin{pmatrix} 4 & \boxed{0} & 3 & 6 \\ 5 & 3 & \boxed{0} & 3 \\ 2 & 6 & 3 & \boxed{0} \\ \boxed{0} & 0 & 3 & 2 \end{pmatrix}$$

The set S for this matrix is the set of metrics that are indicated by the distribution of boxes:

$$S(L^H_{(bVI^r7|I^r\o 7\triangle)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

Using the calculated box distribution we find the pairing between classes that corresponds to the optimal link E^o.

$$\left[E^o_{(bVI_c7|I_c\o 7)}\right]_\nabla = \left[\begin{pmatrix} Gb & Gb \\ Eb & Eb \\ C & C \\ Ab & Bb \end{pmatrix}\right]_\nabla \tag{G.56}$$

We have already calculated the optimal link, which in this case represents the only pairing with a minimum value of the nabla function. We continue with the process using the T transformation .

Using the T transformation where $T(L_E) = C_{\mathbb{E}}$ we reach the matrix $C_{\mathbb{E}}$ where in this case we do not require multiple transformations since the distribution of boxes that make up the solution is unique. This is how we write the matrix equation for this case:

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$

$$C_{\mathbb{E}}(\psi(bVI_c7)) \longrightarrow \psi(I_c\o 7)$$

$$\begin{pmatrix} s^{\Delta_{14}} & 0 & 0 & 0 \\ 0 & s^{\Delta_{21}} & 0 & 0 \\ 0 & 0 & s^{\Delta_{32}} & 0 \\ 0 & 0 & 0 & s^{\Delta_{43}} \end{pmatrix} \cdot \begin{pmatrix} Gb_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Ab_{z_4} \end{pmatrix} = \begin{pmatrix} Gb_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{G.57}$$

From the previous matrix, performing the pertinent calculations, we reach its characteristic polynomial. With said polynomial we are already in a position to classify it in a tonal area to understand the tonal function.

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)(s^0 - \lambda)(s^0 - \lambda)(s^0 - \lambda)$$

Inspecting the previous polynomial, we study the algebraic multiplicities, where three formulas appear whose sum coincides with the dimension of the frequency space in this case. Thus the convergent algebraic multiplicity is $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$. The static algebraic multiplicity has value $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$. Finally, the divergent algebraic multiplicity has value $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 1$. With this information, we can now apply the polynomial criterion and establish in which area the tonal function is. In this case, applying this criterion we see that it will be the tonic area. We continue studying the case in the formal context to classify this function and be able to assign a direction to the edge in the graph of tonal functions.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVI7)$ or $\psi(I\o 7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$ when a set of numbers $z_j \in \mathbb{Z}$ with $1 \leq j \leq 4, j \in \mathbb{N}$.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVI7}(t) = \psi_{Gb_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Ab_{z_4}}(t) \\ \psi_{I\o 7}(t) = \psi_{Gb_{z_1}}(t) + \psi_{Eb_{z_2}}(t) + \psi_{C_{z_3}} + \psi_{Bb_{z_4}}(t) \end{cases}$$

We write below, using the notation of parentheses, the wave functions of each voicing. We see that as a function of the aperture $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$ where $\epsilon = (z_1, ..., z_4)$ each of the functions that assign each class to a particular frequency will assign said fundamental frequency as a function of ϵ, keeping, for the optimization conditions, the value of absolute perception to a minimum.

$$\psi_{bVI_c7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVI_c7)} - e^{-2\pi tki\psi_j(bVI_c7)}}{2i}$$

$$\longrightarrow \psi_{I_c\o 7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I_c\o 7)} - e^{-2\pi tki\psi_j(I_c\o 7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 3$ then following the polynomial criterion we obtain the function of the degree $bVI7$ related to the Locrian tonal center. In this case is unique and it can be represented only by one polynomial $\in \mathbb{R}[\lambda]$

$$\boxed{\Phi[E_{(bVI7|I\o 7)}] \in T^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-s^2 - 3)\ \lambda^3 + (3\,s^2 + 3)\ \lambda^2 + (-3\,s^2 - 1)\ \lambda + s^2$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2}{4} - \frac{3}{4}\right)\ \lambda^4 + (s^2 + 1)\ \lambda^3 + \left(-\frac{3\,s^2}{2} - \frac{1}{2}\right)\ \lambda^2 + s^2\,\lambda$

