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AN ANTIMAXIMUM PRINCIPLE FOR A DEGENERATE PARABOLIC PROBLEM

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Abstract. We obtain an antimaximum principle for the following quasilinear parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = \lambda\, |u|^{p-2} u + f(x,t), & (x,t) \in \Omega \times (0,T); \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T); \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{P}$$

which involves the p-Laplace operator $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (with Dirichlet boundary conditions, $1 < p < \infty$) and a spectral parameter $\lambda \in \mathbb{R}^N$ taking values near the first eigenvalue λ_1 of $-\Delta_p$. We show that *any* weak solution $u : \Omega \times [0,T) \to \mathbb{R}$ of problem (P) (suitably defined in a standard way) eventually becomes positive for all $x \in \Omega$ and all times $t \geq T_+$, provided, for instance, $f(x,t) \geq \underline{f}(x) > 0$ for some function $\underline{f} \in L^\infty(\Omega)$, $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and $\lambda_1 < \lambda < \lambda_1 + \delta$. Here, the "key" constants $\delta \equiv \delta(\underline{f}, u_0) > 0$ and $T_+ \equiv T_+(f, u_0) \in (0,T)$ depend on f (or $\underline{f}$ only) and u_0. In particular, a solution u eventually becomes

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positive even if the initial data u_0 are "arbitrarily" negative as long as they are smooth enough.

1. Introduction

Beginning with the work of Clément and Peletier [9], various kinds of anti-maximum principles have been established for linear and nonlinear *elliptic* operators; for linear elliptic operators, see e.g. Alziary, Fleckinger, and Takáč [3], Sweers [25], Takáč [26], and the references therein. In the case of the Dirichlet p-Laplacian Δ_p $(1 < p < \infty)$, $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, which we deal with throughout the present article, the antimaximum principle takes the following form; see Arcoya and Gámez [6] or Fleckinger et al. [14]: Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with a connected C^2-boundary $\partial\Omega$. Denote by λ_1 the first (smallest) eigenvalue of $-\Delta_p$. Then, given any $f \in L^\infty(\Omega)$ with $0 \leq f \not\equiv 0$ in Ω, there exists a constant $\delta \equiv \delta(f) > 0$ such that, if $\lambda_1 < \lambda < \lambda_1 + \delta$, then every solution $u \in W_0^{1,p}(\Omega)$ of the boundary-value problem

$$-\Delta_p u = \lambda\, |u|^{p-2} u + f(x) \ \text{ in } \Omega; \qquad u = 0 \ \text{ on } \partial\Omega, \tag{1.1}$$

satisfies $u < 0$ in Ω and $\partial u/\partial\nu > 0$ on $\partial\Omega$. In contrast, if $-\infty < \lambda < \lambda_1$, then $u > 0$ in Ω and $\partial u/\partial\nu < 0$ on $\partial\Omega$. As usual, $\partial/\partial\nu$ denotes the outer normal derivative on the boundary $\partial\Omega$. In this setting one has $u \in C^1(\overline{\Omega})$.

An antimaximum principle for (only) linear *parabolic* operators has been obtained in the work of Díaz and Fleckinger [10, Theorem 2.1]. The main result of our present work is an analogue for a nonlinear parabolic operator with Δ_p in the initial-boundary-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = \lambda\, |u|^{p-2} u + f(x,t), & (x,t) \in \Omega \times (0,T); \\ \quad u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T); \\ \quad u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

$T = T_\infty$, where T_∞ $(0 < T_\infty \leq \infty)$ denotes the *maximum time* for existence of a weak solution $u : \Omega \times (0,T) \to \mathbb{R}$. Of course, $T_\infty \equiv T_\infty(f, u_0)$ may depend on f and $u_0 \in W_0^{1,p}(\Omega)$, especially when $p > 2$. We can state our antimaximum principle as follows.

Theorem 1.1. *Assume that $f \in L^\infty(\Omega \times \mathbb{R}_+)$ satisfies $f(x,t) \geq \underline{f}(x)$ in $\Omega \times \mathbb{R}_+$, where $\underline{f} \in L^\infty(\Omega)$ is a function with $0 \leq \underline{f} \not\equiv 0$ in Ω, and $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then there exist constants $\delta \equiv \delta(\underline{f}, u_0) > 0$ and $T_+ \equiv T_+(f, u_0) \in (0, T_\infty)$ with the following property: If $\lambda_1 < \lambda < \lambda_1 + \delta$ and if*

$u : \Omega \times (0, T_\infty) \to \mathbb{R}$ is a weak solution of problem (1.2) *that is bounded in $\Omega \times (0, T')$ for every $T' \in (0, T_\infty)$, then u satisfies $u(x,t) > 0$ for all $(x,t) \in \Omega \times [T_+, T_\infty)$ and $(\partial u/\partial \nu)(x,t) < 0$ for all $(x,t) \in \partial\Omega \times [T_+, T_\infty)$.*

This means that even if the initial data u_0 are large negative, satisfying only the smoothness condition $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, the solution $u(\,\cdot\,, t)$ eventually becomes positive for all times $t \in [T_+, T_\infty)$. By analogy with the corresponding elliptic problem (1.1) above, one has $u(\,\cdot\,, t) \in C^1(\overline{\Omega})$ for $0 < t < T_\infty$.

Let φ_1 denote the eigenfunction associated with λ_1 and normalized by $\varphi_1 > 0$ in Ω and $\int_\Omega \varphi_1^p \, \mathrm{d}x = 1$. The hypothesis $0 \le \underline{f} \not\equiv 0$ in Ω can be weakened to $\int_\Omega \underline{f} \varphi_1 \, \mathrm{d}x > 0$ provided the resonant elliptic problem (1.1) with $\lambda = \lambda_1$ and $f = \underline{f}$ has *no* weak solution. For problem (1.1) this generalization is due to [6, Theorem 27, page 1908].

2. Preliminaries

All Banach and Hilbert spaces used in this article are real. We work with the standard inner product in $L^2(\Omega)$ defined by $\langle u, v\rangle \stackrel{\text{def}}{=} \int_\Omega uv \, \mathrm{d}x$ for $u, v \in L^2(\Omega)$. The orthogonal complement in $L^2(\Omega)$ of a set $\mathcal{M} \subset L^2(\Omega)$ is denoted by $\mathcal{M}^{\perp, L^2}$,

$$\mathcal{M}^{\perp, L^2} \stackrel{\text{def}}{=} \{u \in L^2(\Omega) : \langle u, v\rangle = 0 \text{ for all } v \in \mathcal{M}\}.$$

The inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ in $L^2(\Omega)$ induces a duality between the Lebesgue spaces $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $1 \le p, p' \le \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and between the Sobolev space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$, as well. We keep the same notation also for the duality between the Cartesian products $[L^p(\Omega)]^N$ and $[L^{p'}(\Omega)]^N$. We sometimes emphasize the duality pair involved by indicating it in the subscript; for instance, $\langle\,\cdot\,,\,\cdot\,\rangle_{W_0^{1,p} \times W^{-1,p'}}$ for the pair $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. We use exclusively the norm

$$\|u\|_{W_0^{1,p}(\Omega)} \stackrel{\text{def}}{=} \left(\int_\Omega |\nabla u|^p \, \mathrm{d}x \right)^{1/p}$$

in $W_0^{1,p}(\Omega)$ and its dual norm $\|\cdot\|_{W^{-1,p'}(\Omega)}$ in $W^{-1,p'}(\Omega)$. The closure, interior and boundary of a set $S \subset \mathbb{R}^N$ are denoted by $\overline{S}$, $\operatorname{int}(S)$ and ∂S, respectively, and the characteristic function of S by $\chi_S : \mathbb{R}^N \to \{0, 1\}$. We write

$$|S|_N \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \chi_S(x) \, \mathrm{d}x$$

if S is also Lebesgue measurable. As usual, we set $\mathbb{R}_+ = [0, \infty)$.

We always assume the following.

Hypothesis (H1). *If $N \geq 2$, then Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact manifold of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, and Ω satisfies also the interior sphere condition at every point of $\partial\Omega$ (see Gilbarg and Trudinger* [17, page 33]). *If $N = 1$, then Ω is a bounded open interval in $\mathbb{R}^1$.*

For $N \geq 2$, **(H1)** is satisfied if $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary.

We denote $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and always take $1 < p < \infty$. Let λ_1 denote the first (smallest) eigenvalue of the positive Dirichlet p-Laplacian Δ_p; that is,

$$-\Delta_p \varphi_1 = \lambda_1 |\varphi_1|^{p-2}\varphi_1 \ \text{ in } \Omega; \qquad \varphi_1 = 0 \ \text{ on } \partial\Omega, \tag{2.1}$$

holds with an eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega) \setminus \{0\}$. The eigenvalue λ_1 is simple, by a result due to Anane [4, Théorème 1, page 727] or Lindqvist [20, Theorem 1.3, page 157], and it is given by the Rayleigh quotient

$$\lambda_1 \stackrel{\text{def}}{=} \inf \Big\{ \int_\Omega |\nabla u|^p \, \mathrm{d}x : \ u \in W_0^{1,p}(\Omega) \ \text{ with } \ \int_\Omega |u|^p \, \mathrm{d}x = 1 \Big\}, \tag{2.2}$$

$\lambda_1 > 0$. Moreover, a minimizer – the corresponding eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega) \setminus \{0\}$ – can be normalized by $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_{L^p(\Omega)} = 1$, owing to the strong maximum principle [30, Proposition 3.2.1 and 3.2.2, page 801] or [33, Theorem 5, page 200] (see also [4, Théorème 1, page 727] or [20, Theorem 1.3, page 157]). We have $\varphi_1 \in L^\infty(\Omega)$ by [5, Théorème A.1, page 96]. Consequently, recalling hypothesis **(H1)** , we get even $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, \alpha)$, by a regularity result due to [11, Theorem 2, page 829] and [31, Theorem 1, page 127] (interior regularity), and to [18, Theorem 1, page 1203] (regularity near the boundary). The constant β depends solely on α, N, and p. We keep the meaning of the constants α and β throughout the entire article and denote by $\beta' \in (0, \beta)$ an arbitrary, but fixed number. Finally, the Hopf maximum principle [30, Proposition 3.2.1 and 3.2.2, page 801] or [33, Theorem 5, page 200] renders

$$\varphi_1 > 0 \ \text{ in } \Omega \qquad \text{and} \qquad \frac{\partial \varphi_1}{\partial \nu} < 0 \ \text{ on } \partial\Omega. \tag{2.3}$$

We set

$$U \stackrel{\text{def}}{=} \{x \in \Omega : \nabla\varphi_1(x) \neq \mathbf{0}\}, \ \text{ hence } \ \Omega \setminus U = \{x \in \Omega : \nabla\varphi_1(x) = \mathbf{0}\},$$

and observe that $\Omega \setminus U$ is a compact subset of Ω, by (2.3).

Later, beginning with Lemma 4.6 in Section 4, we use also Hölder spaces of type $C^{1+\beta,(1+\beta)/2}\left(\overline{\Omega}\times[0,T]\right)$ of functions on (the closure of) space-time domains like $\overline{\Omega}\times[0,T]\subset\mathbb{R}^N\times\mathbb{R}_+$. Therefore, to avoid possible confusion with $C^{1,\beta}(\overline{\Omega})$, from now on we prefer to use the notation $C^{1+\beta}(\overline{\Omega})\equiv C^{1,\beta}(\overline{\Omega})$ for the latter, where $0<\beta<1$.

Often, a function $u\in L^1(\Omega)$ will be decomposed as the orthogonal sum $u=u^{\|}\cdot\varphi_1+u^{\top}$ according to

$$u^{\|}\stackrel{\text{def}}{=}\|\varphi_1\|^{-2}_{L^2(\Omega)}\int_\Omega u\,\varphi_1\,\mathrm{d}x\quad\text{ and }\quad\int_\Omega u^{\top}\,\varphi_1\,\mathrm{d}x=0. \tag{2.4}$$

Given a linear subspace $\mathcal{M}$ of $L^1(\Omega)$ with $\varphi_1\in\mathcal{M}$, we write

$$\mathcal{M}^{\top}\stackrel{\text{def}}{=}\left\{u\in\mathcal{M}:\int_\Omega u\,\varphi_1\,\mathrm{d}x=0\right\}.$$

In particular, we will find it convenient to work with the orthogonal sum $L^2(\Omega)=\operatorname{lin}\{\varphi_1\}\oplus L^2(\Omega)^{\top}$ and the direct sum $W_0^{1,p}(\Omega)=\operatorname{lin}\{\varphi_1\}\oplus W_0^{1,p}(\Omega)^{\top}$.

We are interested in weak solutions to the evolutionary problem (1.2) in a cylindrical domain $\Omega\times(0,T)$ with some $0<T\le\infty$.

Definition 2.1. Let $u_0\in W_0^{1,p}(\Omega)$ and $0<T\le\infty$. We say that $u:\Omega\times(0,T)\to\mathbb{R}$ is a *weak solution* of problem (1.2) in $\Omega\times(0,T)$ if it satisfies

$$u\in C\left([0,T']\to L^2(\Omega)\right)\cap L^p\left((0,T')\to W_0^{1,p}(\Omega)\right) \tag{2.5}$$

for every $T'\in(0,T)$, together with

$$\begin{aligned}&\int_\Omega u(x,T')\,\phi(x,T')\,\mathrm{d}x-\int_0^{T'}\langle u,\,\frac{\partial\phi}{\partial t}\rangle_{W_0^{1,p}\times W^{-1,p'}}\,\mathrm{d}t\\&+\int_0^{T'}\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t-\lambda\int_0^{T'}\int_\Omega|u|^{p-2}u\,\phi\,\mathrm{d}x\,\mathrm{d}t\\&=\int_0^{T'}\int_\Omega f(x,t)\,\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega u_0(x)\,\phi(x,0)\,\mathrm{d}x\end{aligned} \tag{2.6}$$

for all $\phi\in L^p((0,T')\to W_0^{1,p}(\Omega))\cap W^{1,p'}((0,T')\to W^{-1,p'}(\Omega))$.

We remark that

$$L^p((0,T')\to W_0^{1,p}(\Omega))\cap W^{1,p'}((0,T')\to W^{-1,p'}(\Omega))\hookrightarrow C([0,T']\to L^2(\Omega)) \tag{2.7}$$

is a continuous embedding; see e.g. Barbu [7], proof of Lemma 4.1, page 168. Moreover, for every $\phi\in L^p((0,T')\to W_0^{1,p}(\Omega))\cap W^{1,p'}((0,T')\to W^{-1,p'}(\Omega))$

the function $t \mapsto \|\phi(\,\cdot\,,t)\|^2_{L^2(\Omega)}$ is absolutely continuous on the interval $[0,T']$ and satisfies

$$\int_\Omega \phi(x,s)^2\,\mathrm{d}x - \int_\Omega \phi(x,r)^2\,\mathrm{d}x = 2\int_r^s \langle \phi,\, \frac{\partial\phi}{\partial t}\rangle_{W_0^{1,p}\times W^{-1,p'}}\,\mathrm{d}t \tag{2.8}$$

for $0 \le r \le s \le T'$. Applying this fact to equation (2.6) we deduce $u \in W^{1,p'}((0,T') \to W^{-1,p'}(\Omega))$ for any weak solution u of problem (1.2) in $\Omega \times (0,T)$, whenever $T' \in (0,T)$; in particular, $\frac{\partial u}{\partial t} \in L^{p'}((0,T') \to W^{-1,p'}(\Omega))$ for every $T' \in (0,T)$.

Remark 2.2. For a technical reason (regularity of solutions [19, Theorem 0.1]), we need to work only with weak solutions of problem (1.2) that are *(essentially) bounded* in $\Omega \times (0,T')$ for every $T' \in (0,T)$. If $2 \le p < \infty$, this property will follow from a well-known result due to Porzio [23], Theorem 2.1, page 1095. If $1 < p < 2$, it will be obtained by constructing (spatially constant) sub- and supersolutions combined with the weak parabolic maximum principle. These sub- and supersolutions will be defined for all times $t \ge 0$, thus allowing for T arbitrarily large, $0 < T \le \infty$.

Remark 2.3. Under the hypotheses $f \in L^\infty(\Omega \times \mathbb{R}_+)$ and $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, the (unique local) solution of the initial-value problem

$$\begin{cases} \dfrac{\mathrm{d}\nu}{\mathrm{d}t} = \lambda\,\nu(t)^{p-1} + \|f\|_{L^\infty(\Omega\times\mathbb{R}_+)}, \quad t \in (0,\overline{T}_\infty); \\ \nu(0) = \nu_0 \stackrel{\text{def}}{=} \max\left\{\|u_0\|_{L^\infty(\Omega)},\, 1\right\} \end{cases} \tag{2.9}$$

easily provides (spatially constant) sub- and supersolutions to problem (1.2), namely, $-\nu(t)$ and $\nu(t)$, respectively. Notice that $\overline{T}_\infty = \infty$ if $1 < p \le 2$, whereas $0 < \overline{T}_\infty < \infty$ if $p > 2$.

For $T_\infty \stackrel{\text{def}}{=} \sup\{T > 0 :\ u \text{ is a weak solution in } \Omega \times (0,T)\}$ we say that $[0,T_\infty)$ is the *maximal (time) interval of existence* of a weak solution u to problem (1.2). More precisely, $u : \Omega \times (0,T_\infty) \to \mathbb{R}$ and T_∞ $(0 < T_\infty \le \infty)$ satisfy:

(i) u is a weak solution of (1.2) in $\Omega \times (0,T_\infty)$, and
(ii) if $\tilde{u}$ is a weak solution of (1.2) in $\Omega \times (0,T)$, $T \ge T_\infty$, such that $\tilde{u} = u$ in $\Omega \times (0,T_\infty)$, then $T = T_\infty$.

For any weak solution, two alternatives are possible: either it exists for all times t, $0 \le t < T_\infty = \infty$, or else it blows up in finite time as $t \nearrow T_\infty < \infty$. We will see later that the latter case (blow-up in finite time) may occur only when $p > 2$ (by Corollary 5.3) and is characterized by $\|u(t)\|_{L^p(\Omega)} \to \infty$ as $t \nearrow T_\infty$ (see Corollary 5.4).

Local (in time) existence and/or uniqueness of a weak solution to problem (1.2) can be obtained by several (somewhat) different methods; see e.g. J.-L. Lions [21], I. I. Vrabie [34], or H. W. Alt and S. Luckhaus [2]. All of them use (at least one of) the facts that the operator $-\Delta_p$ is continuous and maximal monotone from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ (see e.g. Barbu [7], Chapter II, §1, Section 3, pages 48–50) and m-accretive in both $L^2(\Omega)$ and $L^p(\Omega)$, provided its domain is suitably chosen, such that Δ_p is the infinitesimal generator of a strongly continuous semigroup of (nonlinear) contractions on $L^2(\Omega)$ or $L^p(\Omega)$, respectively (cf. [7], Chapter II, §3, Section 2, pages 87–89). In the appendix (Appendix A, §A.1) we sketch the main ideas of a proof of local (in time) existence and/or uniqueness of a weak solution to problem (1.2). Greater details about this topic can be found in the monographs [21], Chapter 2, §5, pages 207–221 (using Faedo-Galërkin approximation), and [34], Theorem 3.8.1, page 178 (using time delay approximation), and in the article [2].

Finally, we apply Theorem 3.8.2 from [34, page 180] to conclude that any local (in time) weak solution to problem (1.2) can be continued to a maximal interval of existence $[0, T_\infty)$. More precisely, [34] deals with *mild solutions* to problem (1.2) satisfying only $u \in C([0, T'] \to L^p(\Omega))$ if $p \geq 2$, and $u \in C\big([0, T'] \to L^2(\Omega)\big)$ if $1 < p < 2$, for every $T' \in (0, T_\infty)$. Such solutions are obtained from a combination of m-accretiveness of the positive Dirichlet p-Laplacian $-\Delta_p$ as an operator on $L^p(\Omega)$ if $p \geq 2$, and $L^2(\Omega)$ if $1 < p < 2$, with the theory of nonlinear perturbations of m-accretive operators. Here, Theorem 2.6 from [7], Chapter III, §2, Section 3, pages 139–143, needs to be applied for $p \geq 2$, with an obvious adjustment for $1 < p < 2$. Taking advantage of some standard a priori estimates from [21], in our setting one can show that every mild solution is in fact a weak solution.

