RACSAM (2021) 115:191
https://doi.org/10.1007/s13398-021-01137-w

ORIGINAL PAPER

On Sobolev bilinear forms and polynomial solutions of second-order differential equations

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Received: 1 July 2021 / Accepted: 5 September 2021 / Published online: 17 September 2021

Abstract

Given a linear second-order differential operator $\mathcal{L} \equiv \phi\, D^2 + \psi\, D$ with non zero polynomial coefficients of degree at most 2, a sequence of real numbers $\lambda_n, n \geq 0$, and a Sobolev bilinear form

$$\mathcal{B}(p, q) = \sum_{k=0}^{N} \left\langle \mathbf{u}_k,\ p^{(k)}\, q^{(k)} \right\rangle, \quad N \geq 0,$$

where $\mathbf{u}_k$, $0 \leq k \leq N$, are linear functionals defined on polynomials, we study the orthogonality of the polynomial solutions of the differential equation $\mathcal{L}[y] = \lambda_n\, y$ with respect to $\mathcal{B}$. We show that such polynomials are orthogonal with respect to $\mathcal{B}$ if the Pearson equations $D(\phi\, \mathbf{u}_k) = (\psi + k\, \phi')\, \mathbf{u}_k$, $0 \leq k \leq N$, are satisfied by the linear functionals in the bilinear form. Moreover, we use our results as a general method to deduce the Sobolev orthogonality for polynomial solutions of differential equations associated with classical orthogonal polynomials with negative integer parameters.

Keywords Classical orthogonal polynomials · Sobolev orthogonal polynomials · Nonstandard parameters

Mathematics Subject Classification 42C05 · 33C45

1 Introduction

For a fixed integer $N \geq 0$, let $\mathbf{u}_k$, $0 \leq k \leq N$, be linear functionals defined on polynomials. Then, a symmetric bilinear form $\mathcal{B}$ can be defined as

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$$\mathcal{B}(p,q)=\sum_{k=0}^{N}\left\langle \mathbf{u}_k, p^{(k)}q^{(k)}\right\rangle. \tag{1.1}$$

A bilinear form involving derivatives, such at (1.1), is said to be a Sobolev bilinear form.

The case when $\mathbf{u}_N$ is a regular (or quasi-definite) linear functional and $\mathbf{u}_k$, $0 \le k \le N-1$, are Dirac deltas supported on convenient points of the real line has been of great interest. In [9], the following discrete-continuous Sobolev bilinear form was studied:

$$\mathcal{B}_I(p,q)=(p(c_0),\ldots,p(c_{N-1}))\,\mathbf{A}\begin{pmatrix} q(c_0)\\ \vdots \\ q(c_{N-1})\end{pmatrix}+\left\langle \mathbf{u}_N, p^{(N)}\,q^{(N)}\right\rangle \tag{1.2}$$

where $c_0,\ldots,c_{N-1}$ are distinct real numbers, N is a fixed positive integer, $\mathbf{u}_N$ is a regular linear functional, and $\mathbf{A}$ is an $N\times N$ quasi-definite symmetric real matrix, that is, its principal submatrices are nonsingular. Notice that (1.2) can be written as

$$\mathcal{B}_I(p,q)=\mathcal{D}_I(p,q)+\mathcal{C}_I(p,q),$$

where $\mathcal{D}_I$ and $\mathcal{C}_I$ are bilinear forms associate with the discrete and continuous parts of $\mathcal{B}_I$, respectively. Using the fact that $\mathbf{A}$ is quasi-definite, it was shown in [9] that if the monic polynomials $\{Q_n(x)\}_{n\ge 0}$ are orthogonal with respect to $\mathcal{B}_I$, then, for $n\ge N$:

- $Q_n(c_j)=0,\ 0\le j\le N-1,$
- $Q_n(x)=h(x)R_{n-N}(x)$, where $h(x)=\prod_{j=0}^{N-1}(x-c_j)$ and R_{n-N} is a monic polynomial of degree $n-N$,
- $Q_n^{(N)}(x)=\dfrac{n!}{(n-N)!}P_{n-N}(x)$, where $\{P_n(x)\}_{n\ge 0}$ are monic orthogonal polynomials associated with $\mathbf{u}$,
- $\{Q_n(x)\}_{n=0}^{N-1}$ are orthogonal with respect to $\mathcal{D}_I$.

Similar results were obtained in [10], where the bilinear form (1.1) was considered with $\mathbf{u}_N$ regular, and taking $\mathbf{u}_k$, $0\le k\le N-1$, to be derivatives of Dirac deltas supported on some points of the real line.

Special attention has been given to the case when $\mathbf{u}_N$ is associated with the classical Jacobi or Laguerre orthogonal polynomials, and $\mathbf{u}_k$, $0\le k\le N-1$, are derivatives of Dirac deltas supported on the endpoints of the interval of orthogonality of $\mathbf{u}_N$ (see [1–3,5,15,18,19] and the references therein). Such bilinear forms arise when studying the orthogonality of Jacobi or Laguerre polynomials with the so-called *non standard* parameters, that is, negative integer parameters such that the coefficient c_n in the corresponding three-term recurrence relation

$$x\,p_n(x) = a_n\,p_{n+1}(x)+b_n\,p_n(x)+c_n\,p_{n-1}(x),$$

vanishes. In this case, discrete-continuous Sobolev orthogonality must be considered since, due to Favard's Theorem (see [6], Theorem 4.4), there is no regular linear functional associated with these families of polynomials.

The main strategy for dealing with Jacobi and Laguerre polynomials with non-standard parameters (Gegenbauer polynomials included) is to use the fact that the derivatives of the classical orthogonal polynomials are again polynomials of the same family with a shift in their parameters. For instance, in [15], the Sobolev orthogonality for the Laguerre polynomials $\{L_n^{(-N)}(x)\}_{n\ge 0}$ was established through the bilinear form (1.1) with $\mathbf{u}_N$ being the classical linear functional associated with the Laguerre polynomials $\{L_n^{(0)}(x)\}_{n\ge 0}$ and the rest of the functionals being derivatives of Dirac deltas supported on the origin of the real line. Later, in [18], a unified approach to the orthogonality of the generalized Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ was given for any real value of the parameter α, by proving their orthogonality with respect to a more general Sobolev non-diagonal inner product that reduces to (1.1) when $-\alpha = N \in \mathbb{N}$.

In [3], a general approach was given for dealing with the orthogonality of Gegenbauer polynomials $\{C_n^{(-m+1/2)}(x)\}_{n\geq 0}$ with non standard parameters, that is, for $m \in \mathbb{N}$. Therein, the bilinear form (1.1) is expressed as the sum of two bilinear forms: a discrete form $\mathcal{D}$ ($\mathbf{u}_k$, $0 \leq k \leq 2m$, are derivatives of Dirac deltas supported on the endpoints of the interval of orthogonality $[-1, 1]$), and a continuous form $\mathcal{C}$ defined by means of $\mathcal{C}(p, q) = \langle \mathbf{u}_{2m}, p^{(2m)}\, q^{(2m)}\rangle$, where $\mathbf{u}_{2m}$ is the regular linear functional associated with the Gegenbauer polynomials $\{C_n^{(m+1/2)}(x)\}_{n\geq 0}$. We must remark that the strategy followed in [3] is to prove the existence of a $2m \times 2m$ symmetric positive definite matrix $\mathbf{A}$ such that

$$\mathcal{D}(p, q) = \mathbf{P}\,\mathbf{A}\,\mathbf{Q}^t, \tag{1.3}$$

where $\mathbf{F} = (f(1), f'(1), \ldots, f^{(m-1)}(1), f(-1), f'(-1), \ldots, f^{(m-1)}(-1))$, and, therefore,

$$\mathcal{B}(p, q) = \mathbf{P}\,\mathbf{A}\,\mathbf{Q}^t + \langle \mathbf{u}_{2m}, p^{(2m)}\, q^{(2m)}\rangle.$$

Observe that $\mathcal{B}$ acts on polynomials of degree $\leq 2m-1$ only through $\mathcal{D}$, and, for polynomials of degree $\geq 2m$, $\mathcal{B}$ acts only through $\mathcal{C}$. In the same way, in [19], the Jacobi polynomials $\{P_n^{(-N,\beta)}(x)\}_{n\geq 0}$ with $N \in \mathbb{N}$ and $-\beta \notin \mathbb{N}$, were proven to be orthogonal with respect to

$$\widetilde{\mathcal{B}}(p, q) = \widetilde{\mathbf{P}}\,\mathbf{A}\,\widetilde{\mathbf{Q}}^t + \langle \mathbf{u}_N, p^{(N)}\, q^{(N)}\rangle,$$

where $\widetilde{\mathbf{F}} = (f(1), f'(1), \ldots, f^{(N-1)}(1))$, $\mathbf{A}$ is a symmetric positive definite matrix and $\mathbf{u}_N$ is the regular linear functional associated with the Jacobi polynomials $\{P_n^{(0,\beta+N)}(x)\}_{n\geq 0}$.

The study of the Jacobi polynomials $\{P_n^{(-k,-\ell)}(x)\}_{n\geq 0}$, $k, \ell \in \mathbb{N}$, presents an additional challenge. As shown in [20], the hypergeometric expression for $P_n^{(\alpha,\beta)}(x)$ is valid for arbitrary complex values of the parameters α and β. Nevertheless, a reduction of the degree of $P_n^{(\alpha,\beta)}(x)$, $n \geq 1$, occurs if and only if $n = -\alpha - \beta - j$ for a certain integer $1 \leq j \leq n$. In [1], this issue is dealt with by defining the monic *generalized Jacobi polynomial* $\mathcal{P}_n^{(-k,-\ell)}(x)$ by means of the following formula:

$$\mathcal{P}_n^{(-k,-\ell)}(x) = \frac{1}{2}\left[\lim_{\alpha\to -k} P_h^{(\alpha,-\ell)}(x) + \lim_{\beta\to -\ell} P_h^{(-k,\beta)}(x)\right], \tag{1.4}$$

where $h = k+\ell-n-1$ if $(k+\ell)/2 \leq n < \max\{k, \ell\}$, and $h = n$ otherwise. The polynomials defined in (1.4) inherit several of the important properties of the Jacobi polynomials and satisfy $\deg \mathcal{P}_n^{(-k,-\ell)} = n$, $n \geq 0$. Moreover, these polynomials are orthogonal with respect to the bilinear

$$\widehat{\mathcal{B}}(p, q) = \widehat{\mathcal{C}}(p, q) + \langle \mathbf{u}, p^{(k+\ell)}\, q^{(k+\ell)}\rangle,$$

where $\widehat{\mathcal{C}}$ is a discrete bilinear form as in (1.3) with $\mathbf{A}$ being an $N \times N$ symmetric matrix, and $\mathbf{u}$ is the regular linear functional associated with $\{P_n^{(\ell,k)}(x)\}_{n\geq 0}$.

Using some ideas from [17], a construction of discrete-continuous bilinear forms is presented in [19], that is valid for general orthogonal polynomials $\{p_n(x)\}_{n\geq 0}$, not necessarily the Jacobi and Laguerre polynomials. Therein, the discrete part $\mathcal{D}$ is defined by $\mathcal{D}(p_n, p_m) = k_n\, \delta_{n,m}$, $n, m \leq N - 1$, with arbitrary positive constants k_n. Therefore, $\mathcal{D}$

acts on the linear space of polynomials of degree at most $N-1$, and its Gram matrix associated with the basis $\{p_n(x)\}_{n=0}^{N-1}$ is a diagonal matrix $\mathbf{K} = \operatorname{diag}\{k_0, k_1, \ldots, k_{N-1}\}$. A non singular matrix $\mathbf{H}$ is defined as the inverse of a matrix whose entries are the polynomials $p_0(x), p_1(x), \ldots, p_{N-1}(x)$ and possibly their derivatives evaluated at the roots of polynomials $p_N(x)$ (taking into account their multiplicities) which are known. Then,

$$\mathcal{D}(p,q) = \mathbf{P}\,\mathbf{H}\,\mathbf{K}\,\mathbf{H}^t\,\mathbf{Q}^t,$$

where $\mathbf{P}$ and $\mathbf{Q}$ are vectors whose entries are the evaluations of $p(x)$ and $q(x)$, and possibly their derivatives, at the roots of $p_N(x)$. Note that $\mathbf{A} = \mathbf{H}\,\mathbf{K}\,\mathbf{H}^t$ is the Gram matrix of $\mathcal{D}$ with respect to the basis of Lagrange interpolation polynomials associated with the roots of $p_N(x)$. In this way, $\mathcal{D}$ is the restriction of the bilinear form $\mathcal{B}$ to the linear space of polynomials of degree at most $N-1$, and, thus, $\mathcal{B} = \mathcal{D}+\mathcal{C}$ for some appropriate continuous bilinear form $\mathcal{C}$.

Finally, [16] studies the orthogonality of the polynomial solutions of the linear second-order differential equation

$$\mathcal{L}[y] \equiv \phi(x)\,y'' + \psi(x)\,y' = \lambda_n\,y, \tag{1.5}$$

where ϕ and ψ are non zero polynomials of degree at most 2, and λ_n are constants. Therein, we find the following result.

Theorem ([16, Theorem 3.3.]) *Let $\{P_n(x)\}_{n\geq 0}$ be a sequence of orthogonal polynomials associated with the (quasi-definite) bilinear form* (1.1) *(with $N = 1$). Then the following statements are equivalent:*

(a) *$\{P_n(x)\}_{n\geq 0}$ satisfies the differential equation $\mathcal{L}[P_n] = \lambda_n\,P_n$.*

(b) *The differential operator $\mathcal{L}$ is symmetric with respect to $\mathcal{B}$, that is,*

$$\mathcal{B}(\mathcal{L}[p], q) = \mathcal{B}(p, \mathcal{L}[q]),$$

for all polynomials p, q.