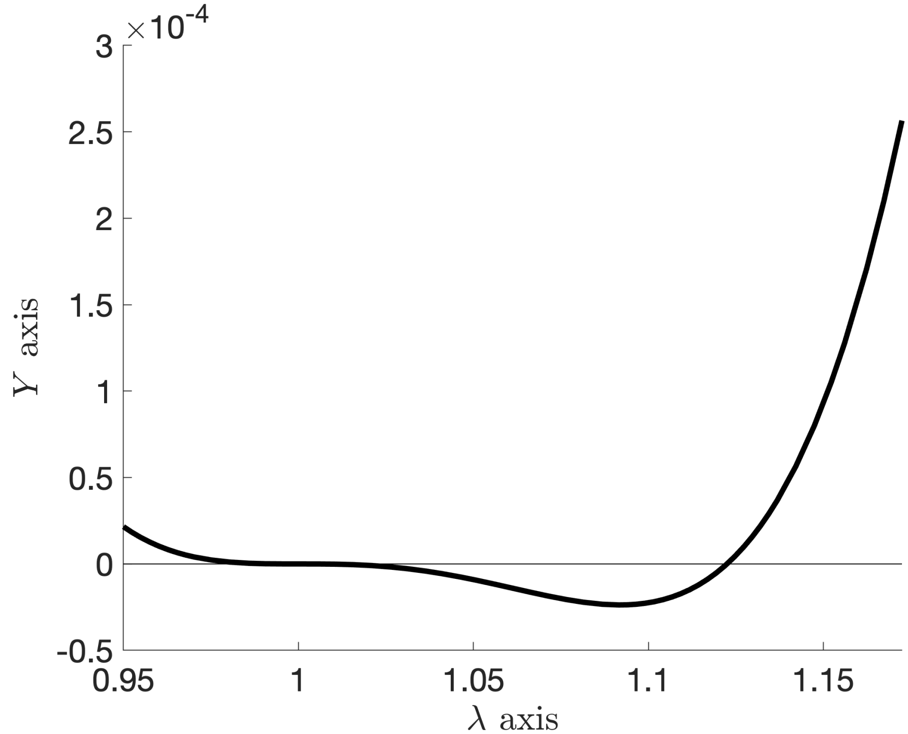


Figure G.11: Characteristic polynomial associated to the P bVI7⟶Iø7 cadence

G.3.6. bVII-7⟶Iø7 cadence

We end the chapter calculating the tonal function of the seventh degree of the Locrian mode, in such a way that we give meaning to the last edge of the graph of tonal functions. Thus we have a complete graph that covers the tonal functions that appear with some regularity and we make it clear that in case of any doubt, we can put the method into practice and by means of the Hungarian algorithm, infinite arimetic, the T transformation and the specified mechanics In this work, reach the solution.

$$E_{(bVII_c^r-7|I_c^r\o 7)} = \begin{pmatrix} Ab & Bb \\ F & Gb \\ Db & Eb \\ Bb & C \end{pmatrix} \tag{G.58}$$

Once we have the link E, as usual, we build the matrix L in such a way that we can use the whole process to find a distribution of boxes or several that indicate how the tonal centers behave. As we have seen, the method is so general that it works for any tonal center X in comparison with another Y where the tuning of the centers is not relevant since we can always build a matrix L that represents the metrics that are drawn from that tuning.

