Universidad Politécnica
de Madrid

**Escuela Técnica Superior de
Ingenieros Informáticos**

Grado en Matemáticas e Informática

Trabajo Fin de Grado

Structural Insights into Higher-Dimensional Homotopy Groups of Spheres

Autor: Alfonso Mateos Vicente
Tutor(a): Alfonso Zamora Saiz

Madrid, June 2024

Este Trabajo Fin de Grado se ha depositado en la ETSI Informáticos de la Universidad Politécnica de Madrid para su defensa.

Trabajo Fin de Grado
Grado en Matemáticas e Informática

Título: Structural Insights into Higher-Dimensional Homotopy Groups of Spheres

June 2024

Autor: Alfonso Mateos Vicente
Tutor: Alfonso Zamora Saiz
Departamento de Matemática Aplicada a las TIC
Escuela Técnica Superior de Ingenieros Informáticos
Universidad Politécnica de Madrid

Agradecimientos

En primer lugar, quiero expresar mi más profundo agradecimiento a mis padres y a mi hermana. Vuestra constante ayuda y comprensión han sido esenciales para llegar hasta aquí. Sin vosotros, este logro no habría sido posible.

A mi pareja, gracias por tu paciencia, cariño y por creer en mí. Tu apoyo ha sido indispensable en este camino.

A mis profesores, especialmente a mi tutor de TFG, Alfonso Zamora Saiz, quiero expresar mi sincera gratitud. Gracias por tu guía y por los cursos que despertaron en mí el interés por el tema de este proyecto. Tu dedicación ha sido crucial para la realización de este trabajo.

Este trabajo refleja no solo mi esfuerzo, sino también el apoyo y la inspiración que he recibido de todos vosotros. A todos, muchas gracias.

Alfonso Mateos Vicente

Resumen

Este manuscrito, titulado "Structural Insights into Higher-Dimensional Homotopy Groups of Spheres", profundiza en el ámbito de la topología algebraica, centrándose particularmente en los grupos de homotopía de dimensiones superiores de esferas. Se basa en varios conceptos y teoremas fundamentales que forman el núcleo de su marco de investigación.

El estudio comienza con una breve introducción al Teorema de Seifert-Van Kampen. Aunque solo se aborda de manera ligera, este teorema establece una base esencial para comprender las propiedades conectivas de los espacios en la topología, preparando el escenario para discusiones más complejas que siguen.

Una parte significativa del manuscrito se dedica a los Complejos CW. Aquí, se explora en detalle el Teorema de Aproximación Celular, formando un componente central de la discusión. El estudio también aborda temas críticos como el Teorema de Aproximación de CW y el Teorema de Whitehead, subrayando su importancia en la comprensión de los Complejos CW.

El concepto de suspensión, particularmente el Teorema de Suspensión de Freudenthal, emerge como otro enfoque clave. Esta parte del estudio examina cómo se pueden analizar y entender los espacios topológicos complejos a través de la suspensión, destacando sus propiedades intrínsecas.

Además, el manuscrito proporciona un análisis en profundidad de las secuencias exactas y las fibraciones. Examina minuciosamente las secuencias exactas para grupos de homotopía y Complejos CW, añadiendo profundidad a la comprensión de la topología algebraica. El estudio de las fibraciones se centra especialmente en las fibraciones de Serre, culminando en un examen de la fibración de Hopf. Esta exploración no solo enriquece la comprensión de la topología algebraica, sino que también revela los patrones y conexiones intrincados inherentes en la estructura de los espacios topológicos.

A través de su examen exhaustivo de los Complejos CW, la suspensión, las secuencias exactas y las fibraciones, el manuscrito presenta un recorrido completo y esclarecedor por la topología algebraica, ofreciendo nuevas perspectivas sobre la naturaleza compleja de los grupos de homotopía de dimensiones superiores de esferas.

Abstract

This manuscript, titled "Structural Insights into Higher-Dimensional Homotopy Groups of Spheres", delves deeply into the realm of algebraic topology, focusing particularly on the higher-dimensional homotopy groups of spheres. It builds on several foundational concepts and theorems that form the core of its investigative framework.

The study begins with a brief introduction to the Seifert-Van Kampen Theorem. Although touched upon only lightly, this theorem lays essential groundwork for understanding the connective properties of spaces in topology, setting the stage for more complex discussions that follow.

A significant portion of the manuscript is devoted to CW Complexes. Here, the Cellular Approximation Theorem is explored in detail, forming a central component of the discussion. The study also addresses critical topics like the CW Approximation Theorem and the Whitehead Theorem, underscoring their importance in understanding CW Complexes.

The concept of suspension, particularly the Freudenthal Suspension Theorem, emerges as another key focus. This part of the study examines how complex topological spaces can be analyzed and understood through suspension, highlighting their intrinsic properties.

Moreover, the manuscript provides an in-depth analysis of exact sequences and fibrations. It thoroughly examines exact sequences for homotopy groups and CW complexes, adding depth to the understanding of algebraic topology. The study of fibrations focuses especially on Serre fibrations, culminating in an examination of the Hopf bundle. This exploration not only enriches the understanding of algebraic topology but also reveals the intricate patterns and connections inherent in the structure of topological spaces.

Through its comprehensive examination of CW Complexes, suspension, exact sequences, and fibrations, the manuscript presents a thorough and enlightening journey into algebraic topology, offering new insights into the complex nature of higher-dimensional homotopy groups of spheres.

Contents

Chapter 1

Introduction

The exploration of higher-dimensional homotopy groups of spheres, standing at the crossroads of algebraic topology, represents an intriguing interplay of mathematical history, theory, and practical application. This research area, rooted in the groundbreaking efforts of the 19th and 20th centuries, continues to be a vast field of exploration and discovery.

The journey towards understanding the homotopy groups of spheres dates back to Henri Poincaré, whose revolutionary ideas in algebraic topology at the end of the 19th and the beginning of the 20th centuries established a new paradigm in mathematics. Poincaré's vision of unraveling topological spaces using algebraic tools paved the way for advanced study of homotopy groups. These groups, initially emerging as abstract concepts, gradually revealed their fundamental importance in understanding the nature of space.

The 20th century witnessed significant advancements in the field, thanks to the contributions of mathematicians like Heinz Hopf and Jean-Pierre Serre. These scholars not only deepened our understanding of homotopy groups but also established vital connections with other mathematical areas, including fibrations theory and algebraic geometry. Their findings opened new horizons and significantly boosted research in topology and related areas.

The principal objective of this work, titled "Structural Insights into Higher-Dimensional Homotopy Groups of Spheres," is to further the study of these groups by providing an in-depth view of their structure and significance. The manuscript offers a robust foundation through a detailed analysis of CW complexes and the Cellular Approximation Theorem. From this point, it delves deeper into suspension and fibrations, highlighting how these topological operations aid in understanding the homotopy groups of spheres.

The study of higher-dimensional homotopy groups of spheres is crucial, not just for its complexity and mathematical aesthetics but also for its role in understanding fundamental spatial structures. The results presented here are significant not only from a theoretical standpoint; they also have implications in how we model and comprehend spaces across multiple dimensions.

For quick and useful reference, below is a table summarizing the known homotopy groups of spheres:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$

This summary illustrates the relationships known to date and sets a framework for the detailed exploration that follows in the manuscript.

In conclusion, this manuscript is not just a mathematical treatise but also a testament to the ongoing quest for understanding and knowledge in a fundamental field of mathematics. It invites a new generation of mathematicians to immerse themselves in the mysteries of homotopy groups and to contribute to the continuous advancement of this exciting field of study.

Chapter 2

Fundamental Group

This chapter is devoted to laying the essential groundwork for the thesis, with a specific focus on constructing a robust foundation in the realm of algebraic topology. This foundation is critical for the in-depth study of Higher Homotopy Groups presented in subsequent chapters.

The chapter commences with Section 2.1. This section is designed to provide the foundational elements and concepts necessary for understanding the more complex topics that follow. Progressing to Section 2.2, the discussion shifts to introduce the concept of homotopy. In this section, the widely recognized and central concept of the fundamental group is defined and explored in detail.

The narrative then advances to Section 2.3, focusing on the concept of covering spaces. Here, not only is the definition of covering spaces introduced, but also the crucial lifting properties are examined and proven. These properties are of paramount importance in the study of Algebraic Topology. This section also marks the achievement of the thesis's first objective: demonstrating that the fundamental group of the 1-sphere is isomorphic to $\mathbb{Z}$.

The chapter culminates in Section 2.4 with an exploration of the Seifert-Van Kampen Theorem. This theorem is instrumental in computing the fundamental groups of various intriguing topological spaces. Additionally, this section demonstrates why the fundamental group is not necessarily abelian.

2.1 Basic Constructions

In this section, we introduce foundational concepts that are essential for understanding the more complex structures and mappings in topological spaces. These basic constructions lay the groundwork for our study.

Definition 2.1.1 (Pointed Space (X, x_0))**.** *A* pointed space *is a pair* (X, x_0)*, where* X *is a topological space and* $x_0 \in X$ *is a distinguished point, known as the* base point*.*

Definition 2.1.2 (Pointed Map)**.** *A* pointed map *between two pointed spaces*

(X, x_0) *and* (Y, y_0) *is a continuous map* $f : X \to Y$ *that preserves the base points, i.e.,* $f(x_0) = y_0$.

Definition 2.1.3 (Map of pairs). *Let* X, Y *be topological spaces and let* $A \subset X$, $B \subset Y$ *be topological subspaces. A map* $f : (X, A) \to (Y, B)$ *is said to be a map of pairs if it is a map* $f : X \to Y$ *such that* $f(A) \subset B$.

Definition 2.1.4 (n-Sphere $\mathbb{S}^n$ and n-Disk $\mathbb{D}^n$). *The* n-disk *is the set of points in* $\mathbb{R}^{n+1}$ *that are at a unit euclidean distance or less from the origin, this is* $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. *The* n-sphere $\mathbb{S}^n$ *is the boundary of the* $(n+1)$*-disk, i.e.* $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.

For instance, the 1-sphere is the circle and the 2-sphere is the boundary of the unit euclidean ball in $\mathbb{R}^3$.

Definition 2.1.5 (Path in a Pointed Space and Inverse path). *Given a pointed space* (X, x_0), *a* path *in* (X, x_0) *is a continuous map* $f : [0, 1] \to X$ *such that* $f(0) = x_0$. *The* inverse path of f *is defined as* $f^{-1}(t) = f(1 - t)$.

Definition 2.1.6 (Loop in a Pointed Space). *In a pointed space* (X, x_0), *a* loop *is a continuous map* $f : [0, 1] \to X$ *for which* $f(0) = f(1) = x_0$.

Definition 2.1.7 (Loop Space $\Omega(X, x_0)$). *The* loop space $\Omega(X, x_0)$ *of a pointed space* (X, x_0) *is the collection of all loops in* (X, x_0), *i.e.,* $\Omega(X, x_0) = \{f : [0, 1] \to X \mid f \text{ continuous and } f(0) = f(1) = x_0\}$.

Example 2.1.8 (Mapping cylinder, M_f). *Consider a map two topological spaces* X, Y *and a map* $f : X \to Y$, *the mapping cylinder is the quotient space of the disjoint union* $(X \times I) \sqcup Y$ *obtained by identifying each* $(x, 1) \in X \times I$ *with* $f(x)$.

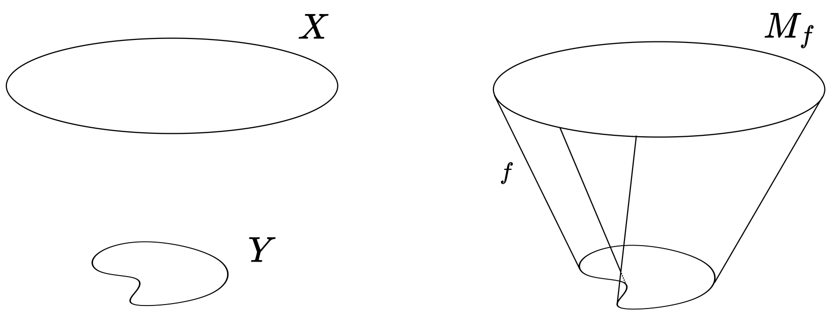

Figure 2.1: Illustration of a mapping cylinder, M_f.

2.2 Homotopy and the Fundamental Group

The purpose of this chapter is to incrementally construct an understanding of fundamental groups and their properties. Before delving into these complex concepts, it is essential to introduce and motivate the idea of the fundamental group.

Consider an infinite sheet of paper. On this sheet, draw a circle and imagine shrinking it continuously until it becomes a single point. This process represents a specific type of topological transformation. Next, replace this sheet with a sphere, akin to a regular ball. Draw a circle on it and gradually move it towards the sphere's pole, where it similarly reduces to a point. However, contrast this with an infinite sheet that has a permanent hole in its center. When drawing a circle around this hole and attempting to shrink it, the task becomes impossible due to the presence of the hole. This scenario highlights that while the transformations on the sheet and sphere share similarities, the case with the perforated sheet presents a distinct situation. These differences are fundamental to the understanding of the concept of the fundamental group in the field of topology.

To utilize this example in a mathematical context, we must first formalize the notion of *stretching*. This formalization will allow us to define and prove the properties of the fundamental group rigorously.

Definition 2.2.1 (Homotopy, Function Homotopy and Homotopy relation $\simeq$)**.** *Consider two pointed spaces (X, x_0) and (Y, y_0), and two pointed maps $f, g : (X, x_0) \to (Y, y_0)$. Maps f and g are defined as homotopic if there exists a continuous map $F : X \times [0, 1] \to Y$ fulfilling $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $t \in [0, 1]$. This map F is referred to as a homotopy between f and g. The relation of two applications f and g being homotopic is noted as $f \simeq g$.*

Example 2.2.2 (Linear Homotopies)**.** *Consider any two paths $f_0, f_1 : I \to (\mathbb{R}^n, x_1)$. A homotopy $F(x, t) = tf_0(x) + (1 - t)f_1(x)$ can be constructed between them. At each instance t, this function represents a linear interpolation of the paths f_0 and f_1. The continuity of this function is guaranteed by the continuous nature of vector addition and scalar multiplication.*

Definition 2.2.3 (Homotopy relative to A)**.** *Consider X, Y two topological spaces and two maps $f, g : X \to Y$ such that $f(a) = g(a)$ for all $a \in A$. Then a homotopy between f and g relative to A is a homotopy $f_t : f \simeq g$ such that it is constant in A i.e.*

$$f_t(a) = f(a) = g(a), \quad t \in I, \ a \in A.$$

An homotopy f_t between f and g relative to A is generally denoted as $f_t : f \simeq g$ rel A, i.e. $f_t : X \to Y$ rel A

Definition 2.2.4 (Homotopy of pairs)**.** *Consider X, Y two topological spaces, two maps of pairs $f, g : (X, A) \to (Y, B)$ with $A \subset X$ and $B \subset Y$. Then f_t is a homotopy of pairs if f_t is a homotopy between f and g such that*

$$f_t(a) \in B, \quad t \in I, \ a \in A.$$

Returning to our initial example, we have established a kind of *stretching* movement between *circles*. Note that while circles were used for illustration, the concept of homotopy applies to all pointed maps. The next step is to examine whether this homotopy relation indeed constitutes an equivalence relation, enabling us to classify pointed maps accordingly.

Proposition 2.2.5. *The homotopy relation $\simeq$ of Definition 2.2.1 is an equivalence relation.*

Proof. Consider three pointed maps α, β, and γ between two pointed spaces (X, x_0) and (Y, y_0).

1. **Reflexivity:** Define $F : X \times [0,1] \to Y$ by $F(x,t) = f(x)$. This constructs a homotopy between f and itself.
2. **Symmetry:** Suppose $F : X \times [0,1] \to Y$ is a homotopy between f and g. Then, the map $G : X \times [0,1] \to Y$ defined by $G(x,t) = F(x, 1-t)$ forms a homotopy between g and f.
3. **Transitivity:** If $F : X \times [0,1] \to Y$ and $G : X \times [0,1] \to Y$ are homotopies between f and g, and g and h respectively, then the map $H : X \times [0,1] \to Y$, defined by $H(x,t) = F(x, 2t)$ for $t \in [0, 1/2]$ and $H(x,t) = G(x, 2t-1)$ for $t \in [1/2, 1]$, establishes a homotopy between f and h.

□

A direct lemma of the homotopy relation is that reparametrizing a path preserves its homotopy class.

Lemma 2.2.6. *Let $\varphi : I \to I$ a continuous map such that $\varphi(0) = 0$ and $\varphi(1) = 1$ and α a path in (X, x_0) then $\alpha \circ \varphi \simeq \alpha$.*

Proof. Let α a path in (X, x_0) and $\varphi : I \to I$ a continuous map such that $\varphi(0) = 0$ and $\varphi(1) = 1$, we can consider $H(s,t) = (1-t)\varphi(s) + ts$ so that $H(s,0) = \varphi(s)$ and $H(s,1) = s$. Note that the image of H lies between $\varphi(s)$ and s, hence is in I, so the composition $\alpha \circ H$ is well defined and continuous. So $\alpha \circ H$ is a homotopy between the original and the reparameterized, i.e $\alpha \circ \varphi \simeq \alpha$. □

Now we want to define a binary operation which allows us to work directly with loops, since recalling our example, the idea was to classify spaces using its loops.

Definition 2.2.7 (Concatenation of loops). *Let (X, x_0) be a pointed space and let $f, g : [0,1] \to (X, x_0)$ be two loops in (X, x_0). The concatenation of f and g is the loop $f \otimes g : [0,1] \to (X, x_0)$ defined as*

$$(f \otimes g)(t) = \begin{cases} f(2t) & \textit{if } t \in [0, 1/2] \\ g(2t-1) & \textit{if } t \in [1/2, 1] \end{cases}$$

Proposition 2.2.8. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ loops such that $\alpha_1 \simeq \alpha_2$ and $\beta_1 \simeq \beta_2$ then $\alpha_1 \otimes \beta_1 \simeq \alpha_2 \otimes \beta_2$*

Proof. Consider the homotopies $f_t : \alpha_1 \simeq \alpha_2, g_t : \beta_1 \simeq \beta_2$, then $f_0(1) = g_0(0)$ so that we can define $f_0 \otimes g_0$, and equivalently we can define $f_1 \otimes g_1$, therefore $f_t \otimes g_t$ defines an homotopy between $f_0 \otimes g_0$ and $f_1 \otimes g_1$. Hence $\alpha_1 \otimes \beta_1 \simeq \alpha_2 \otimes \beta_2$. □

Also, we need to introduce a lemma that will be important to show that the fundamental group which will be formally defined in 2.2.10 is indeed, a group.

Lemma 2.2.9 (Pasting lemma)**.** *Let X, Y be both closed (or both open) subsets of a topological space Z such that $Z = X \cup Y$ and let W be also a topological space. If $f : Z \to W$ is a function such that $f|_X$ and $f|_Y$ are continuous, then f is continuous.*

We are not including the proof of Lemma 2.2.9 since it is an exercise of general topology.

Proposition 2.2.10 (Fundamental group of (X, x_0))**.** *Let (X, x_0) a pointed space, the fundamental group of (X, x_0), denoted by $\pi_1(X, x_0)$, is a group given by the set of homotopy classes of loops in (X, x_0), i.e. $\pi_1(X, x_0) = \Omega(X, x_0)/\simeq$, together with the concatenation of loops operation $[f] \otimes [g] = [f \otimes g]$.*

Proof. By Proposition 2.2.8 we know that the concatenation respects the homotopy so, indeed, the operation $[f]\otimes[g] = [f\otimes g]$ is well-defined and does not depend on the chosen representative of the equivalence class.

It remains to check the following properties:

Associativity: Let $\alpha, \beta, \gamma \in \pi_1(X, x_0)$, we want to prove that $[\alpha \cdot (\beta \cdot \gamma)] = [(\alpha \cdot \beta) \cdot \gamma]$.

$$\alpha \otimes (\beta \otimes \gamma) = \begin{cases} \alpha(2t) & t \in [0, 1/2) \\ \beta(4t - 2) & t \in [1/2, 3/4) \\ \gamma(4t - 3) & t \in [3/4, 1) \end{cases}$$

$$(\alpha \otimes \beta) \otimes \gamma = \begin{cases} \alpha(4t) & t \in [0, 1/4) \\ \beta(4t - 1) & t \in [1/4, 1/2) \\ \gamma(2t - 1) & t \in [1/2, 1) \end{cases}$$

Considering the following reparametrization:

$$\varphi(t) = \begin{cases} \frac{t}{2} & if\ t \in [0, 1/2) \\ t - \frac{1}{4} & if\ t \in [1/2, 3/4) \\ 2t - 1 & if\ t \in [3/4, 1) \end{cases}$$

We observe that $((\alpha\otimes\beta)\otimes\gamma)(\varphi(t)) = \alpha\otimes(\beta\otimes\gamma)$, so $\alpha\otimes(\beta\otimes\gamma)$ is a reparametrization of $(\alpha \otimes \beta) \otimes \gamma$, therefore, by Lemma 2.2.6, $\alpha \otimes (\beta \otimes \gamma) \simeq (\alpha \otimes \beta) \otimes \gamma$.