(c) *The linear functionals $\mathbf{u}_0$ and $\mathbf{u}_1$ satisfy the distributional differential equations*

$$\phi\,\mathbf{u}_0' = \psi\,\mathbf{u}_0, \quad (\phi\,\mathbf{u}_1)' = \psi\,\mathbf{u}_1.$$

Using the fact that the classical orthogonal polynomials are characterized by satisfying the differential equation (1.5) (see [4] as well as [16, Theorem 3.5.]), [16] deals with some examples of classical orthogonal polynomials with non standard parameters, including the Laguerre polynomials with parameter $\alpha = -1$; and the Jacobi polynomials with parameters $\alpha = \beta = -1$, $\alpha = -1$ and $-\beta \notin \mathbb{N}$, and $-\alpha \notin \mathbb{N}$ and $\beta = -1$.

The general aim of this paper is motivated by Theorem 3.3 of [16] in the sense that we study the orthogonality of polynomial solutions of (1.5) when $\mathcal{L}$ is symmetric with respect to (1.1) for arbitrary $N \geq 0$. In this way, we show that the discrete-continuous Sobolev orthogonality of classical orthogonal polynomials with non standard parameters arises from the symmetry of the operator $\mathcal{L}$ and, in fact, we recover the matrix representation of the discrete part of the corresponding bilinear forms. In contrast with [1], we deal with the reduction in degree of the Jacobi polynomials $P_n^{(-k,-\ell)}(x)$, $k, \ell \in \mathbb{N}$, by explicitly constructing a polynomial solution of (1.5) of degree n for each $n \geq 0$.

In Sect. 2, we present basic facts about orthogonal polynomials and, in particular, the classical Hermite, Laguerre, Bessel, and Jacobi polynomials. The main results of this paper are collected in Sect. 3, where we study conditions for a linear second-order differential operator to be symmetric with respect to the bilinear form (1.1). We show that the linear functionals in $\mathcal{B}$ must satisfy the related distributional Pearson equations. Hence, in this section we also study the solutions of the Pearson equations. Lastly, in Sect. 4, we apply our results to deduce the Sobolev orthogonality for polynomial solutions of the differential equations corresponding to the classical orthogonal polynomials with non-standard parameters.

2 Preliminaries

In this section, we present the basic facts needed to establish our main results.

2.1 Orthogonal polynomials

We denote by $\mathbb{R}$ the set of real numbers, and $\mathbb{N}$ denotes the set of positive integers. For each integer $n \geq 0$, we denote by Π_n the linear space of real polynomials of degree at most n, and let $\Pi = \bigcup_{n\geq 0} \Pi_n$ be the collection of all such polynomials.

We call any linear functional $\mathbf{u} : \Pi \to \mathbb{R}$ a moment functional, and the image of a polynomial $p \in \Pi$ under $\mathbf{u}$ will be denoted by $\langle \mathbf{u}, p\rangle$. We denote by Π' the linear space of moment functionals, that is, $\Pi' = \{\mathbf{u} : \Pi \to \mathbb{R} \,|\, \mathbf{u} \text{ is linear}\}$. Observe that Π' is the algebraic dual of Π. Given a sequence of real numbers $\{\mu_n\}_{n\geq 0}$, a moment functional $\mathbf{u}$ can be defined by means of its moments as

$$\langle \mathbf{u}, x^n\rangle = \mu_n, \quad n \geq 0,$$

and extended by linearity to all of Π.

Let $D : \Pi' \to \Pi'$ denote the distributional differential operator. Given a moment functional $\mathbf{u}$, its derivative $D\mathbf{u}$ is the moment functional defined by

$$\langle D\mathbf{u}, p\rangle = -\langle \mathbf{u}, p'\rangle, \quad \forall p \in \Pi,$$

and the left multiplication of $\mathbf{u}$ by a polynomial $q \in \Pi$ is the moment functional $q\mathbf{u}$ defined by

$$\langle q\mathbf{u}, p\rangle = \langle \mathbf{u}, q\, p\rangle, \quad \forall p \in \Pi.$$

Moreover, the product rule holds, that is, $D(q\mathbf{u}) = q'\mathbf{u} + q\, D\mathbf{u}$.

Lemma 1 *Let* $\mathbf{u}$ *be a moment functional. Then* $\mathbf{u} = \mathbf{0}$ *if and only if* $D\mathbf{u} = \mathbf{0}$, *where* $\mathbf{0}$ *is the null moment functional defined as* $\langle \mathbf{0}, q\rangle = 0$, *for all* $q \in \Pi$.

Proof On one hand, assume that $\mathbf{u} = \mathbf{0}$. Then, $\langle D\mathbf{u}, 1\rangle = -\langle \mathbf{u}, 0\rangle = 0$, and, for $n \geq 1$, $\langle D\mathbf{u}, x^n\rangle = -\langle \mathbf{u}, n\, x^{n-1}\rangle = 0$. On the other hand, if $D\mathbf{u} = \mathbf{0}$, then, for $n \geq 0$,

$$\langle \mathbf{u}, x^n\rangle = \left\langle \mathbf{u}, \frac{(x^{n+1})'}{n+1}\right\rangle = -\frac{1}{n+1}\left\langle D\mathbf{u}, x^{n+1}\right\rangle = 0.$$

□

We call any map $\mathcal{B} : \Pi \times \Pi \to \mathbb{R}$ a bilinear form if for all $p, q, r \in \Pi$ and $\lambda \in \mathbb{R}$, it holds that

$$\mathcal{B}(\lambda\, p + q, r) = \lambda\, \mathcal{B}(p, r) + \mathcal{B}(q, r) \quad \text{and} \quad \mathcal{B}(r, \lambda\, p + q) = \lambda\, \mathcal{B}(r, p) + \mathcal{B}(r, q),$$

that is, $\mathcal{B}$ is linear in each one of its entries. Additionally, if $\mathcal{B}(p, q) = \mathcal{B}(q, p)$ for all $p, q \in \Pi$, then we say that $\mathcal{B}$ is *symmetric*.

Let $\{p_n(x)\}_{n\geq 0}$ be a sequence in Π such that $\deg p_n = n$ for $n \geq 0$. Then $\{p_n(x)\}_{n\geq 0}$ is a basis for Π. In addition, if $\mathcal{B}(p_n, p_m) = 0$ for $n \neq m$ and $\mathcal{B}(p_n, p_n) \neq 0$ for $n \geq 0$, we say that $\{p_n(x)\}_{n\geq 0}$ is an orthogonal polynomial sequence (OPS) associated with $\mathcal{B}$.

Given a symmetric bilinear form $\mathcal{B}$, there is not always an OPS associated with $\mathcal{B}$. If an OPS associated with $\mathcal{B}$ exists, then $\mathcal{B}$ is called *quasi-definite*. A bilinear form $\mathcal{B}$ is *positive definite* if $\mathcal{B}(p, p) > 0$ for every non-zero polynomial $p \in \Pi$. Positive definite bilinear forms are quasi-definite and, thus, an OPS associated with $\mathcal{B}$ exists.

Following [16], the quasi-definite and positive definite character of a symmetric bilinear form $\mathcal{B}$ can be characterized in terms of the principal minors of the Gram matrix defined from the elements $\mu_{i,j} := \mathcal{B}(x^i, x^j)$, $i, j \geq 0$. Indeed, $\mathcal{B}$ is quasi-definite (resp. positive definite) if and only if

$$\Delta_n(\mathcal{B}) := \begin{vmatrix} \mu_{0,0} & \mu_{0,1} & \cdots & \mu_{0,n} \\ \mu_{1,0} & \mu_{1,1} & \cdots & \mu_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n,0} & \mu_{n,1} & \cdots & \mu_{n,n} \end{vmatrix} \neq 0, \quad \forall n \geq 0,$$

(resp. $\Delta_n(\mathcal{B}) > 0$, for all $n \geq 0$). Observe that, since $\mathcal{B}$ is symmetric, $\mu_{i,j} = \mu_{j,i}$, $i, j \geq 0$, and, thus, $\Delta_n(\mathcal{B})$, $n \geq 0$, are the principal minors of a symmetric Gram matrix.

Given $\mathbf{u} \in \Pi'$, we can define the bilinear form $\mathcal{B}(p, q) = \langle \mathbf{u}, p\,q\rangle$, $p, q \in \Pi$. Then, we will say that $\mathbf{u}$ is quasi-definite (resp. positive definite) if and only if $\mathcal{B}$ is quasi-definite (resp. positive definite), and that there is an OPS associated with $\mathbf{u}$. In this case, $\mathcal{B}(x^i, x^j) = \langle \mathbf{u}, x^{i+j}\rangle$ and, thus, $\Delta_n(\mathcal{B})$, $n \geq 0$, are the principal minors of a Gram matrix with Hankel structure.

2.2 Classical moment functionals

A quasi-definite moment functional $\mathbf{u}$ is called *classical* if there are non-zero polynomials ϕ and ψ, with $\deg \phi \leq 2$ and $\deg \psi = 1$, such that $\mathbf{u}$ satisfies the distributional Pearson equation

$$D(\phi\,\mathbf{u}) = \psi\,\mathbf{u},$$

or, equivalently,

$$\langle \mathbf{u}, \phi\,p' + \psi\,p\rangle = 0, \quad \forall p \in \Pi.$$

An OPS associated with a classical moment functional is called a classical OPS.

Although there are several properties that characterize the classical moment functionals, we focus our attention on two of them: the first one given in 1929 by Bochner [4] and the second one given in 1935 by Hahn [11]. We state both characterizations in the following theorem.

Theorem 1 *Let* $\mathbf{u}$ *be a quasi-definite functional and* $\{P_n(x)\}_{n\geq 0}$ *an OPS associated with* $\mathbf{u}$. *The following statements are equivalent.*

1. $\mathbf{u}$ *is classical.*
2. [4] *There are non zero polynomials* ϕ *and* ψ *with* $\deg \phi \leq 2$ *and* $\deg \psi = 1$, *such that, for all* $n \geq 0$, $P_n(x)$ *satisfies*

$$\phi\,P_n'' + \psi\,P_n' = \lambda_n\,P_n, \tag{2.1}$$

where $\lambda_n = n(\frac{n-1}{2}\phi'' + \psi')$.

3. [11] *There is a non zero polynomial* ϕ *with* $\deg \phi \leq 2$*, such that* $\left\{P'_{n+1}\right\}_{n\geq 0}$ *is a sequence of orthogonal polynomials associated with the moment functional* $\mathbf{v} = \phi\, \mathbf{u}$.

It is well known (see [4] as well as [12,17]) that, up to affine transformations of the independent variable, the only families of classical orthogonal polynomials are the Hermite, Laguerre, Jacobi, and Bessel polynomials. The classical orthogonal polynomials are classified according to the canonical form of the polynomial $\phi(x)$.

Family	$\phi(x)$	$\psi(x)$	λ_n
Hermite	1	$-2\,x$	$-2\,n$
Laguerre	x	$\alpha+1-x$	$-n$
Jacobi	$1-x^2$	$\beta-\alpha-(\alpha+\beta+2)\,x$	$-n\,(n+\alpha+\beta+1)$
Bessel	x^2	$a\,x+2$	$n\,(n+a-1)$

We remark that when $\phi(x) \equiv 0$, the polynomial solutions of (2.1) are the elements of the sequence $\{x^n\}_{n\geq 0}$, which can not be associated with a quasi-definite moment functional.

The following explicit expressions for the Hermite, Laguerre, and Jacobi polynomials appear in [20], and the explicit expression for the Bessel polynomials is found in [13, p. 108].

Hermite: $$H_n(x) = n!\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \frac{(-1)^k}{k!\,(n-2k)!}(2\,x)^{n-2k},$$

Laguerre: $$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{1}{k!}x^k,$$

Jacobi: $$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(k+\alpha+1)_{n-k}(k+\alpha+\beta+1)_k\left(\frac{x-1}{2}\right)^k,$$

Bessel: $$B_n^{(a)}(x) = \sum_{k=0}^{n}\binom{n}{k}\,(n+a-1)_k\,\left(\frac{x}{2}\right)^k,$$

where, as usual,

$$(\nu)_0 = 1, \quad (\nu)_k = \nu\,(\nu+1)\cdots(\nu+k-1), \quad k \in \mathbb{N},$$

denotes the Pochhammer symbol, and

$$\binom{\nu}{k} = \frac{(\nu-k+1)_k}{k!}, \quad k \geq 0,$$

denotes the extended binomial coefficient.

As shown in [20], the expression for $P_n^{(\alpha,\beta)}(x)$ is valid for arbitrary complex values of the parameters α and β. Nevertheless, a reduction of the degree of $P_n^{(\alpha,\beta)}(x)$, $n \geq 1$, occurs if and only if $n = -\alpha-\beta-k$ for a certain integer $1 \leq k \leq n$. The expression for $L_n^{(\alpha)}(x)$ is valid for arbitrary values of the parameter α (see [20]), and no reduction in the degree ever occurs. For $-a \in \mathbb{N}$, the expression for $B_n^{(a)}(x)$ suffers a reduction in degree if and only if $n = 1-a-k$ for some $0 \leq k \leq n-1$.

For arbitrary values of the parameters, the above expressions for H_n, $L_n^{(\alpha)}$, $P_n^{(\alpha,\beta)}$, and $B_n^{(a)}$ are polynomials solutions of the differential equation (2.1) with the corresponding polynomials $\phi(x)$ and $\psi(x)$. Namely, they satisfy the following linear second-order differential equations:

$$
\begin{array}{ll}
P_n^{(\alpha,\beta)}: & (1-x^2)\,y'' + [\beta-\alpha-(\alpha+\beta+2)\,x]\,y' \;=\; -n\,(n+\alpha+\beta+1)\,y \\
B_n^{(a)}: & x^2\,y'' + (a\,x+2)\,y' \;=\; n\,(n+a-1)\,y \\
L_n^{(\alpha)}: & x\,y'' + (\alpha+1-x)\,y' \;=\; -n\,y \\
H_n: & y'' - 2\,x\,y' \;=\; -2\,n\,y
\end{array}
$$

It is important to note that if either $\alpha,\ \beta$, or $\alpha+\beta+1$ are negative integers, then the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ can not be associated with a quasi-definite moment functional. On the other hand, if $-\alpha,\ -\beta,\ -(\alpha+\beta+1)\notin\mathbb{N}$, then $\{P_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ are associated with a quasi-definite moment functional $\mathbf{u}_{\alpha,\beta}$ satisfying the Pearson equation

$$D\left[(1-x^2)\,\mathbf{u}_{\alpha,\beta}\right] \;=\; [\beta-\alpha-(\alpha+\beta+2)x]\;\mathbf{u}_{\alpha,\beta}.$$

Furthermore, for $\alpha,\beta>-1$, $\mathbf{u}_{\alpha,\beta}$ is positive definite and has the integral representation

$$\langle \mathbf{u}_{\alpha,\beta}, p\rangle \;=\; \int_{-1}^{1} p(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,dx, \quad \forall p\in \varPi.$$

For $-\alpha\notin\mathbb{N}$, the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ are associated with a quasi-definite moment functional $\mathbf{u}_{\alpha}$ satisfying the Pearson equation

$$D\,(x\,\mathbf{u}_{\alpha}) \;=\; (\alpha+1-x)\;\mathbf{u}_{\alpha},$$

and for $\alpha>-1$, $\mathbf{u}_{\alpha}$ is positive definite with integral representation

$$\langle \mathbf{u}_{\alpha}, p\rangle \;=\; \int_{0}^{+\infty} p(x)\,x^{\alpha}e^{-x}dx, \quad \forall p\in \varPi.$$

The polynomials $\{L_n^{(-m)}(x)\}_{n\geq 0}$, with $m\in\mathbb{N}$ can not be associated with a quasi-definite moment functional.