Throughout this work we have considered the fixed tuning to the 12-note western tempered system, then the U matrix is built in this context by calculating those inputs where $dim(U) = 12^2$. This does not limit us to the fact that by providing another set of different classes, a new matrix U is calculated using the same procedure.

$$L_{(bVII^r - |I^r ø7)} = \begin{pmatrix} 2 & 2 & 5 & 4 \\ 5 & 1 & 2 & 5 \\ 3 & 5 & 2 & 1 \\ 0 & 4 & 5 & 2 \end{pmatrix} \tag{G.59}$$

Following the steps of the Hungarian algorithm we develop the L matrix until we reach to the S set:

$$L_{(bVII^r - |I^r ø7)} = \begin{pmatrix} 2 & 2 & 5 & 4 \\ 5 & 1 & 2 & 5 \\ 3 & 5 & 2 & 1 \\ 0 & 4 & 5 & 2 \end{pmatrix} \longrightarrow L^F_{(bVII^r - |I^r ø7)} = \begin{pmatrix} 0 & 0 & 3 & 2 \\ 4 & 0 & 1 & 4 \\ 2 & 4 & 1 & 0 \\ 0 & 4 & 5 & 0 \end{pmatrix}$$
$$\longrightarrow L^H_{(bVII^r - |I^r ø7)} = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 4 & 0 & 0 & 4 \\ 2 & 4 & 0 & 0 \\ 0 & 4 & 4 & 0 \end{pmatrix}$$

By applying the Hungarian algorithm on the matrix L we understand that we arrive at a matrix L^H where in this case there are several solutions.

G.3.7. The Zero method over $L^H_{(bVII^r - 7|I^r ø7)}$

Using the matrix L^H we see that it admits multiple solutions, then we begin to fix zeros. If we fix zero 1, zero 3, zero 5 or zero 8 we reach a first solution:

$$L^H_{(bVII^r - |I^r ø7)} = \begin{pmatrix} \boxed{0} & 0 & 2 & 2 \\ 4 & \boxed{0} & 0 & 4 \\ 2 & 4 & \boxed{0} & 0 \\ 0 & 4 & 4 & \boxed{0} \end{pmatrix} \tag{G.60}$$

Using the matrix L^H we distinguish two solutions, the second one is achieved either by setting zero 2, zero 4, zero 6 or zero 7 we will reach said solution.

$$L^H_{(bVII^r-|I^r\o 7)} = \begin{pmatrix} 0 & \boxed{0} & 2 & 2 \\ 4 & 0 & \boxed{0} & 4 \\ 2 & 4 & 0 & \boxed{0} \\ \boxed{0} & 4 & 4 & 0 \end{pmatrix} \tag{G.61}$$

Observing the two distributions of boxes obtained, we write two sets S that we will develop separately:

$$S_1(L^H_{(bVII^r-7|I^r\o 7)}) = \{\Delta_{11}, \Delta_{22}, \Delta_{33}, \Delta_{44}\}$$
$$S_2(L^H_{(bVII^r-7|I^r\o 7)}) = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{41}\}$$

G.3.8. $S_1(L^H_{(bVII^r-7|I^r\o 7))})$

Following the first distribution of boxes and using the position of these boxes we can calculate an optimal link to calculate the first tonal function:

$$\left[E^1_{(bVII_c-7|I_c\o 7)}\right]_\nabla = \left[\begin{pmatrix} Ab & Bb \\ F & Gb \\ Db & Eb \\ Bb & C \end{pmatrix}\right]_\nabla \tag{G.62}$$

With this link written, it is enough to use the transformation T for the first distribution of boxes in order to find the endomorphism matrix between two frequency vectors, which collects the information of the transformation between voices when the link is optimal.

$$C_{\mathbb{E}} : \Phi^4 \longrightarrow \Phi^4$$
$$C_{\mathbb{E}}(\psi(bVII_c - 7)) \longrightarrow \psi(I_c\o 7)$$
$$\begin{pmatrix} s^2 & 0 & 0 & 0 \\ 0 & s^1 & 0 & 0 \\ 0 & 0 & s^2 & 0 \\ 0 & 0 & 0 & s^2 \end{pmatrix} \cdot \begin{pmatrix} Ab_{z_1} \\ F_{z_2} \\ Db_{z_3} \\ Bb_{z_4} \end{pmatrix} = \begin{pmatrix} Bb_{z_1} \\ Gb_{z_2} \\ Eb_{z_3} \\ C_{z_4+1} \end{pmatrix} \tag{G.63}$$