Identity element: Taking $e(t) = x_0 \in \pi_1(X, x_0)$ we can show that e is the identity element, for this, consider $\alpha \in \pi_1(X, x_0)$, then we have to prove that $e\otimes\alpha \simeq \alpha\otimes e \simeq \alpha$. We can note that $e \cdot \alpha$ is just a reparametrization of α with:

$$\varphi(t) = \begin{cases} 0 & if\ t \in [0, 1/2] \\ 2t - 1 & if\ t \in [1/2, 1] \end{cases}$$

So that $\alpha \circ \varphi = e \otimes \alpha$, hence $\alpha \simeq e \otimes \alpha$. To prove that $\alpha \otimes e \simeq alpha$ is the same but taking the reparametrization:

$$\varphi(t) = \begin{cases} 2t & if\ t \in [0, 1/2] \\ 1 & if\ t \in [1/2, 1] \end{cases}$$

Hence $e \otimes \alpha \simeq \alpha \otimes e \simeq \alpha$.

Inverse element: Consider f^{-1} as the inverse path of f, to show that the element $f \circ f^{-1}$ is homotopic to a constant path, we have to consider the homotopy $h_t = f_t \otimes g_t$ where:

$$f_t = \begin{cases} f(s) & if\ s \in [0, 1-t) \\ f(1-t) & if\ t \in [1-t, 1] \end{cases}$$

and g_t is the inverse of f_t. With this definition, we can observe that $f_0 = f$ and $f_1 = f(0) = x_0$, so f_1 is a the identity element e. Therefore, since the inverse of f_0 is f^{-1} and the inverse of e is e then h is a homotopy from $f \circ f^{-1}$ to $e \circ e \simeq e$. Hence, $f \circ f^{-1} \simeq e$.

Therefore we have just proved that $\pi_1(X, x_0)$ is indeed a group. □

Proposition 2.2.11. *If X and Y are topological spaces and $f : X \to Y$ is a homeomorphism, then for all $n \geq 0$, the homotopy groups of X and Y are isomorphic. That is, $\pi_n(X) \cong \pi_n(Y)$.*

Proof. Consider $f : X \to Y$ the homeomorphism. f induces $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$. Our aim is to prove that f_* is an isomorphism.

Since f is a homemorphism, it is a continuous bijection with a continuous inverse. Consider $[g] \in \pi_n(X, x_0)$, then

$$((f^{-1})_* \circ f_*)([g]) = (f^{-1})_*([f(g)]) = [f^{-1}(f(g))] = [g]$$

Hence $(f^{-1})_* \circ f_*$ is the identity map on $\pi_n(X, x_0)$. Doing the same for $f_* \circ (f^{-1})_*$ it follows that $f_* \circ (f^{-1})_*$ is the identity map on $\pi_n(Y, x_0)$. Hence f_* is indeed an isomorphism. □

Definition 2.2.12 (Path-connected). *X is path-connected if for every two points $x_0, x_1 \in X$, there exists a path $\alpha : I \to X$ sucht that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.*

Although the selection of the x_0 is important for the definition of the fundamental group, it turns out that, if X is path-connected, the fundamental groups obtained by changing the base point are isomorphic.

Proposition 2.2.13. *Let $x_0, x_1 \in X$, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

Proof. Let $h : I \to X$ be a path from x_0 to x_1, then we can associate each loop of $[f] \in \pi_1(X, x_0)$ to $[h \otimes f \otimes h^{-1}] \in \pi_1(X, x_1)$, consider this transformation as $\Phi_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$, we have to prove that Φ_h is an isomorphism.

Since $\Phi_h([\alpha\otimes\beta]) = [h\otimes\alpha\otimes\beta\otimes h^{-1}] = [h\alpha\otimes h^{-1}\otimes h\otimes\beta\otimes h^{-1}] = [h\alpha\otimes h^{-1}]\otimes[h\otimes\beta\otimes h^{-1}] = \Phi_h([\alpha]) \otimes \Phi_h([\beta])$, hence Φ_h is a homomorphism.

Now, to show that Φ_h is an isomorphism, we want to check if it is injective, or if there exists an inverse. We can show that $\Phi_{h^{-1}}$ is the inverse we are looking for. $(\Phi_h \circ \Phi_{h^{-1}})([\alpha]) = \Phi_h([h^{-1} \otimes \alpha \otimes h]) = [h \otimes h^{-1} \otimes \alpha \otimes h \otimes h^{-1}] = [\alpha]$, therefore Φ_h is an isomorphism, hence $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$. □

So, from now on we will assume that our topological spaces are path connected and we will denote the fundamental group of (X, x_0) as $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition 2.2.14 (Simply connected)**.** *X is defined as simply connected if it satisfies two conditions: first, it must be path-connected; second, any path between two points in X can be continuously deformed into any other path between the same points, with the endpoints remaining fixed throughout the deformation.*

Note that if X is simply connected, then every path can be deformed into the constant path. Hence $\pi_1(X) = 0$.

2.3 Covering Spaces

This section is dedicated to demonstrating that the fundamental group of the circle $\pi_1(\mathbb{S}^1)$, is isomorphic to the group of integers $\mathbb{Z}$. To achieve this, we will delve into the concept of covering spaces, a pivotal element in homotopy theory. The focus will be on establishing and understanding the lifting properties, which are essential tools that will be recurrently employed throughout this manuscript.

Definition 2.3.1 (Covering space, covering map, evenly covered and sheets)**.** *Let X be a topological space and let $p : \tilde{X} \to X$ be a continuous surjective map. The space $\tilde{X}$ is called a covering space of X if for every $x \in X$ there exists an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by p. The map p is called a covering map. Such a neighborhood U is called evenly covered by p and the disjoint sets in $\tilde{X}$ that project homeomorphically to U by p are called sheets of $\tilde{X}$ over U.*

Example 2.3.2. *Consider the construction of a covering space for the unit circle $\mathbb{S}^1$. A suitable covering space can be visualized as a helix in $\mathbb{R}^1$. The covering map $p : \mathbb{R}^1 \to \mathbb{S}^1$ is defined by $p(t) = e^{it}$. This setup ensures that for any point (x, y) on $\mathbb{S}^1$, and for any neighborhood U containing (x, y), the preimage $p^{-1}(U)$ under this map will be a disjoint union of open segments of the real line. Each segment, when projected onto $\mathbb{S}^1$, corresponds to the neighborhood U.*

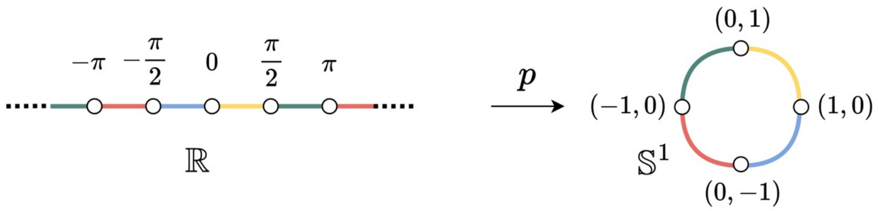

Figure 2.2: Depiction of the covering space transformation of $\mathbb{S}^1$ from $\mathbb{R}^1$ employing p.

2.3.1 Lifting properties

In this section, we introduce a theorem central to algebraic topology, which leads to two essential lemmas. First, consider this definition.

Definition 2.3.3 (Path's lift). *Let $p : \tilde{X} \to X$ be a covering map and let $\alpha : [0, 1] \to X$ be a path in X. A lift of α is a path $\tilde{\alpha} : [0, 1] \to \tilde{X}$ such that $p \circ \tilde{\alpha} = \alpha$.*

A natural question arises: given a point x_0 and a path α originating from x_0, is the path commencing from $p^{-1}(x_0) = \tilde{x}_0$, within the set of paths $p^{-1}(\alpha)$, unique? If this is true, then it logically follows that given a homotopy between two paths, the lifted homotopy should also be unique. This idea is intuitively plausible, and we can illustrate it using Example 2.3.2, where the pre-images of each path within the circle are a series of segments on the real line. By selecting one of the lifted starting points, we ensure the uniqueness of the lifting, and thus, for homotopies starting at a fixed point, the lifted homotopies should also be unique. These two lemmas, and their implications, are supported by a more general theorem, which we present now.

Lemma 2.3.4 (Homotopy lifting property). *Given a map $F : Y \times I \to X$ and a map $\tilde{F} : Y \times \{0\} \to \tilde{X}$ lifting $F|_{Y\times\{0\}}$, then there is a unique map $\tilde{F} : Y \times I \to \tilde{X}$ lifting F with a covering map p and restrictive to the given $\tilde{F}$ on $Y \times \{0\}$.*

Note that this is a really general theorem while Y can be be any conceivable set. This versatility becomes particularly relevant as we proceed to discuss two key properties that modify our understanding of Y.

Proof. This proof is divided into three claims: the first section aims to proof the existence of $\tilde{F}$, the second one will show that $\tilde{F}$ is unique if Y is a point, and the third one will conclude the proof for any Y.

Existence of $\tilde{F}$: Let $(y_0, t) \in Y \times I$, due to the continuity of F, we can find a neighborhood $N_t \times (a_t, b_t)$ such that $F(N_t \times (a_t, b_t))$ is contained in an evenly covered neighborhood of $F(y_0, t)$. Now, we know that by compactness of $y_0 \times I$, we can take a finitely products of $N_t \times (a_t, b_t)$ such that evenly covers $\{y_0\} \times (a_t, b_t)$. Therefore, we can take a single neighborhood N of y_0 and a finite partition of I, $0 = t_0 < t_1 < t_2 < ... < t_n = 1$ so that for each i, $F(N \times [t_i, t_{i+1}])$ is contained in an evenly covered U_i. Let say $\tilde{F}$ has been constructed on $N \times [0, t_i]$ starting with the given $\tilde{F}|_{N\times\{0\}}$. Since U_i is evenly covered, there exists an open set $\tilde{U}_i \subset \tilde{X}$ such that $p(\tilde{U}_i) = U_i$ and containing the point $\tilde{F}(y_0, t_i)$. Taking a neighborhood of y_0 i.e. $y_0 \in N_{y_0} \subset N$ we can assume that $\tilde{F}(N_{y_0}, t_i)$ is contained in $\tilde{U}_i$. Now we can define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ to be the composition of F with the homeomorphism $p^{-1} : U_i \to \tilde{U}_i$. After a finite number of steps we eventualy get the desired $\tilde{F} : N \times I \to \tilde{X}$ for some neighborhood N of y_0.

Uniqueness of $\tilde{F}$ if Y is a point: Suppose two lifts $\tilde{F}$ and $\tilde{F}'$ exist, both of which lift $F : I \to X$ and agree at the starting point, i.e. $\tilde{F}(0) = \tilde{F}'(0)$. Choose a partition of I as before. Assume, inductively, that $\tilde{F}$ and $\tilde{F}'$ coincide over $[0, t_i]$. Now consider the interval $[t_i, t_{i+1}]$. Since the image under $\tilde{F}$ and $\tilde{F}'$ is connected, $\tilde{F}([t_i, t_{i+1}])$ and $\tilde{F}'([t_i, t_{i+1}])$ each lie within a single sheet $\tilde{U}_i$. As both $\tilde{F}(t_i)$ and $\tilde{F}'(t_i)$ are the same by the inductive assumption, they both must lie in the same sheet of $\tilde{U}_i$. Given that the projection is injective on this sheet, and the two lifts project down to the same map F, it follows that $\tilde{F}$ and $\tilde{F}'$ are identical over $[t_i, t_{i+1}]$. Repeating this argument for all such intervals proves the uniqueness of the lift over the entire interval I.

Uniqueness of $\tilde{F}$ over Y: Finally, the constructed lifts $\tilde{F}$ on sets like $N \times I$ are unique on each segment $\{y\} \times I$ and agree on overlaps, resulting in a well-defined lift $\tilde{F}$ on $Y \times I$. Since $\tilde{F}$ it's continuous on each $N \times I$, therefore, $\tilde{F}$ is continuous using the Lemma 2.2.9. And $\tilde{F}$ is unique since it is unique on each segment $y \times I$. □

Lemma 2.3.5 (Path lifting property for covering spaces)**.** *Given a covering map $p : \tilde{X} \to X$, a path $\alpha : I \to X$ with $\alpha(0) = x_0$ and $\tilde{x}_0 = p^{-1}(x_0)$ then there is a unique lift $\tilde{\alpha} : I \to \tilde{X}$ starting at $\tilde{x}_0$.*

Proof. Define a function $F : \{0\} \times I \to X$ where $F(0, t) = \alpha(t)$, ensuring $F(0, 0) = x_0$. In parallel, define $\tilde{F} : \{0\} \times \{0\} \to \tilde{X}$ by setting $\tilde{F}(0, 0) = \tilde{x}_0$. By applying Theorem 2.3.4, it is inferred that a unique map $\tilde{F} : \{0\} \times I \to \tilde{X}$ exists, which satisfies $\tilde{F}(0, 0) = \tilde{x}_0$. Therefore, by defining $\tilde{\alpha}(t) = \tilde{F}(0, t)$, the unique lift of α that begins at $\tilde{x}_0$ is obtained, thus substantiating the lemma's claim. □

Lemma 2.3.6 (Homotopy lifting property for covering spaces)**.** *For each homotopy $F : I \times I \to X$ of paths starting at a point $x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde{F} : I \times I \to \tilde{X}$ of paths starting at $\tilde{x}_0$*

Proof. Define the path $\alpha : I \to X$ by $\alpha(t) = F(t, 0)$. Since $\alpha(0) = x_0$, and $\tilde{x}_0$ is a pre-image of x_0 under the covering map p, one can apply Lemma 2.3.5 to assert the existence of a unique path $\tilde{\alpha} : I \to \tilde{X}$ starting at $\tilde{x}_0$. Consequently, define $\tilde{F} : I \times \{0\} \to \tilde{X}$ such that $\tilde{F}(t, 0) = \tilde{\alpha}(t)$. By invoking Theorem 2.3.4, one obtains $\tilde{F} : I \times I \to \tilde{X}$, which is a homotopy. This homotopy is valid since the restrictions $F|_{I\times\{0\}}$ and $F|_{I\times\{1\}}$ lift constant paths, which must also be constant, due to the uniqueness established when applying Lemma 2.3.5. This is the desired result, as $p(\tilde{F}(t, \cdot)) = F(t, \cdot)$, thus completing the proof. □

Prior to progressing further, it is necessary to introduce new definitions that will play crucial roles throughout this thesis.

Definition 2.3.7 (Induced Homomorphism f_*)**.** *Consider X and Y as topological spaces, with $x_0 \in X$ and $y_0 \in Y$, and let $f : X \to Y$ be a continuous map satisfying $f(x_0) = y_0$. The induced homomorphism of f is defined as $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $f_*([x]) = [f(x)]$.*

Definition 2.3.8 (Pointed Covering Space)**.** *A pointed covering space is a covering space (Y, y_0) of (X, x_0) with covering $p : (Y, y_0) \to (X, x_0)$ such that $p(y_0) = x_0$.*

Here is a direct consequence of the previous lemmas.

Proposition 2.3.9. *The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ induced by the covering $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is injective.*

Proof. Let $[\tilde{\alpha}], [\tilde{\beta}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ such that $p_*([\tilde{\alpha}]) = p_*([\tilde{\beta}])$. Since $p_*([\tilde{\alpha}]) = p_*([\tilde{\beta}]) \Rightarrow [p(\tilde{\alpha})] = [p(\tilde{\beta})] \Rightarrow [\alpha] = [\beta]$, there exists f_t homotopy between α and β. Appealing to Lemma 2.3.5, $\tilde{\alpha}$ and $\tilde{\beta}$ are unique, therefore $p^{-1} \circ f_t \circ p$ is an homotopy between $\tilde{\alpha}$ and $\tilde{\beta}$. Hence $[\tilde{\alpha}] = [\tilde{\beta}]$. □

We now turn our attention to the lifting criterion, a fundamental concept that will play a pivotal role in future proofs.

Lemma 2.3.10 (Lifting criterion)**.** *Consider a covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$, where Y is both path-connected and locally path-connected. A lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ of f exists if and only if the induced map on the fundamental groups, $f_*(\pi_1(Y, y_0))$, is contained within $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.*

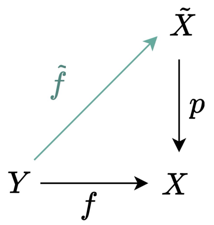

Figure 2.3: Illustration of the lifting criterion scheme.

Proof. In the (⇒) direction, assume the existence of a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ for f. It is observed that $f_* = p_* \circ \tilde{f}_*$. For any element $[\gamma]$ in $\pi_1(Y, y_0)$, the objective is to show that $f_*([\gamma])$ is a member of the image of p_* acting on $\pi_1(\tilde{X}, \tilde{x}_0)$. Noting that $f_*([\gamma]) = p_*\left(\tilde{f}_*([\gamma])\right)$, it suffices to demonstrate that $\tilde{f}_*([\gamma])$ belongs to $\pi_1(\tilde{X}, x_0)$. This follows directly from the definition of $\tilde{f}_*$ as a map from $\pi_1(Y, y_0)$ to $\pi_1(\tilde{X}, \tilde{x}_0)$.

In the (⇐) direction, suppose $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. To construct $\tilde{f}$, consider any point y in Y with a basepoint y_0. The task is to define $\tilde{f}(y)$ such that it resides in $\tilde{X}$ and ensures that $\tilde{f}$ is a continuous function.

Take an arbitrary element $y \in Y$ and select a path γ within Y that connects y_0 to y. Consequently, the composition $f \circ \gamma$ forms a path in X starting at x_0 and ending at $f(y)$. Utilizing Lemma 2.3.5, this path can be lifted, resulting in a corresponding path $(f \tilde{\circ} \gamma)$ in $\tilde{X}$. The function $\tilde{f}$ is then defined at y by setting $\tilde{f}(y) = (f \tilde{\circ} \gamma)(1)$. The remaining task involves establishing that this definition of $\tilde{f}$ is both well-defined and continuous.

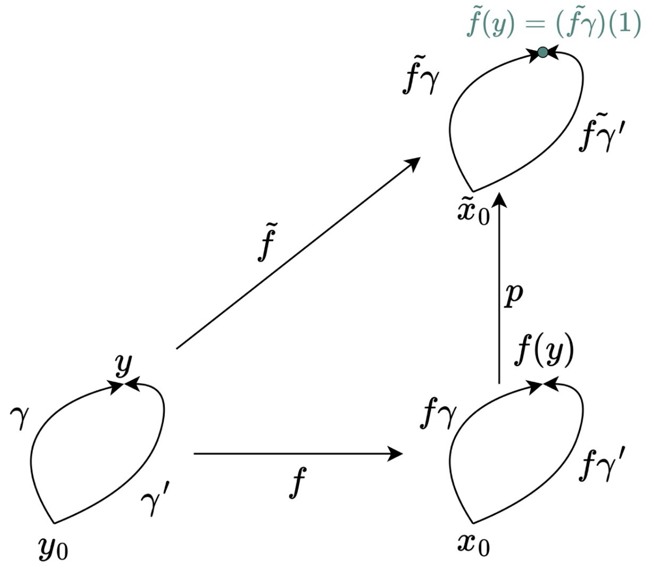

Figure 2.4: Construction diagram of $\tilde{f}$.

To establish that $\tilde{f}$ is well-defined, consider a different path γ' in Y that also connects y_0 to y. The loop formed by concatenating $f \circ \gamma'$ and the inverse of $f \circ \gamma$ at x_0 is denoted as loop α. It follows that $[\alpha]$ is an element of $f_*(\pi_1(Y, y_0))$, which is a subset of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Consequently, there exists a homotopy F connecting α to another loop α', which can be lifted to $\tilde{\alpha}'$. By applying Lemma 2.3.6, a lifted homotopy $\tilde{F}$ is obtained. Given that $\tilde{F}(\cdot, 1)$ is a loop at $\tilde{x}_0$, it implies that $\tilde{F}(\cdot, 0)$ equals $\tilde{\alpha}$, which is also a loop at $\tilde{x}_0$. By the construction of $\tilde{\alpha}$, it is deduced that $(f \circ \gamma')(1) = (f \circ \gamma)(1) = \tilde{f}(y)$, thereby confirming that $\tilde{f}$ is well-defined.