The Hermite polynomials $\{H_n(x)\}_{n\geq 0}$ are associated with the positive definite moment functional $\mathbf{u}$ defined by

$$\langle \mathbf{u}, p\rangle \;=\; \int_{-\infty}^{+\infty} p(x)\,e^{-x^2}dx, \quad \forall p\in \varPi,$$

and satisfying the Pearson equation

$$D\mathbf{u} \;=\; -2\,x\,\mathbf{u}.$$

For $-a\notin\mathbb{N}$, the Bessel polynomials $\{B_n^{(a)}(x)\}_{n\geq 0}$ are only associated with a non-positive definite moment functional $\mathbf{u}^{(a)}$ defined by

$$\langle \mathbf{u}^{(a)}, p\rangle \;=\; \frac{1}{2\pi\, i}\int_{c} p(z)\,z^{a-2}e^{-2/z}dz, \quad \forall p\in \varPi,$$

where c is the unit circle oriented in the counter-clockwise direction, satisfying the Pearson equation

$$D\left(x^2\,\mathbf{u}^{(a)}\right) \;=\; (a\,x+2)\;\mathbf{u}^{(a)}.$$

3 Sobolev bilinear forms and differential operators

From Theorem 1, we have that a family of classical orthogonal polynomials associated with a quasi-definite moment functional $\mathbf{u}$ are solutions of the linear second-order differential equation (2.1) and are simultaneously orthogonal with respect to the symmetric Sobolev bilinear form

$$\mathcal{B}(p,q) = \langle \mathbf{u}, p\,q\rangle + \langle \phi\,\mathbf{u}, p'\,q'\rangle, \quad \forall p,q \in \Pi,$$

where ϕ is a non zero polynomial of degree at most 2. In this section, we study the orthogonality of polynomial solutions of (2.1) with respect to a Sobolev bilinear form involving derivatives up to a fixed but arbitrary order.

Fix an integer $N \geq 0$, and let $\mathbf{u}_k$, $0 \leq k \leq N$ be moment functionals in Π'. A symmetric Sobolev bilinear form can be defined in the following way:

$$\mathcal{B}_N(p,q) = \sum_{k=0}^{N} \left\langle \mathbf{u}_k,\ p^{(k)}\,q^{(k)} \right\rangle, \quad \forall p,q \in \Pi. \tag{3.1}$$

We say that two polynomials $p, q \in \Pi$ are orthogonal with respect to $\mathcal{B}_N$ if

$$\mathcal{B}_N(p,q) = 0.$$

We also define the linear second-order differential operator $\mathcal{L}$ acting on Π as

$$\mathcal{L}[p] = \phi\,p'' + \psi\,p', \quad \forall p \in \Pi, \tag{3.2}$$

where ϕ and ψ are polynomials such that $\deg\phi \leq 2$ and $\deg\psi \leq 1$. Its formal Lagrange adjoint $\mathcal{L}^*$ is the operator acting on Π' satisfying

$$\langle \mathbf{u}, \mathcal{L}[p]\rangle = \langle \mathcal{L}^*[\mathbf{u}], p\rangle, \quad \forall \mathbf{u} \in \Pi',\ \forall p \in \Pi,$$

and, thus,

$$\mathcal{L}^*[\mathbf{u}] = D^2(\phi\,\mathbf{u}) - D(\psi\,\mathbf{u}), \quad \forall \mathbf{u} \in \Pi'.$$

In the sequel, we will be interested in the polynomial solutions $\{p_n(x)\}_{n\geq 0}$ of the differential equation

$$\mathcal{L}[y] = \lambda_n\,y, \quad \lambda_n \in \mathbb{R}. \tag{3.3}$$

The differential equation (3.3) is called *admissible* ([14]) if

$$\lambda_n \neq \lambda_m, \quad n \neq m.$$

Remark 1 It was shown in [16] that (3.3) is admissible if and only if it has a unique linearly independent polynomial solution of degree n for all $n \geq 0$, and

$$\lambda_n = n\left(\frac{n-1}{2}\phi'' + \psi'\right).$$

It is straightforward to verify that the differential equation is admissible if and only if, for all $n \geq 0$,

$$d_n := \frac{1}{2}\,n\,\phi'' + \psi' \neq 0.$$

Definition 1 We say that the operator $\mathcal{L}$ is symmetric with respect to $\mathcal{B}_N$ if

$$\mathcal{B}_N(\mathcal{L}[p], q) = \mathcal{B}_N(p, \mathcal{L}[q]), \quad \forall p,q \in \Pi.$$

The symmetry of the operator $\mathcal{L}$ and an additional hypothesis imply that the polynomial solutions of the differential equation (3.3) are orthogonal with respect to $\mathcal{B}_N$.

Theorem 2 *Let $\mathcal{L}$ be the differential operator defined in (3.2), and assume that it is symmetric with respect to the Sobolev bilinear form (3.1). If $p, q \in \Pi$ are polynomials satisfying*

$$\mathcal{L}[p] = \lambda\, p, \quad \mathcal{L}[q] = \mu\, q,$$

with $\lambda \neq \mu$, then

$$\mathcal{B}_N(p, q) = 0,$$

that is, p and q are orthogonal with respect to $\mathcal{B}_N$.

Proof From the symmetry of $\mathcal{L}$ with respect to $\mathcal{B}_N$, we have

$$\lambda\, \mathcal{B}_N(p, q) = \mathcal{B}_N(\mathcal{L}[p], q) = \mathcal{B}_N(p, \mathcal{L}[q]) = \mu\, \mathcal{B}_N(p, q),$$

or, equivalently, $(\lambda - \mu)\, \mathcal{B}_N(p, q) = 0$. Since $\lambda \neq \mu$, we have $\mathcal{B}_N(p, q) = 0$. □

Corollary 1 *Let $\mathcal{L}$ be the differential operator defined in (3.2). Let $\{p_n(x)\}_{n\geq 0}$ be a sequence of polynomials satisfying the differential equation*

$$\mathcal{L}[p_n] = \lambda_n\, p_n, \quad n \geq 0,$$

and assume that the differential equation is admissible.

If $\mathcal{L}$ is symmetric with respect to $\mathcal{B}_N$, then

$$\mathcal{B}_N(p_n, p_m) = 0, \quad n \neq m,$$

that is, two distinct elements of $\{p_n(x)\}_{n\geq 0}$ are orthogonal with respect to $\mathcal{B}_N$.

The following result provides a partial reciprocal of the previous corollary.

Theorem 3 *Let $\mathcal{B}_N$ be the bilinear form defined in (3.1). Assume that $\mathcal{B}_N$ is quasi-definite, and denote by $\{p_n(x)\}_{n\geq 0}$ an OPS associated with $\mathcal{B}_N$. Let $\mathcal{L}$ be the operator defined in (3.2), and assume that $d_n := \frac{1}{2}\, n\, \phi'' + \psi' \neq 0$, for all $n \geq 0$. Then, the following statements are equivalent.*

(i) *The operator $\mathcal{L}$ is symmetric with respect to $\mathcal{B}_N$.*
(ii) *For each $n \geq 0$, there is a constant λ_n such that $\mathcal{L}[p_n] = \lambda_n\, p_n$.*

Proof (i)⇒(ii) For $n \geq 0$, observe that since $d_n \neq 0$, the operator $\mathcal{L}$ preserves the degree, thus $\mathcal{L}[p_n]$ is a polynomial of degree n. Therefore,

$$\mathcal{L}[p_n] = \sum_{k=0}^{n} c_k\, p_k,$$

where

$$c_k = \frac{\mathcal{B}_N(\mathcal{L}[p_n], p_k)}{\mathcal{B}_N(p_k, p_k)}, \quad 0 \leq k \leq n.$$

Using the symmetry of $\mathcal{L}$, we get $\mathcal{B}_N(\mathcal{L}[p_n], p_k) = \mathcal{B}_N(p_n, \mathcal{L}[p_k]) = 0$ for $0 \leq k \leq n-1$, which implies that $c_k = 0, 0 \leq k \leq n-1$. Then, $\mathcal{L}[p_n] = c_n\, p_n$ and (ii) holds with $\lambda_n = c_n$.

(ii)⇒(i). Since $\mathcal{B}_N$ is bilinear, it suffices to show (i) for a basis of polynomials. For $n, m \geq 0$, we have

$$\mathcal{B}_N(\mathcal{L}[p_n], p_m) - \mathcal{B}_N(p_n, \mathcal{L}[p_m]) = (\lambda_n - \lambda_m)\, \mathcal{B}_N(p_n, p_m) = 0.$$

Clearly, (i) follows. □

We are interested in deducing the necessary and sufficient conditions for the differential operator (3.2) to be symmetric with respect to $\mathcal{B}_N$ in (3.1). With this in mind, we state the following preliminary result.

Lemma 2 *Let $n \geq 0$, and let $\mathcal{L}$ be the operator defined in* (3.2)*. Then, for all $p \in \Pi$ and all $\mathbf{u} \in \Pi'$, the following relation holds,*

$$\begin{aligned} &D^n\left((\mathcal{L}[p])^{(n)}\,\mathbf{u}\right) - \mathcal{L}^*\left[D^n\left(p^{(n)}\,\mathbf{u}\right)\right] \\ &\quad = -D^n\left(p^{(n)}\,D\left[D(\phi\,\mathbf{u}) - (\psi + n\,\phi')\,\mathbf{u}\right] + 2\,p^{(n+1)}\left[D(\phi\,\mathbf{u}) - (\psi + n\,\phi')\,\mathbf{u}\right]\right). \end{aligned}$$

Proof For $n = 0$, it can be easily verified that, for all $p \in \Pi$ and all $\mathbf{u} \in \Pi'$,

$$\mathcal{L}[p]\,\mathbf{u} - \mathcal{L}^*[p\,\mathbf{u}] = -p\,D\,[D(\phi\,\mathbf{u}) - \psi\,\mathbf{u}] - 2\,p'\,[D(\phi\,\mathbf{u}) - \psi\,\mathbf{u}]. \tag{3.4}$$

For $n \geq 1$, using Leibniz product rule and the fact that $\deg\phi \leq 2$ and $\deg\psi \leq 1$, we compute

$$\begin{aligned} &D^n\left((\mathcal{L}[p])^{(n)}\,\mathbf{u}\right) \\ &\quad = D^n\left[\left(\phi\,p^{(n+2)} + n\,\phi'\,p^{(n+1)} + \frac{n(n-1)}{2}\phi''\,p^{(n)} + \psi\,p^{(n+1)} + n\,\psi'\,p^{(n)}\right)\mathbf{u}\right] \\ &\quad = D^n\left[\left(\mathcal{L}[p^{(n)}] + n\,\phi'\,p^{(n+1)} + \frac{n(n-1)}{2}\phi''\,p^{(n)} + n\,\psi'\,p^{(n)}\right)\mathbf{u}\right]. \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathcal{L}^*\left[D^n\left(p^{(n)}\,\mathbf{u}\right)\right] \\ &\quad = D^2\left[\phi\,D^n(p^{(n)}\,\mathbf{u})\right] - D\left[\psi\,D^n(p^{(n)}\,\mathbf{u})\right] \\ &\quad = D^2\left[D^n(\phi\,p^{(n)}\,\mathbf{u}) - n\,\phi'\,D^{n-1}(p^{(n)}\,\mathbf{u}) - \frac{n(n-1)}{2}\phi''\,D^{n-2}(p^{(n)}\,\mathbf{u})\right] \\ &\qquad - D\left[D^n(\psi\,p^{(n)}\,\mathbf{u}) - n\,\psi'\,D^{n-1}(p^{(n)}\,\mathbf{u})\right] \\ &\quad = D^n\left(\mathcal{L}^*[p^{(n)}\,\mathbf{u}] + n\,\psi'\,p^{(n)}\,\mathbf{u} - \frac{n(n-1)}{2}\phi''\,p^{(n)}\,\mathbf{u}\right) - n\,D^2\left[\phi'\,D^{n-1}(p^{(n)}\,\mathbf{u})\right]. \end{aligned}$$

Using (3.4) and

$$D^2\left(\phi'\,D^{n-1}(p^{(n)}\,\mathbf{u})\right) = D^n\left[D\left(\phi'\,p^{(n)}\,\mathbf{u}\right) - (n-1)\phi''\,p^{(n)}\,\mathbf{u}\right],$$

we obtain

$$\begin{aligned} &D^n\left((\mathcal{L}[p])^{(n)}\,\mathbf{u}\right) - \mathcal{L}^*\left[D^n\left(p^{(n)}\,\mathbf{u}\right)\right] \\ &\quad = D^n\left[\mathcal{L}[p^{(n)}]\,\mathbf{u} - \mathcal{L}^*[p^{(n)}\,\mathbf{u}] + n\,\phi'\,p^{(n+1)}\mathbf{u} + n\,D\left(\phi'\,p^{(n)}\,\mathbf{u}\right)\right] \\ &\quad = -D^n\left(p^{(n)}\,D\left[D(\phi\,\mathbf{u}) - (\psi + n\,\phi')\,\mathbf{u}\right] + 2\,p^{(n+1)}\left[D(\phi\,\mathbf{u}) - (\psi + n\,\phi')\,\mathbf{u}\right]\right). \end{aligned}$$

□

Now, we are ready to state the main result of this section.