3. Main result

We assume that $\Omega \subset \mathbb{R}^N$ satisfies hypothesis **(H1)** . If $2 < p < \infty$, we need to impose another technical hypothesis on Ω. To this end, we first introduce a new norm on $W_0^{1,p}(\Omega)$ by

$$\|v\|_{\varphi_1} \stackrel{\text{def}}{=} \Big(\int_\Omega |\nabla \varphi_1|^{p-2} |\nabla v|^2 \,\mathrm{d}x \Big)^{1/2} \quad \text{for } v \in W_0^{1,p}(\Omega), \tag{3.1}$$

and denote by $\mathcal{D}_{\varphi_1}$ the completion of $W_0^{1,p}(\Omega)$ with respect to this norm. That the seminorm (3.1) is in fact a norm on $W_0^{1,p}(\Omega)$ follows from an inequality in Takáč [27, inequality (4.7), page 200]. The Hilbert space $\mathcal{D}_{\varphi_1}$ coincides with the domain of the closure of the quadratic form $\mathcal{Q}_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$2 \cdot \mathcal{Q}_0(\phi) = \int_\Omega |\nabla\varphi_1|^{p-2} \left\{ |\nabla\phi|^2 + (p-2) \left| \frac{\nabla\varphi_1}{|\nabla\varphi_1|} \cdot \nabla\phi \right|^2 \right\} \mathrm{d}x - \lambda_1(p-1) \int_\Omega \varphi_1^{p-2} \phi^2 \,\mathrm{d}x, \quad \phi \in W_0^{1,p}(\Omega). \tag{3.2}$$

For $2 < p < \infty$ we impose the following additional hypothesis on the domain Ω.

Hypothesis (H2). *If $N \geq 2$ and $\partial\Omega$ is not connected, then there is* no *function $v \in \mathcal{D}_{\varphi_1}$, $\mathcal{Q}_0(v) = 0$, with the following four properties:*

- (i) *$v = \varphi_1 \cdot \chi_S$ a.e. in Ω, where $S \subset \Omega$ is Lebesgue measurable, $0 < |S|_N < |\Omega|_N$;*
- (ii) *$\overline{S}$ is connected and $\overline{S} \cap \partial\Omega \neq \emptyset$;*
- (iii) *if V is a connected component of U, then either $V \subset S$ or else $V \subset \Omega \setminus S$;*
- (iv) *$(\partial S) \cap \Omega \subset \Omega \setminus U \quad (= \{x \in \Omega : \nabla\varphi_1(x) = \mathbf{0}\})$.*

It has been *conjectured* in [27, Section 2.1] that **(H2)** always holds true provided **(H1)** is satisfied. The cases when Ω is either an interval in $\mathbb{R}^1$ or else $\partial\Omega$ is connected if $N \geq 2$ have been covered within the proof of Proposition 4.4 in [27, pages 202–205] which claims the following.

Proposition 3.1. *Let $2 < p < \infty$ and assume both hypotheses* **(H1)** *and* **(H2)** *. Then a function $u \in \mathcal{D}_{\varphi_1}$ satisfies $\mathcal{Q}_0(u) = 0$ if and only if $u = \kappa\varphi_1$ for some constant $\kappa \in \mathbb{R}$.*

In particular, there is no function $v \in \mathcal{D}_{\varphi_1}$, $\mathcal{Q}_0(v) = 0$, with properties (i)–(iv). This proposition is the only place where **(H2)** is needed explicitly. All other results in this article depend solely on the conclusion of the proposition which, in turn, implies **(H2)** .

For $1 < p < 2$ we further require hypothesis **(H1)** , but need to redefine the Hilbert space $\mathcal{D}_{\varphi_1}$ as follows. We define $v \in \mathcal{D}_{\varphi_1}$ if and only if $v \in W_0^{1,2}(\Omega)$, $\nabla v(x) = \mathbf{0}$ for almost every $x \in \Omega \setminus U$, and

$$\|v\|_{\varphi_1} \stackrel{\text{def}}{=} \left(\int_U |\nabla\varphi_1|^{p-2} |\nabla v|^2 \,\mathrm{d}x \right)^{1/2} < \infty. \tag{3.3}$$

Recall that $U = \{x \in \Omega : \nabla\varphi_1(x) \neq \mathbf{0}\}$. Consequently, $\mathcal{D}_{\varphi_1}$ endowed with the norm $\|\cdot\|_{\varphi_1}$ is continuously embedded into $W_0^{1,2}(\Omega)$. We *conjecture* that $\mathcal{D}_{\varphi_1}$ is dense in $L^2(\Omega)$. This conjecture would immediately follow from $|\Omega \setminus U|_N = 0$. The latter holds true if Ω is convex; then also $\Omega \setminus U$ is a convex set in $\mathbb{R}^N$ with empty interior, and hence is of zero Lebesgue measure, see Fleckinger et al. [15, Lemma 2.6, page 55].

If the conjecture is false, we need to consider also the orthogonal complement

$$\mathcal{D}_{\varphi_1}^{\perp,L^2} = \{v \in L^2(\Omega) : \langle v, \phi\rangle = 0 \text{ for all } \phi \in \mathcal{D}_{\varphi_1}\}.$$

Notice that $v \in \mathcal{D}_{\varphi_1}^{\perp,L^2}$ implies $v = 0$ almost everywhere in U, as U is open. This means that $\mathcal{D}_{\varphi_1}^{\perp,L^2}$ is isometrically isomorphic to a closed linear subspace of $L^2(\Omega \setminus U)$. Moreover, $\chi_{\Omega\setminus U} \notin \mathcal{D}_{\varphi_1}^{\perp,L^2}$ since $\Omega \setminus U$ is a compact subset of Ω; hence, there is a C^1 function $\phi \in \mathcal{D}_{\varphi_1}$, $0 \leq \phi \leq 1$, with compact support in Ω such that $\phi = 1$ in an open neighborhood of $\Omega \setminus U$.

Hypothesis **(H2)** always holds true for $1 < p < 2$; see [27, Section 8, page 225].

Remark 3.2. It is not difficult to verify that the conclusion of Proposition 3.1 remains valid also for $1 < p < 2$, by [27, Remark 8.1, page 225].

We write $\underline{f} \equiv \zeta\varphi_1 + \underline{f}^\top$ with $\zeta \in \mathbb{R}$ and $\underline{f}^\top \in L^\infty(\Omega)^\top$; recall that $L^\infty(\Omega) = \mathrm{lin}\{\varphi_1\} \oplus L^\infty(\Omega)^\top$.

The main result of our present article is the following *antimaximum principle* for problem (1.2) with any $1 < p < \infty$. This is a more general version of Theorem 1.1 stated in the Introduction (Section 1); here, function $f(x,t)$ does not need to be nonnegative.

Theorem 3.3. (Antimaximum Principle). *Let $1 < p < \infty$ and assume that $\Omega \subset \mathbb{R}^N$ satisfies hypothesis* **(H1)** *. If $p > 2$, assume that Ω satisfies also hypothesis* **(H2)** *. Let $f \in L^\infty(\Omega \times \mathbb{R}_+)$ satisfy*

$$f(x,t) \geq \underline{f}(x) \quad \text{in } \Omega \times \mathbb{R}_+, \tag{3.4}$$

where $\underline{f} \in L^\infty(\Omega)$ is some function such that $\int_\Omega \underline{f}\varphi_1 \,\mathrm{d}x > 0$ and the resonant problem

$$-\Delta_p u = \lambda_1 |u|^{p-2}u + \underline{f}(x) \ \text{ in } \Omega; \qquad u = 0 \ \text{ on } \partial\Omega, \tag{3.5}$$

has no *weak solution. Finally, assume that $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then there exist constants $\delta \equiv \delta(\underline{f}, u_0) > 0$ and $T_+ \equiv T_+(f, u_0) \in (0, T_\infty)$ with the* *following property: If* $\lambda_1 < \lambda < \lambda_1 + \delta$ *and if* $u : \Omega \times (0, T_\infty) \to \mathbb{R}$ *is a weak solution of problem* (1.2)*, then* u *satisfies*

$$\begin{cases} u(x,t) > 0 \ \text{ for all } \ (x,t) \in \Omega \times [T_+, T_\infty) \quad \text{ and} \\ \dfrac{\partial u}{\partial \nu}(x,t) < 0 \ \text{ for all } \ (x,t) \in \partial\Omega \times [T_+, T_\infty). \end{cases} \tag{3.6}$$

Notice that we do not assume that u is bounded in $\Omega \times (0, T')$ for each $T' \in (0, T_\infty)$, cf. Theorem 1.1. Nonetheless, we will show this boundedness in Proposition 5.5, Part (a). Then Lieberman's regularity result [19, Theorem 0.1, page 552] (see Lemma 4.6 in Section 4) guarantees $u(\cdot, t) \in C^1(\overline{\Omega})$ for every $t \in (0, T_\infty)$. In fact, this regularity result will be employed to establish

$$\tau_*(t) \stackrel{\text{def}}{=} \inf_{x \in \Omega} \frac{u(x,t)}{\varphi_1(x)} \geq \tau_+ (\equiv \text{const}) > 0 \ \text{ for all } \ t \in [T_+, T_\infty), \tag{3.7}$$

where $\tau_*(t) \to +\infty$ as $t \nearrow T_\infty$ $(0 < T_+ < T_\infty \leq \infty)$. This inequality entails (3.6) in Theorem 3.3 above.

Concerning function $\underline{f}$ in the resonant problem (3.5) above, we have the following remark on the solvability for $p \neq 2$.

Remark 3.4. Let $\underline{f} \in L^\infty(\Omega)$. If $0 \leq \underline{f} \not\equiv 0$ in Ω, then problem (3.5) has no weak solution $u \in W_0^{1,p}(\Omega)$, by a result in Fleckinger et al. [14, Théorème 1, page 731] or [15, Theorem 2.3, page 54]. Now let us decompose $\underline{f}$ as $\underline{f} = \underline{f}^\parallel \cdot \varphi_1 + \underline{f}^\top$ according to (2.4) and assume $\underline{f}^\top \not\equiv 0$ in Ω. Then, by Theorems 3.1 (for $p > 2$) and 3.5 (for $1 < p < 2$) from Takáč [28, pages 314–315], there exist two constants ζ_*, ζ^*, depending on $\underline{f}^\top$, such that $-\infty < \zeta_* < 0 < \zeta^* < \infty$ and the elliptic problem (3.5) has a weak solution $u \in W_0^{1,p}(\Omega)$ if and only if $\underline{f}^\parallel \in [\zeta_*, \zeta^*]$.

Theorem 3.3 will be proved in a number of steps below. In fact, we obtain a much more precise result if the constant $\delta > 0$ (the time $T_+ < T_\infty$, respectively) in this theorem is chosen small (large) enough. We distinguish between the cases $T_\infty = \infty$ and $T_\infty < \infty$.

Corollary 3.5. ("Large" Positive Solutions for $T_\infty = \infty$). *In the situation of* Theorem 3.3 *above, assume that* $T_\infty = \infty$*. Then we can choose* $T_+ \equiv T_+(f, u_0) \in (0, \infty)$ *large enough, such that if* $\lambda_1 < \lambda < \lambda_1 + \delta$*, then every weak solution* $u : \Omega \times (0, \infty) \to \mathbb{R}$ *of problem* (1.2) *satisfies* $u(x,t) \geq \underline{u}(x,t)$ *for all* $(x,t) \in \Omega \times (T_+, \infty)$*, where* $\underline{u} : \Omega \times (0, \infty) \to \mathbb{R}$ *is the* minimal *weak* *solution of the initial-boundary-value problem*

$$\begin{cases} \dfrac{\partial \underline{u}}{\partial t} - \Delta_p \underline{u} = \lambda\, |\underline{u}|^{p-2}\underline{u} + \underline{f}(x), & (x,t) \in \Omega \times (0,\infty); \\ \quad \underline{u}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty); \\ \quad \underline{u}(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.8}$$

Furthermore, $\underline{u}$ *takes the form*

$$\underline{u}(x,t) = \underline{\tau}(t)\big(\varphi_1(x) + \underline{v}^{\top}(x,t)\big) \quad \text{for all } (x,t) \in \Omega \times (T_+,\infty), \tag{3.9}$$

where the functions $\underline{\tau}$ *and* $\underline{v}^{\top}$ *have the following properties:*

(a) $\underline{\tau} : [T_+,\infty) \to (0,\infty)$ *is locally absolutely continuous with* $\mathrm{d}\underline{\tau}/\mathrm{d}t \in L^{p'}(T_+,T')$ *for every* $T' \in (T_+,\infty)$*, and* $\underline{\tau}(t) \to +\infty$ *as* $t \nearrow \infty$*; and*

(b) $\underline{v}^{\top} \in C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [T_+,T'])$ *for every* $T' \in (T_+,\infty)$*, with*

$$\int_{\Omega} \underline{v}^{\top}(x,t)\, \varphi_1 \,\mathrm{d}x = 0 \quad \text{for all } t \geq T_+.$$

Moreover, given $0 < \beta' < \beta$ *and* $\gamma > 0$*, the constant* $\delta \equiv \delta(\underline{f}, u_0, \beta', \gamma) > 0$ ($T_+ \equiv T_+(\underline{f}, u_0, \beta', \gamma) \in (0,\infty)$*, respectively) can be chosen small (large) enough, such that whenever* $\lambda_1 < \lambda < \lambda_1 + \delta$ *we have also* $|\underline{v}^{\top}(x,t)| \leq \frac{1}{2}\varphi_1(x)$ *for all* $x \in \Omega$ *and* $t \geq T_+$*, together with* $\|\underline{v}^{\top}(\,\cdot\,,t)\|_{C^{1+\beta'}(\overline{\Omega})} \leq \gamma$ *for all* $t \geq T_+$.

The reader is referred to Section 6.1, especially Remark 6.3, for the definition and existence of the *minimal* weak solution of problem (3.8).

Loosely speaking, Corollary 3.5 states that the solution $u(\,\cdot\,,t)$ eventually becomes positive for all times $t \in [T_+,\infty)$ and stays above the function $\underline{u}(\,\cdot\,,t)$, where the latter behaves asymptotically like $\underline{\tau}(t)\varphi_1$ in the $C^{1+\beta'}(\overline{\Omega})$--norm as $t \nearrow \infty$. The asymptotic behavior of $\underline{\tau}(t)$ as $t \nearrow \infty$ is determined by the (positive) solution $z : [T_+,\infty) \to (0,\infty)$ of the ordinary differential equation

$$\|\varphi_1\|^2_{L^2(\Omega)} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, z(t) = (\lambda - \lambda_1 + \Upsilon(t)) z(t)^{p-1} + \langle \underline{f}, \varphi_1 \rangle, \quad T_+ \leq t < \infty, \tag{3.10}$$

with a suitable initial condition at $t = T_+$, $z(T_+) > 0$, where $\Upsilon : [T_+,\infty) \to \mathbb{R}$ is some continuous function that satisfies $|\Upsilon(t)| \leq c(\lambda_1 + \lambda)\gamma$ whenever $T_+ \leq t < \infty$; cf. equation (6.24) and inequalities (6.25) and (6.26). The constant $c > 0$ is independent from γ, λ, and t, such that $0 < \gamma \leq \gamma_1$ and $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$ (with both $\gamma_1 > 0$ and $\delta_1(\gamma) > 0$ small enough), and

$t \geq T_+ \geq T_{\gamma,\lambda}$ (with $T_{\gamma,\lambda} > 0$ large enough). For instance, given $\varepsilon \in (0,1)$, we may choose $\gamma_1 > 0$ small enough, such that $c(2\lambda_1 + \delta_1(\gamma))\gamma_1 \leq \varepsilon$, whence

$$|\Upsilon(t)| \leq \varepsilon \quad \text{ whenever } \; T_+ \leq t < \infty.$$

Since λ ($\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$) depends on the choice of γ ($0 < \gamma \leq \gamma_1$), we are unable to determine the sign of the coefficient $\lambda - \lambda_1 + \Upsilon(t)$ in equation (3.10). Notice that $\langle f(\,\cdot,t), \varphi_1\rangle \geq \langle \underline{f}, \varphi_1\rangle > 0$ holds by (3.4).

Corollary 3.6. ("Large" Positive Solutions for $T_\infty < \infty$) *In the situation of* Theorem 3.3 *above, assume that* $T_\infty < \infty$*. Then* $p > 2$ *and the solution* $u : \Omega \times (0, T_\infty) \to \mathbb{R}$ *of problem* (1.2) *is unique. Furthermore, we can choose* $T_+ \equiv T_+(f, u_0) \in (0, T_\infty)$ *large enough and independent from* $\lambda \in \mathbb{R}$ *with* $\lambda_1 < \lambda < \lambda_1 + \delta$*, such that* u *takes the form*

$$u(x,t) = \tau(t)(\varphi_1(x) + v^\top(x,t)) \quad \text{ for all } (x,t) \in \Omega \times (T_+, T_\infty), \tag{3.11}$$

where the functions τ *and* $v^\top$ *have the following properties:*

(*a*) $\tau : [T_+, T_\infty) \to (0,\infty)$ *is locally absolutely continuous with* $\mathrm{d}\tau/\mathrm{d}t \in L^{p'}(T_+, T')$ *for every* $T' \in (T_+, T_\infty)$*, and* $\tau(t) \to +\infty$ *as* $t \nearrow T_\infty$*; and*

(*b*) $v^\top \in C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [T_+, T'])$ *for every* $T' \in (T_+, T_\infty)$*, with*

$$\int_\Omega v^\top(x,t)\, \varphi_1 \,\mathrm{d}x = 0$$

and $|v^\top(x,t)| \leq \frac{1}{2}\varphi_1(x)$ *for all* $x \in \Omega$ *and* $T_+ \leq t < T_\infty$*, together with* $\|v^\top(\,\cdot,t)\|_{C^{1+\beta'}(\overline{\Omega})} \to 0$ *as* $t \nearrow T_\infty$*, whenever* $0 < \beta' < \beta$*.*

This means that the solution $u(\,\cdot,t)$ eventually becomes positive for all times $t \in [T_+, T_\infty)$ and behaves asymptotically like $\tau(t)\varphi_1$ in the $C^{1+\beta'}(\overline{\Omega})$-norm as $t \nearrow T_\infty$. The asymptotic behavior of $\tau(t)$ as $t \nearrow T_\infty$ is determined by the (positive) solution $z : [T_+, T_\infty) \to (0,\infty)$ of the ordinary differential equation

$$\|\varphi_1\|^2_{L^2(\Omega)} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\, z(t) = \big(\lambda - \lambda_1 + \Upsilon(t)\big) z(t)^{p-1} + \langle f(\,\cdot,t), \varphi_1\rangle, \quad T_+ \leq t < T_\infty, \tag{3.12}$$

with a suitable initial condition at $t = T_+$, $z(T_+) > 0$, where $\Upsilon : [T_+, T_\infty) \to \mathbb{R}$ is some continuous function that satisfies $|\Upsilon(t)| \leq c(\lambda_1 + \lambda)\gamma$ whenever $T_+ \leq t < T_\infty$; cf. equation (6.45) and inequalities (6.46) and (6.47). The constant $c > 0$ is independent from γ, λ, and t, such that $0 < \gamma \leq \gamma_1$ and $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$ (with both $\gamma_1 > 0$ and $\delta_1(\gamma) > 0$ small enough), and $T_{\gamma,\lambda} \leq T_+ \leq t < T_\infty$ (with $T_{\gamma,\lambda} > 0$ large enough). As a consequence, equation (3.12) forces $p > 2$. This claim follows from the fact that $z(t) \nearrow$

$+\infty$ as $t \nearrow T_\infty < \infty$. As in the previous case ($T_\infty = \infty$), we are unable to determine the sign of the coefficient $\lambda - \lambda_1 + \Upsilon(t)$ in equation (3.12).

4. Auxiliary lemmas

We recall the orthogonal decomposition $u = u^\| \cdot \varphi_1 + u^\top$ defined by (2.4), e.g. in

$$L^2(\Omega) = \mathrm{lin}\{\varphi_1\} \oplus L^2(\Omega)^\top \quad \text{or} \quad W_0^{1,p}(\Omega) = \mathrm{lin}\{\varphi_1\} \oplus W_0^{1,p}(\Omega)^\top .$$

Of course, we assume $1 < p < \infty$.

