

Fundamentals of
Structural Theories
and Their Numerical Analysis

Jaime Cervera Bravo

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Jaime Cervera Bravo

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Presentation

This volume, that is the english edition of Cervera Bravo 2025, seeks to serve as support for part of the presentations corresponding to the course “Fundamentals of Structural Theories and Their Numerical Calculation” of the University Master’s degree in Building Structures (*Master Universitario en Estructuras de Edificación* MUEE) taught at ETSAM, in its version updated since the 2020-2021 academic year, an update which, after considering the experience of previous courses, brought a notable reform to the trajectory accumulated since the start of this master’s degree, in the 2007-2008 academic year, under the direction of Ricardo Aroca.

The objective of the course is to present, in a format that tries to combine compactness with clarity, the fundamentals of deterministic and probabilistic methods of structural analysis, with special emphasis on the two most common approaches to deterministic static structural analysis: elastic analysis and limit analysis, showing the parallels and differences between them, as well as the techniques used to discretize a problem

that is generally defined in the continuum. Given its relevance in the professional framework, the techniques employed in the finite element method (FEM) are explored with special interest.

The course contains a brief introduction to probabilistic methods, although it is fundamentally focused on the deterministic approach.

The text includes theoretical bases and the development of some numerical examples, as well as, occasionally, the statement of groups of questions whose answer should be evident to those who have assimilated the content of the preceding sections.

The course, and this text, presuppose a prior basic training in mathematics (differential and integral calculus, as well as algebra and matrix calculus), physics (statics and kinematics), strength of materials (beam bending), and elements of elasticity (stress and strain states), which are usual in the training curricula of the degrees that have access to the master's.

Chapter 1

Basic Theory of Structures

1.1 Introduction

1.1.1 Theories in the Analysis and Simulation of Complex Phenomena

Every complex object or phenomenon manifests observable (measurable) behaviors that, in general, are not immediately or easily predictable. Examples in the field of structures could be the probability of breaking a bar in a building, or the “usability” of a novel design walkway in a gorge subjected to snow and strong winds. And although, for many questions and for a long time,

humanity has employed as a method the combination of emulating and modifying pre-existing objects along with the trial and correction of errors, that is, the prototyping and testing method, both curiosity and efficiency in resource use have led to the progressive creation of theories that aim to facilitate the explanation and prediction of real-world behavior through its theoretical simulation, thereby reducing the number of prototypes necessary to reach a correct final result.

Moreover, from the moment theories are available that allow relating simple observations to be obtained with much more complex behaviors, the realization of physical prototypes ceases to make sense, being replaced by their theoretical modeling. It is this modeling or theoretical “prototyping” that we address in this text.

1.1.2 The Systemic Content of Theories

Every theory is a mental construction, abstract and free, whose object is to emulate the observed behavior of elements or systems of the real world. As a theory, it is neither more nor less than an autonomous system, freely constructed from the minds of its authors, and radically separate and different from the real system it seeks to simulate.

Between interacting systems, *structural coupling* (Luhmann 1996) is called the process by which these configure their behavior in such a way that changes in one of the systems entail predictable or parallel changes in the other (or others). This coupling or adaptation occurs in its long-term interaction process. From this perspective, a good theory is none other than

the one that has achieved a considerable degree of structural coupling with the real-world system it seeks to emulate. And to do this, in its construction process, it has sought to relate elements of the theory with observable elements of the real system, and has progressively adapted their mutual relationship, so that the response of both systems can be placed in parallel, and with this, the future real response to a given action can be predicted from its theoretical emulation. Steps of this coupling can be both alterations in the theory to better approach its emulation objective, and alterations in reality that try to make produced objects better resemble their theoretical parallels: an exemplary case of this second step is that of trusses that, in some cases throughout history, have been made with members linked with physical articulations in their real joints to ensure a real behavior similar to their theoretical articulated emulation.

Every theory contains elements directly related to observable real elements, but it also contains theoretical elements of a more “obscure” nature, not directly observable in reality, for which there might be a temptation to attribute a physical reality. Although such elements are only intelligible from within the theory itself, they can have a crucial value in its interpretation and therefore in its explanatory capacity, whether or not there are real parallels to these elements. We will not be concerned with these issues here, despite their undoubted interest in theoretical physics.

Equally, every theory “ignores” elements of the real world that it considers irrelevant to its objectives, but which, nonetheless, could be important in particular situations. An example may be such of “color” in structural theory: it is not an element

of this. But, however, it can become a relevant issue for a structure located in a situation exposed to intense solar radiation.

It is for this reason that the two systems must never be confused: the real and the one constructed by the theory, always leaving room for doubt, or the question of whether we are adequately representing the problem to be addressed.

1.1.3 Models in Structural Analysis Theories

Models in structural theories are, like in any other theory, information models. They present, in analytical format (functions, etc.) or numeric format (lists of variables and their values), qualities associated with the geometric descriptions representing the emulated object. These models contain, as crucial elements, the following components, ordered from the most to the least easily relatable to real observable elements:

- *Geometry*: defines the spatial configuration of the structure, identifying the position of every point in the structure using a Cartesian *coordinate* system.
- *Kinematics*: describes the movement performed by the structure due to the action of loads, given by *displacements* or movements derived from *internal deformations*. This description explicitly or implicitly includes constraints on movements in the structure's support regions.
- *Statics*: describes the different *loads* and *external forces* (including *reactions*) applied to the structure, as well as the *internal stresses* by which the structure links and balances these forces.

- *Material Relations or Constitutive Relations*: those that establish some type of relationship between the static and kinematic components of the model through hypotheses about the behavior of the structure's materials, such as Hooke's law in elastic models.

For each structural type, appropriate elements for this representation are chosen, which we will detail more precisely in subsequent sections. Before delving into the matter, it is interesting to identify the two professional activity areas supported by this theoretical representation of the structural field.

1.1.4 Professional Objectives: Analysis and Conception, Assessment and Design

In engineering and architecture tasks, two types of objectives are usually addressed: on one side, evaluating the suitability of a specific object for its use (in the case of structures, the reliability or degree of confidence with which its safety of use, resistance, and stability will be maintained, etc.) and on the other side, the definition and proposal of new objects sufficiently reliable for production. We are therefore talking about two different, yet linked, activity areas:

- *Analysis*, or assessment of an already conceived or pre-existing object,
- Conception, or the *project* or *design* of a new, non-existent object.

These are two completely differentiated objectives: in the first case, the object exists to some extent, and the goal is to

evaluate its validity or suitability, its capacity to be used without damage, the reliability of its future behavior. In the second case, the aim is to propose (specify) a new object that will prove sufficiently reliable in its future use.

It is evident that in the project process, analysis episodes of different versions or prototypes of the proposed object may be intercalated, in order to validate its suitability, but the objective is not the analysis itself, but the proposal that will be derived from all the activity.

To the extent that *analysis* is capable of replacing the evaluation of physical prototypes with mathematical (geometric and numeric) models, a good part of the *project* process uses analysis as an auxiliary tool, but this should not obscure the radically different objectives in both activities. And this becomes manifest when we consider the very different perspective from which analysis is being used depending on the intended objective. Let's consider, for example, this difference when we are trying to propose a new structure for a footbridge, or evaluating the suitability of the structure of an existing footbridge: in the first case, the model will try to represent the anticipated structure, and we will seek that the reliability of the result surpasses or at least equals the socially required standard. In this case, it would suffice if some of the acceptable models representing the projected object provides justification of such reliability. Perhaps the model employs simplifications that provide added over-safety, but this would not be relevant: it might involve project cost overruns compared to results that could be validated by more sophisticated models, but the objective would be covered: simplifying, a single model would be enough to justify

the proposal.

However, in the second case, we are trying to justify the suitability of a pre-existing object. Here too we can analyze this object using a simplified method, and likewise, if verified according to it, we could validate that object. A single model would have sufficed. But if that model does not achieve the intended validation, it is not necessarily concluded that the object is invalid: perhaps a second, more sophisticated model could indeed ensure such validity: the first model can provide relevant clues about the object's validity, but may not be sufficient to decree its invalidity.

Therefore, although *analysis* is part of the workflow in both types of activities, the way of interpreting its information is not necessarily equivalent. Not only that, we will also see later that the way in which the different variables involved in the theoretical modeling are handled can be very different depending on whether the work objective is *analysis* or *project*.

1.2 Models and Fundamental Equations

1.2.1 Basic Model Parameters

As we have seen in 1.1.3, we consider the following elements in a structural model: (figure 1.1, page 8)

- *Geometry*: defines the spatial configuration of the structure and, therefore, identifies the position of every point of the structure using a Cartesian *coordinate* system, symbolically

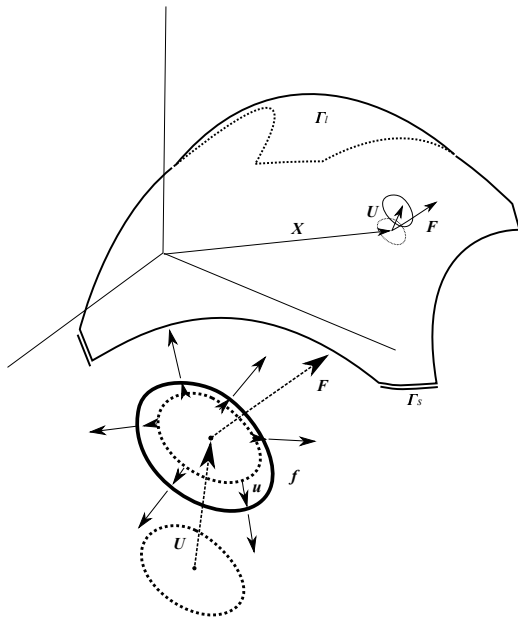


Figure 1.1: Structural models: geometry, and global or local parameters for kinematics and statics. Global view and enlargement of a local view.

represented¹ with $\mathbf{X} = \vec{X} = [x_1, x_2, x_3]^T$. This configura-

1. Symbols in normal typography correspond to single-valued variables or functions of one dimension, while bold symbols are used to identify multidimensional elements, whether vectors or matrices. Additionally, up-

tion occupies a *volume* Ω ; $\mathbf{X} \in \Omega$, whose *boundary* or outer surface is represented by Γ , a surface that can be separated into two parts, $\Gamma = \Gamma_l \cup \Gamma_s$, where Γ_l collects the *free* part, perhaps loaded, but not impeded in its movement, and Γ_s identifies the *supported* part, situated in contact with elements external to the structure that restrict or prevent its movement (foundation). Based on this geometry, the following are also defined

- *Kinematics*: describes the movement performed by the structure due to the action of loads, identified by the *displacements* $\mathbf{U}$ of each point of interest in the structure –within the volume Ω , or on its surface Γ –. These are potentially *free* (only constrained by the rest of the structure) in the region Γ_l , but are *impeded* or constrained in the *support*, Γ_s . These are movements that respond to *internal deformations* $\mathbf{u}$: elongations or shortenings, relative rotations of sections near each other, etc.
- *Statics*: describes on one side the different *external forces* applied to the structure, $\mathbf{F}$, either as *loads* in free regions Γ_l : $\mathbf{F}_l$, or as *reactions* $\mathbf{F}_s$ in support regions Γ_s ; on the other side it considers the *internal stresses* $\mathbf{f}$ through which the structure relates and balances loads and reactions.
- *Material Relations*, also called “constitutive” relations or the relations of the “constitution” of the materials: those

uppercase symbols will be used for elements described in general or global coordinates, while lowercase symbols refer to elements described in local coordinates, thus viewed from a local perspective.

that establish some type of linkage between the values of static and kinematic components of the model through hypotheses about the behavior of the materials in each region or point of the structure. For example, in the elastic approach, we would be talking about stress-strain relations in some version of Hooke's law.

In figure 1.2, some examples of these basic parameters and their relationships are detailed for different observation scales.

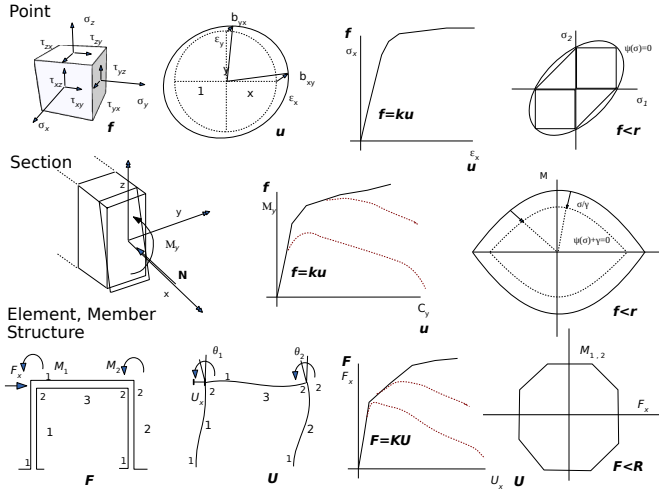


Figure 1.2: Typical Structural Models

Several points of interest can be added here.

- Although we have not yet detailed how the kinematic vari-

ables $\mathbf{U}, \mathbf{u}$ or the static variables $\mathbf{F}, \mathbf{f}$ should be identified, in this notation we are considering a *discrete* description, that is, given by lists of values, perhaps long but finite, such as for $\mathbf{U}$, the horizontal and vertical displacement values of N nodes in a plane truss:

$$\mathbf{U} = [U_{1x} \ U_{1y} \ U_{2x} \ U_{2y} \ \dots \ U_{Nx} \ U_{Ny}]^T,$$

and the corresponding “elongations” of the n bars of this:

$$\mathbf{u} = [u_1 \ u_2 \ \dots \ u_n]^T.$$

The detail will depend on how each structural type is modeled, but in this detail the following is crucial:

- Modeling implies a parallel definition of static and kinematic variables, in such a way that the scalar products $\mathbf{U}^T \mathbf{F}$ and $\mathbf{u}^T \mathbf{f}$ make sense, that is, they are ordered in such a way that these formulas respectively express the *external work* W and the *internal work* w that the *forces* or *stresses* $\mathbf{F}, \mathbf{f}$ would do in the *movement* and *deformation* described by $\mathbf{U}, \mathbf{u}$. In other words, the static and kinematic variables must be energetic conjugates of each other. To do this, variables and their ordering must be appropriately matched, so for the truss in the preceding paragraph we would have as the “vector” of loads applied to the nodes: $\mathbf{F} = [F_{1x} \ F_{1y} \ F_{2x} \ F_{2y} \ \dots \ F_{Nx} \ F_{Ny}]^T$, and as the “vector” of internal efforts the truss bar tensions, $\mathbf{f} = [f_1 \ f_2 \ \dots \ f_n]^T$, using negative values if they were compressions.
- If we intend to model a continuous distribution for kinematic or static variables, we would be using point functions both for movements $\mathbf{v}(\mathbf{X})$ and deformations $\boldsymbol{\epsilon}(\mathbf{X})$ as for external forces, loads and reactions, $\boldsymbol{\tau}(\mathbf{X})$ and internal efforts

$\boldsymbol{\sigma}(\mathbf{X})^2$; in this case, work would be measured by transforming sums into integrals, with $W = \int_{\Gamma} \mathbf{v}^T \boldsymbol{\tau} \, d\Gamma$ for external work if we consider that all external forces are applied to the outer surface, and $w = \int_{\Omega} \boldsymbol{\epsilon}^T \boldsymbol{\sigma} \, d\Omega$ for internal work.

- Finally, the feasibility of the structure requires not exceeding certain limit conditions, both in the kinematic and static components. It is usual to consider limits $\mathbf{Y}$ for global displacements $\mathbf{U}$, as well as strength constraints $\mathbf{r}$ for internal efforts $\mathbf{f}$, using various formats to define such limits: $\mathbf{U} < \mathbf{Y}$, $\mathbf{f} < \mathbf{r}$. In general, kinematic constraints are posed on global movements while static constraints (strengths) are posed on local efforts, although they can obviously be rewritten in terms of other parameters: an example is the usual case of rewriting strength constraints in terms of limit loads: $\mathbf{F} \leq \mathbf{R}$. We will see some of these conditions in more detail later.

It is worth reiterating that the adopted nomenclature applies uppercase to external or global variables (loads, reactions, structural displacements, or external work) and lowercase to internal or local variables (efforts, deformations, internal work).

1.2.2 Fundamental Equations

These are the result of establishing correspondence conditions between the three model components, although the first two can be established without considering the local material behavior of the structure, which only affects the third group.

2. we use Greek letters for the nomenclature of such functions

- *Kinematics*: ensuring correspondence between displacements $\mathbf{U}$ and deformations $\mathbf{u}$ gives rise to the *compatibility equations* (COM) established so that the deformations of a region can be obtained from the differences between displacements of nearby points. Its general format is therefore of the form

$$\mathbf{u} = \mathbf{B}\mathbf{U} \quad (1.1)$$

where $\mathbf{B}$ is a coefficient matrix that expresses these “differences”. See examples in section 1.3.7.2.

- *Statics*: ensuring correspondence between forces $\mathbf{F}$ and internal efforts $\mathbf{f}$ gives rise to the *equilibrium equations* (EQU), by which the forces applied to a region must be in equilibrium with the effort differences between the previous and subsequent sections delimiting that region. It is therefore an equation in the form

$$\mathbf{F} = \mathbf{H}\mathbf{f} \quad (1.2)$$

where $\mathbf{H}$ expresses these differences. See examples in section 1.3.7.1.

- *Material Relations*: also called *constitutive* relations of the material (MAT), are those that establish the correspondence between the internal effort values $\mathbf{f}$ associated with the internal deformations $\mathbf{u}$ of the points or regions considered for the adopted material model. A general form could be

$$\mathbf{f} = \mathbf{f}(\mathbf{u}) \quad (1.3)$$

although the detail of this equation and its treatment is what will differentiate one model from another. We will see examples in section 1.3.7.3.

It is usual to consider free and supported regions separately, so that in the compatibility and equilibrium equations the action or reaction variables can be identified and made independent:

$$\mathbf{u} = \begin{bmatrix} \mathbf{B}_l & \mathbf{B}_s \end{bmatrix} \begin{bmatrix} \mathbf{U}_l \\ \mathbf{U}_s \end{bmatrix} \quad (1.4)$$

$$\begin{bmatrix} \mathbf{F}_l \\ \mathbf{F}_s \end{bmatrix} = \begin{bmatrix} \mathbf{H}_l \\ \mathbf{H}_s \end{bmatrix} \mathbf{f} \quad (1.5)$$

In general $\mathbf{F}_l$ and $\mathbf{U}_s$ are known conditions, data, being loads applied in free regions, and movements (typically impeded, generally zero) in the support, while $\mathbf{F}_s$ and $\mathbf{U}_l$ are unknowns: the reactions in support and the displacements of the structure. Each variable in the list of $\mathbf{U}_l$ corresponds to what is called a *degree of freedom* as it describes a possible independent movement, such that the general movement is the combination of all these possibilities, the more detailed the more variables are in that list.

The number of variables in the list of $\mathbf{U}_s$ establishes the number of *constraints* or *support conditions* of the model.

1.2.3 Work and Energy

We have seen that we organize static and kinematic variables so that their scalar products have the meaning of work, external or internal, depending on the components considered. If we consider two independent states, with kinematics $\bar{\mathbf{U}}, \bar{\mathbf{u}}$ for the first and statics $\mathbf{F}, \mathbf{f}$ for the second, we have that the total work of the second's forces performing the first's movements can be

measured, from external or internal perspectives, with:

$$W = \bar{\mathbf{U}}^T \mathbf{F} \quad (1.6)$$

$$w = \bar{\mathbf{u}}^T \mathbf{f} \quad (1.7)$$

where, in the first expression, we can separate the load and reaction components with

$$W = \begin{bmatrix} \bar{\mathbf{U}}_l^T & \bar{\mathbf{U}}_s^T \end{bmatrix} \begin{bmatrix} \mathbf{F}_l \\ \mathbf{F}_s \end{bmatrix} \quad (1.8)$$

In the case where $\mathbf{U}, \mathbf{u}$ and $\mathbf{F}, \mathbf{f}$ respond to the same system, we can consider this total work $W = \mathbf{U}^T \mathbf{F}$ as a measure of the loss of *potential energy* of the loads from the undeformed to the deformed position of the structure, even though the consideration of how the load is actually applied may involve differences in the deformation process.

For example, suppose the load is applied gradually to a linear elastic structure, through a scale factor γ , varying between 0 and 1. As the deformation is proportional to the load, the *elastic deformation energy* accumulated in each process step will be equal to the actual potential energy loss of the existing load, and considering the proportionality by which $d\mathbf{U} = \mathbf{U}d\gamma$, such energy would reach the value:

$$W_d = \int_0^{\mathbf{U}} d\mathbf{U}^T \gamma \mathbf{F} = \int_0^1 \mathbf{U}^T \mathbf{F} \gamma d\gamma = \frac{1}{2} \mathbf{U}^T \mathbf{F} \quad (1.9)$$

or also, viewed from internal variables in the same elastic case

$$w_d = \int_0^{\mathbf{u}} d\mathbf{u}^T \mathbf{f}(\mathbf{u}) = \frac{1}{2} \mathbf{u}^T \mathbf{f} \quad (1.10)$$

The complementary work or *complementary energy* is the magnitude that, when added to the deformation work, leads to the total work. It is the integral now performed with loads as an independent function. The differentials are now

$$dW_c = \mathbf{U}^T(\mathbf{F})d\mathbf{F} \quad (1.11)$$

$$dw_c = \mathbf{u}^T(\mathbf{f})d\mathbf{f} \quad (1.12)$$

The differential of total work will be the sum of the differentials of deformation and complementary work. Considering internal expressions, we have

$$w = \mathbf{u}^T \mathbf{f} \quad (1.13)$$

$$dw = d\mathbf{u}^T \mathbf{f} + \mathbf{u}^T d\mathbf{f} \quad (1.14)$$

$$w = \int_0^{\mathbf{u}} d\mathbf{u}^T \mathbf{f}(\mathbf{u}) + \int_0^{\mathbf{f}} \mathbf{u}(\mathbf{f})^T d\mathbf{f} \quad (1.15)$$

$$w = w_d + w_c \quad (1.16)$$

It is clear that in linear elastic systems both values w_d and w_c are equal, each being half of the total w , but this will not be the case in systems where material behavior is not linear.

1.2.4 Work Theorems

Although there are many precedents, we will only handle the principle of virtual work (principle, or theorem, depending on which theoretical basis is chosen as fundamental: that is, whether it is a principle that is not doubted and on which the rest of the

discourse is based, or whether it is a deduction obtained from more elementary principles).

Among the antecedents would be the principle of least action, which leads to considering balanced states as those corresponding to minimum total energy, as well as classical theorems, among which Castigliano's theorems (1873) are perhaps the most extensive, with the definitions:

$$\mathbf{f} = \frac{\partial w_d}{\partial \mathbf{u}}; \quad \mathbf{u} = \frac{\partial w_c}{\partial \mathbf{f}} \quad (1.17)$$

These would express:

- 1 if we are able to describe complementary work in terms of internal deformation variables, each effort can be obtained as partial derivatives of such complementary work with respect to these variables,
- 2 if we are able to describe deformation work in terms of internal effort variables, each deformation is obtained from the partial derivatives of such deformation work with respect to such efforts.

The *principle of virtual work* establishes that an equilibrium system $(\tilde{\mathbf{F}}; \tilde{\mathbf{f}})_{EQU}$ is subjected to a compatible movement $(\bar{\mathbf{U}}; \bar{\mathbf{u}})_{COM}$ if and only if the work developed in such movement is zero:

$$-W + w = -\bar{\mathbf{U}}^T \tilde{\mathbf{F}} + \bar{\mathbf{u}}^T \tilde{\mathbf{f}} = 0; \quad (1.18)$$

$$W = \bar{\mathbf{U}}^T \tilde{\mathbf{F}} = w = \bar{\mathbf{u}}^T \tilde{\mathbf{f}}. \quad (1.19)$$

That is, regardless of what the interpretation of the signs is, which are established so that energy lost by loads results stored

as internal deformation energy, the principle postulates that external work equals internal work.

We adopt this as a principle, although there are classic demonstrations such as Jean Bernoulli's letter to Varignon from 1717: *if a system of particles is in equilibrium, each particle is in equilibrium, so the resultant force on each of them is zero, and the work done by displacing such particles will be zero.* The demonstration handled in classical elasticity theory is also classic, which uses the Green–Gauss theorem to transform the volume integral of internal deformation work to the surface integral of work of applied loads.

This principle is of maximum utility in structural theory formulation, as we will be able to verify later.

1.2.5 Static-Kinematic Duality

If we consider the intersection of the virtual work principle expressions, 1.19, with the compatibility (COM) and equilibrium (EQU) conditions expressed by equations 1.1 and 1.2 we have

$$\begin{aligned} \bar{\mathbf{U}}^T \tilde{\mathbf{F}} = \bar{\mathbf{U}}^T \mathbf{H} \tilde{\mathbf{f}} = \bar{\mathbf{u}}^T \tilde{\mathbf{f}} = \bar{\mathbf{U}}^T \mathbf{B}^T \tilde{\mathbf{f}} \\ \forall (\bar{\mathbf{U}}, \bar{\mathbf{u}})_{\text{COM}}, (\tilde{\mathbf{F}}, \tilde{\mathbf{f}})_{\text{EQU}} \end{aligned} \quad (1.20)$$

from which it is deduced that, necessarily,

$$\mathbf{H} = \mathbf{B}^T \quad (1.21)$$

This expression shows us that the equilibrium or compatibility versions of a system's behavior are nothing more than two

alternative views or representations of the same phenomenon, once we have decided to represent that phenomenon through the scalar magnitude we call work. It is this duality that will allow us to approach structural analysis problems from one or the other perspective. And even though the equilibrium perspective is one of the most common in training, often the compatibility perspective is the simplest, insofar as it involves magnitudes or phenomena that can be drawn, visible in physical objects, whereas forces and efforts are usually interpreted as less evident magnitudes to visualize.

1.3 Elastic Analysis and Design

In this section we will consider the solution to problems of *analysis* or *design(project)* under linear-elastic behavior conditions, that is, in cases where material laws establish stresses directly proportional to deformations. This corresponds to cases in which

- small deformations and displacements are considered, such that, it can be considered approximately that the displacement of loads in these movements does not affect equilibrium conditions
- material responses are strictly following Hooke's law, so that stresses vary in the same proportion as deformations, and can be determined from these using appropriate elastic constants.

This therefore requires restricting the material equations 1.3 to their linear form.

1.3.1 Material or Constitutive Relations

With a linear relationship between internal stresses $\mathbf{f}$ and deformations $\mathbf{u}$, it can be expressed in the classic form corresponding to the “Hooke's law”, where expression 1.3 takes the linear form

$$\mathbf{f} = \mathbf{k}\mathbf{u} \tag{1.22}$$

In this expression, the coefficients of $\mathbf{k}$ will be given by the characteristics of the material and sections used, measuring the contribution of the unit value of each term of $\mathbf{u}$ to each term

of $\mathbf{f}$. Generally we will be considering as variables for $\mathbf{u}$ and $\mathbf{f}$ parameters of connected but physically independent regions of the structure —bars, for example— so that, for the complete structure, $\mathbf{k}$ will be a block diagonal matrix, with non-zero terms only in positions relating deformation and stress parameters of each of these independent regions.

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{bmatrix} = \begin{bmatrix} \mathbf{k}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{k}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = \mathbf{k}\mathbf{u} \quad (1.23)$$

In the case of truss structures with pinned bars, each term f_i of $\mathbf{f}$ would be the axial force of bar i , and each term u_i of $\mathbf{u}$ would be the total elongation of bar i , so that each term $\mathbf{k}_i$ would be nothing more than the corresponding axial stiffness coefficient of the bar considered, $E_i A_i / l_i$, product of the elastic modulus by the area of said bar divided by its length. In section 2.1.1 we will see modeling cases for different types of bars.

1.3.2 Summary of Fundamental Variables, Equations and Their Solution

We have seen that variables can be associated, either with the components of the structure considered isolately or locally (elements or pieces), with $\mathbf{u}, \mathbf{f}$ for kinematic and static variables respectively, or with the complete structural model, or global model, with $\mathbf{U}, \mathbf{F}$ for such variables. In our global model, the unknowns are the displacements of “free” points $\mathbf{U}_l$ and the

reactions —forces— $\mathbf{F}_s$ of the supported points. If for the moment we assume that the supported points do not move, thus with external work being zero at these points, we can disregard them, identifying $\mathbf{U}, \mathbf{F}$ with $\mathbf{U}_l, \mathbf{F}_l$. Now, the list, or the number of elements, whether of $\mathbf{U}$ or of $\mathbf{F}$, identifies the DOF of the model.

This list, or this number of DOFs, limits the different possibilities of the deformed geometry configuration, or of the equilibrium conditions, while each isolated element of this —each movement $\mathbf{U}_i = [0, 0, \dots, 0, U_i, 0 \dots 0]^T$ — can be considered an element of the base that describes the (vector) space of all possible movements the model can represent. A space of DOF dimensions.

We have seen the three groups of equations that relate the fundamental variables of the model, the local and global for the first two groups, and the static with the kinematic for the third, namely:

- Compatibility for kinematic variables, equation 1.1
- Equilibrium for static variables, equation 1.2
- Material constitution, equation 1.3, or its linear form, as in 1.22.

Finally, we have seen that the energy formulation (work) viewed from the local variable group or from the global variables allows us to ensure the duality described in equation 1.21.

With this, we have sufficient means for elastic analysis or design: we would only need to be able to solve the $\mathbf{U}$ unknowns of the problem.

To do this we postulate a non-singular linear relation $\mathbf{K}$,

still to be determined, between displacements $\mathbf{U}$ and loads $\mathbf{F}$, we consider jointly the equations expressing duality 1.21 along with equilibrium 1.2 and compatibility 1.1, and can therefore write successively

$$\begin{aligned}\mathbf{F} &= \mathbf{K}\mathbf{U} \\ \mathbf{F} &= \mathbf{B}^T \mathbf{f} \\ \mathbf{F} &= \mathbf{B}^T \mathbf{k}\mathbf{u}\end{aligned}\tag{1.24}$$

so that for any arbitrary movement $\bar{\mathbf{U}}$ independent of $\mathbf{U}$ must result

$$\bar{\mathbf{U}}^T \mathbf{F} = \bar{\mathbf{U}}^T \mathbf{B}^T \mathbf{k}\mathbf{u}$$

Substituting both $\mathbf{F}$ and $\mathbf{u}$ by their expressions in $\mathbf{U}$ we have

$$\bar{\mathbf{U}}^T \mathbf{K}\mathbf{U} = \bar{\mathbf{U}}^T \mathbf{B}^T \mathbf{k}\mathbf{B}\mathbf{U}\tag{1.25}$$

and as this must be valid for any movement conditions $\bar{\mathbf{U}}$ and $\mathbf{U}$, this equality is only possible if, finally,

$$\mathbf{K} = \mathbf{B}^T \mathbf{k}\mathbf{B}\tag{1.26}$$

$\mathbf{K}$ is the *stiffness matrix* of the structure.

Therefore the movement unknowns $\mathbf{U}$, for a model with stiffnesses $\mathbf{k}$ and geometry leading to compatibility conditions $\mathbf{B}$ and equilibrium $\mathbf{H} = \mathbf{B}^T$, can be determined with

$$\mathbf{U} = \mathbf{K}^{-1} \mathbf{F}\tag{1.27}$$

where we call the *flexibility matrix* the matrix $\mathbf{K}^{-1}$, inverse of the stiffness matrix $\mathbf{K}$. Once the displacements $\mathbf{U}$ are obtained,

the deformations $\mathbf{u}$ are determined by compatibility (1.1), the stresses $\mathbf{f}$ from the material conditions for these deformations (1.22), and the reactions $\mathbf{F}_s$ by equilibrium with the stresses of the support regions, as we now see.

Note that if we considered the total global “nodes”, both free and supported, we could order the global material equations in the form:

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_l \\ \mathbf{F}_s \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ll} & \mathbf{K}_{ls} \\ \mathbf{K}_{sl} & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{U}_l \\ \mathbf{U}_s \end{bmatrix} = \mathbf{K}\mathbf{U} \quad (1.28)$$

In this equation, the different blocks express the relationships between loads (or reactions) and movements of free or support regions. Up to now we have considered that supports do not move, with $\mathbf{U}_s = \mathbf{0}$, which has allowed us to ignore the components with subscript s : we have solved $\mathbf{U}_l$ from the expression $\mathbf{F}_l = \mathbf{K}_{ll}\mathbf{U}_l$, implicitly identifying $\mathbf{K}$ with $\mathbf{K}_{ll}$ where dimension l is the number of degrees of freedom DOF of the model.

The determination of free node displacements $\mathbf{U}_l$ now allows us to determine the reactions

$$\mathbf{F}_s = \mathbf{K}_{sl}\mathbf{U}_l + \mathbf{K}_{ss}\mathbf{U}_s \quad (1.29)$$

where again the second term will be cancelled.

It should be noted that the complete matrix $\mathbf{K}$, including support terms, is NOT invertible since rigid body displacements or rotations could occur without deformation: these would include movements at supported points and therefore movements for situations without load. The complete matrix would be singular, with null determinant. Models must necessarily include sufficient movement constraint conditions to prevent such

rigid movements. These constraints can be such that they completely prevent movement, $\mathbf{U}_s = \mathbf{0}$, or can be alternatively established through what might be called non-null forced movements $\mathbf{U}_s \neq \mathbf{0}$.

In such a case the movements are problem data, and what must be done is to rearrange the “free” part of the 1.28 equation in the following way

$$\mathbf{F}_l - \mathbf{K}_{ls}\mathbf{U}_s = \mathbf{K}_{ll}\mathbf{U}_l \quad (1.30)$$

In practical terms, the second (negative) term of the first term in the equation is nothing more than the set of forces that would be necessary to apply locally to the regions in contact with the supports so that, without any additional deformation in the rest of the structure, they can accumulate the deformation demanded by the anticipated forced displacements $\mathbf{U}_s$. In such a way that they are treated as “equivalent loads” without the need to alter the model.

1.3.3 The Elastic Model at a Glance

With all the above, the set of considerations made up to here can be condensed into a diagram that allows putting them all

into relation.

$$\begin{array}{ccccc}
 \mathbf{u} & \longleftarrow & \text{COM}(\mathbf{B}) & \longleftarrow & \mathbf{U} \langle (\Phi, \mathbf{P}) \mathbf{\Upsilon} \\
 & \searrow & \updownarrow & \swarrow & \\
 \text{MAT } \mathbf{k} \downarrow & \text{Internal work} & = & \text{External work} & \downarrow \mathbf{K}_{\text{GLOB\_MAT}} \\
 & \nearrow & \updownarrow & \nwarrow & \\
 \mathbf{r} \rangle (d, \psi) \mathbf{f} & \longrightarrow & \text{EQU}(\mathbf{H} = \mathbf{B}^T) & \longrightarrow & \mathbf{F}
 \end{array} \tag{1.31}$$

A diagram to which has been added the possible global movement restrictions $\mathbf{\Upsilon}$ as well as local strength restrictions $\mathbf{r}$ that are used to compare movements or efforts using linear expressions of the form

$$\Phi \mathbf{U} \leq \mathbf{P} \mathbf{\Upsilon} \tag{1.32}$$

$$\psi \mathbf{f} \leq d \mathbf{r} \tag{1.33}$$

The restrictions are established for displacements $\mathbf{U}$ or internal efforts $\mathbf{f}$ selected individually or in some of their combinations (Φ, ψ) and bounded by predetermined limit values $(\mathbf{\Upsilon}, \mathbf{r})$, which are assigned individually or combinedly to each comparison (each equation) through attribution coefficients $\mathbf{P}, d$.

In the diagram, the symbols must be interpreted according to the following equivalences:

$$\mathbf{x} \langle (\mathbf{f}, \mathbf{g}) \mathbf{y} \equiv \mathbf{f} \cdot \mathbf{x} \leq \mathbf{g} \cdot \mathbf{y}; \tag{s.1}$$

which expresses the restrictions for $\mathbf{x}$ from the constants $\mathbf{y}$,

$$\mathbf{x} \longrightarrow \text{TAG}(\mathbf{y}) \longrightarrow \mathbf{z} \equiv \begin{array}{c} \mathbf{x} \\ \text{TAG}(\mathbf{y}) \downarrow \\ \mathbf{z} \end{array} \equiv \mathbf{y} \cdot \mathbf{x} = \mathbf{z}, \text{ (TAG); } \tag{s.2}$$

which expresses the deduction of $\mathbf{z}$ from $\mathbf{x}$ through $\mathbf{y}$, and where $\text{TAG} \in [\text{EQU}, \text{COM}, \text{MAT}]$,

$$\begin{array}{ccc} \mathbf{x} & & \\ & \searrow & \\ & z & \equiv \mathbf{x}^T \cdot \mathbf{y} = z; \\ & \nearrow & \\ \mathbf{y} & & \end{array} \quad (\text{s.3})$$

where z is the scalar product of vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\begin{array}{ccc} \text{COM}(\mathbf{x}) & & \\ \updownarrow & & \\ w = W & \equiv & \left\{ \begin{array}{l} \text{COM} \ \& \ w = W \implies \text{EQU} \\ \text{EQU} \ \& \ w = W \implies \text{COM} \\ \text{COM} \ \& \ \text{EQU} \implies w = W \end{array} \right. \\ \updownarrow & & \\ \text{EQU}(\mathbf{y}) & & \end{array} \quad (\text{s.4})$$

an expression in which the validity of two of the relations ensures the validity of the third.