Theorem 4 *Let $\mathcal{B}_N$ be the bilinear form defined in (3.1), and let $\mathcal{L}$ be the second-order differential operator defined in (3.2). Then $\mathcal{L}$ is symmetric with respect to $\mathcal{B}_N$ if and only if the moment functionals $\mathbf{u}_k$, $0 \le k \le N$, satisfy the Pearson equations*

$$D(\phi\, \mathbf{u}_k) \,=\, (\psi + k\, \phi')\, \mathbf{u}_k, \quad 0 \le k \le N. \tag{3.5}$$

Proof Clearly, $\mathcal{L}$ is symmetric with respect to $\mathcal{B}_N$ if and only if, for all $p, q \in \Pi$, $\mathcal{B}_N(\mathcal{L}[p], q) - \mathcal{B}_N(p, \mathcal{L}[q]) \,=\, 0$. Notice that,

$$\mathcal{B}_N(\mathcal{L}[p], q) - \mathcal{B}_N(p, \mathcal{L}[q]) = \sum_{k=0}^{N} (-1)^k \left\langle D^k\left((\mathcal{L}[p])^{(k)} \mathbf{u}_k\right) - \mathcal{L}^*\left[D^k(p^{(k)}\mathbf{u}_k)\right], q\right\rangle.$$

Then, $\mathcal{L}$ is symmetric with respect to $\mathcal{B}_N$ if and only if, for all $p \in \Pi$,

$$\sum_{k=0}^{N} (-1)^k \left(D^k\left((\mathcal{L}[p])^{(k)} \mathbf{u}_k\right) - \mathcal{L}^*\left[D^k(p^{(k)}\mathbf{u}_k)\right]\right) \,=\, \mathbf{0},$$

or, equivalently, by Lemma 2,

$$\sum_{k=0}^{N} (-1)^k D^k\Big(p^{(k)} D\left[D(\phi\mathbf{u}_k) - (\psi + k\phi')\mathbf{u}_k\right] + 2p^{(k+1)}\left[D(\phi\mathbf{u}_k) - (\psi + k\phi')\mathbf{u}_k\right]\Big) = \mathbf{0}. \tag{3.6}$$

On one hand, assume that, for $0 \le k \le N$, $\mathbf{u}_k$ satisfies $D(\phi\, \mathbf{u}_k) \,=\, (\psi + k\, \phi')\, \mathbf{u}_k$. Then, (3.6) holds for all $p \in \Pi$ and therefore $\mathcal{L}$ is symmetric with respect to $\mathcal{B}_N$.

On the other hand, assume that (3.6) holds for all $p \in \Pi$. Then, letting $p(x) \,=\, 1$, (3.6) reads

$$D\left[D(\phi\, \mathbf{u}_0) - \psi\, \mathbf{u}_0\right] \,=\, \mathbf{0}.$$

From Lemma 1, we deduce that $D(\phi\, \mathbf{u}_0) - \psi\, \mathbf{u}_0 \,=\, \mathbf{0}$. Consequently, the term corresponding to $k = 0$ in (3.6) is $\mathbf{0}$, and we can sum from $k = 1$ to $k = N$.

Now, letting $p(x) \,=\, x$, (3.6) reads

$$-D^2\left[D(\phi\, \mathbf{u}_1) - (\psi + \phi')\, \mathbf{u}_1\right] \,=\, \mathbf{0}.$$

Using Lemma 1 twice, we deduce that $D(\phi\, \mathbf{u}_1) - (\psi + \phi')\, \mathbf{u}_1 \,=\, \mathbf{0}$, and, consequently, the summation in (3.6) goes from $k = 2$ to $k = N$. Repeating this process for $3 \le m \le N$ by letting $p(x) \,=\, x^m$, we deduce that $D(\phi\, \mathbf{u}_m) - (\psi + m\, \phi')\, \mathbf{u}_m \,=\, \mathbf{0}$ for $0 \le m \le N$. □

We turn our attention to the solutions of the Pearson equations in Theorem 4.

Proposition 1 *Let ϕ and ψ be two polynomials with* $\deg \phi \le 2$ *and* $\deg \psi \le 1$.

(i) *The moment functional* $\mathbf{u}$ *satisfies the Pearson equation*

$$D(\phi\, \mathbf{u}) \,=\, \psi\, \mathbf{u}$$

if and only if its sequence of moments $\{\mu_n\}_{n\ge0}$ satisfies the three-term relation

$$d_n\, \mu_{n+1} + e_n\, \mu_n + n\, \phi(0)\, \mu_{n-1} = 0, \quad n \ge 0, \tag{3.7}$$

where

$$d_n = \frac{1}{2} n\, \phi'' + \psi', \quad e_n = n\, \phi'(0) + \psi(0).$$

(ii) *If* $d_n \neq 0$ *for all* $n \geq 0$, *then the Pearson equation* $D(\phi\,\mathbf{u}) = \psi\,\mathbf{u}$ *has a unique solution up to a constant factor.*

Proof (i) If the moment functional $\mathbf{u}$ satisfies $D(\phi\,\mathbf{u}) = \psi\,\mathbf{u}$, then, for all $n \geq 0$,

$$\langle D(\phi\,\mathbf{u}), x^n \rangle = \langle \psi\,\mathbf{u}, x^n \rangle,$$

equivalently,

$$\langle \mathbf{u}, n\,\phi\,x^{n-1} + \psi\,x^n \rangle = 0.$$

Writing

$$\phi(x) = \frac{1}{2}\phi'' x^2 + \phi'(0)\,x + \phi(0), \quad \psi(x) = \psi'\,x + \psi(0),$$

and substituting in the previous equation, we obtain

$$\left\langle \mathbf{u}, \frac{1}{2} n\,\phi'' x^{n+1} + n\,\phi'(0)\,x^n + n\,\phi(0)\,x^{n-1} + \psi'\,x^{n+1} + \psi(0)\,x^n \right\rangle = 0,$$

and (3.7) follows. It is easy to verify that the implication in the opposite direction holds by inverting each of the previous steps.

(ii) If $d_n \neq 0, n \geq 0$, then (3.7) is a linear recurrence relation. It follows that (3.7) is uniquely solvable for μ_{n+1}, $n \geq 0$, once μ_0 is fixed. Then, the moment functional $\mathbf{u}$ satisfying $D(\phi\,\mathbf{u}) = \psi\,\mathbf{u}$ is uniquely determined up to a constant factor.

□

Remark 2 Observe that if the differential equation (3.3) is admissible, then it follows from Remark 1 that $d_n \neq 0$ for all $n \geq 0$, and, thus, the Pearson equation $D(\phi\,\mathbf{u}) = \psi\,\mathbf{u}$ is uniquely solvable up to a constant factor.

Proposition 2 *Let* $\mathcal{B}_N$ *be the bilinear form defined in* (3.1) *and assume that the moment functionals* $\mathbf{u}_k$, $0 \leq k \leq N$, *satisfy the Pearson equations* (3.5). *Define*

$$d_n = \frac{1}{2}\,n\,\phi'' + \psi', \quad n \geq 0. \tag{3.8}$$

If there is some $0 \leq m \leq N$ *such that* $d_{n+2m} \neq 0$ *for* $n \geq 0$, *then* $\mathbf{u}_{m+i} = a_i\,\phi^i\,\mathbf{u}_m$ *for* $0 \leq i \leq N-m$ *(*$a_0 = 1$*), and* $\phi^j\,\mathbf{u}_{m-j} = b_j\,\mathbf{u}_m$ *for* $1 \leq j \leq m$, *where* a_i *and* b_j *are constants.*

Proof Observe that, for $0 \leq i \leq N-m$, the moment functionals $\mathbf{u}_{m+i}$ and $\phi^i\,\mathbf{u}_m$ satisfy the same Pearson equation. Indeed,

$$D(\phi\,\phi^i\,\mathbf{u}_m) = i\,\phi'\,\phi^i\,\mathbf{u}_m + \phi^i\,D(\phi\,\mathbf{u}_m) = [\psi + (m+i)\,\phi']\,\phi^i\,\mathbf{u}_m.$$

Moreover, $d_{n+2m} \neq 0$ for $n \geq 0$, implies $d_{n+2(m+i)} \neq 0$, $n \geq 0$. Then, by Proposition 1, each Pearson equation $D(\phi\,\mathbf{u}_{m+i}) = [\psi + (m+i)\,\phi']\,\mathbf{u}_{m+i}$, $0 \leq i \leq N-m$, is uniquely solvable up to constant factor. Hence, $\mathbf{u}_{m+i} = a_i\,\phi^i\,\mathbf{u}_m$ for $0 \leq i \leq N-m$.

Similarly, for $1 \leq j \leq m$, $\mathbf{u}_m$ and $\phi^j\,\mathbf{u}_{m-j}$ satisfy the same Pearson equation $D(\phi\,\mathbf{u}_m) = [\psi + 2\,m\,\phi']\,\mathbf{u}_m$. Since this equation is uniquely solvable up to a constant factor, we have $\phi^j\,\mathbf{u}_{m-j} = b_j\,\mathbf{u}_m$ for $1 \leq j \leq m$. □

4 Sobolev orthogonal polynomial solutions of differential equations

In this section, we focus our attention on the differential equations satisfied by the Hermite, Laguerre, Jacobi, and Bessel polynomials. In particular, we study in great detail the Sobolev orthogonality of the polynomial solutions, including the case when the parameters in the Jacobi and Laguerre polynomials have negative integer values.

As we have already mentioned in Sect. 2, though valid for arbitrary complex values of the parameters, the expression for $P_n^{(\alpha,\beta)}$ suffers a reduction in degree for certain values of the parameters. In particular, $\{P_n^{(-m,-\ell)}(x)\}_{n\geq 0}$ with $m, \ell \in \mathbb{N}$, fails to constitute a polynomial solution of the differential equation

$$(1-x^2)\,y'' + [m-\ell+(m+\ell-2)\,x]\,y' = -n\,(n-m-\ell+1)\,y,$$

satisfying $\deg P_n^{(-m,-\ell)} = n$ for all $n \geq 0$. Nevertheless, it is shown in the sequel that there is a polynomial sequence $\{p_n(x)\}_{n\geq 0}$ with $\deg p_n = n$ satisfying this differential equation. The following proposition will be used to construct polynomial solutions of this differential equation and, in general, of (3.3).

Proposition 3 *Let $\mathcal{L}$ be the differential operator defined in (3.2). Assume there is a non zero polynomial ρ satisfying the differential equations*

$$\mathcal{L}[\rho] = \phi\,\rho'' + \psi\,\rho' = \mu\,\rho, \quad \mu \in \mathbb{R}, \tag{4.1}$$

and

$$\phi\,\rho' = h\,\rho, \tag{4.2}$$

where h is a polynomial of degree ≤ 1. Let $k = \deg \rho$.

Moreover, let $\{p_n(x)\}_{n\geq 0}$ be a sequence of polynomials satisfying the differential equation

$$\mathcal{M}[p_n] \equiv \mathcal{L}[p_n] + 2\,h\,p_n' = \lambda_n\,p_n, \quad \lambda_n \in \mathbb{R}.$$

Then the polynomials $q_n(x) = \rho(x)\,p_{n-k}(x)$, $n \geq k$, satisfy the differential equation

$$\mathcal{L}[q_n] = (\mu + \lambda_{n-k})\,q_n.$$

Proof Using (4.1) and (4.2), we obtain

$$\begin{aligned}\mathcal{L}[q_n] &= (\phi\,\rho'' + \psi\,\rho')\,p_{n-k} + \rho\,(\phi\,p_{n-k}'' + \psi\,p_{n-k}') + 2\,\phi\,\rho'\,p_{n-k}' \\ &= \mu\,\rho\,p_{n-k} + \rho\,(\phi\,p_{n-k}'' + \psi\,p_{n-k}') + 2\,h\,\rho\,p_{n-k}' \\ &= (\mu + \lambda_{n-k})\,q_n.\end{aligned}$$

□

4.1 Laguerre polynomials

As it was explained in Sect. 2, for arbitrary values of the parameter α, the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ constitute a sequence of polynomial solutions of the admissible differential equation

$$x\,y'' + (\alpha + 1 - x)\,y' = -n\,y, \quad n \geq 0,$$

satisfying $\deg L_n^{(\alpha)} = n$, $n \geq 0$.

Define the linear second-order differential operator

$$\mathcal{L}^{(\alpha)}[p] \equiv x\, p'' + (\alpha + 1 - x)\, p', \quad \forall p \in \Pi.$$

Let N be a fixed non-negative integer. By Theorem 4, $\mathcal{L}^{(\alpha)}$ is symmetric with respect to the Sobolev bilinear form $\mathcal{B}_N$ defined in (3.1) if and only if the moment functionals $\mathbf{u}_k$, $0 \le k \le N$, satisfy the Pearson equations

$$D(x\, \mathbf{u}_k) = (\alpha + k + 1 - x)\, \mathbf{u}_k, \quad 0 \le k \le N.$$

In this case, (3.8) reads $d_n = -1$, $n \ge 0$.

We will proceed to study the Sobolev orthogonality for Laguerre polynomials with standard and non-standard parameters, that is, for $-\alpha \notin \mathbb{N}$ and $-\alpha \in \mathbb{N}$, respectively.

4.1.1 Sobolev orthogonality for standard parameter

For $-\alpha \notin \mathbb{N}$, $\mathbf{u}_0$ can be chosen to be the quasi-definite moment functional defined as

$$\langle \mathbf{u}_0, p \rangle = \int_0^{+\infty} p(x)\, x^{\alpha}\, e^{-x}\, dx, \quad \forall p \in \Pi.$$

Moreover, from Proposition 2, we deduce that $\mathbf{u}_k = a_k\, x^k\, \mathbf{u}_0$, $1 \le k \le N$, where a_k are constants. Hence, (3.1) reads

$$\mathcal{B}_N(p, q) = \sum_{k=0}^{N} a_k \int_0^{+\infty} p^{(k)}(x)\, q^{(k)}(x)\, x^{\alpha+k}\, e^{-x}\, dx, \quad \forall p, q \in \Pi,$$

where $a_0 = 1$. Since $\mathcal{L}^{(\alpha)}$ is symmetric with respect to $\mathcal{B}_N$, the Laguerre polynomials satisfy $\mathcal{B}_N(L_n^{(\alpha)}, L_m^{(\alpha)}) = 0$ for $n \neq m$.