To demonstrate the continuity of $\tilde{f}$, we need to verify that for any given point $y \in Y$ and a neighborhood $U \subseteq X$ of $f(y)$, there exists a neighborhood V of y for which $\tilde{f}(V) \subseteq \tilde{U}$. Consider U as an open neighborhood around $f(y)$, which corresponds to $\tilde{U} \subseteq \tilde{X}$ such that the mapping $p : \tilde{U} \to U$ is a homeomorphism. Given the continuity of f, we can select a path-connected open neighborhood V around y satisfying $f(V) \subseteq U$. Our task now is to establish that $\tilde{f}(V) \subseteq \tilde{U}$. For any element $y' \in V$, we aim to show that $f(y') \in \tilde{U}$. To do this, we construct a path γ within V that connects y_0 to y, and another path α from y to y'. The concatenated path $(f \circ \gamma) \cdot (f \circ \alpha)$ can be lifted to $(f \tilde{\circ} \gamma) \cdot (f \tilde{\circ} \alpha)$ where $(f \tilde{\circ} \alpha) = p^{-1} \circ f \circ \alpha$. Considering that $\tilde{f}(y') = ((f \tilde{\circ} \gamma) \cdot (f \tilde{\circ} \alpha))(1) = (p^{-1} \circ f \circ \alpha)(1)$, and since $p^{-1} : U \to \tilde{U}$, it follows that $\tilde{f}(y') \in \tilde{U}$. Therefore, the restriction $\tilde{f}|_V = p^{-1} \circ f$

ensures that $\tilde{f}(V) \subseteq \tilde{U}$, confirming the continuity of $\tilde{f}$.

Thus, $\tilde{f}$ is a well-defined, continuous map lifting f, as desired. □

This theorem is a cornerstone in algebraic topology as it provides a necessary and sufficient condition for when a continuous map into a space can be lifted to a map into a covering space of that space. It is widely used in the study of fundamental groups and covering spaces.

2.3.2 Fundamental Group of the 1-Sphere

Now we are prepared to prove one of the most important theorems of the homotopy theory: the fundamental group of the 1-sphere is isomorphic to $\mathbb{Z}$. But, first of all, for illustrative purposes, we want introduce the idea of the proof: Consider $\mathbb{S}^1$ as the unit 1-sphere in the real plane. Given a loop based on $x_0 = (1, 0) \in \mathbb{S}^1$ one can imagine how many times the loop winds around the 1-sphere, either in positive or negative direction. This *winding number* can be any integer: 0 if the loop does not wind around the 1-sphere at all, 1 if it winds once in positive direction, -1 if it winds once in negative direction, 2 if it winds twice in positive direction, etc. This is the idea of the proof.

Theorem 2.3.11. *The fundamental group of the 1-sphere is isomorphic to a free group on one generator i.e.* $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Proof. Consider $\mathbb{R}^1$ as a covering space of $\mathbb{S}^1$ with the covering map $p : (\mathbb{R}^1, \tilde{x}_0) \to (\mathbb{S}^1, x_0)$ defined by $p(s) = (\cos(2\pi s), \sin(2\pi s))$ with $x_0 = (1, 0)$ and $\tilde{x}_0 = 0$. Define loops $w_n : I \to \mathbb{S}^1$ such that $w_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$. The objective is to demonstrate that $\pi_1(\mathbb{S}^1)$ is an infinite cyclic group generated by the homotopy class of the loop w_n.

Let $[\alpha] \in \pi_1(\mathbb{S}^1)$. Appealing to the Lemma 2.3.5 there exists a unique lift $\tilde{\alpha}$ starting at 0. Since $p(\tilde{\alpha}(1)) = \alpha(1) = x_0$ and $p^{-1}(x_0) = n \in \mathbb{Z}$, then $\tilde{\alpha}(1) = n \in \mathbb{Z}$. Since $\tilde{\alpha}(0) = 0$ and $\tilde{\alpha}(1) = n \in \mathbb{Z}$, $f_t = t\tilde{\alpha} + (1-t)\tilde{w}_n$ is a homotopy between $\tilde{\alpha}$ and $\tilde{w}_n$, so that $g_t = p \circ f_t$ is an homotopy between α and w_n, so $[\alpha] = [w_n]$.

It remains to prove that n is unique. Consider w_m such that $[\alpha] = [w_m]$. Thus $w_n \simeq w_m$, therefore there exists an homotopy f_t between w_n and w_m. Appealing to Lemma 2.3.6 there exists a unique lift of f_t, $\tilde{f}_t$, so that $\tilde{f}_0 = \tilde{w}_n$ and $\tilde{f}_1 = \tilde{w}_m$. Since $\tilde{f}_t$ is a homotopy of paths $f_0(1) = f_1(1)$. Hence $n = m$.

At this point, we have already proven that $\pi_1(\mathbb{S}^1) = \mathbb{W} = \{[w_n] : n \in \mathbb{Z}\}$, as sets. To finalize we have to show that $(\mathbb{W}, \otimes) \cong (\mathbb{Z}, +)$ as groups, where $\otimes$ is the operation defined in Proposition 2.2.10. Consider the following application.

$$\Phi : \mathbb{W} \to \mathbb{Z}, \ \Phi([w_n]) = n$$

It is injective since $\Phi([w_n]) = \Phi([w_m]) \to n = m \to [w_n] = [w_m]$. And it is surjective by definition of $\mathbb{W}$, i.e. if $n \in \mathbb{Z}$ then $[w_n] \in \mathbb{W}$. Finally $\Phi([w_n] \otimes [w_m]) = \Phi([w_n \otimes w_m]) = \Phi([w_{n+m}]) = n + m$. Thus Φ is an isomorphism $(\mathbb{W}, \otimes)$ and $(\mathbb{Z}, +)$, i.e. $(\mathbb{W}, \otimes) \cong (\mathbb{Z}, +)$. Hence $(\pi_1(\mathbb{S}^1), \otimes) \cong (\mathbb{Z}, +)$. □

2.3.3 Fundamental Group of n-Spheres for n greater than 1

Finally, we are going to prove that the fundamental group of the n-sphere is trivial for all $n \geq 2$. First we have to introduce a small lemma.

Lemma 2.3.12. *If a space X is the union of a collection of path-connected open sets A_α each of them containing the basepoint x_0 and for each intersection $A_\alpha \cap A_\beta$ is path-connected, then every loop is homotopic to a concatenation of loops each of which is contained on a single A_α.*

Proof. Consider a loop $f : I \to X$, we want a partition of I, $0 = s_1 < s_2 < ... < s_m = 1$ such that the image of $f_i = f|_{[s_{i-1}, s_i]}$ is contained in A_α.

Consider A_i as the A_α which contains the image of f_i, then for each pair A_i and A_{i+1}, consider the map g_i which goes from x_0 to $f(s_i) \in A_i \cap A_{i+1}$ since $A_i \cap A_{i+1}$ is path-connected. Now consider the following loop.

$$h = (f_1 \otimes g_1^{-1}) \otimes (g_1 \otimes f_2 \otimes f_2^{-1}) \otimes ... \otimes (g_m \otimes f_m)$$

This loop can be visualized using the Figure 2.5.

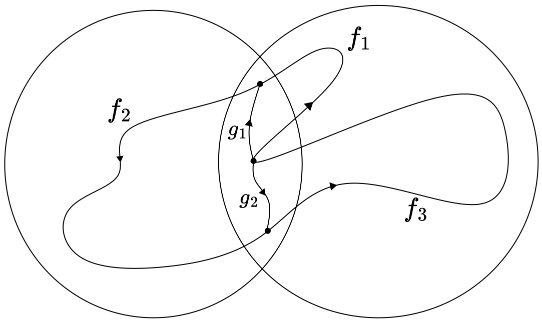

Figure 2.5: Visualization of each f_i and g_i.

Then h is homotopic to f by construction and h is constructed by a composition of loops each lying in a single A_i. □

Theorem 2.3.13. *The fundamental group of the n-sphere is trivial for all $n \geq 2$, i.e. $\pi_1(\mathbb{S}^n) = 0$, $n \geq 2$.*

Proof. Consider two different points $x_1, x_2 \in \mathbb{S}^n$ and define $A_1 = \mathbb{S}^n \backslash \{x_1\}$ and $A_2 = \mathbb{S}^n \backslash \{x_2\}$. Then $\mathbb{S}^n = A_1 \cup A_2$ and $A_1 \cap A_2$ is path-connected. Note that A_1 and A_2 are both homeomorphic to $\mathbb{R}^n$. Then for each loop $f : I \to X$, then there exists f_1, f_2 such that $[f] = [f_1 \otimes f_2]$ by 2.3.12. Since A_1 and A_2 are both homeomorphic to $\mathbb{R}^n$ and $\pi_1(\mathbb{R}^n) = 0$, then $[f_1] = [0]$ and $[f_2] = [0]$. Therefore $[f] = [0]$. Hence $\pi_1(\mathbb{S}^n) = 0$. □

2.4 Seifert-Van Kampen Theorem

The Seifert-Van Kampen theorem is a fundamental result in algebraic topology. It enables the computation of the fundamental group of a space utilizing the fundamental groups of its constituent subspaces. This section is dedicated to introducing Seifert-Van Kampen theorem. Additionally, it will be demonstrated that the fundamental group of the 2-sphere is trivial and that the fundamental group is not necessarily abelian. Thus, this section aims to address the final two objectives of the chapter.

To commence the exploration of the theorem, an illustrative example is presented. Consider A and B as two circles intersecting at a single, unique point x_0. Based on prior calculations, it is understood that $\pi_1(A) \simeq \pi_1(B) \simeq \mathbb{Z}$. Consequently, each fundamental group is isomorphic to a cyclic group generated by a single element. Denote these elements as a and b, representing the loop representatives for the cyclic groups corresponding to A and B, respectively. A loop encompassing both circles can be represented by each product of powers of a and b, indicating that such products are elements of $\pi_1(A \cup B)$. This leads to the preliminary conclusion that $\pi_1(A \cup B) \simeq \mathbb{Z} \times \mathbb{Z}$ which is the direct product of two copies of $\mathbb{Z}$. The Seifert-Van Kampen theorem will be employed to verify this hypothesis. To advance in this discussion, it is crucial to delineate some fundamental concepts.

Definition 2.4.1 (Free Product $\times_\alpha G_\alpha$)**.** *The free product of a collection of groups $\{G_\alpha\}_{\alpha \in I}$ forms a group composed of all words in the format:*

$$g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_n^{\epsilon_n}$$

where for each index i, g_i belongs to $G_{\alpha(i)}$ and $\epsilon_i = \pm 1$, indicating the inclusion of either the group element or its inverse. It is stipulated that if g_i and g_{i+1} are elements of the same group G_α, then $g_i g_{i+1}$ undergoes reduction based on the group operation in G_α. This implies that if $g_i = e_\alpha$ (the identity element in G_α), it does not appear in the word. Furthermore, if $g_{i+1} = g_i^{-1}$, then both elements are omitted.

With these foundational concepts established, it is now possible to introduce Seifert-Van Kampen theorem as follows:

Theorem 2.4.2 (Seifert-Van Kampen theorem)**.** *If X is a path-connected topological space formed by the union of path-connected open sets A_α, then the homomorphism $\Phi : \times_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective and the kernel of Φ is the normal subgroup N generated by all elements of form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism $\pi_1(X) \simeq \times_\alpha \pi_1(A_\alpha)/N$.*

Proof. A detailed proof can be found in [1, Theorem 1.20]. □

Example 2.4.3 (The Fundamental Group of the wedge of Two Circles)**.** *Consider A and B as two circles with their intersection at $x_0 = A \cap B$. It is already established that $\pi_1(A) \simeq \pi_1(B) \simeq \mathbb{Z}$, and there exist elements a and b such that*

$\pi_1(A) = \langle a \rangle$ *and* $\pi_1(B) = \langle b \rangle$*. Utilizing the Seifert-Van Kampen theorem, it can be deduced that* $\pi_1(A \cup B) = \langle a, b \rangle$*, which implies* $\pi_1(A \cup B) \simeq \mathbb{Z} \times \mathbb{Z}$*.*

This example serves as a demonstration of how the fundamental group is not necessarily abelian, since the free product $\mathbb{Z} \times \mathbb{Z}$ is not an abelian group.

Chapter 3

CW Complexes

In this chapter, we explore the concept of CW complexes, crucial structures in topological spaces. We aim to demonstrate the Cellular Approximation Theorem. Our exploration starts with the basics of CW complexes in Section 3.1, leading to a deeper analysis and examples, culminating in the detailed examination of the Cellular Approximation Theorem in Section 3.2.

3.1 Exploring CW Complexes

CW Complexes offer a framework for constructing topological spaces, specifically by attaching the boundary of k-dimensional disks to n-dimensional spheres, where $k < n$.

Definition 3.1.1 (n-cell)**.** *A n-cell e^n_α is a space homeomorphic to $\mathbb{D}^n \backslash \partial\mathbb{D}^n$ i.e. $e^n_\alpha \approx \mathbb{D}^n \backslash \partial\mathbb{D}^n$.*

Definition 3.1.2 (Attaching map and characteristic map)**.** *Consider a space X, a n-cell e^n_α and a map $\phi_\alpha : \partial e^n_\alpha \to X$, we can build a new space $X \cup_{\phi_\alpha} e^n_\alpha$ defined as the disjoint union $X \sqcup e^n_\alpha$ by identifying each $y \in \partial e^n_\alpha \subset e^n_\alpha$ with $\phi_\alpha(e^n_\alpha) \in X$ for all $y \in \partial e^n_\alpha$. We refer to this process as attaching e^n_α to X, and we call $\phi_\alpha : \partial e^n_\alpha \to X \cup_{\phi_\alpha} e^n_\alpha$ as the attaching map and $e^n_\alpha \to X \cup_{\phi_\alpha} e^n_\alpha$ as the characteristic map of e^n_α.*

A CW Complex is a space that has been formed by joining smaller spaces, so first we want to formalize the concept of joining these spaces.

Definition 3.1.3 (CW decomposition)**.** *A* CW decomposition *of a space X is a sequence of spaces $X^0 \subset X^1 \subset ... \subset X^n \subset ...$ such that it satisfies that*

1. *Start with a discrete collection of points. The result is a set of points called* 0-skeleton, X^0.
2. *X^n is obtained by X^{n-1} by attaching a set of n-cells $\{e^n_\alpha\}_{\alpha\in\Lambda}$ via the attaching maps $\phi_\alpha : \partial e^n_\alpha \to X^{n-1}$*
3. *$X = \bigcup_{n\geq 0} X^n$ with the weak topology i.e. $U \subset X$ is open if and only if $U \cap X^n$ is open in X^n for all $n \geq 0$.*

Definition 3.1.4 (CW Complex and n-skeleton)**.** *A CW Complex X is a space equipped with a CW decomposition. X^n is called the n-skeleton of X.*

To illustrate these definitions, several examples are provided.

Example 3.1.5 (Torus as a CW Complex)**.** *The process begins with a single point x, representing the 0-skeleton X^0. Subsequently, two circles that emanate from X^0 are considered, forming the 1-skeleton X^1. The final step involves covering these circles with a sphere, similar to a blanket covering X^1, creating the 2-skeleton X^2. The graphical representation of this construction is depicted in Figure 3.1.*

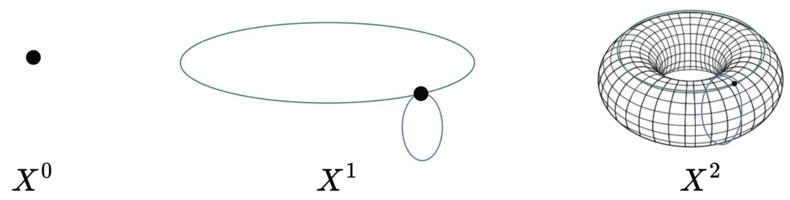

Figure 3.1: Skeletons of the Torus

Example 3.1.6 ($\mathbb{S}^n$ as a CW Complex)**.** *Consider the case of $\mathbb{S}^2$, a sphere in three-dimensional space. The construction commences with a unique point $x = e^0_\beta$, which constitutes the 0-skeleton X^0. The 1-skeleton is not involved in this particular instance. Instead, the focus shifts to a cell e^2_α designed such that its boundary ∂e^2_α maps to the point e^0_β. This cell uniquely represents the 2-skeleton X^2. Extending this concept to $\mathbb{S}^n$, the framework entails a single, unique point at X^0 and a unique cell at X^n. The cell's boundary is associated directly with the lone point at X^0, completing the structure of the n-dimensional sphere as a CW Complex.*

In the field of Algebraic Topology, a fundamental concept is the *topological invariant*, a key attribute for characterizing topological spaces. A topological invariant is a number that remains constant under homeomorphisms. When employing Cell Complexes, a significant outcome is the derivation of the *Euler Characteristic*.

Definition 3.1.7 (Euler Characteristic)**.** *Given a CW Complex X, the Euler Characteristic is defined by the formula:*

$$\mathcal{X}(X) = \sum_\alpha (-1)^n |e^n_\alpha|$$

Here, $|e^n|$ represents the cardinal of the set e^n, i.e. the number of n-cells.

Applying this to the torus example and the $\mathbb{S}^n$ example previously discussed, the Euler Characteristic is calculated as follows:

$$\mathcal{X}_{\text{torus}} = 1 - 2 + 1 = 0$$

$$\mathcal{X}_{\mathbb{S}^n} = \begin{cases} 1 - 0 + 0 - ... + 1 = 2 & \text{if } n \text{ is even} \\ 1 - 0 + 0 - ... - 1 = 0 & \text{if } n \text{ is odd} \end{cases}$$

Definition 3.1.8 (Subcomplex and CW pair)**.** *A subcomplex of a cell X is a closed subspace $A \subset X$ that is a union of cells of X. A pair (X, A) consisting of a CW complex X and a subcomplex A will be called CW pair.*

Lemma 3.1.9. *Consider a CW Complex X. $U \subset X$ is open if and only if $\phi_\alpha^{-1}(U) \subset e_\alpha^n$ is open for all e_α^n.*

Proof. A detailed proof can be found in [3, Lecture 8, Lemma 6]. □

Lemma 3.1.10. *Let X be a CW Complex and $A \subset X$ a subspace. If A has at most one point in each open cell then A is closed in X and the subspace topology on A is discrete.*

Proof. We check this by induction. We want to prove that $A \cap X^n$ is closed and discrete.

For $n = 0$, X^0 is discrete and closed in X therefore $A \cap X^0$ is discrete and closed in X. Now assume that $A \cap X^{n-1}$ is discrete and closed in X. For n, we divide $A \cap X^n = B \sqcup C$ with $B = A \cap X^{n-1}$ and $C = A \cap X^n \backslash X^{n-1}$. First of all, C is open since the n-cells in A because the open n-cells are open in X^n, and by the same reason, C is discrete. Consider $x \in \overline{C}$, then x overlies in the same cell as any other point $c \in C$ near x, and $c = x$, hence C is closed in X. Hence $A \cap X^n$ is discrete and closed in X. □

Proposition 3.1.11. *Any compact subset of CW Complex is contained in finitely many open cells.*

Proof. Consider a CW Complex X and a compact subset $B \subset X$. Now consider one point $p_\alpha \in X \cap e_\alpha^n$ for each n-cell e_α^n such that $e_\alpha^n \cap X \neq \emptyset$, so that we build $S = \{p_\alpha\}_{\alpha \in \Lambda}$. Using Lemma 3.1.10, S is closed in X and the subspace topology on S is discrete, since S is a closed subset contained in a compact set X, then S is compact. Since S is compact and discrete, S has to be finite. □

This latter proposition allows us to explain the terminology of the CW Complex given by J. H. C. Whitehead:

- C *'closure finite'*: The closure of every cell lies in a finite subcomplex.
- W *'weak topology'*: U is open if and only if $U \cap X^n$ is open for all $n \geq 0$.