The following lemma is an extension of Lemma 5.1 from Fleckinger and Takáč [16, page 963].

Lemma 4.1. *Given any $0 < \gamma < \infty$, define*

$$\Lambda_\gamma \stackrel{\mathrm{def}}{=} \inf \left\{ \frac{\int_\Omega |\nabla(\varphi_1 + v^\top)|^p \,\mathrm{d}x}{\int_\Omega |\varphi_1 + v^\top|^p \,\mathrm{d}x} : v^\top \in W_0^{1,p}(\Omega)^\top \ \textit{ and } \ \|v^\top\|_{W_0^{1,p}(\Omega)} \geq \gamma \right\}.$$

Then $\lambda_1 < \Lambda_\gamma \leq \Lambda_{\gamma'} < \infty$, whenever $0 < \gamma < \gamma' < \infty$, and also $\Lambda_\gamma \searrow \lambda_1$ as $\gamma \searrow 0$.

Proof. The inequality $\Lambda_\gamma > \lambda_1$ is proved in [16], Lemma 5.1, page 963. It is obvious that $0 < \gamma < \gamma' < \infty$ implies $\Lambda_\gamma \leq \Lambda_{\gamma'}$. To verify $\Lambda_\gamma \searrow \lambda_1$ as $\gamma \searrow 0$, let us fix any function $\phi \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \phi\, \varphi_1 \,\mathrm{d}x = 0 \quad \text{and} \quad \int_\Omega |\phi| \,\mathrm{d}x = 1.$$

Next, define $\psi_\varepsilon \stackrel{\mathrm{def}}{=} \varphi_1 + \varepsilon\phi$ for $0 < \varepsilon < 1$ and observe that if γ and ε are chosen such that $0 < \gamma \leq \varepsilon \cdot \|\phi\|_{W_0^{1,p}(\Omega)} = \|\psi_\varepsilon^\top\|_{W_0^{1,p}(\Omega)}$, then we have

$$\Lambda_\gamma \leq \Big(\int_\Omega |\nabla\varphi_1 + \varepsilon\nabla\phi|^p \,\mathrm{d}x \Big) \Big/ \Big(\int_\Omega |\varphi_1 + \varepsilon\phi|^p \,\mathrm{d}x \Big)$$

with the right-hand side tending to λ_1 as $\varepsilon \searrow 0$. In particular, taking $\varepsilon = \gamma / \|\phi\|_{W_0^{1,p}(\Omega)}$ and letting $\gamma \searrow 0$, we arrive at $\Lambda_\gamma \searrow \lambda_1$ as claimed. $\square$

The next lemma is an interpolation inequality of Gagliardo-Nirenberg type.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary $\partial\Omega$ is a $C^{1,\alpha}$-manifold for some $0 < \alpha < 1$. Assume $0 < \beta' < \beta < \alpha$ and $1 \leq p_0 < \infty$. Then there exist constants $\theta \in (0,1)$ and $C > 0$ such that*

$$\|u\|_{C^{\beta'}(\overline{\Omega})} \leq C\, \|u\|^\theta_{C^\beta(\overline{\Omega})}\, \|u\|^{1-\theta}_{L^{p_0}(\Omega)} \quad \textit{for all } u \in C^\beta(\overline{\Omega}). \tag{4.1}$$

Proof. We begin with the following standard version of the Gagliardo-Nirenberg inequality; see e.g. Triebel [32], Remark 2.4.2/2(a), equation (8), page 185, combined with Theorem 1.3.3(f), equation (5), page 25:

$$\|u\|_{W^{s,p}(\Omega)} \leq C_1 \, \|u\|^{\theta}_{W^{s_1,p_1}(\Omega)} \, \|u\|^{1-\theta}_{L^{p_0}(\Omega)} \quad \text{for all } u \in W^{s_1,p_1}(\Omega), \tag{4.2}$$

where $C_1 > 0$ is a constant and the numbers s, p, p_0, s_1, p_1, and θ satisfy $0 \leq s_1 < \alpha$, $1 \leq p_0 \leq p_1 < \infty$, $0 < \theta < 1$, and $s = \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. To derive the desired inequality (4.1) from (4.2), we first observe that if we can choose $s \geq 0$ and $p \geq p_0$ such that $s - \frac{N}{p} = \beta'$ (> 0), then there is a constant $C_2 > 0$ such that

$$\|u\|_{C^{\beta'}(\overline{\Omega})} \leq C_2 \, \|u\|_{W^{s,p}(\Omega)} \quad \text{for } u \in W^{s,p}(\Omega), \tag{4.3}$$

by Morrey's embedding theorem (Gilbarg and Trudinger [17, Theorem 7.17, page 163]). On the other hand, if $s_1 \leq \beta$, then there is another constant $C_3 > 0$ such that

$$\|u\|_{W^{s_1,p_1}(\Omega)} \leq C_3 \, \|u\|_{C^{\beta}(\overline{\Omega})} \quad \text{for } u \in C^{\beta}(\overline{\Omega}). \tag{4.4}$$

Applying the last two inequalities to (4.2) we arrive at (4.1) with $C = C_1 C_2 C_3^{\theta}$.

Numbers s, p, s_1, and p_1 in inequalities (4.3) and (4.4) need to be chosen as follows. We simply fix $s_1 = \beta$ in (4.4). To guarantee also (4.3), we first take p_1 $(p_1 \geq p_0)$ such that $p_1 > \frac{N}{\beta - \beta'}$, then set

$$\theta = \Big(\beta' + \frac{N}{p_0}\Big) \Big/ \Big[\beta + N\Big(\frac{1}{p_0} - \frac{1}{p_1}\Big)\Big].$$

Clearly, $0 < \theta < 1$. Finally, taking $s = \theta\beta$ and p such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we observe that indeed the numbers s and p satisfy $s \geq 0$ and $p \geq p_0$ together with

$$s - \frac{N}{p} = \theta\beta - N\Big(\frac{1-\theta}{p_0} + \frac{\theta}{p_1}\Big) = \theta\Big[\beta + N\Big(\frac{1}{p_0} - \frac{1}{p_1}\Big)\Big] - \frac{N}{p_0} = \beta'.$$

The proof is now complete. □

The following is a special case of a more general L^∞-regularity result shown in Anane's thesis [5, Théorème A.1, page 96].

Lemma 4.3. *Let $p > 1$ and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $g(\,\cdot, s) \in L^1_{\mathrm{loc}}(\Omega)$ for every $s \in \mathbb{R}$, and the following inequality holds with some constants $a > 0$ and $b \geq 0$:*

$$s \cdot g(x,s) \leq a|s|^p + b|s| \quad \textit{for all } s \in \mathbb{R} \textit{ and a.e. } x \in \Omega.$$

Assume that $u \in W_0^{1,p}(\Omega)$ *satisfies*

$$\int_\Omega |\nabla u|^{p-2} (\nabla u \cdot \nabla \phi)\, \mathrm{d}x = \int_\Omega g(x, u(x))\, \phi \, \mathrm{d}x \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Then $u \in L^\infty(\Omega)$ *and there exists a constant* $c > 0$ *such that* $\|u\|_{L^\infty(\Omega)} \le c$, *where* c *depends solely upon* a, b, N, p, *and the norm* $\|u\|_{L^{p_0}(\Omega)}$ *with*

$$p_0 = \begin{cases} p^* = \frac{Np}{N-p} & \text{if } p < N; \\ 2p & \text{if } p \ge N. \end{cases}$$

A corresponding parabolic L^∞-regularity result for $1 < p \le 2$ is stated in the following lemma.

Lemma 4.4. *Let* $1 < p \le 2$, $0 < T < \infty$, $\lambda \in \mathbb{R}_+$, *and let* $f \in L^\infty(\Omega \times \mathbb{R}_+)$ *and* $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. *Assume that* $u : \Omega \times (0, T) \to \mathbb{R}$ *is a weak solution of problem* (1.2) *in* $\Omega \times (0, T)$. *Then* $u \in L^\infty(\Omega \times (0, T))$ *and, moreover,*

$$\|u(\,\cdot, t)\|_{L^\infty(\Omega)} \le C(t) = c_2\, e^{c_1 t}, \quad 0 \le t < T, \tag{4.5}$$

where $c_1, c_2 > 0$ *are constants depending only on* λ, $\|f\|_{L^\infty(\Omega \times \mathbb{R}_+)}$, *and* $\|u_0\|_{L^\infty(\Omega)}$, *but not on* $p \in (1, 2]$ *or* T. *More precisely, we have* $|u(x, t)| \le \nu(t)$ *for almost every* $x \in \Omega$ *and for every* $t \in [0, T)$, *where* $\nu : \mathbb{R}_+ \to [1, \infty)$ *is the* (*unique global*) *solution of the initial-value problem* (2.9) *with* $\overline{T}_\infty = \infty$.

Proof. First, we observe that the (spatially constant) functions $-\nu(t)$ and $\nu(t)$, respectively, defined by (2.9) are sub- and supersolutions to problem (1.2). This follows by inserting $\mp\nu(t)$ into the initial-boundary-value problem (1.2) and using $\Delta_p \nu(t) = 0$, $-\|f\|_{L^\infty(\Omega\times\mathbb{R}_+)} \le f(x, t) \le \|f\|_{L^\infty(\Omega\times\mathbb{R}_+)}$, $\nu(t) \ge 0$, and $-\nu_0 \le u_0(x) \le \nu_0$.

Second, we show how to apply the weak parabolic maximum principle to problem (1.2) in order to derive $|u(x, t)| \le \nu(t)$ for almost every $x \in \Omega$ and for every $t \in [0, T)$. We concentrate on proving the inequality $u(x, t) \le \nu(t)$; the proof of the other inequality $u(x, t) \ge -\nu(t)$ is analogous. Since $u(x, t)$ is a solution and $\nu(t)$ a supersolution to problem (1.2), the difference $u(x, t) - \nu(t)$ satisfies the inequalities

$$\begin{cases} \dfrac{\partial}{\partial t}(u - \nu) - (\Delta_p u - \Delta_p \nu) \le \lambda\, (|u|^{p-2} u - |\nu|^{p-2} \nu), & (x, t) \in \Omega \times (0, T); \\ u(x, t) - \nu(t) = -\nu(t) \le 0, & (x, t) \in \partial\Omega \times (0, T); \\ u(x, 0) - \nu(0) = u_0(x) - \nu_0 \le 0, & x \in \Omega, \end{cases} \tag{4.6}$$

in the sense of distributions on $\Omega \times [0,T)$; cf. the definition of a weak solution to problem (1.2) where one has to take only such ϕ that satisfy $\phi \geq 0$ almost everywhere in $\Omega \times (0,T')$, for every $T' \in (0,T)$. By our hypotheses on u and ν, combined with the embedding in (2.7), we have $u, \nu \in C\big([0,T'] \to L^2(\Omega)\big)$ for every $T' \in (0,T)$. Let $w = (u-\nu)^+$ denote the positive part of $u-\nu$; that is, $(u-\nu)^+ = \max\{u-\nu,\, 0\}$. This function belongs to $L^p\big((0,T') \to W_0^{1,p}(\Omega)\big) \cap W^{1,p'}\big((0,T') \to W^{-1,p'}(\Omega)\big)$, by a standard result from Gilbarg and Trudinger [17, Lemma 7.6, page 152]. Consequently, $w \in C\big([0,T'] \to L^2(\Omega)\big)$ as above, by the embedding in (2.7). We multiply the first inequality in (4.6) by $w = (u-\nu)^+$ and integrate the product over Ω, thus arriving at

$$\begin{aligned} &\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\,\|w(\,\cdot,t)\|^2_{L^2(\Omega)} + \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \nu|^{p-2}\nabla \nu)\cdot \nabla(u-\nu)^+\,\mathrm{d}x \\ &\leq \lambda \int_\Omega (|u|^{p-2}u - |\nu|^{p-2}\nu)(u-\nu)^+\,\mathrm{d}x \end{aligned}$$

for almost every $t \in (0,T)$. Since the integral on the left-hand side is non-negative, we finally get

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\,\|w(\,\cdot,t)\|^2_{L^2(\Omega)} \leq \lambda \int_\Omega (|u|^{p-2}u - |\nu|^{p-2}\nu)(u-\nu)^+\,\mathrm{d}x \tag{4.7}$$

for almost every $t \in (0,T)$. Next, we write

$$|u|^{p-2}u - |\nu|^{p-2}\nu = (p-1)\Big(\int_0^1 |\nu + s(u-\nu)|^{p-2}\,\mathrm{d}s\Big)(u-\nu) \tag{4.8}$$

and estimate the integral on the right-hand side by the second inequality in (4.8) from the Appendix, §A.2:

$$\int_0^1 |\nu + s(u-\nu)|^{p-2}\,\mathrm{d}s \leq C_p \cdot \Big(\max_{0\leq s\leq 1} |\nu + s(u-\nu)|\Big)^{p-2},$$

where $C_p > 0$ is a constant depending only on $p \in (1,2]$; the case $p=2$ is trivial. Clearly, ν being a solution of problem (2.9), it is monotone increasing with $\nu(t) \geq \nu_0 \geq 1$ for all $t \geq 0$. It follows that (recall $1 < p \leq 2$)

$$\int_0^1 \Big|\nu(t) + s(u(x,t) - \nu(t))\Big|^{p-2}\,\mathrm{d}s \leq C_p\,\nu(t)^{p-2} \leq C_p\,\nu_0^{p-2} \leq C_p.$$

We apply these inequalities to (4.8) to obtain

$$(|u|^{p-2}u - |\nu|^{p-2}\nu)(u-\nu)^+ \leq (p-1)\,C_p\,[(u-\nu)^+]^2 \leq C_p\,[(u-\nu)^+]^2.$$

Finally, we use this inequality to estimate the integrand on the right-hand side in (4.7):

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \|w(\,\cdot, t)\|^2_{L^2(\Omega)} \le \lambda\, C_p \|w(\,\cdot, t)\|^2_{L^2(\Omega)}$$

for almost every $t \in (0, T)$. The function $t \mapsto \|w(\,\cdot, t)\|^2_{L^2(\Omega)}$ being absolutely continuous on the interval $[0, T']$ for every $T' \in (0, T)$, by equation (2.8), we thus get

$$\|w(\,\cdot, t)\|^2_{L^2(\Omega)} \le e^{2\lambda C_p t} \|w(\,\cdot, 0)\|^2_{L^2(\Omega)} = 0 \quad \text{for every } t \in [0, T'],$$

by Gronwall's lemma and the fact that $u(x, 0) - \nu(0) \le 0$ for almost every $x \in \Omega$. This entails $u(\,\cdot, t) \le \nu(t)$ almost everywhere in Ω for every $t \in [0, T)$.

We have proved $|u(x, t)| \le \nu(t)$ for almost every $x \in \Omega$ and for every $t \in [0, T)$, as desired. Consequently, (4.5) follows from (2.9). The lemma is proved. □

For $2 \le p < \infty$, an analogue of Lemma 4.4 is a special case of an L^∞_{loc}-regularity result from Porzio [23], Theorem 2.1, page 1095, stated below. (We refer also to O'Leary [22], Theorem 1, page 436.) A formal way for passing from local L^∞-regularity to global is outlined in Porzio [24], Theorem 1.1 (page 437) and Remark 1.2 (page 438). To state this regularity result, we replace (1.2) by a more general initial-boundary-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = g(x, t, u), & (x, t) \in \Omega \times (0, T); \\ \quad u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T); \\ \quad u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{4.9}$$

Lemma 4.5. *Let $2 \le p < \infty$, $0 < T < \infty$, and let $g : \Omega \times (0, T) \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $g(\,\cdot, \cdot, s) \in L^1_{\mathrm{loc}}(\Omega \times (0, T))$ for every $s \in \mathbb{R}$, and the following inequality holds with some constants $a > 0$ and $b \ge 0$:*

$$|g(x, t, s)| \le a|s|^{p-1} + b \quad \text{for all } s \in \mathbb{R} \text{ and a.e. } (x, t) \in \Omega \times (0, T).$$

Assume that $u : \Omega \times (0, T) \to \mathbb{R}$ is a weak solution of problem (4.9) in $\Omega \times (0, T)$. Then $u \in L^\infty(\Omega \times (\varepsilon, T'))$ whenever $0 < \varepsilon < T' < T$. In addition, if the initial data $u_0 \in L^\infty(\Omega)$, then $u \in L^\infty(\Omega \times (0, T'))$ for $0 < T' < T$.

For problem (1.2) one can improve the regularity of a bounded weak solution as follows; see Lieberman [19, Theorem 0.1, page 552].

Lemma 4.6. *Let $1 < p < \infty$, $\lambda \in \mathbb{R}$, and assume that $\Omega \subset \mathbb{R}^N$ satisfies hypothesis* **(H1)** . *Let $f \in L^\infty(\Omega \times (0, T))$ where $0 < T < \infty$. Assume* *that* $u : \Omega \times (0,T) \to \mathbb{R}$ *is a bounded weak solution of problem* (1.2)*. Then* $u \in C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [\varepsilon, T])$ *for every* $\varepsilon \in (0,T)$*, where* $\beta \in (0,1)$ *is a constant independent of* u*. In addition, if the initial data* $u_0 \in C^{1+\beta}(\overline{\Omega})$*, then* $u \in C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [0,T])$*.*

Here, $C^{1+\beta,(1+\beta)/2}(\overline{\Omega}\times[0,T])$ denotes the standard parabolic Hölder space defined, e.g., in [19, formula 0.6, page 552], where $\beta \in (0,1)$ is a given number. It follows that a function $u \in C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [0,T])$ satisfies $\frac{\partial u}{\partial x_i}(\,\cdot, t) \in C^{\beta}(\overline{\Omega})$ uniformly in $t \in [0,T]$ for $i = 1, 2, \dots, N$ together with $u(x, \,\cdot\,) \in C^{(1+\beta)/2}([0,T])$ uniformly in $x \in \overline{\Omega}$.

Corollary 4.7. *Let* p*,* λ*,* Ω*, and* f *be as in* Lemma 4.6 *above. Assume that* $u : \Omega \times (0,T) \to \mathbb{R}$ *is a bounded weak solution of problem* (1.2)*. Then* $u(\,\cdot, t) \in C^{1+\beta}(\overline{\Omega})$ *for every* $t \in (0,T]$*, where* $\beta \in (0,1)$ *is a constant independent from* u*. In particular, we have*

$$u(x,t) \geq -c(t)\,\varphi_1(x) \quad \text{ in } \Omega, \quad 0 < t \leq T, \tag{4.10}$$

where $c(t) > 0$ *is some constant.*

Remark 4.8. The norm $\|u(\,\cdot, t)\|_{C^{1+\beta}(\overline{\Omega})}$ and the constant $c(t) > 0$ in inequality (4.10) above depend on the solution u solely through its L^∞-norm on $\Omega \times (0,t)$; i.e., on an upper bound for $\|u\|_{L^\infty(\Omega\times(0,t))}$, by the arguments employed in the proof of Lemma 4.4 combined with the dependencies of constants in Lieberman's regularity result [19, Theorem 0.1, page 552].

Our last auxiliary result is a simple variant of the parabolic *weak comparison principle* for problem (1.2).