1.3.4 Elastic Analysis

An analysis problem is simply to determine, for a given structure $(\mathbf{k}, \mathbf{B})$, subjected to loads $\mathbf{F}_l$ and known support conditions $\mathbf{U}_s$, the displacements of free points $\mathbf{U}_l$, and the deformations $\mathbf{u}$ and efforts $\mathbf{f}$ of its elements, and verify that they meet the established requirements, which can be done according to the sequence already seen in 1.3.2, which we review here:

- construct $\mathbf{B}$ and $\mathbf{k}$.
- determine the global stiffness matrix, equation 1.26

- determine free displacements $\mathbf{U}$ from loads $\mathbf{F}$ which may include equivalent forces from forced support movements, equations 1.27 (and perhaps 1.30)
- determine deformations $\mathbf{u}$, equation 1.1
- determine efforts $\mathbf{f}$, equation 1.22
- determine reactions $\mathbf{F}_s$, equation 1.29
- check (verify) that strength 1.33 and displacement or stiffness 1.32 constraints are met.

The analysis result will subsequently be used to validate or invalidate the solution.

It can be added that the sequence up to the determination of the $\mathbf{f}$ stresses can be summarized in a “solution matrix” $\mathbf{S}$ in the form

$$\mathbf{f} = \mathbf{S}\mathbf{F} \quad (1.34)$$

$$\mathbf{f} = \mathbf{k}\mathbf{u} = \mathbf{k}\mathbf{B}\mathbf{U} = \mathbf{k}\mathbf{B}\mathbf{K}^{-1}\mathbf{F} \quad (1.35)$$

$$\mathbf{S} = \mathbf{k}\mathbf{B} \left(\mathbf{B}^T \mathbf{k} \mathbf{B} \right)^{-1} \quad (1.36)$$

It can be noted that, in an isostatic problem, or with an equal number of unknowns and equations, since $\mathbf{F} = \mathbf{H}\mathbf{f}$, having the solution $\mathbf{f} = \mathbf{H}^{-1}\mathbf{F}$, it results that $\mathbf{S} = \mathbf{H}^{-1}$.

1.3.5 Elastic Design

A design problem, from the elastic perspective, is to determine with the elastic model the required dimensions to solve the problem corresponding to a given geometry and loads and support

conditions. Therefore, $\mathbf{B}$ could be constructed and both $\mathbf{F}_l$ and $\mathbf{U}_s$ would be known. In many cases, the problem can now be approached with the following procedure:

- construct $\mathbf{B}$
- simultaneously decide
 - a set of efforts $\mathbf{f}$ that resolve equilibrium 1.2.
 - a deformed configuration $\mathbf{U}$ that meets the constraints 1.32, and whose compatible deformations $\mathbf{u}$ (equation 1.1) are coherent with the previous efforts (have equivalent signs).
- determine the necessary sections so that strengths $\mathbf{r}$ and stiffnesses $\mathbf{k}$ are respectively satisfied and concordant with applicable material conditions, equations 1.22 and 1.33.

The result will be valid upon fulfilling the three relevant equation groups as well as the problem constraints. An example can be the distribution in plan of wind actions in a braced system with multiple planes (and therefore statically indeterminate in plan): it will suffice to

- distribute (almost) arbitrarily the loads between the braced systems, always ensuring their resultant is in plan in equilibrium with the action resultant
- assume the same drift ratio for all stories in all bracing planes and determine the resulting deformations in the diagonals.
- determine the necessary sections so that there is agreement between the resultant efforts from load distribution across different planes and the deformations derived from the anticipated deflection.

1.3.6 Assumptions, Values, and Limitations of the Elastic Model

The elastic approximation, formulated with small-magnitude displacements and deformations, results in a linear formulation, solvable with reduced resources, and in which the superposition principle is applicable: *the solution to the linear combination of several problems is the linear combination of the solutions for each problem separately.*

Among the necessary assumptions for its validity, the following can be noted

- the deformed geometry does not influence the equilibrium, which does not take into account either the possible displacement of loads relative to their initial position, nor the influence of deformation on the equilibrium configuration.
- the behavior of materials is reduced to linear relations, without the ability to model either hardening or relaxation.

It is important to consider that solutions depend locally on

- the specific conditions with which the movement restrictions of the support are materialized
- the specific form of load application
- the way in which geometric changes are materialized (nodes, etc. . .)

In relation to this, it is interesting to consider the *Saint-Venant principle* which indicates that the elastic solution in regions far from singular points –load application, supports, or geometric changes– does not depend on the form of load application or

the local geometry of these points but only on the loads or reactions themselves. These “distant” regions are those located at distances greater than slightly more than the depth of these singular points. For regions near these points, however, the ways of applying the load or changes in geometry must be analyzed carefully –the nodes, therefore, will always be included in this reservation–

On the other hand, it is a model very sensitive to the initial conditions adopted for the model. Among these should be considered

- the stress states derived from the existence or forced elimination of gaps or discontinuities,
- the qualities of materials in the case of coexistence of several of these

For all the above reasons, it is a relatively limited approximation in handling uncertainty conditions.

The appearance of precision and rigor of the numerical model should not cause one to lose sight of the imprecision of the model associated with the sources of uncertainty mentioned.

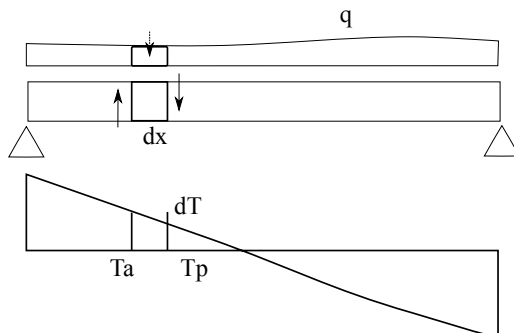


Figure 1.3: Equilibrium of shear forces and loads in a beam segment

1.3.7 Application examples

1.3.7.1 Equilibrium

If among the equilibrium equations in a beam we consider those of shear, figure 1.3, we have the classic differential equation for any unit length segment, where q is the load density (load per unit length), T the shear force, and x the position along the bar:

$$q = -\frac{\partial T}{\partial x}$$

This expression, for segments of arbitrary length between cuts a (anterior) and p (posterior) takes the following form, where

F_{ap} is the total load contained between cuts a and p :

$$F_{ap} = \int_a^p q \, dx = -(T_p - T_a) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} T_a \\ T_p \end{bmatrix}$$

Thus, we can express these equilibrium conditions in the form $\mathbf{F} = \mathbf{H}\mathbf{f}$, where $\mathbf{F}$ contains the loads applied to each of the regions chosen for the model, $\mathbf{f}$ the forces corresponding to the cuts that isolate these regions, and $\mathbf{H}$ the coefficients that allow expressing the equilibrium between them in each of such regions. These coefficients express the differences between forces “before” the region and those “after” the region, as indeed, depending on whether the considered cut corresponds to the “decreasing” or negative face, or to the “increasing” or positive face, the same internal force (for example a tension, or the shear in the example) subtracts or adds, having opposite directional effect when we refer it to the global axes for the region. In the case where the region is “unitary” we have that the equation takes the form $\mathbf{F}_* = \partial \mathbf{f}$ where now $\mathbf{F}_*$ (which we have denoted as $\boldsymbol{\tau}$ in 1.2.1) expresses the mean *load density*, that is, the unit load considered, and $\mathbf{f}$ (which we denoted there as $\boldsymbol{\sigma}$) still contains the internal forces at the cuts, with ∂ being the differential operator with which we write the equilibrium conditions.

For a truss, like the one in figure 1.4, the equilibrium equations of any of its joints will express, in the axes chosen for that purpose at the joint, the equilibrium of the load $\mathbf{F}$ on the joint with the projections $\mathbf{H}$ of the member forces $\mathbf{f}$ on said axes. In the case of a pin-jointed truss model, each member will only have an axial force, and each joint a point load with as many components as dimensions the model has (2D or 3D).

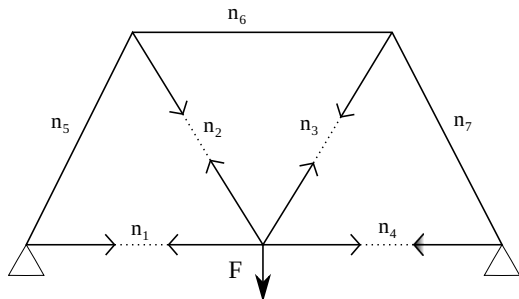


Figure 1.4: Equilibrium of member forces and loads at a truss joint

It should be noted that the axes chosen for the regions (local axes) can be specific to each region, but this can also be applied to global movements, which can be identified by specific directions or axes for each region, joint or movement considered in the basis of the movements $\mathbf{U}$ to describe: we could choose, for example, in a truss, specific axes for each joint without affecting the description made: it would only affect how loads and displacements are measured, and their relationships with forces and deformations, as given by the matrices $\mathbf{H} = \mathbf{B}^T$.

1.3.7.2 Compatibility

For small movements, the relationships are linear. If we consider unit deformations or deformation densities (deformation per unit length for each point of infinitesimal regions) as shown

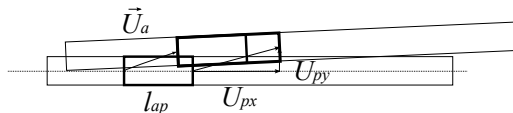


Figure 1.5: Compatibility between displacements and longitudinal elongation of a bar.

in figure 1.5 we have

$$\begin{aligned} \mathbf{u}_* &= \partial \mathbf{U} \\ \varepsilon_x &= \frac{\partial u_x}{\partial x} \end{aligned}$$

while if we encompass the accumulated deformations in finite regions we will have

$$\begin{aligned} \mathbf{u} &= \mathbf{B}\mathbf{U} \\ \varepsilon_x &= \frac{U_{px} - U_{ax}}{l_{ap}} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}_{ap} \begin{bmatrix} U_{ax} \\ U_{px} \end{bmatrix} \\ u_{ap} = l_{ap} \varepsilon_x &= U_{px} - U_{ax} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} U_{ax} \\ U_{px} \end{bmatrix} \end{aligned}$$

Now $\mathbf{B}$ expresses how to obtain these deformations $\mathbf{u}$ of the regions into which we divide the structure (u_{ap} in the drawing, corresponding to the region between cuts a and p) from the relevant movements $\mathbf{U}$ of said cuts, ends, or global reference *nodes* of the structure, (in this case the longitudinal displacements of such cuts, namely U_{ax} and U_{px}).

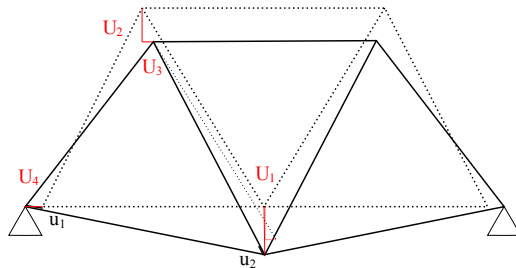


Figure 1.6: Bar deformations compatible with joint displacements in a truss.

In pin-jointed bar structures (figure 1.6) the end movements will be projected onto the relevant directions for each bar, obtaining the deformations as differences between these. Matrix $\mathbf{B}$ will express, for each row i corresponding to the considered deformation u_i in each column j the effect of a unit displacement for U_j . That is, B_{ij} would be the deformation u_i derived from a movement³ such as

$$\mathbf{U}_{j,u} = \begin{bmatrix} U_1 & U_2 & \dots & U_{j-1} & U_j & U_{j+1} & \dots & U_n \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}^T.$$

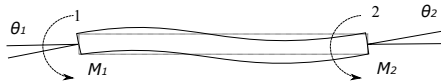


Figure 1.7: Bending moment-rotation stiffness relationships in a beam.

1.3.7.3 Material relations (or material constitution) elastic

If we consider the end rotations as parameters for deformations for a beam in bending, and therefore the end moments for the forces, as in figure 1.7, the corresponding stiffness relationships condensed in the stiffness matrix will be the known ones

$$\mathbf{f} = \mathbf{k}\mathbf{u} \equiv \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{l} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

1.3.7.4 Rigid body supported by several pin-jointed bars

This involves the problem of supporting a **rigid** body with mass m , with dimensions (in two-dimensional projection) of $2a \times a/2$ at a height a on four pin-jointed inclined supports, of cross-section A , and made of material with elasticity modulus (Young's modulus) E . The body may be subjected to horizontal

3. $\mathbf{U}_{j,u}$ indicates a displacement vector for the entire structure in which, however, there would be only one unit movement (u) in the j degree of freedom.

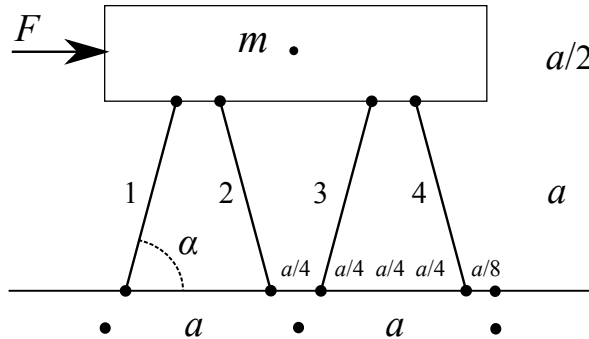


Figure 1.8: Body supported by four pin-jointed columns

forces F , in addition to the usual gravitational action or vertical load.

The elastic response without considering buckling requires a global model: $\mathbf{U}$ and $\mathbf{F}$ and local models $\mathbf{u}$ and $\mathbf{f}$ for the elastically deformable bars. The decision for the model design is free. Here we select the components of the model.

- Since the body is considered rigid, the global behavior has only three possible movements (degrees of freedom), and given that the ground connection points cannot move, it will be sufficient to choose three independent movements to represent it. We choose the movements that can be referenced around the center of gravity: displacements and rotation: $\mathbf{U} = [U_x \ U_y \ \Theta]^T$. We could have chosen three different parameters, such as the horizontal and vertical displacement

of the body's lower right corner plus the vertical displacement of the lower left corner (the horizontal movement will coincide for small movements, and the rest could be derived from these). Consistent with our choice of $\mathbf{U}$, the forces $\mathbf{F}$ are the actions applied to the body referenced to its center of gravity, forces and moment $\mathbf{F} = [F_x \ F_y \ M]^T$, which are those that perform work for each isolated unit movement, that is, for each element of the basis that describes the possible movements of the body: $\mathbf{U}_{1,u} = [1 \ 0 \ 0]^T$, $\mathbf{U}_{2,u} = [0 \ 1 \ 0]^T$, $\mathbf{U}_{3,u} = [0 \ 0 \ 1]^T$.

- Since only the columns are deformable, and given that the pinned ends require that there be only axial forces in them, with no bending moments, the local model must represent only these axial forces and the elongations or shortenings of the corresponding bars. Again we can freely choose the relevant parameters, as long as they are consistent from the work perspective: it is usual (and we will do so here) to consider as force parameters the forces of each bar, considering tension as positive: $\mathbf{f} = [f_1 \ f_2 \ f_3 \ f_4]^T$. In this case, the deformation parameters $\mathbf{u} = [u_1 \ u_2 \ u_3 \ u_4]^T$ must necessarily be the total elongations of each of the bars (the difference between the longitudinal displacements of both ends), in order to measure the work in the bar with the product $\mathbf{u}^T \mathbf{f}$. An interesting alternative would be to establish as deformation parameters the unit deformations of each of the bars: $\mathbf{u} = [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3 \ \varepsilon_4]^T$. In this case, as the work for a bar i with an axial force f_i that elongates in all its sections ε_i

is $w_i = \int_l f_i \varepsilon_i dx = \varepsilon_i f_i l$, the appropriate force parameters would be $\mathbf{f} = [f_1 l_1 \quad f_2 l_2 \quad f_3 l_3 \quad f_4 l_4]^T$. That is, instead of forces, the stress volume of each bar would be used as parameters: $f_i l_i = \sigma_i A_i l_i = \sigma_i V_i$. Another alternative used in the early days of matrix analysis of pin-jointed bars is to consider as force parameter the so-called stress coefficient in (Parkes 1974, cited by Pellegrino and Calladine 1985), called *force density* in tensioned cable network problems, in *form finding* problems⁴, and defined as the quotient f_i/l_i . This definition allows using the differences between node coordinates in the compatibility matrices instead of the direction cosines of the bars, but in exchange it will imply that the deformation parameters must be got as the product of the total elongation by the bar length.

- Once the static and kinematic parameters of the bar are decided, the stiffness relations $\mathbf{k}$ are those that relate for each element i the static terms with the kinematic ones. Here there is only one of such relationships per member so we write $f_i = E_i A_i u_i / l_i$ being therefore $\mathbf{k}_i = [E_i A_i / l_i]$, a “block” of only 1x1 rows and columns. In the case of having used ε_i as deformation parameter, and $f_i l_i$ as the corresponding force parameter, now we would have $f_i l_i = E_i A_i \varepsilon_i l_i$ and we should use for the stiffness the form $\mathbf{k}_i = [E_i A_i l_i] =$

4. The alignment between force and bar means that the force density coincides with any of its projections. If values for force densities –usually in their horizontal projection– are adopted a priori by prestressing, the non-linearity of the equilibrium equations for non-infinitesimal shape changes transforms into a set of linear relationships that allow rapid determination of the equilibrium shape.

$[E_i V_i]$. It should be noted that this latter approach holds considerable theoretical interest, as the three resulting measurements, the deformation ε_i , the “force” $f_i l_i$ and the stiffness $E_i V_i$ use variables of great theoretical significance: the unit deformation, the *structural quantity* defined in chapter 5, and the product of the bar volume by its Hooke’s modulus.

- With global and local parameters established, the equilibrium or compatibility relationships will be given by the matrices $\mathbf{H} = \mathbf{B}^T$. These can be constructed from the compatibility or the equilibrium perspectives, it will depend on the model and particular preferences. Since movements and deformations are drawable, the kinematic approach (compatibility) is usually simpler although it is not mandatory and although in many cases the analyst’s prior training may make the static approach seem simpler. In the kinematic approach, each column j of $\mathbf{B}$ is obtained by successively determining the deformations that would result for each isolated unit global displacement $\mathbf{U}_j$. That is, B_{ij} is the deformation u_i that occurs if only the unit movement $\mathbf{U}_j$ is given. Alternatively, in the static approach (equilibrium), each row i of matrix $\mathbf{H}$ expresses the contribution of all forces $\mathbf{f}$ for the equilibrium of the global force F_i so that each term H_{ij} is the component (projection) contributed to F_i by a unit force in f_j .

The equations considered and the results of the construction of the equilibrium and the compatibility matrices are included below, with references to the drawings used for this purpose.

Equilibrium: figure 1.9.

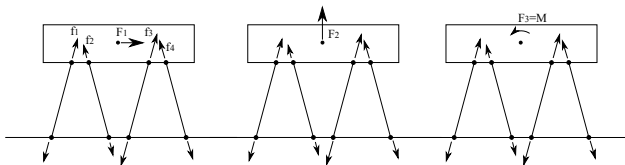


Figure 1.9: Equilibrium geometries.

The equations for each degree of freedom (rows of $\mathbf{H}$) are

$$F_1 = f_1 \cos \alpha - f_2 \cos \alpha + f_3 \cos \alpha - f_4 \cos \alpha$$

$$F_2 = f_1 \sin \alpha + f_2 \sin \alpha + f_3 \sin \alpha + f_4 \sin \alpha$$

$$F_3 = M = \cos \alpha (f_1 - f_2 + f_3 - f_4) \frac{a}{4} + \sin \alpha \left(-f_1 \frac{5a}{8} - f_2 \frac{3a}{8} + f_3 \frac{3a}{8} + f_4 \frac{5a}{8} \right)$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} c\alpha & -c\alpha & c\alpha & -c\alpha \\ s\alpha & s\alpha & s\alpha & s\alpha \\ \frac{a}{4}c\alpha - \frac{5a}{8}s\alpha & -\frac{a}{4}c\alpha - \frac{3a}{8}s\alpha & \frac{a}{4}c\alpha + \frac{3a}{8}s\alpha & -\frac{a}{4}c\alpha + \frac{5a}{8}s\alpha \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$$\mathbf{F} = \mathbf{H}\mathbf{f} = \mathbf{B}^T \mathbf{f}$$

where the expressions $s\alpha = \sin \alpha$ and $c\alpha = \cos \alpha$ have been used.

Compatibility; figure 1.10

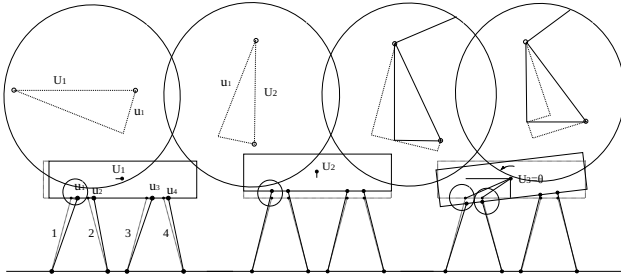


Figure 1.10: Compatibility geometries.

The deformations activated by the composition of displacements (sum of the effects of all of them) are

$$\begin{aligned}
 u_1 &= \cos \alpha U_1 + \sin \alpha U_2 + \left(-\frac{5a}{8} \sin \alpha + \frac{a}{4} \cos \alpha \right) U_3 \\
 u_2 &= -\cos \alpha U_1 + \sin \alpha U_2 + \left(-\frac{3a}{8} \sin \alpha - \frac{a}{4} \cos \alpha \right) U_3 \\
 u_3 &= \cos \alpha U_1 + \sin \alpha U_2 + \left(+\frac{3a}{8} \sin \alpha + \frac{a}{4} \cos \alpha \right) U_3 \\
 u_4 &= -\cos \alpha U_1 + \sin \alpha U_2 + \left(+\frac{5a}{8} \sin \alpha - \frac{a}{4} \cos \alpha \right) U_3
 \end{aligned}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} c\alpha & s\alpha & -\frac{5a}{8}s\alpha + \frac{a}{4}c\alpha \\ -c\alpha & s\alpha & -\frac{3a}{8}s\alpha - \frac{a}{4}c\alpha \\ c\alpha & s\alpha & \frac{3a}{8}s\alpha + \frac{a}{4}c\alpha \\ -c\alpha & s\alpha & \frac{5a}{8}s\alpha - \frac{a}{4}c\alpha \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{B}\mathbf{U}$$

Stiffness: The set of elastic relationships can be expressed as

$$f_i = \frac{E_i A_i}{l_i} u_i$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \frac{E_1 A_1}{l_1} & 0 & 0 & 0 \\ 0 & \frac{E_2 A_2}{l_2} & 0 & 0 \\ 0 & 0 & \frac{E_3 A_3}{l_3} & 0 \\ 0 & 0 & 0 & \frac{E_4 A_4}{l_4} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\mathbf{f} = \mathbf{k}\mathbf{u}$$

Therefore, the resulting global stiffness matrix $\mathbf{K} = \mathbf{B}^T \mathbf{k} \mathbf{B}$ for this problem formulation would be:

$$\mathbf{K} = \frac{EA}{l} \begin{bmatrix} 4c^2 & 0 & -ac^2 \left(\frac{s}{2c} - 1 \right) \\ 0 & 4s^2 & 0 \\ -ac^2 \left(\frac{s}{2c} - 1 \right) & 0 & a^2 c^2 \left(\frac{17s^2}{16c^2} - \frac{s}{4c} + \frac{1}{4} \right) \end{bmatrix}$$

where $c = \cos \alpha$ and $s = \sin \alpha$ are used.

If a different formulation had been employed, now using unit strains to determine the “local” deformations of bars, and therefore using the stress volume for the forces, the matrices $\mathbf{H}$ and $\mathbf{B}$ would include the division by the lengths of the bars

$$\mathbf{B} = \begin{bmatrix} \frac{c\alpha}{l_1} & \frac{s\alpha}{l_1} & -\frac{1}{l_1} \left(\frac{5a}{8} s\alpha + \frac{a}{4} c\alpha \right) \\ -\frac{c\alpha}{l_2} & \frac{s\alpha}{l_2} & -\frac{1}{l_2} \left(\frac{3a}{8} s\alpha - \frac{a}{4} c\alpha \right) \\ \frac{c\alpha}{l_3} & \frac{s\alpha}{l_3} & \frac{1}{l_3} \left(\frac{3a}{8} s\alpha + \frac{a}{4} c\alpha \right) \\ -\frac{c\alpha}{l_4} & \frac{s\alpha}{l_4} & \frac{1}{l_4} \left(\frac{5a}{8} s\alpha - \frac{a}{4} c\alpha \right) \end{bmatrix}$$

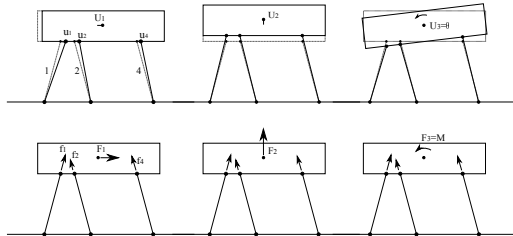


Figure 1.11: Gap in a column, which does not begin to take load

and the local stiffness matrix $\mathbf{k}$ would change to

$$\mathbf{k} = \begin{bmatrix} E_1 A_1 l_1 & 0 & 0 & 0 \\ 0 & E_2 A_2 l_2 & 0 & 0 \\ 0 & 0 & E_3 A_3 l_3 & 0 \\ 0 & 0 & 0 & E_4 A_4 l_4 \end{bmatrix}$$

The global stiffness matrix $\mathbf{K}$ would be identical, as the global representation has not been modified.

However, if we consider potential sources of uncertainty, the uniqueness of the model is no longer evident: for example, if we consider the existence of a gap in the third column, such that it does not begin to take load when the others are already loaded (figure 1.11), both the necessary model and the results change: now the stiffness and compatibility will only take into account deformations in the members that are under load, so we have a different system, which will give rise to a different stiffness

matrix and solution:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} c\alpha & s\alpha & -\frac{5a}{8}s\alpha + \frac{a}{4}c\alpha \\ -c\alpha & s\alpha & -\frac{3a}{8}s\alpha - \frac{a}{4}c\alpha \\ -c\alpha & s\alpha & \frac{5a}{8}s\alpha - \frac{a}{4}c\alpha \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_4 \end{bmatrix} = \begin{bmatrix} \frac{EA}{l} & 0 & 0 \\ 0 & \frac{EA}{l} & 0 \\ 0 & 0 & \frac{EA}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix}$$

$$\mathbf{K}^* = \frac{EA}{l} \begin{bmatrix} c\alpha^2 & -c\alpha s\alpha & -a c\alpha \left(-\frac{5s\alpha}{8} + \frac{c\alpha}{4}\right) \\ -c\alpha s\alpha & s\alpha^2 & a s\alpha \left(-\frac{3s\alpha}{8} - \frac{c\alpha}{4}\right) \\ -a c\alpha \left(-\frac{5s\alpha}{8} + \frac{c\alpha}{4}\right) & a s\alpha \left(-\frac{3s\alpha}{8} - \frac{c\alpha}{4}\right) & a^2 \left(\frac{5s\alpha}{8} - \frac{c\alpha}{4}\right)^2 \end{bmatrix}$$

It should be added that in this particular case, being an iso-static condition, the solution matrix could be obtained simply by inverting matrix $\mathbf{H}$.

It is left to the reader as an exercise to numerically determine and interpret results for various conditions in the stiffness relationships between inner and outer columns, or applied loads, among which the following figures show a sample for horizontal load only and equal stiffness in all columns, with or without gaps in the third one. The extensions of the column lines mark the passing points of the resultants of any combinations of each pair of forces in these, which are drawn in some cases as sum of vectors from these points.

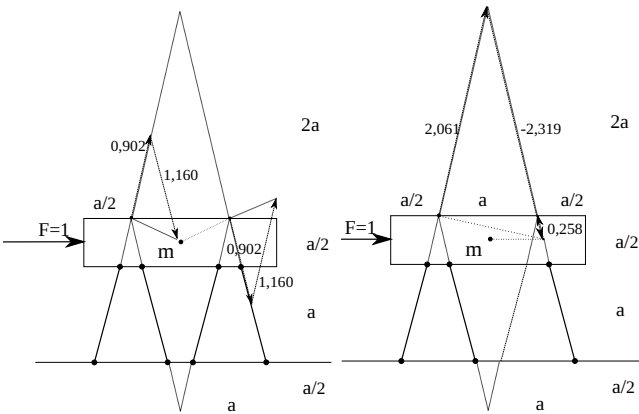


Figure 1.12: Elastic response for horizontal load, equal columns and initial conditions without gaps or with gap in the third column.

1.4 Plastic analysis and design

Plastic analysis, or limit analysis Plastic analysis, or limit analysis, is based on the hypothesis of sufficient ductility of the structure, that is, on the assumption that, once the deformation corresponding to the limit of the strength is reached in any section, said section continues to deform without loss (or gain) of strength. In the load-resistance diagrams considered in figure 1.2, such assumption would correspond to having horizontal branches in the regions where this resistance limit is reached. In a more general version of the model, the resistance evolves in the plastification process slightly increasing or decreasing in a process governed by a hardening parameter that allows considering inclined branches in such diagrams. Here we will nevertheless stick to the plastic model without hardening.

This hypothesis allows, as we will see next, a very different approach to static problems. In this new approach, the progressive increase of the load is considered, which through redistribution in the internal forces generates successive regions in plastic regime, until the structure transforms into a mechanism and collapses. The model is based on the analysis of the collapse mechanism, its kinematics and the equilibrium between the plastification forces and the load that causes it.

1.4.1 Collapse modes: kinematics and equilibrium

To approach a limit analysis, the characteristics of the geometries and mobilized resistances in collapse conditions must be

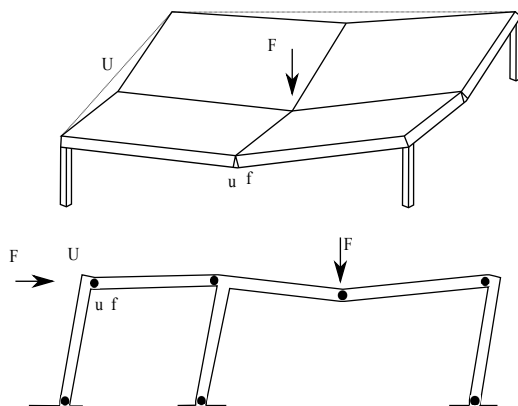


Figure 1.13: Collapse modes

considered. Figure 1.13 exemplifies a couple of typical modes, which correspond to the failures of plates or flexed bars. In the case of bars in flexion or flexo-compression, the adopted model asserts the formation of plastic hinges in the sections of maximum moment—which are often located at the ends of the bars, at their connection to the nodes—. In the case of plates, the most common hypothesis asserts the formation of straight failure lines that separate (almost) flat sections of the plate, and in which the bending moment strength limits have been reached.

In both cases, what turns out to be fundamental is:

- the deformation concentrates in the yielding element, line or hinge,

- said element maintains the limit strength throughout the (beginning of the) collapse process.

Thus, we consider in the modeling the relevant regions, susceptible to break (or not) in the possible structural collapse movements, and their kinematics and forces, and we have that

- just as in the elastic model, we can establish a set of global displacement parameters $\mathbf{U}$ and possible collapse deformations $\mathbf{u}$ associated with a set of external load parameters $\mathbf{F}$ and internal forces $\mathbf{f}$ which may each have reached the failure limit or not, but which must, in the collapse situation, satisfy the kinematic compatibility equations for that collapse movement, equation 1.1, and those of the equilibrium reached in that same instant of collapse initiation, equation 1.2.
- but unlike the elastic model, now the material relationships consist of establishing whether the strength limit has been reached or not, thus having that regions at the strength limit will have non-zero deformations $\mathbf{u}$, while, as we explain in section 1.4.5, zero deformations must be assumed for regions that have not reached said strength limit.

The key, therefore, of the appropriate modeling method will be in the definition of the conditions that express the strength limit, together with the different interpretation of the meaning of the kinematic parameters: even though there can be used the same ones as for the elastic model, now they will be the concentrated deformations, linked to the collapse movement, located in the yielding regions.

Thus, in bar structures, the models will be associated with

the considered failure forms:

- if it is moment failure, we will be talking about “hinges” (failure lines in plates) at the points of interest (bar ends at their connection to the node, at the center, or under loads in beams, etc.) and the rotation in the hinge associated with the plastic moment of the considered section.
- if it is axial force failure without bending, we will be considering the yield elongation in any of the sections of the bar (now it would be indifferent which one it is that section and we only need one force parameter and another for the deformation, as if it were a telescopic damper in the bar that only activates upon reaching the yield limit)
- it is common to build more sophisticated models, which we will see in section 1.4.2, for the failure conditions in situations of possible combined forces.

Although the compatibility and equilibrium equations are identical to those considered in the elastic case, it must be repeated that they correspond to the failure condition. In general, we will have a structure with its possible dimensions, and some initial loads $\mathbf{F}_I$ but we do not know the loads needed to break the structure, so we will consider that the *failure loads* $\mathbf{F}$ are the initial loads $\mathbf{F}_I$ affected by the load factor γ necessary to increase them to the failure condition. The equilibrium equation at the failure limit takes therefore the form

$$\mathbf{F} = \gamma \mathbf{F}_I = \mathbf{H} \mathbf{f}. \quad (1.37)$$

On the other hand, the failure situation means that the “kinematics” is really that of a continued movement, in which the

displacements $\mathbf{U}$ will be associated with the plastic deformations $\mathbf{u}$, changing with time while maintaining the proportion. We will really be talking about displacement or deformation velocities $\dot{\mathbf{U}}$ and $\dot{\mathbf{u}}$, that is, the displacements or deformations accumulated in a given period of time, although this period is indifferent as the geometries are proportional for different periods. We can speak, then, of both displacements and plastic deformations, as well as displacement or deformation “velocities”, and although this latter form is more rigorous, neither form should be confusing:

$$\dot{\mathbf{u}} = \mathbf{B}^T \dot{\mathbf{U}}. \quad (1.38)$$

1.4.2 Material relationships: strength conditions.

The material conditions in limit analysis are defined by establishing the resistance or strength capacity that bounds the limits that can be reached, individually or in combination, by the internal forces defined in the model, $\mathbf{f}$. The parameters of $\mathbf{f}$ may or may not interact in failure situations: for example, if they refer to forces in independent sections, it is obvious that the force values of one section do not influence whether in another the capacity limits have been reached, but in cases where the parameters refer to force components of the same section, moment and axial force, for example, failure depends on the interaction between both. For the solution of the problem that we will see later, it is useful to define these *interaction diagrams* for strength capacities using linear relationships, so that

in general we will establish the comparison terms through linear combinations of the forces $\psi \mathbf{f}$ that will be compared with the corresponding strength or resistance limits $\mathbf{r}$, so that the expressions of whether or not the resistance limit is reached will have the form

$$\psi \mathbf{f} \leq \mathbf{r} \quad (1.39)$$

where each equation in this list corresponds to each of the limit conditions that may have been established for the forces or their combinations.

Often it happens that, although each piece p has its own resistance parameters $\mathbf{r}_p$, constructive rationalization makes the same cross-sections be selected for different pieces, so that the catalog of limit capacities is repeated. In this case it is better to consider that we use a list of reference limit capacities $\mathbf{r}$ and that their attribution to each of the equations of the previous system is made through their assignment via an expression $\mathbf{d}\mathbf{r}$ that simply attributes to each equation the resistance capacity relevant to the applied design criteria: for example in a frame with all beams equal and all columns equal we would have in $\mathbf{r}$ the resistance capacity references of the typical beam and typical column, and $\mathbf{d}$ would be nothing more than a matrix of zeros and ones that would apply to each equation the beam or column resistance, as appropriate. Thus the resistance condition will be expressed in general as

$$\psi \mathbf{f} \leq \mathbf{d}\mathbf{r} \quad (1.40)$$

It is very common to consider separately the limits to the forces of different sections in expressions $\psi_i \mathbf{f}_i \leq \mathbf{d}_i \mathbf{r}$, where $\mathbf{r}$ contains

all available resistance qualities, $\mathbf{d}_i$ the selection for the corresponding section i and where the internal forces $\mathbf{f}_i$ of these are combined with $\boldsymbol{\psi}_i$ in equations appropriate to such sections. In that case, the restrictions for the entire structure are, given the independence in section checks, simply the following:

$$\begin{bmatrix} \boldsymbol{\psi}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\psi}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\psi}_n \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{bmatrix} \leq \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_n \end{bmatrix} \mathbf{r} \quad (1.41)$$

In any case, we can use indistinctly any of equations 1.39 or 1.40, of which we present examples in 1.4.13.2.