Definition 3.1.12 (Homotopy Extension Property)**.** *The pair (X, A) has the homotopy extension property if given any space Y, a homotopy $f_t : A \to Y$, and a map $g_0 : X \to Y$ such that $f_0 = g_0|_A$, there exists a homotopy $g_t : X \to Y$ that starts from the given map g_0 and extends the homotopy f_t, in the sense that $f_t = g_t|_A$ for all t.*

Proposition 3.1.13. *If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the homotopy extension property.*

Proof. A detailed proof can be found in [1, Proposition 0.16]. □

To illustrate the latter proposition, consider X the torus and A one of the circles from the construction of the torus as a CW Complex detailed in Example 3.1.5. Then if we try to construct $X \times I$ the resulting space can be visualized as a solid torus from which a smaller torus has been removed, and if we try to construct $X \times \{0\} \cup A \times I$ we get the same torus surface but with a disc around the circle A. This whole process is illustrated in Figure 3.2. We can see that $X \times I \simeq X \times \{0\} \cup A \times I$.

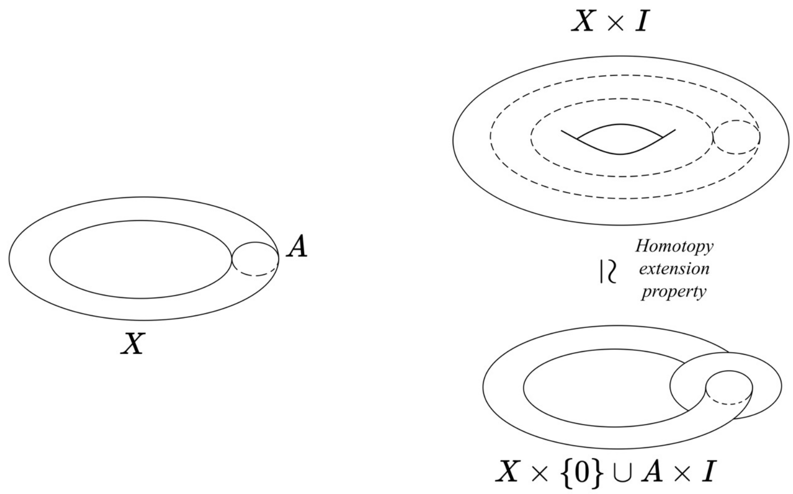

Figure 3.2: Homotopy extension property illustrated.

3.2 Cellular Approximation Theorem

Consider a mapping $f : \mathbb{S}^1 \to \mathbb{S}^2$ where f projects the elements of $\mathbb{S}^1$ onto the equator of $\mathbb{S}^2$. This setup intuitively suggests a relationship between the topological structures of $\mathbb{S}^1$ and $\mathbb{S}^2$, particularly in how the mapping f aligns with the cellular structures of these spheres. The *Cellular Approximation Theorem* formalizes this relationship.

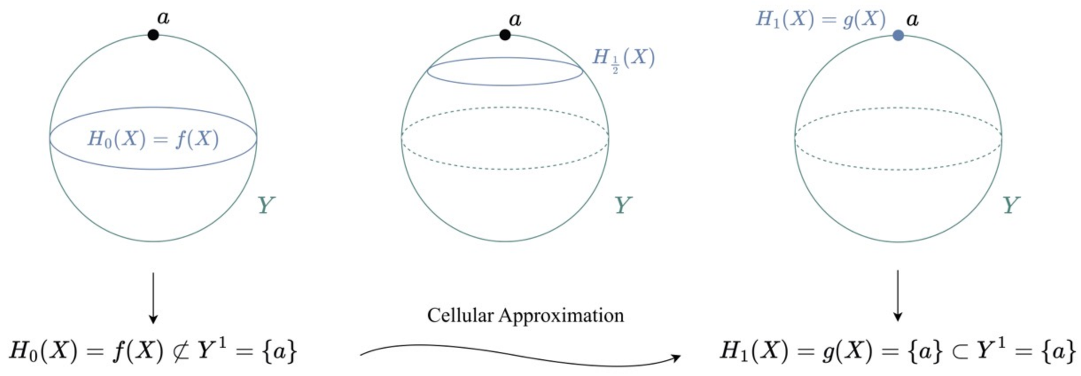

Figure 3.3: Illustration of a cellular approximation between $\mathbb{S}^1$, $\mathbb{S}^2$ and $f : \mathbb{S}^1 \to \mathbb{S}^2$.

Figure 3.3 visually illustrates the concept of cellular approximation. This diagram shows the initial space $X = \mathbb{S}^1$, mapped as the equator onto a secondary space $Y = \mathbb{S}^2$ through the function f. Cellular Approximation theory suggests an alternative function g, where $f \simeq g$ as demonstrated by the homotopy H_t, ensuring that the image of the k-skeleton of X is within the k-skeleton of Y, i.e., $g(X^k) \subset Y^k$. In our example, let $g : X \to Y$, $g(x) = a$. Consider the linear retraction from f to g over Y, denoted H_t. Figure 3.3 demonstrates that for H_0, which represents f, $f(X^1)$ corresponds solely to the equator so that it is not part of the 1-skeleton of Y, consisting only of the point $\{a\}$. However, $g(X^1) = \{a\}$, which indeed lies within $Y^1 = \{a\}$.

Definition 3.2.1 (Cellular map). *Let X, Y be CW complexes and let $f : X \to Y$ be a continuous map. We say that f is a* cellular map *if for every n-skeleton, X^n, its image is contained in the n-skeleton of Y, Y^n i.e. $X^n \subset X, f(X^n) \subset Y^n$.*

Definition 3.2.2 (Polyhedron and piecewise linear map). *A* polyhedron *in $\mathbb{R}^n$ is defined as a subspace that is the union of finitely many convex polyhedra. Each convex polyhedron is a compact set obtained by intersecting finitely many half-spaces, defined by linear inequalities of the form $\sum_i a_i x_i \leq b$. A* piecewise linear map *from a polyhedron to $\mathbb{R}^k$ is a map which is linear when restricted to each convex polyhedron.*

Lemma 3.2.3. *Let $f : I^n \to Z$ be a continuous map, where Z is obtained from a subspace W by attaching a cell e^k. Then there is a homotopy $f_t : (I^n, f^{-1}(e^k)) \to (Z, e^k) \text{rel} f^{-1}(W)$ from $f = f_0$ to a map f_1 for which there is a polyhedron $K \subset I^n$ such that:*

- *$f_1(K) \subset e^k$ and $f_1|_K$ is picewise linear with respect to an identification of e^k with $\mathbb{R}^k$.*
- *$f_1^{-1}(U) \subset K$ for some nonempty open set $U \subset e^k$.*

Proof. Identify $e^k \to \mathbb{R}^k$, consider two closed balls B_1, B_2 of radius 1 and 2 respectively, centered at $0 \in \mathbb{R}^n$. Since B_2 is compact in $\mathbb{R}^k$, $f^{-1}(B_2) \subset I^n$ will be compact, hence f will be uniformly continuous in $f^{-1}(B_2)$. Thus there exists $\epsilon > 0$, such that if $|x-y| < \epsilon$ implies $|f(x)-f(y)| < \frac{1}{2}$ for all $x, y \in f^{-1}(B_2)$. Now subdivide I so that I^n subdivides into cubes lying in a ball of diameter less than ϵ. Consider K_1 as the union of all cubes meeting $f^{-1}(B_1)$ and K_2 as the union of all cubes meeting K_1. The we have

$$f^{-1}(B_1) \subset K_1 \subset K_2 \subset f^{-1}(B_2)$$

We may assume that $\epsilon < \frac{1}{2}\text{dist}(f^{-1}(B_1), I^n \backslash f^{-1}(\text{int}B_2))$. Now, we subdivide each cube in K_2 into simplices inductively. The idea is to take all the faces of each cube in K_2 and add the center point of the face as a vertex to each simplex.

Let $g : K_2 \to e^k$ such that $g = f$ on vertices and g is extended linearly on each simplex. Let $\Phi : K_2 \to [0,1]$ such that $\Phi = 1$ on vertices in K_1, $\Phi = 0$ on vertices in $K_2 \backslash K_1$, and Φ is extended linearly in each simplex. Therefore

$$\begin{cases} \Phi(\sum_{i=0}^n t_i v_i) = \sum_{i=0}^n t_i \Phi(v_i) = \sum_{i=0}^n t_i = 1 & \text{if simplex } \sigma \in K_1 \\ \Phi(\sum_{i=0}^n t_i v_i) = \sum_{i=0}^n t_i \Phi(v_i) = \sum_{i=0}^n t_i \cdot 0 = 0 & \text{if simplex } \sigma \in K_2 \backslash K_1 \end{cases}$$

Thus $\Phi(K_1) = 1$ and $\Phi(K_2) = 0$. Finally we define a homotopy

$$f_t : K_2 \to e^k, \quad f_t = (1 - \Phi t)f + (\Phi t)g$$

so that $f_0 = f$, $f_1|_{K_1} = g|_{K_1}$ and f_t is the constant homotopy on simplices contained between K_1 and K_2. In fact, since $K_1 \subset \text{int}K_2$, we may extend f_t to be the constant homotopy of f on $I^n \backslash K_2$.

Now we want to prove that $f_1(\overline{I^n \backslash K_1})$ is disjoint from the centerpoint 0 of B_1 and hence from a neighborhood U of 0. This verification has two steps.

1. Since $f^{-1}(B_1) \subset K_2$, then $f(I^n \backslash K_2)$ must be disjoint from B_1.
2. Consider $\sigma \in K_2$ not entirely contained in K_1, then there exists a ball B_σ of radius $\frac{1}{2}$ such that $f(\sigma) \in B_\sigma$ since $K_2 \subset f^{-1}(B_2)$. Since f_t is an affine combination of f and g on K_2 and g and f are in B_σ and B_σ is convex, it follows that $f_t(\sigma) \subset B_\sigma$ for all t and in particular, $f_1(\sigma) \subset B_\sigma$. We know that $B_\sigma \not\subset B_1$ since

$$\left.\begin{array}{l} f^{-1}(B_1) \subset K_1 \\ \sigma \not\subset K_1 \\ f(\sigma) \subset B_\sigma \end{array}\right\} \Rightarrow \left.\begin{array}{l} f(\sigma) \not\subset B_1 \\ f(\sigma) \subset B_\sigma \end{array}\right\} \Rightarrow B_\sigma \not\subset B_1$$

   Since the radius of B_σ is $\frac{1}{2}$, which is the same of B_1, this implies that $0 \notin B_\sigma$ and thus $0 \notin f_1(\sigma)$.

These two facts imply that $0 \notin f_1(\overline{I^n \backslash K_1})$. Since f_1 must be continuous and $\overline{I^n \backslash K_1}$ is compact, then $f_1(\overline{I^n \backslash K_1})$ must be disjoint from some open set U containing 0. Setting $K = K_1$ we have that $f_1^{-1}(U) \subset K_1 = K$. □

With this foundational result established, we can now proceed to delve into the Cellular Approximation Theorem.

Theorem 3.2.4 (Cellular Approximation Theorem)**.** *Every continuous map $f : X \to Y$ between CW complexes X and Y is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, then this homotopy may be taken to be constant on A.*

Proof. We are going to prove this theorem inductively, so that we build up the map $g : X \to Y$ and the homotopy $H_t : f \simeq g$.

If $n = 0$, then $f(X^0)$ represents a discrete collection of points within Y. Consider an element e^0_α in X^0. Suppose that $f(e^0_\alpha)$ does not belong to Y^0. However, given that $f(e^0_\alpha)$ is an element of Y and, based on the construction principles of CW complexes, each n-cell in Y includes path-components terminating at some 0-cell in Y. Therefore, we can associate $f(e^0_\alpha)$ with the endpoint of the 0-cell of the path-component that intersects with $f(e^0_\alpha)$. This association enables us to construct the homotopy to a cellular map as postulated.

Suppose that $f : X \to Y$ is already cellular on the skeleton X^{n-1} i.e. $f(X^{n-1}) \subset Y^{n-1}$ and consider a n-cell e^n_α of X. Then $\overline{e^n_\alpha}$ is compact, so that $f(\overline{e^n_\alpha})$ is also compact in Y. By appealing the Proposition 3.1.11 then $f(e^n_\alpha) \subset f(\overline{e^n_\alpha})$ meets a finite number of cells, so that we can take e^k_β as the cell of largest dimension that meets $f(e^n_\alpha)$. Assume that $k > n$, otherwise f is already cellular, so that the idea is to prove that we can deform $f|_{X^{n-1}\cup e^n_\alpha}$, staying fixed on X^{n-1}, so that $f(e^n_\alpha)$ misses some point $p \in e^k_\beta$.

We now want to apply the Lemma 3.2.3. Consider $Z = Y^k$, $W = Y^k \setminus \text{int}(e^k)$ and composing f with the characteristic map $\Phi : I^n \to X^{n-1} \cup e^n$, so that $f \circ \Phi : I^n \to Y$, then we can apply the Lemma 3.2.3. We then obtain a homotopy

$$g_t : (I^n, (f \circ \Phi)^{-1}(e^k_\beta)) \to (Y^k, e^k_\beta) \text{ rel } (f \circ \Phi)^{-1}(Y^k \setminus \text{int}(e^k_\beta))$$

Since $\Phi(\partial I^n) \subset X^{n-1} \subset f^{-1}(Y^{n-1}) \subset f^{-1}(Y^k \backslash e^k_\beta)$, then g_t induces an homotopy

$$g'_t : (I^n, (f \circ \Phi)^{-1}(e^k_\beta)) \to (Y^k, e^k_\beta) \text{ rel } \partial I^n$$

Therefore $f_t = g'_t \circ \Phi^{-1}$ is an homotopy

$$f_t : X^{n-1} \cup e^n \to Y^k \text{ rel } X^{n-1}$$

And also the Lemma 3.2.3 tells us that g'_1 is picewise linear on some polyhedron $K \subset I^n$, so $g_1(K)$ is convex and is thus contained in the union of finitely many hyperplanes of dimension $n < k$. Also, since $U \subset e^k_\beta$ is open and $g_1^{-1}(U) \subset K$, there must be points in U that g_1 misses. Therefore f_1 misses points in U. Consider $p \in e^k_\beta$ be one of these points and compose a deformation retraction of $Y^k \backslash \{p\}$ onto $Y^k \backslash e^k_\beta$ with our homotopy f_t. Since $f(e^n_\alpha)$ intersects finitely many cells, we may iterate this process a finite number of times, until $f_1(e^n_\alpha)$ does not meet any cells of dimension greater than n.

Doing this on each n-cell $e^n_\alpha \subset X$, except those contained in A, gives a homotopy of $f|_{X^n}$ rel $X^{n-1} \cup A^n$ to a cellular map. Using the Proposition 3.1.13 we can extend this homotopy to all of X, constant on $X \backslash X^n$, we may apply each homotopy $f_t : X^n \to Y$ on the interval $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$, and then we let n go to ∞. □

Chapter 4

Higher Homotopy Groups

This chapter expands the discussion to higher homotopy groups, extending the foundational concepts established for the fundamental group to higher-dimensional spheres.

Initially, the chapter revisits the fundamentals of the fundamental group, defined through loops or continuous functions from the interval $[0, 1]$ to a space with fixed endpoints. This framework is extended to higher dimensions through continuous functions from the n-dimensional cube to a space, setting the groundwork for defining higher homotopy groups. These concepts are explored in detail in Section 4.1.

Further, the chapter discusses the properties and structures within higher homotopy groups, illustrating their implications and applications in topology. This structured approach clarifies the theoretical underpinnings of higher homotopy groups and sets the stage for advanced discussions on homotopical connectivity and the relationships between different homotopy groups, detailed in Section 4.2.

4.1 Definitions and Foundational Constructions

In pursuit of extending the fundamental group to higher dimensions (specifically $n \geq 2$), this section first establishes the homotopy relation in higher dimensions followed by the definition of the requisite operations before defining the higher-dimensional analogue of the fundamental group. This approach aligns with the methodology employed in Section 2.2 for the definition of the fundamental group.

Definition 4.1.1 (Homotopy relation in n-dimensions). *Let (X, x_0) be a pointed space. Let $\Omega(X, x_0)$ be the set of the continuous maps $\alpha : I^n \to X$ such that $\alpha|_{\partial I^n} = c_{x_0}$. Two maps $\alpha, \beta \in \Omega(X, x_0)$ are said to be homotopic if there exists a homotopy H_t relative to ∂I^n, i.e. $H_0 = \alpha$, $H_1 = \beta$ and $H_t(\partial I^n) = \{x_0\}$.*

Definition 4.1.2 (Concatenation). *Consider a pointed space (X, x_0). The concatenation of two mappings $f, g : I^n \to X$ such that $f|_{\partial I^n} = g|_{\partial I^n} = c_{x_0}$ is defined by the* *operation $f \otimes g$, where:*

$$(f \otimes g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & \textit{for } t_1 \in [0, 1/2], \\ g(2t_1 - 1, t_2, \dots, t_n) & \textit{for } t_1 \in [1/2, 1]. \end{cases}$$

With the relation and the operation defined, it becomes possible to construct the n-th homotopy group of a pointed space (X, x_0), denoted as $\pi_n(X, x_0)$.

Definition 4.1.3 (n-th homotopy group $\pi_n(X, x_0)$)**.** *The set of homotopy classes of $\Omega(X, x_0)$ is called the n-th homotopy group of (X, x_0) and is denoted by $\pi_n(X, x_0)$.*

This definition establishes the n-th homotopy group $\pi_n(X, x_0)$ as the set of homotopy classes of $\Omega(X, x_0)$. When $n = 1$, this n-th homotopy group coincides with the fundamental group. This equivalence arises because $\partial I = \{0, 1\}$ implies that the condition $f(\partial I) = x_0$ is equivalent to $f(0) = f(1) = x_0$.

Proposition 4.1.4. *The operation $\otimes$ is well-defined.*

Proof. Consider $f_1 \simeq f_2$ and $g_1 \simeq g_2$ in $\pi_n(X, x_0)$. We want to prove that $[f_1 \otimes g_1] = [f_2 \otimes g_2]$. Consider $F_t : f_1 \simeq f_2$ rel ∂I^n and $G_t : g_1 \simeq g_2$ rel ∂I^n. Since $F_t(\partial I^n) = G_t(\partial I^n) = \{x_0\}$, we can take $H_t : I^n \to X$ as $H_t = F_t \otimes G_t$. Therefore $H_t : f_1 \otimes g_1 \simeq f_2 \otimes g_2$ rel ∂I^n. Hence $[f_1 \otimes g_1] = [f_2 \otimes g_2]$. □

Proposition 4.1.5. *$\pi_n(X, x_0)$ is a group for all $n \geq 0$.*

Proof. Consider $[f], [g], [h] \in \pi_n(X, x_0)$ with $x_0 > 0$.

Associativity: We want to show that $[(f \otimes g) \otimes h] = [f \otimes (g \otimes h)]$. Considering the reparametrization φ built in Proposition 2.2.10, observe that $((f \otimes g) \otimes h)(\varphi(t)) = f \otimes (g \otimes h)$. Hence $[(f \otimes g) \otimes h] = [f \otimes (g \otimes h)]$.

Identity element: Consider $e(t_1, ..., t_n) = x_0$ we can show that e is the identity element by using the same reparametrization of Proposition 2.2.10.

Inverse element: The inverse element of f is $f^{-1}(s_1, ..., s_n) = f(1 - s_1, ..., 1 - s_n)$. Consider the homotopy

$$H_t(s_1, ..., s_n) = \begin{cases} f \otimes f^{-1} & \text{if } s_1 \in [0, 1-t) \\ x_0 & \text{if } s_1 \in [1-t, 1] \end{cases}$$

Hence $H_0 = f \otimes f^{-1}$, $H_1 = x_0$ and $H_t(\partial I^n) = \{x_0\}$. □

Proposition 4.1.6. *$\pi_n(X, x_0)$ is abelian $\forall n > 1$.*

Proof. Given any two loops f, g in $\pi_n(X, x_0)$, we want to show that their concatenation $f \otimes g$ is homotopic to $g \otimes f$, i.e., $f \otimes g \simeq g \otimes f$.

Consider the case when $n = 2$. This will give us an intuition that can be generalized to higher dimensions.

Consider the unit square I^2 where both f and g are defined. The product $f \otimes g$ can be visualized as first traversing the loop f in the left half of the square and then g in the right half. Similarly, $g \otimes f$ would involve traversing g in the top half and f in the bottom half.

To prove that these two are homotopic, we are going to construct an explicit homotopy between them.

Start by shrinking the domains of both f and g to smaller sub-cubes within I^2. Ensure that the boundaries of these sub-cubes map to the base point x_0 and extend the homotopy we are constructing by the constant x_0 to maintain continuity. After this contraction, we will have some free space within the unit square. Now, with the available free space, we can slide the two sub-cubes around as long as they do not overlap. The key observation here is that we can slide one sub-cube past the other to interchange their positions without them overlapping. Once the positions of the sub-cubes have been interchanged, expand both f and g to fill their original domains in I^2.

