



Universidad Politécnica
de Madrid

**Escuela Técnica Superior de
Ingenieros Informáticos**



Grado en Matemáticas e Informática

Trabajo Fin de Grado

**Valuation of Financial Derivatives with
the Black-Scholes Method**

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Madrid, Junio 2024

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Resumen

La valoración de derivados financieros se encuentra a la vanguardia de las matemáticas financieras, presentando un desafío complejo que entrelaza marcos teóricos con aplicaciones prácticas. Esta investigación estudia las bases matemáticas y las implicaciones prácticas de la valoración de derivados, centrándose en modelos fundamentales como el cálculo de Itô y la fórmula de Black-Scholes, junto con métodos de modelización financiera y estrategias de optimización de riesgos.

Esta investigación gira en torno unos objetivos generales establecidos, que incluyen examinar el modelo binomial de Cox-Ross-Rubinstein dentro de un marco de tiempo discreto, evaluar el Modelo de Media-Varianza para el desarrollo de estrategias de inversión basadas en el equilibrio riesgo-retorno, analizar el cálculo de Itô para entender su aplicación en la modelización de movimientos de precios en los mercados financieros, y estudiar la derivación y aplicabilidad de la fórmula de Black-Scholes en la valoración de opciones europeas.

El clímax del trabajo será el estudio y demostración de la fórmula de Black-Scholes. También se explorarán aplicaciones prácticas de estos conceptos teóricos, a través de ejercicios prácticos, con el objetivo de ofrecer una comprensión sólida del tema desde perspectivas teórica y práctica.

A través de una investigación detallada, se busca mejorar el entendimiento académico y práctico de la valoración de derivados financieros. Además, se conecta conceptos teóricos con la toma de decisiones financieras y la gestión de riesgos en el mundo real, resaltando la evolución de la modelización y sus importantes implicaciones para la industria financiera. El objetivo es ofrecer una comprensión clara y accesible de las matemáticas financieras y su aplicación práctica, proporcionando una base sólida para comprender las técnicas utilizadas en el análisis y las estrategias de inversión en el ámbito financiero.

Abstract

The valuation of financial derivatives is at the forefront of financial mathematics, presenting a complex challenge that combines theoretical frameworks with practical applications. This research studies the mathematical foundations and practical implications of derivative valuation, focusing on fundamental models such as Itô calculus and the Black-Scholes formula, along with financial modeling methods and risk optimization strategies.

This research revolves around established general objectives, which include examining the Cox-Ross-Rubinstein binomial model within a discrete time framework, assessing the Mean-Variance Model for developing investment strategies based on risk-return balance, analyzing Itô calculus to understand its application in modeling price movements in financial markets, and studying the derivation and applicability of the Black-Scholes formula in the valuation of European options.

The climax of the work will be the study and demonstration of the Black-Scholes formula. Practical applications of these theoretical concepts will also be explored through practical exercises, with the objective of providing a solid understanding of the subject from both theoretical and practical perspectives.

Through detailed research, this study aims to enhance both the academic and practical understanding of financial derivative valuation. Moreover, it connects theoretical concepts with financial decision-making and risk management in the real world, highlighting the evolution of modeling and its significant implications for the financial industry. The objective is to provide a clear and accessible understanding of financial mathematics and its practical application, establishing a solid foundation for understanding the techniques used in the analysis and investment strategies within the financial sector.

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Chapter 1

Introduction

One of the most dynamic branches of applied mathematics that has seen considerable growth in recent decades is financial mathematics. This expansion has been driven by the increasing complexity of financial markets and the demand for sophisticated analysis of investment strategies, risk management, and derivative pricing. Financial mathematics mixes rigorous theory with real-world applications, incorporating a synthesis of probability, statistics, economic theory, and computational techniques to solve financial problems and enhance decision-making processes in markets.

Continuous financial models stand as the critical point where each financial product must be valued at every moment in time t . Such models necessitate mathematical tools that can cope with quantities not fixed but rather subject to variability, risk, or chance. This necessity has established stochastic calculus as a key component in modern finance. In this domain, functions are not solely deterministic; they are inherently stochastic random variables defined by their probabilistic behavior. This field's significance surged with the pioneering work of Louis Bachelier in 1900 [1], who first applied stochastic processes to model stock market prices, laying the groundwork for the subsequent development of financial mathematics. Kiyosi Itô later expanded on these ideas, giving rise to the Itô Calculus in 1942 [14], which would become integral to modeling the random motion of financial instruments. It is these mathematical innovations that enable the precise valuation of the increasing variety of financial products in today's markets, underscoring the blend of probability with economic theory to articulate the concept of risk and its quantification.

In 1973, the field of financial mathematics was revolutionized with the creation of the Black-Scholes model by Fischer Black and Myron Scholes [9]. This formula, a proof of the capability of applied mathematics in finance, provided the first robust analytical tool for valuing European options, leading to more effective and efficient markets. It computes an option's theoretical price by considering the volatility of the stock, the time to expiration, the risk-free interest rate, and the strike price of the option. For this groundbreaking contribution, Scholes and Robert C. Merton, who extended the model, were honored with the Nobel Prize in Economic Sciences in 1997. Although Fischer Black was not awarded the Nobel

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Prize because he passed away two years prior, he was posthumously recognized during the 1997 ceremony's laudatio, underscoring the formula's monumental impact on both theoretical and applied finance, which laid the groundwork for the rapidly growing field of quantitative finance [15].

In this dissertation, the study of Itô's Stochastic Calculus and the Black-Scholes Formula is proposed for the valuation of financial derivatives, particularly European options. The exploration will involve a detailed examination of the first three chapters of 'Option Pricing and Portfolio Optimization' by Ralf Korn and Elke Korn [18].

This investigation will be divided into four pivotal chapters, each playing a crucial role in the analysis and comprehension of financial options valuation through advanced mathematical models and risk optimization strategies.

Chapter 2, 'Strategic Financial Modeling: The Mean-Variance Approach to Risk and Optimization', examines the evolution of financial modeling, starting with Bachelier's dissertation, which established the groundwork for stochastic models in finance. This chapter delves into the Mean-Variance Approach [20], detailing its formulation, solution methods, and its essential role in portfolio optimization. It also discusses the integration with the Cox-Ross-Rubinstein Model [6], illustrating the practical application of these theories in enhancing financial decision-making and risk management strategies.

Chapter 3, 'Exploring Itô Calculus: The Mathematical Foundation of Financial Modeling,' explores Itô Calculus, beginning with the modeling of security prices and progressing through Brownian motion, martingales, and Wiener processes [30]. It examines the basic principles of Itô Calculus, including derivatives, integrals, and the stochastic integral, leading to an in-depth discussion on the Itô Formula and its applications. The narrative expands to cover the stock price equation and continuous-time market models [4], providing insights into sophisticated financial models and theories.

Chapter 4, 'Deciphering the Black-Scholes Model: A Comprehensive Guide to Option Pricing', immerses into the Black-Scholes Model, a fundamental framework for option pricing. Beginning with the basics of option pricing, it then explores the replication principle, concluding with an in-depth study and demonstration of the Black-Scholes Formula for valuing European options.

This exploration aims to provide an exhaustive understanding of the model's role and its application in financial markets, highlighting its importance in both theoretical and practical financial decision-making.

Chapter 2

Strategic Financial Modeling: The Mean-Variance Approach to Risk and Optimization

This chapter explores the evolution of financial modeling, starting with Bachelier's Dissertation, which laid the foundation for stochastic models in finance. It then details the Mean-Variance Approach, its formulation, solutions, and key role in portfolio optimization. Finally, it examines the integration with the Cox-Ross-Rubinstein Model, highlighting the practical applications of these theories in improving financial decision-making and risk management.

2.1 Introduction to Economic Foundations

In this section, foundational economic concepts are introduced, serving as the cornerstone for comprehending future financial structures and models.

Definition 2.1.1 (Security). A security is a financial instrument that represents an ownership position in a publicly-traded corporation (stock), a creditor relationship with a governmental body or a corporation (bond), or rights to ownership as represented by an option.

Definition 2.1.2 (Return). The term return refers to the gain or loss generated by a security over a defined period, often expressed as a percentage of the investment's initial value. The return R_t on an investment over a time period t can be calculated as:

$$R_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}} \times 100\% \quad (2.1)$$

where P_{t-1} is the initial price of the security, P_t is the price at time t , and D_t represents any dividends or interest payments received during the period.

In contrast, the expected return μ_i of an asset is the mean of its historical returns, representing what an investor can expect to earn on average if the asset

Chapter 2. Strategic Financial Modeling: The Mean-Variance Approach to Risk and Optimization

was held over many similar time periods. It is calculated using the formula:

$$\mu_i = \frac{1}{N} \sum_{t=1}^N R_t \quad (2.2)$$

Definition 2.1.3 (Risk). Risk is defined as the uncertainty associated with the future returns of a security. It is typically quantified by the standard deviation (σ) of the security's historical returns, reflecting its volatility. The standard deviation of returns for a security is given by:

$$\sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (R_i - \bar{R})^2} \quad (2.3)$$

where R_i is the return in period i , \bar{R} is the average return over N periods, and N is the total number of periods.

Definition 2.1.4 (Portfolio). A financial portfolio is the collection of investments an individual owns, usually including stocks, bonds and cash.

Example 2.1.1 (Example of Portfolio). *An investor's possible portfolio might consist of 50% stocks, such as shares in companies like Microsoft and Coca-Cola, 30% in bonds, such as U.S. Treasury bonds or corporate bonds from IBM, and 20% in cash or cash equivalents like money market funds or certificates of deposit.*

Definition 2.3.5 (Portfolio Vector). The portfolio vector, denoted as $\pi = (\pi_1, \dots, \pi_d)$, represents the proportional allocations of capital across a range of assets within an investor's portfolio. Each component π_i is calculated by the expression:

$$\pi_i = \frac{\varphi_i \cdot p_i}{x} \quad \sum_{i=1}^n \pi_i = 1 \quad i = 1, \dots, n \quad (2.4)$$

Example 2.1.2 (Example of Portfolio Vector). *The portfolio vector of the previous example can be expressed as:*

$$\pi = (0.50, 0.30, 0.20)$$

where $\pi_1 = 0.50$ represents the proportion of the portfolio invested in stocks, $\pi_2 = 0.30$ represents the proportion invested in bonds, and $\pi_3 = 0.20$ represents the proportion held in cash or cash equivalents.

Definition 2.3.6 (Portfolio Return). The portfolio return, expressed as R^π , is a measure of the total return of the portfolio over a given time period T calculated as:

$$R^\pi = \sum_{i=1}^n \pi_i \cdot R_i(T) \quad (2.5)$$

where $R_i(T)$ represents the return of asset i over the time period T .

2.1. Introduction to Economic Foundations

The mean of the portfolio return is:

$$E(R^\pi) = \sum_{i=1}^n \pi_i \cdot \mu_i \quad (2.6)$$

The variance of the portfolio return is given by:

$$Var(R^\pi) = \sum_{i=1}^n \sum_{j=1}^n \pi_i \cdot \sigma_{ij} \cdot \pi_j \quad (2.7)$$

where σ_{ij} represents the covariance between the returns of asset i and asset j .

Definition 2.1.5 (Diversification). Diversification is a strategy employed in portfolio management that involves spreading investments across various asset to reduce the impact of volatility on any single investment. The goal of a diversified portfolio is to achieve a balance that minimizes risk exposure, ensuring its performance over time ¹.

Definition 2.1.6 (Portfolio Optimization). Portfolio optimization is the process of selecting an optimal asset distribution out of a set of potential portfolios with respect to a specific objective. The objective function can be expressed mathematically as:

$$\max (E[R^\pi]) - \lambda \cdot Var(R^\pi) \quad (2.8)$$

where λ is the risk aversion coefficient, representing the investor's preference for risk. The goal typically involves maximizing returns while minimizing risks, resulting in a multi-objective optimization problem ².

Typical values of λ range from 0 to 10. A lower value indicates lower risk aversion, meaning the investor is more willing to accept higher risk for potentially higher returns. Conversely, a higher λ indicates higher risk aversion, meaning the investor prefers to minimize risk even if it means accepting lower returns.

Definition 2.1.7 (Efficient Frontier). The efficient frontier refers to a set of portfolios which offers the highest expected return for a defined level of risk or vice versa. Portfolios that lie below the efficient frontier are considered sub-optimal while those that lie above it are not feasible under the given assumptions. The efficient frontier curve represents the boundary between these optimal and sub-optimal portfolios.

Definition 2.1.8 (Capital Market Line (CML)). The capital market line depicts the range of portfolios that yield the highest expected return for a given risk level, integrating the risk-free rate with the market portfolio of risky assets. It defines the optimal trade-off between risk and return, guiding investors to allocate assets efficiently ³.

¹Investopedia, Diversification, <https://cuts.top/F2Me>. Accessed on March 13, 2024

²Investopedia, Portfolio Optimization, <https://cuts.top/H3Qw>. Accessed on March 17, 2024.

³Investopedia, Capital Market Line, <https://cuts.top/H3R5>. Accessed on March 19, 2024.

⁴Image available at: <https://cuts.top/H30j>. Accessed on March 21, 2024.

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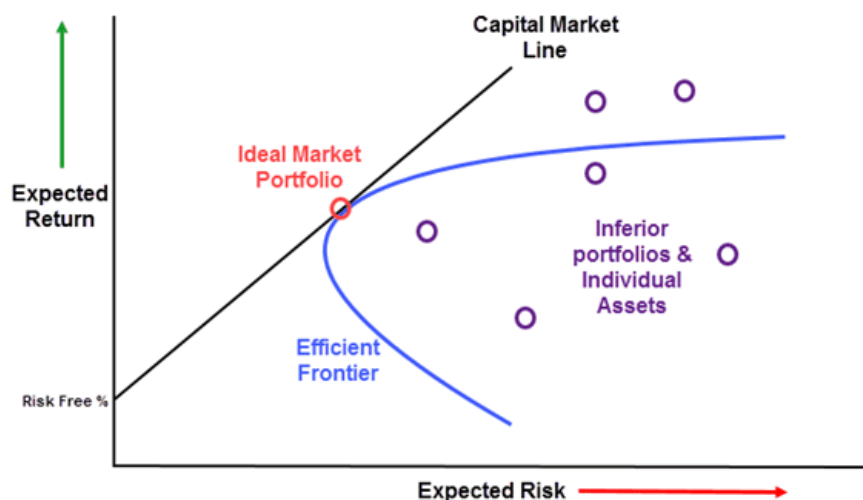


Figure 2.1: Illustration of the Efficient Frontier and the Capital Market Line⁴.

Remark. The blue curve represents the Efficient Frontier, showing the best expected return for a given risk level. Portfolios below this curve are inefficient, yielding lower returns for the same risk. The 'Ideal Market Portfolio,' tangent to the Efficient Frontier, indicates the best risk-return combination. The red straight line is the Capital Market Line (CML), representing portfolios that optimally combine the risk-free asset and the market portfolio.

2.1.1 Valuation and Characteristics of European Call and Put Options

Definition 2.1.9 (Option Contract). An option contract is an agreement between two parties to facilitate a potential transaction on an underlying security at a specified price on or before a certain date. The specified price is known as the strike price, and the date on which the option expires is called the expiration date. The maturity of an option refers to the point in time when the option reaches its expiration date.

In financial markets, the two main types of options are American and European. American options provide the flexibility to be exercised at any time before expiration, allowing holders to capitalize on favorable market conditions. Conversely, European options can only be exercised on their expiration date, restricting their use to that specific moment, regardless of prior market fluctuations.

In this dissertation, we will focus on European options due to their simplicity and tractability in mathematical modeling and valuation.

Definition 2.1.10 (European Option). A European option is a type of option contract that can only be exercised on its expiration date. Unlike American options, which can be exercised at any time before maturity, European options limit the exercise to a single, specific date.

2.1. Introduction to Economic Foundations

The key characteristics of the European Options are:

- **Exercise Date:** Can only be exercised on its expiration date, not before.
- **Premium:** The buyer of the option pays a premium to the seller for the rights conveyed by the option.
- **No Obligation to Exercise:** The holder is not obligated to exercise the option if it is not beneficial.
- **Underlying Asset:** European options can be written for a variety of underlying assets, including stocks, indexes, commodities, and currencies.
- **Strike Price:** The price at which the holder can buy (call) or sell (put) the underlying asset.
- **Market Trading:** These options are often traded on exchanges and can be bought and sold through broker accounts.
- **Option Premium:** Cost to the buyer for acquiring the option rights.

Example 2.1.3 (Option Contract). *Consider a European option contract on Tesla Inc. (TSLA) stock with the following characteristics:*

- **Underlying Asset:** Tesla Inc. (TSLA) stock.
- **Option Type:** European option, which can only be exercised at the expiration date.
- **Purchase Date of the Contract:** April 1, 2024.
- **Expiration Date:** June 15, 2024.
- **Strike Price:** 900 € per share.
- **Option Premium:** 50 € per share.

This option contract grants the holder the right, but not the obligation, to transact the underlying asset (TSLA stock) at the strike price (900 € per share) on the expiration date (June 15, 2024). The holder paid a premium of 50 € per share for this right.

Profitability depends on the TSLA stock price on the expiration date. If the market price exceeds the strike price, the holder can exercise the option for a profit, excluding the premium and fees. On the contrary, it expires worthless, and the holder's loss is limited to the premium paid.

There are two types of options contracts: puts and calls. Both are used to speculate on security prices or hedge exposure, and can be sold for income.

Definition 2.1.11. (Call Option) A European call option grants the holder the right to buy the underlying asset at a specified strike price, denoted by K on the expiration date. If the market price of the underlying asset is above the strike price at expiration, the holder can profit by exercising the option.

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The value of a European call option at maturity is determined by the formula:

$$C(T) = \max(V(T) - K, 0) \quad (2.9)$$

Here $C(T)$ is the call option value at time T and $V(T)$ is the underlying asset price at T . If $V(T) > K$, the option is 'in the money', and the holder profits from the difference. If $V(T) \leq K$, the option is not exercised and is worth zero.

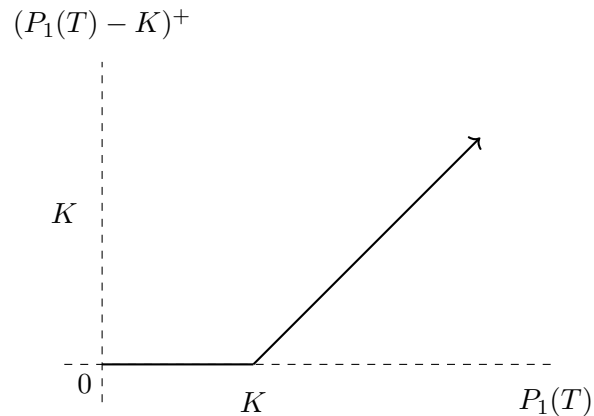


Figure 2.2: Payoff diagram of a call

Example 2.1.4 (Real Call Option). Suppose we purchased a European call option on Apple Inc. (AAPL) stock 3 months ago with the following parameters:

- Underlying Asset: Apple Inc. (AAPL) stock
- Purchase Date: 3 months ago
- Expiration Date: Today
- Strike Price (K): 150 €
- Market Price of AAPL at Expiration ($V(T)$): 169 €

The value of this call option at maturity is calculated using the Equation 2.9. Given the current market price of AAPL at expiration ($V(T) = 169\text{€}$), we find:

$$C(T) = \max(169 - 150, 0) = 19 \text{ €}$$

Therefore, since the market price of the stock ($V(T)$) is above the strike price (K) at expiration, the option is "in the money," and exercising it would yield a profit of 19 € per share.

Definition 2.1.12 (Put Option). A European put option gives the holder the right to sell the underlying asset at a specified strike price on the expiration date. If the market price of the underlying asset is below the strike price at expiration, the holder can profit by exercising the option.

The value at maturity for a European put option is calculated with the formula:

$$P(T) = \max(K - V(T), 0) \quad (2.10)$$

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Where $P(T)$ is the put option value at time T . The put option is valuable when the asset's price falls below the strike price K , providing a gain equal to $K - V(T)$. If the market price exceeds K , the option is not exercised and is worth zero.

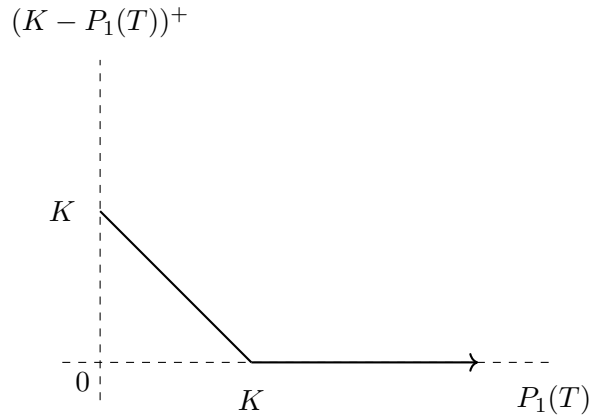


Figure 2.3: Payoff diagram of a put

Example 2.1.5 (Put Option Example). Suppose we purchased a European put option on Microsoft Corporation (MSFT) stock 3 months ago with the following parameters:

- Underlying Asset: Microsoft Corporation (MSFT) stock
- Purchase Date: 3 months ago
- Expiration Date: Today
- Strike Price (K): 250 €
- Market Price of MSFT at Expiration ($V(T)$): 270 €

The value of this put option at maturity is calculated using the Equation 2.10. Given the current market price of MSFT at expiration ($V(T) = 270$ €), we find:

$$P(T) = \max(250 - 270, 0) = 0 \text{ €}$$

Since the market price of MSFT ($V(T)$) is above the strike price (K) at expiration, the option is "out of the money" and exercising it would not be profitable. Therefore, the option expires worthless, and the loss is limited to the premium paid for the option at the time of purchase.