Equation 1.40 is nothing more than a set of limits not to be reached if collapse is to be avoided: the i th expression or condition would be the expression $\boldsymbol{\psi}_i \mathbf{f} \leq \mathbf{d}_i \mathbf{r} \equiv \sum_j \psi_{ij} f_j \leq \sum_k d_{ik} r_k$, or in Einstein notation, implicit sum if indices are repeated: $\psi_{ij} f_j \leq d_{ik} r_k$. Each of these expressions represents a (hyper)plane in the space of forces $\mathbf{f}$, defined by the coefficients $\boldsymbol{\psi}_i$ and the limit that cannot be exceeded $\mathbf{d}_i \mathbf{r}$. The expression remains valid if both terms are multiplied by the same factor, and can, therefore, be put in “canonical form” if we divide both terms by the modulus of said coefficients $\|\boldsymbol{\psi}_i\| = \sqrt{\sum_j \psi_{ij}^2}$ that is, if we use each “row” i in the form

$$\frac{1}{\|\boldsymbol{\psi}_i\|} \boldsymbol{\psi}_i \mathbf{f} \leq \frac{1}{\|\boldsymbol{\psi}_i\|} \mathbf{d}_i \mathbf{r} \quad (1.42)$$

In this case, the coefficients become the *versor* of the plane, that is, a unit vector perpendicular to that plane; the product

of the first term is, therefore, the scalar product of the force state by said versor, thus the projection of the force on the perpendicular to the plane; and the term on the right becomes the distance from the plane to the origin (the point of zero forces); the expression thus shows that the limit is reached if the projection of the force on the perpendicular to the plane equals the distance of this one to the origin, and is not reached if said projection is smaller. This will be valid for each and every one of the expressions that, therefore, envelop the space or set of admissible forces by a convex hyperpolyhedron whose faces coincide with each of the limit planes. Said surface is the *limit surface* or failure surface. In said hyperpolyhedron, the points of the faces will represent possible forces of a collapse mode governed by a single condition, while the edges will represent forms of collapse in which the limits of two of such conditions are simultaneously reached, etc. In the first case, the collapse mode is identical for all points of the plane, and its kinematics is described by the versor of the plane, common to all. For the second, the failure will combine in some compatible way the collapse geometry corresponding to the different planes concerned.

1.4.3 Collapse condition

Any of the expressions contained in 1.40 expresses a different *collapse mode*, which is not reached if the inequality is strictly met, but which is reached if equality is met. That is,

- If $\psi_{ij}f_j < d_{ik}r_k \equiv \psi_{ij}f_j - d_{ik}r_k < 0$ there is *slack*, or resistance margin, and plastification is not reached; therefore there is no deformation for that mode.

- If $\psi_{ij}f_j = d_{ik}r_k \equiv \psi_{ij}f_j - d_{ik}r_k = 0$ there is no slack, plastification has been reached, and plastic deformations associated with the collapse process corresponding to that mode can occur.

In this way we can conceive a *multiplier* λ_i for each condition or check, positive or null, depending on whether collapse has been reached or not. That is, the multiplier will be an indicator of whether the corresponding mode is active in the collapse or not. We will see that we can interpret this multiplier as a measure of the plastification velocity of that mode.

In any case, we will have that the product $(\psi_{ij}f_j - d_{ik}r_k)\lambda_i$ is always null: if there is plastification because the slack is zero, if there isn't, because the multiplier is zero. Therefore, a key expression of the collapse model is now

$$(\boldsymbol{\psi}\mathbf{f} - \mathbf{d}\mathbf{r})^T\boldsymbol{\lambda} = 0 \quad (1.43)$$

1.4.4 Work and energy

If we now consider the collapse process in terms of energies, we have that the measures of work done per unit time are

$$W = \dot{\mathbf{U}}^T \mathbf{F} \quad (1.44)$$

$$w = \dot{\mathbf{u}}^T \mathbf{f} \quad (1.45)$$

where W is the energy loss of the loads in collapse (external energy) and w is the energy dissipated in the plastic flow of the sections at the resistance limit and which, therefore, are yielding (internal energy)

To characterize the failure condition we will use Von Mises' *flow rule* which establishes that *the plastic forces in collapse are those that maximize the energy dissipated in the deformation*.

Let's consider that condition in a collapse process in which the mode and its kinematics are defined by the values of λ , $\mathbf{u}$ and $\mathbf{U}$. In two successive instants of the collapse process, the forces may be varying, taking $\mathbf{f}$ to $\mathbf{f} + \partial\mathbf{f}$.

Now, if the energy dissipated per unit time is the maximum possible, the variations in that behavior environment must be null, or what is the same

$$\frac{\partial w}{\partial \mathbf{f}} \partial \mathbf{f} = \mathbf{u}^T \partial \mathbf{f} = 0. \quad (1.46)$$

In addition to this, as it is a collapse process, the forces will remain in some region of the limit surface (at the resistance limit) and therefore the differences between forces between two successive instants $\partial \mathbf{f}$ must be tangent to the active plane, a , corresponding and therefore orthogonal to the versor that corresponds to said active plane, represented by the row of coefficients ψ_a of the corresponding limit expression.

We have, then, that

$$\psi_a \partial \mathbf{f} = 0 \quad (1.47)$$

If we compare expressions 1.46 and 1.47 we see that $\mathbf{u}$ and ψ_a^T must be proportional, that is, the versor of the failure plane is parallel to the vector that represents the velocity of deformation in the collapse process, or what is the same, the deformation vector is orthogonal to the plane that represents the limit condition reached.

We can therefore express a relationship between the deformation velocities in failure and the resistance conditions using the plastic multipliers defined in the previous section:

$$\mathbf{u} = \boldsymbol{\psi}^T \boldsymbol{\lambda} \quad (1.48)$$

Indeed if there is only one active plane, the corresponding multiplier λ_a is the only non-zero one, and therefore selects the column of $\boldsymbol{\psi}^T$ or the row of $\boldsymbol{\psi}$ corresponding to the coefficients that determine said plane. If there are several active planes, it corresponds to an encounter, an intersection or a vertex between planes. The vector that expresses the deformation can be contained in the angle or cone formed between the versors of the concerned planes, which is adequately represented if positive λ values are used for the corresponding planes: the expression will provide a linear combination of their versors contained therefore in said cone.

This is *associated plasticity*: the gradient of the *dissipation* energy plastic function coincides with (is associated with) the failure function or limit surface:

$$\frac{\partial w}{\partial \mathbf{f}} = \boldsymbol{\lambda}^T \boldsymbol{\psi} \quad (1.49)$$

A *non-associated* plasticity condition will be one in which the dissipation potential function and the failure function are not related, and is common in friction problems, such as in soils:

$$\frac{\partial w}{\partial \mathbf{f}} = \boldsymbol{\lambda}^T \boldsymbol{\phi}; \quad \boldsymbol{\phi} \neq \boldsymbol{\psi}. \quad (1.50)$$

1.4.5 Fundamental theorems of limit analysis in associated plasticity

Only plastic deformation varies during collapse

We establish as an auxiliary tool a theorem that validates the rigid-plastic model for analysis, in the presence of sufficient ductility and, therefore, as long as strength is not reduced in the plastic deformation process.

In the deformation process, the parameters expressing the deformation rate $\dot{\mathbf{u}}$ can be considered as composed of the sum of the elastic $\dot{\mathbf{u}}_e$ and the plastic $\dot{\mathbf{u}}_p$ parts. If we consider two successive instances of the collapse process and the variations in stresses and loads between them, with $\dot{\mathbf{F}} = \partial \mathbf{F}$, $\dot{\mathbf{f}} = \partial \mathbf{f}$, the energy variations will be

$$\dot{\mathbf{U}}^T \dot{\mathbf{F}} - \dot{\mathbf{u}}^T \dot{\mathbf{f}} = 0 \quad (1.51)$$

but since loads do not vary, $\dot{\mathbf{F}} = \mathbf{0}$.

Decomposing the deformations between elastic and plastic components, we have

$$(\dot{\mathbf{u}}_e + \dot{\mathbf{u}}_p)^T \dot{\mathbf{f}} = 0 \quad (1.52)$$

$$\dot{\mathbf{u}}_e^T \dot{\mathbf{f}} + \dot{\mathbf{u}}_p^T \dot{\mathbf{f}} = 0 \quad (1.53)$$

But since this is associated plasticity, the second sum must be zero, which implies that the first is also zero. Given that stresses could be varying —within the “limit surface”— this implies that elastic deformations cannot vary.

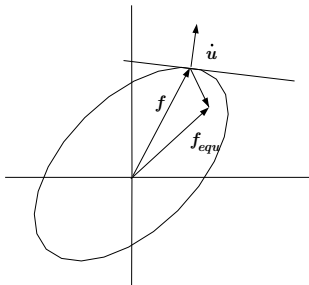


Figure 1.14: Static theorem. The limit of local resistances $\mathbf{f}$ is represented along with one of the imaginable cases for the failure load, as well as the case of stresses in equilibrium with the loads $\mathbf{f}_{\text{equ}}$.

Therefore, during collapse only plastic deformation varies, and consequently, any pre-existing elastic deformations are not relevant to the analysis.

Lower bound theorem, or static theorem

If in a structure, and for a given load system, which we will denote as $\mathbf{F}_{\text{equ}}$,

- a system of stresses $\mathbf{f}_{\text{equ}}$ in equilibrium with the loads can be established, and
- for these stresses the failure condition, or complete plastification, is not reached

it can be assured that the structure will not collapse under these loads, or equivalently, the loads $\mathbf{F}_{\text{equ}}$ are a lower bound to the failure loads $\mathbf{F}$ of that structure.

The proof is by contradiction: we assume the theorem is incorrect and verify that this assumption leads to a manifestly erroneous conclusion.

We will assume, therefore, that there is a collapse under loads $\mathbf{F}_{\text{equ}}$, collapse described by a compatible kinematics given by $[\mathbf{u}, \mathbf{U}]$ for which the associated stresses $\mathbf{f}$ would only be capable of equilibrating a smaller load, $\gamma\mathbf{F}_{\text{equ}}$, with $\gamma < 1$.

We now have a compatible kinematics $[\mathbf{u}, \mathbf{U}]$ and two possible equilibrated states $[\mathbf{f}, \gamma\mathbf{F}_{\text{equ}}]$ and $[\mathbf{f}_{\text{equ}}, \mathbf{F}_{\text{equ}}]$ for which, by the principle of virtual work, we can write

$$\mathbf{u}^T \mathbf{f} = \gamma \mathbf{U}^T \mathbf{F}_{\text{equ}} \quad (1.54)$$

$$\mathbf{u}^T \mathbf{f}_{\text{equ}} = \mathbf{U}^T \mathbf{F}_{\text{equ}} \quad (1.55)$$

therefore

$$\mathbf{u}^T (\mathbf{f} - \gamma \mathbf{f}_{\text{equ}}) = 0 \quad (1.56)$$

However, since the vector $(\mathbf{f} - \gamma \mathbf{f}_{\text{equ}})$ must be oriented towards the interior of the limit surface, see figure 1.14, it is not tangent to it and, therefore, cannot be orthogonal to the vector $\mathbf{u}^T$ corresponding to collapse. This implies that the conclusion is incorrect and, therefore, the assumption is incorrect.

In conclusion

$$\mathbf{F}_{\text{equ}} \leq \mathbf{F} \quad (1.57)$$

As corollaries, we can point out

- It will suffice to establish a set of stresses in equilibrium with the loads, and ensure resistance for these stresses to ensure that the loads do not exceed the structure's failure loads, provided there is sufficient ductility.
- Solutions based on elastic analysis are safe, despite uncertainties related to unknown imperfections in geometry or initial self-stress states.

The theorem is qualified in many texts as the *safety theorem*.

Upper bound theorem, or kinematic theorem

If in a structure,

- for an arbitrarily chosen compatible collapse system [$\mathbf{u}_{\text{com}}$, $\mathbf{U}_{\text{com}}$]
- loads are determined, which we will denote as $\mathbf{F}_{\text{com}}$, that lose as much potential energy in that collapse as is dissipated in the plastic deformation,

it can then be affirmed that the structure will collapse under these loads, or equivalently, that the loads $\mathbf{F}_{\text{com}}$ are an upper bound to the structure's failure load, $\mathbf{F}$. It is also demonstrated by contradiction.

If we assume there is no collapse, there would exist a state of internal stresses $\mathbf{f}_s$ different from those associated with the collapse kinematics. We can denote the latter as $\mathbf{f}_{\text{com}}$. For the stresses $\mathbf{f}_s$, the failure surface would not yet have been reached, and they would be in equilibrium with the loads $\mathbf{F}_{\text{com}}$.

Therefore, the following expressions are simultaneously sat-

isfied

$$\mathbf{u}_{\text{com}}^T \mathbf{f}_{\text{com}} = \mathbf{U}_{\text{com}}^T \mathbf{F}_{\text{com}} \quad (1.58)$$

$$\mathbf{u}_{\text{com}}^T \mathbf{f}_s = \mathbf{U}_{\text{com}}^T \mathbf{F}_{\text{com}} \quad (1.59)$$

The first is derived from the equality of internal and external energies deduced from the way we determine the loads $\mathbf{F}_{\text{com}}$. The second will be satisfied by the virtual work theorem, considering the supposed equilibrated situation, in which the stresses do not reach failure, together with the compatible collapse deformation arbitrarily chosen at the start.

If we subtract the second expression from the first, we will have

$$\mathbf{u}_{\text{com}}^T (\mathbf{f}_{\text{com}} - \mathbf{f}_s) = 0 \quad (1.60)$$

But this is not possible since the stress difference vector cannot be orthogonal to the deformation vector (parallel to the failure surface) as $\mathbf{f}_s$ is interior to the failure surface. Thus the assumption is incorrect and, therefore,

$$\mathbf{F}_{\text{com}} \geq \mathbf{F}. \quad (1.61)$$

The immediate corollary is that if we determine the failure load from any kinematics by equating dissipated energies between the mechanism and the loads, we obtain an unsafe situation. It should be noted that in this process, only the stresses of the points assumed to plastify are considered in practice, assuming that the others do not plastify and therefore do not dissipate energy locally, which means they are attributed stresses lower

than the available resistances in those positions. But the mechanism may be erroneous, so there will be points or positions not considered in the assumed mechanism where the corresponding stresses (not considered in plastic deformation) will have exceeded those of failure before those of the considered points. Correctly solving the problem by this approach would require enumerating all possible mechanisms and determining the lowest failure load among all mechanisms.

In any case, one could also, from the obtained solution, reconstruct the complete equilibrium situation, thereby determine the positions where the resistance condition is violated, and redesign such points to avoid it. This would be resorting to the static theorem as a means of obtaining safety.

Uniqueness theorem

We have seen two bounds that limit the failure load: a lower bound and an upper bound: $\mathbf{F}_{\text{equ}} \leq \mathbf{F} \leq \mathbf{F}_{\text{com}}$, so that if we link the set of conditions considered, we could eliminate the space between both bounds. This is what the uniqueness theorem does. Now we consider that

- we establish a state of stresses $\mathbf{f}$ in equilibrium with the loads $\mathbf{F}$
- in all regions of the structure the stresses are less than or equal to those of plastification, or equivalently, the resistance conditions $\psi \mathbf{f} \leq d\mathbf{r}$ are satisfied
- there exists a kinematically compatible collapse state $[\mathbf{u}, \mathbf{U}]$ in which the plastic flow regions correspond to the regions

where the stress equals that of plastification

In this case, it can be affirmed that the considered load is the structure's failure load.

Indeed, by the first and second conditions, and the static theorem, it will result that $\mathbf{F}$ is a lower bound of the failure load, but also, by the second and third conditions, and the virtual work principle, the dissipated energy will coincide with the work done by external loads, thus satisfying, by the kinematic theorem, that $\mathbf{F}$ is also an upper bound to the failure load, so that, when both bounds coincide, it turns out to be the failure load itself.

1.4.6 Fundamental variables and equations

We have from all the above a set of equations to approach problems with this approximation, in which we consider a set of initially prescribed loads $\mathbf{F}_I$: first the general trio of static, kinematic, and material relationship conditions, equations 1.37, 1.21, 1.38 and 1.40:

$$\begin{aligned}\mathbf{F} &= \gamma \mathbf{F}_I = \mathbf{H} \mathbf{f}, \quad \text{with } \mathbf{H} = \mathbf{B}^T \\ \dot{\mathbf{u}} &= \mathbf{B}^T \dot{\mathbf{U}} \\ \psi \mathbf{f} &\leq d\mathbf{r}\end{aligned}$$

Secondly, the expressions derived from Von Mises flow rule and collapse conditions, characterized by indicators or multipliers λ ,

equations 1.48 and 1.43:

$$\begin{aligned} \mathbf{u} &= \boldsymbol{\psi}^T \boldsymbol{\lambda} \\ (\boldsymbol{\psi} \mathbf{f} - d\mathbf{r})^T \boldsymbol{\lambda} &= 0 \end{aligned}$$

And lastly, the limit conditions for the failure load derived from the fundamental theorems of limit analysis:

$$\mathbf{F}_{\text{equ}} \left(\mathbf{f} | \mathbf{F}_{\text{equ}} = \mathbf{B}^T \mathbf{f} \ \& \ \boldsymbol{\psi} \mathbf{f} \leq d\mathbf{r} \right) \leq \mathbf{F} \quad (1.62)$$

$$\mathbf{F}_{\text{com}} \left(\mathbf{f}_{\mathbf{u}} | \dot{\mathbf{u}} = \mathbf{B}\dot{\mathbf{U}} \ \& \ \mathbf{f}_{\mathbf{u}} = \mathbf{r}_{\mathbf{u}} \ \& \ \dot{\mathbf{u}}^T \mathbf{f}_{\mathbf{u}} = \dot{\mathbf{U}}^T \mathbf{F}_{\text{com}} \right) \geq \mathbf{F} \quad (1.63)$$

equations that can both be read as “the load obtained from stresses that satisfy the indicated conditions is a bound of the failure load” (The first: the load F_{equ} , obtained from stresses $\mathbf{f}$ in equilibrium with the loads and satisfying the resistance condition is a lower bound of the failure load $\mathbf{F}$. The second: the load F_{com} obtained starting from a compatible mechanism, in which the resistant capacities of the pieces are established by the procedure of equating the energy dissipated by the plastification stresses in said mechanism with the energy lost by the loads, is an upper bound of the failure load)

With these resources, we can now proceed to both analysis and design of structures from the plastic or limit perspective.

1.4.7 Plastic analysis

Given the duality of structural problems (the conjugate “static” and “kinematic” perspectives), we can tackle the problems in-

dependently from each of them, which we do in the following sections.

In both cases, it involves an already defined and dimensioned structure, which is subjected to certain known loads, and for which we want to determine whether or not the failure load has been reached, as well as the stresses of all pieces.

1.4.7.1 The static approach

We consider applied loads (initially) to the structure $\mathbf{F}_I$, as well as a sizing for which sections $\mathbf{r}$ have been chosen that seek to ensure resistance conditions for the structure defined by the expression $\boldsymbol{\psi} \mathbf{f} \leq \mathbf{d} \mathbf{r}$, where the symbols $\boldsymbol{\psi}$ and $\mathbf{d}$ are known matrices although the internal forces $\mathbf{f}$ in the structure are (still) unknown.

We do not know the failure load for this sizing, so we formulate the problem as finding the load factor Γ capable of taking the initial loads $\mathbf{F}_I$ to failure, along with the internal forces $\mathbf{f}$ corresponding to the equilibrium reached at the collapse condition.

Therefore, for the limit load $\mathbf{F} = \Gamma \mathbf{F}_I$, both equilibrium conditions and resistance conditions must be satisfied.

Since the static theorem is applicable, the problem can be formally formulated as *finding the situation for which the highest positive value for the load factor Γ is obtained using the decision variables $\mathbf{f}$ and Γ and ensuring that both resistance and equilib-*

rium conditions are met. This can be formally established:

$$\begin{aligned}
 & \max_{\mathbf{f}, \Gamma}(\Gamma) \\
 & \psi \mathbf{f} \leq \mathbf{d} \mathbf{r} \\
 & -\mathbf{B}^T \mathbf{f} + \Gamma \mathbf{F}_I = 0 \\
 & \Gamma > 0
 \end{aligned} \tag{1.64}$$

This is a “linear programming” problem: obtaining values for certain variables that

- ensure a maximum for a linear combination of such variables
- satisfying a set of linear inequality or equality constraints for these variables

In this case, the “linear combination” being maximized involves only one of the design variables, although the constraints involve the entire set of these: $\mathbf{f}$ and Γ .

Since the solutions developed by Dantzig (see Dantzig 1963) and already used in logistical tasks during World War II, this type of problem has standard solutions among which the *simplex* is the most common, being therefore available in algebraic libraries, or in commonly used spreadsheets.

The equation written in matrix format takes the following

form⁵:

$$\begin{aligned} \max_{\mathbf{f}, \Gamma}(\Gamma) &= \max_{\mathbf{f}, \Gamma} \left(\begin{bmatrix} \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \Gamma \end{bmatrix} \right) \\ \begin{bmatrix} \psi & \mathbf{0} \\ -\mathbf{B}^T & \mathbf{F}_I \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \Gamma \end{bmatrix} &\leq \begin{bmatrix} d\mathbf{r} \\ \mathbf{0} \end{bmatrix} \\ \Gamma &> 0 \end{aligned} \quad (1.65)$$

1.4.7.2 The kinematic approach

The task now would be to determine the configuration of compatible movements-deformations $\dot{\mathbf{U}}$, $\dot{\mathbf{u}}$ that corresponds to collapse, as well as the corresponding failure load $\mathbf{F} = \gamma \mathbf{F}_I$, ensuring that the criteria for the collapse condition are met, along with those of the flow rule. In this case, since a collapse process can be equally described with any solution proportional to any of the compatible solutions (we can multiply the $\dot{\mathbf{U}}$ and $\dot{\mathbf{u}}$ of one solution to have values that are also solutions) we must add a *normalization* condition to obtain a unique solution.

We establish the following normalization condition:

$$\dot{\mathbf{U}}^T \mathbf{F}_I = \mathbf{F}_I^T \dot{\mathbf{U}} = 1 \quad (1.66)$$

Likewise, we use the compatibility condition (1.1) in which we include the restriction imposed by the flow rule (1.48), considering that our *decision variables* (the unknowns whose values

5. In the notation, and for simplification purposes, the rows corresponding to the two sets of expressions are separately identified, whether inequalities (resistance equations) or equalities (equilibrium equations).

must be determined) will be both the displacements $\dot{\mathbf{U}}$ and the plastic multipliers λ .

As we are considering collapse kinematics, the kinematic theorem is applicable, so we must find the minimum load factor for which the energy lost by the loads in the collapse equals that dissipated by the yielding sections.

The collapse condition 1.43 and the flow rule 1.48 allow us to write

$$\begin{aligned} (\psi \mathbf{f} - \mathbf{d}\mathbf{r})^T \lambda &= 0 \\ \mathbf{f}^T \psi^T \lambda &= \mathbf{r}^T \mathbf{d}^T \lambda \\ \mathbf{f}^T \mathbf{u} &= \mathbf{r}^T \mathbf{d}^T \lambda \end{aligned} \tag{1.67}$$

and by the kinematic theorem (external dissipation less than or equal to internal for the chosen mechanism)

$$\mathbf{F}^T \mathbf{U} = \gamma \mathbf{F}_I^T \mathbf{U} \leq \mathbf{f}^T \mathbf{u} = \mathbf{r}^T \mathbf{d}^T \lambda \tag{1.68}$$

equation from which results, if we use the normalization 1.66, the following

$$\gamma \leq \mathbf{r}^T \mathbf{d}^T \lambda \tag{1.69}$$

Thus the problem can be posed as finding the load factor, along with the collapse geometry and the corresponding positive collapse multipliers, that correspond to the minimum value of the product $\mathbf{r}^T \mathbf{d}^T \lambda$ for the conditions imposed by the combina-

tion of compatibility and flow rule and normalization:

$$\begin{aligned}
 \gamma &= \min_{U, \lambda} ((d\mathbf{r})^T \cdot \boldsymbol{\lambda}) = \min_{U, \lambda} \gamma(\boldsymbol{\lambda}) \\
 \boldsymbol{\psi}^T \boldsymbol{\lambda} - \mathbf{B}\mathbf{U} &= \mathbf{0} \\
 \mathbf{F}_I^T \mathbf{U} &= 1 \\
 \boldsymbol{\lambda} &\geq \mathbf{0}
 \end{aligned} \tag{1.70}$$

Again, this is the problem of finding an extreme (now minimum) in a linear combination of the decision variables, maintaining linear equality or inequality constraints for these same variables, that is, a linear programming problem solvable with standard tools, such as “simplex”.

The following will be the matrix format for the equation:

$$\begin{aligned}
 \gamma &= \min_{\lambda, U} \left([(\mathbf{d}\mathbf{r})^T \quad \mathbf{0}^T] \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{U} \end{bmatrix} \right) = \min_{\lambda, U} \gamma(\boldsymbol{\lambda}) \\
 \begin{bmatrix} \boldsymbol{\psi}^T & -\mathbf{B} \\ \mathbf{0}^T & \mathbf{F}_I^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{U} \end{bmatrix} &= \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \\
 \boldsymbol{\lambda} &\geq \mathbf{0}
 \end{aligned} \tag{1.71}$$

It is of great interest, as we will see in section 2.2.2.3, the transposed version of this equation, which has the following for-

mat:

$$\begin{aligned} \gamma &= \min_{\lambda, U} \left([\lambda^T \quad U^T] \begin{bmatrix} (dr) \\ \mathbf{0} \end{bmatrix} \right) = \min_{\lambda, U} \gamma(\lambda) \\ [\lambda^T \quad U^T] \begin{bmatrix} \psi & \mathbf{0} \\ -\mathbf{B}^T & \mathbf{F}_I \end{bmatrix} &= [\mathbf{0}^T \quad 1] \\ \lambda &\geq \mathbf{0} \end{aligned} \tag{1.72}$$

We see here an evident relationship with the equation from the static approach, 1.65, a relationship derived from the duality described later, in section 2.2.2.3.

1.4.8 The Plastic Project

The static theorem provides a powerful tool for design: *if there is sufficient ductility and stability, any stress distribution in equilibrium with the loads for which there is sufficient resistance is safe*. That is, it is possible to arbitrarily select the balanced stress distribution, and dimension to ensure sufficient resistance for such distribution in all regions of the structure.

We have already seen that this theorem validated elastic stress distributions, but even more, it validates the sizing of parts for this or any other balanced distribution if there is sufficient resistance.

From this perspective, it is possible to approach the problem of sizing in such a way that the cost of the solution is as low as possible, while maintaining load capacity, that is, the problem of obtaining the distribution of stresses and resistances that, supporting the load, that is, maintaining equilibrium with the

maximum anticipated load under conditions that do not violate the resistance condition, turns out to be of minimum cost.

The solution is simple and immediate if it were possible to establish cost indices per unit of resistant capacity, that is, if the cost of the structure can be established through linear laws associated with the resistant capacities $\mathbf{r}$ susceptible to use in the project.

That is, laws of $\mathbf{f}$ and $\mathbf{r}$ are sought that fulfill equilibrium and resistance considering that the cost of the structure to be minimized can be computed through the expression $C = \mathbf{c}_u^T \mathbf{r}$, for known cost coefficients $\mathbf{c}_u$.

The problem turns out to be the following linear programming problem:

$$\begin{aligned} \min_{\mathbf{f}, \mathbf{r}} C(\mathbf{r}) &= \min_{\mathbf{f}, \mathbf{r}} \mathbf{c}_u^T \mathbf{r} \\ \psi \mathbf{f} - d \mathbf{r} &\leq \mathbf{0} \\ \mathbf{B}^T \mathbf{f} &= \mathbf{F} \\ \mathbf{r} &\geq \mathbf{0} \end{aligned} \tag{1.73}$$

or, in matrix format

$$\begin{aligned} \min_{\mathbf{f}, \mathbf{r}} C(\mathbf{r}) &= \min_{\mathbf{f}, \mathbf{r}} \left(\begin{bmatrix} \mathbf{0}^T & \mathbf{c}_u^T \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{r} \end{bmatrix} \right) \\ \begin{bmatrix} \psi & -d \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{r} \end{bmatrix} &\leq \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \end{bmatrix} \\ \mathbf{r} &\geq \mathbf{0} \end{aligned} \tag{1.74}$$

It is a problem solvable, again, through simplex.

1.4.9 Generalization to Other Problems

Solutions 1.65, 1.71 in analysis problems, or 1.74 for design problems can be easily generalized to different problems using the same means.

A typical case is to consider that the load is composed of two parts, one invariant, and another that increases until reaching failure—for example in seismic loading, the gravitational load versus the horizontal load—

The failure load $\mathbf{F}$ is now the sum of two parts, of which only one is affected by the load factor. The “decision variables” remain the same, so it suffices to reorder the two parts of the load in the equilibrium equations.

$$\mathbf{F} = \mathbf{F}_0 + \Gamma \mathbf{F}_1 \implies -\mathbf{B}^T \mathbf{f} + \Gamma \mathbf{F}_1 = -\mathbf{F}_0. \quad (1.75)$$

The solution by “Simplex” to the maximum load factor Γ is thus formulated as follows:

$$\begin{aligned} \max_{\mathbf{f}, \Gamma} & \left(\Gamma = [\mathbf{0}^T \quad 1] \begin{bmatrix} \mathbf{f} \\ \Gamma \end{bmatrix} \right) \\ \begin{bmatrix} \psi & \mathbf{0} \\ -\mathbf{B}^T & \mathbf{F}_1 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \Gamma \end{bmatrix} & \leq \begin{bmatrix} d\mathbf{r} \\ -\mathbf{F}_0 \end{bmatrix} \\ \Gamma & \geq 0 \end{aligned} \quad (1.76)$$

using $\mathbf{F}_0$ for the constant load part and $\mathbf{F}_1$ for the part of load that is amplified.

In the case of sizing problems, it may be the condition that part of the sections are prefixed, seeking only the optimal ones

for the pending part of the structure. If the prefixed resistances are $\mathbf{r}_f$ and those to be determined are $\mathbf{r}_d$ it can be established

$$\mathbf{d}\mathbf{r} = \mathbf{d} \cdot \mathbf{r} = \mathbf{d}_f \cdot \mathbf{r}_f + \mathbf{d}_d \cdot \mathbf{r}_d \quad (1.77)$$

so that the resistance constraints can be formulated as

$$\psi \cdot \mathbf{f} - \mathbf{d}_d \cdot \mathbf{r}_d \leq \mathbf{d}_f \cdot \mathbf{r}_f \quad (1.78)$$

and the “Simplex” solution for minimum cost in the pending decision region will be

$$\begin{aligned} \min_{\mathbf{f}, \mathbf{r}_d} & \left(C = [\mathbf{0}^T \quad \mathbf{c}_{r_d}^T] \cdot \begin{bmatrix} \mathbf{f} \\ \mathbf{r}_d \end{bmatrix} \right) \\ \begin{bmatrix} \psi & -\mathbf{d}_d \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{f} \\ \mathbf{r}_d \end{bmatrix} & \leq \begin{bmatrix} \mathbf{d}_f \cdot \mathbf{r}_f \\ \mathbf{F} \end{bmatrix} \\ \mathbf{r} & \geq \mathbf{0} \end{aligned} \quad (1.79)$$

using $\mathbf{c}_{r_d}$ for the unit costs associated with the resistance laws of the pieces to be designed.

1.4.10 Summary and Generalization of Fundamental Variables and Their Relationships

We can summarize the set of variables handled in the model to define the structure and to determine analysis solutions:

The structure is defined:

- Globally, by the static and kinematic characterization of degrees of freedom: $\mathbf{F}$, $\mathbf{U}$.

- Locally
 - by the static and kinematic characterization of the bars: $\mathbf{f}, \mathbf{u}$
 - by the characterization of local failure conditions according to resistant capacities: $\psi, \mathbf{r}$
- In its geometry and connectivity: $\mathbf{B}$.

Each load condition imposes an initial “vector” $\mathbf{F}_I$

The analysis determines for each initial “load direction” $\mathbf{F}_I$

- the load factor $\gamma = \Gamma$ that leads to the failure load $\mathbf{F}$ for that “direction”
- the corresponding collapse configuration $\lambda, \mathbf{U}$
- the stresses $\mathbf{f}$ and the plastic component $\mathbf{u}$ of the deformation for such stresses.

Each “load direction”⁶ would activate some mechanism i of the n collapse mechanisms that could be enumerated in the structure. This mechanism could be represented by its corresponding kinematics: $\mathbf{U}_i$, to which would correspond a “hinge” configuration and plastification geometries given by $\lambda_i, \mathbf{u}_i$. In some regions p , identified by non-zero values of $\lambda_i, \mathbf{u}_i$, the stress would be equal to the resistant capacity, being less than this in the remaining (regions s).

Considering the linearity of the problem, to each limit plane described by ψ_i in the stress space $\mathbf{f}$, one could associate the corresponding limit plane Ψ_i in the load space $\mathbf{F}$ so that the

6. for example in a frame with gravitational and wind load, different “directions” would be the corresponding combinations of different gravitational and wind loads.

set of such “limit regions” would be nothing more than a hyperpolyhedron projection in $\mathbf{F}$ of the limit surface ϕ considered for the $\mathbf{f}$. Or in other words, the resistance constraints could be declared in the load space in a format similar to 1.40:

$$\Psi \mathbf{F} \leq \mathbf{R} \quad (1.80)$$

If the plastification rate $\dot{\mathbf{u}}$ for each local collapse mechanism is linked to the multipliers λ , by virtue of the flow rule 1.48, one could associate multipliers Λ to the collapse rate of each global mechanism considered, positive if that mechanism occurs, null otherwise, in such a way that the space of possible Λ combinations would describe all possible mechanisms, and each possible direction in that space one of such mechanisms or combination.

We can easily imagine that the local multipliers λ can be constructed from the global Λ through linear expressions, so that we would have:

$$\lambda = \mathbf{L} \Lambda \quad (1.81)$$

For the collapse condition in the global space, we could write, considering 1.43

$$(\Psi \mathbf{F} - \mathbf{R})^T \Lambda = 0 \quad (1.82)$$

If additionally we use 1.81 in a variant of 1.43 where we use $\mathbf{d} = \mathbf{I}$ we have

$$(\psi \mathbf{f} - \mathbf{r})^T \mathbf{L} \Lambda = 0 \quad (1.83)$$

where, introducing $\mathbf{L}$ in the term in parentheses, results

$$(\mathbf{L}^T \psi \mathbf{f} - \mathbf{L}^T \mathbf{r})^T \Lambda = 0 \quad (1.84)$$

Comparing this last expression with 1.82 it becomes possible to suggest the following identifications:

$$\Psi \mathbf{B}^T \equiv \mathbf{L}^T \psi \quad (1.85)$$

$$\mathbf{R} \equiv \mathbf{L}^T \mathbf{r} \quad (1.86)$$

Finally we can affirm, in analogy with 1.48 that:

$$\mathbf{U} = \Psi^T \mathbf{\Lambda} \quad (1.87)$$

because indeed, if we multiply both terms by $\mathbf{B}$, we obtain that same expression:

$$\mathbf{B}\mathbf{U} = \mathbf{B}\Psi^T \mathbf{\Lambda}$$

$$\mathbf{u} = \psi^T \mathbf{L}\mathbf{\Lambda}$$

$$\mathbf{u} = \psi^T \boldsymbol{\lambda}$$

In them, it is interesting to see (equation 1.84) that the product matrix $\Psi \mathbf{B}^T \equiv \mathbf{L}^T \psi$ allows determining, starting from the stresses $\mathbf{f}$, its comparison effect with the global load capacity against collapse, $\mathbf{R} \equiv \mathbf{L}^T \mathbf{r}$, while its transpose $\mathbf{B}\Psi^T \equiv \psi^T \mathbf{L}$ determines the deformations $\mathbf{u}$ corresponding to the collapse configuration $\mathbf{\Lambda}$.

With all these expressions we can finally condense the model as a whole, as we did with the elastic model, and as we do in the following section.

1.4.11 The Plastic Model at a Glance

The set of variables and relationships of the plastic model, considering its local or global versions, as well as its static and

kinematic components, is summarized in the following symbolic scheme, which is created using criteria analogous to those used for the elastic analysis summary, 1.31

$$\begin{array}{ccccc}
 u & \xleftarrow{\psi^T} & \text{COMPATIBILITY}(\mathbf{B}) & \xleftarrow{\Psi^T} & U \\
 \downarrow & \searrow \lambda \downarrow & \psi^T \mathbf{L} = \mathbf{B} \Psi^T & \downarrow \Lambda & \downarrow \\
 \text{Int. Work} & = \text{Internal Dissipation} & \geq \text{External Dissipation} & = & \text{Ext. Work} \\
 \uparrow & \nearrow r \uparrow & \longrightarrow \mathbf{L}^T \longrightarrow & \uparrow \mathbf{R} & \uparrow \\
 f & \longrightarrow & \text{EQUILIBRIUM}(\mathbf{H} = \mathbf{B}^T) & \longrightarrow & F
 \end{array} \quad (1.88)$$

1.4.12 Assumptions, Values, and Limitations of the Limit Plastic Model

The plastic approximation is, by virtue of the static theorem, of exceptional utility as it allows validating solutions based on simple equilibrium conditions endowed with sufficient strength, including the equilibrium conditions obtained by elastic methods.

The basic assumptions are

- it involves *associated plasticity*, that is, the orthogonality condition is met between plastic strain rates and the (tangent) plane representing the limit surface, which contains the stress variations from the onset of collapse, encompassing all possible stresses for a given collapse mode.
- it considers that there is sufficient ductility in all sections to

ensure that strength is maintained throughout the deformation necessary to reach the collapse state,

The first assumption does not apply in non-associated plasticity situations, such as in all problems where friction phenomenon is involved in collapse: soils, sliding in structures, etc. It does hold, however, in linear piece problems in masonry structures (arches) when failure responds to rotations without sliding. We will see this case in section 2.2.1.4.

The second assumption requires carefully considering the relationship between ductility demand and capacity in the problems considered, taking into account both material and geometry. While demand is linked to global deformation, it is potentially greater in slender structures. Likewise, such demand will be greater the further the equilibrium scheme reached in failure is from the probable elastic scheme for the structure, considering realistic support conditions.