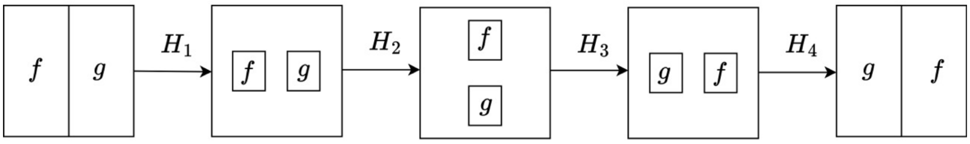

Figure 4.1: Construction of the Homotopy between $f + g$ and $g + f$.

The continuous transformation from $f \otimes g$ to $g \otimes f$ through these steps gives us the desired homotopy.

For higher dimensions, the same principle applies. Instead of considering sub-cubes within a square, we'd be considering sub-hypercubes within a hypercube I^n. The idea of shrinking, sliding, and expanding remains consistent, allowing us to construct a homotopy between $f \otimes g$ and $g \otimes f$ in any dimension $n > 1$. Hence, $\pi_n(X, x_0)$ is abelian for all $n > 1$. □

Note that the method used for the proof before cannot be applied for the fundamental group as there is not any space available while trying to shrink the domain.

At this moment, one could have seen that the idea of the loops in higher dimensions are more like "spheres" than something that maps a "square", so now we want to introduce a reinterpretation of the definition of the higher homotopy groups in terms of spheres.

Definition 4.1.7 (Higher homotopy groups in terms of spheres). *Let (X, x_0) be a pointed space. Let $\Omega(X, x_0)$ be the set of the continuous maps $\alpha : (\mathbb{S}^n, s_0) \to (X, x_0)$ such that $\alpha(s_0) = x_0$ where $s_0 = (1, 0, 0, ..., 0) \in \mathbb{S}^n$ is the base point of $\mathbb{S}^n$. Two maps $\alpha, \beta \in \Omega(X, x_0)$ are said to be homotopic if there exists a homotopy $H_t : \alpha \simeq \beta$ rel $\{s_0\}$. The set of homotopy classes of $\Omega(X, x_0)$ is called the n-th homotopy group of (X, x_0) and is denoted by $\pi_n(X, x_0)$.*

These two definitions are equivalent, but the second one is more intuitive. The sum of $f \otimes g$ can be interpreted as the composition $\mathbb{S}^n \xrightarrow{c} \mathbb{S}^n_1 \vee \mathbb{S}^n_2 \xrightarrow{f \vee g} X$ where c collapses the equator $\mathbb{S}^{n-1}$ in S^n to a point and we can choose the base point s_0 to lie in this $\mathbb{S}^{n-1}$, being $\mathbb{S}^n_1 \vee \mathbb{S}^n_2$ the quotient topological space given by identifying two points in two n-spheres and $f \vee g : \mathbb{S}^n_1 \vee \mathbb{S}^n_2 \to X$ defined as

$$(f \vee g)(s_1, ..., s_n) = \begin{cases} f(s_1, ..., s_n) & \text{if } (s_1, ..., s_n) \in \mathbb{S}^n_1 \\ g(s_1, ..., s_n) & \text{if } (s_1, ..., s_n) \in \mathbb{S}^n_2 \end{cases}$$

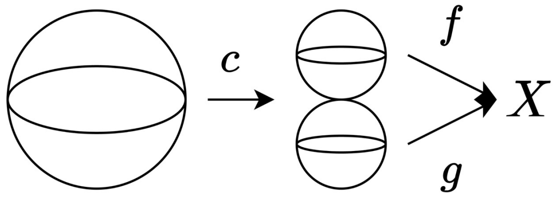

Figure 4.2: Construction of the concatenation between f and g, i.e. $f \otimes g$.

Even the fact that this definition is more difficult at the time of defining the concatenation, it is more intuitive and it is easier to imagine the transformations when working with it.

Another property of the fundamental group that is preserved in higher dimensions is that for a path-connected space X, the homotopy groups corresponding to different base points are isomorphic. This property facilitates a simplification in notation; henceforth, we shall denote the k-th homotopy group of the n-dimensional sphere simply as $\pi_k(\mathbb{S}^n)$, without specifying a particular base point.

Proposition 4.1.8. *If X is path connected, then $\pi_n(X, x_0) \approx \pi_n(X, x_1)$ $\forall\, x_0, x_1 \in X$.*

Proof. To prove that $\pi_n(X, x_0)$ is isomorphic to $\pi_n(X, x_1)$ we have to build an isomorphism, so for this proof, we will first define the map and, then, prove that indeed it is an isomorphism.

Since X is path-connected, then there exists a path between x_0 and x_1, i.e. $\gamma : I \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$, so that the map we were looking for is $\varphi : \pi_n(X, x_0) \to \pi_n(X, x_1)$ such that $\varphi([f]) = [\gamma f]$ with γf defined as follows: First shrink the domain of f to an smaller concentric cube. Then insert γ on each radial inside the gap formed. Resulting to something similar to the following Figure 4.3.

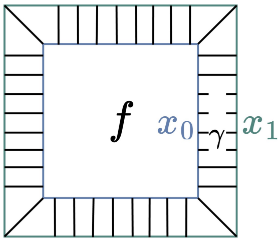

Figure 4.3: Construction of γf.

Now we have to prove that φ is a isomorphism.

Injectivity. Consider $[f], [g] \in \pi_n(X, x_0)$ such that $\varphi([f]) = \varphi([g])$, then $[\gamma f] = [\gamma g]$, consider the homotopy $H_t : \gamma f \simeq \gamma g$. We can take the inverse of γ as $\gamma^{-1} : I \to X$ such that $\gamma^{-1}(t) = \gamma(1-t)$ and consider $G_t = \gamma^{-1} H_t$. Then $G_0 = \gamma^{-1} H_0 = f$ and $G_1 = \gamma^{-1} H_1 = g$. Hence $[f] = [g]$.

Surjectivity. Consider $[g] \in \pi_n(X, x_1)$. Then $[\gamma^{-1} g] \in \pi_n(X, x_0)$ and $\varphi([\gamma^{-1} g]) = [\gamma(\gamma^{-1} g)] = [g]$.

Finally, we want to prove that $\gamma(f \otimes g) = \gamma f \otimes \gamma g$. We first deform f and g to be constant to the right and to the left respectively, we can define this as $(f \otimes e_0)$ and $(e_0 \otimes g)$, being e_0 the identity element of $\pi_n(X, x_0)$.

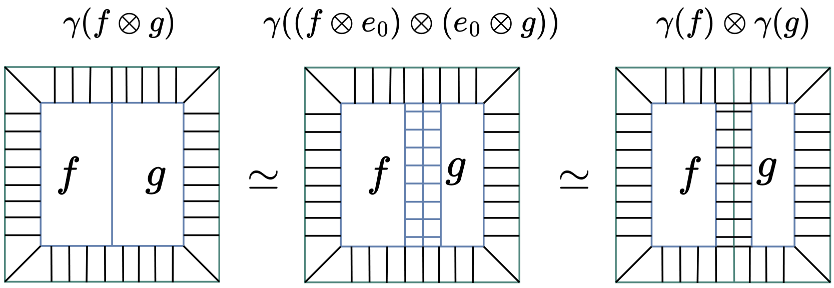

Figure 4.4: Illustration of the proof of $\gamma(f \otimes g) = \gamma f \otimes \gamma g$.

Thus $\gamma f \otimes \gamma g \simeq \gamma(f \otimes e_0) \otimes \gamma(e_0 \otimes g) \simeq \gamma((f \otimes e_0) \otimes (e_0 \otimes g)) \simeq \gamma(f \otimes g)$.

Hence φ is an isomoporhism between $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$. $\square$

With this proposition we can continue noting $\pi_n(X, x_0)$ as $\pi_n(X)$ as we did with the fundamental group.

Proposition 4.1.9. *A covering space $p : \tilde{X} \to X$ induces an isomorphism $p : \pi_n(\tilde{X}) \to \pi_n(X)$ for all $n \geq 2$*

Proof. p_* is surjective by the lifting criterion 2.3.10.

Observe that an element of p_* is represented by a loop $\tilde{f}_0 : I \to \tilde{X}$ with a homotopy f_t from $f_0 = p \circ \tilde{f}_0$ to the trivial loop f_1. Applying the Lemma 2.3.6 there exists a lifted homotopy $\tilde{f}_t$ starting in $\tilde{f}_0$ and ending in the constant loop. Hence $[\tilde{f}_0] = [0]$ in $\pi_n(\tilde{X})$. Hence p_* is injective.

Hence p_* is an isomorphism. □

Theorem 4.1.10. *The n-th homotopy group of the 1-sphere is trivial for $n \geq 2$, i.e. $\pi_n(\mathbb{S}^1) = 0$.*

Proof. Consider the following cover $p : \mathbb{R} \to \mathbb{S}^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$. Apply Proposition 4.1.9 to p, hence $\pi_n(\mathbb{S}^1) \approx \pi_n(\mathbb{R}) = 0$ for all $n \geq 2$. □

We can now move on to one of the objectives of this work.

Theorem 4.1.11. *The n homotopy group of a sphere of lower dimension $k < n$ is trivial, i.e. $\pi_n(\mathbb{S}^k) = 0$ for $n < k$*

Proof. Consider the CW Complex form of $\mathbb{S}^k$ as one point in $(\mathbb{S}^k)^0 = e^0_\alpha$ and the whole k-cell in $(\mathbb{S}^k)^k$ as illustrated in Figure 4.5.

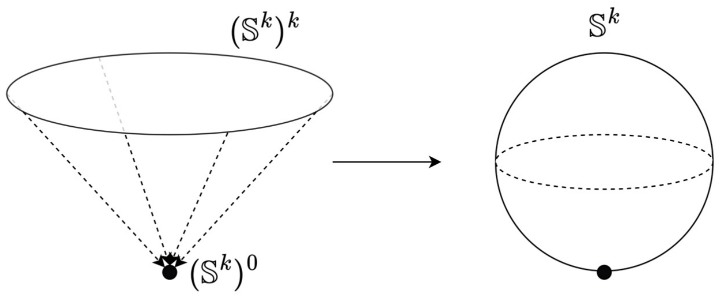

Figure 4.5: Cell Skeleton of $\mathbb{S}^k$

Let $[f] \in \pi_n(\mathbb{S}^k)$ where f is of the form $f : \mathbb{S}^n \to \mathbb{S}^k$. Appealing to the Cellular Approximation Theorem (Theorem 3.2.4), there exists g cellular such that $f \simeq g$. Since g cellular, $g((\mathbb{S}^n)^n) \subset (\mathbb{S}^k)^n$ and since $n < k$

$$(\mathbb{S}^k)^n = \{e^0_\alpha\} \Rightarrow g((\mathbb{S}^n)^n) = \{e^0_\alpha\} \Rightarrow g = c_{e^0_\alpha}$$

Therefore, f is homotopic to a constant map, and hence f is null-homotopic. This implies that $\pi_n(\mathbb{S}^k) = 0$ for $n < k$. □

4.2 Homotopical Connectivity and Relative Homotopy Groups

We aim to introduce the concept of connectivity within higher-order homotopy groups. This concept endeavors to generalize the property of path-connectedness.

Definition 4.2.1 (n-connected). *Consider a topological space X, it is said to be n-connected for $n \geq 0$ if $\pi_k(X) = 0$ for all $k \leq n$.*

If we consider a topological space X, employing the definition of $\pi_n(X, x_0)$, we can observe that $\pi_0(X, x_0)$ is interpreted as the group of homotopy classes of maps f of the form $f : (\mathbb{S}^0, s_0) \to (X, x_0)$. Therefore, we see that if X is path-connected, then $\pi_0(X, x_0) = 0$. Hence X being path-connected is equivalent to X being 0-connected. Likewise, if we consider X being simply connected, then $pi_0(X, x_0) = 0$ since X is path-connected and $\pi_1(X) = 0$, therefore X is 1-connected.

Example 4.2.2 (1-sphere). *From previous results, we have proven that $\pi_1(\mathbb{S}^1) \approx \mathbb{Z}$, hence $\mathbb{S}^1$ is 0-connected but not 1-connected.*

Given a pointed space (X, x_0) and and a subspace $x_0 \in A \subset X$, then we refer to (X, A, x_0) as a *pointed pair of spaces*. Since $A \subset X$ we can take the inclusion $i : (A, x_0) \to (X, x_0)$ which induces a map $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$.

Definition 4.2.3 (Map between pointed pair of spaces). *Given pointed pairs of spaces (X, A, C) and (Y, B, y_0), where $C \subset A \subset X$, we define a map between them $f : (X, A, C) \to (Y, B, y_0)$ to be a continuous map $f : X \to X = Y$ such that that $f(A) \subset B$ and $f(C) = y_0$. When C is a singleton $\{x_0\}$ last condition means $f(x_0) = y_0$.*

Definition 4.2.4 (Relative Homotopy Group of (X, x_0) based in $x_0 \in A \subset X$). *We defined the n-th Homotopy Group of (X, x_0) based in $x_0 \in A \subset X$ with $n \geq 1$ as*

$$\pi_n(X, A, x_0) = \{f, f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\} / \simeq_{\pi_n(X, x_0)}$$

being $J^n = I^n \times \{0\} \cup \partial I^n \times I$.

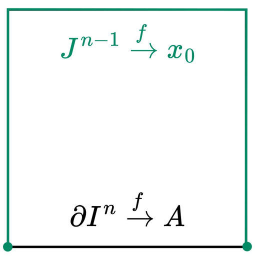

Figure 4.6: Illustration of ∂I^n and J^{n-1}.

We will simply write $\pi_n(X, A)$ instead of $\pi_n(X, A, x_0)$ unless there is a risk of ambiguity.

We can define the concatenation between maps $f, g : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ as

$$(f \otimes g)(t_1, t_2, ..., t_n) = \begin{cases} g(2t_1, t_2, ..., t_n) & if 0 < t_1 < \frac{1}{2} \\ f(2t_1 - 1, t_2, ..., t_n) & if \frac{1}{2} < t_1 < 1 \end{cases}$$

One checks that $\pi_n(X, A)$ is indeed a group by using the same arguments as in Proposition 4.1.5.

Finally, note that we can use the homotopical connectivity in the sense of relative homotopy groups, i.e. (X, A) is said to be n-connected with $n \geq 0$ if $\pi_k(X, A) = 0$ for all $k \leq n$.

Lemma 4.2.5 (Compression Criterion). *Consider a map $f : (\mathbb{D}^n, \mathbb{S}^{n-1}, s_0) \to (X, A, x_0)$. $[f] = [c_{x_0}] \in \pi_n(X, A, x_0)$ if and only if f is homotopic relative $\mathbb{S}^{n-1}$ to a map $g : (\mathbb{D}^n, s_0) \to (A, x_0)$.*

Proof. ($\Rightarrow$) Assume that $[f] = [c_{x_0}] \in \pi_n(X, A, x_0)$. Therefore, we cant consider $F : (\mathbb{D}^n \times I, \mathbb{S}^{n-1} \times I, \{s_0\} \times I) \to (X, A, x_0)$ the homotopy from f to the constant map to x_0. Now we can deform this homotopy to another homotopy H_t by shifting the top of the disk down and keeping $H_t|_{\mathbb{S}^{n-1}} = f|_{\mathbb{S}^{n-1}}$ as described in Figure 4.7.

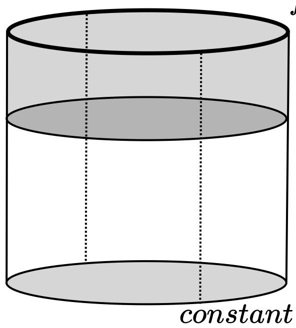

Figure 4.7: Illustration of the homotopy H_t.

Hence the image of H_1 is $H_1(\mathbb{D}^n) = H_1(\partial\mathbb{D}^n) \cup \{x_0\} = H_1(\mathbb{S}^{n-1}) \cup \{x_0\} = F(\mathbb{S}^{n-1} \times I) \subset A$ and $H_t|_{\mathbb{S}^{n-1}} = f|_{\mathbb{S}^{n-1}}$.

($\Leftarrow$) Suppose that f is homotopic to g relative $\mathbb{S}^{n-1}$ and $g(\mathbb{D}^n) \subset A$. Since $\mathbb{D}^n$ is contractible to s_0, consider the homotopy $H_t : \mathbb{D}^n \to s_0$ and now consider $G_t := g \circ H_t$. Observe that G_t is a homotopy between g and a constant map and $G_t(\mathbb{S}^{n-1}) \subset A$ since $g(\mathbb{D}^n) \subset A$, hence $[f] = [g] = [c_{x_0}]$ in $\pi_n(X, A, x_0)$. $\square$

Chapter 5

Freudenthal Suspension Theorem

This aims to demonstrate the Freudenthal Suspension Theorem. We begin by defining exact sequences, focusing on their role in homotopy groups in Section 5.1. This foundation supports our proofs of the Whitehead's Theorem and the CW Approximation Theorem in Sections 5.2 and 5.3, respectively. Finally, we rigorously present and prove the Freudenthal Suspension Theorem in Section 5.4.

5.1 Exact sequences

In the context of group theory, a sequence $G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} ... \xrightarrow{f_n} G_n$ is said to be *exact at* G_i if $im(f_i) = ker(f_{i+1})$. The sequence is called $exact$ if the sequence is exact for all G_i.

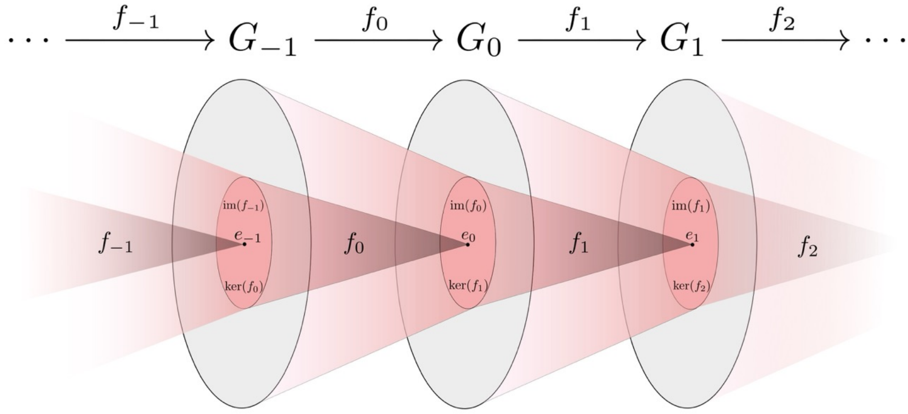

Figure 5.1: Illustration of an Exact Sequence of Groups extracted from [5]

Proposition 5.1.1. *Some properties of the maps are derived within some exact sequences*

1. *If* $A \xrightarrow{f} B \to 0$ *is exact then* f *is surjective.*
2. *If* $0 \to A \xrightarrow{f} B$ *is exact then* f *is injective.*
3. *If* $0 \to A \xrightarrow{f} B \to 0$ *is exact then* f *is an isomorphism.*

Proof. For the sequence $A \xrightarrow{f} B \to 0$ to be exact, the image of f must be the kernel of the zero map, which is B, hence f is surjective. For the sequence $0 \to A \xrightarrow{f} B$ to be exact, the kernel of f must be the image of the zero map, which is 0, hence f is injective. Finally, if $0 \to A \xrightarrow{f} B \to 0$ is exact, then f is both surjective and injective, thus an isomorphism. □

Lemma 5.1.2 (Five lemma)**.** *Consider the following conmutative diagram of abelian groups*

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{k} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{k'} & E' \end{array}$$

if the rows are exact then

1. γ *is surjective if* β *and* δ *are surjective and* ε *is injective, and,*
2. γ *is injective if* β *and* δ *are injective and* α *is surjective.*

Proof. A detailed proof can be found in [1, The Five-Lemma]. □

Theorem 5.1.3 (Exact Sequence for Relative Homotopy Groups)**.** *Let* i_* *and* j_* *be the homomorphisms induced by the inclusions* $i : (A, x_0) \hookrightarrow (X, x_0)$ *and* $j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ *respectively, and let* $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ *be given by* $\partial([f]) = [f|_{I^{n-1}\times 0}]$*. Then the following sequence is exact*

$$\ldots \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \ldots$$

Proof. ($im(i_*) = ker(j_*)$) Consider $[f] \in \pi_n(A, x_0)$, since $im(f) \subset A$, by the Compression Criterion 4.2.5, $(j_* \circ i_*)([f]) = [f] \in \pi_n(X, A, x_0)$ which is indeed $[f] = [0]$ since f is homotopic to a map $f : (\mathbb{D}^n, s_0) \to (A, x_0)$ relative to $\mathbb{S}^{n-1}$, hence $im(i_*) \subset ker(j_*)$. Now consider $[f] \in \pi_n(X, x_0)$ such that $j_*([f]) = [0]$. Since $j_*([f]) = [0] \in \pi_n(X, A, x_0)$

by the compression criterion 4.2.5, there exists $H_t : f \simeq g$ rel $\mathbb{S}^{n-1}$ such that $im(g) \subset A$, therefore $i_*([g]) = [i(g)] = [g] = [f] \in \pi_n(A, x_0)$, hence $ker(j_*) \subset im(i_*)$.