4.1.2 Sobolev orthogonality for non standard parameter

For $-\alpha = m \in \mathbb{N}$ and $N \ge m$, let the moment functional $\mathbf{u}_m$ satisfy $D(x\, \mathbf{u}_m) = (1 - x)\, \mathbf{u}_m$, and, therefore, $\mathbf{u}_m$ can be chosen to be the positive definite moment functional associated with the Laguerre polynomials $\{L_n^{(0)}(x)\}_{n \ge 0}$, given by

$$\langle \mathbf{u}_m, p \rangle = \int_0^{+\infty} p(x)\, e^{-x}\, dx, \quad \forall p \in \Pi.$$

From Proposition 2, we deduce that $\mathbf{u}_{m+i} = a_i\, x^i\, \mathbf{u}_m$, $0 \le i \le N - m$, where a_i are constants (with $a_0 = 1$).

For $0 \le j \le m-1$, $\mathbf{u}_j$ must satisfy the Pearson equation $D(x\, \mathbf{u}_j) = (-m+j+1-x)\, \mathbf{u}_j$, or, equivalently,

$$\left\langle \mathbf{u}_j, (-m + j + n + 1)\, x^n - x^{n+1} \right\rangle = 0, \quad n \ge 0. \tag{4.3}$$

Therefore, we get

$$\langle \mathbf{u}_j, p \rangle = \sum_{\ell=0}^{m-j-1} (-1)^{\nu}\, b_{j,\nu}\, p^{(\nu)}(0), \quad 0 \le j \le m - 1, \quad \forall p \in \Pi,$$

where $b_{j,\nu}$ are constants. Using (4.3), we get

$$b_{j,\nu} = \binom{m-j-1}{\nu} b_j, \quad 0 \le \nu \le m-j-1,$$

where the constants $b_j = b_{j,0}$ are free parameters. Then (3.1) reads

$$\begin{aligned} \mathcal{B}_N(p,q) = & \sum_{j=0}^{m-1} \sum_{\nu=0}^{m-j-1} (-1)^\nu \binom{m-j-1}{\nu} b_j \left(p^{(j)} q^{(j)}\right)^{(\nu)}(0) \\ & + \sum_{i=0}^{N-m} a_i \int_0^{+\infty} p^{(m+i)}(x)\, q^{(m+i)}(x)\, x^i\, e^{-x}\, dx. \end{aligned} \tag{4.4}$$

If we define the discrete bilinear form $\mathcal{D}$ as

$$\mathcal{D}(p,q) = \sum_{j=0}^{m-1} \sum_{\nu=0}^{m-j-1} (-1)^\nu \binom{m-j-1}{\nu} b_j \left(p^{(j)} q^{(j)}\right)^{(\nu)}(0), \quad \forall p, q \in \Pi,$$

and the continuous bilinear form $\mathcal{C}$ as

$$\mathcal{C}(p,q) = \sum_{i=0}^{N-m} a_i \int_0^{+\infty} p^{(m+i)}(x)\, q^{(m+i)}(x)\, x^i\, e^{-x}\, dx, \quad \forall p, q \in \Pi,$$

then (4.4) can be expressed as

$$\mathcal{B}_N(p,q) = \mathcal{D}(p,q) + \mathcal{C}(p,q), \quad \forall p, q \in \Pi.$$

Additionally, observe that the polynomial $\rho(x) = x^m$ satisfies the differential equations

$$\mathcal{L}^{(-m)}[\rho] = -m\,\rho, \quad \text{and} \quad x\,\rho' = m\,\rho.$$

Taking into account that $\{L_n^{(m)}(x)\}_{n\ge 0}$ satisfy the admissible differential equation

$$\mathcal{L}^{(m)}[y] = \mathcal{L}^{(-m)}[y] + 2m\,y' = -n\,y, \quad n \ge 0,$$

then, from Proposition 3, it follows that the polynomials $\{q_n(x)\}_{n\ge 0}$ with

$$\begin{aligned} q_n(x) &= L_n^{(-m)}(x), \quad 0 \le n \le m-1, \\ q_n(x) &= x^m\, L_{n-m}^{(m)}(x), \quad n \ge m, \end{aligned}$$

constitute a polynomial solution of $\mathcal{L}^{(-m)}[q_n] = -n\,q_n$ such that $\deg q_n = n$, and, thus, satisfy $\mathcal{B}_N(q_n, q_m) = 0$ for $n \ne m$. From Remark 1 and the expression for the Laguerre polynomials, we deduce that

$$L_n^{(-m)}(x) = \frac{(-1)^m}{(n-m+1)_m}\, x^m\, L_{n-m}^{(m)}(x), \quad n \ge m.$$

Remark 3 The matrix expression of the discrete bilinear form $\mathcal{D}$ can be written as follows. For $0 \le j \le m-1$, let $\mathtt{F}(j)$ be the symmetric matrix of size $(m-j)\times(m-j)$ with entries given by

$$(\mathtt{F}(j))_{\varsigma,\nu} = \begin{cases} (-1)^{\nu+\varsigma} \binom{\nu+\varsigma}{\nu} \binom{m-j-1}{\nu+\varsigma} b_j, & 0 \le \nu+\varsigma \le m-j-1, \\ 0, & \text{otherwise,} \end{cases}$$

and, for $0 \leq k \leq m-1$, let $\mathbf{G}(k)$ be the symmetric matrix of size $m \times m$ defined as $\mathbf{G}(0) = \mathtt{F}(0)$, and

$$\mathbf{G}(k) = \left(\begin{array}{c|c} \mathtt{F}(k) & 0 \\ \hline 0 & 0 \end{array}\right), \quad 1 \leq k \leq m-1.$$

Then,

$$\begin{aligned} \mathcal{D}(p,q) &= \sum_{j=0}^{m-1} \left(p^{(j)}(0), \ldots, p^{(m-1)}(0)\right) \mathtt{F}(j) \begin{pmatrix} q^{(j)}(0) \\ \vdots \\ q^{(m-1)}(0) \end{pmatrix} \\ &= P(0) \left(\sum_{k=0}^{m-1} \mathbf{G}(k)\right) Q(0)^\top, \end{aligned}$$

where $F(0) = \left(f(0),\ f'(0),\ \ldots,\ f^{(m-1)}(0)\right)$.

Moreover, let $\mathbf{L}(0)$ be the non-singular lower triangular matrix of derivatives of Laguerre polynomials

$$\mathbf{L}(0) \;=\; \left((L_n^{(-m)})^{(k)}(0)\right)_{n,k=0,1,\ldots,m-1},$$

given by

$$(L_n^{(-m)})^{(k)}(0) \;=\; (-1)^k \binom{n-m}{n-k}, \quad n \geq k.$$

Then, we have the $m \times m$ diagonal matrix

$$\mathbf{H}^{(-m)} \;:=\; \mathbf{L}(0) \left(\sum_{k=0}^{m-1} \mathbf{G}(k)\right) \mathbf{L}(0)^\top \;=\; \begin{pmatrix} h_0^{(-m)} & & & \bigcirc \\ & h_1^{(-m)} & & \\ & & \ddots & \\ \bigcirc & & & h_{m-1}^{(-m)} \end{pmatrix},$$

where $h_n^{(-m)} \;=\; \mathcal{B}_N(L_n^{(-m)}, L_n^{(-m)})$, $0 \leq n \leq m-1$.

Since the numbers $h_n^{(-m)}$ depend on the free parameters b_j, given an arbitrary $m \times m$ diagonal matrix $\mathbf{H}$, $\mathcal{D}$ can be explicitly constructed by means of

$$\mathcal{D}(p,q) \;=\; P(0) \left[\mathbf{L}(0)^{-1}\, \mathbf{H} \left(\mathbf{L}(0)^{-1}\right)^\top\right] Q(0)^\top.$$

Clearly, if $\mathcal{B}_N$ is quasi-definite, then $\mathbf{H}$ must be chosen to be a positive definite matrix.

4.2 Jacobi polynomials

For arbitrary complex values of α and β, the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ are polynomial solutions of the differential equation

$$(1-x^2)\, y'' + [\beta - \alpha - (\alpha+\beta+2)\, x]\, y' \;=\; \lambda_n\, y, \quad \lambda_n \;=\; -n(n+\alpha+\beta+1). \tag{4.5}$$

Define the differential operator

$$\mathcal{L}^{(\alpha,\beta)}[p] \;\equiv\; (1-x^2)\, p'' + [\beta - \alpha - (\alpha+\beta+2)\, x]\, p', \quad \forall p \in \mathit{\Pi},$$

and fix a non negative integer N. We know from Theorem 4 that $\mathcal{L}^{(\alpha,\beta)}$ is symmetric with respect to the Sobolev bilinear form $\mathcal{B}_N$ defined in (3.1) if and only if the moment functionals $\mathbf{u}_k$, $0 \le k \le N$, satisfy the Pearson equations

$$D\left[(1-x^2)\,\mathbf{u}_k\right] = [\beta-\alpha-(\alpha+\beta+2k+2)\,x]\,\mathbf{u}_k. \tag{4.6}$$

In this case, (3.8) reads

$$d_n = -(n+\alpha+\beta+2), \quad n \ge 0. \tag{4.7}$$

Here, we study the Sobolev orthogonality for the polynomial solutions of (4.5) when the values of the parameters fall in one of the following four cases:

1. Standard values: $-\alpha,\ -\beta,\ -(\alpha+\beta+1) \notin \mathbb{N}$.
2. Type I non standard values: $-\alpha \in \mathbb{N}, -\beta \notin \mathbb{N}$.
3. Type II non standard values: $-\alpha \notin \mathbb{N}, -\beta \in \mathbb{N}$.
4. Type III non standard values: $-\alpha,\ -\beta \in \mathbb{N}$.
5. Type IV non standard values: $-\alpha = -\beta \in \mathbb{N}$.

4.2.1 Sobolev orthogonality for standard parameters

For $-\alpha,\ -\beta,\ -(\alpha+\beta+1) \notin \mathbb{N}$, let $\mathbf{u}_0 \in \Pi'$ be the quasi-definite moment functional defined as

$$\langle \mathbf{u}_0, p\rangle = \int_{-1}^{1} p(x)\,(1-x)^{\alpha}\,(1+x)^{\beta}\,dx, \quad \forall p \in \Pi,$$

and, for $1 \le k \le N$, let $\mathbf{u}_k = a_k\,(1-x^2)^k\,\mathbf{u}_0$, where a_k are constants. Then, $\mathcal{B}_N$ in (3.1) with the moment functionals $\mathbf{u}_k$, $0 \le k \le N$, reads

$$\mathcal{B}_N(p,q) = \sum_{k=0}^{N} a_k \int_{-1}^{1} p^{(k)}(x)\,q^{(k)}(x)\,(1-x)^{\alpha+k}\,(1+x)^{\beta+k}\,dx, \quad \forall p,q \in \Pi,$$

with $a_0 = 1$. Moreover, since each moment functional $\mathbf{u}_k$ satisfies the Pearson equation (4.6), the operator $\mathcal{L}^{(\alpha,\beta)}$ is symmetric with respect to $\mathcal{B}_N$.

From the symmetry of $\mathcal{L}^{(\alpha,\beta)}$ with respect to $\mathcal{B}_N$, we have that

$$\mathcal{B}_N\left(P_n^{(\alpha,\beta)}, P_k^{(\alpha,\beta)}\right) = 0, \quad n \ne k.$$

4.2.2 Sobolev orthogonality for type I non-standard parameters

For $-\alpha = m \in \mathbb{N}$ and $-\beta \notin \mathbb{N}$, we have that

$$d_{n+2m} = -(n+m+\beta+2) \ne 0, \quad n \ge 0.$$

and that the differential equation satisfied by the Jacobi polynomials $\{P_n^{(-m,\beta)}(x)\}_{n\ge 0}$,

$$\mathcal{L}^{(-m,\beta)}[y] = -n(n-m+\beta+1)\,y, \tag{4.8}$$

is admissible.

In order to give the Sobolev orthogonality for $\{P_n^{(-m,\beta)}(x)\}_{n\ge 0}$, we construct solutions for the Pearson equations (4.6) with $N \ge m$. From Proposition 1 we know that,

$$D[(1-x^2)\,\mathbf{u}_m] = [m+\beta-(m+\beta+2)\,x]\,\mathbf{u}_m.$$

is uniquely solvable up to a constant factor. Then $\mathbf{u}_m$ can be chosen to be the quasi-definite moment functional associated with the Jacobi polynomials $\{P_n^{(0,\beta+m)}(x)\}_{n\geq 0}$, that is,

$$\langle \mathbf{u}_m, p\rangle = \int_{-1}^{1} p(x)\,(1+x)^{\beta+m}\,dx, \quad \forall p \in \Pi.$$

By Proposition 2,

$$\mathbf{u}_{m+i} = a_i\,(1-x^2)^i\,\mathbf{u}_m, \quad 0 \leq i \leq N-m,$$

where a_i are constants ($a_0 = 1$).

For $1 \leq j \leq m$, let the moment functional $\mathbf{u}_{m-j}$ satisfy the Pearson equation

$$D[(1-x^2)\,\mathbf{u}_{m-j}] = [m+\beta-(m+\beta-2j+2)\,x]\,\mathbf{u}_{m-j},$$

or, equivalently,

$$\left\langle \mathbf{u}_{m-j},\, [m+\beta-n-(m+\beta+n-2j+2)\,x]\,(x-1)^n \right\rangle = 0, \quad n \geq 0.$$

Hence, we deduce that

$$\langle \mathbf{u}_{m-j}, p\rangle = c_j \sum_{\nu=0}^{j-1} \frac{2^\nu}{(\beta+m+2-2j)_\nu} \binom{j-1}{\nu} p^{(\nu)}(1), \quad \forall p \in \Pi,$$

where c_j is a free parameter.