Taking the matrix $C_{\mathbb{E}}$ and calculating its characteristic polynomial we arrive at the tonal function, which we have to study for its classification. In this way, the polynomial would remain as:

$$p_{C_{\mathbb{E}}}(\lambda) = (s^2 - \lambda)^3(s^1 - \lambda)$$

By observing said polynomial we have three equations for each of the algebraic multiplicities, which are nothing more than a classification of the algebraic multiplicity based on the position of each root with respect to $E(M)$. So we have these three equations: $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 4$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 0$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 0$. Having reached the tonal function, and having written its multiplicities, we are now in a position to classify said function in the corresponding area. The classification by areas is one of the key elements to study a link effectively. This is how we determine a clear direction between the tonal centers when intuition is not able to facilitate a result.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVII-7)$ or $\psi(I\o7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum_{j=1}^{n} \psi_{X_j}(t)$. We write the decomposed functions note by note. In other words, we write them as the sum of each wave function of each note. We carry out this very explicit process for a matter of depth of concept. We see here, not differently from the rest of the cases, that we leave each function expressed while waiting for a selection of subscripts, which will not alter the minimum value of perception if voice-leading is optimal.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII-7}(t) = \psi_{Ab_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{Db_{z_3}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I\o7}(t) = \psi_{Bb_{z_1}}(t) + \psi_{Gb_{z_2}}(t) + \psi_{Eb_{z_3}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

For completeness and order we write the general wave functions for each voicing where each function is composed of $n \times h$ harmonics, each harmonic has an amplitude determined by the distribution Γ which is a vector in $\mathbb{R}^h$ and the arrow $\longrightarrow$ indicates the temporal order of each of the functions. It is understood that the openness of these functions depends on the selection of subscripts in a permissive way and that the selection of fundamental frequencies respects the optimization conditions.

$$\psi_{bVII-7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII-7)} - e^{-2\pi tki\psi_j(bVII-7)}}{2i}$$
$$\longrightarrow \psi_{I\o7}(t) = \sum_{j=1}^{n}\sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\o7)} - e^{-2\pi tki\psi_j(I\o7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 4$ then following the polynomial criteria we obtain the one function of the degree $bVII-7$ related to the Locrian tonal center. This case is specially interesting because it has multiple functions that share nabla value although the polynomials are different. For the first solution, we have that the tonal function falls within the subdominant area, so we finally state it. In this case it can be represented by a polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E^1_{(bVII-7|I\o7)}] \in S^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + \left(-\frac{1}{s} - \frac{2}{s^2} - 1\right) \lambda^3 + \left(\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} + \frac{1}{s^4}\right) \lambda^2 + \left(-\frac{2}{s^3} - \frac{1}{s^4} - \frac{1}{s^5}\right) \lambda + \frac{1}{s^5}$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s^2+s+2}{s^2\,4}\right) \lambda^4 + \frac{s^3+2\,s^2+2\,s+1}{s^4\,3} \lambda^3 + \left(-\frac{2\,s^2+s+1}{s^5\,2}\right) \lambda^2 + \frac{\lambda}{s^5}$