2.2 Bachelier's Legacy: The Precursor to Modern Financial Theory

In this section, we will explore the impact of Louis Bachelier's 'The Theory of Speculation' on contemporary financial theory aiming to highlight the significance of his work in the development of financial economics. Moreover, fundamental concepts for the comprehension of this dissertation such as the stochastic processes will be introduced and explained.

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2.2.1 Louis Bachelier's Theory of Speculation: Stochastic Processes

Bachelier's thesis, 'The Theory of Speculation' (1900) [1], introduced stochastic processes to model the unpredictable fluctuations of stock prices, now recognized as Wiener processes or Brownian motion. His work has significantly influenced modern quantitative finance, impacting financial economics, risk management, and the valuation of financial derivatives [28].

Definition 2.2.1 (Probability Space). A probability space, (Ω, \mathcal{F}, P) , constructs a framework for a stochastic process. This space encompasses three pivotal components:

1. A sample space, Ω , including all possible outcomes.
2. An event space, denoted by \mathcal{F} , a collection of events, each a set of outcomes from the sample space.
3. A probability measure, P , allocating to each event in the event space a probabilistic value ranging from 0 to 1, inclusively.

Definition 2.2.2 (Almost Surely). In probability theory, an event is said to occur *almost surely* if it happens with probability 1. Formally, given a probability space (Ω, \mathcal{F}, P) , an event $A \in \mathcal{F}$ occurs almost surely if $P(A) = 1$.

This concept is used to describe events that are certain to happen in a probabilistic sense, although it does not imply that they will happen in every possible outcome. There may still be outcomes in Ω where the event does not occur, but the set of such outcomes has probability zero.

Definition 2.2.3 (Random Variable). A random variable X is a mathematical function that assigns a real number to each outcome of a sample space, denoted by Ω , associated with a random experiment.

There are two primary types of random variables:

Definition 2.2.4 (Discrete Random Variable). A discrete random variable takes values from a finite or countably infinite set. For instance, the result of rolling a dice. The probability distribution of a discrete random variable, denoted by $p(x)$, is a list of probabilities associated with each of its possible values.

Definition 2.2.5 (Continuous Random Variable). A continuous random variable takes values from an uncountable set, typically represented by an interval on the real line, \mathbb{R} . For example, the time it takes for an event to occur. The probability density function, often denoted by $f(x)$ describes the likelihood of the random variable falling within a specific range of values.

Definition 2.2.6 (Stochastic Process). A stochastic or random process represents a sequence of random variables indexed by another variable that usually represents time. It is used to model systems that evolve in a probabilistic manner, where the randomness is characterized through probability distributions.

Given a probability space (Ω, \mathcal{F}, P) , a stochastic process $\{V(\omega, t) : \omega \in \Omega, t \in T\}$ is a collection of random variables $V(\cdot, t)$ for each $t \in T$, where T is an index set representing time. Here, ω represents a specific outcome within the sample

2.2. Bachelier's Legacy: The Precursor to Modern Financial Theory

space Ω , and $V(\omega, t)$ denotes the real or discrete value taken by the process at time t for the outcome ω . Specifically, for a given time t , $V(\cdot, t)$ is a random variable that describes the distribution of possible values the process can take at time t . And for a given outcome ω , $V(\omega, t)$ is the actual value, either real or discrete (depends on the associated variable $V(\cdot, t)$), that the process assumes.

A more formal definition of a stochastic process V involves its joint probability distributions. For any finite sequence of time points $\{t_1, t_2, \dots, t_n\} \subseteq T$, the joint distribution of V at these times is given by $p(V(\omega_1, t_1), V(\omega_2, t_2), \dots, V(\omega_n, t_n))$, which specifies the probability of the process taking a specific sequence of values at the corresponding times.

In this framework, ω signifies a particular event in the sample space Ω , which includes all possible outcomes. Fixing ω and varying t yields the realization of the stochastic process $\{V(\omega, t) : t \in T\}$, which describes how the process evolves over time for that specific outcome. Conversely, at a given time point t , varying ω across Ω shows the distribution of possible values $V(t)$ can take.

Remark (Joint Distribution). The joint probability distribution of a stochastic process, denoted as $p(V(\omega_1, t_1), V(\omega_2, t_2), \dots, V(\omega_n, t_n))$, describes how the values of the process at different times $\{t_1, t_2, \dots, t_n\}$ are interrelated. This distribution helps determine the likelihood of the process taking a specific sequence of values at these times within the period T .

Definition 2.2.7 (Stochastic Function). A stochastic or random function $X(t)$, is a function characterized by its dependency on a variable t , usually representing time, such that for each fixed $t \in T$, $X(t)$ defines a random variable. This means that $X(t)$ associates each outcome in a probability space (Ω, \mathcal{F}, P) with a value, which can be real or discrete, depending on the nature of the random variable. Thus, the function's value at t is dependent on chance, and the set of values $X(t)$ can take for any t is determined by the probability distribution of the random variable at that moment.

Definition 2.2.8 (Mean Value and Covariance of a Stochastic Process). The mean value and covariance are fundamental statistical properties that describe the behavior of a stochastic process $V(t)$ over time.

The *mean value* function, $m_V(t)$, of the process at time t is given by the expectation of $V(t)$:

$$m_V(t) = E[V(t)] \tag{2.11}$$

The *covariance function*, $C_V(t_1, t_2)$, between two times t_1 and t_2 is defined as the expectation of the product of deviations of $V(t_1)$ and $V(t_2)$ from their respective mean values:

$$C_V(t_1, t_2) = E[(V(t_1) - m_V(t_1))(V(t_2) - m_V(t_2))] \tag{2.12}$$

Each asset within the portfolio, π_i , is modeled using a stochastic function with an associated mean $m_V(t)$ and variance $C_V(t_1, t_2)$. These parameters are used to calculate the expected return and variability of the asset, essential for the mean-variance approach.

2.3 The Mean Variance Approach

In this section, we focus on the Mean-Variance Approach, a fundamental concept in portfolio optimization pioneered by Harry Markowitz in 1952. This method revolutionized investment strategy by introducing a quantitative balance between risk and return, contributing to Markowitz's Nobel Prize in 1990.

Markowitz's framework helps investors craft portfolios that optimize expected return for a given risk level, balancing risk aversion and financial reward. Despite its simplicity and widespread use, we will also examine the limitations and critiques of the Mean-Variance Approach, which have led to the development of continuous-time models [18].

2.3.1 Description of the Model

The model underlying the mean-variance approach is the one-period model. This means investment decisions are made solely at the start of a period, with outcomes assessed at its close, without intervening actions in between. The consequences of these decisions will then be observed at the end of the period. This is the reason why this model is also called the static market model.

All market models used in this dissertation operate under the assumption that securities can be fractionally divided to any extent required and that there are no transaction costs associated with either the purchase or sale of these securities.

Definition 2.3.1 (One-Period Model). A one-period model is a financial model in which a market comprising n distinct securities is examined exclusively at two distinct points in time: the initial time $t = 0$ and the final time $t = T$.

At $t = 0$, the securities are priced at p_1, \dots, p_n , which are known prices. However, the final prices P_1, \dots, P_n at $t = T$ are uncertain and unpredictable. To model this uncertainty, the final prices are represented as non-negative random variables on a probability space (Ω, \mathcal{F}, P) , acknowledging that their exact values cannot be determined until the time $t = T$ arrives [18].

Definition 2.3.2 (Return of the Securities). The return of the security i at time T , denoted as $R_i(T)$, is defined as the ratio of the final price $P_i(T)$ to the initial price p_i , for $i = 1, \dots, n$. That is,

$$R_i(T) := \frac{P_i(T)}{p_i}, \quad i = 1, \dots, n. \quad (2.13)$$

Definition 2.3.3 (Mean, Variance and Covariance of a Set of Returns). For a set of returns $\{R_1(T), R_2(T), \dots, R_d(T)\}$ corresponding to securities $1, 2, \dots, d$ at time T , the expected mean, variance, and covariance are defined as follows:

The *expected mean* of the return for security i , denoted by μ_i , is the average return expected for security i at time T . It is calculated using the formula:

$$E[R_i(T)] = \mu_i \quad (2.14)$$

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The *variance* of the return for security i , denoted by σ_i^2 , measures the dispersion of the returns around the expected mean μ_i , indicating the degree of risk associated with the security. It is defined as:

$$\text{Var}(R_i(T)) = E[(R_i(T) - \mu_i)^2] = \sigma_i^2 \quad (2.15)$$

The *covariance* between the returns of securities i and j , denoted by σ_{ij} , measures how their returns move together. A positive covariance indicates that returns move in the same direction, while a negative covariance indicates they move in opposite directions. It is calculated as:

$$\text{Cov}(R_i(T), R_j(T)) = E[(R_i(T) - \mu_i)(R_j(T) - \mu_j)] = \sigma_{ij} \quad (2.16)$$

2.3.2 Portfolio Analysis

In the context of portfolio analysis, understanding the strategies investors employ is essential for effectively navigating the financial markets. Two such strategies, fundamentally opposite in their approach and expectations regarding asset price movements, are long buying and short selling.

Definition 2.3.4 (Long Buying). Long buying is an investment strategy where an investor purchases a security with the expectation that its price will rise over time. This approach involves buying and holding the security, benefiting from any potential price appreciation. It reflects an optimistic outlook on the security's future value, with the investor aiming to sell it at a higher price in the future for a profit.

Definition 2.3.5 (Short Selling). Short selling is an investment strategy where an investor borrows a security and sells it on the open market, planning to buy it back later for less money. However, if the security's price rises, the investor will face losses when buying it back at the higher price to return it to the lender.

Definition 2.3.6 (Initial Wealth). The initial wealth x of a given investor, which is strictly positive ($x > 0$), is defined as the aggregate value of the investor's holdings. It is calculated as the sum of the products of the number of shares held in each security and the respective price per share of these securities.

That can be simplified in the following formula known as the *budget equation*:

$$\sum_{i=1}^n \varphi_i \cdot p_i = x \quad (2.17)$$

Remark. For the integrity of the one-period model, we consider $\varphi_i \geq 0$ for all securities. Should it be negative, it would imply short selling, which could potentially lead to a negative wealth at period's end.

In the mean-variance approach, two criteria are considered: expected return and risk (variance). The goal is to balance higher returns against return fluctuations. This involves either setting a minimum return and finding the portfolio with the lowest variance or setting a maximum variance and seeking the highest return within that risk level.

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2.3.3 Formulating the Mean-Variance Optimization Problem

To formulate the optimization problem within the mean-variance approach framework, we have two options: one of maximization and one of minimization.

The first approach involves selecting portfolios $\pi \in \mathbb{R}^n$ with a variance below a specified threshold. Among these, we choose the one with the highest expected return. This means maximizing the portfolio's expected return $E(R^\pi)$ under a given upper bound c_1 for the variance $Var(R^\pi)$.

Mathematically, this problem can be written as:

$$\begin{aligned} & \max_{\pi \in \mathbb{R}^d} E(R^\pi) \\ & \text{subject to } \pi_i \geq 0 \text{ for } i = 1, \dots, n, \quad \sum_{i=1}^n \pi_i = 1, \quad \text{Var}(R^\pi) \leq c_1 \end{aligned} \quad (2.18)$$

The problem can be expressed in matrix form as follows

$$\begin{aligned} & \max_{\pi} \quad \mu^T \pi \\ & \text{subject to } \quad \pi^T C \pi \leq c_1, \quad \mathbf{1}^T \pi = 1, \quad \pi \geq \mathbf{0}, \end{aligned} \quad (2.19)$$

where

- μ is the vector of expected returns of the assets.
- C is the covariance matrix of the asset returns.
- π is the portfolio weight vector.
- $\mathbf{1}$ is a vector of ones (for the condition that the sum of the weights is 1).
- $\mathbf{0}$ is a vector of zeros (for the condition that the weights are non-negative, i.e., no short selling).

Remark (Full Investment). The condition $\sum_{i=1}^n \pi_i = 1$ ensures full investment, meaning all capital is allocated across the portfolio without surplus, fully utilizing the investor's budget.

The second approach involves selecting portfolios $\pi \in \mathbb{R}^n$ with an expected return above a specified threshold. Among these, we choose the one with the smallest variance. This means minimizing the portfolio's return variance $Var(R^\pi)$ under a given lower bound c_2 for the expected return $E(R^\pi)$.

Mathematically, this problem can be written as:

$$\begin{aligned} & \min_{\pi \in \mathbb{R}^d} \text{Var}(R^\pi) \\ & \text{subject to } \quad \pi_i \geq 0 \text{ for } i = 1, \dots, n, \quad \sum_{i=1}^n \pi_i = 1, \quad \text{Var}(E^\pi) \geq c_2 \end{aligned} \quad (2.20)$$

And in matrix form:

$$\begin{aligned} & \min_{\pi} \quad \pi^T C \pi \\ & \text{subject to } \quad \mu^T \pi \geq c_2, \quad \mathbf{1}^T \pi = 1, \quad \pi \geq \mathbf{0}, \end{aligned} \quad (2.21)$$

2.3.4 Solving the Mean-Variance Optimization Problem

A primal optimization problem involves an objective function aimed at maximization or minimization, constrained by specific conditions. The dual problem transforms these constraints into variables and the variables into constraints [17]. In convex programming, strong duality ensures that the optimal values of both primal and dual problems converge.

Although problems presented by the *Equation 2.18* and *Equation 2.20* are not duals of each other, they are intimately connected by the following theorem:

Theorem 1. *Let's consider the optimization problems of minimization and maximization and assume the covariance matrix C is positive definite. If the constraints c_1, c_2 satisfy:*

$$\min_{1 \leq i \leq n} \mu_i < c_2 < \max_{1 \leq i \leq n} \mu_i \quad (2.22)$$

$$\min_{\pi \geq 0, \mathbf{1}^T \pi = 1} C^2(\pi) \leq c_1 \leq \max_{\pi \geq 0, \mathbf{1}^T \pi = 1} C^2(\pi) \quad (2.23)$$

where $\mathbf{1}$ is the vector of ones in \mathbb{R}^d . Then we have the following results:

1. If π^* solves the minimization problem with $E(R^{\pi^*}) = \sum_{i=1}^d \pi_i^* \mu_i = c_2$, then π^* also solves the maximization problem with $c_1 = \text{Var}(R^{\pi^*})$.
2. If π^{**} solves the maximization problem with $\text{Var}(R^{\pi^{**}}) = \sum_{i=1}^d \sum_{j=1}^d \pi_i^{**} \pi_j^{**} \sigma_{ij} = c_1$, then π^{**} also solves the minimization problem with $c_2 = E(R^{\pi^{**}})$.

Proof. Let's start by considering the minimization problem. Assuming π^* is the solution to this problem, meaning it minimizes $\pi^T C \pi$ under the given constraints. We want to prove that π^* is the solution for the maximization problem too. By definition, π^* must satisfy:

1. π^* is non-negative: $\pi^* \geq \mathbf{0}$.
2. The weights sum to one: $\mathbf{1}^T \pi^* = 1$.
3. The expected return constraint is met: $\mu^T \pi^* \geq c_2$.

Given that π^* minimizes the variance $\pi^T C \pi$, it must also satisfy the variance constraint if it is used in the maximization problem. Since π^* is already a feasible solution for this problem and π^* minimizes $\pi^T C \pi$, it must be the case that: $\pi^{*T} C \pi^* \leq c_1$

This implies π^* meets the variance constraint of the maximization problem. Additionally, since π^* achieves the minimum variance, it must also maximize the expected return $\mu^T \pi$ within the feasible region defined by $\pi^T C \pi \leq c_1$, $\mathbf{1}^T \pi = 1$, and $\pi \geq \mathbf{0}$.

Hence, π^* also solves the maximization problem.

Now, we will operate the same way: let π^{**} be the solution to the maximization problem, let's see if π^{**} is also the solution for the minimization problem. If π^{**} is the solution, means that it maximizes $\mu^T \pi$ under the given constraints. Therefore, π^{**} must satisfy:

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1. π^{**} is non-negative: $\pi^{**} \geq \mathbf{0}$.
2. The weights sum to one: $\mathbf{1}^T \pi^{**} = 1$.
3. The variance constraint is met: $\pi^{**T} C \pi^{**} \leq c_1$.

Given that π^{**} maximizes the expected return $\mu^T \pi$ under the constraint $\pi^T C \pi \leq c_1$, it achieves the highest possible expected return for any portfolio within the feasible set.

Now consider the minimization problem again: since π^{**} is already a feasible solution for this problem and π^{**} maximizes $\mu^T \pi$, it must be the case that: $\mu^T \pi^{**} \geq c_2$

This implies π^{**} meets the expected return constraint of the minimization problem. Additionally, since π^{**} achieves the maximum expected return within the feasible region defined by $\pi^T C \pi \leq c_1$, $\mathbf{1}^T \pi = 1$, and $\pi \geq \mathbf{0}$, it must also minimize the variance $\pi^T C \pi$ within the feasible set for the given c_2 .

Therefore, π^{**} also solves the minimization problem. □

2.3.5 Optimizing Portfolios: The Role of the Efficient Frontier

The Efficient Frontier defines portfolios that optimize expected return for a given risk or minimize risk for a given return, offering the best diversification and risk-return trade-off [11]. Portfolios below the frontier are sub-optimal, yielding lower returns for their risk. Those to the right involve greater risk without proportionate return, making them less efficient.

Definition 2.3.7 (Efficiency). Efficiency in terms of portfolio optimization means offering the highest possible returns for any level of risk.

The key points of this concept emphasize that:

- Maximizing returns for a given risk is synonymous with locating a portfolio on the Efficient Frontier.
- Investment risk is quantified by the standard deviation of returns, whereas the covariance between assets informs the collective risk.
- The Efficient Frontier demonstrates the profound benefits of diversification within portfolio construction.

However, the Efficient Frontier has setbacks. Assumptions like normally distributed returns and rational investor behavior may not hold in reality, causing discrepancies between theoretical and actual performance.

The shape of the Efficient Frontier shows that risk doesn't increase linearly with return. It highlights the optimal balance between risk and return, tailored to each investor.

2.3.6 Strategic Asset Allocation and the Capital Market Line

The Capital Market Line (CML) emerges as an extension of the Efficient Frontier when a risk-free asset is introduced into the market. It represents the superior equilibrium condition where investors balance their portfolios between the risk-free asset and the market portfolio of risky assets [31].

- The CML is characterized by the optimal trade-off between risk and return, as per the Capital Asset Pricing Model (CAPM).
- Its slope signifies the reward per unit of risk.
- Portfolios on the CML, particularly the tangency portfolio, represent the most efficient combination of risk and return.

Definition 2.3.8 (Risk-free Asset). A risk-free asset refers to an investment with a guaranteed return and no chance of financial loss like government-issued securities. The risk-free rate is a foundational benchmark because it serves as a standard for assessing an investment's return against its risk.

Definition 2.3.9 (Capital Asset Pricing Model). The Capital Asset Pricing Model (CAPM) is a financial theory that quantifies the relationship between the expected return of an investment and its risk, offering a method to assess an asset's expected return based on its risk premium over the risk-free rate.

Mathematically, we can compute the CML as follows:

$$R^\pi = r_f + \frac{R_M - r_f}{\sigma_M} \sigma_{R^\pi} \quad (2.24)$$

where:

- R^π = portfolio return
- r_f = risk-free rate (theoretical rate of return of an investment with zero risk)
- R_M = market return (total return of the market portfolio)
- σ_M = standard deviation of market returns
- σ_{R^π} = standard deviation of portfolio returns

Remark. Portfolios on the CML are presumed to be more advantageous due to their incorporation of the risk-free rate, giving rise to the tangency portfolio which is central to the CAPM's equilibrium scenario.

2.3.7 Understanding the Sharpe Ratio

The Sharpe Ratio, named after Nobel laureate William F. Sharpe, is a measure for calculating risk-adjusted returns. It is used to understand the return of an investment compared to its risk. The higher the Sharpe Ratio, the better the investment's historical risk-adjusted performance when comparing similar portfolios [8].

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Definition 2.3.10 (Sharpe Ratio). The Sharpe Ratio is defined as the difference between the returns of the investment and the risk-free return, divided by the standard deviation of the investment returns (which represents the risk):

$$\text{Sharpe Ratio} = \frac{E(R^{\pi^*}) - r_f}{\sigma_{R^{\pi}}} \quad (2.25)$$

where:

- $E(R^{\pi^*})$ is the expected return of the portfolio,
- r_f is the risk-free rate of return,
- $\sigma_{R^{\pi}}$ is the standard deviation of the portfolio's excess return.

Remark. A higher Sharpe ratio indicates superior risk-adjusted returns. A negative Sharpe ratio means the portfolio's returns are less than the benchmark or expected to be below zero.

2.3.8 The Diversification Effect

The diversification effect in portfolio management is based on the principle that a varied asset portfolio yields higher returns and poses less risk than any single asset. This occurs because the lack of perfect correlation between assets reduces overall portfolio volatility.