The plastic approach, as seen so far, also does not consider stability issues associated with the consequences of structure deformation (load displacements, etc.) and is therefore inapplicable in cases where stability is the critical condition: buckling, etc.

As a non-linear approximation, the superposition principle does not apply.

It is therefore a tool that allows validating designs, and a good reference for the last stages of deformation if sufficient ductility exists and no instability phenomena are present.

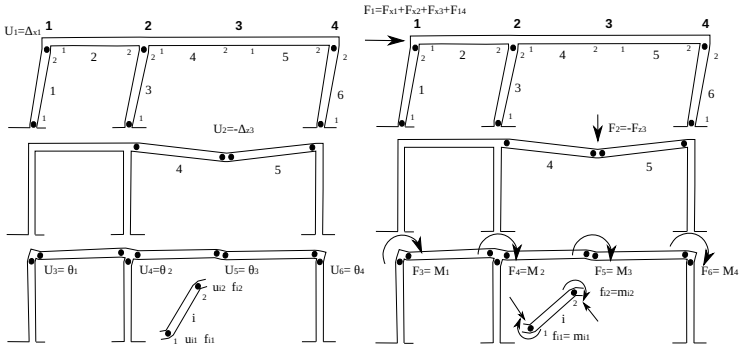


Figure 1.15: Plastic Modeling of a Two-Bay, One-Story Frame: Kinematics (left) and Statics (right). The isolated bar corresponds to an arbitrary bar i , and represents the criteria for end numbering and positive rotation and end stress values.

1.4.13 Application Examples

1.4.13.1 Establishing Parameters for Limit Stresses and Deformations

The example presented corresponds to a frame that could be analyzed using both elastic and limit approaches.

In figure 1.15 the parameters are represented for possible independent collapse kinematics and the corresponding load parameters, energetically conjugate to the former: 1: frame collapse by lateral displacement; 2: deck collapse; 3, 4, 5 and 6: uncontrolled joint rotations. Combined collapse conditions can

occur: for example, the combination of 1 and 2 would imply simultaneous lateral displacement and deck sinking, with the hinge around the right joint occurring in one of the ends, 2 of beam 5, or 2 of column 6, at the point of least resistance for an identical bending moment.

The fact that the model allows considering both possibilities for hinge location requires (and allows) independence of joint rotation as a parameter, which permits, from the perspective of equations summarized in $\mathbf{B}^T = \mathbf{H}$, the model to coincide with the approach used in the elastic approximation.

In this model we will have (some of) the compatibility conditions in the following expression:

$$\begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ \dots \\ u_{51} \\ u_{52} \\ \dots \end{bmatrix} = \begin{bmatrix} -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{1}{l_5} & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{l_5} & 0 & 0 & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix}$$

and for the equilibrium conditions:

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{h} & -\frac{1}{h} & 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{l_5} & \frac{1}{l_5} & \dots \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \dots \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \\ \dots \\ f_{51} \\ f_{52} \\ \dots \end{bmatrix}$$

But now we will have that the collapse load must be determined at the limit by an increase factor γ on the initial known load vector $\mathbf{F}_I$: The equilibrium considered is that of the breaking stresses with the load $\mathbf{F}_I$ amplified until reaching the breaking limit: $\mathbf{F} = \gamma \mathbf{F}_I = \mathbf{H} \mathbf{f}$.

1.4.13.2 Limit Material Relations: Strength Conditions

Figure 1.16 shows a bar subjected to end moments, where hinges can form if the plastic moment is reached, with three possible diagrams for a rigid-plastic model, that is, a model with no deformation until plastification, and indefinite deformation at that point. Each graphic is described on different axes:

- the one relating moment to rotation at end 1. For a bar i : $f_{i1}(u_{i1})$
- the one relating moment to rotation at end 2: $f_{i2}(u_{i2})$ or simply $f_2(u_2)$ if referring to a single bar.

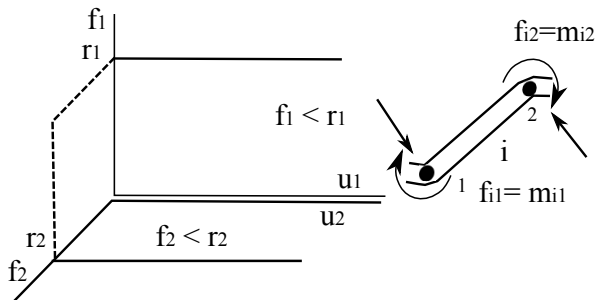


Figure 1.16: Plastic Moment-Rotation Relations and Limit Strengths for a Bar with Bending Moments at Both Ends.

- the one establishing the plastic moment strength limits appropriate to each end: $f_1 = m_1 \leq r_1$; $f_2 = m_2 \leq r_2$

The last figure is defined on the pair of axes f_1 and f_2 where end moments are identified, although there is no interaction between both efforts for the strength limit condition, since the limits for each end are independent.

However, if we consider the potential interaction of end moment with axial force, the figure containing the strength limit conditions for that same bar must now include three axes (three parameters), namely, the two end moments and the axial force. The figure is 1.17. For that figure, the conditions for zero axial force remain as in the previous figure, but now the rest of the scenario changes.

If we consider each end in isolation, the strength limit will

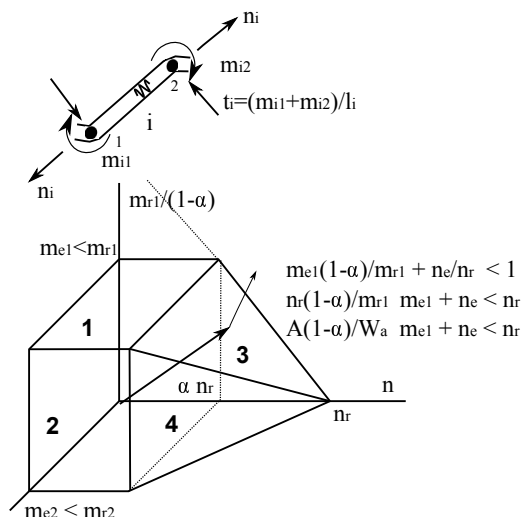


Figure 1.17: Limit Strengths of a Bar with Axial Force and Bending Moments at Both Ends.

depend on the proportions between bending moment and axial force.

A usual approximation in rolled sections is to consider that flexural strength is provided by the flanges and that, as long as the axial force does not exceed the web's capacity, the flexural strength remains unchanged, reducing linearly from that point until nullifying as the axial force approaches the section's resistance limit. This gives rise to bilinear limit graphics for each

end, m_1, n and m_2, n , which we can extend to perpendicular limit planes as the strengths of both ends are independent.

Thus, we have the following equations for the strength limit (considering the subscript difference between load effect values e and resistance capacity values r , using W_a for the flange resistance module, A for the section area, αA for the web area portion, and f_y for the steel yield strength):

$$\begin{aligned} m_{e1} &\leq m_{r1} = W_a f_y \\ m_{e2} &\leq m_{r2} = W_a f_y \\ \frac{m_{e1}(1-\alpha)}{m_{r1}} + \frac{n_e}{n_r} &\leq 1 \\ \frac{m_{e2}(1-\alpha)}{m_{r2}} + \frac{n_e}{n_r} &\leq 1 \end{aligned}$$

The first two reflect the limit for small axial force, and the last two for axial force above what can be resisted by the web alone.

If we look at either of the last two, we can successively rewrite them:

$$\begin{aligned} n_r \frac{m_{ei}(1-\alpha)}{m_{ri}} + n_e &\leq n_r \\ \frac{A(1-\alpha)}{W_a} m_{ei} + n_e &\leq n_r \end{aligned}$$

and therefore, substituting n_r by Af_y and m_r by W_af_y the

complete set can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{(1-\alpha)A}{W_a} & 0 & 1 \\ 0 & \frac{(1-\alpha)A}{W_a} & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ n \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} W_a f_y \\ A f_y \end{bmatrix}$$

Expression in which we can identify the general format of equation 1.40, where the first matrix ψ expresses the interactions considered in each check, $\mathbf{f}$ the stress parameters considered, $\mathbf{r}$ the resistance capacities defined for the pieces, and $\mathbf{d}$ the way of attributing these capacities to each check.

It is important to observe now that each row of matrix ψ contains the coefficients of the limit plane for the corresponding check (here a 3D plane, although for more stress parameters it would be a hyperplane with as many dimensions as parameters). A property of such coefficients is that in this space (here in 3D) they define vectors orthogonal to said limit plane, so that if the equations are scaled so that these coefficients have a unit module, they would be the *versors* of these limit planes.

In the figure, these 4 limit planes are identified and numbered according to the resistance limit equation they represent.

1.4.13.3 Indeformable Body Supported on Several Articulated Bars

We continue here with the limit version of the problem solved elastically in section 1.3.7.4, according to figure 1.8.

The task is to characterize and resolve the problem of load and failure stresses for that same structure.

The most sensible approach consists of reinterpreting the parameters used in the elastic analysis in a plastic key, considering the same stresses for the local view, and deformations as somehow accumulated in the appropriate plastic region: in the case we had considered bending stresses, we would have hinges at the node joints, which would continue to allow locating the global displacement and load variables. Since we only have axial stresses, deformation will be interpreted as a yield elongation in some part of each bar —its precise position is not important— so that it remains the energetically conjugate of the axial stress.

The equilibrium in the limit condition can still be represented as in figure 1.9, and the same applies to the compatibility of displacements and deformations, described in figure 1.10, so that the equilibrium and compatibility equations are the same as those we used for the elastic approximation, although now they will be interpreted as part of the collapse configuration qualities.

In any case, it is of interest to emphasize again the way of constructing the equations and corresponding matrices. Each of the equilibrium figure drawings corresponds to one of the rows of matrix $\mathbf{H} = \mathbf{B}^T$ as it isolates the equilibrium of a single load component, while each of the compatibility figure drawings corresponds to one of the columns of matrix $\mathbf{B}$, as it represents the deformation induced in some of the structure elements by each of the global displacement components considered in isolation.

Therefore, the only thing that changes now in the numerical format of the model is the group of equations defining material conditions: now these will be strength limit conditions.

Since we are talking about axial stresses in a material with

similar behavior in tension and compression, we can write them in the following form:

$$|f_1| \leq r_1 = r_e$$

$$|f_2| \leq r_2 = r_i$$

$$|f_3| \leq r_3 = r_i$$

$$|f_4| \leq r_4 = r_e$$

where $r_e = A_e f_y$ will be the resistance of exterior columns and $r_i = A_i f_y$ that of interior columns, the areas being the effective ones if the possibility of buckling must be considered —plastic in this case— Given the complications of the numerical treatment of the absolute value, the following format is more useful, in which the limit for such absolute value is replaced by the pairs of limits for the positive or negative sign of the effort, considering the resistant capacities as positive values.

$$f_1 \leq r_e$$

$$-f_1 \leq r_e$$

$$f_2 \leq r_i$$

$$-f_2 \leq r_i$$

$$f_3 \leq r_i$$

$$-f_3 \leq r_i$$

$$f_4 \leq r_e$$

$$-f_4 \leq r_e$$

These equations, considering that there are only two variables for the design of the strengths, namely, the strengths of exterior

and interior pillars, as well as only four stress variables, can be rewritten in the following form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_e \\ r_i \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

expression that responds to the general format of 1.40

$$\psi f - dr \leq 0$$

Since we have all the elements necessary for the formulation of the analysis problem, either in the static approximation (see 1.65) or in the kinematics (see 1.71), we can obtain the solutions for the load factor and the associated stresses, or for the geometry of the collapse with the help of any of the resources that have the “simplex” among their tools. The same could be said in relation to the design problem addressed with the solution 1.74.

The following screenshots show the formulation and solutions developed with the LibreOffice spreadsheet, and using the “Solver”, using Simplex as the solution algorithm, the step-by-step process being very simple.

1.4. PLASTIC ANALYSIS AND DESIGN

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	A	B	C	D	E	F	G	H	I	J	K	L	M
1	a	alpha	coseno	seno	longitud	p interior	p exterior						
2		1.325817664	0.242535625	0.9701425	1.030776406	1.15	0.85	A					
3		75.96379553				1.115663979	0.824621129x						
4	B				H=B*Y								
5		0.242535625	0.9701425	-0.54570516		0.242535625	-0.242535625	0.242535625	-0.24253563				
6		-0.24253563	0.9701425	-0.42443734		0.9701425	0.9701425	0.9701425	0.9701425				
7		0.242535625	0.9701425	0.42443734		-0.54570516	-0.42443734	0.42443734	0.545705156				
8		-0.24253563	0.9701425	0.545705156									
22	ANALISIS (PERITAJE)												
28	ESTÁTICO												
29													
30	Psi					DECISION	operacion	restriccion u objetivo (estático)					
31		1	0	0	0	f	f	d					
32		1	0	0	0	0	0.85	0.85			1	0	0.85 pe
33		1	0	0	0	0	-1.092857143	-0.85			1	0	1.15 pi
34		0	1	0	0	0	1.092857143	-1.09285714			0	1	
35		0	1	0	0	0	-0.85	1.092857143			0	1	
36		0	0	1	0	0	0.942424143	1.092857143			0	1	
37		0	0	1	0	0	0	-1.09285714			0	1	
38		0	0	0	1	0	0	-0.85			1	0	
39		-0.24253563	0.242535625	-0.24253563	0.242535625	1	-2.56017E-13				1	0	
40		-0.9701425	-0.9701425	-0.9701425	-0.9701425	-1	-0.94242414						
41		0.545705156	0.42443734	-0.42443734	-0.54570516		-3.42171E-13						
42		-H				F, J							

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3		75.96379553				1.115663979	0.824621129x							
9					k									
23	ANALISIS (PERITAJE)													
24	CINEMÁTICO													
25		0	0	0	0	0	0	0						
26		0.554367143	0	0	0	0	0	0.554367143			1	0	-0.57142857	
27	lambda													
28														
29	ESTÁTICO													
30						DECISION	operacion	restriccion u objetivo (estático)						
31	Psi					f	f	d						
32		1	0	0	0	0	0.85	0.85			1	0	0.85 pe	
33		1	0	0	0	0	-1.092857143	-0.85			1	0	1.15 pi	
34		0	1	0	0	0	1.092857143	-1.09285714			0	1		
35		0	1	0	0	0	-0.85	1.092857143			0	1		
36		0	0	1	0	0	0.942424143	1.092857143			0	1		
37		0	0	1	0	0	0	-1.09285714			0	1		
38		0	0	0	1	0	0	-0.85			1	0		
39		0	0	0	-1	0	0	0.85			1	0		
40		-0.24253563	0.242535625	-0.24253563	0.242535625	1	-2.56017E-13							
41		-0.9701425	-0.9701425	-0.9701425	-0.9701425	-1	-0.94242414							
42		0.545705156	0.42443734	-0.42443734	-0.54570516		-3.42171E-13							
43		-H				F, J								
44		6.04516E-14	-1.27676E-14	1.27676E-14	-6.04516E-14		1 Operacion (Enfoque Cinemático)							
45		0	0	0	0		1 Objetivo Restriccion (enfoque cinemático)							

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H59 $\text{F} \times \text{E} = (\text{MMULT}(\text{TRANSPOSE}(\text{H49:D54}); \text{H49:H54}))$

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2		1.358817662	0.942355625	0.330776400	1.135	0.8								
3		75.06375653			1.115663675	0.824621125x								
4					H=PT									
5		0.242535625	0.9701425	-0.54570516		0.242535625	-0.242535625	0.242535625	-0.24253563					
6		-0.24253563	0.9701425	-0.42443734		0.9701425	0.9701425	0.9701425	0.9701425					
7		0.242535625	0.9701425	0.42443734		-0.54570516	-0.42443734	-0.42443734	-0.42443734					
8		-0.24253563	0.9701425	0.54570516										
46														
47		DIMENSIONADO				DECISION				MULTIP.			OBJETIVO	
48						f/i								
49						0.901829356	0					1.11022E-16		
50		-1	0	0	0	-1.67901166	0					1.80389871		
51		0	1	0	0	-1.064423624	0					-3.30023382		0
52		-1	-1	0	0	-1.901829356	0					0	<=	0
53		0	0	1	0	-1.901829356	1					-1.03077641		0
54		0	0	-1	0	-1.67901166	1					-2.31504691		0
55		0	0	0	-1	>=	-1					1.80389871		0
56		0	0	0	-1	-1	0					3.58491E-13		0
57		0.242535625	-0.24253563	0.242535625	-0.24253563	0	0					1	=	1
58		0.9701425	0.9701425	0.9701425	0.9701425	0	0					1	=	1
59		-0.54570516	-0.42443734	-0.42443734	-0.54570516	0	0					5.01832E-13		0
60														
61														
62														

Hoja1

Buscar Buscar todo Cadena formateada Distinguir mayúsculas y minúsculas

Elastic analysis.

- Input of data (geometry, loads $\mathbf{F} \dots$);
- construction of matrix $\mathbf{B}$ and its transpose in $\mathbf{H}$, or vice versa; the numbers of rows or columns correspond to the number of variables in the local space (number of stress parameters) or global space (number of degrees of freedom).
- construction of matrix $\mathbf{k}$;
- matrix $\mathbf{K} = \mathbf{H}\mathbf{k}\mathbf{B}$ with the expression `MMULT(H;MMULT(k;B))` applied in the selection of the region that will occupy $\mathbf{K}$, where the matrices are referenced by the rectangle of their corresponding cells, and where the instruction is inserted by pressing simultaneously **Ctrl+Shift+Enter**.
- Construction of the inverse $\mathbf{K}^{-1}$
- solutions are immediate with $\mathbf{U} = \mathbf{K}^{-1}\mathbf{F}$; $\mathbf{u} = \mathbf{B}\mathbf{U}$; $\mathbf{f} = \mathbf{k}\mathbf{u}$.

Plastic Analysis, static approach.

- Construct the *constraint coefficients* matrix with its 4 parts $\boldsymbol{\psi}$, $\mathbf{0}$, $-\mathbf{B}^T$, $\mathbf{F}$, where only $\boldsymbol{\psi}$ is new.
- Space for the decision variables (column) vector $\mathbf{f}$, Γ , which is filled with arbitrary values
- Product of constraint coefficient matrices and decision variables: these will be the “result” values for comparison, consisting of two parts:
 - conditions effect of stresses corresponding to strength constraints ($\leq$)

- conditions corresponding to equilibrium constraints (=)
- For the purpose of constructing the column corresponding to the *comparison objectives* vector for the above:
 - establish the strength vector $\mathbf{r}$ to consider, considering exclusively those of independent decision (for example if all pieces must have the same strength there would be only one value)
 - construct matrix $\mathbf{d}$ that assigns strengths to the strength inequalities in which they are applicable,
 - Product $\mathbf{d}\mathbf{r}$ for capacity or strength conditions (the target limit capacity values)
 - extend the column thus obtained with $-\mathbf{F}_0$, where $\mathbf{F}_0$ is the part of the load to which the load factor Γ is not applied, in this case null, and therefore adding as many zeros as equilibrium equations.
- use of the Solver (Spreadsheet linear solver)
 - Indicating that the target cell is the box with Γ
 - Seeking to maximize
 - Varying the values of the cells we have called decision variables
 - With strength constraints, comparison ($\leq$) between result values and target
 - With equilibrium constraints, comparison ($=$) between result values and target
 - Ensuring $\Gamma > 0$ (comparison with a cell where a zero has been placed)

- Options: Linear solver, no additional options needed
- SOLVE.

The result of the operation is the updating by the “Solver” of the values of the decision variables $\mathbf{f}$, Γ to those that solve the problem.

Plastic Analysis, kinematic approach. For the kinematic approach, the same technique is used, either for the original comparison, or for the transposed comparison of equation 1.72.

In the first option, a transposed copy of the constraint coefficients matrix is created, using as column of decision variables the $\boldsymbol{\lambda}$ and $\mathbf{U}$ and arranging the comparison terms in columns.

In the second option, which requires less storage space, the same constraint coefficients matrix is used as in the static approach, using rows for both the decision variables and for the values corresponding to each of the two constraint terms: the results derived from the values adopted for the decision variables and the target values for these.

In this case, the limit value, now to be minimized, will be placed in a cell containing the result of the dot product $\boldsymbol{\lambda}^T(d\mathbf{r})$.

Chapter 2

Fundamentals and techniques of analysis

Having developed the basic concepts that are handled in the theories used to describe, analyze and design structures, the structural theory, this chapter develops the relationship of these concepts with traditional formulations for the most common structural types, fundamentally structures conceptualized as collections of bars, perhaps more complex pieces like blocks, connected to each other by nodes or joints that allow delimiting them and describing their interaction with precision.

The formulation seeks to conceptualize the internal behavior of such pieces as well as the conditions of their joint work in the structure in their interaction through their links, nodes or joints.

2.1 The elastic approach

2.1.1 Modeling of bars

This section describes how to model the mechanical and kinematic qualities of bars conceived as a continuous sequence of slices described along a linear directrix, usually straight. We are talking, therefore, about structures formed by bars. We seek to precisely describe the internal behavior of these bars and its relationship with the *boundary conditions*, that is, with the conditions imposed at the nodes, usually defined at their ends.

2.1.1.1 Section properties; normal stresses

We consider a slice extracted from a bar, in which three coordinate axes x, y, z are defined, with the origin at the directrix, located at the center of gravity of the section, the first axis in the direction of the directrix, and the other two perpendicular to this and to each other. We have two faces corresponding to two consecutive sections. We generally consider the face corresponding to the increasing direction in the x axis and the *stresses on* this, and the *movements of* this face in relation to those of the face in the decreasing direction.

Figure 2.1 describes the two cuts, decreasing and increasing, of a slice (a section) with the three reference axes, and in which the normal stresses and the corresponding forces —balanced and therefore equal and opposite except for slight variation between both cuts— are represented by the action on the slice on its

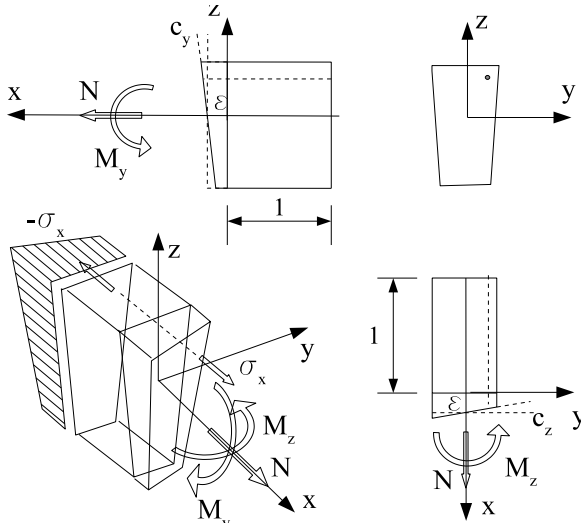


Figure 2.1: Normal stresses in a bar section and their corresponding forces and deformations

increasing face, using positive signs for actions in the positive direction of the axes or their respective rotations according to the right-hand rule: from y to z around x , from z to x around y , and from x to y around z .

The 3D image shows the stress at a generic point (y, z) and the complete stress block, as well as their resultants referred to the axes. The front and side images show the section, as well as its deformations, understood as the relative displacements of

one face with respect to the preceding one, located at a unit distance. The Navier-Bernoulli model is considered, whereby the plane section remains plane after deformation and where the section normal to the directrix remains normal to the elastic deformed shape of said directrix. This makes it possible to describe the deformation of the slice by the longitudinal deformation of the directrix ε and its two rotations around the z and y axes which, for the unit of length, can be described by the curvatures c_z and c_y , which in the figure are drawn in the positive direction, according to the criteria of the previous paragraph. The corresponding forces associated with such deformations are represented, namely, the normal force N_x and the bending moments on the section, M_y, M_z .

The other three possible movements if we consider the six rigid movements of the face considered, that is, the rotation around x , and the displacements parallel to the axes y, z , are associated with shear stresses, their resulting forces being the torsional moment M_x and the shear with its two components parallel to such axes, T_y, T_z , components that are not considered for now in the model.

The normal stress at any point in the section is $\sigma_x(y, z)$, with none on the outer faces of the bar, and is elastically associated with the longitudinal deformation of the corresponding fiber $\epsilon_x(y, z)$, so that, according to figure 2.1, we have:

$$\sigma_x(y, z) = E \epsilon_x(y, z) = E (\varepsilon + z c_y - y c_z) \quad (2.1)$$

This stress distribution assumes the following resultants (or aggregate values), associated with the three deformations con-

sidered in the slice $[\varepsilon \ c_y \ c_z]$:

$$N_x = \int \sigma_x \, dA = \int E (\varepsilon + z c_y - y c_z) \, dA = A \sigma_0 \quad (2.2)$$

$$\begin{aligned} My &= \int z \sigma_x \, dA = \int z \epsilon_x E \, dA \\ &= \int z E (\varepsilon + z c_y - y c_z) \, dA = c_y E \int z^2 \, dA \\ &= EI_y c_y = I_y \frac{\sigma_{x,z_{\max}} - \sigma_o}{z_{\max}} = I_y \frac{\Delta \sigma_{x,z}}{z} = W_y (\sigma_{x,z_{\max}} - \sigma_o) \end{aligned} \quad (2.3)$$

$$M_z = - \int y \sigma_x \, dy \, dz = -I_z \frac{\Delta \sigma_{x,y}}{y} = -W_z (\sigma_{x,y_{\max}} - \sigma_o) \quad (2.4)$$

In the previous expressions, we have used the fact that the static moments of odd order in any of the variables are zero since the origin corresponds to the center of gravity. On the other hand, in the moment expressions appears the constant ratio between the stress difference (at any point on the axis) with respect to that at the center of gravity and the distance to said center, which is usually translated in the expressions to their maximum values, giving rise to the section modulus as the ratio between the moment of inertia and the distance from the “furthest fiber” to the center of gravity.

From the previous expressions, we can deduce the normal stress at any point from the resultant forces, as well as the relationships between the resultant forces and the deformations of

the slice

$$\sigma_x(y, z) = \frac{N}{A} + z \frac{M_y}{I_y} - y \frac{M_z}{I_z} \quad (2.5)$$

$$\varepsilon = \frac{N}{EA}; \quad c_y = \frac{M_y}{EI_y}; \quad c_z = \frac{M_z}{EI_z}. \quad (2.6)$$

These latter expressions can be easily rewritten in the format of *force equals stiffness times deformation*.

Let us now consider the formulation of the three remaining movement parameters, associated with the torsional moment and shear forces.

2.1.1.2 Section properties; shear stresses

The three remaining movement components for the positive face of the slice (figure 2.1, projection yz) would correspond to the displacements in directions y, z , associated with the shear forces T_y, T_z , and the rotation around the x axis, associated with the torsional moment M_x , all these forces being associated with the distribution of shear stresses in the section.

In the case of lateral displacements and corresponding shear forces, equilibrium considerations added to the hypothesis that the section remains orthogonal to the deformed axis allow us to determine the shear forces as derived functions of the corresponding bending moments, while determining the displacements y, z as an effect of the rotations θ_z, θ_y , whose derivatives are the already considered curvatures c_z, c_y , so that, in this model, such deformation and force components can be disregarded.

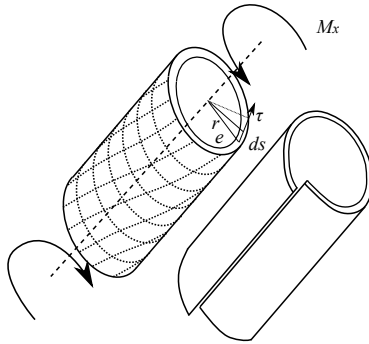


Figure 2.2: Torsional behavior of a hollow closed section.

In the case of the torsional moment and the corresponding section rotation, such simplification is not possible, although the formulation is simple for closed hollow sections with limited thickness in relation to the section dimensions, which we address next.

In figure 2.2, we see a segment of a hollow member subjected to torsion and its deformation, where lines parallel to the axis deform helically. We can also see that if we were to cut the member longitudinally, both edges of the cut would slide to reduce the deformation: such sliding does not occur in the member due to the shear stresses parallel to the axis, conjugate to the shear stresses contained in the section's cutting plane. The latter are contained within the thickness and are parallel to the mid-line of said thickness. For each position on the mid-line s that defines

the section, both the thickness $e(s)$ and the average stress $\tau(s)$ can vary. For each position, we consider that the stress across the entire thickness is the same, equal to that average stress.

By longitudinal equilibrium in a slice segment between two sections where we would make two longitudinal cuts, the product τe must be constant along the entire section path. This constancy allows us to assimilate the stress state to a flow that circulates in a closed path around the entire section.

The torsional moment M_x can be determined by integrating the moments contributed by that flow along the entire path, which we do by considering the dot product of the radius vector $\vec{r}$ from the center of rotation to the position in the section, with the force flow vector $\vec{\tau}e$ located at that position. This integral can be related to the corresponding area dA_ϕ of the triangle formed by two successive radius vectors and the section length element ds that separates them, and considering the constancy of τe throughout the section, which directly relates the moment to the area enclosed by the section's mid-line:

$$M_x = \oint \vec{r} \cdot \vec{\tau}e \, ds = \oint 2\tau e \, dA_\phi = 2\tau e A_\phi \quad (2.7)$$

To determine the (unit) deformation linked to said moment, that is, the torsional rotation per unit length or torsional “curvature”, c_x , we consider the identity between the deformation work that would occur in the rotation c_x between two sections separated by a unit distance, viewed from the external perspective as work done by the moment M_x , or viewed from the internal perspective as deformation work derived from the product of shear stresses by the angular distortions γ corresponding to

said rotation. In the process, we can consider the same elastic behavior from both perspectives, so that, ignoring the factor 1/2 corresponding to slow loading, we have:

$$\begin{aligned} M_x c_x &= \iiint \tau \gamma \, dV = \int dh \oint \tau \gamma e \, ds \\ &= \int_0^1 dh \frac{\tau^2}{G} e^2 \oint \frac{ds}{e} = \frac{M_x^2}{4G A_\phi^2} \oint \frac{ds}{e} \end{aligned} \quad (2.8)$$

where we have used the constancy of τe , the elastic relationship $\gamma = \tau/G$, being G the shear modulus, and the equation 2.7. We can therefore determine the moment-to-curvature relationship as:

$$\frac{M_x}{c_x} = G \frac{4 A_\phi^2}{\oint ds/e} = G I_t \quad (2.9)$$

Such a model is not appropriate for describing the behavior of an open section, where the stress “flow”, parallel to the section edges, can no longer be considered constant across the thickness, and where deformations involve warping: consider a double-fixed beam with an H-section subjected to a torsional moment at the center. In this case, what is relevant is rather that the opposite lateral displacements of the flanges impose opposite bending in them, and the shear associated with that bending, parallel to the flange and opposite in direction between both, is what, in its product with the lever arm or distance between flanges, balances the corresponding torsional moment.

In this case, we are talking about torsion as *bi-shear*, and to the extent that the flange shear T_f can be understood as

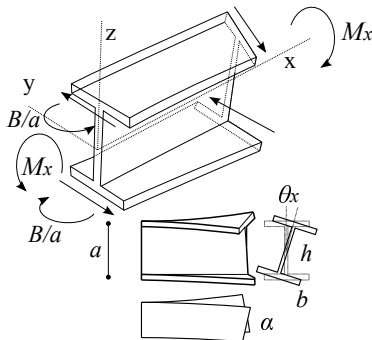


Figure 2.3: Torsional behavior with warping of an open section member

a derivative of the local flange bending moment, opposite between both flanges, we can speak of bi-shear as a derivative of that *bi-moment*, associated with the corresponding normal stresses. These normal stresses bend the flanges in plan view in opposite directions, which implies warping of the section, which can be measured by the rotation between the flanges α relative to their distance $a = h - t_f$ (the depth minus the flange thickness) for two successive sections one unit apart. Thus the force and deformation parameters become the bi-moment B whose longitudinal derivative is the bi-shear $T_f a = M_x = \partial B / \partial x$ and the warping coefficient $\omega = \alpha / a = 2\theta_{f,z} / a$, where $\theta_{f,z}$ is the rotation of a flange around the vertical, therefore derivative of the lateral flange displacement $\delta_{f,y}$, which can be analyzed from

the bending imposed on it by the bi-moment, and where the torsional rotation θ_x can likewise be determined from that lateral displacement $\delta_{f,y}$, as could be deduced from figures such as 2.3. The relationships existing between these variables allow determining the warping modulus as a component of the relationship between bi-moment and warping, which would correspond to the pair of conjugate static and kinematic variables of the model. This is a model that we will not detail here, but which leads to the warping torsion formulations contained in the regulations for open sections.

In summary, we are representing the deformation conditions of the member by a set of functions ϵ , the corresponding internal forces, energy conjugates of those, by another set of functions, σ , and the “constitutive” relationships or material behavior by the stiffness functions κ that relate forces to deformations. In the straight Bernoulli-Navier member, this model is as follows:

$$\epsilon = \begin{bmatrix} \varepsilon \\ c_x \\ c_y \\ c_z \end{bmatrix}; \quad \sigma = \begin{bmatrix} N_x \\ M_x \\ M_y \\ M_z \end{bmatrix}; \quad \kappa = \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & GI_t & 0 & 0 \\ 0 & 0 & EI_y & 0 \\ 0 & 0 & 0 & EI_z \end{bmatrix}$$

$$\sigma = \kappa \epsilon. \quad (2.10)$$

In this model, the different stiffnesses are uncoupled –there are only terms in the diagonal of κ , although this is not necessarily so in all possible models, as we will see later.

2.1.1.3 Properties of the element: variables and equations

In the previous sections, we have seen that we summarize the behaviors of the element's sections, for each position x in it, through the following kinematic and static variables:

- kinematic, of deformation ϵ :
 - elongation $\varepsilon(x)$,
 - vertical bending curvature $c_y(x)$ and lateral curvature $c_z(x)$,
 - torsional curvature $c_x(x)$,
- static, or internal force σ :
 - axial force $N(x)$,
 - “vertical” bending moment $M_y(x)$ and lateral moment $M_z(x)$,
 - torsional moment $M_x(x)$,

for which we have obtained the “stiffness” relations κ in expressions 2.6 and 2.9, condensed in 2.10. Now we need to characterize the behavior of the element based on its sections.

For this, we consider the six possible movements of any section, according to figure 2.4, which will define the kinematic variables as functions described over the length x of the element ($0 \leq x \leq l$), corresponding to the components of displacements (including rotations) in three dimensions:

- displacements $u(x), v(x), w(x)$
- rotations $\theta_x(x), \theta_y(x), \theta_z(x)$

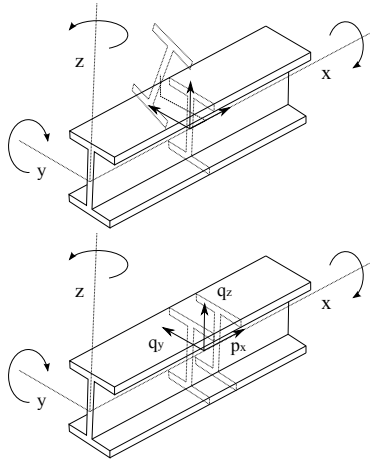


Figure 2.4: Section movements and distributed loads.

Associated with these movements, we would have the static variables defined by the corresponding energy conjugates, that is, the applied loads, namely

- the longitudinal distributed load $p_x(x)$ and transverse loads lateral $q_y(x)$ and vertical $q_z(x)$,
- the moments applied axially (as torsion $m_x(x)$) or applied as bending, lateral $m_y(x)$ or vertical $m_z(x)$,

all these values measured per unit length.

We can now establish the three classical groups of equations between the corresponding parameters, namely, the compati-

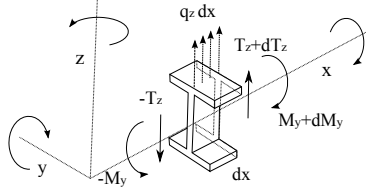


Figure 2.5: Equilibrium of vertical forces and vertical bending moments of a slice

bility equations that link displacements with deformations and the equilibrium equations between forces and loads, to which the material behavior equations will be added, in this case elastic, between forces and deformations.

Each of the equilibrium equations is established for the differential element in the form observed in figure 2.5, where the equilibrium of vertical forces implies $q_z dx + dT_z = 0$, which can also be written as $q_z + \partial T_z / \partial x = q_z + T'_z = 0$. For moment equilibrium, in this case results in $-T_z dx + m_y dx + dM_y = 0$, where m_y would be the external moment applied per unit length, which is usually assumed zero, as in the figure. In such case, $-T_z + dM_y/dx = -T_z + M'_y = 0$. In the figure, the positive directions of the forces, according to the chosen criteria, are assigned to the face of increasing x coordinates.

$$\begin{aligned}
q_y + T'_y &= 0 \\
q_z + T'_z &= 0 \\
T_y + M'_z &= 0 \\
-T_z + M'_y &= 0
\end{aligned} \tag{2.11}$$

Among all these equations, those corresponding to the moment equilibrium between the shear forces and the variation of the bending moments will be used in those corresponding to the equilibrium of the transverse load with the shear forces to eliminate these from the group of variables to consider.

The compatibility equations between displacements outside the axis and the corresponding rotations and curvatures in the slice, following the naming criteria of figure 2.5, will be:

$$\begin{aligned}
c_z &= \frac{\partial \theta_z}{\partial x} = \theta'_z \\
c_y &= \frac{\partial \theta_y}{\partial x} = \theta'_y \\
\theta_z &= \frac{\partial v}{\partial x} = v' \\
\theta_y &= -\frac{\partial w}{\partial x} = -w'
\end{aligned} \tag{2.12}$$

where the expressions are ordered to consider first the lateral bending movement and then the vertical.