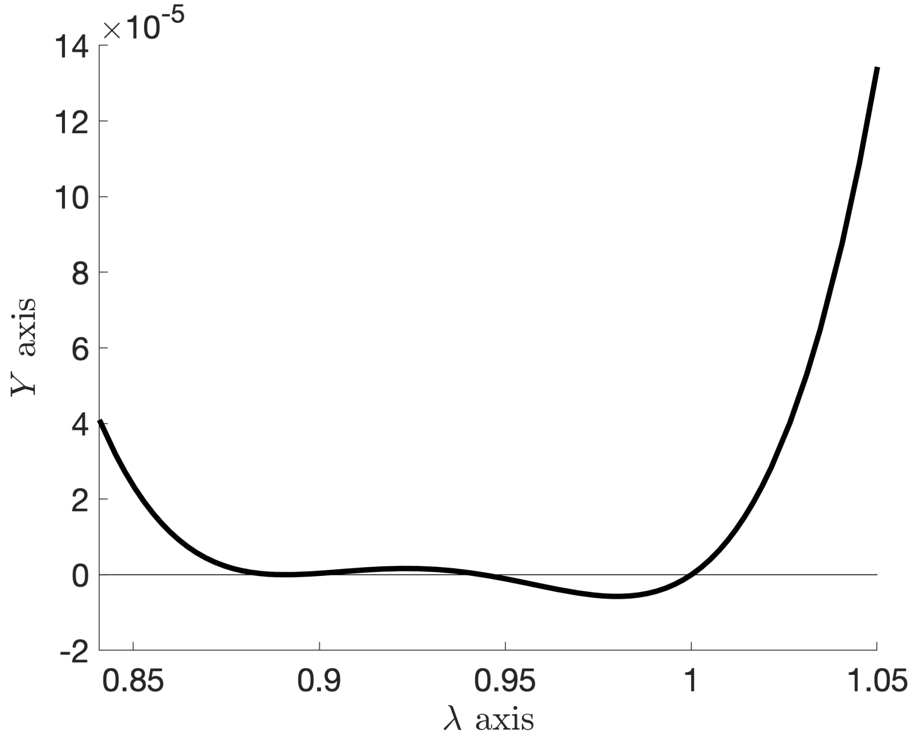


Figure G.12: Characteristic polynomial associated to the P bVII-7⟶Iø7 cadence (I)

G.3.9. $S_2(L^H_{(bVII^r-7|I^r\o7))})$

Following the second distribution of boxes provided by the results of the Zero method, we reach a second optimal pairing that allows us to find a second optimal link, and therefore another tonal function different from the previous one.

$$\left[E^2_{(bVII_c-7|I_c\o7)}\right]_\nabla = \left[\begin{pmatrix} C & Gb \\ Ab & Eb \\ F & C \\ Db & Bb \end{pmatrix}\right]_\nabla \tag{G.64}$$

Since there is a second solution where the value of the nabla function is kept to a minimum but the link is different and this distinction is not limited exclusively to the permutation of its rows, then we see that we are dealing with a dual tonal function. We now have to study whether this duality means that more than one tonal functions appear in different areas. For this we will use the polynomial criterion on the algebraic multiplicities of said function. We already know what are the implications of the position of the roots of the pitch function with respect to $E(M)$. These implications are essentially the movement of the voices and the value of perception due to said movement.

Using the transformation $T(L_E) = C_\mathbb{E}$ we find the endomorphism matrix that transforms one frequency vector into another optimally for any $\epsilon \in \mathbb{Z}^{dim(\Phi^n)}$ where $\epsilon = (z_1, ..., z_4)$. This is how the endomorphism matrix collects the transformation ratios between voices dimension by dimension.

$$C_\mathbb{E} : \Phi^4 \longrightarrow \Phi^4$$
$$C_\mathbb{E}(\psi(bVII_c - 7)) \longrightarrow \psi(I_c\o7)$$
$$\begin{pmatrix} s^{-2} & 0 & 0 & 0 \\ 0 & s^{-2} & 0 & 0 \\ 0 & 0 & s^{-1} & 0 \\ 0 & 0 & 0 & s^0 \end{pmatrix} \cdot \begin{pmatrix} Ab_{z_1} \\ F_{z_2} \\ Db_{z_3} \\ Bb_{z_4} \end{pmatrix} = \begin{pmatrix} Gb_{z_1} \\ Eb_{z_2} \\ C_{z_3} \\ Bb_{z_4} \end{pmatrix} \tag{G.65}$$

Taking the matrix $C_\mathbb{E}$ and calculating its characteristic polynomial we arrive at the tonal function, which we have to study for its classification. In this way, the polynomial would remain as:

$$p_{C_\mathbb{E}}(\lambda) = (s^{-2} - \lambda)^2(s^{-1} - \lambda)(s^0 - \lambda)$$

By observing said polynomial we have three equations for each of the algebraic multiplicities, which are nothing more than a classification of the algebraic multiplicity based on the position of each root with respect to $E(M)$. So we have these three equations: $\sum_{i=1}^{|\lambda^+|} m_{\lambda_i^+} = 0$, $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ and $\sum_{i=1}^{|\lambda^0|} m_{\lambda_i^0} = 1$. We see that in the second solution, the tonal function has changed and therefore the algebraic multiplicities of said polynomial. Thus we have a new case that shares the nabla function but places the tonal function in a different area.