($im(j_*) = ker(\partial)$) Consider $[f] \in \pi_n(X, x_0)$, so that $f(I^{n-1}) = f(\partial I^n) = \{x_0\}$, therefore $(\partial \circ j_*)([f]) = \partial([f]) = [f|_{I^{n-1}\times 0}] = [x_0]$. Hence we can consider the homotopy between x_0 and 0, so that $im(j_*) \subset ker(\partial)$. Now consider $[f] \in ker(\partial)$, since $\partial([f]) = [f|_{I^{n-1}\times 0}] = [0]$ we can take the homotopy $H_t : f|_{I^{n-1}\times 0} \simeq 0$. Now we can consider G_t as

$$\begin{cases} G_0 = f \\ G_t|_{I^{n-1}} = H_t \\ G_t = x_0 \text{ otherwise} \end{cases}$$

Therefore $j_*([G_1]) = [G_1] = [G_0] = [f] \in \pi_n(X, A, x_0)$, hence $ker(\partial) \subset im(j_*)$.

($im(\partial) = ker(i_*)$) Consider $[f] \in \pi_n(X, A, x_0)$, therefore $(i_* \circ \partial)([f]) = [f|_{I^{n-1}} \times 0] \in \pi_{n-1}(X, x_0)$ but note that $[f|_{I^{n-1}} \times 0] = [0] \in \pi_{n-1}(X, x_0)$ via the homotopy $f_t(\bar{x} \in I^{n-1}) = f(\bar{x}_1 \cdot (1-t), ..., \bar{x}_{n-1} \cdot (1-t), 0)$, so that $f_0 = f$ and $f_1 = x_0$, thus $(i_* \circ \partial)([f]) = [f_1] = [x_0] = [0] \in \pi_{n-1}(X, x_0)$, hence $im(\partial) \subset ker(i_*)$. Now consider $[f] \in ker(i_*)$, since $i_*([f]) = [f] = [x_0] \in \pi_{n-1}(X, x_0)$, we can take the homotopy $H_t : f \simeq x_0 \in \pi_{n-1}(X, x_0)$. Now we can take another homotopy G_t such that $G_t|_{\partial I^{n-1}} = \{x_0\}$ and $G_t = H_t$ otherwise, so that $imG_0 = imf \subset A$, $G_1 = x_0 \in A$ and $G_t(\partial I^{n-1}) = \{x_0\} \subset A$. Now considering $h : I^n \to X$, $h(\bar{x}) := H_{\bar{x}_n}(\bar{x}_1, ..., \bar{x}_{n-1})$, we can conclude that $h(I^n) \subset X$, $h(\partial I^n) \subset X$ and $h(J^{n-1}) = x_0$, thus $\partial([h]) = [f] \in ker(i_*)$. Hence $ker(i_*) \subset im(\partial)$. □

5.2 Whitehead's Theorem

Lemma 5.2.1 (Compression lemma). *Consider a CW pair (X, A) and a pair (not necessarily CW) (Y, B) such that $B \neq \emptyset$. Consider $n \geq 0$, if there exists $e^n_\alpha \in X \backslash A$ and $\pi_n(Y, B) = 0$ then every map $f : (X, A) \to (Y, B)$ is homotopic relative A to a map $g : X \to B$, i.e. there exists $H_t : f \simeq g$ rel A.*

Proof. We are going to prove the lemma by induction on X^n, the idea is to prove that fixing $n \geq 0$, if there exists $e^{n+1}_\alpha \in X^{n+1} \backslash A$ and $\pi_{n+1}(Y, B) = 0$, then there exists $H_t : f \simeq g$ rel A.

For the case base, consider $n = 0$. Suppose there exists $e^0_\alpha \in X^1 \backslash A$ and $\pi_0(Y, B) = 0$. Now consider $f : (X^0, A) \to (Y, B)$, since e^0_α is just a single point, then $\partial f|_{e^0_\alpha}$ does not exists, therefore $f|_{e^0_\alpha} \in \pi_0(Y, B)$. Thus $[f|_{e^0_\alpha}] = [0]$ in $\pi_0(Y, B)$, so that we can apply the compression criterion 4.2.5 getting that there exists $g : X^0 \to B$ such that there exists $H_t : f \simeq g$ rel A. This process can be applied for all 0-cells that exist in $X^0 \backslash A$.

Assume that the lemma works for $n - 1$. Suppose there exists $e^n_\alpha \in X^n \backslash A$ and $\pi_n(Y, B) = 0$. Then $f|_{e^n_\alpha} \subset Y$ and since $\partial e^n_\alpha \in X^{n-1}$, and in X^{n-1} is assumed the lemma, then there can be two cases, first if $\partial e^n_\alpha \in A$ then $\partial f|_{e^n_\alpha} \in B$, otherwise f is homotopic to a map g such that $im(g) \subset B$. Hence $[f|_{e^n_\alpha}] \in \pi_{n-1}(Y, B)$. Therefore, we have that $[f|_{e_\alpha^n}] = [0]$ in $\pi_{n-1}(Y, B)$, so that we can apply the compression criterion 4.2.5 getting that there exists $g : X^n \to B$ such that there exists $H_t : f \simeq g$ rel A. This process can be applied for all n-cells that exist in $X^n \backslash A$. □

Definition 5.2.2 (Weak Homotopy Equivalence). *A map $f : X \to Y$ is a weak homotopy equivalence if it induces an isomorphism $f_* : \pi_n(X) \to \pi_n(Y)$ for all n.*

Example 5.2.3. *Consider the Warsaw circle, $\mathbb{S}_W$, defined as*

$$\mathbb{S}_W = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid -\frac{1}{2}\pi < x \leq \frac{1}{2}\pi, x \neq 0 \right\} \cup \{(0, y) \mid -1 \leq y \leq 1\} \cup C$$

where C is an arc in $\mathbb{R}^2$ joining $\left(\frac{1}{2}\pi, 0\right)$ and $\left(-\frac{1}{2}\pi, 0\right)$, disjoint from the other two subsets specified above except at its endpoints. This space looks like Figure 5.2.

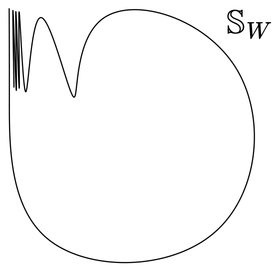

Figure 5.2: Illustration of the Warsaw circle, $\mathbb{S}_W$.

Then, the Warsaw circle is weak homotopy equivalent to a point p but it is not possible to retract the Warsaw circle to a point, i.e. p is not homotopy equivalent to $\mathbb{S}_W$.

Whitehead´s theorem asserts that weak homotopy equivalence implies homotopy equivalence for CW complexes.

Theorem 5.2.4 (Whitehead's Theorem). *If a map $f : X \to Y$ between CW Complexes induces an isomorphism $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In the case that X is a CW subcomplex of Y and f is the inclusion of X in Y, X is a deformation by retraction of Y.*

Proof. First, consider X as a CW subcomplex of Y and f as the inclusion of X in Y. Since $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism, then using the long exact sequence of homotopy groups 5.1.3, we know that $im(\partial) = ker(f_*) = 0$, thus $\pi_n(Y, X) = 0$. We can apply the Compression Lemma 5.2.1 so that $id : (Y, X) \to (Y, X)$ is homotopy relative X to a map g such that $im(g) \subset X$. Hence X is a retraction of Y.

To prove the first part of the lemma, consider the mapping cylinder M_f, defined in Example 2.1.8. We know that Y is a retraction of M_f, so taking the inclusion $i : Y \hookrightarrow M_f$, induces an isomorphism $i_* : \pi_n(Y) \to \pi_n(M_f)$, now using the exact sequence of homotopy groups 5.1.3 we see that $\pi_n(M_f, X) = 0$. If f is a cellular map then X is a CW subcomplex of M_f, therefore X is a retraction of M_f, which means that X is homotopy equivalent to M_f. If f is not a cellular map, then we can apply the Cellular Approximation Theorem 3.2.4 to get $g \simeq f$ with g cellular. Hence X is homotopy equivalent to Y. □

5.3 CW Approximation

Definition 5.3.1 (CW model)**.** *Let X be a space and $A \subset X$ a CW Complex. An n-connected CW model for (X, A) is an n-connected CW pair (Z, A) and a map $f: Z \to X$ such that $f|_A$ is the identity map and $f_*: \pi_i(Z) \to \pi_i(X)$ is an isomorphism for all $i > n$ and an monomorphism when $i = n$.*

Proposition 5.3.2. *Consider a space X and a CW Complex $A \subset X$ such that $A \neq \emptyset$, then for all $n \geq 0$, there exists a n-connected CW model (Z, A) of (X, A). This model can be chosen by attaching cells to A of dimension greater than n.*

Proof. The idea for the proof is to construct Z by attaching cells of dimension greater than n to A. Therefore, we are constructing

$$A = Z_n \subset Z_{n+1} \subset Z_{n+2} \subset ...$$

where Z_{k+1} is obtained by attaching $k+1$-cells to Z_k. We are going to prove this inductively.

First of all, the base case $k = n$ is trivial since $Z_n = A$ so the map f we are looking for is just the identity.

Assume that we can construct a $k-1$-connected CW model (Z_{k-1}, A) of (X, A) by attaching cells to A of dimension greater than $k-1$. Consider $f: Z_{k-1} \to X$ as the map of the CW model and consider $\{[\phi_\alpha]\} := ker(f_*)$. Then for each $\alpha \in \{[\phi_\alpha]\}$, attach a k-cell e^k_α such that $\partial e^k_\alpha = \phi_\alpha$ to Z_{k-1} resulting in a CW Complex denoted as Y_k. Since (Y_k, Z_{k-1}) is a CW pair, we can extend f to $f': Y_{k+1} \to X$. Note that the map $f'_*: \pi_i(Y_k) \to \pi_i(X)$ is still injective for $n \leq i < k-1$ because f_* was injective and the extension is generated by attaching cells of superior dimension. It is surjective when $n \leq i \leq k-1$ since $f_* = f'_* \circ j_*$ with $j: Z_{k-1} \hookrightarrow Y_k$.

Now consider $\{[\varphi_\beta]\}$ as the set of generators of $\pi_k(X)$ and attach an sphere $\mathbb{S}^k$ to Y_k resulting in a CW complex denoted as Z_k. In the same way as before, extend f' to $f'': Z_k \to X$. Therefore, the map $f''_*: \pi_i(Z_k) \to \pi_i(X)$ is surjective for $i = k$ by construction and it is surjective when $n < i \leq k-1$ and injective when $n \leq i \leq k-1$ since $f'_* = j_* \circ f''_*$, being $j: Y_k \to Z_k$.

Finally, define $Z := \bigcup_{i \geq n} Z_i$ and $g: Z \to X$ with $g|_{Z_i} = g_i$, being g_i the map defined on each step of the inductive construction of Z_i. First we know that Z is constructed only attaching cells of dimension larger than n to A. Indeed, (Z, A) is n-connected since $i_*: \pi_i(A) \to \pi_i(Z)$ is an isomorphism for $i \leq n$, therefore we can use the long exact sequence for homotopy groups 5.1.3. $g_*: \pi_i(Z) \to \pi_i(X)$ is an isomorphism when $i > n$ and an injection when $i = n$ since $(f_i)_*$ are also injective.

Finally, observe that $g|_A$ is the identity map. Hence g defines an n-connected CW model (Z, A) of (X, A). □

Corollary 5.3.3 (CW Approximation Theorem)**.** *Consider a n-connected CW pair (X, A), then there exists a CW pair (Z, A) such that $Z \backslash A$ only contains cells of dimension greater than n and there exists an homotopy equivalence $g : Z \to X$ such that $g|_A = id$.*

Proof. Applying Proposition 5.3.2 we get a n-connected CW model (Z, A) such that $Z \backslash A$ only contains cells of dimension greater than n. Considering the asscoiated map of the CW model, $g : Z \to X$, we already know that $g|_A = id$. We just have to prove that $g_* : \pi_i(Z) \to \pi_i(X)$ is an isomorphism for all i, and then, by applying the Whitehead's Theorem 5.2.4 we will get that g is a homotopy equivalence.

We already know by definition of CW model that g_* is an isomorphism for $i > n$ and an injection for $i = n$. Hence, we just have to show that it is an isomorphism for $i < n$ and a surjection for $i = n$. Consider the following diagram.

$$\begin{array}{ccc} \pi_i(Z) & \xrightarrow{j_*} & \pi_i(X) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i'_*} \\ \pi_i(A) & \xleftarrow[id_*]{} & \pi_i(A) \end{array}$$

where i and i' are induced by the natural inclusions. When $i \leq n$, $\pi_i(X, A) = 0$ and $\pi_i(Z, A) = 0$, so by their long exact sequences for homotopy groups, i_* and i_* are isomorphisms for $i < n$ and are surjections for $i = n$. Hence $g_* : \pi_i(Z) \to \pi_i(X)$ is an isomorphism for all i.

As said before, we apply the Whitehead's Theorem 5.2.4 to g, so that g is an homotopy equivalence. □

5.4 Freudenthal Suspension Theorem

Proposition 5.4.1 (Homotopy Excision Theorem)**.** *Let X be a CW Complex decomposed as the union of subcomplexes A and B such that $X = A \cup B$ and $C = A \cap B$ is connected and non empty. If (A, C) is m-connected, (B, C) is n-connected and $j : (A, C) \to (X, B)$ is the inclusion map then $j_* : \pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism when $i < m + n$ and is a surjection when $i = m + n$.*

Proof. We are going to do this proof by dividing it into three cases.

Case 1. Assume that A is constructed by attaching $m + 1$-cells e_α^{m+1} to C and B is constructed by attaching one $n + 1$-cell e^{n+1} to C. We want to prove that $j_* : \pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism when $i < m + n$ and is a surjection when $i = m + n$.

First, we prove that j_* is suprajective. Consider $f : (I^i, \delta I^i, J^{i-1}) \to (X, B, x_0)$ so that $[f] \in \pi_i(X, B, x_0)$. Since f is continuous and compact, then $im(f)$ intersects with a finite number of cells. Applying Lemma 3.2.3 we obtain that there exist a homotopy between f and a map g such that there exist simplexes $\Delta_\alpha^{m+1} \subset e_\alpha^{m+1}$ and $\Delta^{n+1} \subset e^{n+1}$ so that $g^{-1}(\Delta_\alpha^{m+1})$ and $g^{-1}(\Delta^{n+1})$ are polyhedra and g is picewise linear. Before continuing, we have to introduce a claim.

Claim. If $i \leq m+n$ then there exist points $p_\alpha \in e_\alpha^{m+1}$, $q \in e^{n+1}$ and a map $\varphi : I^{i-1} \to [0,1]$ such that

- $\varphi|_{\delta I^{i-1}} = 0$
- $f^{-1}(q)$ is below the graph of φ in I^i
- $f^{-1}(p_\alpha)$ is above the graph of φ in I^i

Proof. Consider $q \in e^{n+1}$, since $f^{-1}(\Delta^{n+1})$ is a finite union of convex polyhedra, then $g^{-1}(q)$ is a finite union of convex polyhedra of dimension lower or equal than $i-(n+1)$. Let $T = \pi^{-1}(\pi(g^{-1}(q)))$ being $\pi : I^i \to I^{i-1}$ the projection map. Note that T has dimension at most $i-n$. Hence $f(T) \cap \Delta_\alpha^{m+1}$ is also of dimension at most $i-n$, since $T \cap g^{-1}(\Delta_\alpha^{m+1})$ is also of dimension at most $i-n$ and $g|_{T \cap g^{-1}(\Delta_\alpha^{m+1})}$ is linear. Thus if $i-n < m+1$ there is some $p_\alpha \in \Delta_\alpha^{m+1}$ that is not in $g(T)$, so that $g^{-1}(p_\alpha) \cap T = \emptyset$. Now choose all the p_α that satisfies this, since T and $g^{-1}(p_\alpha)$ are closed subspaces of I^i they all have to be compact. Thus we can choose a ε-neighborhood around $\pi(g^{-1}(q))$, U, disjoint from $\pi(g^{-1}(p_\alpha))$ for all α. Hence we can construct $\varphi : I^{i-1} \to [0,1]$ satisfying the claimed elements. □

Consider the homotopy F_t of f such that F_1 is the restriction of f to the space above the graph of φ given by applying the claim. Hence $F_1(I^i) \cap \{q\} = \emptyset$. Let $P = \bigcup_\alpha g^{-1}(p_\alpha)$ and $Q = g^{-1}(q)$. Then we have the following conmutative diagram

$$\begin{array}{ccc} \pi_i(A,C) & \xrightarrow{j_*} & \pi_i(X,B) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i'_*} \\ \pi_i(X\backslash Q, (X\backslash Q)\backslash P) & \xrightarrow{i''_*} & \pi_i(X, X\backslash P) \end{array}$$

being i_*, i'_* and i''_* induced by the obvious inclusion maps. Note that i'_* and i_* are isomorphisms and since $[g]$ is image of i''_* and $[f] = [g]$, then $[f] \in im(j_*)$. Hence j_* is suprajective.

Now we prove that j_* is injective. Consider $[f], [g] \in \pi_i(A, C)$ such that $j_*([f]) = j_*([g])$. Thus there exists an homotopy $F_t : f \simeq g$. Using the claim, we can retract this homotopy just above the graph of φ, so that we can remove $F^{-1}(q)$ from the domain of F. Hence $[f] = [g]$.

Case 2. Assume that A is constructed by attaching $m+1$-cells e_α^{m+1} to C and B is constructed by attaching cells of dimension lower or equal than $n+1$ to C. We want to prove that $j_* : \pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism when $i < m+n$ and is a surjection when $i = m+n$.

First, we prove that j_* is suprajective. Consider $f : (I^i, \delta I^i, J^{i-1}) \to (X, B, x_0)$ so that $[f] \in \pi_i(X, B, x_0)$. Since f is continuous and compact, then we can apply the same procedure as for Case 1. and we get that $[f] \in im(j_*)$. The injectivity is also analogous to Case 1.

Case 3. Assume that A is constructed by attaching cells of dimension greater or equal than $m+1$ to C and B is constructed by attaching cells of dimension lower or equal than $n+1$ to C. We want to prove that $j_* : \pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism when $i < m+n$ and is a surjection when $i = m+n$.

Note that we may assume the cells $A \backslash C$ have dimension at most $m+n+1$ because adding cells of dimension greater than $m+n+1$ will have no effect on π_i for $i \leq m+n+1$.

Let $A_k \subset A$ the union of C with all the cells of A of dimension k at most and let $X_k = A_k \cup B$. We will prove by induction on k that $(j|_{A_k})_* : \pi_i(A_k, C) \to (X_k, B)$ is an injection when $i < n+m$ and a surjection when $i \leq n+m$.

Note that our base case is when $k = m+1$ which is indeed the Case 3. We thus need only to prove when $k = n+m+1$. Consider the following diagram

$$\begin{array}{ccccccccc}
\pi_{i+1}(A_k, A_{k-1}) & \xrightarrow{f} & \pi_i(A_{k-1}, C) & \xrightarrow{g} & \pi_i(A_k, C) & \xrightarrow{h} & \pi_i(A_k, A_{k-1}) & \xrightarrow{k} & \pi_{i-1}(A_{k-1}, C) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle (j|_{A_{k-1}})_*} & & \downarrow{\scriptstyle (j|_{A_k})_*} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle (j|_{A_{k-1}})_*} \\
\pi_{i+1}(X_k, X_{k-1}) & \xrightarrow{f'} & \pi_i(X_{k-1}, B) & \xrightarrow{g'} & \pi_i(X_k, B) & \xrightarrow{h'} & \pi_i(X_k, X_{k-1}) & \xrightarrow{k'} & \pi_{i-1}(X_{k-1}, B)
\end{array}$$

If $i < n+m$ then α and δ are isomorphisms by the Case 2. and by inductive hypothesis $(j|_{A_{k-1}})_*$ is an isomorphism. Applying Lemma 5.1.2, $(j|_{A_k})_*$ is an isomorphism.