Example 2.3.1 (Diversification Effect). *Let us assume a portfolio where securities A and B have a positive covariance, indicating they tend to move together, whereas security C has zero covariance with A and B, indicating independence.*

For a portfolio π with equal investments in A, B, and C (i.e., the portfolio vector $\pi = (1/3, 1/3, 1/3)$, the variance of the portfolio return $\text{Var}(R^{\pi})$ is given by:

$$\text{Var}(R^{\pi}) = \text{Var}\left(\frac{1}{3}R_A + \frac{1}{3}R_B + \frac{1}{3}R_C\right)$$

Suppose the variances of A, B, and C are σ_A^2 , σ_B^2 , and σ_C^2 respectively, and the covariances are σ_{AB} , $\sigma_{AC} = 0$, and $\sigma_{BC} = 0$. The formula for the portfolio variance simplifies to:

$$\text{Var}(R^{\pi}) = \frac{1}{9} (\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + 2\sigma_{AB})$$

If σ_{AB} is positive but smaller than the individual variances, and since σ_{AC} and σ_{BC} are zero, the overall variance of the portfolio is reduced compared to a portfolio consisting only of A and B. This illustrates the diversification effect even when some securities are correlated, highlighting the benefit of including uncorrelated assets in a portfolio.

2.4 Examples

2.4.1 Optimization of a Two-Asset Portfolio: Minimizing Variance with a Given Return Threshold

An investor considers two assets for their portfolio. Asset A has an 8% expected return and 10% variance, while asset B has a 12% expected return and 30% variance, with a 5% covariance. The investor seeks the optimal combination that minimizes portfolio risk, ensuring the expected return is at least 8%.

Step 1: Formulating the Optimization Problem. Define x as the fraction of capital invested in Asset A and $1 - x$ in Asset B. The optimization problem can be formulated as follows:

$$\min_{\pi \in \mathbb{R}^d} \text{Var}(R^\pi) = (0.1x^2) + (0.3(1-x)^2) + (2 \cdot 0.05 \cdot x \cdot (1-x))$$

subject to $\pi_i \geq 0$ for $i = 1, \dots, n$. $\sum_{i=1}^n \pi_i = 1$, $E(R^\pi) = (0.08x + 0.12(1-x)) \geq 0.08$

Step 2: Setting up the Equations. To solve this, we set up equations with the given data and constraints, then apply optimization methods to find the value of x that minimizes variance while meeting the return constraint.

The variance equation for the portfolio is:

$$\text{Var}(R^\pi) = 0.1x^2 + 0.3(1-x)^2 + 2 \cdot 0.05 \cdot x \cdot (1-x)$$

The expected return constraint is:

$$E(R^\pi) = 0.08x + 0.12(1-x) \geq 0.08$$

Step 3: Solving the Optimization Problem. First let's compute the first derivative of the objective function. We simplify and rewrite the function:

$$\text{Var}(R^\pi) = 0.1x^2 + 0.3(1-2x+x^2) + 0.1x - 0.1x^2 = 0.3x^2 - 0.5x + 0.3$$

We calculate the first derivative with respect to x to find the critical points:

$$\frac{d}{dx} \text{Var}(R^\pi) = \frac{d}{dx} (0.3x^2 - 0.5x + 0.3) = 0.6x - 0.5$$

We set the derivative equal to zero to find the critical points:

$$0.6x - 0.5 = 0, \quad 0.6x = 0.5, \quad x = \frac{0.5}{0.6} \approx 0.8333$$

We calculate the second derivative to determine the concavity and whether the critical point is a minimum or maximum:

$$\frac{d^2}{dx^2} \text{Var}(R^\pi) = \frac{d}{dx} (0.6x - 0.5) = 0.6$$

Since the second derivative is positive, $\frac{d^2}{dx^2} \text{Var}(R^\pi) > 0$, the critical point $x \approx 0.8333$ is a local minimum.

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Let's check if $x \approx 0.8333$ satisfies the constraint.

$$E(R^\pi) = 0.08 \cdot 0.8333 + 0.12(1 - 0.8333) = 0.0866 \approx 0.087 \geq 0.08$$

Step 4: Solution. The critical point $x \approx 0.8333$ suggests allocating 83.33% to Asset A and 16.67% to Asset B. This strategy minimizes the portfolio's variance given the asset parameters and constraints.

To compute the minimum variance at this optimal allocation, we substitute $x = 0.8333$ back into the variance formula:

$$\text{Var}(R^\pi) = 0.3 \times 0.8333^2 - 0.5 \times 0.8333 + 0.3$$

Simplifying the calculation:

$$\text{Var}(R^\pi) = 0.3 \times 0.6944 - 0.41665 + 0.3 = 0.20832 - 0.41665 + 0.3 = 0.09167$$

Thus, the minimum variance of the portfolio at this allocation is approximately 0.09167, or 9.167%, which confirms the effectiveness of this allocation in reducing the portfolio's overall risk.

2.4.2 Advanced Portfolio Optimization: Integrating a Risk-Free Asset with Mean-Variance Analysis and the Capital Market Line

An investor considers two risky assets and a risk-free asset. Asset A has an 8% expected return and 10% variance, Asset B has a 12% expected return and 30% variance, and the risk-free asset offers a 3% return. The covariance between Assets A and B is 5%.

Step 1: Maximizing the Sharpe Ratio. To maximize the Sharpe Ratio, defined by Equation 2.25, we solve the following optimization problem:

$$\max_{x_A, x_B, x_C} \frac{0.08x_A + 0.12x_B + 0.03x_C - 0.03}{\sqrt{0.10x_A^2 + 0.30x_B^2 + 2 \cdot 0.05 \cdot x_A \cdot x_B}}$$

subject to:

$$x_A + x_B + x_C = 1, \quad x_A, x_B, x_C \geq 0$$

Step 2: Solving the Optimization Problem Using an online optimization tool, we find the following optimal proportions for the assets:

$$x_A = \frac{63}{208} \approx 0.3029, \quad x_B = \frac{3}{16} = 0.1875, \quad x_C = \frac{53}{104} \approx 0.5096$$

The optimal proportions suggest allocating 30.29% to Asset A, 18.75% to Asset B, and 50.96% to the risk-free Asset C. This maximizes the Sharpe Ratio, providing the highest return per unit of risk.

Step 3: Constructing the Capital Market Line (CML) and Evaluating the Efficient Frontier. The Efficient Frontier is constructed by combining proportions of risky assets (A and B) and the risk-free asset (C). Portfolios on the CML represent optimal risk-return combinations.

First, using Equation 2.24, we calculate the expected return and standard deviation of the market portfolio, the optimal mix of risky assets that maximizes the Sharpe Ratio.

$$\begin{aligned} R_m &= 0.08 \cdot x_A + 0.12 \cdot x_B + 0.03 \cdot x_C = 0.08 \cdot \frac{63}{208} + 0.12 \cdot \frac{3}{16} + 0.03 \cdot \frac{53}{104} \\ &= 0.08 \cdot 0.3029 + 0.12 \cdot 0.1875 + 0.03 \cdot 0.5096 \approx 0.02423 + 0.0225 + 0.01529 = \\ &0.06202 = 6.202\% \end{aligned}$$

Next, we calculate the standard deviation of the market portfolio:

$$\begin{aligned} \sigma_m &= \sqrt{0.10 \cdot \left(\frac{63}{208}\right)^2 + 0.30 \cdot \left(\frac{3}{16}\right)^2 + 2 \cdot 0.05 \cdot \frac{63}{208} \cdot \frac{3}{16}} \\ \sigma_m &= \sqrt{0.10 \cdot 0.09175 + 0.30 \cdot 0.01172 + 0.01518} = \sqrt{0.009175 + 0.003516 + 0.01518} \\ \sigma_m &\approx 0.167 = 16.7\% \end{aligned}$$

Using these values, we can construct the CML equation:

$$R_p = 0.03 + \frac{0.06202 - 0.03}{0.167} \sigma_p = 0.03 + 0.1927 \sigma_p$$

The Efficient Frontier is constructed by combining proportions of the assets. We will consider different combinations of weights:

Portfolio	Weight A (%)	Weight B (%)	Weight C (%)
1	100	0	0
2	75	25	0
3	50	50	0
4	25	75	0
5	0	100	0
6	30.29	18.75	50.96

Table 2.1: Portfolio Weights

Portfolio 1

$$\begin{aligned} w_A &= 1, w_B = 0, w_C = 0 \\ E(R_p) &= 1 \cdot 0.08 + 0 \cdot 0.12 + 0 \cdot 0.03 = 0.08 \\ \sigma_p^2 &= 1^2 \cdot 0.10 + 0^2 \cdot 0.30 + 2 \cdot 1 \cdot 0 \cdot 0.05 = 0.10 \\ \sigma_p &= \sqrt{0.10} \approx 0.3162 \end{aligned}$$

Portfolio 2

$$\begin{aligned} w_A &= 0.75, w_B = 0.25, w_C = 0 \\ E(R_p) &= 0.75 \cdot 0.08 + 0.25 \cdot 0.12 + 0 \cdot 0.03 = 0.09 \\ \sigma_p^2 &= 0.75^2 \cdot 0.10 + 0.25^2 \cdot 0.30 + 2 \cdot 0.75 \cdot 0.25 \cdot 0.05 = 0.09375 \\ \sigma_p &= \sqrt{0.09375} \approx 0.3062 \end{aligned}$$

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Portfolio 3

$$w_A = 0.50, w_B = 0.50, w_C = 0$$

$$E(R_p) = 0.50 \cdot 0.08 + 0.50 \cdot 0.12 = 0.10$$

$$\sigma_p^2 = 0.50^2 \cdot 0.10 + 0.50^2 \cdot 0.30 + 2 \cdot 0.50 \cdot 0.50 \cdot 0.05 = 0.125$$

$$\sigma_p = \sqrt{0.125} \approx 0.3536$$

Portfolio 4

$$w_A = 0.25, w_B = 0.75, w_C = 0$$

$$E(R_p) = 0.25 \cdot 0.08 + 0.75 \cdot 0.12 = 0.11$$

$$\sigma_p^2 = 0.25^2 \cdot 0.10 + 0.75^2 \cdot 0.30 + 2 \cdot 0.25 \cdot 0.75 \cdot 0.05 = 0.19375$$

$$\sigma_p = \sqrt{0.19375} \approx 0.4401$$

Portfolio 5

$$w_A = 0, w_B = 1, w_C = 0$$

$$E(R_p) = 0 \cdot 0.08 + 1 \cdot 0.12 = 0.12$$

$$\sigma_p^2 = 0^2 \cdot 0.10 + 1^2 \cdot 0.30 = 0.30$$

$$\sigma_p = \sqrt{0.30} \approx 0.5477$$

Portfolio 6

$$w_A = \frac{63}{208} \approx 0.3029, w_B = \frac{3}{16} = 0.1875, w_C = \frac{53}{104} \approx 0.5096$$

$$E(R_p) = 0.3029 \cdot 0.08 + 0.1875 \cdot 0.12 + 0.5096 \cdot 0.03 = 0.06202$$

$$\sigma_p^2 = 0.3029^2 \cdot 0.10 + 0.1875^2 \cdot 0.30 + 2 \cdot 0.3029 \cdot 0.1875 \cdot 0.05 = 0.0254$$

$$\sigma_p = \sqrt{0.0254} \approx 0.1593$$

Results Table

Portfolio	Weight A (%)	Weight B (%)	Weight C (%)	Return (%)	Risk (%)
1	100	0	0	8.00	31.62
2	75	25	0	9.00	30.62
3	50	50	0	10.00	35.36
4	25	75	0	11.00	44.01
5	0	100	0	12.00	54.77
6	30.29	18.75	50.96	6.20	15.93

Table 2.2: Portfolio Risks and Returns

2.5 The Cox-Ross-Rubinstein Model

In the following section, we explore the Cox-Ross-Rubinstein (CRR) model, formalized by John Cox, Stephen Ross, and Mark Rubinstein in 1979. This discrete-time framework uses discrete random variables to understand asset price dynamics.

Known as the binomial options pricing model, the CRR model assumes the stock price either goes up or down by a fixed factor each period. This framework allows for step-by-step valuation of options, creating a multi-period market model fundamental to understanding options and complex derivatives.

2.5.1 Description of the Model

In the CRR framework, the price of a stock at each subsequent time step can either increase by a factor of u or decrease by a factor of d . These factors are constants, implying that the percentage change in stock price remains the same for each up or down movement [22].

Let's define $V(0)$ as the initial stock price. The probability of an upward movement, denoted as p , is assumed to be constant for each period and independent of past price movements.

For the consideration of the model, we must take into account the following constraints: $u > 1$, $0 < d < 1$, and $V(0) > 0$.

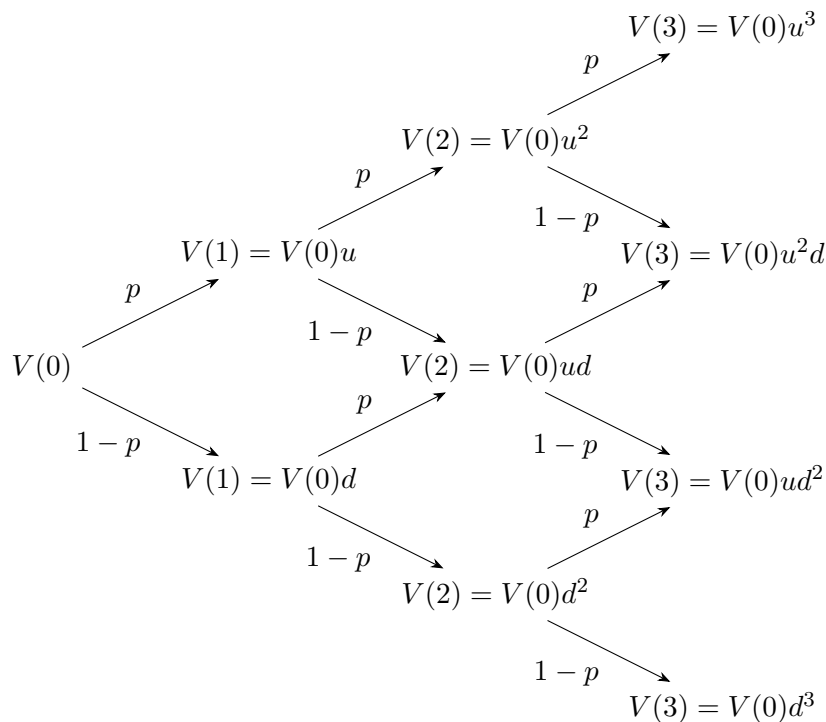


Figure 2.4: Lattice representation of three periods of the CRR Model

Chapter 2. Strategic Financial Modeling: The Mean-Variance Approach to Risk and Optimization

Starting with the initial stock price $V(0)$, each node splits into two outcomes: an increase by u with probability p , or a decrease by d with probability $1 - p$. After three periods, there are four distinct stock prices, each resulting from a unique sequence of movements. The likelihood of each path follows a binomial distribution based on the number of upward and downward movements [22].

Definition 2.5.1 (Bernoulli Process). A stochastic process $V = V(t)_{t=1, \dots, T}$ defined on a probability space (Ω, \mathcal{F}, P) is said to be a Bernoulli Process with parameter $0 < p < 1$ if the random variables $V(1), \dots, V(T)$ are independent and have the common probability distribution:

$$P(V(t) = 1) = 1 - P(V(t) = 0) = p, \quad t \in \{1, \dots, T\} \quad (2.26)$$

In the Cox-Ross-Rubinstein Model, each upward movement is independent of past price movements. The sample space Ω has a cardinality of 2^T , consisting of all possible stock price paths over time. Each path is represented by a vector of 'ups' (1) and 'downs' (0).

The probability measure P is given by $P(\omega) = p^n(1-p)^{T-n}$, where ω is any elementary outcome corresponding to n 'ups' and $T - n$ 'downs'. Setting this probability measure on Ω is equivalent to stating that the random variables $V(1), \dots, V(T)$ are independent and identically distributed.

Definition 2.5.2 (Bernoulli Counting Process). The Bernoulli Counting Process $N = N(t)_{t=1, \dots, T}$ is defined in terms of the Bernoulli Process V by setting $N(0) = 0$ and

$$N(t, \omega) = V(1, \omega) + \dots + V(T, \omega), \quad t \in \{1, \dots, T\}, \quad \omega \in \Omega \quad (2.27)$$

Now we can compute the mean and variance of the return of an option with initial price $V(0)$. The expected return after T periods, $E(R_T)$, is calculated considering all possible price movements. Given the probability p of an upward movement and $1 - p$ for a downward movement, the expected return is:

$$E(R_T) = \sum_{k=0}^T \binom{T}{k} p^k (1-p)^{T-k} \cdot \left(\frac{u^k d^{T-k} V(0)}{V(0)} \right) \quad (2.28)$$

The variance of the option's return after T periods, $Var(R_T)$, accounts for the variability in possible price paths and is calculated as:

$$Var(R_T) = \sum_{k=0}^T \binom{T}{k} p^k (1-p)^{T-k} \cdot \left(\left(\frac{u^k d^{T-k} V(0)}{V(0)} \right) - E(R_T) \right)^2 \quad (2.29)$$

This formulation encapsulates the dispersion of returns around the expected return, highlighting the risk associated with the option over the specified period.

If we had a simplified case where there was only one period, the previous equations would be:

$$E(R_1(T)) = E\left(\frac{V(T)}{V(0)}\right) = p \cdot \frac{u \cdot V(0)}{V(0)} + (1-p) \cdot \frac{d \cdot V(0)}{V(0)} = p \cdot u + (1-p) \cdot d \quad (2.30)$$

2.5. The Cox-Ross-Rubinstein Model

$$\text{Var}(R_1(T)) = \text{Var}\left(\frac{V(T)}{V(0)}\right) = p \cdot u^2 + (1-p) \cdot d^2 - (p \cdot u + (1-p) \cdot d)^2 \quad (2.31)$$

After n periods, the security has the price: $V(t) = V(0) \cdot u^{N(t)} \cdot d^{t-N(t)}$. Including t , the stock price goes up $N(t)$ times and down $t - N(t)$ times.

2.5.2 Risk-Neutral Valuation in the CRR Model

The concept of risk-neutral valuation is crucial in pricing derivatives. In a risk-neutral world, investors are indifferent to risk, simplifying derivative pricing by allowing expected payoffs to be discounted at the risk-free rate. In the Cox-Ross-Rubinstein (CRR) model, this involves calculating a risk-neutral probability p that aligns the asset's expected return with the risk-free rate r . The derivative price is then found by discounting its expected payoffs at r using the binomial tree method.

For the CRR model, the risk-neutral probability is given by:

$$e^{r\Delta t} = pu + (1-p)d \quad (2.32)$$

Here, u and d represent the up and down factors of the asset price, and Δt is the time interval between steps. This probability is used to compute the expected payoffs of the derivative at each node of the tree.

2.5.3 Example

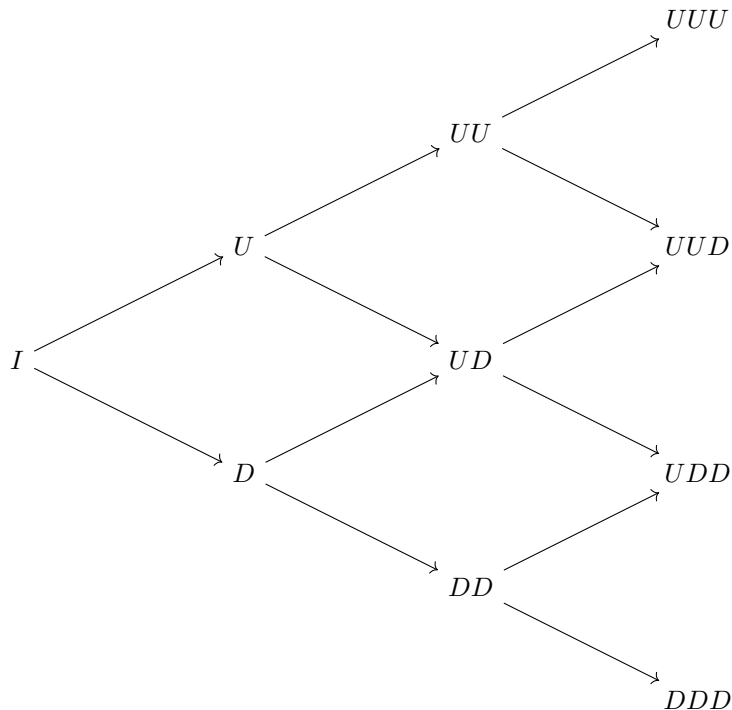
Example 2.5.1. Consider a stock currently priced at 100€. We evaluate a European call option with a strike price of 100€ maturing in three periods. The stock price increases by $u = 1.1$ or decreases by $d = 0.9$ each period. The risk-free interest rate is 5% per period, and the probability of an upward movement is $p = 0.5$. Calculate the following:

1. The expected value of the call option's return at the end of the third period.
2. The variance of the call option's return at the end of the third period.
3. The current price of the call option utilizing the risk-neutral valuation approach.

Step 1: Constructing the binomial tree

The Lattice representation of the problem is:

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The nodes represent possible stock prices at the end of each period, with up and down movements indicated by arrows.

Step 2: Computing the Progression of Stock Prices

Given $V(0) = 100\text{€}$, $u = 1.1$, $d = 0.9$, and $r = 0.05$ we can compute:

- After one period: $V(1) = 100 \times 1.1 = 110$ (U), $V(1) = 100 \times 0.9 = 90\text{€}$ (D).
- After two periods: $V(2) = 110 \times 1.1 = 121\text{€}$ (UU), $V(2) = 110 \times 0.9 = 99\text{€}$ (UD), $V(2) = 90 \times 0.9 = 81\text{€}$ (DD).
- After three periods: $V(3) = 121 \times 1.1 = 133.1\text{€}$ (UUU), $V(3) = 121 \times 0.9 = 108.9\text{€}$ (UUD), $V(3) = 99 \times 0.9 = 89.1\text{€}$ (UDD), $V(3) = 81 \times 0.9 = 72.9\text{€}$ (DDD).