Taking $\mathbf{u}_k$, $0 \leq k \leq N$, as above, (3.1) reads

$$\begin{aligned}\mathcal{B}_N(p,q) &= \sum_{j=1}^{m}\sum_{\nu=0}^{j-1} \frac{2^\nu c_j}{(\beta+m+2-2j)_\nu} \binom{j-1}{\nu} \left(p^{(m-j)}\,q^{(m-j)}\right)^{(\nu)}(1)\\ &\quad + \sum_{i=0}^{N-m} a_i \int_{-1}^{1} p^{(m+i)}(x)\,q^{(m+i)}(x)\,(1-x)^i\,(1+x)^{\beta+m+i}\,dx,\end{aligned}$$

with $a_0 = 1$. Since the Jacobi polynomials $\{P_n^{(-m,\beta)}(x)\}_{n\geq 0}$ satisfy the admissible differential equation (4.8), then it follows from the symmetry of $\mathcal{L}^{(-m,\beta)}$ with respect to $\mathcal{B}_N$ and Corollary 1 that

$$\mathcal{B}_N\left(P_n^{(-m,\beta)}, P_k^{(-m,\beta)}\right) = 0, \quad n \neq k.$$

Observe that $\mathcal{B}_N$ can be expressed as the sum of two bilinear forms: a discrete bilinear form

$$\mathcal{D}_I(p,q) = \sum_{j=1}^{m}\sum_{\nu=0}^{j-1} \frac{2^\nu c_j}{(\beta+m+2-2j)_\nu} \binom{j-1}{\nu} \left(p^{(m-j)}\,q^{(m-j)}\right)^{(\nu)}(1),$$

and a continuous bilinear form

$$\mathcal{C}_I(p,q) = \sum_{i=0}^{N-m} a_i \int_{-1}^{1} p^{(m+i)}(x)\,q^{(m+i)}(x)\,(1-x)^i\,(1+x)^{\beta+m+i}\,dx.$$

In this way

$$\mathcal{B}_N(p,q) = \mathcal{D}_I(p,q) + \mathcal{C}_I(p,q), \quad \forall p, q \in \Pi.$$

In light of Proposition 3, it is possible to deduce an expression for $P_n^{(-m,\beta)}(x)$, $n \geq m$, in terms of $P_{n-m}^{(m,\beta)}$, which is a Jacobi polynomial with standard parameters. Observe that the polynomial $\rho(x) = (1-x)^m$ satisfies the differential equations

$$\mathcal{L}^{(-m,\beta)}[\rho] = -(\beta+1)\, m\, \rho, \quad \text{and} \quad (1-x^2)\, \rho' = -m\,(1+x)\, \rho,$$

and that

$$\begin{aligned} &\mathcal{L}^{(m,\beta)}[P_n^{(m,\beta)}] \\ &\quad = \mathcal{L}^{(-m,\beta)}[P_n^{(m,\beta)}] - 2\,m(1+x)\Big(P_n^{(m,\beta)}\Big)' = -n(n+m+\beta+1)\, P_n^{(m,\beta)}. \end{aligned}$$

Then the polynomials $q_n(x) = (1-x)^m\, P_{n-m}^{(m,\beta)}(x), n \geq m$, satisfy the admissible differential equation (4.8). By Remark 1, $P_n^{(-m,\beta)}(x)$ must be a constant multiple of $q_n(x)$. From the expression of the Jacobi polynomials, we deduce that

$$P_n^{(-m,\beta)}(x) = \frac{(-1)^m}{2^m}\, \frac{(2\,n - 2\,m + \beta + 1)_m}{(n-m+1)_m}\, (1-x)^m\, P_{n-m}^{(m,\beta)}(x), \quad n \geq m. \tag{4.9}$$

Remark 4 The discrete bilinear form $\mathcal{D}_I$ can be written in matrix form. Indeed, for $1 \leq j \leq m$, let $\mathtt{A}(j)$ be the symmetric matrix of size $j \times j$ with entries given by

$$(\mathtt{A}(j))_{\varsigma,\nu} = \begin{cases} \dbinom{\nu+\varsigma}{\nu}\dbinom{j-1}{\nu}\dfrac{2^{\nu}\, c_j}{(\beta+m+2-2\,j)_{\nu}}, & 0 \leq \nu+\varsigma \leq j-1, \\ 0, & \text{otherwise}, \end{cases}$$

and, for $0 \leq k \leq m-1$, let $\mathbf{B}(k)$ be the symmetric matrix of size $m \times m$ defined as $\mathbf{B}(0) = \mathtt{A}(m)$, and

$$\mathbf{B}(k) = \left(\begin{array}{c|c} \mathtt{0} & \mathtt{0} \\ \hline \mathtt{0} & \mathtt{A}(m-k) \end{array}\right), \quad 1 \leq k \leq m-1, \tag{4.10}$$

where $\mathtt{0}$ is the zero matrix of appropriate size. Then,

$$\begin{aligned} \mathcal{D}_I(p,q) &= \sum_{j=1}^{m} \Big(p^{(m-j)}(1), \ldots, p^{(m-1)}(1)\Big)\, \mathtt{A}(j) \begin{pmatrix} q^{(m-j)}(1) \\ \vdots \\ q^{(m-1)}(1) \end{pmatrix} \\ &= P(1) \left(\sum_{k=0}^{m-1} \mathbf{B}(k)\right) Q(1)^{\top}, \end{aligned}$$

where $F(1) = \big(f(1),\ f'(1),\ \ldots,\ f^{(m-1)}(1)\big)$.

Moreover, taking into account that $\mathcal{D}_I$ coincides with the restriction of $\mathcal{B}_N$ to the linear space of polynomials of degree at most $m-1$, we deduce that

$$\mathbf{H}^{(-m,\beta)} := \mathbf{P}(1) \left(\sum_{k=0}^{m-1} \mathbf{B}(k)\right) \mathbf{P}(1)^{\top} = \begin{pmatrix} h_0^{(-m,\beta)} & & & \bigcirc \\ & h_1^{(-m,\beta)} & & \\ & & \ddots & \\ \bigcirc & & & h_{m-1}^{(-m,\beta)} \end{pmatrix},$$

where $h_n^{(-m,\beta)} = \mathcal{B}_N(P_n^{(-m,\beta)}, P_n^{(-m,\beta)})$, $0 \leq n \leq m-1$, and $\mathbf{P}(1)$ is the non-singular matrix of derivatives of Jacobi polynomials

$$\mathbf{P}(1) = \left((P_n^{(-m,\beta)})^{(k)}(1)\right)_{n,k=0,1,\dots,m-1},$$

given by

$$(P_n^{(-m,\beta)})^{(k)}(1) = \frac{(k-m+1)_{n-k}\,(k-m+\beta+1)_k}{2^k\,(n-k)!}, \quad n \geq k.$$

Observe that the numbers $h_n^{(-m,\beta)}$ must depend on the free parameters c_j. Therefore, given an arbitrary $m \times m$ diagonal matrix $\mathbf{H}$, $\mathcal{D}_I$ can be explicitly constructed by means of

$$\mathcal{D}_I(p,q) = P(1)\left[\mathbf{P}(1)^{-1}\,\mathbf{H}\,\left(\mathbf{P}(1)^{-1}\right)^{\top}\right] Q(1)^{\top}.$$

Clearly, if $\mathcal{B}_N$ is quasi-definite, then $\mathbf{H}$ must be chosen to be a positive definite matrix.

4.2.3 Sobolev orthogonality for type II non-standard parameters

The Jacobi polynomials satisfy the important identity ([20, p. 59])

$$P_n^{(\alpha,\beta)}(x) = (-1)^n\,P_n^{(\beta,\alpha)}(-x). \tag{4.11}$$

Then it is possible to give the Sobolev orthogonality for the polynomial sequence $\{P_n^{(\alpha,-\ell)}(x)\}_{n\geq 0}$ with $-\alpha \notin \mathbb{N}$ and $-\beta = \ell \in \mathbb{N}$, by interchanging the roles of the parameters in Sect. 4.2.2.

Observe that for $-\alpha \notin \mathbb{N}$ and $-\beta = \ell \in \mathbb{N}$, the differential equation

$$\mathcal{L}^{(\alpha,-\ell)}[y] = -n(n+\alpha-\ell+1)\,y, \tag{4.12}$$

is admissible and

$$d_{n+2\,\ell} = -(n+\alpha+\ell+2) \neq 0, \quad n \geq 0.$$

Fix $N \geq \ell$. Constructing the solutions of the Pearson equations (4.6) as in the previous subsection, we obtain

$$\mathbf{u}_{\ell+i} = b_i\,(1-x^2)^i\,\mathbf{u}_\ell, \quad 0 \leq i \leq N-\ell,$$

where

$$\langle \mathbf{u}_\ell, p\rangle = \int_{-1}^{1} p(x)\,(1-x)^{\alpha+\ell}\,dx, \quad \forall p \in \Pi,$$

with b_i being a constant ($b_0 = 1$), and

$$\langle \mathbf{u}_{\ell-j}, p\rangle = e_j \sum_{\nu=0}^{j-1} \frac{(-2)^\nu}{(\alpha+\ell+2-2\,j)_\nu}\binom{j-1}{\nu}\,p^{(\nu)}(-1), \quad \forall p \in \Pi,$$

for $1 \leq j \leq \ell$, where e_j is a free parameter. Therefore, (3.1) with the above moment functionals is written as

$$\begin{aligned}\mathcal{B}_N(p,q) = & \sum_{j=1}^{\ell}\sum_{\nu=0}^{j-1} \frac{(-2)^{\nu}\, e_j}{(\alpha+\ell+2-2\,j)_{\nu}} \binom{j-1}{\nu} \left(p^{(\ell-j)}\, q^{(\ell-j)}\right)^{(\nu)}(-1) \\ & + \sum_{i=0}^{N-\ell} b_i \int_{-1}^{1} p^{(\ell+i)}(x)\, q^{(\ell+i)}(x)\,(1-x)^{\alpha+\ell+i}\,(1+x)^{i}\, dx,\end{aligned}$$

where $b_0 = 1$. Since the Jacobi polynomials $\{P_n^{(\alpha,-\ell)}(x)\}_{n\geq 0}$ satisfy (4.12), then it follows from the symmetry of $\mathcal{L}^{(\alpha,-\ell)}$ with respect to $\mathcal{B}_N$ and Corollary 1, that they satisfy

$$\mathcal{B}_N\left(P_n^{(\alpha,-\ell)}, P_k^{(\alpha,-\ell)}\right) = 0, \quad n \neq k.$$

If we define the discrete bilinear form $\mathcal{D}_{II}$ by

$$\mathcal{D}_{II}(p,q) = \sum_{j=1}^{\ell}\sum_{\nu=0}^{j-1} \frac{(-2)^{\nu}\, e_j}{(\alpha+\ell+2-2\,j)_{\nu}} \binom{j-1}{\nu} \left(p^{(\ell-j)}\, q^{(\ell-j)}\right)^{(\nu)}(-1), \quad \forall p,q \in \Pi,$$

and the continuous bilinear form $\mathcal{C}_{II}$ by

$$\mathcal{C}_{II}(p,q) = \sum_{i=0}^{N-\ell} b_i \int_{-1}^{1} p^{(\ell+i)}(x)\, q^{(\ell+i)}(x)\,(1-x)^{\alpha+\ell+i}\,(1+x)^{i}\, dx, \quad \forall p,q \in \Pi,$$

then we have

$$\mathcal{B}_N(p,q) = \mathcal{D}_{II}(p,q) + \mathcal{C}_{II}(p,q), \quad \forall p,q \in \Pi.$$

Moreover, using (4.9) and (4.11), we obtain

$$P_n^{(\alpha,-\ell)}(x) = \frac{1}{2^{\ell}} \frac{(2\,n-2\,\ell+\alpha+1)_{\ell}}{(n-\ell+1)_{\ell}} (1+x)^{\ell}\, P_{n-\ell}^{(\alpha,\ell)}(x), \quad n \geq \ell.$$

Remark 5 Analogously to Remark 4, for $1 \leq j \leq \ell$, let $\mathsf{C}(j)$ be the symmetric matrix of size $j \times j$ with entries given by

$$(\mathsf{C}(j))_{\varsigma,\nu} = \begin{cases} \binom{\nu+\varsigma}{\nu}\binom{j-1}{\nu} \dfrac{(-2)^{\nu}\, e_j}{(\alpha+\ell+2-2\,j)_{\nu}}, & 0 \leq \nu+\varsigma \leq j-1, \\ 0, & \text{otherwise,} \end{cases}$$

and, for $0 \leq k \leq \ell-1$, let $\mathbf{D}(k)$ be the symmetric matrix of size $\ell \times \ell$ defined as $\mathbf{D}(0) = \mathsf{C}(\ell)$, and

$$\mathbf{D}(k) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathsf{C}(\ell-k) \end{array}\right), \quad 1 \leq k \leq \ell-1. \tag{4.13}$$

Then, $\mathcal{D}_{II}$ can be rewritten in matrix form as

$$\mathcal{D}_{II}(p,q) = \widetilde{P}(-1)\left(\sum_{k=0}^{\ell-1} \mathbf{D}(k)\right) \widetilde{Q}(-1)^{\top}.$$

where $\widetilde{F}(-1) = (f(-1), f'(-1), \ldots, f^{(\ell-1)}(-1))$.