Lemma 4.9. *Let* $1 < p < \infty$*,* $0 \leq \mu < \infty$*,* $0 < T < \infty$*, and assume that* $\Omega \subset \mathbb{R}^N$ *satisfies hypothesis* **(H1)** *. Let* $u, v \in L^p\big((0,T) \to W_0^{1,p}(\Omega)\big) \cap W^{1,p'}\big((0,T) \to W^{-1,p'}(\Omega)\big)$ *be two functions which satisfy*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u + \mu\,|u|^{p-2}u \leq \dfrac{\partial v}{\partial t} - \Delta_p v + \mu\,|v|^{p-2}v \quad \text{ in } \Omega \times (0,T); \\ \qquad\qquad u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T); \\ \qquad\qquad u(x,0) \leq v(x,0), \quad x \in \Omega, \end{cases} \tag{4.11}$$

the first inequality being valid in the sense of distributions on $\Omega \times (0,T)$ *and* $u(\,\cdot, 0) \leq v(\,\cdot, 0)$ *in* $L^2(\Omega)$*. Then* $u, v \in C\big([0,T] \to L^2(\Omega)\big)$ *and* $u(\,\cdot, t) \leq v(\,\cdot, t)$ *in* $L^2(\Omega)$ *holds for every* $t \in [0,T]$*.*

Proof. By our hypotheses on u and v, combined with the embedding in (2.7), we have $u, v \in C\big([0,T] \to L^2(\Omega)\big)$. Let $w = (u-v)^+ = \max\{u-v,\, 0\}$. This function belongs to $L^p\big((0,T) \to W_0^{1,p}(\Omega)\big) \cap W^{1,p'}\big((0,T) \to W^{-1,p'}(\Omega)\big)$ again, by a standard result from Gilbarg and Trudinger [17, Lemma 7.6, page 152]. Consequently, $w \in C\big([0,T] \to L^2(\Omega)\big)$ as above. Multiplying the inequality (which is assumed to hold in the sense of distributions)

$$\frac{\partial}{\partial t}(u-v) - \Delta_p u + \Delta_p v + \mu\big(|u|^{p-2}u - |v|^{p-2}v\big) \le 0, \quad (x,t) \in \Omega \times (0,T),$$

by $w = (u-v)^+$ and integrating the product over Ω, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \|w(\,\cdot\,,t)\|^2_{L^2(\Omega)} \le 0 \;\text{ for a.e. } t \in (0,T).$$

The function $t \mapsto \|w(\,\cdot\,,t)\|^2_{L^2(\Omega)}$ being absolutely continuous on the interval $[0,T]$, by equation (2.8), we thus get $\|w(\,\cdot\,,t)\|^2_{L^2(\Omega)} \le \|w(\,\cdot\,,0)\|^2_{L^2(\Omega)} = 0$ for every $t \in [0,T]$, by Gronwall's lemma. This entails $u(\,\cdot\,,t) \le v(\,\cdot\,,t)$ almost everywhere in Ω for every $t \in [0,T]$. The lemma is proved. □

5. Bounds on a solution, global in time

Throughout this section we assume that $u : \Omega \times (0, T_\infty) \to \mathbb{R}$ is a weak solution of problem (1.2) defined on a maximal (time) interval of existence $[0, T_\infty)$, $0 < T_\infty \le \infty$, with the initial conditions $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We establish *a priori* upper bounds on suitable norms of $u(\,\cdot\,,t)$ for each $t \in (0, T_\infty)$. More precisely, if $1 < p \le 2$, we allow for $T_\infty = \infty$ by taking advantage of the a priori upper bound $\|u(\,\cdot\,,t)\|_{L^\infty(\Omega)} \le C(t)$, $0 < t < \infty$, established in Lemma 4.4, inequality (4.5), which entails also $\|u(\,\cdot\,,t)\|_{W_0^{1,p}(\Omega)} \le \hat{C}(t)$, $0 < t < \infty$. Here, the functions $C, \hat{C} : \mathbb{R}_+ \to (0,\infty)$ are monotone increasing and unbounded, but easy to calculate. On the other hand, if $p > 2$, we estimate (from above) the norm $\|u(\,\cdot\,,t)\|_{W_0^{1,p}(\Omega)}$ in terms of $\|u(\,\cdot\,,t)\|_{L^p(\Omega)}$, for each $t \in (0, T_\infty)$. In order to derive these bounds, we take advantage of a regularity result for a weak solution of (1.2) which is due to Brézis [8], Théorème 3.6, pages 72–73. It is based on the fact that the set

$$\begin{aligned}\mathcal{G}_0 \stackrel{\mathrm{def}}{=} &\Big\{(u,f) \in W_0^{1,p}(\Omega) \times W^{-1,p'}(\Omega) :\, u \in W_0^{1,p}(\Omega) \;\text{ and }\; f = -\Delta_p u\Big\} \\ &\cap \big(L^2(\Omega) \times L^2(\Omega)\big) \end{aligned} \tag{5.1}$$

is the graph of a maximal monotone operator $\mathcal{A}_0 : L^2(\Omega) \to L^2(\Omega)$; see [8, Example 2.3.4, page 25].

To this end, given $1 < p < \infty$ and $\lambda \in \mathbb{R}$, we set

$$\mathcal{E}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u|^p \,\mathrm{d}x \quad \text{for } u \in W_0^{1,p}(\Omega). \tag{5.2}$$

Clearly, for $\lambda = 0$, $\mathcal{E}_0$ is a convex functional on $W_0^{1,p}(\Omega)$. However, we wish to consider $\mathcal{E}_0$ as a convex functional on $L^2(\Omega)$. We define the *effective domain* of $\mathcal{E}_0$ by $\mathcal{D}(\mathcal{E}_0) \stackrel{\text{def}}{=} W_0^{1,p}(\Omega) \cap L^2(\Omega)$. Notice that $\mathcal{D}(\mathcal{E}_0) = W_0^{1,p}(\Omega)$ if $p \geq \frac{2N}{N+2}$. We denote by $\partial\mathcal{E}_0(u)$ the *subdifferential* of $\mathcal{E}_0$ at $u \in \mathcal{D}(\mathcal{E}_0)$; i.e., $\partial\mathcal{E}_0(u)$ is the set of all $\phi \in L^2(\Omega)$ such that

$$\mathcal{E}_0(v) \geq \mathcal{E}_0(u) + \langle \phi, v-u \rangle \quad \text{holds for all } v \in \mathcal{D}(\mathcal{E}_0)$$

(by [8, Example 2.1.4, page 21]). The *domain* of the set-valued mapping $\partial\mathcal{E}_0$ from $L^2(\Omega)$ into itself is defined by

$$\mathcal{D}(\partial\mathcal{E}_0) \stackrel{\text{def}}{=} \big\{ u \in \mathcal{D}(\mathcal{E}_0) : \partial\mathcal{E}_0(u) \neq \emptyset \big\}.$$

It is a matter of a straightforward calculation to show that the graph of $\partial\mathcal{E}_0$ coincides with the graph of the maximal monotone operator $\mathcal{A}_0$ defined in (5.1) above. (This calculation is based on $-\Delta_p$ being continuous and maximal monotone from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$; see e.g. [7], Chapt. II, §1, Section 3, pages 48–50.)

We will use Brézis' regularity result [8], Théorème 3.6, pages 72–73, in the following form.

Lemma 5.1. *Let $1 < p < \infty$, $0 < T < \infty$, and $g \in L^2(Q_T)$ where $Q_T = \Omega \times (0,T)$. Assume that $u : Q_T \to \mathbb{R}$ is a weak solution of problem* (1.2) *with $\lambda = 0$, $f = g$, and $u_0 \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$. Then we have the following statements:*

(i) *The $L^2(\Omega)$-valued function $u : (0,T) \to L^2(\Omega)$, $u(t) \equiv u(\,\cdot\,,t)$ for $t \in (0,T)$, is (strongly) differentiable almost everywhere in $(0,T)$ with the (time-) derivative $\frac{\partial u}{\partial t} \in L^2(Q_T)$, $u(t) \in \mathcal{D}(\partial\mathcal{E}_0)$ for almost every $t \in (0,T)$, and the functional differential equation (cf. problem* (1.2))

$$\frac{\partial u}{\partial t}(t) + \partial\mathcal{E}_0(u(t)) = g(t) \tag{5.3}$$

holds in $L^2(\Omega)$ for almost every $t \in (0,T)$, where also $g(t) \equiv g(\,\cdot\,,t) \in L^2(\Omega)$.

(ii) *The function* $t \mapsto \int_\Omega |\nabla u(x,t)|^p \,\mathrm{d}x$ *is absolutely continuous on* $[0,T]$ *with the derivative satisfying*

$$\int_\Omega \Big|\frac{\partial u}{\partial t}(t)\Big|^2 \,\mathrm{d}x + \frac{1}{p}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u(x,t)|^p \,\mathrm{d}x = \int_\Omega g(t)\,\frac{\partial u}{\partial t}(t)\,\mathrm{d}x \tag{5.4}$$

for almost every $t \in (0,T)$.

(iii) *The* $L^2(Q_T)$*-norms of* $\frac{\partial u}{\partial t}$ *and* g *satisfy*

$$\Big\|\frac{\partial u}{\partial t}\Big\|_{L^2(Q_T)} \le \|g\|_{L^2(Q_T)} + \Big(\frac{1}{p}\int_\Omega |\nabla u_0|^p \,\mathrm{d}x\Big)^{1/2}. \tag{5.5}$$

This lemma easily renders the following identity, by taking the inner product in $L^2(\Omega)$ of equation (5.3) with $u(t)$:

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u(x,t)|^2 \,\mathrm{d}x + \int_\Omega |\nabla u(x,t)|^p \,\mathrm{d}x = \int_\Omega g(x,t)\,u(x,t)\,\mathrm{d}x \tag{5.6}$$

for almost every $t \in (0,T)$.

Now let us return to our standard setting $p \in (1,\infty)$ and $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. If $1 < p \le 2$, then every weak solution u of problem (1.2) is (essentially) bounded in $\Omega\times(0,T_\infty)$, by the a priori estimate (4.5) from Lemma 4.4, with some constants $c_1, c_2 > 0$ depending only on λ, $\|f\|_{L^\infty(\Omega\times\mathbb{R}_+)}$, and $\|u_0\|_{L^\infty(\Omega)}$, but not on $p \in (1,2]$ or $T = T_\infty$. If $p \ge 2$, then u is bounded in $\Omega \times (0,T')$ for every $T' \in (0,T_\infty)$, by Lemma 4.5. Hence, there exists a monotone increasing function $C : [0,T_\infty) \to (0,\infty)$ such that

$$\|u(\,\cdot,t)\|_{L^p(\Omega)} \le C(t), \quad 0 \le t < T_\infty. \tag{5.7}$$

For any $p \in (1,\infty)$, identity (5.4) with $g(x,t) = \lambda\,|u|^{p-2}u + f(x,t)$ which reads

$$\int_\Omega \Big|\frac{\partial u}{\partial t}(t)\Big|^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\;\mathcal{E}_\lambda(u(t)) = \int_\Omega f(t)\,\frac{\partial u}{\partial t}(t)\,\mathrm{d}x \tag{5.8}$$

for almost every $t \in (0,T_\infty)$, by (5.2). We apply the Cauchy-Schwarz inequality to the right-hand side to get

$$\frac{1}{2}\int_\Omega \Big|\frac{\partial u}{\partial t}(t)\Big|^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\;\mathcal{E}_\lambda(u(t)) \le \frac{1}{2}\int_\Omega |f(t)|^2 \,\mathrm{d}x \le E \tag{5.9}$$

for almost every $t \in (0,T_\infty)$, where $E \stackrel{\text{def}}{=} \frac{1}{2}\,|\Omega|_N \cdot \|f\|^2_{L^\infty(\Omega\times\mathbb{R}_+)} < \infty$. The function $t \mapsto \mathcal{E}_\lambda(u(t))$ being absolutely continuous on $[0,T]$, whenever $0 < T < T_\infty$, we may integrate equation (5.9), thus arriving at

$$\mathcal{E}_\lambda(u(t)) + \frac{1}{2}\int_s^t \int_\Omega \Big|\frac{\partial u}{\partial t}(x,t')\Big|^2 \,\mathrm{d}x\,\mathrm{d}t' \le \mathcal{E}_\lambda(u(s)) + E(t-s) \tag{5.10}$$

for all $s, t \in \mathbb{R}$ satisfying $0 \le s \le t < T_\infty$. Recalling (5.2) and $u(0) = u_0 \in W_0^{1,p}(\Omega)$, as a useful consequence of (5.10) we obtain

$$\begin{aligned} &\int_\Omega |\nabla u(x,t)|^p \,\mathrm{d}x + \frac{p}{2} \int_0^t \int_\Omega \left| \frac{\partial u}{\partial t}(x,t') \right|^2 \mathrm{d}x \,\mathrm{d}t' \\ &\le \lambda \int_\Omega |u(x,t)|^p \,\mathrm{d}x + p\,\mathcal{E}_\lambda(u_0) + p\,E\,t \end{aligned} \tag{5.11}$$

for all $0 \le t < T_\infty$.

Finally, we apply (4.5) (for $1 < p \le 2$) and (5.7) (for $p > 2$) to the integral $\int_\Omega |u(x,t)|^p \,\mathrm{d}x$ on the right-hand side in (5.11) to derive the following lemma.

Lemma 5.2. *Let $u : \Omega \times (0, T_\infty) \to \mathbb{R}$ be a weak solution of problem* (1.2) *defined on a maximal (time) interval of existence $[0, T_\infty)$. Then there exists a monotone increasing function $C : [0, T_\infty) \to (0, \infty)$ such that* (4.5) *holds if $1 < p \le 2$, with constants c_1, c_2 independent from $p \in (1, 2]$ and $T = T_\infty$, and* (5.7) *holds if $p > 2$. Furthermore, we have*

$$\left(\int_\Omega |\nabla u(x,t)|^p \,\mathrm{d}x + \frac{p}{2} \int_0^t \int_\Omega \left| \frac{\partial u}{\partial t}(x,t') \right|^2 \mathrm{d}x \,\mathrm{d}t' \right)^{1/p} \le \hat{C}(t) \tag{5.12}$$

for all $0 \le t < T_\infty$, where

$$\hat{C}(t) \stackrel{\text{def}}{=} \left[\lambda\, |\Omega|_N\, C(t)^p + p\,\mathcal{E}_\lambda(u_0) + p\,E\,t \right]^{1/p} \quad \text{if } 1 < p \le 2;$$

$$\hat{C}(t) \stackrel{\text{def}}{=} \left[\lambda\, C(t)^p + p\,\mathcal{E}_\lambda(u_0) + p\,E\,t \right]^{1/p} \quad \text{if } p > 2,$$

with $E = \frac{1}{2} |\Omega|_N \cdot \|f\|^2_{L^\infty(\Omega \times \mathbb{R}_+)} < \infty$. Of course, also $\hat{C} : [0, T_\infty) \to (0, \infty)$ is monotone increasing. In particular, we have

$$\|u(\,\cdot\,, t)\|_{W_0^{1,p}(\Omega)} \le \hat{C}(t), \quad 0 \le t < T_\infty. \tag{5.13}$$

We combine Lemma 5.1 and the estimate in (5.12) with the local existence results mentioned at the end of Section 2 (explained in the Appendix, §A.1) to obtain the following global (in time) existence results for all $t \in [0, T_\infty)$.

Corollary 5.3. *If $1 < p \le 2$, then any local (in time) weak solution of problem* (1.2) *can be continued to the maximal interval of existence $[0, T_\infty)$ with $T_\infty = \infty$. Such a solution u always satisfies* (4.5) *and* (5.12) *for all times $t \ge 0$.*

Corollary 5.4. *If $2 < p < \infty$ and u is a weak solution of problem* (1.2) *defined on a maximal (time) interval of existence $[0, T_\infty)$, with $0 < T_\infty < \infty$, then we have*

$$\lim_{t \to T_\infty -} \|u(\,\cdot\,, t)\|_{L^p(\Omega)} = \infty. \tag{5.14}$$

Proof of Corollary 5.3. It remains to show $T_\infty = \infty$. Suppose the contrary, $0 < T_\infty < \infty$. We apply the estimate in (5.12) to conclude that

$$\int_0^{T_\infty} \int_\Omega \left| \frac{\partial u}{\partial t}(x,t') \right|^2 \mathrm{d}x \, \mathrm{d}t' \leq \tilde{C}_\infty \stackrel{\text{def}}{=} \frac{2}{p} \, \hat{C}(T_\infty)^p < \infty.$$

Using the Cauchy-Schwartz inequality in

$$\|u(t) - u(s)\|_{L^2(\Omega)} \leq \int_s^t \left\| \frac{\partial u}{\partial t}(t') \right\|_{L^2(\Omega)} \mathrm{d}t'$$
$$\leq \left(\int_s^t \left\| \frac{\partial u}{\partial t}(t') \right\|_{L^2(\Omega)}^2 \mathrm{d}t' \right)^{1/2} (t-s)^{1/2} \leq \tilde{C}_\infty^{1/2} (t-s)^{1/2}$$

for $0 \leq s \leq t < T_\infty$, we now observe that the limit

$$u(\,\cdot\,, T_\infty) = \lim_{t \to T_\infty -} u(\,\cdot\,, t) \quad \text{in} \;\; L^2(\Omega)$$

exists. Furthermore, since $W_0^{1,p}(\Omega)$ is reflexive and $L^\infty(\Omega)$ is the dual space of $L^1(\Omega)$, the estimates in (4.5) and (5.12) render also $u(\,\cdot\,, T_\infty) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and both remain valid for $t = T_\infty$ as well.

Finally, the local existence results mentioned at the end of Section 2 guarantee that u can be continued past T_∞ to a weak solution of problem (1.2) in a longer (time) interval of existence $[0, T_\infty + \varepsilon)$ with some $\varepsilon > 0$. But this contradicts the maximality of the interval $[0, T_\infty)$. We have verified $T_\infty = \infty$. □

Proof of Corollary 5.4. On the contrary to (5.14), suppose that

$$C_\infty \stackrel{\text{def}}{=} 1 + \liminf_{t \to T_\infty -} \|u(\,\cdot\,, t)\|_{L^p(\Omega)} < \infty.$$

This means that there is a monotone increasing sequence

$$0 < t_1 < t_2 < \cdots < t_n < \cdots < T_\infty \; (< \infty) \quad \text{such that } t_n \nearrow T_\infty \text{ as } n \nearrow \infty$$

and the sequence $\{u(\,\cdot\,, t_n)\}_{n=1}^\infty$ is bounded in $L^p(\Omega)$ by

$$\|u(\,\cdot\,, t_n)\|_{L^p(\Omega)} \leq C_\infty \quad \text{for all} \;\; n = 1, 2, \dots .$$

It follows from (5.11) that

$$\int_\Omega |\nabla u(x, t_n)|^p \, \mathrm{d}x + \frac{p}{2} \int_0^{t_n} \int_\Omega \left| \frac{\partial u}{\partial t}(x,t') \right|^2 \mathrm{d}x \, \mathrm{d}t' \leq \hat{C}_\infty^p \tag{5.15}$$

for every $n = 1, 2, \dots$, where

$$\hat{C}_\infty \stackrel{\text{def}}{=} \left[\lambda \, C_\infty^p + p \, \mathcal{E}_\lambda(u_0) + p \, E \, T_\infty \right]^{1/p} < \infty.$$

In analogy with our proof of Corollary 5.3 above, from (5.15) we deduce

$$\int_0^{T_\infty} \int_\Omega \left| \frac{\partial u}{\partial t}(x,t') \right|^2 \mathrm{d}x \, \mathrm{d}t' \leq \tilde{C}_\infty \stackrel{\text{def}}{=} \frac{2}{p} \hat{C}_\infty^p < \infty$$

and so the limit

$$u(\,\cdot\,, T_\infty) = \lim_{t \to T_\infty -} u(\,\cdot\,, t) \quad \text{ in } \ L^2(\Omega)$$

exists, by the arguments used in the proof of Corollary 5.3. Furthermore, since $W_0^{1,p}(\Omega)$ is reflexive, the estimate in (5.15) renders also $u(\,\cdot\,, t_n) \rightharpoonup u(\,\cdot\,, T_\infty)$ weakly in $W_0^{1,p}(\Omega)$ as $n \to \infty$, and (5.15) remains valid for T_∞ in place of t_n as well.