After three periods, the stock price reaches a peak of 133.1€. A path with two ups followed by a down (UUD) results in 108.9€. One up followed by two downs (UDD) leads to 89.1€. A full downward trajectory (DDD) drops the price to 72.9€.

Step 3: Calculating the payoffs for a call option

The payoff for a European call option at expiry T is $\max(V(T) - K, 0)$. Given that the strike price $K = 100$ we calculate this for each end node:

- Payoff (UUU): $\max(133.1 - 100, 0) = 33.1\text{€}$
- Payoff (UUD): $\max(108.9 - 100, 0) = 8.9\text{€}$
- Payoff (UDD): $\max(89.1 - 100, 0) = 0\text{€}$
- Payoff (DDD): $\max(72.9 - 100, 0) = 0\text{€}$

2.5. The Cox-Ross-Rubinstein Model

For a European call option with a strike price of 100€ maturing in three periods, three upward movements (UUU) yield a maximum payoff of 33.1€. Two ups followed by a down (UUD) yield a payoff of 8.9€. If the stock ends lower after one up and two downs (UDD) or follows a downward path throughout (DDD), the option yields no payoff.

Step 3: Risk-neutral probabilities

Under the risk-neutral measure, the probability p is adjusted to ensure that the expected stock price growth equals the risk-free rate. This is given by the formula:

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

where r is the risk-free rate, Δt is the time period, u is the up factor, and d is the down factor. Substituting the known values ($u = 1.1$, $d = 0.9$, $r = 0.05$, and $\Delta t = 1$ period),

$$p = \frac{e^{0.05} - 0.9}{1.1 - 0.9} \approx \frac{1.05127 - 0.9}{0.2} = 0.75635 \approx 0.756$$

This computed value ensures that the expected growth rate of the stock, under the risk-neutral measure, matches the exponential growth factor of the risk-free rate over one period.

Step 4: Calculating the expected value and variance of the option's return

Given the payoffs calculated previously and the risk-neutral probability $p \approx 0.75635$, and its complement $1 - p = 0.24365$, the expected value and variance of the option's return at $T = 3$ can be calculated using the binomial model.

$$E(R_T) = V(0) \cdot u^{Tp} \cdot d^{T(1-p)} = 100 \cdot 1.1^{3 \cdot 0.75635} \cdot 0.9^{3 \cdot 0.24365} = 114.94117 \approx 114.94\text{€}$$

The variance of the option's return $\text{Var}(R_T)$ accounts for the variability in possible price paths and is calculated using the formula:

$$\begin{aligned} \text{Var}[R_T] &= V(0)^2 \cdot \left(u^{2Tp} \cdot d^{2T(1-p)} - u^{Tp} \cdot d^{T(1-p)} \right)^2 \\ &= 100^2 \cdot \left(1.1^{2 \cdot 3 \cdot 0.75635} \cdot 0.9^{2 \cdot 3 \cdot 0.24365} - 1.1^{3 \cdot 0.75635} \cdot 0.9^{3 \cdot 0.24365} \right)^2 \\ &= 29.0975364 \approx 29.10 \end{aligned}$$

The expected value of 114.94€ illustrates the potential return from the initial investment, while the variance of 29.10 underscores the inherent risk due to price volatility over the period.

Step 5: Discounting back to find the current price

To find the current price of the European call option, we discount the expected payoff back to the present value using the risk-free interest rate. The formula for the current price C_0 of the option is:

$$C_0 = e^{-rT} \times E(R_T)$$

Where:

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- r is the risk-free interest rate per period,
- T is the number of periods,
- $E(R_T)$ is the expected payoff at the end of the third period.

Using the previously calculated expected return $E(R_T) \approx 1148.94$ and the risk-free rate of 5%, the current price of the call option is:

$$C_0 = e^{-0.05 \times 3} \times 114.94 = 98.92977 \approx 98.93\text{€}$$

Therefore, the present value of the expected payoff of the European call option, when discounted at the risk-free rate, is approximately 98.93€.

2.6 Limitations

The mean-variance approach in a one-period setting, while straightforward and popular for its simplicity, exhibits significant limitations. Its singular trading opportunity at $t = 0$ precludes any adaptation to evolving market conditions, making it static and overly simplistic in modeling security prices.

Moreover, the approach's reliance on variance as the only risk measure does not accommodate dynamic risk considerations, leading to the pursuit of continuous-time market models that can more accurately reflect the complexities of financial markets and trading strategies. This static nature and the absence of temporal flexibility underline the necessity for more sophisticated, dynamic modeling frameworks in financial analysis.

On the other hand, the complexity of discrete-time multi-period models, such as the Cox-Ross-Rubinstein Model, escalates substantially with an increasing number of periods confronting significant challenges in being solved in real time [18].

Chapter 3

Exploring Itô Calculus: The Mathematical Foundation of Financial Modeling

This chapter delves into Itô Calculus, a cornerstone of financial modeling, starting from the modeling of security prices and advancing through Brownian motion, martingales, and the Wiener process. We examine the core principles of Itô Calculus, including derivatives, integrals, and the stochastic integral, leading to an in-depth discussion of the Itô Formula and its applications. The narrative extends to the stock price equation and continuous-time market models.

3.1 Modeling the Security Prices

Definition 3.1.1 (Continuous-Time Market Model). A Continuous-Time Market Model operates in continuous time over the interval $[0, T]$, featuring $n + 1$ securities: n risky assets and one risk-free asset. Initial prices are set at $t = 0$ with constants p_1, p_2, \dots, p_n for risky assets and p_0 for the risk-free asset. Prices for risky assets at any $t > 0$ follow stochastic processes $P_1(t), P_2(t), \dots, P_n(t)$, while the risk-free asset price $P_0(t)$ is deterministic.

Investors can trade continuously, adjusting portfolios dynamically to reflect price changes from $t = 0$ to T . The model examines initial, final, and ongoing price trajectories.

Definition 3.1.2 (Interest). Interest is the payment from a borrower or deposit-taking financial institution to a lender or depositor of an amount above repayment of the principal sum (that is, the amount borrowed), at a particular rate.

Definition 3.1.3 (Interest Rate). The interest rate, denoted as r , represents the proportion of a loan that is charged as interest to the borrower, or the surplus that is given to the depositor. It is typically expressed as an annual percentage of the amount.

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It is important to distinguish between the compound interest and the continuous rate:

Definition 3.1.4 (Compound Interest). The compound interest, denoted as r , is the interest rate on an investment or loan that is compounded once per period. In general, compounding of interest r/n at times i/n , $i = 1, \dots, n$, $n \in \mathbb{N}$, leads to a bank account having a value of

$$K \cdot \left(1 + \frac{r}{n}\right)^n \quad (3.1)$$

being K the initial value of the bank account.

Definition 3.1.5 (Continuous Rate). The continuous rate, denoted as \tilde{r} , is the interest rate that, when compounded continuously, results in the same final amount as that obtained through another specified compounding method over a certain period. Annually is:

$$\tilde{r} = \ln(1 + r) \quad (3.2)$$

When modeling bond prices under continuous compounding with a constant interest rate r , the price of a bond with initial price p_0 at any time t within $[0, T]$ is given by the formula:

$$P_0(t) = p_0 \cdot e^{r \cdot t} \quad (3.3)$$

In a general scenario with a time-dependent interest rate $r(t)$, the bond price is determined by the integral of the interest rate from 0 to t , expressed as:

$$P_0(t) = p_0 \cdot e^{\int_0^t r(s) ds} \quad (3.4)$$

This model can also be represented using a differential equation:

$$\frac{dP_0(t)}{dt} = P_0(t) \cdot r(t), \quad P_0(0) = p_0 \quad (3.5)$$

which states that the rate of change of the bond price is proportional to the product of the current price and the current interest rate. Integrating this equation:

$$P_0(t) = p_0 + \int_0^t P_0(s) \cdot r(s) ds \quad (3.6)$$

We can imagine the stock price as a bond price with stochastic variability. The stock oscillates around a bond-like component with its own interest rate \hat{r} , typically higher than the bond's rate r to compensate for its volatility and risk.

For a bond with a constant interest rate, the relationship of its price over time is log-linear, given by:

$$\ln(P_0(t)) = \ln(p_0) + r \cdot t \quad (3.7)$$

Therefore the logarithm of a stock's price can be modeled as follows:

$$\ln(P_i(t)) = \ln(p_i) + \mu_i \cdot t + \epsilon \quad (3.8)$$

where the randomness, ϵ is characterized by the following properties:

3.2. Brownian Motion and Martingales

- It has zero mean, that is, $E(\epsilon) = 0$.
- It is time-dependent.
- It aggregates all deviations from the expected growth rate $\mu_i \cdot t$.

Assuming that these deviations arise from many small, similar, independent factors, the central limit theorem implies that the randomness follows a normal distribution $\sim N(0, \sigma^2 t)$ for some $\sigma > 0$. Defining the deviation at time t as

$$Y(t) = \ln(P_i(t)) - \ln(p_0) - \mu_i \cdot t \tag{3.9}$$

we identify $Y(t) \sim N(0, \sigma^2 t)$.

Furthermore, considering the increments of $Y(t)$, $Y(t) - Y(s)$ for $s < t$, should depend only on the time interval $t - s$ and be distributed according to $N(0, \sigma^2(t - s))$, illustrating that $Y(t)$ forms a stochastic process.

3.2 Brownian Motion and Martingales

In this section, we explore the pivotal concepts of Brownian Motion and Martingales, foundational in the field of quantitative finance. Originating from Robert Brown's 1827 observation of pollen particles' random motion, these concepts were revolutionized in the 20th-century Louis Bachelier's application to asset price behavior. This groundbreaking approach posited that financial markets' dynamics could be modeled through a Gaussian distribution [29].

3.2.1 Discrete-time stochastic processes

Definition 3.2.1 (Simple Random Walk). A simple random walk is a discrete-time stochastic process $\{X_k\}_{k \geq 0}$ where $X_0 = 0$ and for all $k \geq 1$,

$$X_k = Y_1 + Y_2 + \dots + Y_k \tag{3.10}$$

with each Y_i being an independent and identically distributed random variable taking values ± 1 with equal probability.

Example 3.2.1 (Simple Random Walk). Consider a gambling scenario where an unbiased coin is used. At each turn, the dealer flips the coin:

1. If the outcome is heads, the gambler wins \$1.
2. If the outcome is tails, the gambler loses \$1.

Assuming each coin toss is independent, the gambler's net winnings can be modeled as a simple random walk starting at zero. Each coin toss moves the balance up or down by \$1, capturing the stochastic nature of successive gambles.

Lemma 2. The simple random walk $\{X_k\}$ satisfies the following properties:

1. The expected value of X_k is 0 for all k , i.e., $E[X_k] = 0$.
2. The increments $X_{k_i} - X_{k_{i-1}}$ for $0 \leq k_0 \dots < k_r$ are mutually independent.

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3. The distribution of the increment $X_{k+h} - X_k$ is the same as the distribution of X_h for all $h \geq 1$ and $k \geq 0$.

Proof. Initially we will prove the first proposition of the lemma: since each Y_i has an expected value of 0, the linearity of expectation gives:

$$E[X_k] = E[Y_1 + Y_2 + \cdots + Y_k] = E[Y_1] + E[Y_2] + \cdots + E[Y_k] = 0 + 0 + \cdots + 0 = 0.$$

For the second proposition we have that the increments

$$X_{k_i} - X_{k_{i-1}} = Y_{k_{i-1}+1} + \cdots + Y_{k_i}$$

are sums of disjoint groups of Y_i variables. Since the Y_i variables are independent, the sums of these disjoint groups are also independent.

Finally, for the third proposition we have that the increment

$$X_{k+h} - X_k = Y_{k+1} + \cdots + Y_{k+h}$$

involves a sum of h consecutive Y_i variables, starting at index $k + 1$. Due to the identical distribution and independence of Y_i , this sum has the same distribution as any other sum of h consecutive Y_i variables, such as $X_h = Y_1 + \cdots + Y_h$ \square

Remark. A key characteristic of the simple random walk is that the influence of past events on future outcomes is defined solely by the present state, not by the entire sequence of past events.

Definition 3.2.2 (Markov Chain). A Markov Chain is a stochastic process $\{X_n\}_{n \geq 0}$ defined on a discrete state space S where the probability of moving to the next state depends solely on the current state and not on the sequence of events that preceded it. Mathematically, it is defined as:

$$\begin{aligned} P(X_{n+1} = j \mid X_n = i, X_{n-1} = k, \dots, X_0 = l) \\ = P(X_{n+1} = j \mid X_n = i) \quad \forall n \geq 0; i, j \in S \end{aligned} \quad (3.11)$$

A Markov Chain is said to be time-homogeneous if the transition probabilities are independent of n , and can be expressed by a transition matrix $[p_{ij}]$ where:

$$p_{ij} = P(X_{n+1} = j \mid X_n = i), \quad i, j \in S \quad (3.12)$$

It follows that:

$$\sum_{j \in S} p_{ij} = 1 \quad \text{for all } i \in S \quad (3.13)$$

ensuring that the probabilities of transitioning from any given state to all possible next states sum up to 1.

For a finite Markov chain, the state space S is a finite set, commonly denoted as $S = [m]$ for some integer m , which represents the indices of the possible states.

Remark. A simple random walk is an example of a Markov chain. However, there is no transition probability matrix associated with the simple random walk since the sample space is of infinite cardinality.

3.2. Brownian Motion and Martingales

Example 3.2.2 (Markov Chain for Credit Ratings). Consider credit ratings assigned to companies, categorized into states like Excellent, Good, Fair, and Default. These represent the company's financial health and default probability.

The transition probabilities between these states can be represented as a matrix:

$$P = \begin{pmatrix} p_{\text{Excellent, Excellent}} & p_{\text{Excellent, Good}} & p_{\text{Excellent, Fair}} & p_{\text{Excellent, Default}} \\ p_{\text{Good, Excellent}} & p_{\text{Good, Good}} & p_{\text{Good, Fair}} & p_{\text{Good, Default}} \\ p_{\text{Fair, Excellent}} & p_{\text{Fair, Good}} & p_{\text{Fair, Fair}} & p_{\text{Fair, Default}} \\ p_{\text{Default, Excellent}} & p_{\text{Default, Good}} & p_{\text{Default, Fair}} & p_{\text{Default, Default}} \end{pmatrix}$$

where p_{ij} is the probability of transitioning from rating i to rating j . Each row of the matrix sums to 1, indicating that the probabilities of moving to any state from a given state are collectively exhaustive.

Assume the transition probabilities from Good to other states are:

$$\begin{aligned} p_{\text{Good, Excellent}} &= 0.05 \\ p_{\text{Good, Good}} &= 0.90 \\ p_{\text{Good, Fair}} &= 0.04 \\ p_{\text{Good, Default}} &= 0.01 \end{aligned}$$

This implies that a company rated as Good has a 90% chance of maintaining its rating, a 5% chance of upgrading to Excellent, a 4% chance of downgrading to Fair, and a 1% chance of defaulting to Default next period.

Definition 3.2.3 (σ -Algebra). A σ -algebra on a set Ω is a non-empty collection \mathcal{F} of subsets of Ω that satisfies the following three properties:

1. $\Omega \in \mathcal{F}$ (the sample space is in the σ -algebra).
2. If $A \in \mathcal{F}$, then $A^c = \Omega \setminus A \in \mathcal{F}$ (the complement of any event in the σ -algebra is also in the σ -algebra).
3. If $\{A_i\}_{i=1}^{\infty}$ is a countable collection of sets in \mathcal{F} , then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (the σ -algebra is closed under countable unions).

It is highly important to state the definition filtration which allow us to formalize a model that describes a flow of information over time.

Definition 3.2.4 (Filtration). Let (Ω, \mathcal{F}, P) be a probability space. A *filtration* is a sequence $\{\mathcal{F}_t\}_{t \geq 0}$ of σ -algebras such that $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \mathcal{F}$ for all t , where \mathcal{F}_t represents the collection of events that are observable by time t .

Definition 3.2.5 (Martingale). A discrete-time stochastic process $\{X_t\}_{t \geq 0}$ is called a martingale if for all $t \geq 0$,

$$X_t = E[X_{t+1} | \mathcal{F}_t] \tag{3.14}$$

where $\mathcal{F}_t = \{X_0, \dots, X_t\}$ represents the filtration up to time t , indicating the information available up to and including time t .

This definition implies that the expected value of the next state X_{t+1} , given all prior states, is equal to the current state X_t . Thus, the expected increment or gain in the process at any point is zero, capturing the fairness of the process in a probabilistic sense.

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Lemma 3. For any time $t \geq s$, the condition $X_s = E[X_t | \mathcal{F}_s]$ holds.

Proof. We will prove the proposition by induction on t .

Base case ($t = s$): Trivially, $X_s = E[X_s | \mathcal{F}_s]$ by the definition of conditional expectation, as the condition simplifies to the expectation of a constant.

Inductive step: Assume that $X_s = E[X_t | \mathcal{F}_s]$ holds for some $t \geq s$. We need to show that $X_s = E[X_{t+1} | \mathcal{F}_s]$.

By the tower property of conditional expectation and the martingale property, we demonstrate the following equality:

$$E[X_{t+1} | \mathcal{F}_s] = E[E[X_{t+1} | \mathcal{F}_t] | \mathcal{F}_s] = E[X_t | \mathcal{F}_s].$$

1. First Equality ($E[X_{t+1} | \mathcal{F}_s] = E[E[X_{t+1} | \mathcal{F}_t] | \mathcal{F}_s]$): This step utilizes the property of conditional expectations. The tower property states that if Z is a random variable and \mathcal{G} and \mathcal{H} are σ -algebras such that $\mathcal{H} \subseteq \mathcal{G}$, then:

$$E[E[Z | \mathcal{G}] | \mathcal{H}] = E[Z | \mathcal{H}].$$

In our case, $Z = X_{t+1}$, $\mathcal{G} = \mathcal{F}_t$, and $\mathcal{H} = \mathcal{F}_s$, acknowledging that $\mathcal{F}_s \subseteq \mathcal{F}_t$ as $s \leq t$.

2. Second Equality ($E[E[X_{t+1} | \mathcal{F}_t] | \mathcal{F}_s] = E[X_t | \mathcal{F}_s]$): This step applies the martingale property, which states that for any t , the conditional expectation of X_{t+1} given the filtration at time t (\mathcal{F}_t) equals X_t . Hence,

$$E[X_{t+1} | \mathcal{F}_t] = X_t.$$

Substituting this back into the tower property expression gives us the required equality. □

Remark. A simple random walk is a martingale because its expected future value, given all past information, is equal to its current value. In other words, the increments of a simple random walk are independent and have an expected value of zero, ensuring that the process has no drift.

Example 3.2.3 (Example of Martingale). Consider a gambler playing roulette, betting on outcomes with less than a 50% chance of winning. Let X_k represent the gambler's balance after k rounds. This process is not a martingale because the expected balance after the next game is less than the current balance, reflecting the casino's edge.

$$E[X_{k+1} | \mathcal{F}_k] < X_k.$$

Definition 3.2.6 (Stopping Time). Given a stochastic process $\{X_0, X_1, \dots\}$, a non-negative integer-valued random variable τ is called a stopping time if, for every integer $k \geq 0$, the event $\tau \leq k$ depends only on the events X_0, X_1, \dots, X_k .

3.2. Brownian Motion and Martingales

Example 3.2.4 (Stopping Times in Stock Trading). Consider an investor who buys a stock at \$50. Let τ be the first time the stock price reaches \$100. Assuming the stock follows a random walk, τ is a stopping time.

Now, the investor wishes to sell at the first local maximum, the highest price before any significant decline. Here, τ is not a stopping time, as identifying a local maximum requires future price information, unavailable at time τ .

Theorem 4 (Doob's Optional Stopping Time Theorem). Let $\{X_n\}_{n \geq 0}$ be a martingale and τ be a stopping time such that $\tau \leq T$ for some constant T . Then, $E[X_\tau] = E[X_0]$.

Proof. Consider the representation of X_τ as follows:

$$X_\tau = X_0 + \sum_{i=0}^{\tau-1} (X_{i+1} - X_i).$$

Here, $\tau \leq T$ is a constant, allowing us to express the expectation of X_τ by the linearity of expectation:

$$E[X_\tau] = E[X_0] + \sum_{i=0}^{T-1} E[(X_{i+1} - X_i) \cdot 1_{\{\tau > i\}}]$$

Each term $E[(X_{i+1} - X_i) \cdot 1_{\{\tau > i\}}]$ can be analyzed by considering the information up to time i , \mathcal{F}_i , and using the martingale property:

$$E[(X_{i+1} - X_i) \cdot 1_{\{\tau > i\}}] = E[E[(X_{i+1} - X_i) \cdot 1_{\{\tau > i\}} \mid \mathcal{F}_i]]$$

Since $E[X_{i+1} \mid \mathcal{F}_i] = X_i$, we have:

$$E[(X_{i+1} - X_i) \cdot 1_{\{\tau > i\}} \mid \mathcal{F}_i] = E[X_{i+1} - X_i \mid \mathcal{F}_i] \cdot 1_{\{\tau > i\}} = 0$$

Therefore, each term in the sum evaluates to zero, and we conclude:

$$E[X_\tau] = E[X_0]$$

□

3.2.2 Continuous-time Stochastic Processes

We now explore continuous-time stochastic processes. Unlike discrete-time processes, which are described by increments $X_{k+1} - X_k$, continuous-time processes require defining specific properties of their probability distributions and proving a distribution exists that meets these criteria.