Using the well known principle of sine waves superposition, we know that for every frequency $\phi \in \Phi^+$ as a component of $\psi(bVII-7)$ or $\psi(I\o7)$ for a given tonal center, its clear that the function $\psi_X(t)$ can be created as the sum of every $\psi_{X_j}(t)$ function of the respective component of the frequency vector $\psi_X(t) = \sum\limits_{j=1}^{n} \psi_{X_j}(t)$. We write the decomposed functions note by note. In other words, we write them as the sum of each wave function of each note. We carry out this very explicit process for a matter of depth of concept. We see here, not differently from the rest of the cases, that we leave each function expressed while waiting for a selection of subscripts, which will not alter the minimum value of perception if voice leading is optimal.

$$\mathcal{B}_\epsilon = \begin{cases} \psi_{bVII-7}(t) = \psi_{Ab_{z_1}}(t) + \psi_{F_{z_2}}(t) + \psi_{Db_{z_3}}(t) + \psi_{Bb_{z_3}}(t) \\ \psi_{I\o7}(t) = \psi_{Bb_{z_1}}(t) + \psi_{Gb_{z_2}}(t) + \psi_{Eb_{z_3}}(t) + \psi_{C_{z_3}}(t) \end{cases}$$

For completeness and order we write the general wave functions for each voicing where each function is composed of $n \times h$ harmonics, each harmonic has an amplitude determined by the distribution Γ which is a vector in $\mathbb{R}^h$ and the arrow $\longrightarrow$ indicates the temporal order of each of the functions. It is understood that the openness of these functions depends on the selection of subscripts in a permissive way and that the selection of fundamental frequencies respects the optimization conditions.

$$\psi_{bVII-7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(bVII-7)} - e^{-2\pi tki\psi_j(bVII-7)}}{2i}$$

$$\longrightarrow \psi_{I\o7}(t) = \sum_{j=1}^{n} \sum_{k=1}^{h} \Gamma_k \frac{e^{2\pi tki\psi_j(I\o7)} - e^{-2\pi tki\psi_j(I\o7)}}{2i}$$

As the $\sum_{i=1}^{|\lambda^-|} m_{\lambda_i^-} = 3$ then following the polynomial criterion we obtain the one function of the degree $bVII-7$ related to the Locrian tonal center. In this case it can be represented by a polynomial $\in \mathbb{R}[\lambda]$.

$$\boxed{\Phi[E^2_{(bVII-7|I\o7)}] \in D^{\mathbb{R}[\lambda]}}$$

Characteristic polynomial: $p_{C_{\mathbb{E}}}(\lambda) = \lambda^4 + (-3\,s^2 - s)\ \lambda^3 + (3\,s^4 + 3\,s^3)\ \lambda^2 + (-s^6 - 3\,s^5)\ \lambda + s^7$

Integral of $p_{C_{\mathbb{E}}}(\lambda) : \int p_{C_{\mathbb{E}}}(\lambda) d\lambda = \frac{\lambda^5}{5} + \left(-\frac{s\,(3\,s+1)}{4}\right)\ \lambda^4 + (s^3\ (s+1))\ \lambda^3 + \left(-\frac{s^5\,(s+3)}{2}\right)\ \lambda^2 + s^7\,\lambda$