Finally, the local existence and uniqueness results mentioned at the end of Section 2, combined with (5.15), guarantee that there is a number $\varepsilon > 0$ such that, for each $n = 1, 2, \dots$, u can be continued past t_n to a unique weak solution of problem (1.2) in a longer (time) interval of existence

$$[0, t_n + \varepsilon) = [0, t_n] \cup [t_n, t_n + \varepsilon).$$

Since $t_n \nearrow T_\infty$ as $n \nearrow \infty$, we have $\cup_{n=1}^\infty [0, t_n + \varepsilon) = [0, T_\infty + \varepsilon)$ which contradicts the maximality of the interval $[0, T_\infty)$. We have verified (5.14). $\square$

We summarize the results from Lemma 5.2 and Corollaries 5.3 and 5.4 in the following proposition. Recall that $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proposition 5.5. *Let $u : \Omega \times (0, T_\infty) \to \mathbb{R}$ be a weak solution of problem* (1.2) *defined on a maximal (time) interval of existence $[0, T_\infty)$. Then $1 < p \leq 2$ implies $T_\infty = \infty$, whereas if $2 < p < \infty$ and $0 < T_\infty < \infty$, then the blow-up of the solution $u(\,\cdot\,, t)$ as $t \nearrow T_\infty$ is characterized by* (5.14)*. For any $1 < p < \infty$, regardless of whether $T_\infty = \infty$ or $T_\infty < \infty$, we have the following regularity statements:*

(a) *$u \in L^\infty(\Omega \times (0, T'))$ whenever $0 < T' < T_\infty$, and there exists a monotone increasing function $\check{C} : [0, T_\infty) \to (0, \infty)$ such that*

$$\|u(\,\cdot\,, t)\|_{L^\infty(\Omega)} \leq \check{C}(t), \quad 0 \leq t < T_\infty. \tag{5.16}$$

(b) *$u \in C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [\varepsilon, T'])$ whenever $\varepsilon, T' \in \mathbb{R}$ are such that $0 < \varepsilon < T' < T_\infty$, where $\beta \in (0,1)$ is a constant independent of ε, T' and u, and there exists a monotone increasing function $\check{C}_\varepsilon : [\varepsilon, T_\infty) \to (0, \infty)$ such that*

$$\|u\|_{C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [\varepsilon, T'])} \leq \check{C}_\varepsilon(T'), \quad \varepsilon < T' < T_\infty. \tag{5.17}$$

In addition, if $u_0 \in C^{1+\beta}(\overline{\Omega})$, then one may take $\varepsilon = 0$ above.

Proof. Part (a). The a priori estimate (5.16) follows from Lemma 4.4 for $1 < p \le 2$ and from Lemma 4.5 for $p \ge 2$. In the latter case we need to take into account also the dependencies of constants in the L^∞_{loc}-regularity result from Porzio [23], Theorem 2.1, page 1095 (see also O'Leary [22], Theorem 1, page 436).

Part (b). The a priori estimate (5.17) follows from Part (a) combined with Lemma 4.6. The dependencies of constants are specified in Lieberman [19, Theorem 0.1, page 552]. □

6. Proof of the main result

We are ready to give the proofs of Theorem 3.3 (our main result) and Corollaries 3.5 and 3.6. Throughout this section, $u : \Omega \times (0, T_\infty) \to \mathbb{R}$ denotes a weak solution to the initial-boundary-value problem (1.2) defined on a maximal (time) interval of existence $[0, T_\infty)$.

First, we observe that we may assume $u_0 \in C^{1+\beta}(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$. Indeed, given any initial data $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we deduce from Proposition 5.5, Part (b), followed by Corollary 4.7 and Remark 4.8 that $u(\,\cdot\,, t) \in C^{1+\beta}(\overline{\Omega})$ for every $t \in (0, T_\infty)$. Moreover, the constant $c(t) > 0$ in inequality (4.10) and the upper bound on the norm $\|u(\,\cdot\,, t)\|_{C^{1+\beta}(\overline{\Omega})}$ in (5.17) depend on the solution u solely through the L^∞-norm $\|u_0\|_{L^\infty(\Omega)}$ of the initial value u_0. Hence, it is irrelevant *which* weak solution u of the initial-boundary-value problem (1.2) is considered (since such a solution might not be unique). Consequently, instead of starting with the initial data u_0 at time $t = 0$, we may start with the "initial" data $u(\,\cdot\,, \varepsilon)$ at time $t = \varepsilon \in (0, T_\infty)$ and consider only the restriction of the solution u to $\Omega \times (\varepsilon, T_\infty) \to \mathbb{R}$. Thus, without any loss of generality we may and will assume $u_0 \in W_0^{1,p}(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ throughout this section. In particular, we have

$$u(x, 0) = u_0(x) \ge -c_0 \varphi_1(x) \quad \text{for all } \ x \in \Omega, \tag{6.1}$$

where $c_0 > 0$ is some constant depending only on the norm $\|u_0\|_{C^1(\overline{\Omega})}$.

Next, recall that $\underline{f} \in L^\infty(\Omega)$ is some function such that $\int_\Omega \underline{f} \varphi_1 \,\mathrm{d}x > 0$ and the resonant problem (3.5) has no weak solution in $W_0^{1,p}(\Omega)$. Let us fix another constant $c' > c_0$. We apply a generalized form of the antimaximum principle due to Drábek et al. [13, Theorem 6.9, pages 465–466] to the stationary (elliptic) Dirichlet problem

$$-\Delta_p v_0 = \lambda\, |v_0|^{p-2} v_0 + \underline{f}(x) \ \text{ in } \Omega; \qquad v_0 = 0 \ \text{ on } \partial\Omega, \tag{6.2}$$

to conclude that there exists $\delta' > 0$ small enough, such that for each $\lambda \in (\lambda_1, \lambda_1 + \delta')$, every weak solution $v_0 \in W_0^{1,p}(\Omega)$ of problem (6.2) satisfies $v_0 \leq -c'\varphi_1$ almost everywhere in Ω. (This result generalizes the antimaximum principle from Arcoya and Gámez [6, Theorem 27, page 1908]; see also Takáč [29, Theorem 8.13, pages 459–460] for an alternative proof.)

Remark 6.1. (Drábek et al. [13, Remark 6.10, page 466]). Let us note that for $\lambda < \lambda_1$, problem (6.2) possesses a weak solution by the coercivity of the corresponding energy functional $u \mapsto \mathcal{J}_\lambda(u; \underline{f}) : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{J}_\lambda(u; \underline{f}) &\stackrel{\text{def}}{=} \mathcal{E}_\lambda(u) - \int_\Omega \underline{f}(x) u \, \mathrm{d}x \\ &= \frac{1}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x - \frac{\lambda}{p} \int_\Omega |u|^p \, \mathrm{d}x - \int_\Omega \underline{f}(x) u \, \mathrm{d}x \end{aligned} \tag{6.3}$$

for $u \in W_0^{1,p}(\Omega)$. On the other hand, for $\lambda_1 < \lambda < \lambda_1 + \delta'$, problem (6.2) is solvable by a topological degree argument from Drábek [12, Theorem 12.26].

Given any number $\lambda \in (\lambda_1, \lambda_1 + \delta')$, let $v_0 \in W_0^{1,p}(\Omega)$ be any weak solution of problem (6.2). By the regularity results mentioned above, we get first $v_0 \in L^\infty(\Omega)$ by [5, Théorème A.1, page 96], and then $v_0 \in C^{1+\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$, by [11, Theorem 2, page 829] or [31, Theorem 1, page 127] for interior regularity, and [18, Theorem 1, page 1203] for regularity near the boundary. Recall that $v_0 \leq -c'\varphi_1 < -c_0\varphi_1 \leq u_0$ in Ω.

Finally, let us recall the definition and properties of Λ_γ from Lemma 4.1 for any $\gamma > 0$. We have $\lambda_1 < \Lambda_\gamma \leq \Lambda_{\gamma'} < \infty$ whenever $0 < \gamma < \gamma' < \infty$, and also $\Lambda_\gamma \searrow \lambda_1$ as $\gamma \searrow 0$. We fix $\gamma_1 > 0$ such that $\Lambda_{\gamma_1} \leq \lambda_1 + \delta'$ and define

$$\delta_1(\gamma) \stackrel{\text{def}}{=} \tfrac{1}{2}(\Lambda_\gamma - \lambda_1) > 0 \quad \text{for every } \gamma \in (0, \gamma_1], \tag{6.4}$$

whence

$$0 < \delta_1(\gamma) < \delta' \quad \text{and} \quad \lambda_1 + \delta_1(\gamma) < \Lambda_\gamma \quad \text{for every } \gamma \in (0, \gamma_1]. \tag{6.5}$$

Moreover, we have $\delta_1(\gamma) \leq \delta_1(\gamma')$ whenever $0 < \gamma < \gamma' \leq \gamma_1$, and also $\delta_1(\gamma) \searrow 0$ as $\gamma \searrow 0$.

We distinguish between the cases $T_\infty = \infty$ and $0 < T_\infty < \infty$. Let us recall that $1 < p \leq 2$ implies $T_\infty = \infty$, by Corollary 5.3. We treat the case $T_\infty = \infty$ first.

6.1. **Case $T_\infty = \infty$.** By our hypothesis above, the initial value u_0 satisfies $u_0 \in C^{1+\beta}(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$, whence also (6.1) holds (with a constant $c_0 > 0$).

Given $\gamma \in (0, \gamma_1]$, let $\lambda \in (\lambda_1, \lambda_1 + \delta_1(\gamma))$ and let $v_0 \in W_0^{1,p}(\Omega)$ be any weak solution of problem (6.2). Recall that $v_0 \leq -c'\varphi_1 < -c_0\varphi_1 \leq u_0$ in Ω.

We define a sequence of functions $\underline{u}_n : \Omega \times (0, \infty) \to \mathbb{R}$ for $n = 0, 1, 2, \dots$ as follows: We set $\underline{u}_0(x,t) \stackrel{\text{def}}{=} v_0(x)$ for almost every $x \in \Omega$ and for every $t \geq 0$. For $n = 1, 2, 3, \dots$ we define $\underline{u}_n : \Omega \times (0, \infty) \to \mathbb{R}$ recursively to be the (unique) weak solution of the initial-boundary-value problem

$$\begin{cases} \dfrac{\partial \underline{u}_n}{\partial t} - \Delta_p \underline{u}_n = \lambda \, |\underline{u}_{n-1}|^{p-2} \underline{u}_{n-1} + \underline{f}(x), & (x,t) \in \Omega \times (0, \infty); \\ \quad \underline{u}_n(x,t) = 0, & (x,t) \in \partial\Omega \times (0, \infty); \\ \quad \underline{u}_n(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{6.6}$$

(The existence and uniqueness of $\underline{u}_n$ are discussed in Appendix A, §A.1.) As usual, we abbreviate $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$.

Lemma 6.2. *The function $(x, t; n) \mapsto \underline{u}_n(x,t) : \Omega \times \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ is monotone increasing in the variables $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$, for almost every $x \in \Omega$; i.e., for all $n \in \mathbb{Z}_+$ and $t, h \geq 0$ we have $\underline{u}_n(x,t) \leq \underline{u}_{n+1}(x,t)$ and $\underline{u}_n(x,t) \leq \underline{u}_n(x, t+h)$ for a.e. $x \in \Omega$. Similarly, we have $\underline{u}_n(x,t) \leq u(x,t)$.*

Proof. We begin with the proof of monotonicity of $\underline{u}_n(x,t)$ with respect to $n \in \mathbb{Z}_+$ by induction. For $n = 0$ we have

$$\begin{aligned} \frac{\partial \underline{u}_1}{\partial t} - \Delta_p \underline{u}_1 &= \lambda \, |\underline{u}_0|^{p-2} \underline{u}_0 + \underline{f}(x) \\ &= \lambda \, |v_0|^{p-2} v_0 + \underline{f}(x) = \frac{\partial \underline{u}_0}{\partial t} - \Delta_p \underline{u}_0 \end{aligned}$$

in the weak sense in $\Omega \times (0, \infty)$ with the Dirichlet boundary conditions $\underline{u}_1 = 0$ on $\partial\Omega \times (0, \infty)$ and the initial condition $\underline{u}_1(x,0) = u_0(x)$ for $x \in \Omega$ (see Definition in Section 2).

Here, we have taken advantage of $\lambda_1 < \lambda < \lambda_1 + \delta'$ and $v_0 \leq -c'\varphi_1 < 0$ almost everywhere in Ω. The initial data satisfy

$$\underline{u}_1(x,0) = u_0(x) > \underline{u}_0(x,0) = v_0(x) \quad \text{ for a.e. } x \in \Omega,$$

thanks to $v_0 \leq -c'\varphi_1 < -c_0\varphi_1 \leq u_0$ in Ω. We apply the (parabolic) weak comparison principle (Lemma 4.9 with $\mu = 0$) to conclude that $\underline{u}_0(x,t) \leq \underline{u}_1(x,t)$ for every $t \geq 0$ and almost every $x \in \Omega$. Now let $n \geq 1$ and assume that we already know that $\underline{u}_{n-1}(x,t) \leq \underline{u}_n(x,t)$ is valid for every $t \geq 0$ and almost every $x \in \Omega$. We compute as above, making use of equation (6.6),

$$\frac{\partial \underline{u}_{n+1}}{\partial t} - \Delta_p \underline{u}_{n+1} = \lambda \, |\underline{u}_n|^{p-2} \underline{u}_n + \underline{f}(x)$$

$$\geq \lambda\, |\underline{u}_{n-1}|^{p-2}\underline{u}_{n-1} + \underline{f}(x) = \frac{\partial \underline{u}_n}{\partial t} - \Delta_p \underline{u}_n$$

for $(x,t) \in \Omega \times (0,\infty)$. Again, we apply the weak comparison principle to conclude that $\underline{u}_{n+1}(x,t) \geq \underline{u}_n(x,t)$ for every $t \geq 0$ and almost every $x \in \Omega$. Thus, we have verified that $\underline{u}_n(x,t)$ is monotone increasing with respect to the index $n \in \mathbb{Z}_+$.

Now, let us fix $h \geq 0$ arbitrary. For $n = 0$ and $(x,t) \in \Omega \times \mathbb{R}_+$ we have $\underline{u}_0(x,t+h) = \underline{u}_0(x,t) = v_0(x)$ which shows that $\underline{u}_0(x,t)$ is monotone increasing with respect to $t \geq 0$. Let $n \geq 1$ and assume that $\underline{u}_{n-1}(x,t) \leq \underline{u}_{n-1}(x,t+h)$ holds for every $t \geq 0$ and almost every $x \in \Omega$. As above, we make use of equation (6.6) to compute

$$\frac{\partial \underline{u}_n}{\partial t}(x,t+h) - \Delta_p \underline{u}_n(x,t+h) = \lambda\, |\underline{u}_{n-1}(x,t+h)|^{p-2}\underline{u}_{n-1}(x,t+h) + \underline{f}(x)$$
$$\geq \lambda\, |\underline{u}_{n-1}(x,t)|^{p-2}\underline{u}_{n-1}(x,t) + \underline{f}(x) = \frac{\partial \underline{u}_n}{\partial t}(x,t) - \Delta_p \underline{u}_n(x,t)$$

for $(x,t) \in \Omega \times (0,\infty)$. We apply the weak comparison principle once more to conclude that $\underline{u}_n(x,t) \leq \underline{u}_n(x,t+h)$ for every $t \geq 0$ and almost every $x \in \Omega$. Hence, we have verified that $\underline{u}_n(x,t)$ is monotone increasing also with respect to $t \geq 0$.

Finally, the weak comparison principle guarantees $\underline{u}_n(x,t) \leq u(x,t)$, by induction on $n \in \mathbb{Z}_+$ using similar arguments as above combined with inequality (3.4). We note that, in the proof of $\underline{u}_n(x,t) \leq u(x,t)$, the independence of the constant $c_0 > 0$ from a particular weak solution u of the initial-boundary-value problem (1.2) is used in an essential way; cf. remarks about how to pass from arbitrary initial data $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to $u_0 \in W_0^{1,p}(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ at the beginning of this section. □

Lemma 6.2 now implies that the monotone limit

$$\underline{u}_n(x,t) \nearrow \underline{u}(x,t) \quad \text{as } n \nearrow \infty \tag{6.7}$$

exists for every $t \geq 0$ and almost every $x \in \Omega$ and satisfies $\underline{u}(x,t) \leq u(x,t)$, by Lebesgue's monotone convergence theorem. It follows from Proposition 5.5 that both estimates (5.16) and (5.17) remain valid also for $\underline{u}$ in place of u, with $T_\infty = \infty$ and possibly different monotone increasing functions $\check{C} : [0,T_\infty) \to (0,\infty)$ and $\check{C}_\varepsilon : [\varepsilon,T_\infty) \to (0,\infty)$, respectively. Since $u \in L^\infty(\Omega \times (0,T'))$ whenever $0 < T' < \infty$, we may apply (6.7) directly to the right-hand side in equation (6.6) to conclude that $\underline{u} : \Omega \times (0,\infty) \to \mathbb{R}$ is a weak solution of the initial-boundary-value problem (3.8).

Remark 6.3. Notice that our construction of $\underline{u}$ shows that $\underline{u}$ is the *minimal* weak solution of problem (3.8), by the (parabolic) weak comparison principle (Lemma 4.9 with $\mu = 0$). This means that if $w : \Omega \times (0,T) \to \mathbb{R}$ is another weak solution of problem (3.8), for some $0 < T \le \infty$, then we have $\underline{u}(x,t) \le w(x,t)$ for all $t \in [0,T)$ and almost every $x \in \Omega$. The last claim follows from the fact that $\underline{u}_n(x,t) \le w(x,t)$ for all $n \in \mathbb{Z}_+$ and $t \in [0,T)$ and for almost every $x \in \Omega$, combined with (6.7).

Lemma 6.4. *The function $t \mapsto \underline{u}(\,\cdot,t) : \mathbb{R}_+ \to W_0^{1,p}(\Omega)$ is monotone increasing in the variable $t \in \mathbb{R}_+$; i.e., $\underline{u}(x,t) \le \underline{u}(x,t+h)$ holds for all $t,h \ge 0$ and for almost every $x \in \Omega$. Furthermore, we have*

$$\lim_{t\to\infty} \|\underline{u}(\,\cdot,t)\|_{L^p(\Omega)} = \infty. \tag{6.8}$$

Proof. We apply Lemma 6.2 to equation (6.7) to conclude that $\underline{u}(x,t)$ is monotone increasing with respect to $t \ge 0$.

We show (6.8) arguing by contradiction. Assume that

$$\liminf_{t\to\infty} \|\underline{u}(\,\cdot,t)\|_{L^p(\Omega)} < \infty;$$

that is, there exists an unbounded, monotone increasing sequence of non-negative real numbers $t_k \nearrow \infty$ as $k \nearrow \infty$, such that

$$\sup_{k\in\mathbb{N}} \|\underline{u}(\,\cdot,t_k)\|_{L^p(\Omega)} < \infty. \tag{6.9}$$

Next, note that problem (3.8) without the initial condition is autonomous in the time variable $t \ge 0$. For each $k = 1,2,3,\dots$, let $\underline{u}^{(k)}(x,t) = \underline{u}(x,t+t_k)$ denote the time translation of the unknown function $\underline{u}$ by t_k, for all $t \ge 0$ and almost every $x \in \Omega$. Hence, by (3.8), each $\underline{u}^{(k)}$ satisfies

$$\begin{cases} \dfrac{\partial \underline{u}^{(k)}}{\partial t} - \Delta_p \underline{u}^{(k)} = \lambda\, |\underline{u}^{(k)}|^{p-2}\underline{u}^{(k)} + \underline{f}(x), & (x,t) \in \Omega \times (0,\infty); \\ \underline{u}^{(k)}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty); \\ \underline{u}^{(k)}(x,0) = \underline{u}(x,t_k), & x \in \Omega. \end{cases} \tag{6.10}$$

Since the sequence of functions $\underline{u}(\,\cdot,t_k) \in W_0^{1,p}(\Omega)$ $(k \in \mathbb{N})$ is monotone increasing, (6.9) forces $\underline{u}(\,\cdot,t_k) \nearrow \underline{u}^{(\infty)}$ in $L^p(\Omega)$ as $k \nearrow \infty$, by Lebesgue's monotone convergence theorem, with a limit function $\underline{u}^{(\infty)} \in L^p(\Omega)$. Consequently, $\underline{u}(x,t)$ being monotone increasing with respect to $t \ge 0$, we have even $\underline{u}(\,\cdot,t) \nearrow \underline{u}^{(\infty)}$ in $L^p(\Omega)$ as $t \nearrow \infty$. It follows that, given any

$0 < T < \infty$, we have

$$\sup_{t\in[0,T]} \|\underline{u}(\,\cdot\,, t+t_k) - \underline{u}^{(\infty)}\|_{L^p(\Omega)} \;\longrightarrow\; 0 \quad \text{as } k\to\infty. \tag{6.11}$$

To show that $\underline{u}^{(\infty)}$ is in fact a stationary solution of problem (3.8) without the initial condition; i.e.,

$$-\Delta_p \underline{u}^{(\infty)} = \lambda\, |\underline{u}^{(\infty)}|^{p-2}\underline{u}^{(\infty)} + \underline{f}(x) \;\text{ in } \Omega; \qquad \underline{u}^{(\infty)} = 0 \;\text{ on } \partial\Omega, \tag{6.12}$$

we apply (6.11) to the weak formulation (2.6) of equation (6.10) while passing to the limit equation as $k\to\infty$. More precisely, we first need to recall that, by Proposition 5.5, both estimates (5.16) and (5.17) remain valid also for $\underline{u}$ in place of u, with $T_\infty = \infty$ and possibly different monotone increasing functions $\check{C} : [0, T_\infty) \to (0,\infty)$ and $\check{C}_\varepsilon : [\varepsilon, T_\infty) \to (0,\infty)$, respectively. In particular, (5.17) implies that the set of functions $\underline{u}^{(k)} : \Omega\times[0,T]\to\mathbb{R}$ ($k\in\mathbb{N}$, $t_k \geq \varepsilon > 0$) is relatively compact in the Hölder space $C^{1+\beta',(1+\beta')/2}(\overline{\Omega}\times[0,T])$, for any $\beta'\in(0,\beta)$. Thus, we may pass to a subsequence, if necessary, to obtain the following improvement of (6.11):

$$\|\underline{u}^{(k)} - \underline{u}^{(\infty)}\|_{C^{1+\beta',(1+\beta')/2}(\overline{\Omega}\times[0,T])} \;\longrightarrow\; 0 \quad \text{as } k\to\infty.$$

Now we apply this fact to the weak formulation (2.6) of equation (6.10), while passing to the limit equation as $k\to\infty$, in order to conclude that $\underline{u}^{(\infty)}$ is a stationary solution of problem (3.8) without the initial condition; that is, (6.12) holds.