Definition 3.2.7 (Brownian Motion). A Wiener Process, commonly known as Brownian Motion, is a continuous-time stochastic process $W(t)$ that models random fluctuations. It satisfies the following properties:

1. Initial Condition: $W(0) = 0$.

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2. Independence of Increments: The increments of the process are independent. That is, for any $0 \leq t_1 < t_2 < \dots < t_n$, the random variables $W(t_2) - W(t_1)$, $W(t_3) - W(t_2)$, \dots , $W(t_n) - W(t_{n-1})$ are independent.
3. Stationarity of Increments: The increments are stationary. Specifically, for any $0 \leq s < t$, the increment $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$, denoted as $W(t) - W(s) \sim N(0, t - s)$.
4. Continuity: The paths of $W(t)$ are almost surely continuous, meaning that each sample path $t \mapsto W(t)$ is a continuous function.

A particular instance of a path chosen according to the Brownian motion is referred to as a sample Brownian path.

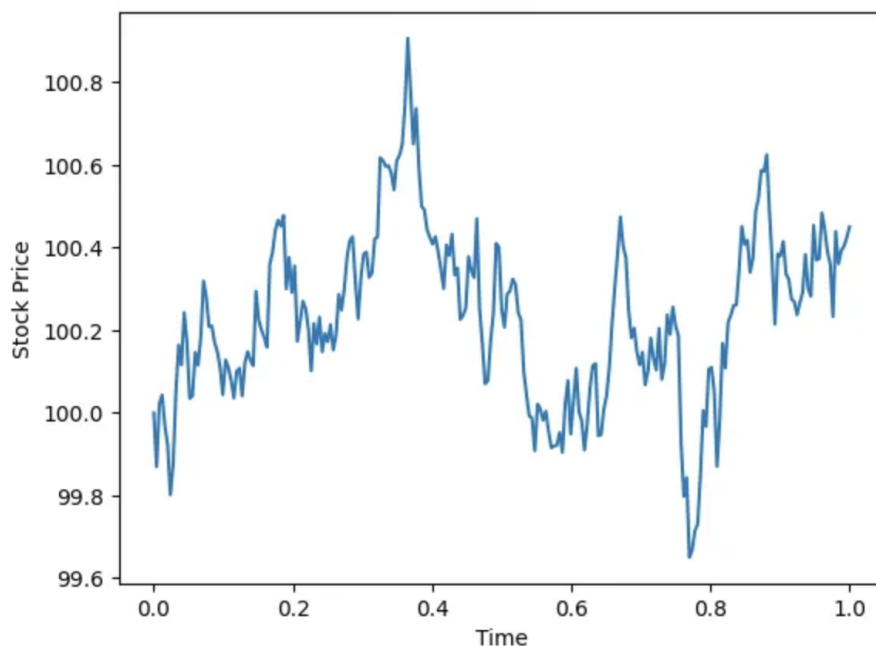


Figure 3.1: Simulated Stock Price Movement Using Brownian Motion¹.

Remark. The graph illustrates a single realization of a Brownian motion path, highlighting the erratic and unpredictable nature of stock price movements. Despite its randomness, Brownian motion is a well-defined mathematical model with properties like independence and stationarity of increments.

Example 3.2.5 (Standard Brownian Motion as a Limit). Consider standard Brownian motion as the limit of simple random walks. Suppose $\{Y_0, Y_1, \dots\}$ is a sequence with increments of mean 0 and variance 1. Define Z as a piece-wise linear function from $[0, 1]$ to \mathbb{R} by:

$$Z\left(\frac{t}{n}\right) = Y_t, \quad \text{for } t = 0, \dots, n$$

¹Image available at: <https://cuts.top/F2H->. Accessed on April 14, 2024.

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and linearly interpolate between these points. As n increases, the distribution of $Z(t)$ approaches that of standard Brownian motion. Specifically, the distribution of $Z(y)$ converges to $N(0, t)$ by the Central Limit Theorem.

Remark (Central Limit Theorem). The Central Limit Theorem states that if a set of random variables X_1, \dots, X_n are independent and identically distributed with a given mean μ and finite variance σ^2 , then the sum $S_n = X_1 + X_2 + \dots + X_n$ converges in distribution to a normal distribution as n approaches infinity. More precisely, the normalized sum:

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \tag{3.15}$$

converges in distribution to a standard normal distribution $N(0, 1)$ as n increases, irrespective of the distribution of the original variables X_j .

Example 3.2.6 (Applying the Central Limit Theorem). Consider a portfolio π formed by options modeled by independent and identically distributed random variables X_1, X_2, \dots, X_n , each with mean μ and variance σ^2 . We aim to estimate the mean of the distribution. A natural estimator is the sample mean, defined as:

$$Y_n = \frac{1}{n}(X_1 + X_2 + \dots + X_n).$$

This estimator Y_n is a random variable. By the Law of Large Numbers, Y_n converges to μ as n approaches infinity. For large n , the distribution of $\sqrt{n}(Y_n - \mu)$ approximates a normal distribution with mean 0 and standard deviation σ , useful for hypothesis testing and confidence intervals.

In practice, we record Brownian motion at discrete intervals due to measurement limits. While Brownian motion is a good theoretical model, observed paths resemble it more closely as we measure at finer time scales. With more frequent data points, the observed behavior increasingly mimics a simple random walk.

Lemma 5. For a Brownian motion $B(t)$ and any $a > 0$, the probability that the maximum $M(t)$ of $B(s)$ for $s \leq t$ exceeds a is given by:

$$P(M(t) \geq a) = 2P(B(t) > a) = 2 - 2\Phi\left(\frac{a}{\sqrt{t}}\right) \tag{3.16}$$

where Φ is the cumulative distribution function of the standard normal distribution.

Proof. Define τ_a as the first hitting time of a , that is: $\tau_a = \min\{s : B(s) = a\}$.

Note that τ_a is a stopping time because its occurrence depends only on the history up to time τ_a .

By the symmetry and independent increments of Brownian motion, for all $0 \leq s < t$, the event $(B(t) - B(s) > 0)$ has the same probability as $(B(t) - B(s) < 0)$, specifically:

$$P(B(t) - B(s) > 0) = P(B(t) - B(s) < 0) = \frac{1}{2}$$

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Applying the Markov Property of Brownian motion, the process after τ_a is still a Brownian motion starting from a . Hence, the event $B(t) > a$ given $\tau_a < t$ has the same probability as $B(t) < a$ given $\tau_a < t$, so:

$$P(B(t) > a \mid \tau_a < t) = P(B(t) < a \mid \tau_a < t) = \frac{1}{2}$$

Combining these results using the Reflection Principle [16], which states that $P(B(t) > a)$ and $P(M(t) \geq a)$ are related through the event of crossing a before t , we get:

$$P(M(t) \geq a) = P(\tau_a < t) = 2P(B(t) > a)$$

Calculating $P(B(t) > a)$ using the normal distribution properties of Brownian increments, we have:

$$P(B(t) > a) = 1 - \Phi\left(\frac{a}{\sqrt{t}}\right)$$

where Φ denotes the cumulative distribution function of the standard normal distribution. Thus, the claim follows as:

$$P(M(t) \geq a) = 2 - 2\Phi\left(\frac{a}{\sqrt{t}}\right)$$

□

Lemma 6. For each $t \geq 0$, the Brownian motion is almost surely not differentiable at t .

Proof. Fix a real t_0 and assume that the Brownian motion B is differentiable at t_0 . Suppose there exist constants A and ϵ_0 such that for all $0 < \epsilon < \epsilon_0$, the inequality $|B(t) - B(t_0)| \leq A\epsilon$ holds for all t satisfying $|t - t_0| \leq \epsilon$. Denote this event by $E_{\epsilon, A}$.

Notice that the probability of $E_{\epsilon, A}$ can be expressed as:

$$P(E_{\epsilon, A}) = P\left(\max_{|t-t_0| \leq \epsilon} |B(t) - B(t_0)| \leq A\epsilon\right) = 2\left(1 - \Phi\left(\frac{A\epsilon}{\sqrt{\epsilon}}\right)\right)$$

where Φ is the cumulative distribution function of the standard normal distribution. As ϵ approaches zero, the right-hand side tends to zero because the argument of Φ tends to infinity. Therefore, $P(E_{\epsilon, A}) \rightarrow 0$ as $\epsilon \rightarrow 0$.

By the countable additivity of probability measures, the probability that B is differentiable at t_0 for some constant A is zero, as it would require $P(\cap_{n=1}^{\infty} E_{1/n, A}) = 0$ since this is true for any A , it follows that B is almost surely not differentiable at t_0 . □

Theorem 7 (Quadratic Variation). Consider a partition $\Pi = \{t_0, t_1, \dots, t_j\}$ of the interval $[0, T]$, where $|\Pi| = \max_i(t_{i+1} - t_i)$. Then, as $|\Pi|$ approaches zero, Brownian motion B_t satisfies the following relation almost surely:

$$\lim_{|\Pi| \rightarrow 0} \sum_i (B_{t_{i+1}} - B_{t_i})^2 = T \tag{3.17}$$

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Proof. For simplicity, we will assume that the partition intervals are uniform, that means, $t_{i+1} - t_i = \frac{T}{n}$ for all i , where n is the number of intervals in Π . Under this setup, the increments $(B_{t_{i+1}} - B_{t_i})$ are independent and identically distributed random variables.

Each increment squared $(B_{t_{i+1}} - B_{t_i})^2$ has an expected value equal to the variance of the increment, which is $t_{i+1} - t_i = \frac{T}{n}$. Hence, the sum of these squared increments is: $\sum_i (B_{t_{i+1}} - B_{t_i})^2$. By the law of large numbers, as the number of intervals n increases (which corresponds to $|\Pi| \rightarrow 0$), this sum converges almost surely to T , because:

$$E \left[\sum_i (B_{t_{i+1}} - B_{t_i})^2 \right] = n \cdot \frac{T}{n} = T$$

□

Remark (Law of Large Numbers). The Law of Large Numbers states that, if identical, independently distributed variables with a finite mean are summed and then divided by the number of variables, the ratio will almost surely converge to the expected value of these variables as the number of trials approaches infinity.

Mathematically, let X_1, X_2, \dots, X_n be a sequence of independent and identically distributed random variables with expected value $E[X_i] = \mu$. We can affirm that:

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mu \quad \text{almost surely as } n \rightarrow \infty$$

Example 3.2.7 (Applying the Law of Large Numbers). Consider a simplified casino game like roulette, where the casino pays \$1.00 for red and the player pays \$1.05 for black. The expected gain for the casino per game is \$0.025, calculated as $0.5 \times (-1.05) + 0.5 \times 1.00 = -0.025$.

For the player, this bias seems negligible over a few games, where randomness dominates. However, for the casino, with many games, the Law of Large Numbers ensures the average gain per game will converge to the expected \$0.025.

Theorem 8. Suppose f is a continuously differentiable function on $[0, T]$. For a partition $\Pi = \{t_0, t_1, \dots, t_j\}$ of $[0, T]$ with $|\Pi| = \max_i (t_{i+1} - t_i)$, the quadratic variation of f along Π is given by:

$$\sum_i (f(t_{i+1}) - f(t_i))^2 \leq \max_{s \in [0, T]} f'(s)^2 \cdot \sum_i (t_{i+1} - t_i)^2 \tag{3.18}$$

As $|\Pi| \rightarrow 0$, this sum tends to zero, illustrating that the function f does not exhibit the erratic behavior typical of Brownian motion, where $\sum_i (B_{t_{i+1}} - B_{t_i})^2$ converges to T .

Proof. By the Mean Value Theorem, for each interval $[t_i, t_{i+1}]$, there exists a $\xi_i \in (t_i, t_{i+1})$ such that

$$f(t_{i+1}) - f(t_i) = f'(\xi_i)(t_{i+1} - t_i)$$

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Squaring both sides, we obtain

$$(f(t_{i+1}) - f(t_i))^2 = f'(\xi_i)^2 (t_{i+1} - t_i)^2$$

Summing over all i , we get

$$\sum_i (f(t_{i+1}) - f(t_i))^2 = \sum_i f'(\xi_i)^2 (t_{i+1} - t_i)^2$$

Using the fact that f' is continuous and $f'(\xi_i)^2$ will be bounded by $\max_{s \in [0, T]} f'(s)^2$, we have

$$\sum_i (f(t_{i+1}) - f(t_i))^2 \leq \max_{s \in [0, T]} f'(s)^2 \cdot \sum_i (t_{i+1} - t_i)^2$$

As $|\Pi| \rightarrow 0$, the term $\sum_i (t_{i+1} - t_i)^2$ also tends to zero because it is equivalent to $|\Pi| \cdot T$ when the partition is uniform. Therefore, the sum of squared increments converges to zero. \square

Definition 3.2.8 (Brownian Motion with Drift). Consider a Brownian motion $B(t)$ and a fixed real b . The process $X(t) = B(t) + bt$ is called a *Brownian motion with drift* b . For this process, the expected trajectory over time is given by $E[X(t)] = bt$, showing that the linear drift bt becomes dominant over the stochastic fluctuation $B(t)$ as t increases.

Indeed, for any fixed $\epsilon > 0$, the path of $X(t)$ will almost surely remain between the lines $y = (b - \epsilon)t$ and $y = (b + \epsilon)t$ for large t .

3.3 Interpretation of the Stock Price Model

When using Brownian motion, denoted as $(W_t, \mathcal{F}_t)_{t \geq 0}$, to model financial securities, the goal is to capture the stochastic nature of stock prices. Particularly, the logarithmic representation of stock prices is commonly employed. Taking as example a simple market with only one stock and a bond, it is appropriate to model the stock price using a single-variable Brownian motion, incorporating volatility σ to account for the uncertainty.

The logarithm of the stock price $P_1(t)$ at time t can be expressed as:

$$\ln P_1(t) = \ln p_1 + b_1 t + \sigma W_t \quad (3.19)$$

where b_1 is the drift of the stock price. Thus, the stock price itself is modeled as: $P_1(t) = p_1 e^{b_1 t + \sigma W_t}$. For a more complex model involving n different securities in a market, we generalize this to a multi-dimensional setting:

$$\ln P_i(t) = \ln p_i + b_i t + \sum_{j=1}^m \sigma_{ij} W_j(t), \quad i = 1, \dots, n \quad (3.20)$$

From which, the prices of the securities are given by:

$$P_i(t) = p_i \exp \left(b_i t + \sum_{j=1}^m \sigma_{ij} W_j(t) \right) \quad (3.21)$$

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where $(W_1(t), \dots, W_m(t))$ represent an m -dimensional Brownian motion, thereby influencing the logarithmic distribution of stock prices:

$$\ln P_i(t) \sim \mathcal{N}(\ln p_i + b_i t, \sum_{j=1}^m \sigma_{ij}^2 t) \quad (3.22)$$

Consequently, we establish that $P_i(t)$ follows a log-normal distribution.

Lemma 9. Define $\tilde{b}_i = b_i + \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2$ for each i , then for $P_i(t)$ as above, we have:

1. The expected value of $P_i(t)$ is:

$$E[P_i(t)] = p_i e^{\tilde{b}_i t} \quad (3.23)$$

2. The variance of $P_i(t)$ is:

$$\text{Var}(P_i(t)) = p_i^2 e^{2\tilde{b}_i t} \left(\exp \left(\sum_{j=1}^m \sigma_{ij}^2 t \right) - 1 \right) \quad (3.24)$$

3. For the stochastic process defined as:

$$X_t := a \exp \left(\sum_{j=1}^m \left(c_j W_j(t) - \frac{1}{2} c_j^2 t \right) \right), \quad a, c_j \in \mathbb{R}, \quad j = 1, \dots, m \quad (3.25)$$

the process X_t is a martingale.

Proof. First we will prove the first point of the lemma for the general case $m > 1$.

Each term $\sigma_{ij} W_j(t)$ is normally distributed with mean 0 and variance $\sigma_{ij}^2 t$ due to the properties of Brownian motion scaled by σ_{ij} . The sum $\sum_{j=1}^m \sigma_{ij} W_j(t)$, therefore, is also normally distributed as the sum of independent normal variables, with mean 0 and variance $\sum_{j=1}^m \sigma_{ij}^2 t$.

We first express the expected value of $P_i(t)$:

$$\begin{aligned} E[P_i(t)] &= E \left[p_i \exp \left(\tilde{b}_i t + \sum_{j=1}^m \sigma_{ij} W_j(t) - \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2 t \right) \right] \\ &= p_i e^{b_i t} E \left[\exp \left(\sum_{j=1}^m \sigma_{ij} W_j(t) - \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2 t \right) \right] \end{aligned}$$

Using the moment-generating function property for a normally distributed variable X with mean μ and variance σ^2 , $E[e^X] = e^{\mu + \frac{1}{2}\sigma^2}$, and noting $\mu = 0$ and $\sigma^2 = \sum_{j=1}^m \sigma_{ij}^2 t$, we find:

$$E \left[\exp \left(\sum_{j=1}^m \sigma_{ij} W_j(t) \right) \right] = \exp \left(0 + \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2 t \right) = \exp \left(\frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2 t \right)$$

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Thus, substituting back, we have:

$$E[P_i(t)] = p_i e^{\tilde{b}_i t} \exp\left(-\frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2 t + \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2 t\right) = p_i e^{\tilde{b}_i t}$$

Therefore, the expected value of $P_i(t)$ is indeed $p_i e^{\tilde{b}_i t}$, as stated in the lemma.

Now let's prove the second point of the lemma. We can compute the expected value of the square of the stock price:

$$\begin{aligned} E[P_i(t)^2] &= E\left[p_i^2 \exp\left(2\left(\tilde{b}_i t + \sum_{j=1}^m \sigma_{ij} W_j(t) - \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2 t\right)\right)\right] \\ &= p_i^2 e^{2\tilde{b}_i t} E\left[\exp\left(2\sum_{j=1}^m \sigma_{ij} W_j(t) - \sum_{j=1}^m \sigma_{ij}^2 t\right)\right] \end{aligned}$$

Since $\sum_{j=1}^m \sigma_{ij} W_j(t)$ is normally distributed with mean 0 and variance $\sum_{j=1}^m \sigma_{ij}^2 t$, the moment-generating function gives us:

$$E\left[\exp\left(2\sum_{j=1}^m \sigma_{ij} W_j(t)\right)\right] = \exp\left(2^2 \cdot \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2 t\right) = \exp\left(2\sum_{j=1}^m \sigma_{ij}^2 t\right)$$

Therefore we can compute:

$$\begin{aligned} E[P_i(t)^2] &= p_i^2 e^{2\tilde{b}_i t} \exp\left(\sum_{j=1}^m \sigma_{ij}^2 t\right) \\ \text{Var}(P_i(t)) &= E[P_i(t)^2] - (E[P_i(t)])^2 = p_i^2 e^{2\tilde{b}_i t} \left(\exp\left(\sum_{j=1}^m \sigma_{ij}^2 t\right) - 1\right) \end{aligned}$$

Thus, we have derived the variance of $P_i(t)$ as specified in the lemma.

Finally, we can prove the third and last point of the lemma. First, we can recognize that for each j , the following process is a martingale:

$$Y_j(t) = \exp\left(c_j W_j(t) - \frac{1}{2} c_j^2 t\right)$$

Given the independence of $W_j(t)$ across different j , the product of independent martingales $X_t = a \prod_{j=1}^m Y_j(t)$ is also a martingale. This stems from the fact that the expectation of a product of independent martingales, conditioned on the past, is the product of their conditional expectations.

We can compute the conditional expectation using the independence of increments of Brownian motions and the martingale property of each $Y_j(t)$:

$$E(X_t | \mathcal{F}_s) = a E\left(\prod_{j=1}^m Y_j(t) | \mathcal{F}_s\right) = a \prod_{j=1}^m E(Y_j(t) | \mathcal{F}_s)$$

Since the increments $W_j(t) - W_j(s)$ are independent of \mathcal{F}_s and $Y_j(t)$ is a martingale, we have: $E(Y_j(t) | \mathcal{F}_s) = Y_j(s)$. Therefore,

$$E(X_t | \mathcal{F}_s) = a \prod_{j=1}^m Y_j(s) = X_s$$

which confirms that X_t is a martingale. □

Following the insights from *Lemma 9*, we can redefine our model for the stock price, $P_i(t)$:

$$P_i(t) := p_i \cdot e^{b_i t} \cdot \exp \left(\sum_{j=1}^m \left(\sigma_{ij} W_j(t) - \frac{1}{2} \sigma_{ij}^2 t \right) \right) \quad (3.26)$$

This model suggests that each stock price $P_i(t)$ is composed of:

1. The mean stock price: $p_i \cdot e^{b_i t}$
2. A martingale with expectation 1 which models the fluctuation of the stock price around its mean value: $\exp \left(\sum_{j=1}^m \left(\sigma_{ij} W_j(t) - \frac{1}{2} \sigma_{ij}^2 t \right) \right)$

Definition 3.3.1 (Volatility Matrix). The volatility matrix captures the volatilities that affect each stock through various market factors

$$\sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nm} \end{pmatrix}$$

Additionally, any stochastic process $P_i(t)$ modeled in this form is referred to as a geometric Brownian motion with drift vector b and a volatility given by each row $\sigma_i = (\sigma_{i1}, \dots, \sigma_{im})$ of the matrix σ .

Definition 3.3.2 (Drift Vector). For a set of financial assets indexed by i in a model with n dimensions, the drift vector is defined as follows:

$$b = (b_1, \dots, b_n)' \quad (3.27)$$

where each component b_i represents the expected rate of return of the i -th asset.