As in Remark 4, we have the $\ell \times \ell$ diagonal matrix

$$\widetilde{\mathbf{H}}^{(\alpha,-\ell)} := \widetilde{\mathbf{P}}(-1)\left(\sum_{k=0}^{\ell-1}\mathbf{D}(k)\right)\widetilde{\mathbf{P}}(-1)^{\top} = \begin{pmatrix} \widetilde{h}_0^{(\alpha,-\ell)} & & & \bigcirc \\ & \widetilde{h}_1^{(\alpha,-\ell)} & & \\ & & \ddots & \\ \bigcirc & & & \widetilde{h}_{\ell-1}^{(\alpha,-\ell)} \end{pmatrix},$$

where $\widetilde{h}_n^{(\alpha,-\ell)} = \mathcal{B}_N(P_n^{(\alpha,-\ell)}, P_n^{(\alpha,-\ell)}), 0 \le n \le \ell-1$, and $\widetilde{\mathbf{P}}(-1)$ is the nonsingular matrix of derivatives of the Jacobi polynomials

$$\widetilde{\mathbf{P}}(1) = \left((P_n^{(\alpha,-\ell)})^{(k)}(-1)\right)_{n,k=0,1,\ldots,\ell-1},$$

given by

$$(P_n^{(\alpha,-m)})^{(k)}(-1) = (-1)^{n-k}\frac{(k-\ell+1)_{n-k}\,(k-\ell+\alpha+1)_k}{2^k\,(n-k)!} \quad n \ge k.$$

Moreover, since the numbers $\widetilde{h}_n^{(\alpha,-\ell)}$ depend on the free parameters e_j, $\mathcal{D}_{II}$ can be explicitly constructed by means of

$$\mathcal{D}_{II}(p,q) = \widetilde{P}(-1)\left[\widetilde{\mathbf{P}}(-1)^{-1}\,\widetilde{\mathbf{H}}\,\left(\widetilde{\mathbf{P}}(-1)^{-1}\right)^{\top}\right]\widetilde{Q}(-1),$$

with $\widetilde{\mathbf{H}}$ being an arbitrary $\ell \times \ell$ diagonal matrix. If $\mathcal{B}_N$ is quasi-definite, then $\widetilde{\mathbf{H}}$ must be chosen to be a positive definite matrix.

4.2.4 Sobolev orthogonality type III non-standard parameters

Here, we consider the polynomial solutions of (4.5) with $-\alpha = m$ and $-\beta = \ell$, $m, \ell \in \mathbb{N}$. In this case, the differential equation is not admissible. Moreover, the polynomial $P_n^{(-m,-\ell)}(x)$ suffers a reduction in degree whenever $n = m + \ell - k$ for some integer $1 \le k \le n$. Hence, we are concerned with constructing polynomial solutions $\{q_n(x)\}_{n\ge 0}$ of (4.5) such that $\deg q_n = n$, and also to give the Sobolev orthogonality for these solutions.

First, we will construct polynomial solutions of (4.5). In [1], the Jacobi polynomials were redefined in order to circumvent the reduction in degree of $P_n^{(-m,-\ell)}(x)$, and there it was shown that these redefined Jacobi polynomials satisfy (4.5) and are orthogonal with respect to a Sobolev bilinear form. We will not use this alternative definition of the Jacobi polynomials, but instead we use Theorem 3 to construct polynomial solutions of (4.5).

Due to (4.11), we can assume without any loss of generality that $\ell \le m$. Observe that for $0 \le n \le \ell - 1$, the polynomial $P_n^{(-m,-\ell)}(x)$ has degree n and satisfies (4.5). No degree reduction occurs in this case.

For $\ell \le n \le m + \ell - 1$, the polynomial $q_n(x) = (1+x)^{\ell}\,P_{n-\ell}^{(-m,\ell)}$ has degree n and satisfies (4.5). Indeed, let $\rho_1(x) = (1+x)^{\ell}$. The polynomial ρ_1 satisfies the differential equations

$$\mathcal{L}^{(-m,-\ell)}[\rho_1] = \ell\,(m-1)\,\rho_1 \quad \text{and} \quad (1-x^2)\rho_1' = \ell\,(1-x)\,\rho_1.$$

Moreover, the polynomial $P_n^{(-m,\ell)}(x)$ satisfies

$$\begin{aligned}&\mathcal{L}^{(-m,\ell)}[P_n^{(-m,\ell)}] \\ &\quad = \mathcal{L}^{(-m,-\ell)}[P_n^{(-m,\ell)}] + 2\,\ell\,(1-x)\left(P_n^{(-m,\ell)}\right)' = -n(n+\ell-m+1)\,P_n^{(-m,\ell)}.\end{aligned}$$

Thus, from Theorem 3, we get that the polynomials $q_n(x)$ satisfy

$$\mathcal{L}^{(-m,-\ell)}[q_n] = -n\,(n-m-\ell+1)\,q_n.$$

For $n \geq m+\ell$, define the polynomials $\tilde{q}_n(x) = (1-x)^m\,(1+x)^\ell\,P_{n-m-\ell}^{(m,\ell)}(x)$. Clearly, $\deg \tilde{q}_n = n$. We will show that these polynomials satisfy (4.5). Define the polynomial $\rho_2(x) = (1-x)^m\,(1+x)^\ell$, and observe that ρ_2 satisfies the differential equations

$$\mathcal{L}^{(-m,-\ell)}[\rho_2] = -(m+\ell)\,\rho_2 \quad \text{and} \quad (1-x^2)\,\rho_2' = [\ell-m-(m+\ell+2)\,x]\,\rho_2.$$

Taking into account that the Jacobi polynomials $P_n^{(m,\ell)}(x)$ satisfy the differential equation

$$\begin{aligned}
&\mathcal{L}^{(m,\ell)}[P_n^{(m,\ell)}]\\
&\quad= \mathcal{L}^{(-m,-\ell)}[P_n^{(m,\ell)}] + 2\,[\ell-m-(m+\ell+2)\,x]\left(P_n^{(m,\ell)}\right)'\\
&\quad= -n\,(n+m+\ell+1)\,P_n^{(m,\ell)},
\end{aligned}$$

we deduce from Theorem 3 that the polynomials $\tilde{q}_n(x)$ satisfy

$$\mathcal{L}^{(-m,-\ell)}[\tilde{q}_n] = -n\,(n-m-\ell+1)\,\tilde{q}_n.$$

Altogether, the sequence of polynomials $\{\mathcal{P}_n^{(-m,-\ell)}(x)\}_{n\geq 0}$ with

$$\mathcal{P}_n^{(-m,-\ell)}(x) = \begin{cases} P_n^{(-m,-\ell)}(x), & 0 \leq n \leq \ell-1,\\ (1+x)^\ell\,P_{n-\ell}^{(-m,\ell)}(x), & \ell \leq n \leq \ell+m-1,\\ (1-x)^m\,(1+x)^\ell\,P_{n-m-\ell}^{(m,\ell)}(x), & m+\ell \leq n, \end{cases} \tag{4.14}$$

constitute polynomial solutions of (4.5) such that $\deg \mathcal{P}_n^{(-m,-\ell)} = n$.

Now that we have the explicit expression of a sequence of polynomials satisfying (4.5), we proceed to determine the moment functionals in (3.1) such that $\mathcal{L}^{(-m,-\ell)}$ is symmetric with respect to $\mathcal{B}_N$, $N \geq m+\ell$. Hence, we find solutions for the Pearson equation (4.6) for $0 \leq k \leq N$.

In this case, (4.7) reads

$$d_n = -(n-m-\ell+2), \quad n \geq 0.$$

Observe that $d_{n+2\,(m+\ell)} \neq 0$ for $n \geq 0$. By Proposition 2, we have

$$\mathbf{u}_{m+\ell+i} = a_i\,(1-x^2)^i\,\mathbf{u}_{m+\ell}, \quad 0 \leq i \leq N-m-\ell,$$

where a_i are constants ($a_0 = 1$), and $\mathbf{u}_{m+\ell}$ satisfies

$$D[(1-x^2)\,\mathbf{u}_{m+\ell}] = [m-\ell-(m+\ell+2)\,x]\,\mathbf{u}_{m+\ell}.$$

Then $\mathbf{u}_{m+\ell}$ can be chosen to be the quasi-definite moment functional associated with the Jacobi polynomials $\{P_n^{(\ell,m)}(x)\}_{n\geq 0}$, that is,

$$\langle \mathbf{u}_{m+\ell}, p\rangle = \int_{-1}^{1} p(x)\,(1-x)^\ell\,(1+x)^m\,dx, \quad \forall p \in \Pi.$$

For $1 \leq j \leq m+\ell$, let the moment functional $\mathbf{u}_{m+\ell-j}$ satisfy the Pearson equation

$$D[(1-x^2)\,\mathbf{u}_{m+\ell-j}] = [m-\ell-(m+\ell-2\,j+2)\,x]\,\mathbf{u}_{m+\ell-j}.$$

For $1 \leq j \leq m$, we obtain

$$\langle \mathbf{u}_{m+\ell-j}, p \rangle = c_j \sum_{\nu=0}^{j-1} \frac{2^\nu}{(m+2-\ell-2\,j)_\nu} \binom{j-1}{\nu} p^{(\nu)}(1), \quad \forall p \in \Pi,$$

where c_j is a free parameter.

Similarly, for $1 \leq r = j - m \leq \ell$, we get

$$\langle \mathbf{u}_{\ell-r}, p \rangle = e_r \sum_{\nu=0}^{r-1} \frac{(-2)^\nu}{(\ell+2-m-2\,r)_\nu} \binom{r-1}{\nu} p^{(\nu)}(-1), \quad \forall p \in \Pi,$$

where e_r is a free parameter.

Taking the moment functionals $\mathbf{u}_k$, $0 \leq k \leq N$, as above, the bilinear form (3.1) reads

$$\begin{aligned} \mathcal{B}_N(p,q) &= \sum_{j=1}^{m} \sum_{\nu=0}^{j-1} \frac{2^\nu\, c_j}{(m+2-\ell-2\,j)_\nu} \binom{j-1}{\nu} \left(p^{(m-j)} q^{(m-j)}\right)^{(\nu)} (1) \\ &\quad + \sum_{r=1}^{\ell} \sum_{\nu=0}^{r-1} \frac{(-2)^\nu\, e_r}{(\ell+2-m-2\,r)_\nu} \binom{r-1}{\nu} \left(p^{(\ell-r)} q^{(\ell-r)}\right)^{(\nu)} (-1) \\ &\quad + \sum_{i=0}^{N-(m+\ell)} a_i \int_{-1}^{1} p^{(m+\ell+i)}(x)\, q^{(m+\ell+i)}(x)\, (1-x)^{m+i}\, (1+x)^{\ell+i}\, dx. \end{aligned} \tag{4.15}$$

Let $\mathcal{D}_{III}$ be the discrete bilinear form defined by

$$\begin{aligned} \mathcal{D}_{III}(p,q) = &\sum_{j=1}^{m} \sum_{\nu=0}^{j-1} \frac{2^\nu\, c_j}{(m+2-\ell-2\,j)_\nu} \binom{j-1}{\nu} \left(p^{(m-j)} q^{(m-j)}\right)^{(\nu)} (1) \\ &+ \sum_{r=1}^{\ell} \sum_{\nu=0}^{r-1} \frac{(-2)^\nu\, e_r}{(\ell+2-m-2\,r)_\nu} \binom{r-1}{\nu} \left(p^{(\ell-r)} q^{(\ell-r)}\right)^{(\nu)} (-1), \end{aligned}$$

and let $\mathcal{C}_{III}$ be the continuous bilinear form defined by

$$\mathcal{C}_{III}(p,q) = \sum_{i=0}^{N-(m+\ell)} a_i \int_{-1}^{1} p^{(m+\ell+i)}(x)\, q^{(m+\ell+i)}(x)\, (1-x)^{\ell+i}\, (1+x)^{m+i}\, dx.$$

Then (4.15) can be written as

$$\mathcal{B}_N(p,q) = \mathcal{D}_{III}(p,q) + \mathcal{C}_{III}(p,q).$$

The differential equation (4.5) with $\alpha = -m$ and $\beta = -\ell$ is not admissible. Indeed, $\lambda_n = \lambda_k$ if and only if $n = 0$ and $k = m + \ell - 1$. Therefore, by the Sobolev orthogonality of $\{\mathcal{P}_n^{(-m,-\ell)}(x)\}_{n\geq 0}$ does not follow from Corollary 1. Nevertheless, from the symmetry of $\mathcal{L}^{(-m,-\ell)}$ with respect to $\mathcal{B}_N$ and Theorem 2, it follows that

$$\mathcal{B}_N(\mathcal{P}_n^{(-m,-\ell)}, \mathcal{P}_k^{(-m,-\ell)}) = 0, \quad n \neq k, \quad \text{where } n \neq 0 \text{ or } k \neq m+\ell-1.$$

On the other hand, we compute

$$\mathcal{B}_N(\mathcal{P}_0^{(-m,-\ell)}, \mathcal{P}_{m+\ell-1}^{(-m,-\ell)}) = c_m\, \mathcal{P}_0^{(-m,-\ell)}(1)\, \mathcal{P}_{m+\ell-1}^{(-m,-\ell)}(1).$$

Therefore, setting $c_m = 0$, we have $\mathcal{B}_N(\mathcal{P}_n^{(-m,-\ell)}, \mathcal{P}_k^{(-m,-\ell)}) = 0$ for $n \neq k$.