In these last equations, the Bernoulli–Navier model allows deriving the bending curvature from the transverse displacements of the deformed shape, whose derivative is the rotation of

both the directrix and the section, as perpendicularity is maintained.

Thus, the resulting equations are the following:
equilibrium

$$\begin{aligned}
 p_x + \frac{\partial N_x}{\partial x} &= p_x + N'_x = 0 \\
 m_x + \frac{\partial M_x}{\partial x} &= m_x + M'_x = 0 \\
 q_z + \frac{\partial^2 M_y}{\partial x^2} &= q_z + M''_y = 0 \\
 q_y - \frac{\partial^2 M_z}{\partial x^2} &= q_y - M''_z = 0
 \end{aligned} \tag{2.13}$$

compatibility

$$\begin{aligned}
 \varepsilon &= \frac{\partial u}{\partial x} = u' \\
 c_x &= \frac{\partial \theta_x}{\partial x} = \theta'_x \\
 c_y &= -\frac{\partial^2 w}{\partial x^2} = -w'' \\
 c_z &= \frac{\partial^2 v}{\partial x^2} = v''
 \end{aligned} \tag{2.14}$$

constitutive or material equations, elastic behaviour for the Bernoulli–

Navier model

$$\begin{aligned}
 N_x &= EA \varepsilon \\
 M_x &= GI_t c_x \\
 M_y &= EI_y c_y \\
 M_z &= EI_z c_z
 \end{aligned} \tag{2.15}$$

where now the order of the equations corresponds successively to axial, vertical, and lateral movements, and where rotations or moments are expressed in relation to the axis around which they act. We have, in short, extended the model described in equations 2.10 through new functions for displacements $\mathbf{v}$ and distributed load $\boldsymbol{\tau}$, whose relationships with the previous deformations $\boldsymbol{\epsilon}$ and forces $\boldsymbol{\sigma}$ are obtained through the differential equations 2.13 and 2.14, which involve applying some type of differential operator $\boldsymbol{\partial}$ to some functions to determine others (deformations as differences between displacements, or loads as differences between forces). In our case, the displacement and load functions are respectively:

$$\mathbf{v} = \begin{bmatrix} u_x \\ \theta_x \\ w \\ v \end{bmatrix}; \quad \boldsymbol{\tau} = \begin{bmatrix} p_x \\ m_x \\ q_z \\ q_y \end{bmatrix}; \tag{2.16}$$

and the differential relationships pointed out before, the follow-

ing:

$$\begin{aligned} \boldsymbol{\epsilon} = \begin{bmatrix} \varepsilon \\ c_x \\ c_y \\ c_z \end{bmatrix} &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} u_x \\ \theta_x \\ w \\ v \end{bmatrix} = \boldsymbol{\partial} \mathbf{v}; \\ \boldsymbol{\tau} = \begin{bmatrix} p_x \\ m_x \\ q_z \\ q_y \end{bmatrix} &= \begin{bmatrix} -\frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} N_x \\ M_x \\ M_y \\ M_z \end{bmatrix} = \boldsymbol{\delta} \boldsymbol{\sigma}. \quad (2.17) \end{aligned}$$

It should be noted that the differential operator $\boldsymbol{\delta}$ applicable to the forces to determine the loads is the adjoint of that applied to the displacements to obtain the deformations ($\boldsymbol{\partial}$). This is because, if we consider the principle of virtual work, for any compatible movement $\bar{\boldsymbol{\epsilon}} = \boldsymbol{\partial} \bar{\mathbf{v}}$ applied to a state of forces in equilibrium with the loads $\boldsymbol{\tau} = \boldsymbol{\delta} \boldsymbol{\sigma}$ it must result that the external work, displacements by loads $\int_{\Gamma} \bar{\mathbf{v}}^T \boldsymbol{\tau} d\Gamma$ must equal the internal work of deformations by forces, $\int_{\Omega} \bar{\boldsymbol{\epsilon}}^T \boldsymbol{\sigma} d\Omega$, so that, being equal both scalars

$$\begin{aligned} \int_{\Gamma} \bar{\mathbf{v}}^T \boldsymbol{\tau} d\Gamma &= \int_{\Gamma} \bar{\mathbf{v}}^T \boldsymbol{\delta} \boldsymbol{\sigma} d\Gamma \\ \int_{\Omega} \bar{\boldsymbol{\epsilon}}^T \boldsymbol{\sigma} d\Omega &= \int_{\Omega} (\boldsymbol{\partial} \bar{\mathbf{v}})^T \boldsymbol{\sigma} d\Omega \end{aligned} \quad (2.18)$$

results¹

$$\boldsymbol{\delta} = \boldsymbol{\partial}^* \quad (2.19)$$

1. In the context of some type of scalar product, the usual definition of adjoint operators requires the equality of scalar products $\langle \bar{u} | \partial v \rangle = \langle \delta \bar{u} | v \rangle$.

In the considered case of the bar, both domains Γ and Ω contain the directrix x , although Γ must include the ends of the considered region.

In short, the set of compatibility equations, equilibrium and material or constitutive behavior are of the form

$$\epsilon = \partial v; \quad \tau = \partial^* \sigma; \quad \sigma = \kappa \epsilon. \quad (2.20)$$

2.1.1.4 Differential equations of the bar

We can now combine the equations of the three groups, which is easy since, being uncoupled from each other, it allows considering each component separately. After successively substituting in the equilibrium equations, 2.13, the material conditions 2.15 and compatibility 2.14 we obtain the set of equations that link the movements of the bar with the loads applied to it.

$$\begin{aligned} (EAu')' &= -p_x \\ (GI_t \theta'_x)' &= -m_x \\ (EI_y w'')'' &= q_z \\ (EI_z v'')'' &= q_y \end{aligned} \quad (2.21)$$

In the general format used in 2.20, such chain of substitutions would lead us to the differential equation

$$\tau = \partial^*(\kappa \partial v). \quad (2.22)$$

2.1.1.5 Element behavior: decomposition

Given the preceding equations, and the principle of superposition implicit in the elastic model, we will employ a decomposition procedure of the element problem to represent its behavior, both in isolation and in its interaction with the set of elements that form the structure. In this procedure, we consider separately the solution of two problems:

- the situation of *perfect clamping*, in which the element is loaded with distributed loads q , and subject to constraints that completely prevent the movement of its ends, resulting in reactions $-\mathbf{f}_q$ applied to these ends. In this situation, the directrix of the bar moves and deforms under the loads, and a distribution of forces in equilibrium with loads and reactions is generated, such that the beam equations described by 2.21 are satisfied.
- the situation of the complete connected structure, containing all bars, unloaded, where we apply as loads at the nodes or connections the actions $\mathbf{F}$ obtained by considering external forces $\mathbf{f}_q$ opposite to the previous reactions of all elements. In this second problem (or second part of the problem) the structure moves by adding displacements, deformation and additional internal forces to the bars, in a movement that releases the constraints of the previous step as the sums of $-\mathbf{f}_q$ and $\mathbf{f}_q$ of the superposition will cancel out at the nodes.

The superposition of both problems solves the general problem in two steps, where in the first there is no consideration of the influence of the rest of the structure on the behavior of the bar, and in the second we do not need to consider the dis-

tributed loads on the different elements, only their connections in the complete structure.

Therefore, for the analysis of the interaction that involves the general movement of the structure, it will be sufficient to consider the connected bars without load.

2.1.1.6 Element properties: shape functions

For any unloaded bar whose ends are subjected to movements and forces imposed by its inclusion in the movements of a structure subjected to loads at its nodes, the equations that describe such movement will be 2.21 where the load terms on the bar (p_x, m_x, q_z, q_y) will be zero. We can therefore consider that such movements can be determined by functions that simultaneously satisfy these equations along with the movement conditions corresponding to their ends. Such functions to describe the displacement can be considered as *shape functions* or *interpolation functions* since, ultimately, they will be nothing more than functions that describe the movement along the bar from the movements of its ends.

If we consider bars of constant section, the fulfillment of these equations is immediate if we select as functions polynomials of such degree that the derivative of the displacement applied in the function becomes zero, that is, first-degree polynomials — linear — for the first two, which describe axial movements, and third-degree polynomials for the remaining two, which describe transverse movements.

That is, we will have

$$\begin{aligned}
 u'' &= 0; \quad u = a_1x + a_0 \\
 \theta_x'' &= 0; \quad \theta_x = b_1x + b_0 \\
 v^{IV} &= 0; \quad v = c_3x^3 + c_2x^2 + c_1x + c_0 \\
 w^{IV} &= 0; \quad w = d_3x^3 + d_2x^2 + d_1x + d_0
 \end{aligned} \tag{2.23}$$

where the coefficients of the polynomials are for the moment unknown, although they can be determined if we consider the fulfillment of the end conditions: they must be those that ensure that said movements are the ones prescribed for the ends of the element.

Thus, for the longitudinal displacement u of the left and right ends of an element of length l we will have

$$\begin{aligned}
 u_i &= u(0) = a_0; \quad u_d = u(l) = a_1l + a_0; \\
 a_0 &= u_i; \quad a_1 = \frac{u_d - u_i}{l} \\
 u(x) &= u_i + \frac{u_d - u_i}{l}x = \left(1 - \frac{x}{l}\right)u_i + \frac{x}{l}u_d
 \end{aligned} \tag{2.24}$$

where we have written at the end the law of displacements at each point of the bar as the sum of the influence on this of the displacements of each end.

Similarly, for transverse displacements we will have to establish four conditions to fix the four coefficients: both the transverse displacement and the rotation of the two ends must coincide in the definition of the function with those prescribed for the left and right ends of the bar. For the case of vertical

movements we have

$$\begin{aligned} w(0) &= w_i; \quad w'(0) = \theta_y(0) = \theta_{y,i}; \\ w(l) &= w_d; \quad w'(l) = \theta_y(l) = \theta_{y,d}. \end{aligned} \quad (2.25)$$

One way to calculate the sought coefficients is to substitute in the polynomial expression written in matrix format the coordinates of the nodes, and equate the resulting expression to the displacements w or rotations $w' = \theta$ of such ends, and since $w' = 3d_3x^2 + 2d_2x + d_1$, we obtain

$$\begin{bmatrix} w_i \\ w'_i \\ w_d \\ w'_d \end{bmatrix} = \begin{bmatrix} 1 & x_i & x_i^2 & x_i^3 \\ 0 & 1 & 2x_i & 3x_i^2 \\ 1 & x_d & x_d^2 & x_d^3 \\ 0 & 1 & 2x_d & 3x_d^2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad (2.26)$$

equation that can be solved for the vector of sought coefficients by doing

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 & x_i & x_i^2 & x_i^3 \\ 0 & 1 & 2x_i & 3x_i^2 \\ 1 & x_d & x_d^2 & x_d^3 \\ 0 & 1 & 2x_d & 3x_d^2 \end{bmatrix}^{-1} \begin{bmatrix} w_i \\ \theta_{y,i} \\ w_d \\ \theta_{y,d} \end{bmatrix} \quad (2.27)$$

so that if the coordinates of left and right ends are respectively 0 and l we will have:

$$\begin{aligned} d_0 &= w_i, \quad d_1 = \theta_{y,i}, \\ d_2 &= -\frac{3}{l^2}w_i - \frac{2}{l}\theta_{y,i} + \frac{3}{l^2}w_d - \frac{1}{l}\theta_{y,d}, \\ d_3 &= \frac{2}{l^3}w_i + \frac{1}{l^2}\theta_{y,i} - \frac{2}{l^3}w_d + \frac{1}{l^2}\theta_{y,d} \end{aligned} \quad (2.28)$$

and therefore we can write the function that represents the transverse movement in the form

$$w = \left(\frac{2x^3}{l^3} - \frac{3x^2}{l^2} + 1 \right) w_i + \left(\frac{x^3}{l^2} - \frac{2x^2}{l} + x \right) \theta_{y,i} + \left(-\frac{2x^3}{l^3} + \frac{3x^2}{l^2} \right) w_d + \left(\frac{x^3}{l^2} - \frac{x^2}{l} \right) \theta_{y,d} \quad (2.29)$$

where again we have expressed the displacement as the sum of the independent effects of each of the end movements, displacement or rotation, considered in isolation. Similarly, it would result for the lateral bending movements $v(x)$.

That is, we have that the displacements along the continuum of the bar can be described in general in a format in which we use known shape or interpolation functions that accumulate the independent effect of a collection of separate parameters, $\mathbf{a}$, which, in this case, correspond to each of the isolated components of the end movements of the considered bar. The function that describes the movement is the dot product of some shape functions known a priori by the coefficients that represent the relevant end movements. Each movement will be associated with the appropriate function:

$$\mathbf{v} = \mathbf{N}(x) \mathbf{a} = \mathbf{N}(x) \mathbf{u} = \mathbf{N} \mathbf{u}. \quad (2.30)$$

2.1.1.7 Models: Load and stiffness matrices

We have seen how to describe the deformation of the element from the end movements.

If for the element we have a list of kinematic parameters that correspond to said end movements, it will be pertinent to consider the corresponding list of static parameters and select for these the corresponding energy conjugates, that is, for each displacement component at each end we will select the corresponding force component applied at said end, and similarly for each rotation component the corresponding moment applied to the end.

In this way we can consider that such end movements characterize the deformations of the element and its end forces therefore characterize the forces that the structure imposes on such element. For this reason we denote the first with $\mathbf{u}$ and the second with $\mathbf{f}$, reserving the symbols $\mathbf{U}$ and $\mathbf{F}$ for the movements of the nodes that connect the elements to each other and the corresponding loads applied to these by virtue of the decomposition discussed in section 2.1.1.5.

However, in order for the description of the element behavior to be considered complete and adequate from these end parameters, we have to ensure that the deformation work for any state of the element, measured in the internal description, continuously along the entire directrix, can be reproduced identically through the work measured through the evolution of the end parameters. Or what is the same, applying the principle of virtual work, that the internal work done by the forces in an arbitrary virtual deformation is identical to the external work done by the end forces in the end movements associated with that virtual deformation. We have then, that we can describe the distributed movements $\mathbf{v}$ from those of the ends $\mathbf{u}$, so that for any compatible movement thus described $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$ that equality

of work implies

$$\begin{aligned}
 \bar{\mathbf{u}}^T \mathbf{f} &= \int \bar{\mathbf{v}}^T \boldsymbol{\tau} \, d\Gamma \\
 &= \int \bar{\mathbf{u}}^T \mathbf{N}^T \boldsymbol{\tau} \, d\Gamma = \bar{\mathbf{u}}^T \int \mathbf{N}^T \boldsymbol{\tau} \, d\Gamma, \quad \forall \bar{\mathbf{u}}^T; \\
 \mathbf{f} &= \int \mathbf{N}^T \boldsymbol{\tau} \, d\Gamma
 \end{aligned} \tag{2.31}$$

and similarly,

$$\begin{aligned}
 \bar{\mathbf{u}}^T \mathbf{f} &= \int \bar{\boldsymbol{\epsilon}}^T \boldsymbol{\sigma} \, d\Omega = \int \bar{\boldsymbol{\epsilon}}^T \boldsymbol{\kappa} \boldsymbol{\epsilon} \, d\Omega \\
 &= \int (\partial(\mathbf{N}\bar{\mathbf{u}}))^T \boldsymbol{\kappa} \partial(\mathbf{N}\mathbf{u}) \, d\Omega \\
 &= \int \bar{\mathbf{u}}^T (\partial\mathbf{N})^T \boldsymbol{\kappa} \partial(\mathbf{N})\mathbf{u} \, d\Omega \\
 &= \bar{\mathbf{u}}^T \left(\int \mathbf{N}'^T \boldsymbol{\kappa} \mathbf{N}' \, d\Omega \right) \mathbf{u}, \quad \forall \bar{\mathbf{u}}^T; \\
 \mathbf{f} &= \mathbf{k}\mathbf{u} \\
 \mathbf{k} &= \int \mathbf{N}'^T \boldsymbol{\kappa} \mathbf{N}' \, d\Omega
 \end{aligned} \tag{2.32}$$

where the parameters $\bar{\mathbf{u}}$, $\mathbf{u}$ that characterize the movements of the arbitrary compatible system or the real deformation linked to the internal forces and deformations $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ have been extracted outside the integrals, and where consequently the relevant differential operator is limited in its application to the shape functions

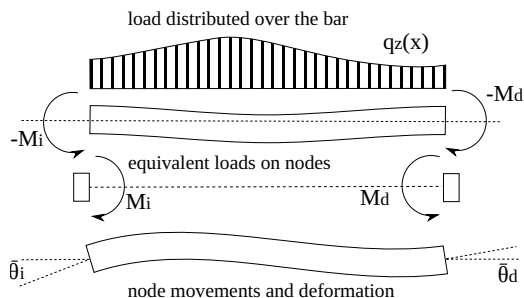


Figure 2.6: Equivalent nodal loads for the distributed load on the element

preset for the problem considered, so that the determination of the relevant stiffness relationship is immediate.

We have, therefore, a general method that we can use to determine the relevant expressions to the problem of the straight element of constant section.

Load: fixed-end values

We apply 2.31 to the straight element to determine the nodal forces M_i, M_d equivalent to the distributed load $q(x)$ on it (figure 2.6), where the end movement parameters are modeled as the rotations of these ends with respect to the straight line connecting them, and therefore the end force parameters are these two moments.

We will have

$$\begin{aligned} \begin{bmatrix} \bar{\theta}_i & \bar{\theta}_d \end{bmatrix} \begin{bmatrix} M_i \\ M_d \end{bmatrix} &= \int \bar{z}(x) q(x) dx \\ &= \int \begin{bmatrix} \bar{\theta}_i & \bar{\theta}_d \end{bmatrix} \begin{bmatrix} N_i(x) \\ N_d(x) \end{bmatrix} q(x) dx \end{aligned} \quad (2.33)$$

obtaining

$$\begin{bmatrix} M_i \\ M_d \end{bmatrix} = \begin{bmatrix} \int q(x) N_i(x) dx \\ \int q(x) N_d(x) dx \end{bmatrix} \quad (2.34)$$

We can thus use this expression to obtain the fixed-end moments of the beam using the shape functions corresponding to the rotations in the vertical plane (associated with the end rotations $\theta_{y,i}, \theta_{y,d}$) that are used in equation 2.29. Let's examine three typical cases:

- for uniform load, $q(x) = q$, constant:

$$\begin{aligned} M_i &= \int_0^l q \left(\frac{x^3}{l^2} - \frac{2x^2}{l} + x \right) dx = \frac{ql^2}{12} \\ M_d &= \int_0^l q \left(\frac{x^3}{l^2} - \frac{x^2}{l} \right) dx = -\frac{ql^2}{12} \end{aligned} \quad (2.35)$$

- for triangular load, $q(x) = q_{\max}x/l$:

$$\begin{aligned} M_i &= \int_0^l \frac{q_{\max}x}{l} \left(\frac{x^3}{l^2} - \frac{2x^2}{l} + x \right) dx = \frac{q_{\max}l^2}{30} \\ M_d &= \int_0^l \frac{q_{\max}x}{l} \left(\frac{x^3}{l^2} - \frac{x^2}{l} \right) dx = -\frac{q_{\max}l^2}{20} \end{aligned} \quad (2.36)$$

- for centered point load, $q(x) = P\delta(x - l/2)$, where the Dirac delta function $\delta(x)$ is used, which is zero throughout the domain except at coordinate 0, where it becomes undefined, and has unit integral when crossing that point:

$$\begin{aligned} M_i &= \int_0^l P \delta\left(x - \frac{l}{2}\right) \left(\frac{x^3}{l^2} - \frac{2x^2}{l} + x\right) dx = \frac{Pl}{8} \\ M_d &= \int_0^l P \delta\left(x - \frac{l}{2}\right) \left(\frac{x^3}{l^2} - \frac{x^2}{l}\right) dx = -\frac{Pl}{8} \end{aligned} \quad (2.37)$$

Element stiffness

Similarly, we apply 2.32 to determine the stiffness of the straight element with constant cross-section, using the shape functions already seen in 2.24 and 2.29, and considering as differential operator $\boldsymbol{\partial}$ the one corresponding to the kinematic relations among those described in 2.17:

- longitudinal stiffness; tension or torsion:

$$\begin{aligned} \begin{bmatrix} \bar{u}_i & \bar{u}_d \end{bmatrix} \begin{bmatrix} f_{x,i} \\ f_{x,d} \end{bmatrix} &= \int \bar{\varepsilon} N_x dV = \int_0^l \bar{\varepsilon} EA \varepsilon dx \\ &= \int_0^l \left(\begin{bmatrix} \bar{u}_i & \bar{u}_d \end{bmatrix} \begin{bmatrix} N_i(x) \\ N_d(x) \end{bmatrix} \right)' EA \left(\begin{bmatrix} N_i(x) & N_d(x) \end{bmatrix} \begin{bmatrix} u_i \\ u_d \end{bmatrix} \right)' dx \end{aligned}$$

where we can extract the numerical parameters corresponding to each deformation field, virtual or real, outside the

integrals, resulting in:

$$\begin{aligned}
 &= [\bar{u}_i \quad \bar{u}_d] \left(\int_0^l \begin{bmatrix} N_i(x)' \\ N_d(x)' \end{bmatrix} EA \begin{bmatrix} N_i(x)' & N_d(x)' \end{bmatrix} dx \right) \begin{bmatrix} u_i \\ u_d \end{bmatrix} \\
 &= [\bar{u}_i \quad \bar{u}_d] \int_0^l \begin{bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{bmatrix} EA \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} dx \begin{bmatrix} u_i \\ u_d \end{bmatrix} \\
 &= [\bar{u}_i \quad \bar{u}_d] \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_d \end{bmatrix}; \quad \forall [\bar{u}_i \quad \bar{u}_d] \quad (2.38)
 \end{aligned}$$

which implies

$$\begin{bmatrix} f_{x,i} \\ f_{x,d} \end{bmatrix} = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_d \end{bmatrix}; \quad (2.39)$$

and as the functions are similar for torsion, we will analogously have

$$\begin{bmatrix} m_{x,i} \\ m_{x,d} \end{bmatrix} = \frac{GI_t}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_{x,i} \\ \theta_{x,d} \end{bmatrix}. \quad (2.40)$$

- For the transverse stiffness we have:

$$\begin{aligned}
 [\bar{\theta}_{y,i} \quad \bar{\theta}_{y,d}] \begin{bmatrix} m_{y,i} \\ m_{y,d} \end{bmatrix} &= \int_0^l \bar{c}_y(x) m_y(x) dx = \int_0^l \bar{w}(x)'' EI w(x)'' dx \\
 &= \int_0^l [\bar{\theta}_{y,i} \quad \bar{\theta}_{y,d}] \begin{bmatrix} N_i(x)'' \\ N_d(x)'' \end{bmatrix} EI \begin{bmatrix} N_i(x)'' & N_d(x)'' \end{bmatrix} \begin{bmatrix} \theta_{y,i} \\ \theta_{y,d} \end{bmatrix} dx \\
 &= [\bar{\theta}_{y,i} \quad \bar{\theta}_{y,d}] \int_0^l \begin{bmatrix} N_i(x)'' \\ N_d(x)'' \end{bmatrix} EI \begin{bmatrix} N_i(x)'' & N_d(x)'' \end{bmatrix} dx \begin{bmatrix} \theta_{y,i} \\ \theta_{y,d} \end{bmatrix} \quad (2.41)
 \end{aligned}$$

therefore

$$\begin{aligned}
 \begin{bmatrix} m_{y,i} \\ m_{y,d} \end{bmatrix} &= \int_0^l \begin{bmatrix} N_i(x)'' \\ N_d(x)'' \end{bmatrix} EI \begin{bmatrix} N_i(x)'' & N_d(x)'' \end{bmatrix} dx \begin{bmatrix} \theta_{y,i} \\ \theta_{y,d} \end{bmatrix}; \\
 \mathbf{k} &= \int_0^l \begin{bmatrix} N_i(x)'' \\ N_d(x)'' \end{bmatrix} EI \begin{bmatrix} N_i(x)'' & N_d(x)'' \end{bmatrix} dx = \\
 \int_0^l \begin{bmatrix} \frac{6x}{l^2} - \frac{4}{l} \\ \frac{6x}{l^2} - \frac{2}{l} \end{bmatrix} EI \begin{bmatrix} \frac{6x}{l^2} - \frac{4}{l} & \frac{6x}{l^2} - \frac{2}{l} \end{bmatrix} dx &= \frac{EI}{l} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \quad (2.42)
 \end{aligned}$$

It should be noted that if the interpolation functions $\mathbf{N}$ used in expressions 2.31 and 2.32 to obtain the equivalent nodal loads or member stiffnesses are exact, that is, if they are solutions of the relevant differential equation, the resulting expressions are also exact. But in cases where they are not exact, but only approximate, the resulting expressions also provide an approximation that can be useful.

We can now consolidate all the stiffness results obtained in 2.39, 2.40 and 2.42. To establish the bending stiffnesses, we have considered in 2.42 only the two parameters associated with rotations, which generates the stiffness relationships between rotations-moments, but these relationships can be easily extended to consider displacement parameters (and end shear forces): on one hand, shear forces can be obtained by summing the end moments divided by the length, which implies dividing by the length the sum of the relevant columns of the moment-rotation stiffnesses for the case of end rotation only. And on the other hand, if we isolate a single end displacement, this can be understood as the sum of an undeformed bar rotation (therefore

without bending) plus the rotations at both ends that bring the nodes to their original directions (therefore both equal to the displacement-length ratio), rotations associated with the moments that result from the moment-rotation stiffnesses already seen. This implies that the moment-displacement relationship is the sum, divided by the length, of the stiffnesses of the rows containing the relevant moment-rotation relationships, given rotations (relative to the geometry displaced without deformation) at both ends.

Therefore, we have the following stiffness relationships for the straight member of constant cross-section:

$$\begin{bmatrix} N_{x,i} \\ N_{x,d} \\ M_{x,i} \\ M_{x,d} \\ M_{y,i} \\ M_{y,d} \\ T_{z,i} \\ T_{z,d} \end{bmatrix} = \begin{bmatrix} \frac{EA}{l} & -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{l} & \frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{GI_t}{l} & -\frac{GI_t}{l} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{GI_t}{l} & \frac{GI_t}{l} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\frac{EI_y}{l} & 2\frac{EI_y}{l} & -6\frac{EI_y}{l^2} & 6\frac{EI_y}{l^2} \\ 0 & 0 & 0 & 0 & 2\frac{EI_y}{l} & 4\frac{EI_y}{l} & -6\frac{EI_y}{l^2} & 6\frac{EI_y}{l^2} \\ 0 & 0 & 0 & 0 & -6\frac{EI_y}{l^2} & -6\frac{EI_y}{l^2} & 12\frac{EI_y}{l^3} & -12\frac{EI_y}{l^3} \\ 0 & 0 & 0 & 0 & 6\frac{EI_y}{l^2} & 6\frac{EI_y}{l^2} & -12\frac{EI_y}{l^3} & 12\frac{EI_y}{l^3} \end{bmatrix} \begin{bmatrix} u_i \\ u_d \\ \theta_{x,i} \\ \theta_{x,d} \\ \theta_{y,i} \\ \theta_{y,d} \\ w_i \\ w_d \end{bmatrix} \quad (2.43)$$

relationships that are commonly found in texts in the equivalent

format

$$\begin{bmatrix} N_{x,i} \\ M_{x,i} \\ M_{y,i} \\ T_{z,i} \\ \hline N_{x,d} \\ M_{x,d} \\ M_{y,d} \\ T_{z,d} \end{bmatrix} = \left[\begin{array}{cccc|cccc} \frac{EA}{l} & 0 & 0 & 0 & \frac{EA}{l} & 0 & 0 & 0 \\ 0 & \frac{GI_t}{l} & 0 & 0 & 0 & -\frac{GI_t}{l} & 0 & 0 \\ 0 & 0 & 4\frac{EI_y}{l^2} & -6\frac{EI_y}{l^3} & 0 & 0 & 2\frac{EI_y}{l^2} & 6\frac{EI_y}{l^3} \\ 0 & 0 & -6\frac{EI_y}{l^2} & 12\frac{EI_y}{l^3} & 0 & 0 & -6\frac{EI_y}{l^2} & 12\frac{EI_y}{l^3} \\ \hline -\frac{EA}{l} & 0 & 0 & 0 & \frac{EA}{l} & 0 & 0 & 0 \\ 0 & -\frac{GI_t}{l} & 0 & 0 & 0 & \frac{GI_t}{l} & 0 & 0 \\ 0 & 0 & 2\frac{EI_y}{l^2} & -6\frac{EI_y}{l^3} & 0 & 0 & 4\frac{EI_y}{l^2} & 6\frac{EI_y}{l^3} \\ 0 & 0 & 6\frac{EI_y}{l^2} & -12\frac{EI_y}{l^3} & 0 & 0 & 6\frac{EI_y}{l^2} & 12\frac{EI_y}{l^3} \end{array} \right] \begin{bmatrix} u_i \\ \theta_{x,i} \\ \theta_{y,i} \\ w_i \\ \hline u_d \\ \theta_{x,d} \\ \theta_{y,d} \\ w_d \end{bmatrix} \quad (2.44)$$

format that can be interpreted as an ordering and partitioning of the stiffness relationships $\mathbf{f} = \mathbf{k}\mathbf{u}$ in which the effects of each node are grouped separately, the left node i , or the right node d :

$$\begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_d \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{ii} & \mathbf{k}_{id} \\ \mathbf{k}_{di} & \mathbf{k}_{dd} \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_d \end{bmatrix}. \quad (2.45)$$

A brief digression should now be added regarding the *coupling* between parameters revealed by the fact that matrix $\mathbf{k}$ has elements outside the diagonal. If we consider the axial forces, normal force or (torsional) moment, the off-diagonal terms only reveal the redundancy in the parameters: indeed, the force at one end is equal and opposite to that at the other, so this redundancy can be eliminated by considering only one axial force or moment and, as displacement, that of the relative movement or rotation of one end with respect to the other. In the case of transverse stiffnesses, the coupling of shear forces with moments has already been discussed; shear forces can be ignored along with lateral displacements, in exchange for considering as rotations not the absolute rotations of the ends, but those rela-

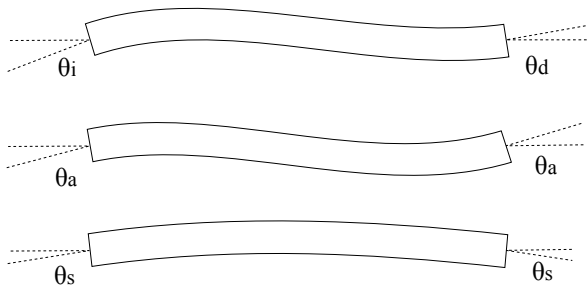


Figure 2.7: Diagonalizing the bending stiffness of beams

tive to the undeformed straight geometry that links the actual positions of the nodes, a condition with which we obtained the bending stiffnesses of 2.42. In this last expression, we are representing the relationship between end moments and (relative) end rotations, and the coupling between rotations at one end and moment at the other persists, as the matrix is not diagonal.

If we consider the problem of diagonalizing this last matrix, that is, establishing a representation for the forces (moments) strictly proportional to the movements (rotations), we would be looking for an expression such as $\mathbf{m} = \mathbf{k}\boldsymbol{\theta} = \lambda\boldsymbol{\theta}$.

It is a classic eigenvalue problem in linear algebra:

$$(\mathbf{k} - \lambda\mathbf{I})\boldsymbol{\theta} = \mathbf{0}, \boldsymbol{\theta} \neq \mathbf{0} \implies |\mathbf{k} - \lambda\mathbf{I}| = 0 \quad (2.46)$$

The solution to this problem allows an alternative representation of moments and rotations, as shown in figure 2.7, where these rotations or moments (at both ends) and their effect throughout

the member are the sum of their symmetric and antisymmetric parts, and where the stiffness relationships for these parts are

$$\begin{bmatrix} M_{y,s} \\ M_{y,a} \end{bmatrix} = \frac{EI_y}{l} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \theta_{y,s} \\ \theta_{y,a} \end{bmatrix} \quad (2.47)$$

It can now be seen that both components of the bending problem are uncoupled, which is extremely useful for addressing problems manually, such that it has been common throughout history to formulate problems of symmetric structures as the composition of their behavior for the symmetric and antisymmetric parts of the load.

From this perspective, the most reduced and clear way to represent the stiffness relationships of the straight member of constant cross-section Bernoulli–Navier for all possible movements in space would be

$$\begin{bmatrix} N_{x,id} \\ M_{x,id} \\ M_{y,a} \\ M_{y,s} \\ M_{z,a} \\ M_{z,s} \end{bmatrix} = \frac{1}{l} \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & GI_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 6EI_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 2EI_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 6EI_z & 0 \\ 0 & 0 & 0 & 0 & 0 & 2EI_z \end{bmatrix} \begin{bmatrix} u_{x,id} \\ \theta_{x,id} \\ \theta_{y,a} \\ \theta_{y,s} \\ \theta_{z,a} \\ \theta_{z,s} \end{bmatrix} \quad (2.48)$$

where the subscript id expresses the movement of one node i relative to the other d .

An additional modification that allows adopting identical units in all terms of matrix κ consists of using as axial kinematic and static parameters of the bar those noted as possible in example 1.3.7.4: the unit strain $\varepsilon_{x,id}$ and the corresponding

stress volume $N_{x,id}l_{id}$, which consequently transforms the diagonal term to the form $EV = EAl = \frac{1}{l}EAl^2$. This format allows identifying the relative importance of the different stiffness terms, with the smaller terms being the most relevant in the response. This justifies the frequent elimination of axial elongation components compared to those of bending, given the high value of the pivot EAl^2 compared to $6EI = 6EAi^2$ as the geometric slenderness l/i of the members usually has a value of several dozens.

2.1.2 Analysis of the complete structure

2.1.2.1 Substructures

We will consider the possibility of substructures existing, that is, regions of the structure that can be considered as a whole, composed of elements of lesser dimension or complexity that integrate it: this could be the case of beam regions with varying depths, which can be represented in two different ways:

- either as the composition of each and every one of its segments, connected to each other through nodes that are considered in the global model
- or as a single element in which all the properties of the assembly are condensed and connected only through a limited set of nodes to the global model.

Although it is very common to consider the first procedure as a general method, despite it potentially involving additional parameters in the global model, it is interesting to consider the second approach, not so much for the reduction in degrees of

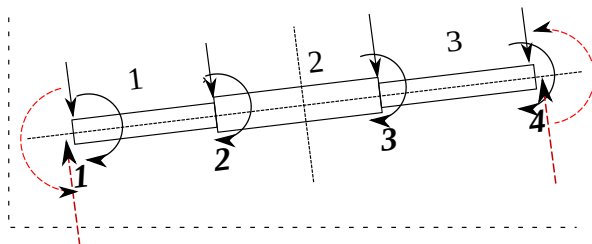


Figure 2.8: Substructure of three elements and four nodes to be condensed into two. Local and global axes do not coincide.

freedom it may represent, but for the generality it provides. Indeed, the procedure could be applied to the conditions of prefabricated pieces of any format, or to the study of the behavior of pieces whose ends must be joined with connections of known stiffness. In this case, the model would allow incorporating the behavior of such nodes into that of the piece itself.

Let us consider one of such substructures, in figure 2.8, where the static parameters corresponding to the connection with the rest of the structure are shown in dashed lines (the reactions of the structure against the substructure, in the latter's axes), and in solid lines the internal parameters linked to movements or loads of the substructure nodes, including both those that only connect elements of the substructure among themselves and those that connect with the nodes of the global model. We will use the subscript c ($1 \leq c \leq c_{\max}$) to identify the parameters corresponding to the connection with the rest of the structure,

and the subscript i ($1 \leq i \leq i_{\max}$) for the unconnected internal ones. The total number of parameters g considered to describe the substructure is the set of all these ($g_{\max} = c_{\max} + i_{\max}$). From the internal perspective, the equivalent accumulated actions of all segments of the substructure on the set of nodes considered in it will be given by parameters $\mathbf{F}$ that integrate the internal i and connected c parameters, in equilibrium with the internal forces $\mathbf{f}$ relevant for each of these segments, while the reactions of the structure $-\mathcal{F}_c$ against the substructure would correspond to those necessary to equilibrate all loads in the latter, taking into consideration all its movements.

We will therefore consider the problem in three phases, and its solution through the superposition of the corresponding states:

- first, we analyze the substructure as if none of its nodes moved, so that the distributed loads are applied to it along with reactions that prevent the movements of all nodes of the substructure, whether connected or not (perfect fixed-end conditions); ...
- secondly, we consider that the internal nodes move due to loads applied to them, which will be equal and opposite to the fixity reactions from the previous phase, but maintaining constraints from the rest of the structure capable of preventing any movement in its connections to the substructure (therefore without movements in the connected nodes or degrees of freedom), adding the necessary reactions at such nodes;
- finally, we consider the joint movement and interaction of substructure and structure, thus releasing the movements in

the connection nodes between both, movements and interaction derived from the loads on the previously constrained nodes, equal and opposite to the reactions previously accumulated in them.

For each phase, we can consider their own axis systems, namely, those specific to the elements that make up the substructure, those of the substructure as a whole, or the global ones of the structure. By the principle of superposition, in the first phase the distributed loads are reduced to equivalent loads on the nodes of the substructure, in the second phase the loads on the internal nodes, which are the ones that begin to move, are reduced to additional loads on the connected nodes of the complete structure, and in the third the accumulated loads act on the structure mobilizing its complete behavior. The sum — superposition — of loads and force laws of the three phases will be the solution of the complete problem since, as in each phase the reactions of the previous phase are, with changed sign, the loads of the new phase, in the superposition all reactions (such as the perfect fixity moments) of all intermediate phases are cancelled out.