3.4 Itô Calculus

In this section, we focus on the Itô Calculus, an indispensable mathematical framework in the field of stochastic analysis. Developed by Kiyosi Itô in 1944, extends traditional calculus into the domain of stochastic processes, providing tools to model and analyze the evolution of random systems over time. This calculus has become a cornerstone in financial mathematics, especially in the modeling of asset prices and the valuation of financial derivatives.

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3.4.1 Introduction

Itô calculus, is essential for modeling stochastic processes with randomness. The Itô integral extends traditional integration into stochastic processes evolving in continuous time. This extension is crucial for converting discrete-time martingales into continuous-time frameworks, allowing for precise modeling of stochastic processes.

For the definition of the Itô integral we will follow a systematic approach divided into three distinct steps:

1. Identify a class of simple integrands for which the definition of the integral is straightforward and intuitive.
2. Use a continuity argument to broaden the definition of the integral.
3. Establish that there is a continuous martingale that corresponds to the value of the Itô integral.

3.4.2 First Step: Defining the Itô Integral

Definition 3.4.1 (Borel Set). A Borel Set is any set in a topological space that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement. For a topological space X , the collection of all Borel sets on X forms a σ -algebra, known as the Borel algebra or Borel σ -algebra. The Borel algebra on X is the smallest σ -algebra containing all open sets (or, equivalently, all closed sets).

Definition 3.4.2 (Measurable Space). A measurable space is a pair (X, \mathcal{A}) consisting of a set X and a σ -algebra \mathcal{A} on X .

Definition 3.4.3 (Measurability of a Function). A function $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ between two measurable spaces is called measurable if for every set B in the σ -algebra \mathcal{B} on Y , the preimage of B under f ,

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \quad (3.28)$$

is in the σ -algebra \mathcal{A} on X .

Example 3.4.1 (Measurable Function). For a real-valued function

$$f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra on \mathbb{R} with the usual topology, f is measurable if the preimage of any Borel set is a Borel set.

Definition 3.4.4 (Adaptability). A function $f(\omega, t)$ is adapted to a filtration $\{F_t\}_{t \geq 0}$ if for each fixed time t , the function $f(\cdot, t)$ is measurable with respect to the σ -algebra F_t at that time.

For the first step of the definition we will consider the following:

1. Let \mathcal{B} denote the Borel σ -algebra on $[0, T]$.

2. Consider $\{F_t\}_{t \geq 0}$ as the standard filtration associated with Brownian motion.
3. Define $\mathcal{F}_T \times \mathcal{B}$ as the smallest σ -algebra containing all product sets $A \times B$ where $A \in F_t$ and $B \in \mathcal{B}$.

We define the space \mathcal{H}^2 as the class of all measurable adapted functions f that are square-integrable over $[0, T]$, formally,

$$\mathcal{H}^2 = \left\{ f \in L^2([0, T] \times \Omega) \mid \mathbb{E} \left[\int_0^T f^2(u, t) dt \right] < \infty \right\} \quad (3.29)$$

For a simple function $f(\omega, t)$ in \mathcal{H}^2 of the form $f(\omega, t) = \sum_{i=0}^{n-1} a_i(\omega) 1_{(t_i, t_{i+1}]}(t)$, where $a_i \in F_{t_i}$, and $0 = t_0 < t_1 < \dots < t_n \leq T$, the Itô integral $I(f)$ is defined by the linear combination of increments of Brownian motion:

$$I(f)(\omega) = \sum_{i=0}^{n-1} a_i(\omega) (B_{t_{i+1}} - B_{t_i}) \quad (3.30)$$

This construction ensures that the Itô integral $I(f)$ is well-defined and captures the stochastic nature of the integrand and integral path. This definition is consistent and unambiguous across equivalent representations of f .

3.4.3 Second Step: Extension and Properties of the Itô Integral

The Itô integral mapping $I : \mathcal{H}_0^2 \rightarrow L^2(dP)$ is continuous, implying it preserves the convergence of sequences. This is crucial as it guarantees the extendibility of I from \mathcal{H}_0^2 to \mathcal{H}^2 .

First, we define the spaces involved:

1. \mathcal{H}_0^2 consists of simpler integrand forms, typically step functions that are adapted to the filtration of the underlying probability space.
2. $L^2(dP)$ is the space of square-integrable random variables with respect to the probability measure P .

The continuity of the Itô integral mapping ensures that this integral can be extended from \mathcal{H}_0^2 to the larger space \mathcal{H}^2 , accommodating a wider class of integrands.

Lemma 10 (Itô's Isometry). *For any function f in \mathcal{H}_0^2 , the Itô integral satisfies:*

$$\|I(f)\|_{L^2(dP)} = \|f\|_{L^2(dP \times dt)} \quad (3.31)$$

The proof of the theorem is available at [21].

This identity indicates that I maintains the norm of f from \mathcal{H}_0^2 in $L^2(dP)$, facilitating the extension to more complex functions.

Lemma 11 (Density in \mathcal{H}^2). *For any function f in \mathcal{H}^2 , there exists a sequence $\{f_n\}$ in \mathcal{H}_0^2 such that:*

$$\|f - f_n\|_{L^2(dP \times dt)} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.32)$$

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The proof of the theorem is available at [23].

This lemma demonstrates the ability to approximate any complex function in \mathcal{H}^2 with simpler functions in \mathcal{H}_0^2 , confirming the robustness of the integral's definition.

To define the Itô integral for any f in \mathcal{H}^2 , we take the limit of integrals of the approximating sequences: $I(f) \equiv \lim_{n \rightarrow \infty} I(f_n)$ where each $I(f_n)$ is defined by simpler forms, ensuring the integral's existence for all f in \mathcal{H}^2 and preserving properties like linearity and isometry.

The previous definition states that $I(f)$ is the unique limit in $L^2(dP)$ such that $\|I(f_n) - I(f)\|_{L^2(dP)}$ converges to zero as n increases. Verifying that the choice of sequence $\{f_n\}$ doesn't affect the limit ensures the integral's consistency.

3.4.4 Third Step: Connecting the Itô Integral with Stochastic Processes

The Itô integral mapping, $I : \mathcal{H}^2 \rightarrow L^2(dP)$, is critical for defining the integral over the interval $[0, T]$. To integrate stochastic processes across this interval, we introduce a method that ensures continuity and adaptability of these integrals.

First we will define a time-dependent indicator function:

$$m_t(\omega, s) = \begin{cases} 1 & \text{if } s \in [0, t], \\ 0 & \text{otherwise} \end{cases} \quad (3.33)$$

For any function f in $\mathcal{H}^2[0, T]$, the product $m_t f$ also belongs to $\mathcal{H}^2[0, T]$, and $I(m_t f)$ is a well-defined element of $L^2(dP)$.

A natural candidate for a process version of Itô's integral is $X'_t(\omega) = I(m_t f)(\omega)$. However, this approach has limitations since $I(m_t f)$ can be defined arbitrarily on sets $A_t \in \mathcal{F}_t$ with $P(A_t) = 0$, affecting continuity.

The value of $I(m_t f)$ can vary across negligible subsets, causing discontinuities in X'_t . Despite these issues, careful application of Doob's Maximal Inequality [12] ensures the existence of a unique continuous martingale X_t , which agrees with X'_t with probability one for all $t \in [0, T]$.

Theorem 12 (Itô Integrals as Martingales). *For any f in $\mathcal{H}^2[0, T]$, there exists a continuous martingale X_t with respect to the Brownian filtration \mathcal{F}_t such that:*

$$P(\{\omega : X_t(\omega) = I(m_t f)(\omega)\}) = 1 \text{ for all } t \in [0, T] \quad (3.34)$$

The proof of this theorem can be found in [27].

This theorem ensures that the Itô integral of f over any interval $[0, t]$ is captured by a well-defined continuous martingale, enhancing its applicability in modeling continuous stochastic processes.

Thus, the Itô integral for any function f over $[0, T]$ is defined as:

$$\int_0^t f(u, t) dB_u := X_t, \quad \text{for } t \in [0, T] \quad (3.35)$$

This definition completes the integral's specification and aligns it with the underlying martingale structure.

3.5 The Itô Formula

The Itô formula is a fundamental tool for analyzing and simulating functions of stochastic processes, encapsulating both drift and diffusion characteristics. We now introduce the most important formula of Itô calculus:

Theorem 13 (Itô's Formula). *Consider a one-dimensional Brownian motion W_t and let X_t be a real-valued Itô process defined by:*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s \quad (3.36)$$

where K_s and H_s are processes adapted to the filtration generated by W_t . If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function that is twice continuously differentiable, then for all $t \geq 0$, the value of f at X_t can be expressed as:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s \quad (3.37)$$

where $\langle X \rangle_s$ denotes the quadratic variation of X .

The formula decomposes as follows:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) K_s ds + \int_0^t f'(X_s) H_s dW_s + \frac{1}{2} \int_0^t f''(X_s) H_s^2 ds \quad (3.38)$$

assuring that each component is well-defined almost surely.

We can notice that the Itô Formula diverges from the Fundamental Theorem of Calculus[26] due to the inclusion of the term: $\frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s$. This term, highlights the adjustments necessary when dealing with stochastic processes.

Remark (Notation). The differential form of Itô's formula for a twice continuously differentiable function f evaluated at a stochastic process X_t is given by:

$$df(X_t) = f'(X_t) dX_t + \frac{1}{2} f''(X_t) d\langle X \rangle_t \quad (3.39)$$

Proof. We will prove the Itô Formula following a sequence of six steps:

Step 1: Localization. This initial step ensures that all necessary expectations exist and can be interchanged smoothly. We define the following stochastic processes:

$$N_t = \int_0^t H_s dW_s, \quad B_t = \int_0^t K_s ds, \quad \hat{B}_t = \int_0^t |K_s| ds$$

The stopping times $\{T_n\}_{n \in \mathbb{N}}$ are defined to ensure the process remains within controllable bounds:

$$T_n = \inf \left\{ t \geq 0 \mid |N_t| \geq n \text{ or } B_t \geq n \text{ or } \int_0^t H_s^2 ds \geq n \right\}$$

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if $|X_0| < n$, and $T_n = \infty$ if not. This definition guarantees that the sequence of stopping times is increasing and converges to infinity almost surely.

For the stopped process, the Itô's formula can be applied as follows: $X_t^{(n)} = X_{t \wedge T_n}$ where $t \wedge T_n$ denotes the minimum of t and T_n . This restricts the process to manageable intervals, simplifying the proof of Itô's formula.

Step 2: Taylor Expansion in Itô's Formula. Consider a fixed time $t > 0$ and a partition $\pi = \{t_0, t_1, \dots, t_m\}$ of the interval $[0, t]$, where $t_0 = 0$ and $t_m = t$. Let's denote the maximum step size in the partition as: $\|\pi\| = \max_{1 \leq k \leq m} |t_k - t_{k-1}|$. By applying a Taylor expansion to each difference in the sum over the partition, we address the expression:

$$f(X_t) - f(X_0) = \sum_{k=1}^m (f(X_{t_k}) - f(X_{t_{k-1}}))$$

Expanding f using Taylor's formula, we find:

$$f(X_t) - f(X_0) = \sum_{k=1}^m f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) + \frac{1}{2} \sum_{k=1}^m f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2$$

where η_k is a point between $X_{t_{k-1}}$ and X_{t_k} , typically chosen according to the mean value theorem for Taylor expansions. The goal is to show that as the partition gets finer ($\|\pi\| \rightarrow 0$), these sums converge to the integrals found in the Itô formula:

$$\int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s$$

Step 3: Analysis of the Linear Term in Itô's Formula. The linear term is decomposed into two parts:

$$\begin{aligned} \sum_{k=1}^m f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) &= \sum_{k=1}^m f'(X_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) + \sum_{k=1}^m f'(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) \\ &= A_1(\pi) + A_2(\pi) \end{aligned}$$

where B_t and M_t are process components of the Itô integral, typically involving drift and stochastic parts, respectively. First we will study the convergence of $A_1(\pi)$. As the partition $\|\pi\| \rightarrow 0$, $A_1(\pi)$ converges almost surely and in L^1 to the Lebesgue integral:

$$\int_0^t f'(X_s) dB_s = \int_0^t f'(X_s) K_s ds$$

Second, to understand the convergence of $A_2(\pi)$, we approximate $f'(X_s)$ using a simple process:

$$Y_t^\pi = f'(X_0) + \sum_{k=1}^m f'(X_{t_{k-1}})1_{(t_{k-1}, t_k]}(s)$$

Applying Itô's isometry, see *Lemma 10*, we find:

$$E \left[\left(\int_0^t (f'(X_s) - Y_t^\pi) H_s dW_s \right)^2 \right] = E \left[\int_0^t (f'(X_s) - Y_t^\pi)^2 H_s^2 ds \right]$$

which, by the dominated Convergence Theorem [24], goes to zero as $\|\pi\| \rightarrow 0$. As a result, we can state that $\|\pi\|$ diminishes, $A_2(\pi)$ converges in L^2 to: $\int_0^t f'(X_s)H_s dW_s$. From the convergence of $A_1(\pi)$ and $A_2(\pi)$, the addition of these terms aligns with Itô's integral, converging in L^1 :

$$\int_0^t f'(X_s)K_s ds + \int_0^t f'(X_s)H_s dW_s$$

Step 4: Analysis of The Quadratic Term. The quadratic term in the Itô formula is decomposed as follows: $\sum_{k=1}^m f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2$ which can be further expressed in terms of the drift and diffusion components:

$$\sum_{k=1}^m f''(\eta_k)(B_{t_k} - B_{t_{k-1}} + M_{t_k} - M_{t_{k-1}})^2$$

Breaking it down, we get:

$$\sum_{k=1}^m f''(\eta_k)[(B_{t_k} - B_{t_{k-1}})^2 + 2(B_{t_k} - B_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) + (M_{t_k} - M_{t_{k-1}})^2]$$

with:

1. $I_1(\pi) = \sum_{k=1}^m f''(\eta_k)(B_{t_k} - B_{t_{k-1}})^2$
2. $I_2(\pi) = \sum_{k=1}^m f''(\eta_k)2(B_{t_k} - B_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})$
3. $I_3(\pi) = \sum_{k=1}^m f''(\eta_k)(M_{t_k} - M_{t_{k-1}})^2$

Given that B_t and M_t are bounded, we affirm that $\sum_{k=1}^m |B_{t_k} - B_{t_{k-1}}| \leq \hat{B}_t < C$ and similarly for M_t . For $\|\pi\| \rightarrow 0$, these sums converge to the integrals:

$$\int_0^t f''(X_s)(dX_s)^2 \approx \int_0^t f''(X_s)(dX_s)^2.$$

Considering the term $I_3(\pi)$ we can define:

$$I_3^*(\pi) = \sum_{k=1}^m f''(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2, \quad I_4(\omega) = \sum_{k=1}^m f''(X_{t_{k-1}}) \int_{t_{k-1}}^{t_k} H_s^2 ds$$

This yields:

$$|I_3(\pi) - I_3^*(\pi)| \leq \max_{1 \leq k \leq m} |f''(\eta_k) - f''(X_{t_{k-1}})| \left(\sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^2 \right)$$

Then the Cauchy-Schwarz Inequality [25] implies:

$$E[(I_3(\pi) - I_3^*(\pi))^2] \leq E \left[\left(\max_{1 \leq k \leq m} |f''(\eta_k) - f''(X_{t_{k-1}})| \right)^2 \right] E \left[\left(\sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^2 \right)^2 \right]$$

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By noting that we have $E[(M_{t_k} - M_{t_{k-1}})^2] = E\left[\int_{t_{k-1}}^{t_k} H_s^2 ds\right]$ the following relation is valid:

$$E[(I_3^*(\pi) - I_4(\pi))^2] \leq \|f\|_\infty^2 \cdot E\left[\left(\sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 ds\right)^2\right]$$

This equals

$$\|f\|_\infty^2 \cdot E\left[\left(\sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 ds\right)^2\right]$$

Due to the inequality $(a - b)^2 \leq 2(a^2 + b^2)$, the above relation results in

$$E[(I_3^*(\pi) - I_4(\pi))^2] \leq 2 \cdot \|f\|_\infty^2 \cdot E\left[\left(\sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^2\right)^2 + \left(\sum_{k=1}^m \int_{t_{k-1}}^{t_k} H_s^2 ds\right)^2\right]$$

The dominated convergence theorem applies here, yielding:

$$E\left[\left(\sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^2\right)^2\right] \xrightarrow{\|\pi\| \rightarrow 0} 0$$

confirming that the differences in the martingale increments M_t vanish in expectation as the partition becomes finer. As $\|\pi\| \rightarrow 0$, $I_4(\omega)$ converges almost surely to $\int_0^t f''(X_s) H_s^2 ds$. Consequently, the expected difference between $I_3(\pi)$ and the integral also tends to zero:

$$E[(I_3(\pi) - I_4(\pi))^2] \xrightarrow{\|\pi\| \rightarrow 0} 0$$

which implies that:

$$I_3(\pi) \xrightarrow{\|\pi\| \rightarrow 0} \int_0^t f''(X_s) H_s^2 ds \quad \text{in } L^2$$

Step 5: Conclusion Convergence of the Itô Integral. As the mesh of the partition $\|\pi\|$ tends to zero, the discrete sums converge to their continuous counterparts in the Itô integral. Specifically, the linear term converges to:

$$\int_0^t f'(X_s) K_s ds + \int_0^t f'(X_s) H_s dW_s$$

and the quadratic term converges to:

$$\int_0^t f''(X_s) H_s^2 ds \quad \text{in } L^1$$

For the stopped process, sub-sequences π_k of partitions of $[0, t]$ ensure that the sums converge correctly to these limits, almost surely. This establishes that the outcomes of Itô's formula at each time t are almost surely equivalent. The continuous nature of both sides makes them indistinguishable for all t , confirming the formula's robustness and precision. \square

3.5.1 Multi-dimensional Itô Formula

Theorem 14 (Multidimensional Itô Formula). *Let $Y(t) = (X_1(t), \dots, X_n(t))$ be an n -dimensional Itô process where each component $X_i(t)$ is defined by:*

$$X_i(t) = X_i(0) + \int_0^t K_i(s) ds + \sum_{j=1}^m \int_0^t H_{ij}(s) dW_j(s) \quad (3.40)$$

with $W(t) = (W_1(t), \dots, W_m(t))$ representing an m -dimensional Brownian motion. Assume $f : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a $C^{1,2}$ -function, continuously differentiable in time and twice continuously differentiable in the spatial variables.

Then, for f applied to $Y(t)$, the value at time t can be expressed as:

$$\begin{aligned} f(t, X_1(t), \dots, X_n(t)) &= f(0, X_1(0), \dots, X_n(0)) \\ &+ \int_0^t \left(f_t + \sum_{i=1}^n f_{x_i} K_i(s) + \frac{1}{2} \sum_{i,j=1}^n f_{x_i x_j} H_{ij}(s) H_{ji}(s) \right) (s, X_1(s), \dots, X_n(s)) ds \\ &+ \sum_{i=1}^n \int_0^t f_{x_i}(s, X_1(s), \dots, X_n(s)) H_i(s) dW_i(s) \end{aligned} \quad (3.41)$$

where f_t , f_{x_i} , and $f_{x_i x_j}$ are the partial derivatives of f with respect to time and the spatial variables, respectively.

This theorem generalizes the one-dimensional Itô formula to an n -dimensional setting, accommodating the complexities introduced by having multiple stochastic drivers in the form of a multidimensional Brownian motion.

Proof. The proof for the multi-dimensional Itô formula closely mirrors that of the one-dimensional case. However, there is a small difference in *Step 2*. Let us consider a simplified scenario where $n = 1$ for clarity:

$$\begin{aligned} f(t_k, X(t_k)) - f(t_{k-1}, X(t_{k-1})) &= [f(t_k, X(t_k)) - f(t_k, X(t_{k-1}))] \\ &+ [f(t_k, X(t_{k-1})) - f(t_{k-1}, X(t_{k-1}))] \\ &= f_t(t_k, X(t_{k-1}))(t_k - t_{k-1}) + f_x(t_k, X(t_{k-1}))(X(t_k) - X(t_{k-1})) \\ &\quad + \frac{1}{2} f_{xx}(t_k, \eta_k)(X(t_k) - X(t_{k-1}))^2 \end{aligned}$$

where η_k lies between t_{k-1} and t_k in accordance with Taylor's theorem applied to the function f . \square

3.6 The Stock Price Equation

We will consider now a simple continuous-time market model with a bond and a single stock, influenced by a one-dimensional Brownian motion. The stock price at time t is modeled by:

$$P(t) = p \cdot \exp \left(\left(b - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \quad (3.42)$$

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To demonstrate an application of Itô's Lemma, we define X_t as follows:

$$X_t = \int_0^t \left(b - \frac{1}{2}\sigma^2 \right) ds + \int_0^t \sigma dW_s \quad (3.43)$$

where $f(x) = p \cdot e^x$ is the function used to transform the linear Brownian motion into the log-normal distribution describing the stock price. Applying Itô's formula to $f(X_t)$, we obtain:

$$\begin{aligned} f(X_t) &= p + \int_0^t f'(X_s) \left(b - \frac{1}{2}\sigma^2 \right) ds + \frac{1}{2} \int_0^t f''(X_s) \sigma^2 ds + \int_0^t f'(X_s) \sigma dW_s \\ f(X_t) &= p + \int_0^t P(s) \left(b - \frac{1}{2}\sigma^2 \right) ds + \frac{1}{2} \int_0^t P(s) \sigma^2 ds + \int_0^t P(s) \sigma dW_s \end{aligned}$$

This results in the expression for $P(t)$:

$$P(t) = p + \int_0^t P(s) b ds + \int_0^t P(s) \sigma dW_s \quad (3.44)$$

which is the stock price equation. The differential form of the stock price is given by: $dP(t) = P(t)(bdt + \sigma dW_t)$, $P(0) = p$. This stochastic differential equation describes the dynamics of the stock price, where $P(t)$ is the solution of this equation. This formulation can also be extended to time-dependent parameters $b(t)$ and $\sigma(t)$, where:

$$X_t = \int_0^t \left(b(s) - \frac{1}{2}\sigma^2(s) \right) ds + \int_0^t \sigma(s) dW_s \quad (3.45)$$

Hence, the stock price equation becomes: $dP(t) = P(t)(b(t)dt + \sigma(t)dW_t)$

Example 3.6.1. Consider an initial stock price $p = 100$ €, a drift coefficient $b = 0.05$, and a volatility coefficient $\sigma = 0.2$. Let W_t at $t = 1$ be 0.3 (a sample value from the standard Brownian motion). We will simulate the stock price at $t = 1$ year with a sample path of Brownian motion W_t .