Finally, the monotonicity of $\underline{u}(x,t)$ with respect to $t\geq 0$ entails also $u_0(x) \leq \underline{u}^{(\infty)}(x)$ for almost every $x\in\Omega$. On the other hand, the anti-maximum principle from Drábek et al. [13, Theorem 6.9, pages 465–466] applied to problem (6.12) forces $\underline{u}^{(\infty)} \leq -c'\varphi_1$ almost everywhere in Ω, by our choice of $\lambda\in\mathbb{R}$ such that $\lambda_1 < \lambda < \lambda_1 + \delta'$. But the fact that $u_0 \leq \underline{u}^{(\infty)} \leq -c'\varphi_1$ almost everywhere in Ω contradicts our choice of $c' > c$ and $u_0 \geq -c\varphi_1$ almost everywhere in Ω. The lemma is proved. $\square$

We combine Remark 6.3 with Lemma 6.4, equation (6.8), to derive also

$$\lim_{t\to\infty} \|u(\,\cdot\,,t)\|_{L^p(\Omega)} = \infty. \tag{6.13}$$

We continue with the asymptotic profile of the function $\underline{u}(\,\cdot\,,t)\in W_0^{1,p}(\Omega)$ as $t\to\infty$. Set

$$\underline{\tau}(t) \stackrel{\text{def}}{=} \frac{1}{\|\varphi_1\|^2_{L^2(\Omega)}} \int_\Omega \underline{u}(x,t)\varphi_1(x)\,\mathrm{d}x \quad \text{for every } \; t\geq 0. \tag{6.14}$$

Lemma 6.5. *Given any $\gamma \in (0, \gamma_1]$ and $\lambda \in \mathbb{R}$ with $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$, there exists a constant $T_{\gamma,\lambda} > 0$ such that*

$$\underline{u}(x,t) = \underline{\tau}(t)\Big(\varphi_1(x) + \underline{v}^\top(x,t)\Big) \quad \text{for all } t \ge T_{\gamma,\lambda} \text{ and a.e. } x \in \Omega, \tag{6.15}$$

where $\underline{v}^\top : [T_{\gamma,\lambda}, \infty) \to W_0^{1,p}(\Omega)$ is a function satisfying

$$\int_\Omega \underline{v}^\top(x,t)\varphi_1(x)\,\mathrm{d}x = 0$$

together with

$$\|\underline{v}^\top(\,\cdot, t)\|_{W_0^{1,p}(\Omega)} < \gamma \quad \text{for all } t \ge T_{\gamma,\lambda}. \tag{6.16}$$

In addition, we have

$$\lim_{t\to\infty} \underline{\tau}(t) = +\infty \tag{6.17}$$

and, for all $t \ge T_{\gamma,\lambda}$,

$$\underline{\tau}(t) \ge (1 + \gamma\lambda_1^{-1/p})^{-1} \ge \underline{\tau}_1 \stackrel{\text{def}}{=} (1 + \gamma_1\lambda_1^{-1/p})^{-1} > 0. \tag{6.18}$$

Recall that the number $\delta_1(\gamma) \in (0, \delta')$ has been defined in (6.4); it satisfies (6.5).

Proof. By Lemma 6.4, there exists $t_1 \ge 0$ such that $\|\underline{u}(\,\cdot, t)\|_{L^p(\Omega)} \ge 1$ for all $t \ge t_1$. Set $U(x,t) = \|\underline{u}(\,\cdot,t)\|_{L^p(\Omega)}^{-1}\,\underline{u}(x,t)$ for $(x,t) \in \Omega \times [t_1, \infty)$; hence, $\|U(\,\cdot,t)\|_{L^p(\Omega)} = 1$ and

$$\int_\Omega |\nabla U(x,t)|^p\,\mathrm{d}x = \frac{\int_\Omega |\nabla \underline{u}(x,t)|^p\,\mathrm{d}x}{\int_\Omega |\underline{u}(x,t)|^p\,\mathrm{d}x} \ge \lambda_1 \quad \text{for all } t \ge t_1.$$

Identitying (5.4) with $g(x,t) = \lambda\,|\underline{u}|^{p-2}\underline{u} + \underline{f}(x)$ reads

$$\int_\Omega \Big|\frac{\partial \underline{u}}{\partial t}(t)\Big|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{J}_\lambda(\underline{u}(t);\underline{f}) = 0 \quad \text{for a.e. } t \in (0,\infty),$$

where the functional $\mathcal{J}_\lambda(\,\cdot;\underline{f}) : W_0^{1,p}(\Omega) \to \mathbb{R}$ has been defined in (6.3). This entails $\mathcal{J}_\lambda(\underline{u}(t);\underline{f}) \le \mathcal{J}_\lambda(u_0;\underline{f}) < \infty$ from which we derive

$$\begin{aligned}\int_\Omega |\nabla \underline{u}(x,t)|^p\,\mathrm{d}x &= p\cdot\mathcal{J}_\lambda(\underline{u}(t);\underline{f}) + \lambda\int_\Omega |\underline{u}(x,t)|^p\,\mathrm{d}x + p\int_\Omega \underline{f}(x)\underline{u}\,\mathrm{d}x\\ &\le p\cdot\mathcal{J}_\lambda(u_0;\underline{f}) + \lambda\int_\Omega |\underline{u}(x,t)|^p\,\mathrm{d}x + p\,\|\underline{f}\|_{L^{p'}(\Omega)}\,\|\underline{u}(\,\cdot,t)\|_{L^p(\Omega)},\end{aligned}$$

by Hölder's inequality, for all $t \geq t_1$, and therefore also

$$\int_\Omega |\nabla U(x,t)|^p \,\mathrm{d}x \leq \lambda + p \cdot \frac{\mathcal{J}_\lambda(u_0; \underline{f}) + \|\underline{f}\|_{L^{p'}(\Omega)} \|\underline{u}(\,\cdot\,,t)\|_{L^p(\Omega)}}{\|\underline{u}(\,\cdot\,,t)\|^p_{L^p(\Omega)}} \tag{6.19}$$
$$\leq \lambda + p\Big(\frac{\mathcal{J}_{\lambda_1}(u_0; \underline{f})}{\|\underline{u}(\,\cdot\,,t)\|^p_{L^p(\Omega)}} + \frac{\|\underline{f}\|_{L^{p'}(\Omega)}}{\|\underline{u}(\,\cdot\,,t)\|^{p-1}_{L^p(\Omega)}} \Big)$$

thanks to $\lambda \geq \lambda_1$.

To finish the proof of inequality (6.16), let $\gamma \in (0, \gamma_1]$; hence, $\eta \stackrel{\text{def}}{=} \Lambda_\gamma - \lambda_1 - \delta_1(\gamma) > 0$ holds by (6.4) and (6.5). Finally, assume $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$. Applying (6.8) we find a constant $T'_\eta \geq t_1$ such that in (6.19) above we have

$$p\Big(\frac{\mathcal{J}_{\lambda_1}(u_0; \underline{f})}{\|\underline{u}(\,\cdot\,,t)\|^p_{L^p(\Omega)}} + \frac{\|\underline{f}\|_{L^{p'}(\Omega)}}{\|\underline{u}(\,\cdot\,,t)\|^{p-1}_{L^p(\Omega)}} \Big) \leq \eta \quad \text{ for all } t \geq T'_\eta.$$

From (6.19) we thus derive

$$\int_\Omega |\nabla U(x,t)|^p \,\mathrm{d}x \leq \lambda + p\Big(\frac{\mathcal{J}_{\lambda_1}(u_0; \underline{f})}{\|\underline{u}(\,\cdot\,,t)\|^p_{L^p(\Omega)}} + \frac{\|\underline{f}\|_{L^{p'}(\Omega)}}{\|\underline{u}(\,\cdot\,,t)\|^{p-1}_{L^p(\Omega)}} \Big)$$
$$< \lambda_1 + \delta_1(\gamma) + \eta = \Lambda_\gamma$$

for all $t \geq T'_\eta$ $(\geq t_1)$. Then Lemma 4.1 forces the desired inequality (6.16); we take $T_{\gamma,\lambda} = T'_\eta$.

In particular, the limit in (6.8) applied to formula (6.15) yields (6.17), with help from inequality (6.16).

Finally, to verify inequality (6.18), recall that $\|\underline{u}(\,\cdot\,,t)\|_{L^p(\Omega)} \geq 1$ for all $t \geq T_{\gamma,\lambda}$. Using $\underline{u} = \underline{\tau}(\varphi_1 + \underline{v}^\top)$ we thus get

$$\lambda_1^{1/p} \leq \lambda_1^{1/p} |\underline{\tau}(t)| \cdot \|\varphi_1 + \underline{v}^\top(\,\cdot\,,t)\|_{L^p(\Omega)} \leq |\underline{\tau}(t)| \cdot \|\varphi_1 + \underline{v}^\top(\,\cdot\,,t)\|_{W_0^{1,p}(\Omega)}$$
$$\leq |\underline{\tau}(t)| \,(\lambda_1^{1/p} + \gamma) \quad \text{ for all } \ t \geq T_{\gamma,\lambda},$$

by inequality (6.16). This (combined with the continuity of $\underline{\tau}$ and (6.17)) yields (6.18) as desired. The proof is finished. □

Functions $\underline{\tau} : [T_{\gamma,\lambda}, \infty) \to \mathbb{R}$ and $\underline{v}^\top : [T_{\gamma,\lambda}, \infty) \to W_0^{1,p}(\Omega)$ introduced in Lemma 6.5 have the following additional properties, where inequality (6.22) is a "regularization" of (6.16).

Corollary 6.6. *There exist constants $k_1 \geq \delta_1(\gamma_1)$ (> 0) and $M_1 > 0$ with the following properties, respectively: In the situation of* Lemma 6.5*, given* *any* $\gamma \in (0, \gamma_1]$ *and* $\lambda \in \mathbb{R}$ *with* $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$*, we have*

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}t}\, \underline{\tau}(t) \leq \|\varphi_1\|_{L^2(\Omega)}^{-2} \Big(k_1\, |\underline{\tau}(t)|^{p-1} + \langle \underline{f}, \varphi_1 \rangle \Big) \quad \text{for a.e. } t \geq T_{\gamma,\lambda}, \tag{6.20}$$

$$\|\underline{v}^{\top}(\,\cdot\,, t)\|_{C^{1+\beta}(\overline{\Omega})} \leq M_1 \quad \text{for every } t \geq T_{\gamma,\lambda}. \tag{6.21}$$

In particular, there are constants $\theta \in (0,1)$ *and* $C > 0$ *such that*

$$\|\underline{v}^{\top}(\,\cdot\,, t)\|_{C^{1+\beta'}(\overline{\Omega})} < C\, M_1^{\theta}\, \gamma^{1-\theta} \quad \text{for all } t \geq T_{\gamma,\lambda}. \tag{6.22}$$

All constants k_1*,* M_1*,* θ*, and* C *are independent of the choice of* $\gamma \in (0, \gamma_1]$ *and* λ *with* $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$*.*

Proof. In view of (6.15) we write $\underline{v} = \varphi_1 + \underline{v}^{\top}$; i.e., $\underline{u}(x,t) = \underline{\tau}(t)\, \underline{v}(x,t)$. Consequently, problem (3.8) yields the equation

$$\frac{\partial}{\partial t}\Big(\underline{\tau}(t)\, \underline{v}(\,\cdot\,, t) \Big) = |\underline{\tau}(t)|^{p-2} \underline{\tau}(t) \Big(\Delta_p \underline{v} + \lambda\, |\underline{v}|^{p-2} \underline{v} \Big) + \underline{f} \qquad \text{in } W^{-1,p'}(\Omega) \text{ for a.e. } t \geq T_{\gamma,\lambda}. \tag{6.23}$$

We take the inner product in $L^2(\Omega)$ of both sides of this equation with φ_1, thus arriving at

$$\begin{aligned} &\|\varphi_1\|_{L^2(\Omega)}^2\, \frac{\mathrm{d}}{\mathrm{d}t}\, \underline{\tau}(t) \\ &= |\underline{\tau}(t)|^{p-2} \underline{\tau}(t) \Big(- \int_\Omega |\nabla \underline{v}|^{p-2} (\nabla \underline{v} \cdot \nabla \varphi_1)\, \mathrm{d}x + \lambda \int_\Omega |\underline{v}|^{p-2} \underline{v}\, \varphi_1\, \mathrm{d}x \Big) \\ &+ \langle \underline{f}, \varphi_1 \rangle \qquad \text{for a.e. } t \geq T_{\gamma,\lambda}. \end{aligned} \tag{6.24}$$

We claim (and prove below) that there exists a constant $c > 0$, independent of the choice of numbers $\gamma \in (0, \gamma_1]$, λ with $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$, and $t \geq T_{\gamma,\lambda}$, such that for all such γ, λ, and t we have

$$\Big| \frac{1}{\lambda_1} \int_\Omega |\nabla \underline{v}|^{p-2} (\nabla \underline{v} \cdot \nabla \varphi_1)\, \mathrm{d}x - 1 \Big| \leq c\, \gamma, \tag{6.25}$$

$$\Big| \int_\Omega |\underline{v}|^{p-2} \underline{v}\, \varphi_1\, \mathrm{d}x - 1 \Big| \leq c\, \gamma. \tag{6.26}$$

We apply the last two inequalities to equation (6.24) to get

$$\|\varphi_1\|_{L^2(\Omega)}^2\, \frac{\mathrm{d}}{\mathrm{d}t}\, \underline{\tau}(t) \leq |\underline{\tau}(t)|^{p-1} \big(\lambda - \lambda_1 + c(\lambda_1 + \lambda)\gamma \big) + \langle \underline{f}, \varphi_1 \rangle$$

for almost every $t \geq T_{\gamma,\lambda}$. Hence, we may take $k_1 = \delta_1(\gamma_1) + c\big(2\lambda_1 + \delta_1(\gamma_1)\big)\gamma_1$ to arrive at the (second) desired inequality in (6.20).

To verify inequality (6.25), we rewrite its left-hand side as

$$\begin{aligned}&\int_\Omega |\nabla \underline{v}|^{p-2}(\nabla \underline{v}\cdot\nabla\varphi_1)\,\mathrm{d}x-\lambda_1\\ &=\int_\Omega\Big[|\nabla\varphi_1+\nabla\underline{v}^\top|^{p-2}(\nabla\varphi_1+\nabla\underline{v}^\top)-|\nabla\varphi_1|^{p-2}\nabla\varphi_1\Big]\cdot\nabla\varphi_1\,\mathrm{d}x\\ &=\int_\Omega\Big[\Big(\int_0^1\mathbf{A}(\nabla\varphi_1+s\nabla\underline{v}^\top)\,\mathrm{d}s\Big)\nabla\underline{v}^\top\Big]\cdot\nabla\varphi_1\,\mathrm{d}x,\end{aligned}$$

where the matrix-valued function $\mathbf{A}$ is defined by formula (A.9) (in Appendix A, §A.2). We apply inequalities (A.4) and (A.7) in order to estimate the integral $\int_0^1\mathbf{A}(\nabla\varphi_1+s\nabla\underline{v}^\top)\,\mathrm{d}s$ above,

$$\begin{aligned}&\Big|\int_\Omega|\nabla\underline{v}|^{p-2}(\nabla\underline{v}\cdot\nabla\varphi_1)\,\mathrm{d}x-\lambda_1\Big|\\ &\leq C_p\int_\Omega\Big(\max_{0\leq s\leq 1}|\nabla\varphi_1+s\nabla\underline{v}^\top|\Big)^{p-2}|\nabla\underline{v}^\top|\,|\nabla\varphi_1|\,\mathrm{d}x,\end{aligned}\tag{6.27}$$

where $C_p>0$ is a constant depending only on $p\in(1,\infty)$; one may take $C_p=p-1$ for $p\geq 2$.

If $1<p\leq 2$, then inequality (6.27) entails

$$\begin{aligned}&\Big|\int_\Omega|\nabla\underline{v}|^{p-2}(\nabla\underline{v}\cdot\nabla\varphi_1)\,\mathrm{d}x-\lambda_1\Big|\leq C_p\int_\Omega|\nabla\varphi_1|^{p-1}\,|\nabla\underline{v}^\top|\,\mathrm{d}x\\ &\leq C_p\,\|\nabla\varphi_1\|_{L^p(\Omega)}^{p-1}\,\|\nabla\underline{v}^\top\|_{L^p(\Omega)}=C_p\,\lambda_1^{1-(1/p)}\,\|\nabla\underline{v}^\top\|_{L^p(\Omega)},\end{aligned}\tag{6.28}$$

by Hölder's inequality together with $\int_\Omega|\nabla\varphi_1|^p\,\mathrm{d}x=\lambda_1$. We combine (6.16) and (6.28) to arrive at (6.25).

On the other hand, if $2\leq p<\infty$, then from inequality (6.27) we derive

$$\begin{aligned}&\Big|\int_\Omega|\nabla\underline{v}|^{p-2}(\nabla\underline{v}\cdot\nabla\varphi_1)\,\mathrm{d}x-\lambda_1\Big|\\ &\leq C_p\int_\Omega\Big(|\nabla\varphi_1|+|\nabla\underline{v}^\top|\Big)^{p-2}|\nabla\underline{v}^\top|\,|\nabla\varphi_1|\,\mathrm{d}x\\ &\leq C_p\Big\||\nabla\varphi_1|+|\nabla\underline{v}^\top|\Big\|_{L^p(\Omega)}^{p-2}\,\|\nabla\underline{v}^\top\|_{L^p(\Omega)}\,\|\nabla\varphi_1\|_{L^p(\Omega)}\\ &\leq C_p\,\lambda_1^{1/p}\Big(\lambda_1^{1/p}+\|\nabla\underline{v}^\top\|_{L^p(\Omega)}\Big)^{p-2}\,\|\nabla\underline{v}^\top\|_{L^p(\Omega)},\end{aligned}\tag{6.29}$$

by Hölder's inequality again. We combine (6.16) and (6.29) to arrive at (6.25).

The proof of inequality (6.26) is analogous to that of (6.25).