The solution to the original problem is, therefore, the superposition of the three states studied.

For a first phase, the equivalent loads (opposite to the fixity reactions) that the distributed loads on the elements exert on the g nodes of the substructure are simple to determine, according to the procedures in section 2.1.1.7, equation 2.31, and will be composed of the $\mathbf{F}_{c1}$ and $\mathbf{F}_{i1}$ applied to all nodes, connected or not, of the substructure. The equivalent loads on the connected nodes require opposite reactions in them ($-\mathbf{F}_{c1}$, $-\mathbf{F}_{i1}$).

In a second phase, the i nodes move due to the loads $\mathbf{F}_{i1}$ while the c nodes remain fixed, with additional reactions appearing in them necessary to equilibrate the forces $\mathbf{F}_{i1}$. In this second phase, the forces and deformations of each segment or element of the substructure, $\mathbf{f}$ and $\mathbf{u}$, are linked by the material relations of each element, $\mathbf{k}$, which are easy to know. The reactions can be determined: if the relationship between the movements of the elements of the substructure to the g movements of its nodes is, for the chosen axes or representation in both, $\mathbf{B}_{eg}$ (that is, $\mathbf{u} = \mathbf{B}_{eg}\mathbf{U}_g$), the relationship between actions on the nodes of the substructure and its movements can be established from the respective relationships in the elements: it is the stiffness matrix of the substructure considered in isolation, which we can order and partition according to the degrees of freedom identified as c or i :

$$\mathbf{K}_g = [\mathbf{B}_{eg}^T \mathbf{k} \mathbf{B}_{eg}] = \begin{bmatrix} \mathbf{K}_{cc} & \mathbf{K}_{ci} \\ \mathbf{K}_{ic} & \mathbf{K}_{ii} \end{bmatrix} \quad (2.49)$$

From this stiffness relationship, the reactions are determined considering zero movements of the “connected” degrees of freedom.

$$\begin{bmatrix} \mathbf{R}_{c2} \\ \mathbf{F}_{i1} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{cc} & \mathbf{K}_{ci} \\ \mathbf{K}_{ic} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{c2} \\ \mathbf{U}_{i2} \end{bmatrix} \quad \text{with } \mathbf{U}_{c2} = \mathbf{0}. \quad (2.50)$$

In this way, we can obtain the movements $\mathbf{U}_{i2}$ required to develop the forces that allow sustaining the loads $\mathbf{F}_{i1}$, and the corresponding reactions $\mathbf{R}_{c2}$ necessary to keep the c nodes without

movement:

$$\mathbf{U}_{i2} = (\mathbf{K}_{ii})^{-1} \mathbf{F}_{i1} \quad (2.51)$$

$$\mathbf{R}_{2c} = \mathbf{K}_{ci} \mathbf{U}_{i2} = \mathbf{K}_{ci} (\mathbf{K}_{ii})^{-1} \mathbf{F}_{i1} \quad (2.52)$$

And therefore, since the equivalent actions of the loads $\mathbf{F}_{i1}$ on the connected nodes are opposite to said reactions, the loads from the substructure on the connected nodes, which will thus be the applicable loads to the structure for its joint movement will be

$$\mathcal{F}_c = \mathbf{F}_{c1} - \mathbf{K}_{ci} (\mathbf{K}_{ii})^{-1} \mathbf{F}_{i1} \quad (2.53)$$

These loads therefore incorporate those considered in the first phase, $\mathbf{F}_{c1}$ plus those resulting from the second phase, $-\mathbf{R}_{c2}$.

In a third phase, the movements $\mathbf{U}$ and loads $\mathbf{F}$ of the connected nodes of the substructure in their interaction with the rest of the structure must be considered.

In this phase, the behavior of the substructure is described as

$$\begin{bmatrix} \mathbf{F}_{c3} \\ \mathbf{F}_{i3} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{cc} & \mathbf{K}_{ci} \\ \mathbf{K}_{ic} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{c3} \\ \mathbf{U}_{i3} \end{bmatrix}. \quad (2.54)$$

From this, we can obtain the *condensed* stiffness matrix, reducing the internal parameters i .

Indeed, in this third phase we need to consider only the effect of end movements (and forces), the $\mathbf{U}_c$, which obviously impose internal movements $\mathbf{U}_i$, but now under conditions where the internal forces of the substructure elements no longer require loads at these interior nodes: $\mathbf{F}_i = \mathbf{0}$. The loads are, therefore, null for this movement phase: their effect was considered in

phases one and two, and now we only need to consider additional deformations of the substructure due to effects that occur only associated with parameters linked to its ends: that is, we are in a position to condense the interior degrees of freedom from the degrees of freedom connected to the rest of the structure.

If in equation 2.54 we consider that the loads $\mathbf{F}_i$ are null, we will have

$$\mathbf{F}_i = \mathbf{0} = \mathbf{K}_{ic}\mathbf{U}_c + \mathbf{K}_{ii}\mathbf{U}_i \quad (2.55)$$

therefore

$$\mathbf{U}_i = -(\mathbf{K}_{ii})^{-1}\mathbf{K}_{ic}\mathbf{U}_c \quad (2.56)$$

and consequently

$$\mathbf{F}_c = (\mathbf{K}_{cc} - \mathbf{K}_{ci}(\mathbf{K}_{ii})^{-1}\mathbf{K}_{ic})\mathbf{U}_c = \mathbf{K}_c\mathbf{U}_c \quad (2.57)$$

which allows us to condense into $\mathbf{K}_c$ the complete stiffness of the substructure for analyzing its interaction with the rest of the structure, based only on the connected end movements and forces.

However, since substructure and structure axes might not coincide, we must consider this possible change in representation (as always, via compatibility and equilibrium). To do this, we proceed to rename from here the parameters that refer to the substructure, keeping the uppercase letters in the denominations applicable to the parameters with which we represent the state of the complete structure, resulting in the stiffness expression already condensed in 2.57: $\mathbf{f}_c = \mathbf{k}_c\mathbf{u}_c$, as if it were any element, as well as equivalent end loads, equal and opposite to the complete fixed-end forces of the substructure considered as a whole, and in its own axes, given by expression 2.53.

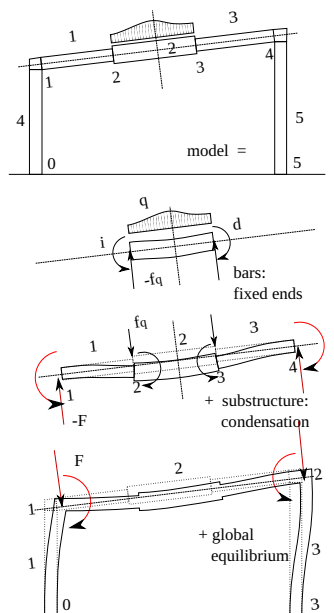


Figure 2.9: The elastic solution: it is that of the problem in the first figure, and will be the sum of the solutions of the following three problems: *a*) perfect fixing of loads against unconnected nodes of the substructure, *b*) movement of the substructure, fixed at its connected nodes by the loads at its unconnected nodes: opposite to the reactions of case *a*), and *c*) general displacement of the structure for loads at its nodes, coming from the reactions of the previous case.

In this way, we have reduced the *substructure* to an element similar to any possible member, characterized by its end parameters or, better expressed, its connection parameters with the rest of the structure, so that the solution to the complete problem (see fig. 2.9) only requires the analysis of the part corresponding to the global interaction of all elements that converge in the structure, characterized by the interaction between the loads that the different members or substructures contribute to the structure's nodes in the case of considering perfect fixing, and the movements that derive from the fact that such fixing does not exist.

2.1.2.2 Assembly of loads and stiffnesses

Up to this point, we have seen how to

- a transform distributed load problems on members into equivalent loads on their ends, that is, on the nodes that connect them with the rest of the structure: these are the forces opposite to the reactions that would appear at said ends in the case of considering them immovable: the case of perfect fixing
- b identify the relationships between the loads applied to said ends and their respective movements, that is, the stiffness relationships of said members, and even of composite elements that we have called substructures (they could be variable section members where each section of the substructure or complete member responds to a different section law).

We have, therefore, identified “member” parameters for forces $\mathbf{f}$ and movements $\mathbf{u}$ and their relationships $\mathbf{f} = \mathbf{k}\mathbf{u}$, as well as

initial values $\mathbf{f}_q, \mathbf{u}_q$, with $\mathbf{u}_q = \mathbf{0}$, if there were no forced initial movements in the structure.

We now want to analyze the behavior of the structure under the effect of all loads $\mathbf{f}_q$ applied to all its members, which cause the generalized movement of all its nodes, ensuring that equilibrium and compatibility conditions are met at these nodes, and that the elastic material conditions of the previous paragraph are equally met in all members.

An issue to consider now is that of the reference axes to be used in equilibrium and compatibility, as we have already pointed out that we will not necessarily have chosen them to coincide for each of the members, or for the nodes of the structure: it is usual to consider axes aligned with their directrix and the principal directions of inertia of their section in the members, while for the structure's nodes it is common to consider global axes associated with directions in a horizontal plane and the vertical... Thus, it becomes necessary to ensure that the relevant equations are posed for a description of the variables made in concordant axes.

Axis transformations.

In figure 2.10 we represent a member end (a node, or reference to the end called n , linked to the reference position of a global node N) and the references to the local axes considered in it, —alphabetic references, (x, y, z) , or numeric $(1, 2, 3)$ — as well as references to the global axes —which can also be alphabetic, (X, Y, Z) , or numeric $(1, 2, 3)$ —. It is also indicated that the positions of said references (of said nodes) may not coincide in

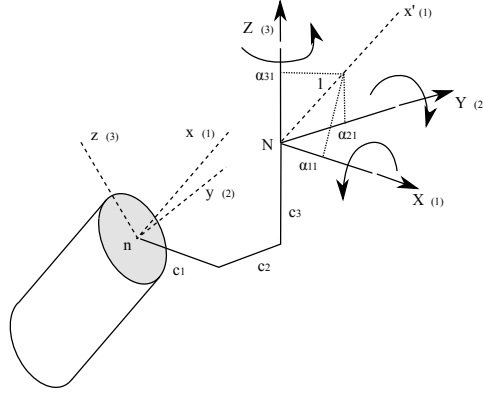


Figure 2.10: Coordinate axes transformation: lowercase references to the node and its local axes, uppercase references to the node and axes in global terms of the complete structure.

local axes (n) or global axes (N), with the displacements between one reference and another being given by the distances ($c1, c2, c3$) measured in global axes. The figure also indicates the direction cosines corresponding to the local axis x , that is, the projections of the point at unit distance from N and in the direction of x on the global axes (X, Y, Z), namely, the values $(\alpha_{X,x}, \alpha_{Y,x}, \alpha_{Z,x})$, or also $(\alpha_{11}, \alpha_{21}, \alpha_{31})$ if we use numeric references instead of alphabetic ones.

The end forces and moments $\mathbf{F}_n$ and $\mathbf{M}_n$, described in axes with the orientations of the global axes, but located in the positions of the **local** axes at n , and which derive from the forces

applied to the member ends, which are described in local axes $\mathbf{f}$, $\mathbf{m}$, can be expressed in the following form, since we are considering only the rotations:

$$\mathbf{F}_n = \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix}_n = \begin{bmatrix} \alpha_{X,x} & \alpha_{X,y} & \alpha_{X,z} \\ \alpha_{Y,x} & \alpha_{Y,y} & \alpha_{Y,z} \\ \alpha_{Z,x} & \alpha_{Z,y} & \alpha_{Z,z} \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}_n = \boldsymbol{\alpha}_{Nn} \mathbf{f}_n \quad (2.58)$$

$$\mathbf{M}_n = \begin{bmatrix} M_X \\ M_Y \\ M_Z \end{bmatrix}_n = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_n = \boldsymbol{\alpha}_{Nn} \mathbf{m}_n \quad (2.59)$$

expressions in which we have used both notation criteria indistinctly, alphabetic or numeric for the direction cosines.

If, additionally, we consider the translations, to bring the references to the positions of the **global** axes at N , we will have that, although the forces do not change, changes must be made in the moment values to represent the same resultants:

$$\mathbf{F}_N = \mathbf{F}_n = \boldsymbol{\alpha}_{Nn} \mathbf{f}_n \quad (2.60)$$

$$\mathbf{M}_N = \begin{bmatrix} M_X \\ M_Y \\ M_Z \end{bmatrix}_N = \begin{bmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{bmatrix} \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix}_N + \begin{bmatrix} M_X \\ M_Y \\ M_Z \end{bmatrix}_n \quad (2.61)$$

$$\mathbf{M}_N = \gamma_{Nn} \boldsymbol{\alpha}_{Nn} \mathbf{f}_n + \boldsymbol{\alpha}_{Nn} \mathbf{m}_n \quad (2.62)$$

or, in a more compact format:

$$\begin{bmatrix} \mathbf{F}_N \\ \mathbf{M}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_{Nn} & \mathbf{0} \\ \boldsymbol{\gamma}_{Nn} \boldsymbol{\alpha}_{Nn} & \boldsymbol{\alpha}_{Nn} \end{bmatrix} \begin{bmatrix} \mathbf{f}_n \\ \mathbf{m}_n \end{bmatrix}; \quad \mathbf{F}_N = \mathbf{H}_{Nn} \mathbf{f}_n \quad (2.63)$$

Due to the duality 1.21, we can similarly determine the relationship between movements described in local axes, which we represent with $\mathbf{u}$, and their description in global axes, which we represent with $\mathbf{U}$:

$$\begin{bmatrix} \mathbf{u}_n \\ \boldsymbol{\theta}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_{Nn}^T & \boldsymbol{\alpha}_{Nn}^T \boldsymbol{\gamma}_{Nn}^T \\ \mathbf{0} & \boldsymbol{\alpha}_{Nn}^T \end{bmatrix} \begin{bmatrix} \mathbf{U}_N \\ \boldsymbol{\Theta}_N \end{bmatrix}; \quad \mathbf{u}_n = \mathbf{B}_{nN} \mathbf{U}_N \quad \mathbf{H}_{Nn} = \mathbf{B}_{nN}^T \quad (2.64)$$

Assembly.

Here we consider how to approach the problem of displacements and forces of the complete structure subjected to loads at the nodes, that is, the last of the problems considered in figure 2.9.

Up to this point, we have the load vectors applied to the connection “nodes” of the substructures, in the case of bars, to their ends, corresponding in each of these to appropriate $\mathbf{f}$ vectors, and we also have their stiffness conditions, given by $\mathbf{k}$ matrices. These vectors and matrices reference components of forces or stiffnesses associated with the nodes $\mathbf{n}$ of such substructures (2 nodes in the case of bars, which we can call left and right: i, d). These nodes are “inserted” into the corresponding ones among the $\mathbf{N}$ of the complete structure, so that for each bar we can identify the N_i, N_d where the i, d of the bar are connected, along with the axis change parameters $\mathbf{c}_i = [c_{i1}, c_{i2}, c_{i3}]^T, \mathbf{c}_d = [c_{d1}, c_{d2}, c_{d3}]^T, \boldsymbol{\alpha}_i \dots$ that describe the

correspondence between local and global geometries. Therefore, we know how to describe $\mathbf{f}, \mathbf{k}$ in global axes: following what was seen in the preceding subsection, we can do this by transforming the expression in local axes

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_d \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{ii} & \mathbf{k}_{id} \\ \mathbf{k}_{di} & \mathbf{k}_{dd} \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_d \end{bmatrix} = \mathbf{k}\mathbf{u} \quad (2.65)$$

to its form in global axes

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{N_i} \\ \mathbf{F}_{N_d} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{N_i N_i} & \mathbf{K}_{N_i N_d} \\ \mathbf{K}_{N_d N_i} & \mathbf{K}_{N_d N_d} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{N_i} \\ \mathbf{U}_{N_d} \end{bmatrix} = \mathbf{K}\mathbf{U} \quad (2.66)$$

and we do this by considering the projections between axes independent for each end:

$$\begin{bmatrix} \mathbf{F}_{N_i} \\ \mathbf{F}_{N_d} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{iN_i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{dN_d}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_d \end{bmatrix} \quad (2.67)$$

$$\begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_d \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{iN_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{dN_d} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{N_i} \\ \mathbf{U}_{N_d} \end{bmatrix}. \quad (2.68)$$

Since by the principle of virtual work it must be

$$\begin{bmatrix} \bar{\mathbf{U}}_{N_i} \\ \bar{\mathbf{U}}_{N_d} \end{bmatrix}^T \begin{bmatrix} \mathbf{F}_{N_i} \\ \mathbf{F}_{N_d} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{u}}_i \\ \bar{\mathbf{u}}_d \end{bmatrix}^T \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_d \end{bmatrix} \quad \forall \begin{bmatrix} \bar{\mathbf{U}}_{N_i} \\ \bar{\mathbf{U}}_{N_d} \end{bmatrix} \quad (2.69)$$

we will necessarily have for each bar that

$$\bar{\mathbf{U}}^T \mathbf{F} = \bar{\mathbf{U}}^T \mathbf{B}^T \mathbf{f} = \bar{\mathbf{U}}^T \mathbf{B}^T \mathbf{k} \mathbf{B} \mathbf{U}; \quad \forall \bar{\mathbf{U}}, \mathbf{U} \quad (2.70)$$

and therefore

$$\mathbf{K} = \mathbf{B}^T \mathbf{k} \mathbf{B}, \quad (2.71)$$

an equation that, as we have seen previously, expresses the projection of stiffness in local axes to the axes chosen for the global representation.

We only need to accumulate the effects of each and every bar in the structure, which is straightforward: from the perspective of loads, the forces applied to each end of each bar must be understood as forces applied from the node of the structure, that is, they are forces applied to the node that are transmitted to the end of the bar. From this perspective, what is applied to a node is, therefore, the sum of all the end forces of all the bars that converge at that node: the “external” forces thus sum nodal resultants of the loads on all the ends of bars (“elements”) that converge at the same node. For a node N , considering all bars e with ends n, m , connected to N ($e \rightarrow N$), where n converges at N , (and where m does so at M) we will have:

$$\mathbf{F}_N = \sum_{e \rightarrow N} \mathbf{F}_{q,e,N} = \sum_{e \rightarrow N} \mathbf{B}_{e,n}^T \mathbf{f}_{q,e,n} \quad (2.72)$$

Therefore, if we traverse the set of “elements” of the model, we can symbolize the assembly of loads as

$$\mathbf{F} = \sum_e \begin{array}{c} N \\ \mathbf{B}_{e,n}^T \mathbf{f}_{q,e,n} \\ M \\ \mathbf{B}_{e,m}^T \mathbf{f}_{q,e,m} \end{array} \quad (2.73)$$

where each bar accumulates its end loads in the corresponding nodes (N and M) of the global model.

Similarly, we can describe the equilibrium at each node as the resultant of all the end forces of the bars that converge at it, which are derived from the displacements of both ends of each bar. If we account for all bars $e \rightarrow N$ we will have

$$\begin{aligned}
 \mathbf{F}_N &= \sum_{e \rightarrow N} \mathbf{F}_{e,N} = \sum_{e \rightarrow N} \mathbf{B}_{e,n}^T \mathbf{f}_{e,n} = \\
 &\quad \sum_{e \rightarrow N} \mathbf{B}_{e,n}^T (\mathbf{k}_{e,nn} \mathbf{u}_{e,n} + \mathbf{k}_{e,nm} \mathbf{u}_{e,m}) = \\
 &\quad \sum_{e \rightarrow N} \mathbf{B}_{e,n}^T (\mathbf{k}_{e,nn} \mathbf{B}_{e,n} \mathbf{U}_N + \mathbf{k}_{e,nm} \mathbf{B}_{e,m} \mathbf{U}_{e,M}) = \\
 &\quad \sum_{e \rightarrow N} (\mathbf{K}_{e,nn} \mathbf{U}_N + \mathbf{K}_{e,nm} \mathbf{U}_{e,M}) \quad (2.74)
 \end{aligned}$$

so that, as before, we can symbolize the assembly process of the global stiffness matrix in the following way:

$$\mathbf{K} = \sum_e \begin{array}{c} N \\ M \end{array} \begin{array}{|c|c|c|c|} \hline & N & M & \\ \hline & \mathbf{K}_{e,nn} & \mathbf{K}_{e,nm} & \\ \hline & \mathbf{K}_{e,mn} & \mathbf{K}_{e,mm} & \\ \hline \end{array} \quad (2.75)$$

where each bar accumulates its stiffness components in the indices of the global matrix corresponding to the nodes to which it is connected.

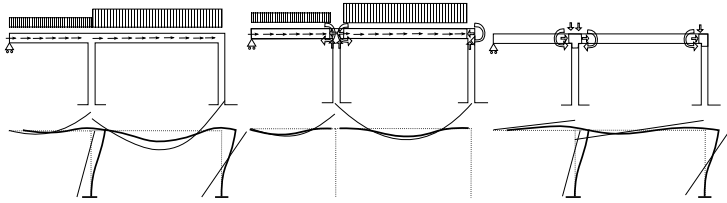


Figure 2.11: Elastic solution (a), as superposition —sum— of the solutions for fixed support (b) and displacement (c).

2.1.2.3 Solution of linear problems

The solution of elastic problems is therefore obtained, as schematized in figure 2.11, as a superposition of the perfectly fixed support and the displacement components.

Indeed, the complete state corresponds to a problem of:

- distributed loads
- with the displaced structure
- with deformation due to both previous effects
- where EQUilibrium, COMpatibility, and MATerial conditions are satisfied

The first part of the decomposition adds loads (reactions) at the ends that prevent global displacements, but, for all loads, distributed plus added, ensures deformations and forces that satisfy the three groups of equations: EQU, COM, and MAT.

The second part of the decomposition considers the displacement conditions (and the resulting deformation and forces) de-

rived from eliminating the reactions from the previous phase (or equivalently, from adding loads opposite to these) while maintaining compliance with the three groups of equations.

Since the principle of superposition is valid in elastic behavior, the sum of values of any function from the two cases of the decomposition provides the values of said function for the original case, that is, the superposition solves the original case.

2.1.2.4 Solution of non-linear problems

Let us recall Newton's method for determining the root (the position for a null value) of a function of one variable: if the value of the function can be approximated from its series development, the root can be approximated from the value of the function and its gradient at a nearby point:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2} + \dots = 0$$

$$\Delta x = x - x_0 \approx -\frac{f(x_0)}{f'(x_0)} \quad (2.76)$$

We can generalize to functions of multiple variables: $f(x) \rightarrow \mathbf{F}(\mathbf{U}) - \mathbf{F}$ where the first term of such difference is the structure's response $\mathbf{F}(\mathbf{U})$ for the displacements $\mathbf{U}$ of a certain time step, and the second, $\mathbf{F}$, is the effective load in that step, with the difference being the load *residual* not yet balanced by the response.

In this case, equilibrium will require a modification $\Delta \mathbf{U}$ in the displacement already achieved which will be approximated,

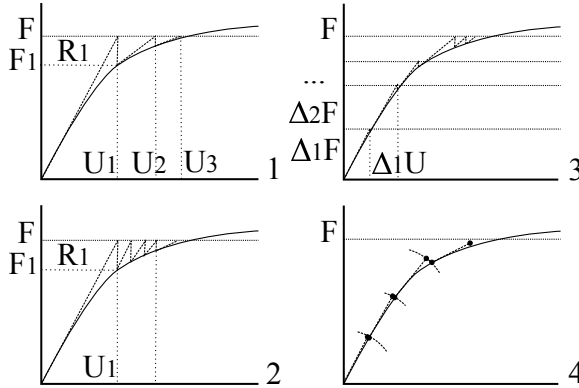


Figure 2.12: Newton, Newton–Raphson, Incremental, or Arc Length methods

considering that $\mathbf{F}'(\mathbf{U}) = \partial \mathbf{F}(\mathbf{U}) / \partial \mathbf{U} = \mathbf{K}(\mathbf{U})$, with the following expression:

$$\Delta \mathbf{U} \approx \frac{\mathbf{F} - \mathbf{F}(\mathbf{U})}{\mathbf{K}(\mathbf{U})} = \frac{\Delta \mathbf{F}}{\mathbf{K}(\mathbf{U})} = \mathbf{K}^{-1}(\mathbf{U}) \Delta \mathbf{F} \quad (2.77)$$

The figures in 2.12 illustrate the possible procedures.

In all of them, the solution to the non-linear problem is reached through a succession of linear problems in which, at each step, the displacement increment required to reach the load residual not yet balanced in the previous step is solved. At each displacement step, the total response achieved is recalculated

and, if it does not balance the load —within the established tolerance margin— the stiffness corresponding to the deformations associated with such displacement is recalculated and the increment needed to define the next step is determined.

That is, the objective is to balance the load $\mathbf{F}$. The initial displacement is $\mathbf{U}_0 = \mathbf{0}$. In Newton's method,

- the initial load step is the total load. Since the initial displacement is zero, this means zero response, not balanced with $\mathbf{F}$, and therefore $\Delta_1 \mathbf{F} = \mathbf{F} = \mathbf{F} - \mathbf{F}(\mathbf{0})$
- The initial stiffness is $\mathbf{K}_0 = \mathbf{K}(\mathbf{U}_0)$
- $\Delta_1 \mathbf{F} = \mathbf{K}_0 \Delta_1 \mathbf{U}$ provides a first approximation to displacement: $\mathbf{U}_1 = \mathbf{U}_0 + \Delta_1 \mathbf{U} = \Delta_1 \mathbf{U}$

The recurrence for the following load steps is straightforward:

- For displacement $\mathbf{U}_i$, the response is $\mathbf{F}_i$, with an unbalanced residual $\mathbf{R}_i = \mathbf{F} - \mathbf{F}_i = \mathbf{F} - \mathbf{F}(\mathbf{U}_i)$. A load $\Delta_{i+1} \mathbf{F} = \mathbf{R}_i$ remains to be balanced. If the residual is small enough, the process stops.
- The stiffness to use is that corresponding to the displacement: $\mathbf{K}_i = \mathbf{K}(\mathbf{U}_i)$
- The displacement increment is estimated through the approximation $\Delta_{i+1} \mathbf{F} = \mathbf{K}_i \Delta_{i+1} \mathbf{U}$. The achieved displacement will be $\mathbf{U}_{i+1} = \mathbf{U}_i + \Delta_{i+1} \mathbf{U}$

It is possible to update the stiffness matrix used only occasionally, a procedure known as the Newton–Raphson method. In this case, the “stiffness” $\mathbf{K}_0$ is maintained throughout the process (or for a given number of steps, after which it is recalculated again)

It is also possible to perform the analysis through a process where the load is increased in successive steps, or in groups of these —incremental procedure—, usually updating the stiffness only at the end of each load step. That is, for a given final load, at each step the load corresponding to an increasing factor is considered: $\lambda_i \mathbf{F} = \mathbf{F}_i = \sum_i \Delta_i \mathbf{F}$, with $\lambda_0 = 0, \lambda_i \in (0, 1]$, and for that load, the necessary displacement increment is determined. In this case, the “residual” for each load increment and the necessary displacement step are $\lambda_{i+1} \mathbf{F} - \mathbf{F}(\mathbf{U}_i) = \Delta_{i+1} \mathbf{F} + \lambda_i \mathbf{F} - \mathbf{F}(\mathbf{U}_i) = \mathbf{K}(\mathbf{U}_i) \Delta_{i+1} \mathbf{U}$.

Finally, it can be considered that each increment is made simultaneously in both displacement $\mathbf{U}$ and load $\mathbf{F}$ spaces, through equivalent arc length increments —for a certain scale factor ψ that allows comparing those of both spaces, so that the arc length of each step δl can be estimated with

$$(\Delta l)^2 = \Delta_i \mathbf{U}^T \Delta_i \mathbf{U} + \psi^2 \Delta_i \mathbf{F}^T \Delta_i \mathbf{F} \quad (2.78)$$

and since, if load factors λ are used to describe the load, we have

$$\Delta_i \mathbf{F}^T \Delta_i \mathbf{F} = \Delta \lambda_i^2 \mathbf{F}^T \mathbf{F} \quad (2.79)$$

the problem will involve obtaining at each step the displacement increment and the corresponding load factor that ensure that both are simultaneously satisfied

$$(\Delta l)^2 = \Delta_{i+1} \mathbf{U}^T \Delta_{i+1} \mathbf{U} + \psi^2 (\Delta \lambda_{i+1})^2 \mathbf{F}^T \mathbf{F}, \quad (2.80)$$

$$\lambda_{i+1} \mathbf{F} - \mathbf{F}(\mathbf{U}_i) = \Delta \lambda_{i+1} \mathbf{F} = \mathbf{K}(\mathbf{U}_i) \Delta_{i+1} \mathbf{U} \quad (2.81)$$

In this approach, the appearance of null or negative values in $\Delta_{i+1} \lambda$ is possible, and even the arc length to be used in each step

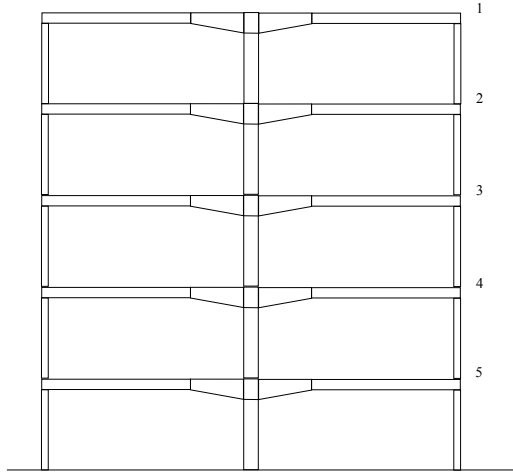


Figure 2.13: Five-story frame with haunched beams

can be adjusted based on the error between the predicted and obtained values in the previous step, which allows the capture of highly non-linear load-displacement processes.

2.1.3 Application example: five-story frame

We aim to model and analyze a five-story frame with haunched beams and steel columns, figure 2.13, in which the facade columns, being slender, are considered double-pinned. Lateral stability is achieved by ensuring indeformability at the central column con-

nections (which in much of the technical documentation is called *rigid connection, of complete rigidity*). The spans are 10 m and the haunch lengths are 2 m, with column heights of 3 m.

IPE 330 beams are considered with haunches up to approximately 1.8 times the beam depth (and approximately quadruple inertia).

Finally, the columns are HEB 180 in floors 1 and 2, and HEB 200, 220, and 240 in floors 3, 4, and 5, using numbering from top to bottom.

The objectives are to:

- Determine the stiffnesses and fixed-end moments of the haunched members.
- Determine the global stiffness of the frame considering as movement parameters the possible horizontal displacement and rotation of the central joints.
- Determine the frame's condensed stiffness matrix in order to *isolate* the direct analysis of the relationship between loads and horizontal displacements.

The procedure will consist of:

- determining the stiffness of the complete haunched beam (substructure), section 2.1.3.1
 - determining the stiffness of the haunched section
 - condensing the stiffness of both constant and haunched sections into the beam
- determining the fixed-end moment diagrams of the complete beam (substructure), section 2.1.3.2

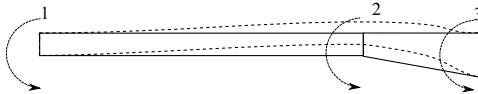


Figure 2.14: Substructure model of the haunched member, formed by a constant section segment and a haunch.

- determining the reactions at all joints assuming the internal joint is immobile
- determining the fixed-end reactions for movements derived from the release of the internal joint
- the equivalent loads of the beam on the frame joints are those balanced by the reactions in the two previous steps and the different diagrams derived from these and from the distributed loads are called fixed-end moments, since they respond to deformations of the loaded member but without end movements.
- For the loads on the structure joints—in any of the possible load hypotheses— determine the displacements and stress diagrams necessary for elastic equilibrium, section 2.1.3.3
 - modeling of degrees of freedom and elastic properties
 - solution for the complete model
 - condensation of degrees of freedom to emphasize only the consequences of horizontal displacements

2.1.3.1 Stiffness of the haunched beam

We will determine the stiffness relationships of this beam, figure 2.14, by condensing (section 2.1.2.1) as a single substructure the stiffness of its two segments: the constant section and the haunch. We already know the stiffness of a constant section segment, according to expression 2.42, but we don't yet know that of the haunched region, so we will construct this using the expressions from section 2.1.1.7, and particularly 2.32.

Stiffness of the haunched section. We will establish the stiffness using functions that describe the stress and deformation laws of the variable section segment without load using `maxima`, Schelter et al. 1982–2022. The procedure will be similar to that applied in constructing the load and stiffness matrices for constant section members in 2.1.1.7, with the exception that the displacement functions we used there, according to 2.1.1.6, are not now applicable, as we must consider the variable section.

Therefore, we must first establish the appropriate shape functions for this case. For this, we assume a haunch segment with ends 0, 1 (at left and right) for which we will deduce the successive laws t , m , g , f of shear forces, moments, (curvatures), rotations and deflections, considering the end values t_0 , m_0 , g_0 , f_0 . We soon see the variable curvatures $c=m/k$, quotients between moment and stiffness. For the latter, we consider its end values defined with k_0 , k_1 .

We have the following interaction with `maxima` for the stress laws, considering that the abscissas vary between $x=0$ and $x=1$,

using l for the length of the haunched section²:

```
t: t0;
m: integrate(t,x) + m0;
```

where the moments follow, therefore, a linear law.

If we consider as the curvature law the interpolation between the end values of the simplest non-linear function, that is, the parabola³, we will have for curvatures, rotations and deflections the following sequence, in which the curvature law depends on three parameters, one known, m_0/k_0 , and two others unknown, a , b , to be established with two conditions for fixing that law:

```
c: a*x**2+b*x+m0/k0;
g: integrate(c,x)+g0;
f: integrate(g,x)+ f0;
```

To fix the values of parameters a , b we establish two conditions:

1. the curvature will be zero at the point where the moment is zero
2. the curvature at the right end will be $c = c_1 = m_1/k_1 = m(l)/k_1$.

2. Pay attention to the difference between the character l and the digit 1.

3. We could seek and use the stiffness law of the haunch shape by deducing the curvature as the quotient between moment and stiffness, but this leads to algebraic complications that do not justify the tiny improvement in precision obtained.

The following expression solves these conditions:

```
solve([ subst([x=1],c=m1/k1),
        subst(solve(m=0,x),c=0) ] , [a,b]);
```

Indeed,

- the first **subst**(itution) constructs the first condition to consider for obtaining the values of **a**, **b** by substituting **x** with **1** in the expression that sets that the curvature is **m1/k1**,
- the second **solve** resolves the coordinate **x** for the condition **m=0**, so that
- the second **subst**(itution) substitutes said expression for **x** in the one that states that at that point there is zero curvature: **c=0**, thus providing the second condition,
- finally, the first **solve** obtains the expressions for **[a,b]** required by the two established conditions **[... , ...]**.

The result has the form **[[a=... ,b=...]]** showing that there is only one solution (first level) that contains two expressions, one for each parameter. The following assignments seek to simplify these expressions: the first substitutes the moment at the right end with its expression dependent on the left end values in the obtained solution (referred to by the symbol **%**), and the second uses the proportion term α to define the right stiffness in terms of the left one, thus defining the expression **cf** that contains the parameters or coefficients obtained and in the simplest possible algebraic format, with which the deformed shape of said section can be defined, **fs**, for the adjustment conditions of the obtained curvature law:

```
cf: ratsimp(part(subst([m1=t0*l+m0],%),1));
cf: ratsimp(subst([k1=alpha*k0],cf));
fs: subst(cf,f);
```

The values of these coefficients are:

$$\left[a = -\frac{(\alpha-1)t_0}{\alpha k_0 l}, b = \frac{\alpha l t_0 + (1-\alpha)m_0}{\alpha k_0 l} \right]$$

and with them, the stiffness law can be established (quotient between moment and curvature, the latter being the second derivative of the deflected shape):

```
k: ratsimp(m/diff(fs,x,2));
```

This results in the following expression for the local moment-curvature stiffness law along the member:

$$k = \frac{\alpha k_0 l}{\alpha l - (\alpha - 1)x}$$

Up to this point, the expressions include generic end conditions, both for movement —displacement and rotation— and for stress —the end forces (moment and shear)— in such a way that they allow describing the corresponding laws for any end condition.

We are now interested in particularizing the conditions that allow us to describe the isolated influence of each end. We therefore consider two cases, both with zero displacement at both

ends, but with unit rotations at end 0 or at end 1 while keeping the other end from rotating, seeking the corresponding end moment and shear conditions.

The conditions for both cases are determined with:

```
mt0: solve(subst([x=1,f0=0,g0=1],[fs=0,diff(fs,x)=0]),
            [m0,t0]);
mt1: solve(subst([x=1,f0=0,g0=0],[fs=0,diff(fs,x)=1]),
            [m0,t0]);
```

Indeed, the first expression seeks to solve $[m0, t0]$ for the conditions $[fs=0, diff(fs, x)=0]$ when $x=1$, and when using the additional conditions $f0=0, g0=1$, that is, the rotation $g0$ is unity, but both the deflection $f0$ (defined for $x=0$) and the deflection and its derivative, the rotation, are zero when $x=1$. The second expression determines the other case: only unit rotation when $x=1$. In both expressions, we must select the first (and only) solution, with the two appropriate end moment $m0$ and shear $t0$ values for each case.