If $i = n+m$ then α and δ are surjective by the Case 2. and by inductive hypothesis $(j|_{A_{k-1}})_*$ is an isomorphism. Applying Lemma 5.1.2, $(j|_{A_k})_*$ is surjective.

Conclusion. Given that the pair (A, C) is n-connected and (B, C) is m-connected, we apply the CW Approximation Theorem 5.3.3 to deduce the homotopy equivalences $g_A : (A', C) \to (A, C)$ rel C and $g_B : (B', C) \to (B, C)$ rel C. Here, $A' \backslash C$ includes only cells of dimension at least $m+1$, and $B' \backslash C$ consists solely of cells with dimension at least $n+1$. Since these homotopy equivalences are relative to $C = A \cap B$, the pasting lemma allows us to construct a homotopy equivalence $f : (A' \cup B', B') \to (A \cup B, B)$. Consequently, $f_* : \pi_i(A' \cup B', B') \to \pi_i(A \cup B, B)$ forms an isomorphism. From Case 3, it follows that $j'_* : \pi_i(Z_A, C) \to \pi_i(Z_A \cup Z_B, Z_B)$, induced by inclusion, is injective for $i < m+n$ and surjective for $i \leq m+n$. Hence, this property must also hold for $j_* : \pi_i(A, C, x_0) \to \pi_i(X, B, x_0)$, as evidenced by the commuting diagram:

$$\begin{array}{ccc} \pi_i(A', C) & \xrightarrow{g_A} & \pi_i(A, C) \\ \downarrow j'_* & & \downarrow j_* \\ \pi_i(A' \cup B', B') & \xrightarrow{f_*} & \pi_i(A \cup B, B) \end{array}$$

□

Definition 5.4.2 (Suspension). *Consider a space X. The suspension of X, SX is defined as the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to a single point and $X \times \{1\}$ to another single point.*

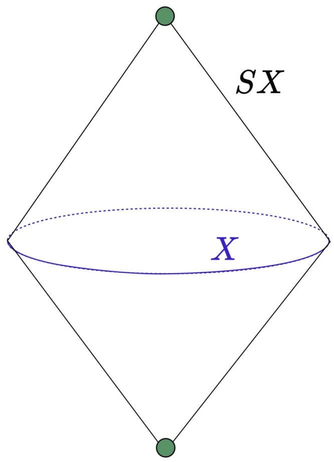

Figure 5.3: Illustration of SX.

One can see SX as a double cone on X, i.e. the union of two copies of the cone $CX = (X \times I)/(X \times \{0\})$.

Example 5.4.3 (Suspension of $\mathbb{S}^n$). *As in Figure 5.3, we can take the space $\mathbb{S}^1$, take two suspension points, which are the green points in the figure, and then draw both cones, so that the resulting space is homotopic to $\mathbb{S}^2$. Indeed, the suspension of $\mathbb{S}^n$ is homotopic to $\mathbb{S}^{n+1}$.*

Suspension is a really important concept in homotopy theory, since not only can we do the suspension of spaces, but we can actually do the suspension of maps i.e. given $f : X \to Y$ its suspension map would be $Sf : SX \to SY$, the quotient map of $f \times I : X \times I \to Y \times I$.

Corollary 5.4.4 (Freudenthal Suspension Theorem). *Let X be an $n-1$-connected CW Complex. Then there is a map $j : \pi_i(X) \to \pi_{i+1}(SX)$ that is an isomorphism when $i < 2n - 1$ and is a surjetion when $i = 2n - 1$*

Proof. We can decompose SX as the two cones C_+X, C_-X intersecting in a copy of X i.e. $C_+X \cap C_-X \approx X$ and $C_+X \cup C_-X \approx SX$. Recalling the exact sequence seen in Theorem 5.1.3, we can build the following exact sequence using the pair (C_+X, X):

$$... \xrightarrow{\partial} \pi_n(X, x_0) \xrightarrow{i_*} \pi_n(C_+X, x_0) \xrightarrow{j_*} \pi_n(C_+X, X, x_0) \xrightarrow{\partial} \pi_{n-1}(X, x_0) \xrightarrow{i_*} ...$$

Since C_+X is contractible, then $\pi_1(C_+X) = \{[x_0]\}$, therefore $ker(\partial_1) = im(i_*) = \{x_0\}$, hence for $C_+X \backslash x_0$, ∂ defines an isomorphism between $\pi_{n+1}(C_+X, X, x_0)$ and $\pi_n(X, x_0)$ and the same can be done for the pair (SX, C_-X).

$$\pi_i(X, x_0) \xrightarrow{\cong} \pi_i(C_+X, x_0) \longrightarrow \pi_i(SX, C_-X, x_0) \xrightarrow{\cong} \pi_{i+1}(SX, x_0)$$

Being the map of the middel just the inclusion. If X is $(n-1)$-connected, the CW pair $(C_\pm X, X)$ is n-connected by using the Theorem 5.1.3 and the same idea of before with the pair $(C_\pm X, X)$. This allows us to apply Proposition 5.4.1, so the map in the middle of the diagram is an isomorphism when $i < 2n-1$ and a surjection when $i = 2n-1$. □

Chapter 6

Fibrations

This chapter delves into the foundational aspects of fibrations, exploring their intricate properties and implications within the field of topology. We begin by introducing the concept of fibrations, particularly focusing on Hurewicz and Serre fibrations, discussed in detail in Section 6.1. This sets the stage for a deeper exploration of fiber bundles in Section 6.2, where we examine their structure and significance. The chapter culminates with an analysis of the Hopf bundle, a specific example of a fiber bundle, which is rigorously treated in Section 6.3. Finally, the chapter ends by showing that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ and $\pi_3(\mathbb{S}^2) = \mathbb{Z}$.

6.1 Hurewicz and Serre fibrations

Definition 6.1.1 (Right Lifting Property (RLP))**.** *Consider a map $p : E \to X$ and $i : A \to B$. The map p is said to have the right lifting property with respect to i if for any two maps $f : A \to E$ and $g : B \to X$, there exists a map $h : B \to E$ with $p \circ h = g$ and $h \circ i = f$.*

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ \downarrow{\scriptstyle i} & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & X \end{array}$$

Definition 6.1.2 (Serre fibration)**.** *A map $p : E \to X$ of topological spaces is a Serre fibration if it has the RLP with respect to all inclusions of the form*

$$i : I^n \times \{0\} \hookrightarrow I^n \times I$$

Definition 6.1.3 (Hurewickz fibration)**.** *A map $p : E \to X$ of spaces is a Hurewickz fibration if it has the RLP with respect to all maps of the form*

$$i : A \times \{0\} \to A \times I$$

Definition 6.1.4 (Fiber)**.** *Consider a map of spaces* $p : E \to X$ *and* $x \in X$*. Then* $p^{-1}(x) \subset E$ *is called the fiber of* p *over* x*.*

Observe that Hurewickz fibrations are all the maps $p : E \to B$ with the Homotopy Lifting Property 2.3.6: Consider an homotopy $H : A \times I \to X$ and a lift of $H_0 = H(\cdot, 0)$, $\tilde{H_0} : A \times \{0\} \to E$ and consider the inclusion $i : A \hookrightarrow A \times I$, if $p : E \to X$ is a Hurewickz fibration, there exists a map $h : A \times I \to E$ such that the following diagram commutes.

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{\tilde{H_0}} & E \\ \downarrow{\scriptstyle i} & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow{H} & X \end{array}$$

Proposition 6.1.5. *If* $p : E \to B$ *is a Hurewickz fibration and* B *is path-connected, then the fibers are homotopy equivalent.*

Proof. Consider two points $b_0, b_1 \in B$. We want to prove that there exist $f : p^{-1}(b_0) \to p^{-1}(b_1)$ and $g : p^{-1}(b_1) \to p^{-1}(b_0)$ such that $f \circ g \simeq id_{p^{-1}(b_1)}$ and that $g \circ f \simeq id_{p^{-1}(b_0)}$.

Since B is path connected, there exists a path $\gamma : I \to B$ such that $\gamma(0) = b_0$ and $\gamma(1) = b_1$. Consider $e \in p^{-1}(b_0)$, and suppose that $\tilde{\gamma}_e(0) = e$ which is something that we can firmly state since $p(e) = b_0 = \gamma(0) = p(\tilde{\gamma}(0))$. Now, since p is a Hurewickz fibration, the following diagram commutes

$$\begin{array}{ccc} \{0\} & \xrightarrow{c_e} & E \\ \downarrow{\scriptstyle i} & \overset{\tilde{\gamma}_e}{\nearrow} & \downarrow{\scriptstyle p} \\ I & \xrightarrow{\gamma} & B \end{array}$$

There is a lift $\tilde{\gamma}_e : I \to E$ such that $p \circ \tilde{\gamma} = \gamma$ starting at e. Now observe that $p(\tilde{\gamma}_e(1)) = \gamma(1) = b_1$, therefore $\tilde{\gamma}_e(1) \in p^{-1}(b_1)$. By doing this procedure to every point in $p^{-1}(b_0)$ it results in a map $f : p^{-1}(b_0) \to p^{-1}(b_1)$ by defining $f(e) := \tilde{\gamma}_e(1)$. Applying the same method to γ^{-1}, it results in $g : p^{-1}(b_1) \to p^{-1}(b_0)$.

It remains to prove that $f \circ g \simeq id_{p^{-1}(b_1)}$ and that $g \circ f \simeq id_{p^{-1}(b_0)}$. First, consider the following map:

$$H_t(e) := \begin{cases} \tilde{\gamma}_e(2t) \text{ if } t \in [0, \frac{1}{2}) \\ f(\tilde{\gamma^{-1}}_e(2t-1)) \text{ if } t \in [\frac{1}{2}, 1] \end{cases}$$

Observe that $H_0(e) = \tilde{\gamma}_e(0) = e$, $H_1(e) = f(\tilde{\gamma^{-1}}_e(1)) = (f \circ g)(e)$ and indeed $H_{\frac{1}{2}^-}(e) = \tilde{\gamma}_e(1) = f(e) = f(\tilde{\gamma^{-1}}_e(0)) = H_{\frac{1}{2}^+}(e)$. Hence $f \circ g \simeq id_{p^{-1}(b_1)}$. The same procedure applies to prove that $g \circ f \simeq id_{p^{-1}(b_0)}$. □

From now on, for a fibration $p : E \to B$, we can discuss a general fiber F without defining a prior point x whose fiber is $F = p^{-1}(x)$, as we now know that all fibers are homotopy equivalent if B is path-connected by Proposition 6.1.5, which we will assume in our subsequent cases.

Theorem 6.1.6 (The long exact sequence of a Serre fibration)**.** *Consider a Serre fibration $p : (E, e_0) \to (X, x_0)$ and $i : (F, e_0) \hookrightarrow (E, e_0)$ the fiber, i.e. $p^{-1}(x_0) = F$. Then there is a long exact sequence of the form:*

$$\ldots \xrightarrow{p_*} \pi_{n+1}(X, x_0) \xrightarrow{\delta} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(X, x_0) \xrightarrow{\delta} \ldots$$

Proof. First we have to construct $\delta : \pi_{n+1}(X, x_0) \to \pi_n(F, e_0)$. Consider c_{e_0} as the constant map in e_0 and $[\alpha] \in \pi_{n+1}(X, x_0)$. Then the following diagram commutes since p is a Serre fibration

$$\begin{array}{ccc} J^n \cong I^n \times \{0\} & \xrightarrow{c_{e_0}} & E \\ \big\downarrow j & \overset{\beta}{\nearrow} & \big\downarrow p \\ I^n \times I & \xrightarrow{\alpha} & X \end{array}$$

Therefore, observe that $c_{e_0} = \beta \circ j$ and $\alpha = p \circ \beta$. Define $\delta([\alpha]) := \beta|_{I^n \times \{1\}}$, so that $\delta|_\alpha : I^n \times \{1\} \to F$ since $p^{-1}(x_0) = F$ and $p \circ \delta|_\alpha = p \circ \beta|_{I^n \times \{1\}} = p \circ \beta|_{J^n} = x_0$. Hence $\delta([\alpha]) \in \pi_n(F, e_0)$.

Now we have to check if δ is well-defined. Consider $[\alpha_1] = [\alpha_2] \in \pi_{n+1}(X, x_0)$ and consider β_1 and β_2 the diagonals of the diagram respectively, we want to prove that $\delta([\alpha_1]) = \delta([\alpha_2]) \in \pi_n(F, e_0)$. Since $[\alpha_1] = [\alpha_2]$ we can take the homotopy $H : I^{n+1} \times I \to X$. Consider $\widetilde{J^{n+1}}$ as J^{n+1} but interchanging the two last coordinates i.e. $\widetilde{j_n} = j_{n+1}$ and $\widetilde{j_{n+1}} = j_n$, and define the map $k : \widetilde{J^{n+1}} \to E$

$$k(\bar{t}) = \begin{cases} \beta_0(\bar{t}) & \text{if } \bar{t} \in I^{n+1} \times \{0\} \\ \beta_1(\bar{t}) & \text{if } \bar{t} \in I^{n+1} \times \{1\} \\ e_0 & \text{otherwise} \end{cases}$$

Then the following diagram commutes since p is a Serre fibration

$$\begin{array}{ccc} \widetilde{J^{n+1}} \cong I^{n+1} \times \{0\} & \xrightarrow{k} & E \\ \big\downarrow j & \overset{l}{\nearrow} & \big\downarrow p \\ I^{n+1} \times I & \xrightarrow{H} & X \end{array}$$

$l|I^n \times \{1\} \times I$ gives a homotopy from $\beta_1|_{I^n \times \{1\}}$ to $\beta_2|_{I^n \times \{1\}}$. Hence $\delta([\alpha_1]) = \delta([\alpha_2])$.

Exactness at $\pi_n(E)$. Consider $[\alpha] \in \pi_n(F, e_0)$, then $(p_* \circ i_*)([\alpha]) = [x_0]$ since $F = p^{-1}(x_0)$. Hence $im(i_*) \subset ker(p_*)$. Now consider $[\alpha] \in ker(p_*) \subset \pi_n(E, e_0)$. We have that $p_*([\alpha]) = [x_0]$, consider the homotopy between x_0 and $[p(\alpha)]$, $H : I^n \times I \to X$ and consider k such that $k|_{I^n \times 0} = \alpha$ and $k = c_{e_0}$ otherwise, then the following diagram commutes since p is a Serre fibration

$$\begin{array}{ccc} J^n & \xrightarrow{k} & E \\ \Big\downarrow j & \overset{l}{\nearrow} & \Big\downarrow p \\ I^n \times I & \xrightarrow{H} & X \end{array}$$

Define $\gamma := l|_{I^n \times \{1\}}$, so that $[\gamma] \in \pi_n(F, e_0)$ and $i_*([\gamma]) = [i(\gamma)] = [\alpha]$ by the homotopy l. Hence $ker(p_*) \subset im(i_*)$.

Exactness at $\pi_n(X)$. Consider $[\beta] \in \pi_n(E)$ and consider the following diagram:

$$\begin{array}{ccc} J^{n-1} & \xrightarrow{c_{e_0}} & E \\ \Big\downarrow j & \overset{\beta}{\nearrow} & \Big\downarrow p \\ I^n & \xrightarrow{\alpha} & X \end{array}$$

Then $\alpha = p \circ \beta$, so taking $\delta([\alpha]) = (\delta \circ p_*)([\beta]) = \beta|_{I^{n-1} \times \{1\}} = c_{e_0}$. Hence $im(p_*) \subset ker(\delta)$.

Now consider $[\alpha] \in ker(\delta) \subset \pi_n(X, x_0)$ and use the same diagram as before. Then, for the diagonal β, we have that $[\beta|_{I^{n-1} \times \{1\}}] = [e_0]$. Consider H the homotopy between $\beta|_{I^{n-1} \times \{1\}}$ and e_0 and define the map G as

$$G(\overline{s}, t) = \begin{cases} \beta(\overline{s}, 2t) & \text{if } t \in [0, \frac{1}{2}) \\ H(\overline{s}, 2t - \frac{1}{2}) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

Then $[G] \in \pi_n(E, e_0)$ and $p \circ G$ is homotopic to $p \circ \beta = \alpha$. Hence $ker(\delta) \subset im(p_*)$.

Exactness at $\pi_{n-1}(F)$. Consider $\alpha \in \pi_n(X)$ and β constructed for $\delta([\alpha])$. Then we know that $\beta|_{I^{n-1} \times \{1\}} \simeq \beta|_{I^{n-1} \times \{0\}} = c_{e_0}$. Therefore $i_*(\delta([\alpha])) = [c_{e_0}]$. Hence $im(\delta) \subset ker(i_*)$. Now consider $[\gamma] \in ker(i_*) \subset \pi_{n-1}(F, e_0)$, so that $i_*([\gamma]) = [i(\gamma)] = [c_{e_0}]$. Consider $H : I^{n-1} \times I \to E$ the homotopy between $i(\gamma)$ and c_{e_0}. Observe the following diagram

$$\begin{array}{ccc} J^{n-1} & \xrightarrow{c_{e_0}} & E \\ \Big\downarrow j & \overset{H}{\nearrow} & \Big\downarrow p \\ I^{n-1} \times I & \xrightarrow{\alpha} & X \end{array}$$

Then $\alpha = p \circ H$ and $\delta([\alpha]) = H|_{I^{n-1} \times \{1\}} \simeq H|_{I^{n-1} \times \{0\}} = i_*([\gamma])$. Hence $ker(i_*) \subset im(\delta)$. $\square$

6.2 Fiber Bundles

Definition 6.2.1 (Fiber Bundle)**.** *A map $p : E \to B$ between topological spaces is a fiber bundle with fiber F if for all $x \in X$, there exists a neighborhood $x \in U \subset X$ for which there exists a homeomorphism $\varphi_U : p^{-1}(U) \cong U \times F$ such that the following map commutes in which the map pr_n is the projection in the $n \in \{1, 2\}$ factor.*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi_U} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U & \end{array}$$

This property is know as local trivalization.

Proposition 6.2.2. *A fiber bundle is a Serre fibration.*

Proof. Consider a fiber bundle $p : E \to B$ with fibre F. Given h and $\tilde{\varphi_0}$, we have to construct $\tilde{h}$ such that the following diagram commutes for all n:

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{\tilde{\varphi_0}} & E \\ \downarrow{\scriptstyle i} & \overset{\tilde{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ I^n \times I & \xrightarrow{h} & B \end{array}$$

First, consider a covering space $\{U_\alpha\}_{\alpha\in\Delta}$ such that each U_α has the local trivialization property, with the attached homeomorphism φ_{U_α} denoted as φ_α. Do a partition of I^n in a finite number of cubes $C_1, ..., C_k$ and do a partition of I in a finite number of intervals $J_1, ..., J_l$ such that $C_i \times J_j \subset U_\alpha$ for some $\alpha \in \Delta$.

Now we will prove that $\tilde{h}$ exists by induction on n:

(1) Base case. The base case can be considered as the following diagram:

$$\begin{array}{ccc} \{0\} & \xrightarrow{\tilde{\varphi_0}} & E \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ I & \xrightarrow{h} & B \end{array}$$

By applying the Lemma 2.3.5, there exists $\tilde{h}$ so that the diagram commutes.

(2) Assume that $\tilde{h}$ exists for $\partial C_i \times I$.

(3) We want to prove that $\tilde{h}$ exists for all $C_i \times I$. Consider C_i and consider J as the interval which contains the 0. Since $\tilde{h}|_{\partial C_i\times J}$ is already defined by hypothesis of induction and $\tilde{h}|_{C\times\{0\}=\tilde{\varphi_0}|_{C_i\times\{0\}}}$, then $\tilde{h}|_{\partial C_i\times J\cup C_i\times\{0\}}$ is already defined. Since $\tilde{h}$ is a lift of h, we have that

$$(p \circ \tilde{h}|_{\partial C_i \times J \cup C_i \times \{0\}})(\partial C_i \times J \cup C_i \times \{0\}) = h(\partial C_i \times J \cup C_i \times \{0\}) \subset h(C_i \times J) \subset U_\alpha$$

$$\tilde{h}|_{\partial C_i \times J \cup C_i \times \{0\}}(\partial C_i \times J \cup C_i \times \{0\}) \subset p^{-1}(U_\alpha)$$

$$(\varphi_\alpha \circ \tilde{h}|_{\partial C_i \times J \cup C_i \times \{0\}})(\partial C_i \times J \cup C_i \times \{0\}) \subset U_\alpha \times F$$

$$(pr_2 \circ \varphi_\alpha \circ \tilde{h}|_{\partial C_i \times J \cup C_i \times \{0\}})(\partial C_i \times J \cup C_i \times \{0\}) \subset F$$

Define $\phi : \partial C_i \times J \cup C_i \times \{0\} \to F$ as $\phi := pr_2 \circ \tilde{\varphi_\alpha} \circ \tilde{h}|_{\partial C_i \times J \cup C_i \times \{0\}}$. Observe that $\partial C_i \times J \cup C_i \times \{0\}$ is a deformation retraction of $C_i \times J$, consider the retraction $r : C_i \times J \to \partial C_i \times J \cup C_i \times \{0\}$. Define the following map:

$$\beta : C_i \times J \to U_\alpha \times F, \quad \beta = (h, \phi \circ r)$$

So now we have defined $\tilde{h}|_{C_i \times J} := \varphi_\alpha^{-1} \circ \beta$, therefore $p \circ \tilde{h}|_{C_i \times J} = p \circ \varphi_\alpha^{-1} \circ \beta = pr_1 \circ \beta = h|_{C_i \times J}$ and $\tilde{h}|_{C_i \times \{0\}} = \tilde{\varphi_0}|_{C_i \times \{0\}}$ by definition.