In order to verify the regularity result (6.21), let us first note that

$$\underline{u} \in C\big([0,T'] \to L^2(\Omega)\big) \cap L^p\big((0,T') \to W_0^{1,p}(\Omega)\big)$$

holds for every $T' \in (0,\infty)$, because $\underline{u}$ is a weak solution to problem (3.8). By remarks following the definition of a weak solution to problem (1.2) in Section 2, one has also $\underline{u} \in W^{1,p'}((0,T') \to W^{-1,p'}(\Omega))$. This implies $\underline{\tau} \in W^{1,p'}(0,T')$ for every $T' \in (0,\infty)$. Since the function $t \mapsto \underline{u}(\,\cdot,t) : \mathbb{R}_+ \to W_0^{1,p}(\Omega)$ is monotone increasing in the variable $t \in \mathbb{R}_+$, also $\underline{\tau} : \mathbb{R}_+ \to \mathbb{R}$ is monotone increasing.

Furthermore, recalling inequality (6.18), we observe that equation (6.23) is equivalent to

$$\begin{aligned}&\underline{\tau}(t)^{-(p-1)}\,\underline{\tau}'(t)\,\underline{v}(\,\cdot,t) + \underline{\tau}(t)^{-(p-2)}\,\frac{\partial}{\partial t}\,\underline{v}(\,\cdot,t)\\ &= \Delta_p\underline{v} + \lambda\,|\underline{v}|^{p-2}\underline{v} + \underline{\tau}(t)^{-(p-1)}\,\underline{f} \quad \text{in } W^{-1,p'}(\Omega) \text{ for a.e. } t \ge T_{\gamma,\lambda}.\end{aligned} \tag{6.30}$$

It follows from equation (6.20) that the coefficient $\kappa(t) \stackrel{\text{def}}{=} \underline{\tau}(t)^{-(p-1)}\,\underline{\tau}'(t)$ in the product $\kappa(t)\,\underline{v}(\,\cdot,t)$ above is (essentially) bounded; more precisely, one has, for almost every $t \ge T_{\gamma,\lambda}$,

$$0 \le \kappa(t) = \underline{\tau}(t)^{-(p-1)}\,\underline{\tau}'(t) \le k_3 \stackrel{\text{def}}{=} \|\varphi_1\|_{L^2(\Omega)}^{-2}\big(k_1 + \underline{\tau}_1^{-(p-1)}\,\langle \underline{f},\varphi_1\rangle\big), \tag{6.31}$$

by inequality (6.18). Next, we make the substitution

$$s \equiv s(t) \stackrel{\text{def}}{=} \int_{T_{\gamma,\lambda}}^t \underline{\tau}(t)^{p-2}\,\mathrm{d}t \quad \text{for all } \ t \ge T_{\gamma,\lambda}. \tag{6.32}$$

We claim that $s(t) \nearrow +\infty$ as $t \nearrow \infty$. Indeed, we first apply (6.31) to (6.32) to estimate

$$s \equiv s(t) \ge \frac{1}{k_3}\int_{T_{\gamma,\lambda}}^t \underline{\tau}(t)^{-1}\,\underline{\tau}'(t)\,\mathrm{d}t = \frac{1}{k_3}\cdot\log\Big(\underline{\tau}(t)/\underline{\tau}(T_{\gamma,\lambda})\Big)$$

for all $t \ge T_{\gamma,\lambda}$. Then (6.17) forces $s(t) \nearrow +\infty$ as $t \nearrow \infty$. Let $\mathfrak{t} : \mathbb{R}_+ \to [T_{\gamma,\lambda},\infty)$ denote the inverse function of $s : t \mapsto s(t)$. After this substitution, equation (6.30) reads

$$\begin{aligned}&\kappa(\mathfrak{t}(s))\,V(\,\cdot,s) + \frac{\partial}{\partial s}\,V(\,\cdot,s)\\ &= \Delta_p V + \lambda\,|V|^{p-2}V + \underline{\tau}(\mathfrak{t}(s))^{-(p-1)}\,\underline{f} \quad \text{in } W^{-1,p'}(\Omega) \text{ for } s \ge 0,\end{aligned} \tag{6.33}$$

with the unknown function $V(x,s) = \underline{v}(x, \mathfrak{t}(s))$ of $(x,s) \in \Omega \times \mathbb{R}_+$. From inequality (6.16) we deduce

$$\|V(\,\cdot\,,s)\|_{W_0^{1,p}(\Omega)} = \|\varphi_1 + \underline{v}^\top(\,\cdot\,,\mathfrak{t}(s))\|_{W_0^{1,p}(\Omega)} < \lambda_1^{1/p} + \gamma \quad \text{for all } s \geq 0.$$

We apply the same regularity arguments as in the proof of Proposition 5.5, but this time to equation (6.33), to conclude that

$$\|V(\,\cdot\,,s)\|_{C^{1+\beta}(\overline{\Omega})} \leq M_1' \equiv \text{const} < \infty \quad \text{for all } s \geq 0.$$

Since $\underline{v}(x,t) = V(x, s(t))$, this proves the desired inequality (6.21) for all $t \geq T_{\gamma,\lambda}$.

Finally, to verify inequality (6.22), we make use of the Gagliardo-Nirenberg inequality (4.1) (Lemma 4.2) to get

$$\|\nabla \underline{v}^\top\|_{C^{\beta'}(\overline{\Omega})} \leq C\, \|\nabla \underline{v}^\top\|_{C^{\beta}(\overline{\Omega})}^{\theta}\, \|\nabla \underline{v}^\top\|_{L^p(\Omega)}^{1-\theta}.$$

(The constants $C \in (0,\infty)$ and $\theta \in (0,1)$ depend solely upon $p \in (1,\infty)$, $0 < \beta' < \beta < 1$, and the domain Ω.) We apply inequalities (6.16) and (6.21) to the right-hand side to obtain (6.22).

The proof of Corollary 6.6 is now complete. □

6.2. **End of the proof (of the main result) for $T_\infty = \infty$.**

Proof of Theorem 3.3 *for* $T_\infty = \infty$. Let $u : \Omega \times (0,\infty) \to \mathbb{R}$ be a weak solution of problem (1.2); hence $T_\infty = \infty$. By Proposition 5.5, Part (b), we know that $u \in C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [\varepsilon, T'])$ whenever $0 < \varepsilon < T' < \infty$, and there exists a monotone increasing function $\check{C}_\varepsilon : [\varepsilon,\infty) \to (0,\infty)$ such that (5.17) is valid. As we have already mentioned right after Theorem 3.3, in order to verify inequalities (3.6), it suffices to establish (3.7). But the latter inequality will follow immediately from

$$\underline{\tau}_*(t) \stackrel{\text{def}}{=} \inf_{x\in\Omega} \frac{\underline{u}(x,t)}{\varphi_1(x)} \geq \underline{\tau}_+(\equiv \text{const}) > 0 \quad \text{for all } t \in [T_+,\infty), \tag{6.34}$$

where $\underline{\tau}_*(t) \nearrow +\infty$ as $t \nearrow \infty$. Here, let us recall that $\underline{u}(x,t) \leq u(x,t)$ for every $t \geq 0$ and almost every $x \in \Omega$, by Lemma 6.2 and the monotone limit in (6.7), as already mentioned before Remark 6.3. Hence, (6.34) $\Longrightarrow$ (3.7) $\Longrightarrow$ (3.6). In fact, we observe that both (6.34) and $\underline{\tau}_*(t) \nearrow +\infty$ $(t \nearrow \infty)$ follow easily from Lemma 6.5 and Corollary 6.6, with help from $|\underline{v}^\top(x,t)| \leq \frac{1}{2}\varphi_1(x)$ for all $x \in \Omega$ and $t \geq T_+$. We prove the last claim below.

First, inequality (6.22) guarantees

$$\sup_{x\in\Omega} \frac{|\underline{v}^{\top}(x,t)|}{\varphi_1(x)} \leq C' \, \|\underline{v}^{\top}(\,\cdot\,,t)\|_{C^{1+\beta'}(\overline{\Omega})} \leq C'CM_1^{\theta}\,\gamma^{1-\theta} \quad \text{for all } \; t \geq T_{\gamma,\lambda}, \tag{6.35}$$

where $C' \equiv C'(\Omega,p,\beta') > 0$ is a constant which depends solely on Ω, p, and β', where $\beta' \in (0,\beta)$. Now set

$$\gamma_2 = (2\,C'CM_1^{\theta})^{-1/(1-\theta)} \quad \text{and} \quad \gamma_3 = \max\{\gamma_1,\, \gamma_2\}.$$

Then, given any $\gamma \in (0,\gamma_3]$ and $\lambda \in \mathbb{R}$ with $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$, from inequality (6.35) above we derive $\sup_{x\in\Omega}\left(|\underline{v}^{\top}(x,t)|/\varphi_1(x)\right) \leq \frac{1}{2}$ for all $t \geq T_{\gamma,\lambda}$ as claimed. Thus, we may take $\gamma = \gamma_3$ and $\delta = \delta_1(\gamma_3)$ and set $T_+ = T_{\gamma_3,\lambda}$, thanks to $T_{\gamma,\lambda} \geq T_{\gamma_3,\lambda} > 0$ for $\gamma \in (0,\gamma_3]$ and $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma_3)$.

The proof of Theorem 3.3 is finished for $T_\infty = \infty$. □

Proof of Corollary 3.5. Formula (3.9) follows directly from (6.15), thanks to $T_+ \geq T_{\gamma,\lambda}$. Part (a) is an easy consequence of equation (6.24) combined with (6.25) and (6.26), supplemented by (6.17). Finally, Part (b) follows from the end of the proof of Theorem 3.3 for $T_\infty = \infty$ given above. □

6.3. **Case** $T_\infty < \infty$**.** As in both previous paragraphs (§6.1 and §6.2), by our hypothesis, the initial value u_0 satisfies $u_0 \in C^{1+\beta}(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$, whence (6.1) also holds (with a constant $c_0 > 0$). We stress that $u : \Omega \times (0,T_\infty) \to \mathbb{R}$ is fixed; it is a weak solution to the initial-boundary-value problem (1.2) defined on a maximal (time) interval of existence $[0,T_\infty)$, $0 < T_\infty < \infty$. By Proposition 5.5, this is possible only if $p > 2$. Thus, we actually treat the case $T_\infty < \infty$ and $2 < p < \infty$. In general, the number T_∞ may depend on *which* weak solution u of the initial-boundary-value problem (1.2) is considered. However, thanks to $p > 2$, the function $u \mapsto \lambda\,|u|^{p-2}u : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on bounded subsets of $\mathbb{R}$. Hence, one easily deduces from Proposition 5.5, Part (a), that the solution u to problem (1.2) is unique; see the discussion in Appendix A, §A.1. It follows that T_∞ depends only on f and $u_0 \in W_0^{1,p}(\Omega)$, but not on the values of $u(\,\cdot\,,t)$ for $t \in (0,T_\infty)$; i.e., one has $T_\infty \equiv T_\infty(f,u_0)$.

In this situation we can replace Lemma 6.5 by the following better result which holds directly for u. We set

$$\tau(t) \stackrel{\text{def}}{=} \frac{1}{\|\varphi_1\|^2_{L^2(\Omega)}} \int_\Omega u(x,t)\varphi_1(x)\,\mathrm{d}x \quad \text{for every } \; t \geq 0.$$

Lemma 6.7. *Given any $\gamma \in (0, \gamma_1]$ and $\lambda \in \mathbb{R}$ with $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$, there exists a constant $T_{\gamma,\lambda} \in (0, T_\infty)$ such that*

$$u(x,t) = \tau(t)\big(\varphi_1(x) + v^\top(x,t)\big) \quad \text{for all } t \in [T_{\gamma,\lambda}, T_\infty) \text{ and a.e. } x \in \Omega, \tag{6.36}$$

where $v^\top : [T_{\gamma,\lambda}, T_\infty) \to W_0^{1,p}(\Omega)$ is a function satisfying

$$\int_\Omega v^\top(x,t)\varphi_1(x)\,\mathrm{d}x = 0$$

together with

$$\|v^\top(\cdot,t)\|_{W_0^{1,p}(\Omega)} < \gamma \quad \text{for all } t \in [T_{\gamma,\lambda}, T_\infty). \tag{6.37}$$

In addition, we have

$$\lim_{t \to T_\infty -} \tau(t) = +\infty, \tag{6.38}$$

$$\tau(t) \geq (1 + \gamma\lambda_1^{-1/p})^{-1} \geq \underline{\tau}_1 > 0 \quad \text{for all } t \in [T_{\gamma,\lambda}, T_\infty), \tag{6.39}$$

where $\underline{\tau}_1 = (1 + \gamma_1\lambda_1^{-1/p})^{-1}$ has been defined in (6.18).

Proof. We follow similar reasoning as in the proof of Lemma 6.5. By Corollary 5.4, equation (5.14), there exists $t_1 \geq 0$ such that $\|u(\cdot,t)\|_{L^p(\Omega)} \geq 1$ for all $t \in [t_1, T_\infty)$. Set $U(x,t) = \|u(\cdot,t)\|_{L^p(\Omega)}^{-1}\, u(x,t)$ for $(x,t) \in \Omega \times [t_1, T_\infty)$; hence, $\|U(\cdot,t)\|_{L^p(\Omega)} = 1$ and

$$\int_\Omega |\nabla U(x,t)|^p\,\mathrm{d}x = \frac{\int_\Omega |\nabla u(x,t)|^p\,\mathrm{d}x}{\int_\Omega |u(x,t)|^p\,\mathrm{d}x} \geq \lambda_1 \quad \text{for all } t \in [t_1, T_\infty).$$

Next, recall that identity (5.4) with $g(x,t) = \lambda\,|u|^{p-2}u + f(x,t)$ reads as (5.8) for almost every $t \in (0, T_\infty)$, where the functional $\mathcal{E}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ has been defined in (5.2). From (5.11) we obtain

$$\int_\Omega |\nabla u(x,t)|^p\,\mathrm{d}x \leq \lambda \int_\Omega |u(x,t)|^p\,\mathrm{d}x + E_\infty \quad \text{for all } 0 \leq t < T_\infty,$$

where $E_\infty \stackrel{\text{def}}{=} p\Big(\mathcal{E}_\lambda(u_0) + E\,T_\infty\Big) < \infty$ with $E = \frac{1}{2}\,|\Omega|_N \cdot \|f\|_{L^\infty(\Omega\times\mathbb{R}_+)}^2$. It follows that, in analogy with inequality (6.19),

$$\int_\Omega |\nabla U(x,t)|^p\,\mathrm{d}x \leq \lambda + E_\infty\,\|u(\cdot,t)\|_{L^p(\Omega)}^{-p} \tag{6.40}$$

holds for all $t \in [t_1, T_\infty)$ and $\lambda \geq \lambda_1$.

The proof may now be completed exactly as that of Lemma 6.5. Let $\gamma \in (0, \gamma_1]$; hence, $\eta \stackrel{\text{def}}{=} \Lambda_\gamma - \lambda_1 - \delta_1(\gamma) > 0$ holds by (6.4) and (6.5). Finally, assume $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$. Applying Corollary 5.4, equation (5.14), we find a constant $T'_\eta \geq t_1$ such that in (6.40) above we have

$$E_\infty \, \|u(\,\cdot, t)\|^{-p}_{L^p(\Omega)} \leq \eta \quad \text{for all } t \geq T'_\eta .$$

From (6.40) we derive

$$\int_\Omega |\nabla U(x,t)|^p \, \mathrm{d}x \leq \lambda + E_\infty \, \|u(\,\cdot, t)\|^{-p}_{L^p(\Omega)} < \lambda_1 + \delta_1 + \eta = \Lambda_\gamma$$

for all $t \geq T'_\eta$ ($\geq t_1$). Then Lemma 4.1 forces the desired inequality (6.37); we take $T_{\gamma,\lambda} = T'_\eta$.

In particular, the limit in (5.14) applied to formula (6.36) yields (6.38).

Inequality (6.39) is verified in the same way as (6.18). Recall that $\|u(\,\cdot, t)\|_{L^p(\Omega)} \geq 1$ for all $t \in [T_{\gamma,\lambda}, T_\infty)$. We factorize $u = \tau(\varphi_1 + v^\top)$ to get

$$\begin{aligned} \lambda_1^{1/p} &\leq \lambda_1^{1/p} \, |\tau(t)| \cdot \|\varphi_1 + v^\top(\,\cdot, t)\|_{L^p(\Omega)} \leq |\tau(t)| \cdot \|\varphi_1 + v^\top(\,\cdot, t)\|_{W_0^{1,p}(\Omega)} \\ &\leq |\tau(t)| \, (\lambda_1^{1/p} + \gamma) \quad \text{for all } \ t \geq [T_{\gamma,\lambda}, T_\infty), \end{aligned}$$

by inequality (6.37). This yields (6.39), thanks to the continuity of τ and (6.38).

The proof is finished. □

Functions $\tau : [T_{\gamma,\lambda}, T_\infty) \to \mathbb{R}$ and $v^\top : [T_{\gamma,\lambda}, T_\infty) \to W_0^{1,p}(\Omega)$ introduced in Lemma 6.7 have the following additional properties, where inequality (6.43) is a “regularization” of (6.37).

Corollary 6.8. *There exist constants $k_1 \geq \delta_1(\gamma_1)$ (> 0) and $M_1 > 0$ with the following properties, respectively: In the situation of* Lemma 6.7, *given any $\gamma \in (0, \gamma_1]$ and $\lambda \in \mathbb{R}$ with $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$, we have*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \, \tau(t) \right| \leq \|\varphi_1\|^{-2}_{L^2(\Omega)} \big(k_1 \, |\tau(t)|^{p-1} + k_2 \big) \quad \text{for a.e. } t \in [T_{\gamma,\lambda}, T_\infty), \tag{6.41}$$

$$\|v^\top(\,\cdot, t)\|_{C^{1+\beta}(\overline{\Omega})} \leq M_1 \quad \text{for every } \ t \in [T_{\gamma,\lambda}, T_\infty), \tag{6.42}$$

where $k_2 \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{t \in [0, T_\infty)} \langle f(\,\cdot, t), \varphi_1 \rangle \in (0, \infty)$.

In particular, there are constants $\theta \in (0,1)$ and $C > 0$ such that

$$\|v^\top(\,\cdot, t)\|_{C^{1+\beta'}(\overline{\Omega})} < C \, M_1^\theta \, \gamma^{1-\theta} \quad \text{for all } \ t \in [T_{\gamma,\lambda}, T_\infty). \tag{6.43}$$

All constants k_1, k_2, M_1, θ, and C are independent of the choice of $\gamma \in (0, \gamma_1]$ and λ with $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$.

Proof. In view of (6.36) we write $v = \varphi_1 + v^\top$; i.e., $u(x,t) = \tau(t)\, v(x,t)$. Consequently, problem (1.2) yields the equation

$$\frac{\partial}{\partial t}\Big(\tau(t)\, v(\,\cdot\,,t)\Big) = |\tau(t)|^{p-2}\tau(t)\Big(\Delta_p v + \lambda\, |v|^{p-2} v\Big) + f(\,\cdot\,,t) \qquad \text{in } W^{-1,p'}(\Omega) \text{ for a.e. } t \in [T_{\gamma,\lambda}, T_\infty). \tag{6.44}$$

We take the inner product in $L^2(\Omega)$ of both sides of this equation with φ_1, thus arriving at

$$\begin{aligned} &\|\varphi_1\|^2_{L^2(\Omega)}\ \frac{\mathrm{d}}{\mathrm{d}t}\,\tau(t) \\ &= |\tau(t)|^{p-2}\tau(t)\Big(-\int_\Omega |\nabla v|^{p-2}(\nabla v\cdot\nabla\varphi_1)\,\mathrm{d}x + \lambda\int_\Omega |v|^{p-2} v\,\varphi_1\,\mathrm{d}x\Big) \\ &+ \langle f(\,\cdot\,,t),\varphi_1\rangle \qquad \text{for a.e. } \ t\in[T_{\gamma,\lambda},T_\infty). \end{aligned} \tag{6.45}$$

Estimates (6.25) and (6.26) from the proof of Corollary 6.6 now read as follows: There exists a constant $c > 0$, independent of the choice of numbers $\gamma \in (0,\gamma_1]$, λ with $\lambda_1 < \lambda < \lambda_1 + \delta_1(\gamma)$, and $t \in [T_{\gamma,\lambda}, T_\infty)$, such that for all such γ, λ, and t we have

$$\Big|\frac{1}{\lambda_1}\int_\Omega |\nabla v|^{p-2}(\nabla v\cdot\nabla\varphi_1)\,\mathrm{d}x - 1\Big| \le c\,\gamma, \tag{6.46}$$

$$\Big|\int_\Omega |v|^{p-2} v\,\varphi_1\,\mathrm{d}x - 1\Big| \le c\,\gamma. \tag{6.47}$$

(In their proofs, inequality (6.16) has to be replaced by (6.37).) We apply the last two inequalities to equation (6.45) to get

$$\|\varphi_1\|^2_{L^2(\Omega)}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\,\tau(t)\Big| \le |\tau(t)|^{p-1}\Big(\lambda - \lambda_1 + c(\lambda_1+\lambda)\gamma\Big) + \langle f(\,\cdot\,,t),\varphi_1\rangle$$

for almost every $t \in [T_{\gamma,\lambda}, T_\infty)$. Hence, we may take $k_1 = \delta_1(\gamma_1) + c\big(2\lambda_1 + \delta_1(\gamma_1)\big)\gamma_1$ to arrive at inequality (6.41).