Using the stock price equation:

$$\begin{aligned} P(1) &= 100 \cdot \exp \left(\left(0.05 - \frac{1}{2}(0.2)^2 \right) \cdot 1 + 0.2 \cdot 0.3 \right) = 100 \cdot \exp \left((0.05 - 0.02) \cdot 1 + 0.06 \right) \\ &= 100 \cdot \exp(0.03 + 0.06) = 100 \cdot \exp(0.09) \approx 100 \cdot 1.09417 = 109.42 \text{€} \end{aligned}$$

Starting with an initial stock price of 100 €, the model predicts a price increase to approximately 109.42 € after one year.

3.7 Market Completeness

In this section, we will consider the scenario where the dimension of the underlying Brownian motion equals the number of stocks ($d = m$), and investors only have access to historical and current data. Formally, this means using the Brownian filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ as our informational basis.

We aim to explore the range of possible final wealth values, $X(T)$, that an investor can achieve by the end of a period, starting with initial capital x . Notably, we will show that any target final wealth $X(T)$ can be attained with a sufficiently large initial capital x . First, we introduce the following notations:

1. $\gamma(t) = \exp\left(-\int_0^t r(s) ds\right)$
2. $\theta(t) = \sigma^{-1}(t)(b(t) - r(t)\mathbf{1})$
3. $Z(t) = \exp\left(\int_0^t \theta(s)' dW(s) - \frac{1}{2} \int_0^t \|\theta(s)\|^2 ds\right)$
4. $H(t) = \gamma(t) \cdot Z(t)$

The function $\theta(t)$ represents a relative risk premium for stock investments. The process $H(t)$, which is positive, continuous, and progressively measurable with respect to the filtration $\{F_t\}_{t \in [0, T]}$, is the unique solution to the stochastic differential equation:

$$dH(t) = -H(t)(r(t) dt + \theta(t)' dW(t)), \quad H(0) = 1 \quad (3.46)$$

Theorem 15 (Completeness of the Market). *This theorem is divided in two parts:*

1. **Admissibility Condition:** *consider a self-financing pair (π, c) , with π being a portfolio process and c a consumption process, which is admissible for an initial wealth $x \geq 0$. This implies that the strategy must not lead to bankruptcy and must satisfy certain risk and return requirements. For an admissible strategy from the set $A(x)$, the wealth process $X(t)$ that evolves satisfies:*

$$E \left[\int_0^t H(s)c(s) ds + H(t)X(t) \right] \leq x \quad \forall t \in [0, T] \quad (3.47)$$

where $H(t)$ is a discounting process that adjusts future cash flows to their present value.

2. **Existence of a Feasible Portfolio Process:** *Let B be a non-negative \mathcal{F}_T -measurable random variable representing a target wealth at time T , and $c(t)$ a consumption process. The condition for the expected total wealth is:*

$$x = E \left[\int_0^T H(s)c(s) ds + H(T)B \right] < \infty \quad (3.48)$$

Under these conditions, there exists a portfolio process $\pi(t)$, for $t \in [0, T]$, such that $(\pi, c) \in A(x)$ and the corresponding wealth process $X(T)$ will almost surely be B at time T .

Proof. First we will prove the first part of the theorem: given a self-financing pair (π, c) from the admissible set $A(x)$ for initial wealth $x \geq 0$, consider the wealth process $X(t)$ and the discounting process $H(t)$ defined as follows:

$$\begin{aligned} dH(t) &= -H(t)(r(t) dt + \theta(t)' dW(t)), \quad H(0) = 1, \\ dX(t) &= (r(t)X(t) - c(t)) dt + X(t)\pi(t)' (\sigma(t) dW(t) + (b(t) - r(t)\mathbf{1}) dt). \end{aligned}$$

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Applying the product rule to the process $H(t)X(t)$, we have:

$$d(H(t)X(t)) = H(t) dX(t) + X(t) dH(t) + d\langle H, X \rangle_t,$$

where: $d\langle H, X \rangle_t = H(t)X(t)\pi(t)'\sigma(t)\theta(t) dt$. Expanding the differentials:

$$\begin{aligned} d(H(t)X(t)) &= H(t) [(r(t)X(t) - c(t)) dt + X(t)\pi(t)'(\sigma(t) dW(t) + (b(t) - r(t)\mathbf{1}) dt)] \\ &\quad + X(t) [-H(t)(r(t)dt + \theta(t)'dW(t))] + H(t)X(t)\pi(t)'\sigma(t)\theta(t) dt \end{aligned}$$

Which simplifies to:

$$d(H(t)X(t)) = -H(t)c(t) dt + \text{other terms involving } dt \text{ and } dW(t).$$

Integrating from 0 to t , we obtain: $H(t)X(t) + \int_0^t H(s)c(s) ds = x + \text{martingale terms}$. Since the martingale terms have expected value zero, taking expectations gives: $E[H(t)X(t) + \int_0^t H(s)c(s) ds] = x$. Therefore, for all t in $[0, T]$, we have:

$$E[H(t)X(t) + \int_0^t H(s)c(s) ds] \leq x,$$

For the second part of the theorem we will define first the wealth process $X(t)$:

$$X(t) = \frac{1}{H(t)} E \left[\int_t^T H(s)c(s) ds + H(T)B \mid \mathcal{F}_t \right].$$

This definition ensures that $X(t)$ is \mathcal{F}_t -measurable, $X(T) = B$ almost surely, and $X(t) \geq 0$. Since $X(0) = x$ almost surely, the initial wealth condition is satisfied, consistent with the unconditional expectation when \mathcal{F}_0 is trivial.

Now define the process $M(t)$ as: $M(t) = X(t) \cdot H(t) + \int_0^t H(s)c(s) ds$. Given that $X(t)$ is defined in terms of expectations involving $H(t)$ and $c(t)$, it follows that:

$$M(t) = E \left[\int_0^T H(s)c(s) ds + H(T)B \mid \mathcal{F}_t \right].$$

$M(t)$ can be shown to be a martingale due to the tower property of conditional expectations and the properties of the Itô integral. We write $M(t)$ in differential form: $dM(t) = d(X(t)H(t)) + H(t)c(t) dt$. Utilizing the differential $d(X(t)H(t))$ and expanding using the product rule, we derive:

$$dM(t) = X(t) dH(t) + H(t) dX(t) + d\langle X, H \rangle_t + H(t)c(t) dt,$$

where $\langle X, H \rangle_t$ is the quadratic covariation of X and H . Since $H(t)$ and $X(t)$ are adapted to the filtration and involve Itô integrals, $M(t)$ simplifies to an integral with respect to $dW(t)$, showing that $M(t)$ is a martingale.

Finally, using the martingale representation theorem, $M(t)$ can be expressed as:

$$M(t) = x + \int_0^t \Psi(s) dW(s),$$

where $\Psi(t)$ is a progressively measurable process with respect to \mathcal{F}_t . The condition $\int_0^T \|\Psi(t)\|^2 dt < \infty$ almost surely ensures that $M(t)$ is well-defined and a true martingale. \square

Chapter 4

Deciphering the Black-Scholes Model: A Comprehensive Guide to Option Pricing

This chapter delves into the Black-Scholes Model, an essential framework for option pricing. Beginning with the fundamentals of option pricing, it progresses to the replication principle. Then the analysis then deepens with the Black-Scholes Formula, detailing its derivation, pivotal remarks, and implications for trading strategies. This exploration aims to provide a comprehensive understanding of the model's role and application in financial markets, highlighting its critical importance in both theory and practical financial decision-making.

Options are financial derivatives deriving their value from underlying assets. Their prominence surged with the establishment of the Chicago Board Options Exchange (CBOE) in 1973, which enhanced liquidity and accessibility for investors. Options are now integral to trading strategies, offering tools for hedging, speculation, and portfolio optimization [18]. Today, options span various assets, including equities, bonds, and commodities, and are traded on major exchanges like NYSE, Euronext, London, Frankfurt, and Tokyo Stock Exchanges. They are crucial in global financial operations. Historically, options date back to ancient Greece, but formal option pricing theory developed in the 20th century. Before modern financial theory, option pricing relied on rudimentary strategies and intuition. Options provide investors with flexibility, leverage, and risk management. They allow control over larger amounts of underlying assets with smaller investments and offer insurance against adverse price movements, ensuring their continued relevance in financial markets [19].

4.1 Replication Principle

In this section, we explore the Replication Principle, a key concept in financial engineering and derivative pricing. This principle shows how to construct a portfolio to replicate a derivative's cash flows, determining its fair market value.

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The goal is to find a logical price, \hat{p} for a financial derivative. A future payout B at time T has a present value of $e^{-rT} \cdot B$, assuming continuous compounding at a constant risk-free rate r .

If the risk-free rate $r(t)$ is a stochastic process, we must calculate the expected present value by integrating $r(s)$ over the interval from 0 to T , leading to the expected present value formula:

$$E\left(e^{-\int_0^T r(s) ds} \cdot B\right) \quad (4.1)$$

Allocating capital equal to this expected present value in the risk-free bond at $t = 0$ ensures a future value of B at T . Thus, the fair price of the option today should equal the expected value of the option's future payout, discounted at the bond price. For a European call option, this is suggested by:

$$\hat{p} = E\left(e^{-\int_0^T r(s) ds} \cdot (P_1(T) - K)^+\right) \quad (4.2)$$

Definition 4.1.1 (Arbitrage). Arbitrage is a strategy that exploits the price differences of the same asset across different markets. Investors buy an asset where it is priced lower and simultaneously sell it where it is priced higher with the purpose of achieving a risk-free profit due to the price disparities.

Example 4.1.1 (Pricing Options in a Discrete-Time Market Setup). Consider a bond with price $B_0(t)$ and a stock with price $P_1(t)$. We neglect the interest rate and focus on a European call option with strike price $K = 2$. At time T , the stock can either quadruple or halve in value, with probabilities a and $1 - a$, respectively. The bond's price remains unchanged. At $t = 0$, $B_0(0) = 1$ and $P_1(0) = 1$.

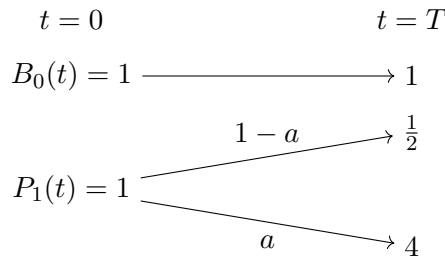


Figure 4.1: Price diagram of the example in a discrete-time setting

The expected payoff of the call option at time T is given by

$$E((P_1(T) - K)^+) = 4a \cdot (P_1(T) - K) + (1 - a) \cdot 0 = 4a$$

assuming $P_1(T) > K$ in the first term, where the stock price quadruples, and $P_1(T) = \frac{1}{2}P_1(0)$ in the second term, where it halves in value.

In this market, the option's price should be determined by a replication strategy using the stock and bond to replicate the option's payoff. To establish the replicating portfolio, we consider the following system based on the two possible outcomes:

$$\varphi_0(0) \cdot B_0(T) + \varphi_1(0) \cdot 4P_1(0) = (4P_1(0) - K)^+$$

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$$\varphi_0(0) \cdot B_0(T) + \varphi_1(0) \cdot \frac{1}{2}P_1(0) = \left(\frac{1}{2}P_1(0) - K\right)^+$$

The solutions to this system will provide the positions $\phi_0(0)$ and $\phi_1(0)$ in the bond and stock that replicate the option's payoff. Such way we will obtain:

$$X(T) = \varphi_0(0)B_0(T) + \varphi_1(0)P_1(T) = (P_1(T) - K)^+$$

And as the option price we set: $p = \varphi_0(0)B_0(0) + \varphi_1(0)P_1(0)$

The option price at $t = 0$ should match the initial investment for the replicating strategy, $(\phi_0(0), \phi_1(0))$, to prevent arbitrage and establish p as the fair price. If the market price \hat{p} differs, it creates arbitrage opportunities where a trader can profit risk-free by buying at \hat{p} and selling at p , or the reverse. In efficient markets, arbitrage is rapidly exploited, making such opportunities short-lived, implying that markets generally lack arbitrage.

For instance, if $K = 1$, these equations:

$$\varphi_0(0) \cdot 1 + \varphi_1(0) \cdot 4 = 2 \quad \varphi_0(0) \cdot 1 + \varphi_1(0) \cdot \frac{1}{2} = 0$$

yield the unique solution : $(\varphi_0(0), \varphi_1(0)) = \left(-\frac{2}{7}, \frac{4}{7}\right)$. Consequently, the fair price of the option is calculated as:

$$\hat{p} = \varphi_0(0) \cdot B_0(0) + \varphi_1(0) \cdot P_1(0) = \frac{4}{7} \cdot 1 - \frac{2}{7} \cdot 1 = \frac{2}{7}$$

The determined price is independent of the specific probability value a . This price is the only logical one under these conditions, ensuring that the financial market, including the stock, bond, and call option, is free from arbitrage.

4.1.1 Arbitrage Opportunities and Contingent Claims

Consider a complete probability space (Ω, \mathcal{F}, P) with an m -dimensional Brownian motion $\{W(t), \mathcal{F}_t\}_{t \geq 0}$. The bond and stock price dynamics in this space are modeled by the following expressions:

1. For the bond price dynamics, we have

$$P_0(t) = p_0 \cdot \exp\left(\int_0^t r(s) ds\right) \tag{4.3}$$

2. For the stock price dynamics of the i -th asset, where $i = 1, \dots, d$, we have

$$P_i(t) = p_i \cdot \exp\left(\int_0^t \left(b_i(s) - \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2(s)\right) ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s) dW_j(s)\right) \tag{4.4}$$

where $b_i(t)$ denotes the drift coefficient

The functions $r(t)$, $b(t) = (b_1(t), \dots, b_d(t))^T$, and $\sigma(t) = (\sigma_{ij}(t))_{ij}$ should be progressively measurable with respect to $\{\mathcal{F}_t\}$ and uniformly bounded in (t, ω) .

We assume the following:

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1. $\sigma(t)\sigma(t)^T$ is uniformly positive definite, meaning there exists a constant $K > 0$ such that $x^T \sigma(t)\sigma(t)^T x \geq K\|x\|^2$ for all $x \in \mathbb{R}^d$ and $t \in [0, T]$.
2. The self-financing pair (π, c) is admissible for an initial capital of $x > 0$.

Definition 4.1.2 (Arbitrage Opportunity). An arbitrage opportunity is described as a self-financing and admissible pair (φ, c) , where φ is a trading strategy and c is a consumption process, that meets the following criteria:

1. The initial wealth is zero, i.e., $X(0) = 0$.
2. The terminal wealth is non-negative, i.e., $X(T) \geq 0$ almost surely.
3. There is probability of the terminal wealth being strictly greater than zero: $P(X(T) > 0) > 0$ or $P\left(\int_0^T c(t) dt > 0\right) > 0$

Corollary 15.1 (Arbitrage Opportunity). *In the complete continuous-time market model, there is no arbitrage opportunity.*

Proof. Suppose (φ, c) constitutes an arbitrage opportunity. Then, since $H(t)$ is strictly positive, we have

$$E \left[H(T)X(T) + \int_0^T H(s)c(s) ds \right] > 0 = X(0) = x$$

However, this leads to a contradiction with *Theorem 15*, which asserts the non-existence of an arbitrage opportunity in a complete market. Therefore, the presumed arbitrage opportunity cannot exist. \square

Definition 4.1.3 (Contingent Claim). A contingent claim, denoted as (g, B) , is a financial derivative that comprises two components: a non-negative payout rate process $g(t)$, adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and a non-negative \mathcal{F}_T -measurable terminal payoff B at maturity T . The claim is considered to be measurable and integrable if it satisfies the following condition for some $\mu > 1$:

$$E \left[\left(\int_0^T g(t) dt + B \right)^\mu \right] < \infty \quad (4.5)$$

Example 4.1.2 (Examples of Contingent Claims). *Some examples of contingent claims are:*

1. **European Call:** $g \equiv 0$ and $B = (P_1(T) - K)^+$. Consider a European call option on Bayer AG stock with a strike price of 55€, expiring in one year. The terminal payoff B is $(P_1(T) - 55)^+$, where $P_1(T)$ is the stock price at maturity.
2. **European Put:** $g \equiv 0$ and $B = (K - P_1(T))^+$. Imagine a European put option on Royal Dutch Shell with a strike price of 20€, expiring in six months. The payoff B is $(20 - P_1(T))^+$, with $P_1(T)$ as the stock price at expiration.
3. **Continuous Dividend Payment:** The payout rate $g(t) \equiv \delta P_1(t)$ for all $t \in [0, T]$, with $\delta > 0$ and $B \equiv 0$. Unilever declares a continuous dividend with an annual yield of $\delta = 3\%$. For an investor holding the stock for the year, the payout rate $g(t)$ is $0.03 \cdot P_1(t)$, and the terminal payoff B is zero.

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Remark. Just as in the case of a discrete-time scenario, we define the fair price of a contingent claim, represented by the pair (g, B) , as the least amount of initial capital required to establish a replicating portfolio for the claim's future payouts.

Definition 4.1.4 (Replication Strategy). A pair (π, c) is defined as a replication strategy for a contingent claim (g, B) if it fulfills the following conditions:

1. The payout process $g(t)$ is replicated exactly by the consumption process $c(t)$ almost surely for all $t \in [0, T]$.
2. At maturity T , the wealth process $X(T)$, which corresponds to (π, c) , matches the contingent claim's terminal value B almost surely.

Definition 4.1.5 (Replication Set). A replication set at cost x for a claim (g, B) is the collection of all admissible strategies (π, c) that can replicate the claim, formally denoted as $\mathcal{D}(x; (g, B))$.

Definition 4.1.6 (Fair Price). The fair price of a contingent claim (g, B) is the lowest initial wealth \hat{p} necessary to finance a replicating strategy. It is given by:

$$\hat{p} = \inf\{p \mid \mathcal{D}(p) \neq \emptyset\} \quad (4.6)$$

Remark. Assuming that the risk-free rate $r(t)$, drift $b(t)$, and volatility $\sigma(t)$ are all bounded and that $\sigma(t)\sigma(t)^\top$ is uniformly positive definite, we can define an upper bound for the expected discounted value of the claim as:

$$\bar{x} = E \left[H(T)B + \int_0^T H(t)g(t)dt \right] < \infty \quad (4.7)$$

According to the Fundamental Theorem of Asset Pricing [7], there exists a replicating portfolio for (g, B) such that the pricing inequality $\hat{p} \leq \bar{x}$ holds true.

Theorem 16 (Fair Price of a Contingent Claim). *The fair price of a contingent claim (g, B) is characterized by the expected discounted value of the claim's future cash flows $\hat{X}(t)$, and it is represented by the following valuation:*

$$\hat{p} = E \left[H(T)B + \int_0^T H(t)g(t)dt \right] < \infty \quad (4.8)$$

and there exists a unique replication strategy $(\hat{\pi}, \hat{c}) \in \mathcal{D}(\hat{p})$ whose wealth process $\hat{X}(t)$ is:

$$\hat{X}(t) = \frac{1}{H(t)} E \left[H(T)B + \int_t^T H(s)g(s)ds \mid \mathcal{F}_t \right] \quad (4.9)$$

Proof. We have established that the fair price \hat{p} is bounded above by \bar{x} . To show $\hat{p} \geq x$, we consider a replication strategy $(\pi, c) \in \mathcal{D}(x)$ that replicates the contingent claim (g, B) . Given the initial capital $x = X(0)$ and the replication property of (π, c) , we apply *Theorem 15* to obtain:

$$x \geq E \left(H(T)X(T) + \int_0^T H(s)c(s) ds \right) = E \left(H(T)B + \int_0^T H(s)g(s) ds \right) = \bar{x}$$

Thus, we conclude that the infimum of prices for non-empty replication sets must satisfy $\hat{p} \geq x$. \square

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Remark. The fair price provided by the theorem is unique and consistent with the absence of arbitrage. Otherwise, there would be arbitrage opportunities in the market consisting of stock, bond, and contingent claim.

Corollary 16.1. Consider now that the payout rate g is zero. The equality:

$$\hat{p} = E(H(T)B) = \int_{\Omega} H(t, \omega)B(\omega)P(d\omega) \quad (4.10)$$

reveals that the option price reflects the expected value of the discounted terminal payment at time T , using a distinct discount factor $H(T)$ instead of the standard $1/P_0(T)$. The function $H(T, \omega)$ represents the present value of one unit of currency paid at time T given the state $\omega \in \Omega$. Thus, $H(T)$ is a state-dependent discount factor, incorporating all fundamental securities in the market.