Finally, do this again but instead of consider $\tilde{\varphi_0}$ in the diagram, consider the previously defined $\tilde{h}|_{C_i \times \{t\}}$ with t being the maximum value of the interval J. Iterating this process a finite number of times, eventually we will get $\tilde{h}|_{C_i \times I}$ fully defined. Now attaching these maps for all $i \in \{1, ..., k\}$ by attaching the adjacent cubes, we will get the aimed $\tilde{h}$ defined for $I^n \times I$. □

Example 6.2.3 (Möbius band)**.** *The Möbius band $\mathcal{M}$ is a surface formed by attaching the ends of a strip of paper together with a half-twist.*

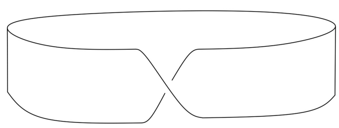

Figure 6.1: Illustration of the Möbius band.

We can understand the Möbius band as fibration p over $\mathbb{S}^1$ where, for each $x \in \mathbb{S}^1$ we can find a neighborhood $x \in U \in \mathbb{S}^1$ so that $p^{-1}(U)$ is isomorphic to an interval I times U. Hence $I \hookrightarrow \mathcal{M} \to \mathbb{S}^1$ is a fiber bundle.

6.3 Hopf bundle

Definition 6.3.1 (Complex projective space, $\mathbb{CP}^n$)**.** *The complex projective space, $\mathbb{CP}^n$, is defined as the quotient space of $\mathbb{C}^{n+1} \backslash \{(0, ..., 0)\}$ via the relation $(z_0, ..., z_n) \sim \lambda(z_0, ..., z_n) = (\lambda z_0, ..., \lambda z_n)$ with $\lambda \in \mathbb{C}$.*

Note that we can build a bijection between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, so that, since $\mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2}$ we can regard $\mathbb{S}^{2n+1}$ as the unit circle inside $\mathbb{C}^{n+1}$. Therefore we can consider $\mathbb{CP}^n$ as the quotient space of $\mathbb{S}^{2n+1}$ under the equivalence relation $(z_0, ..., z_n) \sim \lambda(z_0, ..., z_n) = (\lambda z_0, ..., \lambda z_n)$ with $\lambda \in \mathbb{S}^1 \subset \mathbb{C}$.

Observe that $\mathbb{CP}^n$ is defined as the space of complex lines through the origin in $\mathbb{C}^{n+1}$, therefore each line is determined by a nonzero vector in $\mathbb{C}^{n+1}$, unique up to scalar multiplication. Thus, $\mathbb{CP}^n$ is homeomorphic to the quotient of $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ with $v \sim \lambda v$ for $|\lambda| = 1$. These vectors are of the form $(w, \sqrt{1 - |w|^2}) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$. This is a disk $\mathbb{D}^{2n}_{+}$ bounded by when $|w| = 1$, therefore $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$, thus bounded by the sphere $\mathbb{S}^{2n-1}$. Using this approach, observe that a vector of $\mathbb{D}^{2n}_{+}$ is unique if its last coordinate is nonzero, if it is zero we just have the identifications $v \sim \lambda v$ in $v \in \mathbb{S}^{2n-1}$. It follows that $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a cell e^{2n} via the quotient map $\mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$. Hence $\mathbb{CP}^n = e^0 \cup e^2 \cup ... \cup e^{2n}$.

Proposition 6.3.2. *The spaces* $\mathbb{CP}^1$ *and* $\mathbb{S}^2$ *are homeomorphic.*

Proof. Consider the cell skeleton of $\mathbb{S}^2$ as two cells e^0 and e^2 attached as usual. The cell structure of $\mathbb{CP}^1$ is also constructed by two cells e^0 and e^2 attached as above. Thus $\mathbb{S}^2$ and $\mathbb{CP}^1$ are homeomorphic as CW Complex and since the usual topology and the CW topology coincide, $\mathbb{CP}^1 \cong \mathbb{S}^2$. □

Example 6.3.3. $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ *is a fiber bundle with* $p(z_0, ..., z_n) = [z_0, ..., z_n]$

Proof. We have to prove that $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a fiber bundle so consider a point $X_\alpha = [x_0, ..., x_n] \in \mathbb{CP}^n$. Since $X_\alpha \in \mathbb{CP}^n$, then there must exist $x_i \neq 0$, so any open neighbour of X_α will be of the form $U_\alpha = \{[x_0, ..., x_n] : x_i \neq 0\}$. Hence, we have to prove that there exists a homeomorphism $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{S}^1$.

Note that $p^{-1}(U_\alpha) = \{(z_0, ..., z_n) : z_i \neq 0\}$. Consider the following map

$$\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{S}^1, \quad \varphi_\alpha(z_0, ..., z_n) := ([z_0, ..., z_n], \frac{z_i}{|z_i|})$$

being $|z_i|$ the module of $z_i \in \mathbb{C}\backslash\{0\}$. We have to prove that φ_α is a homemorphism:

Injectivity. Consider $(z_0, ..., z_n), (w_0, ..., w_n) \in p^{-1}(U_\alpha)$ such that $\varphi_\alpha(z_0, ..., z_n) = \varphi_\alpha(w_0, ..., w_n)$. Therefore it follows that $[z_0, ..., z_n] = [w_0, ..., w_n]$ and $\frac{z_i}{|z_i|} = \frac{w_i}{|w_i|}$. Since $[z_0, ..., z_n] = [w_0, ..., w_n]$, then there must exist a $\lambda \in \mathbb{S}^1 \subset \mathbb{C}\backslash\{0\}$ such that $(z_0, ..., z_n) = \lambda(w_0, ..., w_n)$. Thus $z_i = \frac{w_i|z_i|}{|w_i|}$ and $z_i = \lambda w_i$, so $\lambda = \frac{|z_i|}{|w_i|} = |\lambda| = 1$. Hence $(z_0, ..., z_n) = (w_0, ..., w_n)$.

Surjectivity. Consider $([z_0, ..., z_n], k) \in U_\alpha \times \mathbb{S}^1$. Now consider $\lambda = \frac{k|z_i|}{z_i}$, then $\varphi_\alpha(\lambda(z_0, ..., z_n)) = ([\lambda(z_0, ..., z_n)], \frac{\lambda z_i}{|\lambda z_i|}) = ([z_0, ..., z_n], \frac{k|z_i|}{|k z_i|}) = ([z_0, ..., z_n], k)$.

φ_α **is an open map.** Consider $\varphi_\alpha = (\varphi^1_\alpha, \varphi^2_\alpha)$. First of all, φ^2_α is a projection, so it is open. Now consider an open set $V \subset p^{-1}(U_\alpha)$:

$$\begin{aligned} p^{-1}(\varphi_\alpha^1(V)) &= \{(z_0, ..., z_n) \in \mathbb{S}^{2n+1} : p(z_0, ..., z_n) \in \varphi_\alpha^1(V)\} \\ &= \{(z_0, ..., z_n) \in \mathbb{S}^{2n+1} : [z_0, ..., z_n] \in p(V)\} \\ &= \bigcup_{\lambda \in \mathbb{S}^1} \lambda V \end{aligned}$$

Hence $\varphi_\alpha^1(V)$ is open.

This shows that φ_α is an homeomorphism and the following diagram commutes.

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U_\alpha & \end{array}$$

□

Example 6.3.4 (Hopf Bundle)**.** *Consider $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$ is a fiber bundle called the Hopf bundle via the map $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ defined as $\eta := \phi^{-1} \circ p$ being $\phi : \mathbb{CP}^1 \to \mathbb{S}^2$ the homeomorphism between $\mathbb{CP}^1$ and $\mathbb{S}^2$, and $p : \mathbb{S}^3 \to \mathbb{CP}^1$ being $p(z_0, z_1) := [z_0, z_1]$. The map h is called the Hopf map.*

Observe that the Hopf bundle is just the fiber bundle of the Example 6.3.3 for $n = 1$.

We now have the necessary tools to achieve the main objectives of this chapter. First, we will apply the exact long sequence of Serre fibrations to the previously introduced Hopf bundle to demonstrate that $\pi_2(\mathbb{S}^2) = \mathbb{Z}$. This result is crucial for validating a conclusion discussed later in the chapter, where the Freudenthal suspension theorem is examined. The theorem asserts that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$. It is important to note that, in this context, proving $\pi_2(\mathbb{S}^2) = \mathbb{Z}$ is sufficient to support the broader conclusion.

Theorem 6.3.5. *The 2-th homotopy group of the 2-sphere is isomorphic to a free group on one generator, i.e. $\pi_2(\mathbb{S}^2) = \mathbb{Z}$.*

Proof. Since a Hopf bundle is a fiber bundle and a fiber bundle is a Serre fibration, we can apply the long exact sequence of a Serre fibration 6.1.6 to the Hopf bundle i.e.

$$... \xrightarrow{\eta_*} \pi_{n+1}(\mathbb{S}^2, x_0) \xrightarrow{\delta} \pi_n(\mathbb{S}^1, e_0) \xrightarrow{i_*} \pi_n(\mathbb{S}^3, e_0) \xrightarrow{\eta_*} \pi_n(\mathbb{S}^2, x_0) \xrightarrow{\delta} ...$$

Observe that $\pi_2(\mathbb{S}^3) = 0$ and $\pi_1(\mathbb{S}^2) = 0$ by Theorem 4.1.11. It follows that

$$0 = \pi_2(\mathbb{S}^3) \xrightarrow{\eta_*} \pi_2(\mathbb{S}^2) \xrightarrow{\delta} \pi_1(\mathbb{S}^1) \xrightarrow{i_*} \pi_1(\mathbb{S}^2) = 0$$

Observe that δ must be an isomorphism by applying 5.1.1. Hence $\pi_2(\mathbb{S}^2) \cong \pi_1(\mathbb{S}^1) = \mathbb{Z}$. □

As previously mentioned, a critical component in the study of the Freudenthal suspension theorem was the demonstration that $\pi_2(\mathbb{S}^2) = \mathbb{Z}$. Having now established this result, we can confidently proceed to prove that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$.

Theorem 6.3.6. *The n-th homotopy group of the n-sphere is isomorphic to a free group on one generator, i.e. $\pi_n(\mathbb{S}^n) = \mathbb{Z}$.*

Proof. Considering $X = \mathbb{S}$, we already know that the suspension of $\mathbb{S}$ is homotopic to $\mathbb{S}^2$, i.e. $S\mathbb{S} \cong \mathbb{S}^2$. More generally $S\mathbb{S}^n \cong \mathbb{S}^{n+1}$. Apply to X the Freudenthal Suspension Theorem 5.4.4, it follows that in the following sequence

$$\pi_1(\mathbb{S}^1) \to \pi_2(\mathbb{S}^2) \to \pi_3(\mathbb{S}^3) \to ...$$

where each map between successive homotopy groups represents an isomorphism from $\pi_2(\mathbb{S}^2)$ and by Proposition 6.3.5 we know that $\pi_1(\mathbb{S}^1) \cong \pi_2(\mathbb{S}^2) = \mathbb{Z}$. Hence $\pi_n(\mathbb{S}^n) = \mathbb{Z}$. □

In conclusion, one of the most intriguing results derived from the Hopf bundle is that the third homotopy group of the 2-sphere is isomorphic to a free group with one generator. To demonstrate this, we will utilize the exact long sequence of Serre fibrations, previously examined, in the context of the Hopf bundle. This approach yields the acclaimed result, thereby achieving the final objective of our project.

Theorem 6.3.7. *The 3-th homotopy group of the 2-sphere is isomorphic to a free group on one generator, i.e. $\pi_3(\mathbb{S}^2) = \mathbb{Z}$.*

Proof. We can apply the long exact sequence of a Serre fibration to the Hopf bundle i.e.

$$... \xrightarrow{\eta_*} \pi_{n+1}(\mathbb{S}^2, x_0) \xrightarrow{\delta} \pi_n(\mathbb{S}^1, e_0) \xrightarrow{i_*} \pi_n(\mathbb{S}^3, e_0) \xrightarrow{\eta_*} \pi_n(\mathbb{S}^2, x_0) \xrightarrow{\delta} ...$$

Observe that $\pi_3(\mathbb{S}^1) = 0$ by Theorem 4.1.10 and $\pi_1(\mathbb{S}^2) = 0$ by Theorem 4.1.10. It follows that

$$0 = \pi_3(\mathbb{S}^1) \xrightarrow{i_*} \pi_3(\mathbb{S}^3) \xrightarrow{\eta_*} \pi_3(\mathbb{S}^2) \xrightarrow{\delta} \pi_2(\mathbb{S}^1) = 0$$

Since $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ by Corollary 6.3.6, it follows that $\pi_3(\mathbb{S}^2) \cong \pi_3(\mathbb{S}^3) = \mathbb{Z}$. □

Finally, after demonstrating that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ and that $\pi_n(\mathbb{S}^k) = 0$ for $k > n$, we conclude the chapter with a significant and intriguing result: no two spheres of different dimensions are homeomorphic. This result synthesizes and culminates the extensive study and findings developed throughout the project, highlighting the distinct topological properties of spheres in various dimensions.

Corollary 6.3.8. *$\mathbb{S}^n$ and $\mathbb{S}^k$ with $n, k > 0$ are homemorphic if and only if $n = k$.*

Proof. Assume $\mathbb{S}^n$ and $\mathbb{S}^k$ are spheres with $n < k$. By Theorem 6.3.6, we know that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$. On the other hand, Theorem 4.1.11 tells us that $\pi_n(\mathbb{S}^k) = 0$. According to Proposition 2.2.11, which states that the homotopy groups of homeomorphic spaces must be isomorphic, we conclude that $\mathbb{S}^n$ and $\mathbb{S}^k$ cannot be homeomorphic. □

Chapter 7

Conclusions and Future Work

The primary objective of this project has been to explore the sublime beauty and elegance of algebraic topology. This study has drawn inspiration from the seminal works of illustrious mathematicians such as Heinz Hopf and Jean-Pierre Serre. Our aim has been to uncover the unexpected wonders and intricate complexities of a field that, despite its abstract nature, proves to be extraordinarily intuitive and fascinating.

The manuscript begins by establishing the basic tools with which we would subsequently work. Our intention was to clearly and visually present the fundamental concepts of the three-dimensional realm. Thus, we introduced the fundamental group, and as a first encounter with higher dimensions, we addressed the construction of a sphere in a more useful manner that does not rely on metric spaces, leading to the introduction of CW complexes. This provided us with an intuitive perspective on the construction of spaces, prompting us to question whether we could generalize this methodology to other dimensions.

With this foundation, we introduced higher-dimensional homotopy groups, intuitively generalizing the underlying idea of the fundamental group. We observed that these groups exhibited similar behavior in many aspects but also presented crucial differences, which we analyzed in detail. Additionally, the concept of connectivity and relative homotopy groups was introduced, further expanding our theoretical framework.

At this point in the study, we had clearly defined our computational objectives: on one hand, the generalized homotopy groups, and on the other, a clear methodology for defining spheres. Equipped with this knowledge, we ventured to unravel one of the most powerful and elegant tools in algebraic topology: the Freudenthal Suspension Theorem. To this end, we introduced the concept of "suspension" to define new spaces and demonstrated the Whitehead Theorem, culminating in the anticipated proof of the Freudenthal Suspension Theorem. This theorem not only validates the techniques employed but also opens new avenues for the study and understanding of topology in higher dimensions.

Finally, we concluded this exhaustive study by analyzing the fibrations of Serre and Hurewicz, with a particular focus on Serre fibrations. From these emerges one of the most elegant and fundamental structures of modern mathematics: the Hopf Bundle. This analysis allowed us to calculate the homotopy groups of the remaining spheres, thereby achieving the proposed objectives and providing a deeper and more comprehensive understanding of the topological structures studied.

In general, the results of the calculated homotopy groups could be summarized in the following table.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$

The results highlighted with a red background are proven in Theorem 4.1.10, those with a yellow background are substantiated in Theorem 4.1.11, the results with a green background are established using Theorem 6.3.6, and those with a blue background are validated in Theorem 6.3.7.

As a future line of study, we propose conducting a thorough analysis of the results that were not presented in this manuscript. Specifically, these results are the ones against a white background in the latter table. This additional research could provide deeper insights and further validate our initial findings, potentially uncovering new avenues for exploration and application in the field. By focusing on these unexamined results, we aim to enhance the robustness and comprehensiveness of our study, contributing to a more complete understanding of the subject matter.

In conclusion, this journey through algebraic topology has aimed to offer an explanation that, despite its formality and rigor, is also intuitive and accessible. We have navigated a field of extraordinary power and, at times, abstraction, demonstrating that algebraic topology is not only a topic of great depth and complexity but also one of captivating beauty and unparalleled elegance. This journey has expanded our mathematical understanding, illuminating the inherent harmony of mathematical structures, and revealing a world where intuition and abstraction are in perfect harmony.

This work is not only a contribution to mathematical knowledge but also a celebration of the elegance and creativity underlying algebraic topology. It invites us to contemplate mathematics not merely as an academic discipline but as a form of art, where each theorem and each proof is a masterpiece reflecting the intrinsic beauty of human thought.

Chapter 8

Impact Analysis

This project builds upon the foundational work of esteemed mathematicians, as notably encapsulated in the book *Algebraic Topology*, written by Allen Hatcher [1]. While these results are documented in various other sources, our approach introduces a perspective that may significantly influence future research in the field.

Primarily, this project lays the groundwork for delving into homotopy theory and the computation of homotopy groups of high-dimensional spheres. Although similar results exist elsewhere, this manuscript offers a uniquely direct and intuitive exposition, with a coherent progression rooted solely in homotopy theory. Furthermore, the majority of results presented are thoroughly proven within this manuscript, obviating the need for external references. Consequently, we believe this work could serve as an essential resource for anyone aspiring to study the homotopy of spheres.

On a personal level, the implications of this project have been profound. Despite the challenges posed by the complexity of certain results, the immense satisfaction and fulfillment I experienced upon its completion have made every effort worthwhile. Academically, this endeavor has transformed me into a more sensitive and rational mathematician. Prior to this project, I was merely an observer, following a well-trodden path illuminated by my professors. In contrast, this manuscript represents an open exploration, requiring me to navigate dead ends and uncharted territories that initially seemed unpromising but ultimately yielded extraordinary insights. This experience has deepened my passion for the beautiful art of mathematics.

In conclusion, I can unequivocally state that this project has enhanced my mathematical abilities and reaffirmed my belief in the inherent beauty and elegance of the art we call mathematics.

Bibliography

[1] A. Hatcher. (2002) Algebraic topology. [Online]. Available: https://books.google.es/books?id=BjKs86kosqgC

[2] J. H. Sim. (2016) The fundamental group and cw complexes. [Online]. Available: https://math.uchicago.edu/~may/REU2016/REUPapers/Sim.pdf

[3] J. J. Gutierrez. (2014) Homology lecture notes. [Online]. Available: https://www.math.ru.nl/~gutierrez/files/homology/Lecture08.pdf

[4] J. W. Robbin. (2005) Cell complexes. [Online]. Available: https://people.math.wisc.edu/~jwrobbin/751dir/homology

[5] L. A. Lessa. (2020) Illustration of an exact sequence of groups. [Online]. Available: https://commons.wikimedia.org/wiki/File:Illustration_of_an_Exact_Sequence_of_Groups.svg

[6] T. Zhang. (2009) Freudenthal suspension theorem. [Online]. Available: http://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/ZhangTengren.pdf