Therefore, the deformed laws for both cases are easily obtained from the generic form fs by substituting the end conditions, moment, shear, deflection, and rotation:

```
f1: ratsimp(subst(
            [part(mt0,1,1),part(mt0,1,2),f0=0,g0=1],fs));
f2: ratsimp(subst(
            [part(mt1,1,1),part(mt1,1,2),f0=0,g0=0],fs));
```

These deformed shapes can be drawn, in comparison with those used for constant section, with the following expression:

```
plot2d(subst([alpha=4,l=1],
             [f1,f2,x^3/l^2-2*x^2/l+x,x^3/l^2-x^2/l]),
       [x,0,1],[legend,"f1","f2","ni","nd"]);
```

which gives us figure 2.15. It is worth noting in the figure the greater influence of the rotation at the end with higher stiffness on the induced deformation compared to the case of constant section, versus a lesser influence in the case of rotation at the weaker end.

Having established the shape functions, it becomes trivial to determine the stiffness k of the haunched section, according to equation 2.32, which we apply using `maxima`, as before, but also taking into account that the appropriate differential relationship for the problem is the second derivative, in the same way as in the process from 2.41 to 2.42:

```
dfMat: factor(diff(matrix([f1],[f2]),x,2));
kc: factor(integrate((dfMat . transpose(dfMat))*k,x,0,l));
      positive; positive;
```

In the last expression, `maxima` will ask if l or $(\alpha - 1)$ is positive or negative... and we have anticipated the answer, `positive` for both cases.

The stiffness of the haunched section is, therefore,

$$K_c = \frac{EI_0}{l} \frac{6\alpha}{\alpha^2 + 4\alpha + 1} \begin{bmatrix} \alpha + 3 & \alpha + 1 \\ \alpha + 1 & 3\alpha + 1 \end{bmatrix} \quad (2.82)$$

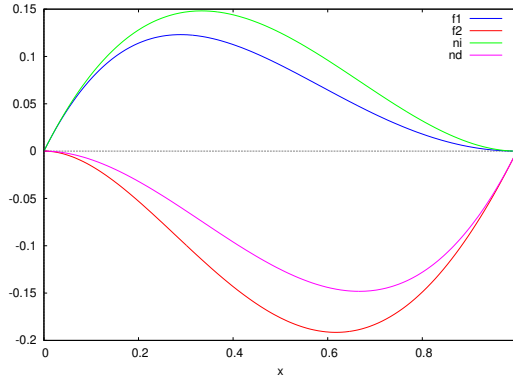


Figure 2.15: Shape functions for variable section (f^* , for $\alpha = 4$) or constant section (n^*) members

which can be specified for each ratio $\alpha = k_1/k_0$ by simply substituting the variable α , as for example, for a relatively probable ratio of 4:

K4: `subst([k0=1,alpha=4],kc )`;

$$K_4 = \frac{EI_0}{11l} \begin{bmatrix} 56 & 40 \\ 40 & 104 \end{bmatrix} \quad (2.83)$$

Stiffness of the complete haunched member, fixed at both ends. Once we know the effects for the two members with three fixed nodes (the constant section segment and the haunched segment), the behavior for the complete member is

obtained by adding the *release* of the internal node, according to the procedure indicated for substructures: this internal node will be the unconnected node capable of moving and whose effects are condensed into the others, as indicated in section 2.1.2.1.

Figure 2.14 shows this substructure, in which, referring the movements capable of flexing the beam to the fixed positions of the end nodes, we would have three possible rotations and one displacement.

We will consider the case of $\alpha = 4$, and a haunch length of 0.2 times the length of the member.

We then have the following stiffness relationship, in which, since we are going to “condense” the movement of node 2, we place its degrees of freedom at the end, and where we apply (add) the stiffnesses of both segments:

$$\begin{bmatrix} M_1 \\ M_3 \\ M_2 \\ P2 \end{bmatrix} = \frac{EI_0}{l} \begin{bmatrix} \frac{4}{0.8} & 0 & \frac{2}{0.8} & -\frac{6}{0.8^2 l} \\ 0 & \frac{104}{11*0.2} & \frac{40}{11*0.2} & \frac{144}{11*0.2^2 l} \\ \frac{2}{0.8} & \frac{40}{11*0.2} & \frac{4}{0.8} + \frac{56}{11*0.2} & -\frac{6}{0.8^2 l} + \frac{96}{11*0.2^2 l} \\ -\frac{6}{0.8^2 l} & \frac{144}{11*0.2^2 l} & -\frac{6}{0.8^2 l} + \frac{96}{11*0.2^2 l} & \frac{12}{0.8^3 l^2} + \frac{240}{11*0.2^3 l^2} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_3 \\ \theta_2 \\ \delta_2 \end{bmatrix} \quad (2.84)$$

Therefore, except for the term EI_0/l , the stiffnesses, according to the format of expression 2.54 are

$$\mathbf{k}_{cc} = \begin{bmatrix} 5 & 0 \\ 0 & 47, 27 \end{bmatrix}; \mathbf{k}_{ci} = \mathbf{k}_{ic}^T = \begin{bmatrix} 2, 5 & -\frac{9,375}{l} \\ 18, 18 & \frac{327,27}{l} \end{bmatrix}; \mathbf{k}_{ii} = \begin{bmatrix} 30, 45 & \frac{208,81}{l} \\ \frac{208,81}{l} & \frac{2750,7}{l^2} \end{bmatrix}, \quad (2.85)$$

where we use lowercase letters as these are local relationships and where, therefore, the condensed stiffness according to 2.57 will be:

$$\mathbf{k}_c = \frac{EI_0}{l} \begin{bmatrix} 4, 26 & 2, 58 \\ 2, 58 & 5, 29 \end{bmatrix}. \quad (2.86)$$

We can obtain this expression with `maxima` with the following sequence, in which we do not include the constant factor $EI_0/l = k_0/l$:

```
kcc: matrix([5,0],[0,520/11]);
kci: matrix([5/2, -6*25/16/1],
            [40*5/11, (12*5)**2/11/1]);
kic: transpose(kci);
kii: matrix([5+5*56/11,-6*25/16/1+25*96/11/1],
            [-6*25/16/1+25*96/11/1,
            12*125/64/1**2+240*125/11/1**2]);
kc: kcc- kci . invert(kii) . kic;
float(%);
```

Stiffness of the haunched member with one hinged end.

This is a new condensation problem, easy to understand in the case of the constant section member, and therefore easy to extend to the haunched case.

For the constant section member, the stiffness relationships establish that

$$\begin{bmatrix} M_i \\ M_d \end{bmatrix} = \frac{EI}{l} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_d \end{bmatrix} \quad (2.87)$$

but if, due to the hinged end i , we know that $M_i = 0$ must be

necessary, we will have

$$\begin{aligned}
 0 &= \frac{EI}{l} (4\theta_i + 2\theta_d) \implies \theta_i = -\frac{2}{4}\theta_d \\
 M_d &= \frac{EI}{l} (2\theta_i + 4\theta_d) = \frac{EI}{l} (-\theta_d + 4\theta_d) \\
 M_d &= 3\frac{EI}{l}\theta_d
 \end{aligned} \tag{2.88}$$

whose result is nothing more than the application of the process described by equation 2.57, where now, ignoring the constant EI/l , we have $\mathbf{K}_{ii} = \mathbf{K}_{cc} = 4$; $\mathbf{K}_{ci} = \mathbf{K}_{ic} = 2$.

If we apply the same procedure to the haunched member, considering the hinge at the end with the smaller depth, we will have that the stiffnesses from 2.86 are now condensed considering that the different terms are $\mathbf{K}_{cc} = 5, 29$; $\mathbf{K}_{ii} = 4, 26$; $\mathbf{K}_{ci} = \mathbf{K}_{ic} = 2, 58$, where we have again ignored the constant EI_0/l , resulting in a relationship that can also be calculated with `maxima`:

```
kac: kc[2,2] - kc[2,1] * kc[1,1]**(-1) * kc[1,2] ;
float(%);
```

$$M = 3,73 \frac{EI_0}{l} \theta \tag{2.89}$$

2.1.3.2 Fixed-end bending moments of the haunched beam

The procedure to condense the reactions of the beam composed of two segments, figure 2.14, that is, the method to obtain the

fixed-end forces at its ends or their negatives, the loads on these ends, considering loads in all its segments, uses, according to section 2.1.2.1, the superposition of reactions from two parts or phases, namely:

- the fixed-end forces of each segment, considering that the intermediate node does not move. This assumes unbalanced loads at said intermediate node. According to the procedures in 2.1.1.7, equation 2.31.
- the end reactions when that intermediate node is released, being subjected to the loads from the preceding step, as detailed in equations 2.50, 2.51, 2.52,

obtaining as a result the equivalent loads described in equation 2.53.

Fixed-end beam: haunched segment. The initial fixing assumes that at node 2 (figure 2.14) there is an applied moment and load, resulting from those coming from both segments. We determine the fixed-end moments of the haunched segment using expression 2.31, as specified in 2.34, using the shape functions already obtained for said segment, and we obtain their shear forces by adding the isostatic and hyperstatic parts⁴:

```
fqcant: integrate(matrix([q*f1],[q*f2]),x,0,1);
fqcant: subst([alpha=4],%);
```

4. The reactions of a simply supported beam plus those derived from the end moments.

$$\begin{aligned} \begin{bmatrix} m_i \\ m_d \end{bmatrix} &= \frac{ql^2}{20(\alpha^2 + 4\alpha + 1)} \begin{bmatrix} \alpha^2 + 6\alpha + 3 \\ -(3\alpha^2 + 6\alpha + 1) \end{bmatrix} = \begin{bmatrix} \frac{43ql^2}{660} \\ -\frac{73ql^2}{660} \end{bmatrix}; \\ \begin{bmatrix} t_i \\ t_d \end{bmatrix} &= \begin{bmatrix} \frac{ql}{2} \\ -\frac{ql}{2} \end{bmatrix} - \begin{bmatrix} \frac{30ql^2}{660} \\ \frac{30ql^2}{660} \end{bmatrix} = \begin{bmatrix} \frac{10ql}{22} \\ -\frac{12ql}{22} \end{bmatrix} \end{aligned} \quad (2.90)$$

Complete fixed-end beam. To the fixed-end forces at the nodes *connected* in the model to the rest of the structure (the $-\mathbf{F}_c$ from section 2.1.2.1, first phase), that is, those of nodes 1 ($-\mathbf{F}_{q1}$) and 3 ($-\mathbf{F}_{q3}$), and which would result from the immobility of the three nodes, 1, 2 and 3, we must add the reactions derived from the displacement of node 2 (second phase of said section), as it is loaded with the equivalent forces of the constant section and haunched segments that converge in it, and which are equal and opposite to the fixed-end forces at that *interior* node ($\mathbf{F}_{q2} = -\sum \mathbf{f}_{i,e}$).

Thus we have at the connected nodes:

```
fq1: subst(4/5*1,1,matrix([-q*1**2/12],[-q*1/2]));
fq3: subst(1/5,1, matrix(-fqcart[2],
      [-q*1/2+(fqcart[1,1]+fqcart[2,1])/1]));
```

and at the “internal” node the loads coming from the constant section region (moment and shear):

```
fq2c: subst(4/5*1,1, matrix([q*1**2/12],[-q*1/2]));
```

which will be added to the corresponding part at the weak end of the haunch

```

fq2a: subst(1/5*1,1, matrix([-fqcart[1,1]],[-q*1/2-
                                (fqcart[1,1] + fqcart[2,1])/1]));
fq2: fq2c+fq2a;

```

That is, the equivalent loads on 1 and 3 if 2 does not move are:

$$\mathbf{F}_{q1} = \begin{bmatrix} -\frac{q(0,8l)^2}{12} \\ -\frac{q(0,8l)}{2} \end{bmatrix}; \quad \mathbf{F}_{q3} = \begin{bmatrix} \frac{73q(0,2l)^2}{660} \\ -\frac{6q(0,2l)}{11} \end{bmatrix} \quad (2.91)$$

The loads acting on 2 to displace it are:

$$\mathbf{F}_{q2} = \begin{bmatrix} \frac{q(0,8l)^2}{12} \\ -\frac{q(0,8l)}{2} \end{bmatrix} + \begin{bmatrix} -\frac{43q(0,2l)^2}{660} \\ -\frac{q(0,2l)}{2} - \frac{43q(0,2l)^2}{660l} + \frac{73q(0,2l)^2}{660l} \end{bmatrix} = \begin{bmatrix} 0,05072ql^2 \\ -0,4909ql \end{bmatrix} \quad (2.92)$$

And therefore the effects on 1 and 3 of these, derived from the reactions for this second phase can be determined according to 2.52, so that considering the rotations as the degrees of freedom of 1 and 3 we will have as equivalent loads of this phase (see second term of 2.53)

```

m13 = - kci . invert(kii) . fq2;

```

$$\begin{bmatrix} M_1 \\ M_3 \end{bmatrix} = - \begin{bmatrix} 2,5 & -\frac{9,375}{l} \\ 18,18 & \frac{327,27}{l} \end{bmatrix} \cdot \begin{bmatrix} 30,45 & \frac{208,81}{l} \\ \frac{208,81}{l} & \frac{2750,7}{l^2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0,05072ql^2 \\ -0,4909ql \end{bmatrix}$$

$$\begin{bmatrix} M_1 \\ M_3 \end{bmatrix} = \begin{bmatrix} -0,0210ql^2 \\ 0,0985ql^2 \end{bmatrix} \quad (2.93)$$

It is interesting to consider an “inverse” procedure, which is used in a slide corresponding to courses of *Fundamentals of Structural Theories* in

the *Master in Building Structures* of the UPM between 2020 and 2022, in the “pill” corresponding to the course presentations, “matrix 03.5,6 ex page 2”: If we consider the complete equilibrium state for that load-displacement condition (which we name ‘d2’) we will have, considering the flexibility $\mathbf{k}^{-1}$:

$$\mathbf{U} = \mathbf{k}^{-1} \mathbf{F} \quad (2.94)$$

$$\begin{bmatrix} \mathbf{U}_{c,d2} \\ \mathbf{U}_{i,d2} \end{bmatrix} = \begin{bmatrix} \mathbf{k}^{-1}_{cc} & \mathbf{k}^{-1}_{ci} \\ \mathbf{k}^{-1}_{ic} & \mathbf{k}^{-1}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{c,d2} \\ \mathbf{F}_{i,d2} \end{bmatrix} \quad (2.95)$$

where the subscripts indicate the references to degrees of freedom of nodes “connected” or “interior” to the substructure, and where the connected nodes do not move although the “interior” ones do, so that $\mathbf{U}_{c,d2} = \mathbf{0}$ and $\mathbf{U}_{i,d2}$ is unknown, while $\mathbf{F}_{i,d2} = \mathbf{F}_{q2}$ is known, with the reactions $\mathbf{R}_{c,d2}$ being unknown. Thus, an alternative way to determine these reactions consists of constructing the flexibility matrix (inverse of the stiffness matrix) of the substructure and operating with it, and, since the equivalent loads are opposite to such reactions, they can be determined as follows: $-\mathbf{R}_{c,d2} = (\mathbf{k}^{-1}_{cc})^{-1} \mathbf{k}^{-1}_{ci} \mathbf{F}_{q2}$

```
kvcart: mat_unblocker(matrix([kcc,kci],[kic,kii]));
fvcart: invert(kvcart);
fvcart . kvcart;
k1cc: submatrix(3,4,fvcart,3,4);
k1ic: submatrix(1,2,fvcart,3,4);
k1ci: submatrix(3,4,fvcart,1,2);
k1ii: submatrix(1,2,fvcart,1,2);
m13:invert(k1cc) . k1ci . fq2;
```

being able to verify the identity of results in both procedures.

Therefore, as effects on nodes 1 and 3 to be added to those of the first phase, we obtain the following moments and shear forces, where the shear forces have been determined considering in 1 the sum of the isostatic part of the reaction plus the hyperstatic part derived from the necessary equilibrium of moments,

and in 3 the remaining necessary to balance all the load:

$$\Delta \mathbf{F}_{q1} = \begin{bmatrix} M_1 \\ F_{v1} \end{bmatrix} = \begin{bmatrix} -0,0210ql^2 \\ -\frac{-0,4909ql}{5} + \frac{(-0,0210-0,0507+0,0985)ql^2}{l} \end{bmatrix};$$

$$\Delta \mathbf{F}_{q1} = \begin{bmatrix} -0,0210ql^2 \\ -0,0714ql \end{bmatrix}; \quad (2.96)$$

$$\Delta \mathbf{F}_{q3} = \begin{bmatrix} 0,0985ql^2 \\ -0.4195ql \end{bmatrix} \quad (2.97)$$

In the maxima version:

```
dfq1: matrix(m13[1],[fq2[2,1]*0.2+
              (m13[1,1]+m13[2,1]-fq2[1,1])/1));
dfq3: matrix(m13[2],[fq2[2,1]*0.8-
              (m13[1,1]+m13[2,1]-fq2[1,1])/1]);
```

In such a way that the total equivalent end loads in the complete haunched member are:

$$\mathbf{F}_{q1} + \Delta \mathbf{F}_{q1} = \begin{bmatrix} -0.0744ql^2 \\ -0.4714ql \end{bmatrix}; \quad \mathbf{F}_{q3} + \Delta \mathbf{F}_{q3} = \begin{bmatrix} 0.103ql^2 \\ -0.5286ql \end{bmatrix} \quad (2.98)$$

```
fq1t: fq1+dfq1;
fq3t: fq3+dfq3;
```

We would arrive at the same end shears considering only the resulting end moments along with the total load distributed uniformly throughout the member, for which the isostatic part at both ends would contribute a shear of $0.5ql$, from which would be subtracted or added the hyperstatic part derived from the end moments: $(-0.0744ql^2 + 0.103ql^2)/l$.

Member hinged at one end. We have the stiffness (eq. 2.86) and the end reactions (eqs. 2.98) of the complete fixed-fixed haunched member.

If the left end is hinged, this implies that the reaction (moment) at that end cannot materialize, which forces its rotation to nullify such reaction and the consequent change in the forces, also at the haunched end, which can be analyzed without difficulty and is left for the reader, who should finally arrive at the following equivalent end forces (moment, shear) at 1 and 3:

$$\mathbf{F}_{q1,a} = \begin{bmatrix} 0 \\ -0.352ql \end{bmatrix}; \mathbf{F}_{q3,e} = \begin{bmatrix} 0.148ql^2 \\ -0.648ql \end{bmatrix} \quad (2.99)$$

These internal forces are therefore the opposite of the reactions of the considered hinged-fixed haunched member subjected to uniformly distributed load.

2.1.3.3 Frame behavior

Being known the elastic properties of the haunched members that constitute the frame beams, and considering constant sections and rigid connections in the central columns, as well as pinned connections in the facade ones, the complete frame model is now elementary: the degrees of freedom would be the displacements and rotations of the nodes but, given the conditions mentioned, the following simplifications are possible:

- considering that the axial stiffness is much greater than the flexural one, the elongations and shortenings of the members can be neglected, so there is only one displacement to consider per story:

- the vertical displacements of the nodes are prevented by the columns, and
- the horizontal displacements are identical in all nodes of the same story due to the axial constraint imposed by the beams.
- considering the hinges in facades, the rotations of these nodes are not necessary in the model as the moments are known —null— in all members that meet at them.

Therefore, only the rotations and displacements of the five central nodes of the frame are part of the model.

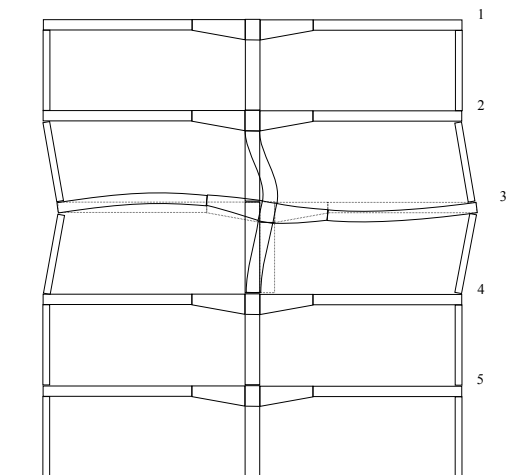


Figure 2.16: Five-story frame: story movements

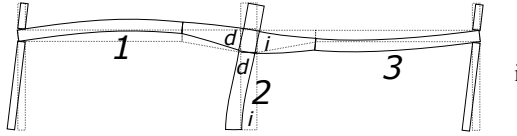


Figure 2.17: Identification of member deformations in a generic story i

In figure 2.16, the movements of one of the stories considered in the whole model can be observed, movements which are reproduced in 2.17 for a generic story, identifying the member numbering and end names and the deformation parameters of its members.

For this model, we will thus have, as parameters for each story i the following:

Global parameters: displacements (and loads)

$$\mathbf{U}_i = \begin{bmatrix} U_{ix} \\ \theta_i \end{bmatrix}; \quad \mathbf{F}_i = \begin{bmatrix} F_{ix} \\ M_i \end{bmatrix};$$

where i refers to the story and x to the direction of movement or load, therefore F_{ix} being the applied horizontal force and M_i the moment applied to the node, for the movements U_{ix} (horizontal displacement) and θ_i (rotation).

Local parameters: deformations (and forces)

$$\mathbf{u}_i = \begin{bmatrix} \theta_{i,1d} \\ \theta_{i,2i} \\ \theta_{i,2d} \\ \theta_{i,3i} \end{bmatrix}; \quad \mathbf{f}_i = \begin{bmatrix} m_{i,1d} \\ m_{i,2i} \\ m_{i,2d} \\ m_{i,3i} \end{bmatrix};$$

where only are considered the rotations or end moments of the members (that are numbered) at their end nodes, that are identified with i (left) or d (right).

Thus, the global displacement parameters form a column of ten parameters that can be ordered in any preferred form, such as any of the formats a or b of the following expressions, in which however the same order must be maintained for both movement and load parameters:

$$\mathbf{U}_a = \begin{bmatrix} U_{x1} \\ \theta_1 \\ U_{x2} \\ \theta_2 \\ U_{x3} \\ \theta_3 \\ U_{x4} \\ \theta_4 \\ U_{x5} \\ \theta_5 \end{bmatrix}; \mathbf{F}_a = \begin{bmatrix} F_{x1} \\ M_1 \\ F_{x2} \\ M_2 \\ F_{x3} \\ M_3 \\ F_{x4} \\ M_4 \\ F_{x5} \\ M_5 \end{bmatrix}; \mathbf{U}_b = \begin{bmatrix} U_{x1} \\ U_{x2} \\ U_{x3} \\ U_{x4} \\ U_{x5} \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}; \mathbf{F}_b = \begin{bmatrix} F_{x1} \\ F_{x2} \\ F_{x3} \\ F_{x4} \\ F_{x5} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix};$$

Since we are going to later condense the model to analyze its behavior subjected only to horizontal wind loads, in what can be called the frame's *shear model*, we will consider the format of variant b . In this model, we will later condense into the c parameters (or also connected parameters) everything that can represent that displacement, keeping in the i parameters the movements that remain integrated or included in the condensation and that, although hidden, will not be ignored as they will be embedded or implicit in the condensed model.

We therefore use the following format, where c refers to the story displacements and i to the rotations of the central nodes of the story:

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_c \\ \mathbf{U}_i \end{bmatrix}; \mathbf{F} = \begin{bmatrix} \mathbf{F}_c \\ \mathbf{F}_i \end{bmatrix}.$$

Thus, the format of the frame's matrix equations can be expressed both in the complete ten-parameter model and in the condensed five-parameter model, as detailed below:

$$\mathbf{F} = \mathbf{K}\mathbf{U}$$

$$\begin{bmatrix} \mathbf{F}_c \\ \mathbf{F}_i \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{cc} & \mathbf{K}_{ci} \\ \mathbf{K}_{ic} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{U}_c \\ \mathbf{U}_i \end{bmatrix} \quad (10 \text{ parameters})$$

$$\mathbf{F}_c = \mathbf{K}_c \mathbf{U}_c \quad (5 \text{ parameters})$$

with

$$\mathbf{K}_c = \mathbf{K}_{cc} - \mathbf{K}_{ci} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ic}$$

according to what expression 2.57 states, and being able to deduce in this model the node rotations, $\boldsymbol{\theta} = \mathbf{U}_i$, resulting from the frame drift from the horizontal displacements $\mathbf{U}_x$, with $\mathbf{U}_c = \mathbf{U}_x$ for the case of null internal loads, $\mathbf{F}_i = \mathbf{M} = \mathbf{0}$, in the form already seen in section 2.1.2.1, equation 2.56:

$$\mathbf{U}_i = -(\mathbf{K}_{ii})^{-1} \mathbf{K}_{ic} \mathbf{U}_c$$

In this way, a model can be constructed, using a spreadsheet for example, in which from the local perspective, the supports have a stiffness defined by that of the fixed-fixed constant section member, equation 2.42, also in 2.87, or its extended format from

2.44, and the beams have the stiffness of the hinged haunched member, equation 2.89.

The assembly of the matrix for the ten-degree-of-freedom model can be done manually, as described in 2.1.2.2, and considering in the columns, for the case of each of the horizontal displacements considered in isolation, the node rotations relative to what would be the displaced and undeformed geometry of the column, rotations implicit in the deformed figure that can be seen in 2.16. Likewise, the condensed, five-degree-of-freedom model can be constructed in the same spreadsheet from the ten-degree model. This reduced model should provide, logically, exactly the same results.

2.2 The plastic approach

2.2.1 Modeling

We will now proceed to model the structural behavior based on the plastic limit condition, assuming that no prior failure occurs, whether due to instability or buckling, or due to brittle failure derived from insufficient ductility. Later, in section 3, we will analyze the first of these issues. On the other hand, for the second one, it is necessary to compare the ductility requirements, which can be studied following Cervera Bravo and Navas-Sánchez 2021, with the available capacities in the elements, described in codes such as ASCE-41 or Eurocode 8, part 3.

The plastic model assumes that, in addition to satisfying

equilibrium conditions until collapse is reached, in every section where the resistance limit is reached, plastification occurs without failure, whereby the local deformation of the section can increase indefinitely without loss of strength. If the structure is hyperstatic, this implies a reconfiguration in the distribution of forces, increasing those of the sections not yet plastified as the load increases until enough sections plastify to transform the structure into a mechanism and therefore lead it to collapse. Hence the term *limit analysis*, and therefore the need to characterize the resistance limits of each and every relevant section of the model.

2.2.1.1 Section resistance or limit capacity

An elementary example of representing resistance limits in metallic sections would be the usual linear combination expressions, typical in regulations, such as the following, where the verification forces would be $\mathbf{f} = [N_{Ed}, M_{Ed}]^T$, obviously capable of having positive or negative values:

$$\frac{|N_{Ed}|}{N_{Rd}} + \frac{|M_{Ed}|}{M_{Rd}} \leq 1 \quad (2.100)$$

In this expression, the coefficients of the verification *limit plane*, a_0, b_0 , are $1/N_{Rd}, 1/M_{Rd}$, coefficients that, to represent the unit vector of the plane, must be divided by the modulus⁵

5. In case doubts arise due to the “mixing” of dimensionally different variables, moment and normal force, it would be worth remembering the alternative representations indicated in example 1.3.7.4. A more detailed discussion of this interesting issue would require excessive space here.

of the vector $[a_0, b_0]^T$:

$$\left\| \frac{\frac{1}{N_R}}{\frac{1}{M_R}} \right\| = \sqrt{\frac{1}{N_R^2} + \frac{1}{M_R^2}} = \frac{\sqrt{N_R^2 + M_R^2}}{N_R M_R} \quad (2.101)$$

so that the verifications for each section adopt the format

$$\boldsymbol{\psi} \mathbf{f} \leq \mathbf{d} \mathbf{r} \rightarrow m_R |N_E| + n_R |M_E| \leq m_R N_R \quad (2.102)$$

where

$$n_R = \frac{N_R}{\sqrt{N_R^2 + M_R^2}}; m_R = \frac{M_R}{\sqrt{N_R^2 + M_R^2}} \quad (2.103)$$

and which, developed to avoid computing the absolute value, can be written in the form

$$\begin{aligned} m_R N_E + n_R M_E &\leq m_R N_R \\ -m_R N_E + n_R M_E &\leq m_R N_R \\ m_R N_E - n_R M_E &\leq m_R N_R \\ -m_R N_E - n_R M_E &\leq m_R N_R \end{aligned} \quad (2.104)$$

or, in matrix format:

$$\begin{bmatrix} m_R & n_R \\ -m_R & n_R \\ m_R & -n_R \\ -m_R & -n_R \end{bmatrix} \begin{bmatrix} N_E \\ M_E \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} m_R N_R; \quad \boldsymbol{\psi} \mathbf{f} \leq \mathbf{d} \mathbf{r}. \quad (2.105)$$

This approach can be generalized to any other forms of representing (plastic) limits, as we do in the following sections.

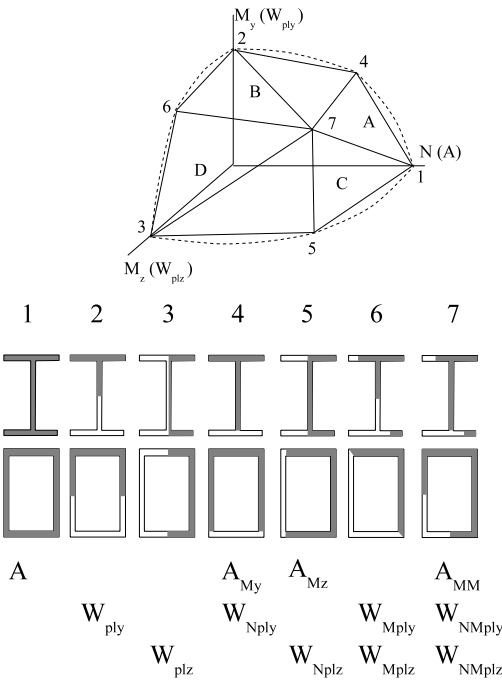


Figure 2.18: Modeling of resistance limits in steel sections

2.2.1.2 Bar structures: section resistance properties

If we analyze the possible collapse conditions of a section subjected to normal stresses exclusively, that is, without considering the possible concurrence of shear forces, such section could collapse by tension or compression exclusively, but also by the combination of tensioned regions while others are compressed, in bending conditions, or axial forces combined with bending.

Sections with tension and compression resistance. Figure 2.18 considers some possible cases in I-sections or rectangular tubes. Cases 1, 2, and 3, where the dark region can be considered as tensions and the white as compression, would correspond respectively to pure axial force, or pure bending in the directions of major and minor inertia. The respective resistances are fixed by the area and yield stress conditions $N = Af_y$ or plastic modulus and yield stress $M_i = W_{pli}f_y$. Such resistances can be represented as points in a space of three axes N, M_y, M_z (or $N/f_y = A, M_y/f_y = W_{ply}, M_z/f_y = W_{plz}$) that allow identifying combined situations. Combinations of axial force and biaxial bending can be considered and, therefore, infinite other limit cases, depending on the position of the neutral line that separates tensions from compressions. However, considering that all that infinity establishes a convex limit in the N, M_y, M_z space, this resistance limit can be simplified by establishing in it some additional points (4 to 7 in the figure), and reducing the limit to the polyhedron of flat faces limited by these vertices, whose coordinates will correspond to the area values or available plastic moduli for the corresponding value of normal force

or bending moment, which can be determined for the considered stress state. Such that the coordinates of collapses like 1, 3, 4, or 7 would be respectively $[A, 0, 0]$, $[0, 0, W_{plz}]$, $[A_{My}, W_{Nply}, 0]$ and $[A_{MM}, W_{NMply}, W_{NMplz}]$, where A or W_{plz} are usual static values from steel section tables, while A_{My} ⁶, W_{Nply} ⁷, A_{MM} ⁸, W_{NMply} ⁹ or W_{NMplz} ¹⁰ would be static values directly calculable from the stress distribution figures, and whose values for usual sections are tabulated in annex B of Cervera 2010a.

The vertices of the limit polyhedron being known in the indicated way, the determination of the limit planes is a mere algebraic operation that is detailed in annex A for the three-dimensional case considered here.

Sections without tension resistance. Figure 2.19 considers the case of sections without tension resistance, as is common in masonry. In this case, and in order to simplify the construction of static values, non-straight neutral lines have been drawn which, although unlikely from the perspective of stress distribution conditions, are however simpler for determining the corresponding static values and, moreover, are on the safe side, as is easily verified by imagining the modification of said values

6. Available area concurrent with major axis bending exhausting the flanges

7. Modulus for major axis bending concurrent with normal force in web

8. Available area concurrent with biaxial bending

9. Modulus in principal direction of biaxial bending concurrent with normal force

10. Modulus in secondary direction of biaxial bending concurrent with normal force

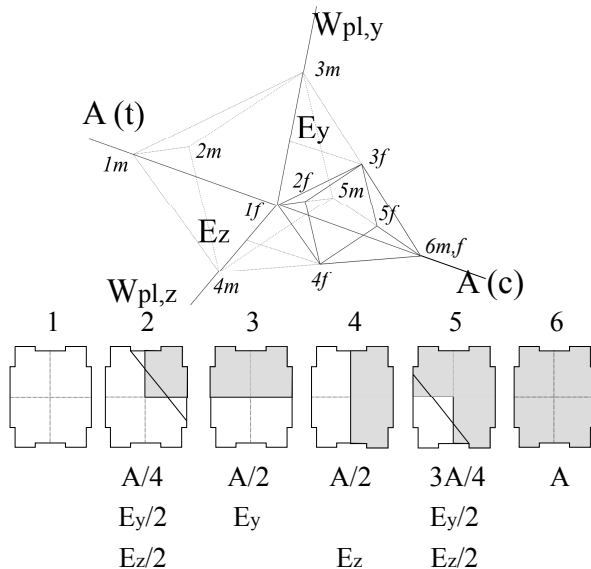


Figure 2.19: Modeling of resistance limits in masonry sections (f), in contrast with the corresponding modeling in the metallic case (m).

if a straight line is assumed to replace the broken one while maintaining the magnitude of shaded and unshaded areas.

The relevant static values are now those of the section area or fractions thereof, as well as the static moments of half section or their fractions, equivalent the former to half of what would

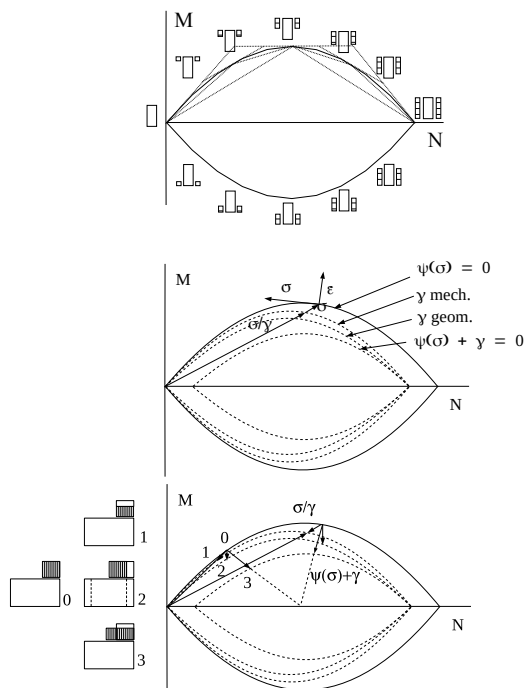


Figure 2.20: Comparison between the three safety models for the case of rectangular sections in flexural-compression without tension resistance. a) eccentric compression states and alternatives, in dotted lines, for limit force combinations. b) contraction of the limit surface according to each safety criterion. c) comparison in stress reduction models between the limit state (0) and those considered safe (1, 2, 3) and the corresponding reduction vectors according to criterion

be the corresponding plastic moduli in sections with tension resistance.

What is relevant now is that the limit surface—or limit resistance surface—is not centered at the origin as was the previous case, but rather said origin—that is, the point of zero force—corresponds to one of the possible limit resistance conditions and, therefore, one of the conditions to avoid: masonry must be subjected to compression to be safe.

And this would lead us to consider the difference between the different safety criteria usually employed.

Safety criteria. Figure 2.20 graphically shows the difference between the three common models for representing safety conditions in relation to the limit states described by the limit surface or failure [states] surface (see Cervera 2010b).

The three criteria could be stated as follows:

- *classical criterion*: The factored forces do not reach the limit surface (in the domain of forces and resistant capacities) corresponding to the reduced strengths. The reduction—proportional, of forces or loads—occurs around the situation corresponding to zero load (zero force). It corresponds to the use of a mechanical safety coefficient γ_m to increase the loads.
- *geometric criterion*: The forces are resisted in a reduced geometry interior to the real one (in the domain or geometric space occupied by the structural volume). The reduction implies a smaller lever arm with equal compression (and less area availability for compression resistance). It corresponds

to the use of a geometric safety coefficient γ_g to reduce the available geometry.

- *robust criterion*: The domain limited by the limit surface (in the space of forces and resistant capacities) is reduced around an interior point to this, centered, or reasonably centered in relation to the complete domain.

In any case, the application of any of these criteria will lead us to characterize the resistance conditions in the format of equation 1.40, as verified in the examples of section 1.4.13.2 and appendix A.

2.2.1.3 Bar structures: member modeling

We have seen both in the previous section, 2.2.1.2 and in the example of the previous chapter, 1.4.13.2, how to characterize the resistant capacities of the sections: it is a design decision of the resistant model for these. The modeling of the member requires establishing said resistant capacity, but also the way in which this limit is reached, and how the corresponding subsequent plastic deformation of the section affects the member and the complete structure, through its connections with the rest of the structure's members.