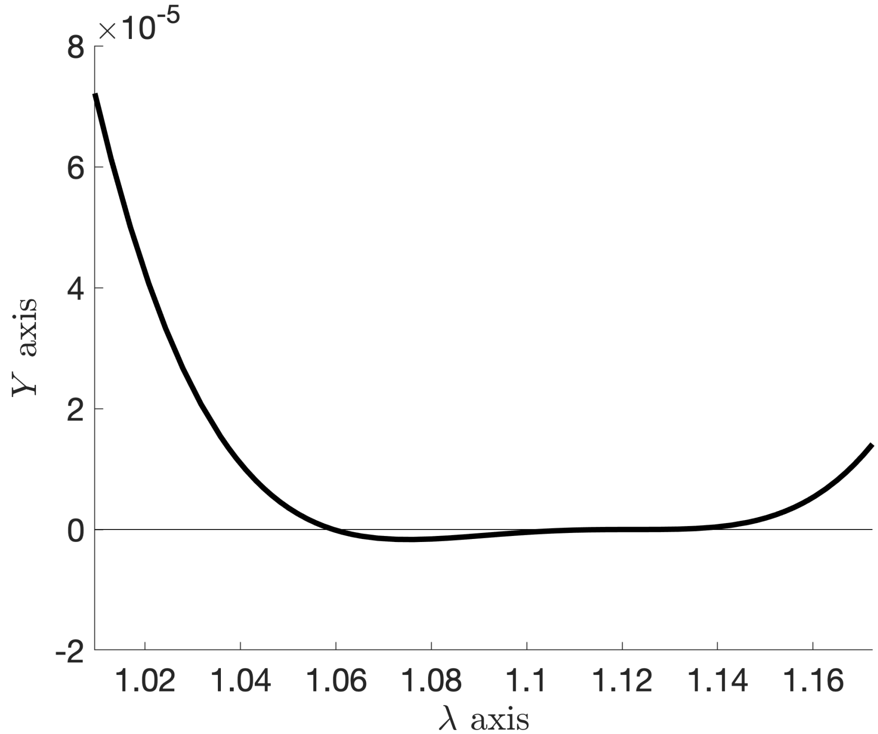


Figure G.13: Characteristic polynomial associated to the P bVII-7⟶Iø7 cadence (II)

G.3.10. Locrian tonal functions

$bII \longrightarrow Io$	$\Phi[E_{(bII\|Io)}] \in D^{\mathbb{R}[\lambda]}$
$bIII \longrightarrow Io$	$\Phi[E_{(bIII\ \|Io)}] \in T^{\mathbb{R}[\lambda]}$
$IV- \longrightarrow Io$	$\Phi[E_{(IV-\|Io)}] \in D^{\mathbb{R}[\lambda]}$
$bV \longrightarrow Io$	$\Phi[E_{(bV\|Io)}] \in S^{\mathbb{R}[\lambda]}$
$bVI \longrightarrow Io$	$\Phi[E_{(bVI\|Io)}] \in T^{\mathbb{R}[\lambda]}$
$bVII- \longrightarrow Io$	$\Phi[E_{(VII-\|Io)}] \in S^{\mathbb{R}[\lambda]}$

$bII\triangle \longrightarrow I\o7$	$\Phi[E_{(bII\triangle\|I\o7)}] \in S^{\mathbb{R}[\lambda]}$
$bIII-7 \longrightarrow I\o7$	$\Phi[E_{(bIII-7\|I\o7)}] \in T^{\mathbb{R}[\lambda]}$
$IV-7 \longrightarrow I\o7$	$\Phi[E_{(IV-7\|I\o7)}] \in S^{\mathbb{R}[\lambda]}$
$bV\triangle \longrightarrow I\o7$	$\Phi[E_{(bV\triangle\|I\o7)}] \in D^{\mathbb{R}[\lambda]}$
$bVI7 \longrightarrow I\o7$	$\Phi[E_{(bVI7\|I\o7)}] \in T^{\mathbb{R}[\lambda]}$
$bVII-7 \longrightarrow I\o7$	$\Phi[E_{(bVII-7\|I\o7)}] \in S^{\mathbb{R}[\lambda]} \cup D^{\mathbb{R}[\lambda]}$

Index