In order to verify the regularity result (6.42), let us first note that $u \in C\big([0,T'] \to L^2(\Omega)\big) \cap L^p\big((0,T') \to W_0^{1,p}(\Omega)\big)$ holds for every $T' \in (0,T_\infty)$, because u is a weak solution to problem (1.2). By remarks following the definition of a weak solution to problem (1.2) in Section 2, one has also $u \in W^{1,p'}\big((0,T') \to W^{-1,p'}(\Omega)\big)$. This implies $\tau \in W^{1,p'}(0,T')$ for every $T' \in (0,T_\infty)$. Furthermore, by inequality (6.39), equation (6.44) is equivalent to

$$\tau(t)^{-(p-1)}\,\tau'(t)\, v(\,\cdot\,,t) + \tau(t)^{-(p-2)}\ \frac{\partial}{\partial t}\, v(\,\cdot\,,t) \tag{6.48}$$

$$= \Delta_p v + \lambda\, |v|^{p-2} v + \tau(t)^{-(p-1)}\, f(\,\cdot, t)$$

in $W^{-1,p'}(\Omega)$ for almost every $t \in [T_{\gamma,\lambda}, T_\infty)$. It follows from equation (6.41) that the coefficient $\kappa(t) \stackrel{\text{def}}{=} \tau(t)^{-(p-1)}\, \tau'(t)$ in the product $\kappa(t)\, v(\,\cdot, t)$ above is (essentially) bounded; more precisely, one has, for almost every $t \in [T_{\gamma,\lambda}, T_\infty)$,

$$|\kappa(t)| = \tau(t)^{-(p-1)}\, |\tau'(t)| \le k_3 \stackrel{\text{def}}{=} \|\varphi_1\|_{L^2(\Omega)}^{-2} \big(k_1 + k_2\, \underline{\tau}_1^{-(p-1)}\big), \tag{6.49}$$

by inequality (6.39). Next, we make the substitution

$$s \equiv s(t) \stackrel{\text{def}}{=} \int_{T_{\gamma,\lambda}}^{t} \tau(t)^{p-2}\, \mathrm{d}t \quad \text{ for all } \ t \in [T_{\gamma,\lambda}, T_\infty). \tag{6.50}$$

We claim that $s(t) \nearrow +\infty$ as $t \nearrow T_\infty$. Indeed, we first apply (6.49) to (6.50) to estimate

$$s \equiv s(t) \ge \frac{1}{k_3} \int_{T_{\gamma,\lambda}}^{t} \tau(t)^{-1}\, \tau'(t)\, \mathrm{d}t = \frac{1}{k_3} \cdot \log\Big(\tau(t)/\tau(T_{\gamma,\lambda})\Big)$$

for all $t \in [T_{\gamma,\lambda}, T_\infty)$. Then (6.38) forces $s(t) \nearrow +\infty$ as $t \nearrow T_\infty$. Let $\mathfrak{t} : \mathbb{R}_+ \to [T_{\gamma,\lambda}, T_\infty)$ denote the inverse function of $s : t \mapsto s(t)$. After this substitution, equation (6.48) reads

$$\begin{aligned} &\kappa(\mathfrak{t}(s))\, V(\,\cdot, s) + \frac{\partial}{\partial s} V(\,\cdot, s) \\ &\quad = \Delta_p V + \lambda\, |V|^{p-2} V + \tau(\mathfrak{t}(s))^{-(p-1)}\, f(\,\cdot, \mathfrak{t}(s)) \end{aligned} \tag{6.51}$$

in $W^{-1,p'}(\Omega)$ for $s \ge 0$, with the unknown function $V(x,s) = v(x, \mathfrak{t}(s))$ of $(x,s) \in \Omega \times \mathbb{R}_+$. From inequality (6.37) we deduce

$$\|V(\,\cdot, s)\|_{W_0^{1,p}(\Omega)} = \|\varphi_1 + v^\top(\,\cdot, \mathfrak{t}(s))\|_{W_0^{1,p}(\Omega)} < \lambda_1^{1/p} + \gamma \quad \text{ for all } \ s \ge 0.$$

We apply the same regularity arguments as in the proof of Proposition 5.5, but this time to equation (6.51), to conclude that

$$\|V(\,\cdot, s)\|_{C^{1+\beta}(\overline{\Omega})} \le M_1' \equiv \text{const} < \infty \quad \text{ for all } \ s \ge 0.$$

Since $v(x,t) = V(x, s(t))$, this proves the desired inequality (6.42) for all $t \in [T_{\gamma,\lambda}, T_\infty)$.

Finally, to verify inequality (6.43), we make use of the Gagliardo-Nirenberg inequality (4.1) (Lemma 4.2) to get

$$\|\nabla v^\top\|_{C^{\beta'}(\overline{\Omega})} \le C\, \|\nabla v^\top\|_{C^{\beta}(\overline{\Omega})}^{\theta}\, \|\nabla v^\top\|_{L^p(\Omega)}^{1-\theta}.$$

(The constants $C \in (0,\infty)$ and $\theta \in (0,1)$ depend solely upon $p \in (1,\infty)$, $0 < \beta' < \beta < 1$, and the domain Ω.) We apply inequalities (6.37) and (6.42) to the right-hand side to obtain (6.43).

The proof of Corollary 6.8 is now complete. □

6.4. **End of the proof (of the main result) for $T_\infty < \infty$.**

Proof of Theorem 3.3 *for* $T_\infty < \infty$. Let $u : \Omega \times (0, T_\infty) \to \mathbb{R}$ be a weak solution of problem (1.2); hence $p > 2$ and u is the unique solution of problem (1.2). By Proposition 5.5, Part (b), we know that $u \in C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [\varepsilon, T'])$ whenever $0 < \varepsilon < T' < T_\infty$, and there exists a monotone increasing function $\check{C}_\varepsilon : [\varepsilon, T_\infty) \to (0,\infty)$ such that (5.17) is valid. As mentioned right after Theorem 3.3, the desired inequalities (3.6) follow from (3.7). Furthermore, combining Lemma 6.7 (inequalities (6.37) and (6.39)) and Corollary 6.8 (inequality (6.43)), we easily arrive at (3.7), while using also the inequality $|v^\top(x,t)| \leq \frac{1}{2}\varphi_1(x)$ for all $x \in \Omega$ and $t \in [T_+, T_\infty)$. We prove the last claim below.

First, inequality (6.43) guarantees

$$\sup_{x\in\Omega} \frac{|v^\top(x,t)|}{\varphi_1(x)} \leq C' \, \|v^\top(\,\cdot\,,t)\|_{C^{1+\beta'}(\overline{\Omega})} \leq C' C M_1^\theta \, \gamma^{1-\theta} \tag{6.52}$$

for all $t \in [T_{\gamma,\lambda}, T_\infty)$, where the constant $C' \equiv C'(\Omega, p, \beta') > 0$ depends solely on Ω, p, and β', where $\beta' \in (0,\beta)$. Now, setting $\gamma_2 = \left(2\, C' C M_1^\theta\right)^{-1/(1-\theta)}$ and $\gamma_3 = \max\{\gamma_1, \gamma_2\}$ again, we can finish the proof of Theorem 3.3 for $T_\infty < \infty$ in much the same way as we have done for $T_\infty = \infty$ in §6.2. By the way, notice that the limit in (6.38) forces $\tau_*(t) \to +\infty$ $(t \to T_\infty-)$. □

Proof of Corollary 3.6. Formula (3.11) follows directly from (6.36), thanks to $T_+ \geq T_{\gamma,\lambda}$. Part (a) is an easy consequence of equation (6.45) combined with (6.46) and (6.47), supplemented by (6.38). Finally, Part (b) follows from the end of the proof of Theorem 3.3 for $T_\infty < \infty$ given above. □

APPENDIX A

A.1. **Local (in time) existence and/or uniqueness.** Here we sketch the main ideas how to obtain local (in time) existence and/or uniqueness of a weak solution to problem (1.2), cf. J.-L. Lions [21], I. I. Vrabie [34], or H. W. Alt and S. Luckhaus [2]:

For $\lambda \leq 0$, the nonlinear operator $\mathcal{A}_\lambda : u \mapsto \mathcal{A}_\lambda u \stackrel{\text{def}}{=} -\Delta_p u - \lambda\, |u|^{p-2} u$ is known to be continuous and maximal monotone from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$; see e.g. Barbu [7], Chapter II, §1, Section 3, pages 48–50. Hence, Theorem

4.2 (if $p \geq 2$) and the remark following its proof (if $1 < p < 2$) in [7], Chapter III, §4, Section 2, pages 166–168, guarantee the existence and uniqueness of a unique weak solution to problem (1.2) on the entire time interval $[0, T)$. Moreover, $\mathcal{A}_\lambda$ is also m-accretive in both $L^2(\Omega)$ and $L^p(\Omega)$, provided its domain is suitably chosen, such that $-\mathcal{A}_\lambda$ is the infinitesimal generator of a strongly continuous semigroup of (nonlinear) contractions on $L^2(\Omega)$ or $L^p(\Omega)$, respectively; cf. [7], Chapter II, §3, Section 2, pages 87–89. Alternatively, one may use the existence and uniqueness results obtained in J.-L. Lions [21], Chapter 2, Théorème 1.1 (page 156) or Théorème 1.2 (page 162) if $p \geq 2$, and Chapter 2, §1.5.2 (page 166) if $1 < p < 2$. Although most of these results are proved only for $\lambda = 0$, they are easily extendible to any $\lambda \leq 0$, by [21, Théorème 1.2, page 162].

For $\lambda > 0$, the nonlinear operator $\mathcal{A}_\lambda$ is no longer monotone or m-accretive and, therefore, we will view the term $\mathcal{B}_\lambda : u \mapsto \mathcal{B}_\lambda u \stackrel{\text{def}}{=} -\lambda\, |u|^{p-2} u$ in $\mathcal{A}_\lambda = \mathcal{A}_0 + \mathcal{B}_\lambda$ as a perturbation of $\mathcal{A}_0$. We consider bounded weak solutions only (locally in time, cf. Remark 2.3); we need to distinguish between the cases $p \geq 2$ and $1 < p < 2$.

The case $p \geq 2$. We take advantage of the facts that $\mathcal{A}_0 : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous and maximal monotone and the function $u \mapsto \mathcal{B}_\lambda u \stackrel{\text{def}}{=} -\lambda\, |u|^{p-2} u : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous. Given $\kappa \in (0, \infty)$, let $T_\kappa : \mathbb{R} \to \mathbb{R}$ denote the *truncation* function

$$T_\kappa(u) \stackrel{\text{def}}{=} \min\{\max\{-\kappa, u\},\, \kappa\} = \begin{cases} -\kappa & \text{if} \quad -\infty < u < -\kappa; \\ u & \text{if} \quad -\kappa \leq u \leq \kappa; \\ \kappa & \text{if} \quad \kappa < u < \infty. \end{cases}$$

Thus, $u \mapsto \mathcal{B}_\lambda(T_\kappa(u)) : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function. Hence, Theorem 4.2 from Barbu [7], Chapt. III, §4, Section2, page 167, can be easily combined with a standard Banach contraction principle for perturbed evolutionary equations, in the Banach space $C([0, T'] \to L^2(\Omega))$, $T' \in (0, \infty)$, in order to obtain a global (also in time) weak solution $u \equiv u_\kappa : \Omega \times (0, \infty) \to \mathbb{R}$ to the following *truncated* version of problem (1.2):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = \lambda\, |T_\kappa(u)|^{p-2} T_\kappa(u) + f(x,t), & (x,t) \in \Omega \times (0, \infty); \\ \quad u(x,t) = 0, & (x,t) \in \partial\Omega \times (0, \infty); \\ \quad u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{A.1}$$

Naturally, the contraction mapping in question is defined by $u \mapsto \tilde{u}$, where $u \in C\Big([0,T'] \to L^2(\Omega)\Big)$ is arbitrary and $\tilde{u}$ is the unique weak solution of

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - \Delta_p \tilde{u} = \lambda\, |T_\kappa(u)|^{p-2} T_\kappa(u) + f(x,t), & (x,t) \in \Omega \times (0,T'); \\ \quad \tilde{u}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T'); \\ \quad \tilde{u}(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{A.2}$$

Finally, we combine the facts that $f \in L^\infty(\Omega \times \mathbb{R}_+)$ and $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with the a priori estimates from Porzio [23], Theorem 2.1, page 1095, which are independent of $\kappa > 0$, to conclude that $|u_\kappa(x,t)| \le M(T) \equiv \text{const} < \infty$ holds for almost all $(x,t) \in \Omega \times (0,T)$ and for all $\kappa > 0$, provided $T \in (0,\infty)$ is small enough. It follows that $u = u_\kappa$ is a weak solution of problem (1.2) whenever $\kappa \ge M(T)$. This solution is unique since it is the unique fixed point of the contraction mapping $u \mapsto \tilde{u} : \mathcal{X} \to \mathcal{X}$ in the closed subset

$$\mathcal{X} = \Big\{ u \in C([0,T] \to L^2(\Omega)) : \ |u| \le M(T) \ \text{ a.e. in } \ \Omega \times (0,T) \Big\}$$

of the Banach space $C([0,T] \to L^2(\Omega))$, provided $T > 0$ is small enough.

The case $1 < p < 2$. In this case we cannot guarantee the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, nor is the operator $\mathcal{B}_\lambda$ locally Lipschitz continuous. Fortunately, in view of the remark made right after the proof of Theorem 4.2 in Barbu [7, page 168], this theorem ([7, page 167]) remains valid also for $1 < p < 2$. We combine this fact with a standard Schauder fixed-point argument for perturbed evolutionary equations, in the Banach space $C([0,T] \to L^p(\Omega))$, $T \in (0,\infty)$, in order to obtain a local (in time) weak solution $u : \Omega \times (0,T) \to \mathbb{R}$ to problem (1.2), provided $T > 0$ is small enough. We use the fixed-point mapping defined by $u \mapsto \tilde{u}$, where $u \in C([0,T] \to L^p(\Omega))$ is arbitrary and $\tilde{u}$ is the unique weak solution of

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - \Delta_p \tilde{u} = \lambda\, |u|^{p-2} u + f(x,t), & (x,t) \in \Omega \times (0,T'); \\ \quad \tilde{u}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T'); \\ \quad \tilde{u}(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{A.3}$$

We combine the facts that $f \in L^\infty(\Omega \times \mathbb{R}_+)$ and $u_0 \in W_0^{1,p}(\Omega)$ with inequality (5.12) in Lemma 5.2 to conclude that the mapping $u \mapsto \tilde{u}$ maps bounded sets from $C([0,T] \to L^p(\Omega))$ into bounded sets in $W^{1,2}([0,T] \to L^2(\Omega)) \cap L^\infty((0,T) \to W_0^{1,p}(\Omega))$. (We do not need $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in this argument.) This intersection being compactly embedded into $C([0,T] \to L^p(\Omega))$, $1 < p < 2$, by standard Sobolev embedding theorems (Adams and Fournier [1]), the mapping $u \mapsto \tilde{u}$ from $C([0,T] \to L^p(\Omega))$ into itself turns out to be *completely continuous* (i.e., continuous and mapping bounded sets into relatively compact sets). Finally, we take advantage of inequality (5.12) again to conclude that $u \mapsto \tilde{u}$ maps the closed subset

$$\mathcal{X} = \left\{ u \in C([0,T] \to L^p(\Omega)) : \, \|u(\,\cdot\,,t)\|_{L^p(\Omega)} \le R \text{ for a.e. } t \in (0,T) \right\}$$

of the Banach space $C([0,T] \to L^p(\Omega))$ into itself, provided $R > 0$ is large enough and $T > 0$ is small enough. Of course, a solution of problem (1.2) obtained by this fixed-point method might not be unique. We have however already seen in Section 6, §6.1, that if $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then we still have a *minimal solution* that is determined uniquely by its construction using a monotone iteration scheme starting from a subsolution to problem (1.2). Our hypotheses on f and u_0 (spelled out in Theorem 3.3) guarantee that the minimal solution is bounded in $\Omega \times (0,T')$ for every $T' \in (0,T_\infty)$, where $T_\infty = \infty$.

A.2. **A few geometric inequalities.** We begin with a few inequalities from [27, Lemma A.1, page 233]. They are used to estimate the (singular) integral in inequality (4.8).

Given $2 < p < \infty$, there exist constants $c_p > 0$ and $c_p' > 0$ such that the following inequalities hold for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$:

$$c_p \Big(\max_{0 \le s \le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2} \le \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2} \, \mathrm{d}s \le \Big(\max_{0 \le s \le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2}, \tag{A.4}$$

$$c_p' \Big(\max_{0 \le s \le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2} \le \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2} (1-s) \, \mathrm{d}s \le \frac{1}{2} \Big(\max_{0 \le s \le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2}. \tag{A.5}$$

For $1 < p < 2$, inequalities (A.4) and (A.5) are reversed [27, Lemma A.1, page 233]. There exist constants $C_p > 0$ and $C_p' > 0$ such that the following inequalities hold for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ with $|\mathbf{a}| + |\mathbf{b}| > 0$:

$$\Big(\max_{0 \le s \le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2} \le \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2} \, \mathrm{d}s \le C_p \cdot \Big(\max_{0 \le s \le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2}, \tag{A.6}$$

$$\frac{1}{2} \Big(\max_{0 \le s \le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2} \le \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2} (1-s) \, \mathrm{d}s \le C_p' \Big(\max_{0 \le s \le 1} |\mathbf{a} + s\mathbf{b}| \Big)^{p-2}. \tag{A.7}$$

Knowing inequalities (A.4) and (A.7) we can easily estimate the difference

$$|\mathbf{a}+\mathbf{b}|^{p-2}(\mathbf{a}+\mathbf{b}) - |\mathbf{a}|^{p-2}\mathbf{a} = \Big(\int_0^1 \mathbf{A}(\mathbf{a}+s\mathbf{b})\,\mathrm{d}s \Big)\mathbf{b} \tag{A.8}$$

for all $\mathbf{a},\mathbf{b} \in \mathbb{R}^N$ and for every $p \in (1,\infty)$, where $\mathbf{A}$ is the matrix-valued function

$$\mathbf{A}(\mathbf{a}) \stackrel{\mathrm{def}}{=} |\mathbf{a}|^{p-2}\Big(I + (p-2)\,\frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2} \Big) \quad \text{for } \ \mathbf{a}\in\mathbb{R}^N,\ \mathbf{a}\neq\mathbf{0}\in\mathbb{R}^N. \tag{A.9}$$

If $p > 2$, we set also $\mathbf{A}(\mathbf{0}) \stackrel{\mathrm{def}}{=} \mathbf{0} \in \mathbb{R}^{N\times N}$. For $\mathbf{a}\neq\mathbf{0}$, $\mathbf{A}(\mathbf{a})$ is a positive definite, symmetric matrix. The spectrum of $|\mathbf{a}|^{2-p}\mathbf{A}(\mathbf{a})$ consists of eigenvalues 1 and $p-1$, whence

$$\min\{1,\,p-1\} \le \frac{\langle \mathbf{A}(\mathbf{a})\mathbf{b},\mathbf{b}\rangle_{\mathbb{R}^N}}{|\mathbf{a}|^{p-2}|\mathbf{b}|^2} \le \max\{1,\,p-1\}, \quad \mathbf{a},\mathbf{b}\in\mathbb{R}^N\setminus\{\mathbf{0}\}. \tag{A.10}$$

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