4.2 The Black-Scholes Formula

This section explores the Black-Scholes Formula, developed by Fischer Black and Myron Scholes in 1973, a cornerstone in the pricing of European-style options that has fundamentally transformed options trading and financial risk management. We will analyze the formula's proof to understand its underpinnings, discuss its broader implications and limitations.

Consider a market model consisting of a single stock and a bond with constant market coefficients, specifically: $d = m = 1$, $r(t) \equiv r$, $b(t) \equiv b$, $\sigma(t) \equiv \sigma > 0$ $\forall t \in [0, T]$, $T > 0$, $r, b, \sigma \in \mathbb{R}$.

We derive the Black-Scholes formula within a market model featuring a stock and a bond, characterized by constant market coefficients. The model includes a steady rate of return and fixed volatility for the asset. Here, d and m indicate the dimensionality of the Brownian motion and the market, respectively, while $r(t)$, $b(t)$, and $\sigma(t)$ represent constant risk-free rate, drift coefficient, and volatility.

Theorem 17 (Black-Scholes Formula for European Call Options). *The valuation of a European call option at any time t before maturity, $X_C(t)$, depends on the current price of the underlying asset, $P_1(t)$, and the present value of the strike price, $K > 0$, discounted at the risk-free rate r . The formula is defined as:*

$$X_C(t) = P_1(t) \cdot \Phi(d_1(t)) - K \cdot e^{-r(T-t)}\Phi(d_2(t)) \quad (4.11)$$

where $d_1(t)$ and $d_2(t)$ are defined as:

$$d_1(t) = \frac{\ln\left(\frac{P_1(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad (4.12)$$

$$d_2(t) = d_1(t) - \sigma\sqrt{T-t} \quad (4.13)$$

Remark. Φ is the distribution function of the standard normal distribution. The variables d_1 and d_2 incorporate market dynamics and option specifics into the model. d_1 represents the standardized log return on the stock price, while d_2 adjusts for the risk-free rate and stock volatility. This distribution function determine the likelihood of the option expiring in the money.

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Theorem 18 (Black-Scholes Formula for European Put Options). *The price of a European put option, $X_P(t)$, is influenced by risk-neutral probabilities and the time value of money, assessing the expected cost of exercising the option against the present value of the asset price. Mathematically, it is expressed as:*

$$X_P(t) = K \cdot e^{-r(T-t)} \Phi(-d_2(t)) - P_1(t) \Phi(-d_1(t)) \quad (4.14)$$

where the variables $d_1(t)$ and $d_2(t)$ are defined as in the call option case but used with negative signs within the cumulative normal distribution functions to reflect the put option's payout characteristics.

4.2.1 Proofs of the Black-Scholes Formula for European Call and Put Options

First, we will prove the Black-Scholes formula for European Call Options. For simplicity, we will restrict our proof to $t = 0$.

Proof. Let θ be defined as $\frac{b-r}{\sigma}$, where b is the drift coefficient and r is the risk-free interest rate. The value of the call option $X_C(0)$ is given by the expected value of the discounted payoff:

$$\begin{aligned} X_C(0) &= E(W(T)(P_1(T) - K)^+) \\ &= E \left[\exp \left(- \left[r + \frac{1}{2} \theta^2 \right] T - \theta \cdot W(T) \right) \left(p_1 \cdot \exp \left(\left[b - \frac{1}{2} \sigma^2 \right] T + \sigma W(T) \right) - K \right)^+ \right] \end{aligned}$$

The positive part is strictly positive if and only if:

$$W(T) > \frac{1}{\sigma} \left(\ln \left(\frac{K}{p_1} \right) - \left(b - \frac{1}{2} \sigma^2 \right) T \right) =: \hat{K}$$

Given that $W(T)$ is normally distributed with mean 0 and variance T , the option's value can be obtained by integrating over the probability density function:

$$\begin{aligned} X_C(0) &= \int_{\hat{k}}^{\infty} e^{-(r+\frac{1}{2}\theta^2)T-\theta x} \left(p_1 \cdot e^{(b-\frac{1}{2}\sigma^2)T+\sigma x} - K \right) \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx \\ &= p_1 \cdot \int_{\hat{k}}^{\infty} \frac{1}{\sqrt{2\pi T}} \exp \left(\left(b - \frac{1}{2} \sigma^2 - r - \frac{1}{2} \theta^2 \right) T + (\sigma - \theta)x - \frac{x^2}{2T} \right) dx \\ &\quad - K \cdot \int_{\hat{k}}^{\infty} \exp(-rT) \cdot \frac{1}{\sqrt{2\pi T}} \exp \left(-\frac{1}{2} \theta^2 T + \theta x - \frac{x^2}{2T} \right) dx \end{aligned}$$

Grouping the exponents we obtain:

$$\begin{aligned} X_C(0) &= p_1 \int_{\hat{k}}^{\infty} \frac{1}{\sqrt{2\pi T}} \exp \left(-\frac{(x - (\sigma - \theta)T)^2}{2T} \right) dx \\ &\quad - K \cdot \exp(-rT) \int_{\hat{k}}^{\infty} \frac{1}{\sqrt{2\pi T}} \exp \left(-\frac{(x + \theta T)^2}{2T} \right) dx \end{aligned}$$

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$$\begin{aligned}
 &= p_1 \cdot \left(1 - \Phi \left(\frac{\hat{k} - (\sigma - \theta)T}{\sqrt{T}} \right) \right) - K \cdot e^{-rT} \cdot \left(1 - \Phi \left(\frac{\hat{k} + \theta T}{\sqrt{T}} \right) \right) \\
 &= p_1 \cdot \Phi \left(\frac{\ln(\frac{p_1}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) - K \cdot e^{-rT} \cdot \Phi \left(\frac{\ln(\frac{p_1}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)
 \end{aligned}$$

□

The Black-Scholes Formula for European Put Options is proved in a similar way:

Proof. Again, let θ be defined as $\frac{b-r}{\sigma}$, where b is the drift coefficient and r is the risk-free interest rate. The value of the put option $X_P(0)$ is given by the expected value of the discounted payoff:

$$X_P(0) = E [e^{-rT}(K - P_1(T))^+] = E \left[\exp(-rT) \left(K - p_1 \cdot \exp(b - \frac{1}{2}\sigma^2)T + \sigma W(T) \right)^+ \right]$$

The positive part is strictly positive if and only if:

$$W(T) < \frac{1}{\sigma} \left(\ln \left(\frac{P_1}{K} \right) + (b - \frac{1}{2}\sigma^2)T \right) =: \hat{K}$$

Given that $W(T)$ is normally distributed with expected value 0 and variance T , the option's value can be obtained by integrating over the probability density function. The integral representation is as follows:

$$\begin{aligned}
 X_P(0) &= \exp(-rT) \int_{-\infty}^{\hat{K}} K \cdot \frac{1}{\sqrt{2\pi T}} \exp \left(-\frac{x^2}{2T} \right) dx \\
 &\quad - \int_{-\infty}^{\hat{K}} p_1 \cdot \frac{1}{\sqrt{2\pi T}} \exp \left(b - \frac{1}{2}\sigma^2 T + \sigma x - \frac{x^2}{2T} \right) dx
 \end{aligned}$$

Applying the cumulative distribution function of the normal distribution to the integrals, we obtain:

$$X_P(0) = K \cdot e^{-rT} \Phi \left(-\frac{\ln(P_1/K) + (b + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} - \sigma\sqrt{T} \right) - p_1 \cdot \Phi \left(-\frac{\ln(P_1/K) + (b + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

□

4.2.2 Implications and Remarks

The Black-Scholes model's valuation of European call and put options is invariant with respect to the drift coefficient b , meaning it supports a "preference-free" valuation. Additionally, the value of a European call option, $C(t)$, is inversely related to time t , and positively correlated with the risk-free interest rate r and the asset's volatility σ . This relationship is described by the following identities:

$$d_2(t) = d_1(t) - \sigma\sqrt{T-t} \tag{4.15}$$

$$P(t)\Phi(d_1(t)) = K e^{-r(T-t)}\Phi(d_2(t)) \tag{4.16}$$

Remark (Black-Scholes Formula for a General t). To demonstrate the Black-Scholes model for any time point within the interval $[0, T]$ it is essential to acknowledge the modified pricing mechanism. The adjusted price at maturity, $P_1(T)$, is defined in relation to $P_1(t)$ by the equation:

$$P_1(T) = P_1(t) \cdot \exp\left(\left(b - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(W(T) - W(t))\right) \quad (4.17)$$

Moreover, the increment in the portfolio's value between T and t is independent of past market information and is normally distributed as $\mathcal{N}(0, T - t)$, illustrating the independent and stationary increments of Brownian motion. This results in:

$$X_C(t) = E\left[\frac{H(T)}{H(t)}(P_1(T) - K)^+ | F_t\right] = E\left[\frac{H(T)}{H(t)}(P_1(T) - K)^+ | W(t)\right] \quad (4.18)$$

In a market with constant coefficients, the value of $W(t)$ is unequivocally determined by $P_1(t)$ and vice versa. In consequence, we can write:

$$X_C(t) = E^{t, P_1(t)}\left[\frac{H(T)}{H(t)}(P_1(T) - K)^+\right] = E^{t, P_1(t)}\left[\frac{H(T)}{H(t)}(P_1(T) - K)^+ | P_1(t)\right] \quad (4.19)$$

Remark (Markov Property). The future trajectory of an option's price relies solely on the current stock price, disregarding historical prices. This concept, known as the Markov property, states that each new price movement is independent of the past.

4.2.3 Change of Measure in Option Pricing: The Cameron-Martin-Girsanov Theorem

This section explores the Cameron-Martin-Girsanov Theorem, crucial in stochastic process theory and financial mathematics. Named after Robert H. Cameron, W.T. Martin, and Ruslan L. Girsanov, it supports risk-neutral valuation, a core component of modern pricing theory, used in the Black-Scholes-Merton model [3]. Risk-neutral valuation in derivative pricing involves shifting the probability measure from the real-world P to the risk-neutral Q , where discounted price processes of traded assets are martingales.

The stock price process under the real-world probability P is modeled by:

$$S_t = S_0 \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \quad (4.20)$$

To transition to a risk-neutral measure Q , we employ the Radon-Nikodym derivative to adjust the drift to the risk-free rate r .

Definition 4.2.1 (Radon-Nikodym Derivative). If μ and ν are two σ -finite measures on a measurable space (X, \mathcal{F}) and ν is absolutely continuous with respect to μ (notated as $\nu \ll \mu$), there exists a μ -integrable function $f : X \rightarrow [0, \infty)$, called the Radon-Nikodym derivative, such that for every measurable set $A \in \mathcal{F}$:

$$\nu(A) = \int_A f d\mu \quad (4.21)$$

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This function f , which is almost everywhere unique, is denoted as $f = \frac{d\nu}{d\mu}$ and represents the density of ν with respect to μ .

Theorem 19 (The Cameron-Martin-Girsanov Theorem). *Let W_t be a standard Brownian motion under the probability measure P and let γ_t be a predictable process that satisfies the integrability condition:*

$$E_P \left[\exp \left(\frac{1}{2} \int_0^T \gamma_t^2 dt \right) \right] < \infty \quad (4.22)$$

Then there exists a probability measure Q equivalent to P under which the process defined by

$$\tilde{W}_t = W_t + \int_0^t \gamma_s ds \quad (4.23)$$

is a standard Brownian motion. The relationship between the measures P and Q is given by the Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp \left(- \int_0^t \gamma_s dW_s - \frac{1}{2} \int_0^t \gamma_s^2 ds \right) \quad (4.24)$$

The proof of the theorem can be found in [10].

For a stochastic process $X_t = \mu t + \sigma W_t$ associated with a Brownian motion W_t under P , the transformed process under Q is:

$$X_t = \sigma \left(\frac{\mu}{\sigma} t + W_t \right) = \sigma \tilde{W}_t \quad (4.25)$$

4.2.4 Applying the Cameron-Martin-Girsanov Theorem to the Black-Scholes Model

The Cameron-Martin-Girsanov Theorem (CMG) allows us to transition from the natural probability measure to a risk-neutral one, essential for deriving the Black-Scholes formula [3]. Consider a financial market model comprising a stock with price dynamics given by the equation:

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad (4.26)$$

The corresponding bond process, assumed to grow at the risk-free rate r , is: $B_t = B_0 e^{rt}$. To employ the CMG Theorem, we first define the discounted stock process Z_t as:

$$Z_t = \frac{S_t}{B_t} = S_0 e^{(\mu-r)t + \sigma W_t} \quad (4.27)$$

The discounted process must be a martingale under the risk-neutral measure Q . We apply Itô's Lemma to obtain the dynamics of Z_t and change the measure using the CMG Theorem.

The new measure Q is constructed so Z_t has no drift, defining the risk-neutral Brownian motion \tilde{W}_t . This allows us to express

$$S_t = S_0 e^{rt} e^{\sigma(\tilde{W}_t - \frac{1}{2}\sigma^2 t)} \quad (4.28)$$

4.3. Example

For a European call option with payoff $(S_T - K)^+$, the price V at time 0 can be calculated using the risk-neutral expectation as:

$$V = e^{-rT} \mathbb{E}^Q [(S_T - K)^+] \quad (4.29)$$

where \mathbb{E}^Q denotes the expected value under the risk-neutral measure Q . This expectation reflects the adjusted dynamics of the stock price under Q , considering all information up to time T and discounting at the risk-free rate r .

Applying the Black-Scholes formula in this framework, the option price is precisely determined, resulting in the renowned formula:

$$V = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2) \quad (4.30)$$

4.2.5 Analysis of the Black-Scholes Model: Benefits and Limitations

The Black-Scholes model is a crucial tool in valuing financial options, providing a framework for calculating fair prices. It aids risk management and portfolio optimization, promoting informed investment decisions by assessing returns and risks. Additionally, it enhances market efficiency and transparency by standardizing option valuation [13].

However, the model is not without its critics, mainly due to its underlying assumptions which, in certain contexts, may limit its applicability and accuracy. The most notable restriction is its exclusive design for pricing European options, ignoring the possibility of early exercise characteristic of US options.

In addition, the model assumes constant volatility and constant interest and dividend rates, which neglects the dynamic nature of financial markets. These simplifications, together with the omission of transaction costs and the assumption of zero arbitrage opportunities, can alter theoretical prices away from those observed in practice, posing challenges in their actual application.

Despite its limitations, the Black-Scholes model remains fundamental in option finance, serving as a basis for more complex models suited to modern financial markets.

4.3 Example

Example 4.3.1. Consider a Glencore (GLEN-L) stock currently priced at $P_1 = 50\text{€}$. We want to price a European call option with a strike price of $K = 55\text{€}$ expiring in one year ($T = 1$ year). The annual risk-free interest rate is $r = 2\%$, and the stock's return volatility is $\sigma = 25\%$. Compute the call option's value step by step.

Step 1: Compute d_1 and d_2

Recalling the Equations 4.12, 4.13 and 4.14, the first thing we need to calculate the values of d_1 and d_2 using the given parameters.

$$d_1 = \frac{\ln\left(\frac{50}{55}\right) + \left(0.02 + \frac{0.25^2}{2}\right)(1 - 0)}{0.25\sqrt{1}} \approx \frac{-0.095 + 0.051}{0.25} = \frac{-0.044}{0.25} = -0.176$$

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$$d_2 = d_1 - 0.25\sqrt{1 - 0} \approx -0.176 - 0.25 = -0.426$$

Step 2: Find $\Phi(d_1)$ and $\Phi(d_2)$

Using the standard normal cumulative distribution function, we find the probabilities corresponding to d_1 and d_2 .

$$\Phi(d_1) = \Phi(-0.176) = 1 - P(X \leq 0.176) \approx 1 - 0.570 = 0.430$$

$$\Phi(d_2) = \Phi(-0.426) = 1 - P(X \leq 0.426) \approx 1 - 0.665 = 0.335$$

Step 3: Calculate the Call Option Price X_C

Now we can calculate the price of the call option using the Black-Scholes formula.

$$\begin{aligned} (X_C(t) &= P_1(t) \cdot \Phi(d_1(t)) - K \cdot e^{-r(T-t)}\Phi(d_2(t))) \\ &= 50 \cdot 0.430 - 55 \cdot \exp(-0.02 \cdot 1) \cdot 0.335 = 21.15 - 18.06 = 3.09\text{€} \end{aligned}$$

Therefore, the theoretical price of the European call option is approximately 3.09€ under the Black-Scholes model with the given parameters.

Chapter 5

Conclusions

In conclusion, this dissertation has provided a thorough exploration of key mathematical models that support the valuation of financial derivatives.

Firstly, *Chapter 2* established the groundwork through the analysis of the Mean-Variance Model, explaining its crucial role in optimizing investment strategies through the evaluation of risk and return. Thereafter, this chapter explored the Cox-Ross-Rubinstein model, demonstrating its utility in a discrete market framework and how it complements the continuous models by offering a step-by-step approach to option valuation.

Furthermore, *Chapter 3* moved into the field of continuous market models, using Itô calculus as a fundamental tool. Within this context, the chapter explored Brownian motion and martingales, laying the foundation for the stochastic behavior of market prices. The Itô Formula was presented as pivotal element, linking the stochastic differential equations with the modeling of complex financial instruments.

Finally, *Chapter 4* delved into the Black-Scholes model, emphasizing its critical role in valuing European options. This section highlighted the model's theoretical foundations and its application in real-world scenarios. It demonstrated how the formula enables accurate option pricing and risk management, becoming essential in the field of quantitative finance.

The purpose of this work was not only to articulate the theoretical aspects of these models but also to demonstrate their practical utility in real-world financial scenarios. Through a meticulous investigation and application of these mathematical models, this dissertation has contributed to the robust understanding of financial derivatives valuation and has laid a foundation for further research and innovation in financial modeling.

Moving forward, the study presented here can extend its applicability beyond European options to the more complex domain of American financial derivatives. Unlike their European counterparts, American securities allow for exercise at any time before expiration, presenting additional challenges and opportunities for valuation and risk management strategies.

Chapter 5. Conclusions

Additionally, this work sets a robust foundation for exploring other advanced topics in financial mathematics. For instance, the principles and models discussed could serve as a basis for analyzing path-dependent options, where the payoff is influenced by the price path of the underlying asset. The application of Monte-Carlo simulation methods could be explored to study and model the complex behaviors of financial instruments in a broader market context.

Moreover, the exploration of binomial and trinomial tree models for option pricing can be advanced to include a variety of exotic options and numerical algorithms, offering a detailed understanding of the variability of options and methods of risk management. The finite-difference methods for option pricing, could also be further refined and extended to a range of financial products.

In essence, the findings and methodologies described in this thesis offer a substantial groundwork and knowledge for future investigations into stochastic control problems and optimization techniques within the financial sector. For further reading and deeper understanding, classical references such as Baxter and Rennie [2] or Brigo and Mercurio [5] offer comprehensive insights into financial mathematics and could guide the continuation of this research.

Chapter 6

Impact Analysis

The valuation of financial derivatives has theoretical and practical implications that affect personal development, business practices, and broader social and economic fields. This analysis highlights the expected benefits and possible adverse effects. Additionally, it references relevant Sustainable Development Goals from the 2030 Agenda. Decisions made during the research, based on impact considerations, will also be discussed.

In the personal context, this dissertation impacts the researcher's academic and professional growth. By studying the valuation of financial derivatives, the researcher gains a deep understanding of complex models like the Black-Scholes formula and Itô calculus. This develops mathematical and analytical skills, improving career prospects in investment banking, risk analysis, and quantitative research. Nonetheless, it can be difficult for the reader to comprehend as it requires a substantial foundational knowledge of economics and limits exposure to broader financial concepts.

In the business context, this research has significant implications for the financial industry, especially in investment, risk management, and derivative valuation. It simplifies complex tools and models used for accurate derivative valuation, leading to better understanding of investment decisions and risk management. However, there is a risk of over-simplification on these models, which may be unfeasible for very complex markets with many changing variables.

Socially, this dissertation has implications for financial literacy and economic stability. Sharing knowledge about financial derivatives can enhance financial understanding among professionals and the general public promoting informed decision-making in day to day economic challenges.

Economically, the impact is profound, affecting markets, investments, and overall economic health. Improved derivative valuation models lead to more efficient and transparent markets, attracting investment and fostering growth. However, poor application or overconfidence in these models can cause market volatility if assumptions are inaccurate, and overemphasis on financial markets can neglect other critical economic sectors.

Chapter 6. Impact Analysis

Culturally, this research can influence attitudes towards finance and risk. It promotes informed risk-taking and innovation in financial markets and enhances finance and economics education, fostering a culture of learning and curiosity.

This work aligns with several Sustainable Development Goals, particularly Decent Work and Economic Growth by improving financial literacy and risk management strategies, Industry, Innovation, and Infrastructure by fostering innovation in financial models, and Partnerships for the Goals by promoting collaboration between academic institutions and financial entities, to achieve common objectives.

Throughout this dissertation, several decisions were based on impact considerations. The selection of models focused on those balancing theoretical rigor with practical applicability to maximize relevance and utility. Practical exercises were included to demonstrate the applicability of theoretical concepts, ensuring the research remained focused on real-world financial practices.

In conclusion, the research conducted in this dissertation has the potential to significantly impact various spheres, from personal and business growth to broader economic, social, and cultural advancements. By aligning with the Sustainable Development Goals and making informed decisions, this work aims to contribute positively to the understanding of the field of financial mathematics and its practical applications.


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