Remark 6 Using (4.10) and (4.13), the bilinear form $\mathcal{D}_{III}$ can be rewritten as

$$\mathcal{D}_{III}(p,q) = \left(\widetilde{P}(-1)\,|\,P(1)\right)\begin{pmatrix}\sum_{k=0}^{\ell-1}\mathbf{D}(k) & \bigcirc \\ \bigcirc & \sum_{k=0}^{m-1}\mathbf{B}(k)\end{pmatrix}\left(\widetilde{Q}(-1)\,|\,Q(1)\right)^{\top}.$$

where

$$\left(\widetilde{F}(-1)\,|\,F(-1)\right) = \left(f^{(0)}(-1),\ldots,f^{(\ell-1)}(-1),f^{(0)}(1),\ldots,f^{(m-1)}(-1)\right).$$

Moreover, we have the $(m+\ell)\times(m+\ell)$ diagonal matrix

$$\begin{aligned}\widehat{\mathbf{H}}^{(-m,-\ell)} &:= (\widetilde{\mathbf{P}}(-1)\,|\,\mathbf{P}(1))\begin{pmatrix}\sum_{k=0}^{\ell-1}\mathbf{D}(k) & \bigcirc \\ \bigcirc & \sum_{k=0}^{m-1}\mathbf{B}(k)\end{pmatrix}(\widetilde{\mathbf{P}}(-1)\,|\,\mathbf{P}(1))^{\top} \\ &= \begin{pmatrix}\widehat{h}_0^{(-m,-\ell)} & & & \bigcirc \\ & \widehat{h}_1^{(-m,-\ell)} & & \\ & & \ddots & \\ \bigcirc & & & \widehat{h}_{m+\ell-1}^{(-m,-\ell)}\end{pmatrix},\end{aligned}$$

where $\widehat{h}_n^{(-m,-\ell)} = \mathcal{B}_N(\mathcal{P}_n^{(-m,-\ell)},\mathcal{P}_n^{(-m,-\ell)})$, $0\le n\le m+\ell-1$,

$$\begin{aligned}\widetilde{\mathbf{P}}(-1) &= \left((\mathcal{P}_n^{(-m,-\ell)})^{(k)}(-1)\right)_{\substack{n=0,1,\ldots,m+\ell-1,\\ k=0,1,\ldots,\ell-1}} \quad \text{and} \\ \mathbf{P}(1) &= \left((\mathcal{P}_n^{(-m,-\ell)})^{(k)}(1)\right)_{\substack{n=0,1,\ldots,m+\ell-1,\\ k=0,1,\ldots,m-1}}.\end{aligned}$$

The matrix $(\widetilde{\mathbf{P}}(-1)\,|\,\mathbf{P}(1))$ is non singular since it can be expressed as a product of two non singular matrices. Indeed, if the explicit expression of the polynomial $\mathcal{P}_n^{(-m,-\ell)}(x)$ is

$$\mathcal{P}_n^{(-m,-\ell)}(x) = \sum_{k=0}^{n} p_{n,k}\,x^k, \quad p_{n,n}\neq 0, \quad n\ge 0, \tag{4.16}$$

then the $(m+\ell)\times(m+\ell)$ matrix

$$\mathcal{P} = \begin{pmatrix} p_{0,0} & 0 & \cdots & 0 \\ p_{1,0} & p_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{m+\ell-1,0} & p_{m+\ell-1,1} & \cdots & p_{m+\ell-1,m+\ell-1}\end{pmatrix},$$

is non singular. Let $\widetilde{M}(-1)$ and $M(1)$ be the matrices obtained from derivatives of the basis $\{x^n\}_{0\le n\le m+\ell-1}$ of $\Pi_{m+\ell-1}$,

$$\begin{aligned}\widetilde{M}(-1) &= \left((x^n)^{(k)}|_{x=-1}\right)_{\substack{n=0,1,\ldots,m+\ell-1,\\ k=0,1,\ldots,\ell-1}}, \\ M(1) &= \left((x^n)^{(k)}|_{x=1}\right)_{\substack{n=0,1,\ldots,m+\ell-1,\\ k=0,1,\ldots,m-1}}.\end{aligned}$$

The matrix $(\widetilde{M}(-1)\,|\,M(1))$ is a generalized Vandermonde matrix and, therefore, non-singular. From (4.16), we deduce that

$$(\widetilde{\mathbf{P}}(-1)\,|\,\mathbf{P}(1)) \;=\; \mathcal{P}\cdot(\widetilde{M}(-1)\,|\,M(1)),$$

and, thus, $(\widetilde{\mathbf{P}}(-1)\,|\,\mathbf{P}(1))$ is non-singular

With this in mind, and taking into account that the numbers $\widehat{h}_n^{(-m,-\ell)}$ depend on free parameters, we have that for an arbitrary $(m+\ell)\times(m+\ell)$ diagonal matrix $\widehat{\mathbf{H}}$, the bilinear form $\mathcal{D}_{III}$ can be explicitly constructed by means of

$$\mathcal{D}_{III}(p,q) \;=\; \big(\widetilde{P}(-1)\,|\,P(1)\big)\;\mathbf{E}\;\big(\widetilde{Q}(-1)\,|\,Q(1)\big)^{\top},$$

where

$$\mathbf{E} \;=\; (\widetilde{\mathbf{P}}(-1)\,|\,\mathbf{P}(1))^{-1}\,\widehat{\mathbf{H}}\,\big((\widetilde{\mathbf{P}}(-1)\,|\,\mathbf{P}(1))^{-1}\big)^{\top}.$$

If the Sobolev bilinear form $\mathcal{B}_N$ is quasi-definite, then $\widehat{\mathbf{H}}$ must be chosen to be a positive definite matrix.

4.2.5 Sobolev orthogonality for type IV non-standard parameters

Now, the case when $-\alpha = -\beta = m \in \mathbb{N}$ is considered. As we will see here, this is not a particular case of the third type of non-standard parameters studied above, in the sense that the continuous part of the Sobolev bilinear form is not directly deduced by setting $m = \ell$ in (4.15).

Indeed, the operator $\mathcal{L}^{(-m,-m)}$ is symmetric with respect $\mathcal{B}_N$ in (3.1) with $N \geq m$, if and only if the moment functionals $\mathbf{u}_k$, $0 \leq k \leq N$, satisfy the Pearson equations (4.6) with $\alpha = \beta = -m$, that is,

$$D[(1-x^2)\,\mathbf{u}_k] \;=\; -2\,(-m+k+1)\,x\,\mathbf{u}_k.$$

In this case, (3.8) reads

$$d_n \;=\; -(n-2\,m+2), \quad n \geq 0,$$

and, thus, $d_{n+2\,m} \neq 0$ for $n \geq 0$. By Proposition 2, we have

$$\mathbf{u}_{m+i} \;=\; a_i\,(1-x^2)^i\,\mathbf{u}_m, \quad 0 \leq i \leq N-m,$$

where a_i are constants and $\mathbf{u}_m$ satisfies $D[(1-x^2)\,\mathbf{u}_m] = -2\,x\,\mathbf{u}_m$. Then $\mathbf{u}_m$ can be chosen as the quasi-definite moment functional

$$\langle \mathbf{u}_m, p\rangle \;=\; \int_{-1}^{1} p(x)\,dx, \quad \forall p \in \Pi,$$

associated with the Jacobi polynomials $\{P_n^{(0,0)}(x)\}_{n\geq 0}$. Therefore, we have that the continuous part of $\mathcal{B}_N$ in this case is the bilinear form $\mathcal{C}_{IV}$ given by

$$\mathcal{C}_{IV}(p,q) \;=\; \sum_{i=0}^{N-m} a_i \int_{-1}^{1} p^{(m+i)}(x)\,q^{(m+i)}(x)\,(1-x^2)^i\,dx,$$

with $a_0 = 1$. On the other hand, $\mathcal{C}_{III}$ with $m = \ell$ reads

$$\mathcal{C}_{III}(p,q) \;=\; \sum_{i=0}^{N-2\,m} a_i \int_{-1}^{1} p^{(2\,m+i)}(x)\,q^{(2\,m+i)}(x)\,(1-x^2)^{m+i}\,dx.$$

Hence, in general, $\mathcal{C}_{III}$ and $\mathcal{C}_{IV}$ do not coincide.

Now, we proceed to find solutions to the remaining Pearson equations. For $1 \le j \le m$, let the moment functional $\mathbf{u}_{m-j}$ satisfy the Pearson equation

$$D[(1-x^2)\,\mathbf{u}_{m-j}] = 2\,(j-1)\,x\,\mathbf{u}_{m-j}.$$

There are two linearly independent solutions. Namely, $\mathbf{u}_{m-j,1}$ and $\mathbf{u}_{m-j,2}$ given by

$$\langle \mathbf{u}_{m-j,1}, p\rangle = \sum_{\nu=0}^{j-1} \frac{2^\nu}{(2-2\,j)_\nu} \binom{j-1}{\nu} p^{(\nu)}(1), \quad \forall p \in \Pi,$$

and

$$\langle \mathbf{u}_{m-j,2}, p\rangle = \sum_{\nu=0}^{j-1} \frac{(-2)^\nu}{(2-2\,j)_\nu} \binom{j-1}{\nu} p^{(\nu)}(-1), \quad \forall p \in \Pi.$$

Then,

$$\langle \mathbf{u}_{m-j}, p\rangle = \sum_{\nu=0}^{j-1} \frac{2^\nu}{(2-2\,j)_\nu} \binom{j-1}{\nu} \Big[c_j\, p^{(\nu)}(1) + (-1)^\nu\, e_j\, p^{(\nu)}(-1)\Big], \quad \forall p \in \Pi,$$

where c_j and e_j are free parameters. Therefore, if we define the discrete bilinear form $\mathcal{D}_{IV}$ as $\mathcal{D}_{III}$ with $m = \ell$, that is,

$$\begin{aligned}\mathcal{D}_{IV}(p,q) = \sum_{j=1}^{m}\sum_{\nu=0}^{j-1} \frac{2^\nu}{(2-2\,j)_\nu} \binom{j-1}{\nu} \Big[c_j\, \left(p^{(m-j)}\,q^{(m-j)}\right)^{(\nu)}(1) \\ +(-1)^\nu\, e_j\, \left(p^{(m-j)}\,q^{(m-j)}\right)^{(\nu)}(-1)\Big],\end{aligned}$$

the Sobolev bilinear form (3.1) reads

$$\mathcal{B}_N(p,q) = \mathcal{C}_{IV}(p,q) + \mathcal{D}_{IV}(p,q).$$

Observe that the polynomial sequence (4.14) with $m = \ell$ satisfies the differential equation

$$\mathcal{L}^{(-m,-m)}[\mathcal{P}_n^{(-m,-m)}] = -n\,(n-2\,m+1)\,\mathcal{P}_n^{(-m,-m)}, \quad n \ge 0,$$

which is not admissible. Similarly to the third type of non-standard parameters, if $c_m = 0$, then $\mathcal{B}_N(\mathcal{P}_n^{(-m,-m)}, \mathcal{P}_k^{(-m,-m)}) = 0$ for $n \ne k$.

Remark 7 Clearly, the matrix representation of $\mathcal{D}_{IV}$ is the same as the matrix representation of $\mathcal{D}_{III}$ with $m = \ell$.

4.3 Hermite polynomials

The Hermite polynomials $\{H_n(x)\}_{n\ge 0}$ constitute a sequence of polynomial solutions of the the admissible equation

$$\mathcal{L}[y] \equiv y'' - 2\,x\,y' = -2\,n\,y.$$

By Theorem 4, we have that for a non-negative integer N, the differential $\mathcal{L}$ is symmetric with respect to the bilinear form $\mathcal{B}_N$ defined in (3.1) if and only if $D(\mathbf{u}_k) = -2x\,\mathbf{u}_k$ for

$0 \leq k \leq N$. Thus, if we take $\mathbf{u}_0$ such that

$$\langle \mathbf{u}_0, p \rangle = \int_{-\infty}^{\infty} p(x)\, e^{-x^2} dx,$$

then from Proposition 2, $\mathbf{u}_k = a_k \mathbf{u}_0$ and therefore (3.1) reads

$$\mathcal{B}_N(p, q) = \sum_{k=0}^{N} a_k \int_{-\infty}^{+\infty} p^{(k)}(x)\, q^{(k)}(x)\, e^{-x^2}\, dx, \quad \forall p, q \in \Pi,$$

where a_k are constants ($a_0 = 1$). From the symmetry of $\mathcal{L}$ with respect to $\mathcal{B}_N$ and Corollary 1, we have $\mathcal{B}_N(H_n, H_m) = 0$ for $n \neq m$.

4.4 Bessel polynomials

Let $\mathcal{L}^{(a)}$ be the differential operator defined by

$$\mathcal{L}^{(a)}[p] \equiv x^2\, p'' + (a\, x + 2)\, p'.$$

Recall from Sect. 2 that for arbitrary complex values of a, the Bessel polynomials $\{B_n^{(a)}(x)\}_{n \geq 0}$ constitute a sequence of polynomial solutions of the differential equation

$$\mathcal{L}^{(a)}[y] = n\,(n + a - 1)\, y, \quad n \geq 0. \tag{4.17}$$

Fix a non-negative integer N. From Theorem 4, $\mathcal{L}^{(a)}$ is symmetric with respect to the bilinear form $\mathcal{B}_N$ in (3.1) if and only if the moment functionals $\mathbf{u}_k$, $0 \leq k \leq N$, satisfy the Pearson equations

$$D(x^2\, \mathbf{u}_k) = [(a + 2\,k)\, x + 2]\, \mathbf{u}_k.$$

In this case, (3.8) reads $d_n = n + a$, $n \geq 0$.

For $-a \notin \mathbb{N}$, we can choose $\mathbf{u}_k = a_k\, x^{2\,k}\, \mathbf{u}_0$, $0 \leq k \leq N$, where

$$\langle \mathbf{u}_0, p \rangle = \frac{1}{2\pi\, i} \int_c p(z)\, z^{a-2} e^{-2/z} dz, \quad \forall p \in \Pi,$$

c is the unit circle oriented in the counter-clockwise direction, and a_k are constants ($a_0 = 1$). Then, we have that $\mathcal{L}^{(a)}$ is symmetric with respect to the bilinear form

$$\mathcal{B}_N(p, q) = \frac{1}{2\pi\, i} \sum_{k=0}^{N} a_k \int_c p^{(k)}(z)\, q^{(k)}(z)\, z^{a+2\,(k-1)} e^{-2/z} dz.$$

In this case, $\deg B_n^{(a)} = n$, $n \geq 0$. Moreover, the differential equation satisfied by the Bessel polynomials is admissible. We know from Corollary 1 that

$$\mathcal{B}_N(B_n^{(a)}, B_k^{(a)}) = 0, \quad n \neq k.$$

For $-a = m \in \mathbb{N}$, the expression for the Bessel polynomial $B_n^{(-m)}(x)$ suffers a reduction in the degree when $n = m + 1 - k$ for some $0 \leq k \leq n - 1$. Indeed,

$$\deg B_n^{(-m)} = m + 1 - n, \quad \left\lceil \frac{m}{2} \right\rceil + 1 \leq n \leq m + 1,$$

where $\lceil \nu \rceil$ is the smallest integer bigger that ν. Unfortunately, in this case the differential equation (4.17) does not have a polynomial solution of degree n for $\lceil \frac{m}{2} \rceil + 1 \leq n \leq m + 1$. Hence, we do not give a Sobolev orthogonality for this case.

Acknowledgements The work of the second author (MEM) has been supported by Ministerio de Ciencia, Innovación y Universidades (MICINN) Grant PGC2018-096504-B-C33.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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