In locally unloaded bars, it will be common to refer all behavior to the end sections, as these are subjected to the greatest forces, so that rotations accumulate in the contact planes of said end sections with the rest of the structure: this leads to establishing as connection nodes the points to which we assimilate such contact planes, being common to choose a single *node* per “joint”, attributing the deformation of the interior of the con-

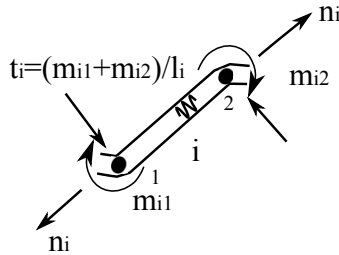


Figure 2.21: Plastic modeling of steel members: the ends can have plastic hinges, and the member could equally be subjected to axial deformations through a yielding mechanism whose position is indifferent.

necting joint between members to the contiguous bars, just as it is done in the elastic approach, and assuming that all deformation is accumulated at the end of the contiguous members to the joint and accepting complete indeformability of the rest of these.

Figure 2.21 reflects the idea of this model, in which the possibility of plastic deformations derived from the coupling of moment and normal force requires the kinematic counterparts of rotation (at both ends) and length change of the member. The latter—whether elongation or shortening—does not require precise determination of the yielding region: having a single normal force in the bar if there is no load applied to it, only one longitudinal deformation parameter is needed which can be associated with the member itself as if it were a spring or exten-

sible mechanism located at any position of said member. On the other hand, it is common to consider the shear force secondary, as being directly deduced from the moment differences at both ends, so that variables are not adopted either for this or for the transverse deformation of the member due to distortion of its sections.

Thus, for each bar the force parameters $\mathbf{f}_i$ are $\begin{bmatrix} m_{i1} & m_{i2} & n_i \end{bmatrix}^T$ for the corresponding deformation parameters $\mathbf{u}_i = \begin{bmatrix} \theta_{i1} & \theta_{i2} & \delta_i \end{bmatrix}^T$ which will be the (only) local parameters to be used in the corresponding equilibrium equations and compatibility between local and global conditions. That is, even when the set of end forces also includes the shears t_i , these are expressed only as a consequence of the end moments ($t_i = (m_{i1} + m_{i2})/l_i$), and are not part, therefore, of the vector $\mathbf{f}_i$.

This approach allows identifying the parameters $\mathbf{f}$ and $\mathbf{u}$ with those of the elastic approach: they are the same end forces or movements, although the interpretation of their meaning differs for deformations: in the elastic case they show the rotations of the member's ends due to the deformations distributed throughout the bar while in the plastic case they derive from the deformation concentrated in hinges or yielding regions. And this identification allows, therefore, the use of the same matrices $\mathbf{H}$ and $\mathbf{B}$ used in the elastic approach to establish the conditions of equilibrium and compatibility, equations 1.2 and 1.1.

It should be noted that this approach requires the consideration in the joints of degrees of freedom for both the displacement and the rotation. The second may seem redundant when there are only two bars and no applied moment, since then the bend-

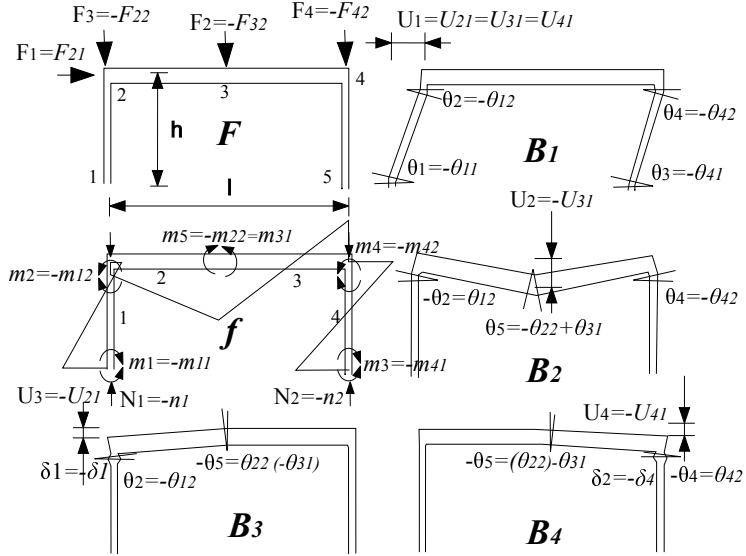


Figure 2.22: Plastic modeling of a steel frame with four global degrees of freedom in displacements considering the possibility of shortening of the columns due to coupling of normal forces and moments in plastification. The load and force vectors are represented, as well as the figures from which the four columns of the compatibility matrix are derived. A single force parameter and rotational deformation per joint is selected versus the alternative used in figure 1.15.

ing moment on both sides of the joint is identical by equilibrium, and the plastic deformation is the sum of those corresponding to both sides of the “joint”, so that a single (plastic) hinge and a single force would suffice. The figure 2.22 shows an example of this latter approach.

2.2.1.4 Structures of rigid blocks

The plastic or limit approach is applicable to the analysis of masonry structures (arches) in cases where Heyman’s conditions, Heyman 1966, can be met exactly or approximately, namely

- masonry has no tensile strength
- compressive strength is “infinite”
- no sliding occurs at the joints.

If these conditions are met exactly, it can be demonstrated that the limit analysis theorems are satisfied and therefore plastic analysis is applicable. The consideration of infinite compressive strength only tries to indicate that failure conditions correspond to collapses derived from the relative rotation between the different pieces or voussoirs, with or without crushing of the contact points. It is a condition that can be relaxed, as local crushing allows considering different rotation centers from the end of the voussoir. Not so the no-sliding condition because, as studied by Magdalena Layos 2013, in case of possibility of sliding the failure model becomes non-associated plasticity (see definition in 1.4.4) and the solution ceases to be unique. In what follows we will consider that the conditions are sufficiently met although we will only accept the analysis results if they agree with them.

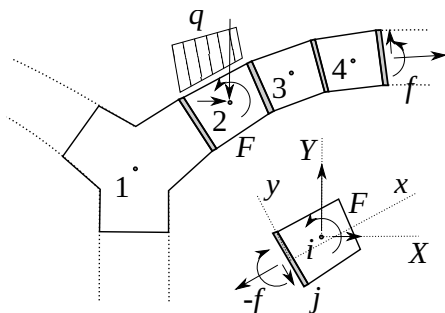


Figure 2.23: Model of structure of rigid blocks linked through deformable joints. The force $\mathbf{f}$ on some joint j resulting from the action exerted on it (by block i) on the face of increasing axes (right face of the joint, in equilibrium with the opposite force on the opposite, negative or left face) is represented.

Parameters: voussoirs and joints. We therefore consider the masonry structure or skeleton constituted by the grouping of a set of rigid or indeformable voussoirs or blocks in mutual contact through joints capable of resisting eccentric compressions but not tensions, as in figure 2.23.

To make the decision through which the global parameters $\mathbf{U}, \mathbf{F}$ or local $\mathbf{u}, \mathbf{f}$ are chosen, we consider the advantage of describing the movements and global equilibria of each of the blocks, while the forces and their limits corresponding to collapse conditions are linked to what happens in the joints. This leads to blocks being what define the global parameters and

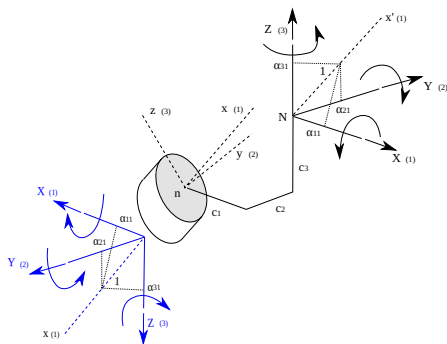


Figure 2.24: Chirality of forces in a section: being equal and opposite the forces applied to the two faces of the section, their description requires either using axis systems of opposite chirality on each face, or differentiating the denominations of the faces for a single axis system associated with one of them. The latter is most common, associating the axes with the positive face, in the growing direction of the perpendicular axis.

joints the local ones.

In this way, the displacements of each of the blocks and the external forces applied to them are $\mathbf{U}$ and $\mathbf{F}$ respectively, in global axes X, Y , while the deformations of the joints —the relative movements between their two faces— and their internal forces will be the parameters $\mathbf{u}$ and $\mathbf{f}$ respectively. These internal forces are described according to local axes x, y , which generally do not coincide with the global ones. Two faces must

be considered in each joint, so that the internal forces are equal to the forces exerted on the increasing face or their opposites on the decreasing face, as shown in figure 2.23 for two different joints. As a general criterion, we choose to represent the force in the joint by the numerical value of the forces applied to its increasing face (in the increasing direction of the axis perpendicular to the face). It is important to note the *chirality* of the forces in a section, meaning that, while the values of the applied forces are equal and opposite depending on the face they are observed from, the axes that would allow using identical positive values on both faces have different chirality: those of the increasing face are left-handed, while those of the decreasing face would be right-handed, as shown in figure 2.24. This requires, therefore, considering both the orientation attributed to the joint and the face with which it contacts each voussoir.

Equilibrium equations of the voussoir. To establish the equilibrium conditions in this case, it systematically happens that the origins of the global and local axes will not coincide, so we must take into consideration the axis change expressions from section 2.1.2.2, figure 2.10, and equations 2.63 and 2.64.

If a joint j connects on its increasing face with a block i ($i|j+$) or if it does so through its decreasing face ($i|j-$), we can consider that with the appropriate axis changes, the relevant terms to add to the equilibrium (to obtain zero resultant) are the global load terms $\mathbf{F}$ to being added to the local $+(-\mathbf{f})$ —or the $+(+\mathbf{f})$ for the negative load—and therefore we will have that the part that said joint contributes to equilibrium, and

the total sum of the joints' contribution to the block could be represented with equations such as the following:

$$\mathbf{F}_{i|j+} = \mathbf{H}_{ij+} \mathbf{f}_{j+}; \quad \mathbf{F}_i = \sum_j \mathbf{F}_{i|j} \quad (2.106)$$

In these expressions, $\mathbf{F}_{i|j+}$ or the possible $\mathbf{F}_{i|j-}$ will indicate the contribution to the equilibrium of the joints connected to the block from the right face (the increasing or positive face) or from the left face (the decreasing or negative face): in the first case the force contributed to the block is $-\mathbf{f}_j$, while in the second it will be $+\mathbf{f}_j$, as shown in figure 2.23, with the consequent implications for matrices $\mathbf{H}_{ij+}$ or $\mathbf{H}_{ij-}$.

In three dimensions, if the position of joint j that connects on its increasing face with block i is at coordinates X, Y, Z , that is, if the eccentricities in figure 2.10 are $[c_1, c_2, c_3] = [-X, -Y, -Z]_j$ (where $[c_1, c_2, c_3]$ are the positive values of the eccentricities in the figure, and $[X, Y, Z]$ being the coordinates of the joint in relation to global axes located at the reference node of the voussoir), the equations of partial contribution to equilibrium can be established¹¹, equations that restricted to the two-dimensional

11.

$$\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}_i = \begin{bmatrix} \alpha_{Xx} & \alpha_{Xy} & \alpha_{Xz} & 0 & 0 & 0 \\ \alpha_{Yx} & \alpha_{Yy} & \alpha_{Yz} & 0 & 0 & 0 \\ \alpha_{Zx} & \alpha_{Zy} & \alpha_{Zz} & 0 & 0 & 0 \\ \alpha_{Zx}Y - \alpha_{Yx}Z & \alpha_{Zy}Y - \alpha_{Yy}Z & \alpha_{Zz}Y - \alpha_{Yz}Z & \alpha_{Xx} & \alpha_{Xy} & \alpha_{Xz} \\ \alpha_{Xx}Z - \alpha_{Zx}X & \alpha_{Xy}Z - \alpha_{Zy}X & \alpha_{Xz}Z - \alpha_{Zz}X & \alpha_{Yx} & \alpha_{Yy} & \alpha_{Yz} \\ \alpha_{Yx}X - \alpha_{Xx}Y & \alpha_{Yy}X - \alpha_{Xy}Y & \alpha_{Yz}X - \alpha_{Xz}Y & \alpha_{Zx} & \alpha_{Zy} & \alpha_{Zz} \end{bmatrix}_{ij} \begin{bmatrix} f_x \\ f_y \\ f_z \\ m_x \\ m_y \\ m_z \end{bmatrix}_j$$

case result in, (for $\mathbf{H}_{ij+}$):

$$\begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix} = \begin{bmatrix} \alpha_{Xx} & \alpha_{Xy} & 0 \\ \alpha_{Yx} & \alpha_{Yy} & 0 \\ \alpha_{Yx}X - \alpha_{Xx}Y & \alpha_{Yy}X - \alpha_{Xy}Y & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ m_z \end{bmatrix} \quad (2.107)$$

where additionally, being the angle $\langle Xx \rangle = \varphi$, it will result that $\alpha_{Xx} = \cos \varphi$, $\alpha_{Yx} = \sin \varphi$, $\alpha_{Xy} = -\sin \varphi$, $\alpha_{Yy} = \cos \varphi$.

For joints that connect from their decreasing face, the terms to consider for the sum that contributes to equilibrium (the zero resultant) are now the local $+\mathbf{f}$, with the corresponding sign changes in the equilibrium matrix $\mathbf{H}_{ij-}$ which now becomes

$$\mathbf{H}_{ij-} = - \begin{bmatrix} \alpha_{Xx} & \alpha_{Xy} & 0 \\ \alpha_{Yx} & \alpha_{Yy} & 0 \\ \alpha_{Yx}X - \alpha_{Xx}Y & \alpha_{Yy}X - \alpha_{Xy}Y & 1 \end{bmatrix} \quad (2.108)$$

but, beware! because while in equation 2.107 the values of coordinates X, Y of the joint are negative in relation to the voussoir axes, now these coordinates become positive, so that although symbolically the matrix is represented as negative of the previous one, the coordinates of the joint with respect to the voussoir are different for each joint and, therefore, the matrix is different.

A simpler alternative consists of using the joint coordinate axes for measuring the eccentricities between global nodes (voussoirs) and local nodes (joints). This is so because these eccentricities only affect the computation of moments and, given that the joint forces are in local axes, their moments at the node (or voussoir) are very simple to determine in these axes. Thus we

have

$$\mathbf{H}_{ij+} = \mathbf{H}_{ij*} = \begin{bmatrix} \alpha_{Xx} & \alpha_{Xy} & 0 \\ \alpha_{Yx} & \alpha_{Yy} & 0 \\ y & -x & 1 \end{bmatrix} \quad (2.109)$$

$$\mathbf{H}_{ij-} = -\mathbf{H}_{ij*} = - \begin{bmatrix} \alpha_{Xx} & \alpha_{Xy} & 0 \\ \alpha_{Yx} & \alpha_{Yy} & 0 \\ y & -x & 1 \end{bmatrix} \quad (2.110)$$

where now x, y are the coordinates of the node that identifies voussoir i in the local axes of the joint considered, j , axes that in the case of the joint connected by its increasing or positive face are oriented towards the interior of the voussoir, while in the case of being connected by the decreasing, or negative face, they are oriented towards the exterior of the voussoir. This difference obviously means different signs in the coordinates x, y of the voussoir for both equations, as happened in the case of 2.107 and 2.108.

Thus the conditions to balance the forces $\mathbf{F}_i$ of a certain voussoir i must include the sum of the contributions of all joints that link with said block, which, considering the face from which they link, can be expressed as

$$\mathbf{F}_i = \sum_{j+(d)} \mathbf{H}_{ij*} \mathbf{f}_{j+} - \sum_{j-(i)} \mathbf{H}_{ij*} \mathbf{f}_{j-} = \sum_j \mathbf{H}_{ij} \mathbf{f}_j \quad (2.111)$$

where the negative sign corresponds to the joints that link from their negative face (or left face of the joint)

Equilibrium equations of the complete structure. Linking in a single expression the equilibrium equations of all vous-

soirs or blocks we will have that, for the complete structure, we can write

$$\begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \vdots \\ \mathbf{F}_N \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \cdots & \mathbf{H}_{1n} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \cdots & \mathbf{H}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{N1} & \mathbf{H}_{N2} & \cdots & \mathbf{H}_{Nn} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{bmatrix} \quad (2.112)$$

where N is the number of blocks, n the number of joints, and using for each $\mathbf{H}_{ij}$ the corresponding contribution of joint j to the equilibrium of block i , null ($\mathbf{0}$) if there is no connection between said joint and block, and considering the sign $+$ or $-$ in the form $\mathbf{H}_{ij*}$ according to the contact face.

Compatibility equations. In the case we need the compatibility equations, it is enough to remember the duality established in 1.2.5, and therefore equation 1.21, so it will suffice to use the transpose of the equilibrium matrix from the previous expression 2.112.

2.2.1.5 Resistance restrictions in masonry

As we have seen for ductile bar structures, 2.2.1.3, the characterization of the resistant properties is a design decision of the resistant model to be used. In the following paragraphs we will present and discuss successively several alternatives for the modeling of plane block structures, starting from the criteria discussed at the beginning of section 2.2.1.4.

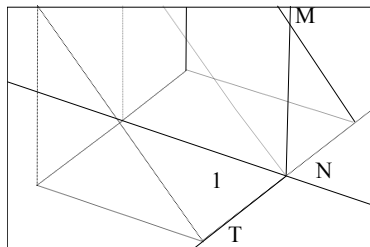


Figure 2.25: “Heyman’s” masonry resistance: infinite compression resistance (normal force), and no possibility of sliding (infinite resistance to tangential force).

The first model, figure 2.25, would respond to the strict compliance with Heyman’s criteria expressed in said section. We would have as resistance conditions the equations

$$\begin{bmatrix} h/2 & 0 & 1 \\ h/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} N \\ T \\ M \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.113)$$

Where h is the joint depth, and where we can consider $h/2 = z_{pl}$ as the plastic lever arm available for a joint of rectangular section. The conditions for possible stresses are not bounded in either N or T .

If we consider limiting the absolute values of stresses N or T , with a limit for T associated with the friction coefficient μ ,

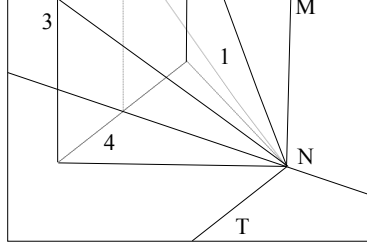


Figure 2.26: “Bounded Heyman” masonry strength: limited compression and tangential strengths.

we will have a second model, figure 2.26, for which we can write:

$$\begin{bmatrix} m & 0 & n \\ m & 0 & -n \\ -1 & 0 & 0 \\ \mu & 1 & 0 \\ \mu & -1 & 0 \end{bmatrix} \begin{bmatrix} N \\ T \\ M \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} |N_m|, \quad (2.114)$$

where

$$n = 1/\sqrt{1 + z_{pl}^2}, \quad m = n z_{pl}; \quad \text{with } z_{pl} = h/2 \text{ in rectangular section.}$$

since effectively, the lateral resistance equations are $|T| \leq \mu|N|$ while for normal stress resistance, which links moment and normal through eccentricity, we have considered the format of table 2 from section C in Cervera 2010a, even without yet limiting the moment capacity: the first two equations are identical to those

in 2.113 except for the factor n which, being a common factor in a homogeneous equation, does not affect the verification. In this equation, we are using z_{pl} to represent the plastic arm of the ductile section, or the distance between the centers of gravity of the two halves of the section, which equals half the depth in rectangular sections, but is equally easy to determine in other sections and, as we will see later, helps normalize the verifications in all types of sections.

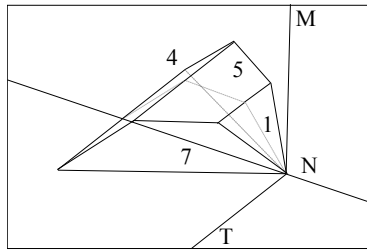


Figure 2.27: Masonry strength: limited compression stresses and tangential force.

This format allows us to clearly express a third model, in which we replace the parabolic curve that would express the flexural-compression resistance in case of limited compression stresses (see figure 2.20) with the circumscribed hexagon, where, therefore, the zero-stress vertex is formed with the same limit lines as in the previous cases, but there is a symmetric vertex for maximum compression, and furthermore the maximum moment corresponds to the situation of half-section under compression.

Thus we have a third model, figure 2.27, corresponding to the resistance condition of the following equation:

$$\begin{bmatrix} m & 0 & n \\ -m & 0 & -n \\ m & 0 & -n \\ -m & 0 & n \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ \mu & 1 & 0 \\ \mu & -1 & 0 \end{bmatrix} \begin{bmatrix} N \\ T \\ M \end{bmatrix} \leq \begin{bmatrix} 0 \\ n \\ 0 \\ n \\ 1/4 \\ 1/4 \\ 0 \\ 0 \end{bmatrix} |N_m|z_{pl}, \quad (2.115)$$

again with $n = 1/\sqrt{1 + z_{pl}^2}$, $m = n z_{pl}$.

In what follows, we will construct a fourth model which, although numerically characterized for rectangular sections, can be used with good approximation for solid sections that are not rectangular. To construct this model, we will approximate the limit curve (parabolic in rectangular sections) through the octagon described in figure 2.28.

For this construction, we consider as resistance conditions the expressions of the different limit lines of the octagon, in the following form:

$$a \frac{N}{|N_m|} + b \frac{M}{|N_m|z_{pl}} \leq d \quad (2.116)$$

In it, a and b are the components of the versor of the considered line for normalized forces, and d is the distance from the line to the origin, deducible from the passage of said line through some of its points. The crucial aspect is, therefore, establishing

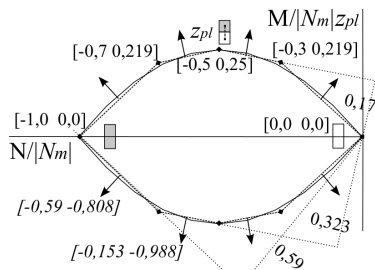


Figure 2.28: Strength of masonry sections: approximation to the limit surface with limited compression capacity.

the coordinates of the octagon vertices. We are using as scale factors the values of maximum normal force $|N_m|$ and reference moment $M_{\text{ref}} = |N_m|z_{pl}$. The vertices are easy to determine and are indicated in square brackets in normal typography in figure 2.28. Those in the lower part can be known by symmetry. Once these vertices are established, it is merely a geometric operation to determine the versors of the segments that join them, and two of them are identified in italics in the same figure. Due to the double symmetry of this, all the remaining ones can be determined. Finally, determining the distances of the different lines to the origin is also mere geometry. In the figure, three of these distances are identified in the extensions with dotted lines of the corresponding segments. Three others can be deduced by symmetry, and the remaining two correspond to lines that pass through the origin.

In this way, we can establish the eight resistance conditions in flexural-compression if we divide equation 2.116 by the modulus of its coefficients shown in the following equation:

$$\left\| \frac{a}{|N_m|}, \frac{b}{|N_m|z_{pl}} \right\| = \frac{|a|}{|N_m|z_{pl}} \sqrt{z_{pl}^2 + \frac{b^2}{a^2}} \quad (2.117)$$

Thus results as resistance condition:

$$\frac{z_{pl}a/|a|}{\sqrt{z_{pl}^2 + \frac{b^2}{a^2}}} N + \frac{b/|a|}{\sqrt{z_{pl}^2 + \frac{b^2}{a^2}}} M \leq \frac{d/|a|}{\sqrt{z_{pl}^2 + \frac{b^2}{a^2}}} |N_m|z_{pl} \quad (2.118)$$

In it, we can ignore the common divisor factor so that, substituting for the values of a, b, d corresponding to each plane and adding the limit equations for friction in the tangential force, result in the ten equations of the following limit expression:

$$\begin{bmatrix} z_{pl} & 1,37 & 0 \\ z_{pl} & 6,46 & 0 \\ -z_{pl} & 6,46 & 0 \\ -z_{pl} & 1,37 & 0 \\ z_{pl} & -1,37 & 0 \\ z_{pl} & -6,46 & 0 \\ -z_{pl} & -6,46 & 0 \\ -z_{pl} & -1,37 & 0 \\ \mu & 0 & 1 \\ \mu & 0 & -1 \end{bmatrix} \begin{bmatrix} N \\ M \\ T \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1,111 \\ 2,111 \\ 1 \\ 0 \\ 1,111 \\ 2,111 \\ 1 \\ 0 \\ 0 \end{bmatrix} |N_m|z_{pl}; \quad \psi \mathbf{f} \leq d\mathbf{r} \quad (2.119)$$

This expression can be used approximately in sections different from the rectangular one insofar as it is an approximation to the

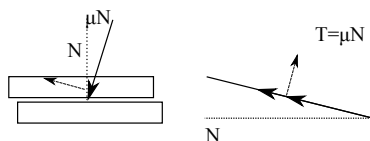


Figure 2.29: Violation of flow rule: on the left, the oblique normal and the displacement vector orthogonal to it that would imply separation between surfaces. On the right, the limit line in the space of forces N, T and the corresponding plastification vector, which should be orthogonal if the flow rule were fulfilled.

limit curve in which four of the eight vertices, corresponding to the maxima or minima of each of the stress components N or M are exact in any circumstance, while the other four respond to a sensible approximation to the different possible curves that define the strength of symmetric solid sections in masonry.

It should be noted again that the model would be invalid in cases where failure occurs with sliding, as the flow rule established in section 1.4.6 for *associated plasticity* models would imply, as indicated in section 1.4.12, a collapse kinematics that would be being violated in the real failure: sliding failure violates the flow rule insofar as it requires a separation between the sliding surfaces that does not occur in reality. Figure 2.29 illustrates the issue.

2.2.2 Verification of the complete structure

Up to this point, we have seen how to construct the resistance conditions for the sections, which limit the stresses through equations 1.40 (in versions for sections such as 2.105 or 2.119, which can be generalized to the complete structure without any difficulty, as in 1.41). We have also seen how to establish the conditions of equilibrium (and implicitly compatibility) between such stresses and the loads that would cause the structure to collapse, through equations 1.37, in which we usually do not know at the beginning of the process the final collapse loads $\mathbf{F}$ or, alternatively, the load factor γ by which the initially applied loads $\mathbf{F}_I$ must be scaled to reach failure. Thus, the solution to the collapse state of the complete structure is nothing more than solving the problem of determining the load factor needed to reach that collapse, along with the corresponding stress state.

2.2.2.1 Limit load factors

As already seen in section 1.4.9, the formulation of equilibrium conditions allows the load factor to be applied only to a part of the loads, such as when permanent loads are known and we want to know how much the live load can be increased, obtaining the collapse load as the sum of both components: $\mathbf{F} = \mathbf{F}_0 + \gamma\mathbf{F}_I$, see 1.75.

We have also seen in section 2.2.1.2 that when limit conditions are close to zero or near-zero stress situations, classical safety criteria are not adequate: in this case, we saw as an alternative the determination of the so-called robust criterion that

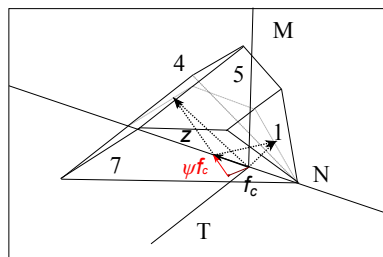


Figure 2.30: Masonry resistance and its “center”: limited compressive and shear stresses. A couple of stress cases are added, corresponding to two limits in planes 1 and 4, and their composition as the sum of central ($\mathbf{f}_c$, common to both, in solid line) and additional components. For one of them, the decomposition of the central stress between parallel and orthogonal components to the corresponding limit plane is added. The latter is the projection of $\mathbf{f}_c$ on the direction of the plane’s versor, aligned with the additional component, a component contained in the N, M plane given the symmetry of the limit surface with respect to this plane.

seeks to determine the form and scale for the most detrimental load alteration.

We saw this for the section, but it is easily generalizable for all of them, if for each one it is possible to determine the stress condition $\mathbf{f}_c$ that we can call central, or stress furthest from any of the failure situations. We can extend, for example, figure

2.27 by adding that center, as we do in 2.30, so that any stress condition can be understood as the sum of that component plus the difference required to reach the considered stress.

What we see for the section is equally possible for the structure because, by equilibrium, we can determine the load vector $\mathbf{F}_c$ that generates this optimal situation for all sections:

$$\mathbf{F}_c = \mathbf{H} \mathbf{f}_c \quad (2.120)$$

so that we can express as a variant or additional part of the load the part of the initial load that moves away from this optimal situation, and therefore characterize the failure based on the load factor that can be applied to this additional part until failure is reached. We would now have, therefore

$$\mathbf{F}_I = \mathbf{F}_c + \mathbf{F}_a \quad (2.121)$$

$$\mathbf{F} = \mathbf{F}_c + \gamma \mathbf{F}_a \quad (2.122)$$

where the load factor considered would therefore coincide with what we have called robust.

The determination of the stresses $\mathbf{f}_c$, necessary to know the optimal load for the structure, $\mathbf{F}_c$ is simple: the resistance equations can be rewritten using positive *slack variables* $\mathbf{z}$:

$$\psi \mathbf{f} \leq \mathbf{d} \mathbf{r} \equiv \psi \mathbf{f} + \mathbf{z} = \mathbf{d} \mathbf{r}; \quad \mathbf{z} \geq \mathbf{0} \quad (2.123)$$

If we choose the denomination z for the smallest of all these, we can again rewrite these equations as

$$\psi \mathbf{f} + \mathbf{1} z \leq \mathbf{d} \mathbf{r}; \quad z \geq 0. \quad (2.124)$$

where we have added the minimum slack z to the combination of stresses $\psi \mathbf{f}$ in all limit equations (hence the use of the column of ones expressed by $\mathbf{1}$). If we want to find the stress conditions furthest from failure, this means finding the values of $\mathbf{f}$ and z that maximize z , so the solution for $\mathbf{f}_c$ is the solution to the following linear programming problem, which can be solved with SIMPLEX:

$$\begin{aligned} & \max_{\mathbf{f}_c, z} z \\ & \psi \mathbf{f}_c + \mathbf{1} z \leq \mathbf{d} \mathbf{r} \\ & z > 0 \end{aligned} \tag{2.125}$$

2.2.2.2 Solution of limit load problems

According to all of the above, it is possible to establish as a verification problem, and using the static approach (sections 1.4.7.1 and 1.4.9), the linear programming problem 1.76 which, according to the static theorem, combines the equations of equilibrium and resistance in determining the maximum admissible load factor:

$$\begin{aligned} & \max_{\mathbf{f}, \Gamma} \Gamma \\ & \begin{bmatrix} \psi & \mathbf{0} \\ -\mathbf{B}^T & \mathbf{F}_1 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \Gamma \end{bmatrix} \leq \begin{bmatrix} \mathbf{d} \mathbf{r} \\ -\mathbf{F}_0 \end{bmatrix} \\ & \Gamma \geq 0 \end{aligned}$$

In this formulation, the initial load $\mathbf{F}_I$ is considered decomposed into two parts, $\mathbf{F}_I = \mathbf{F}_0 + \mathbf{F}_1$, one constant and one

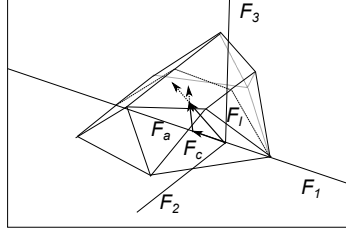


Figure 2.31: Load factors in masonry: the growth vectors to the limit of initial loads $\mathbf{F}_I$ are represented—for a space of three load parameters—depending on whether the traditional load factor is used on the total load, or the robust one.

variable, so that the load factor is only applied to the variable part $\mathbf{F}_1$ therefore the failure load is finally $\mathbf{F} = \mathbf{F}_0 + \Gamma \mathbf{F}_1$.

We thus have the following possibilities, partially illustrated in figure 2.31:

- *traditional load factor*
 - applied to all loads when $\mathbf{F}_0 = \mathbf{0}$
 - applied to live loads when $\mathbf{F}_0$ considers permanent loads—with their own safety factor—and $\mathbf{F}_1$ considers variable loads: in this case, the available coefficient for these is measured.
- *robust load factor* when $\mathbf{F}_0 = \mathbf{F}_c$ are the loads that equilibrate the optimal stress condition $\mathbf{f}_c$, determined through the solution to problem 2.125, and the subsequent equilib-

rium 2.120, and where therefore $\mathbf{F}_1 = \mathbf{F}_a$ expresses the part of the loads that move away from this optimal situation.

2.2.2.3 Duality in linear programming problems

At this point, it is worth considering the implications for plasticity problems of an important duality in linear programming problems. Such problems are those where a linear function of a set of variables to be optimized is maximized or minimized (a *cost* where the coefficients of the linear expression would be the *prices* associated with each variable) and where the variables are subject to linear constraints, expressible with a certain coefficient matrix and limit values for each constraint.

A classical formulation of duality establishes that if there exists a linear programming problem called *primal*, of minimization, there also exists an adjoint or *dual* problem, in another system of variables, of maximization where the linear constraints are expressed with the transpose of the matrix of the first problem, for another set of limit values, and with the particularity that the limit values of one problem are the coefficients (*prices*) of the linear expression that is maximized or minimized in the other, with the limit (*cost*) obtained in the solution being identical for both problems. Expressing it symbolically:

$$\begin{array}{ll} \min_{\mathbf{x}}(\mathbf{c}^T \cdot \mathbf{x}) & \max_{\mathbf{y}}(\mathbf{b}^T \cdot \mathbf{y}) \\ \mathbf{a} \cdot \mathbf{x} \geq \mathbf{b} & \iff \mathbf{a}^T \cdot \mathbf{y} \leq \mathbf{c} \\ \mathbf{x} \geq \mathbf{0} & \mathbf{y} \geq \mathbf{0} \end{array} \quad (2.126)$$

With $\min_{\mathbf{x}} \mathbf{c}^T \cdot \mathbf{x} = \max_{\mathbf{y}} \mathbf{b}^T \cdot \mathbf{y}$.

Problem dual to the limit load. If we compare the right-hand expression in 2.126 with the static plastic or limit load problem in the formulation of equation 1.65, we can obtain as its dual problem the following¹²,

$$\begin{aligned} \min_{\mathbf{v}_1, \mathbf{v}_2} & \left([(\mathbf{d}\mathbf{r})^T \quad \mathbf{0}^T] \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \right) \\ & \begin{bmatrix} \psi^T & -\mathbf{B} \\ \mathbf{0}^T & \mathbf{F}_I^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \\ & \mathbf{v}_1 \geq \mathbf{0} \end{aligned} \quad (2.127)$$

problem in which we have broken down the new set of variables $\mathbf{v}$ into two parts, $\mathbf{v}_1$ and $\mathbf{v}_2$, corresponding to the two components of the expressions, both for the cost and for the definition of the constraints, which are derived from the different groups of components in the equations of the original problem.

We observe that the first group of variables, $\mathbf{v}_1$, are—in the cost expression— multiplying coefficients of the comparison resistance capacity values for each of the resistance equations, while the second group, $\mathbf{v}_2$, are—in the constraint equations— multipliers for the initial load terms.

Kinematic interpretation of the dual problem. We can establish an immediate interpretation of the preceding problem by setting $\mathbf{v}_1 = \boldsymbol{\lambda}_p$ and $\mathbf{v}_2 = \mathbf{U}$, the parameters of the kinematic

12. some of the details of the transformation in the form that the restrictions take would require a more detailed or rigorous process that is not worth considering here

approach studied in section 1.4.7.2, equation 1.71. Indeed, if we observe the second part in the expression of the constraints, we would have $\mathbf{F}_I^T \mathbf{U} = 1$ which is nothing more than the normalization condition of the collapse velocity through a fixed measure for the energy loss of the loads, given that, in a collapse process with kinematics given by $\mathbf{U}$, the displacements are proportional at each step, but indeterminate.

If $\mathbf{U}$ are the displacements, in the first part of the constraints we have the expression $\psi^T \boldsymbol{\lambda}_p - \mathbf{B}\mathbf{U} = \mathbf{0}$ which is nothing more than the flow rule expression 1.48 combined with the compatibility equation 1.1 when $\boldsymbol{\lambda}_p$ are the plastic multipliers associated with each resistance constraint.

Finally, since the collapse load is $\mathbf{F} = \gamma \mathbf{F}_I$, it follows that the scalar product being minimized precisely determines the load factor:

$$\min \left([(\mathbf{d}\mathbf{r})^T \quad \mathbf{0}^T] \begin{bmatrix} \boldsymbol{\lambda}_p \\ \mathbf{U} \end{bmatrix} \right) = \min ((\mathbf{d}\mathbf{r})^T \boldsymbol{\lambda}_p) = \gamma,$$

since $(\mathbf{d}\mathbf{r})^T \boldsymbol{\lambda}_p \geq \mathbf{F}^T \mathbf{U} = \gamma \mathbf{F}_I^T \mathbf{U}$, according to equation 1.68.

In this way, we verify that the kinematic approach is nothing other than the dual of the static approach. It should be noted that this duality that we see repeatedly results from the properties of the scalar product, therefore from the choice of the associated static and kinematic variables and their product, the work.

With this vision, we can approach all the limit load problems already analyzed from the static perspective, and even tackle new problems.

In relation to the limit load problems addressed in section 1.4.9, their dual versions can be easily identified. For example, the dual kinematic version of the static problem in 1.76 would now be

$$\begin{aligned} \min_{\lambda_p, U} \left(\gamma = [(\mathbf{dr})^T \quad -\mathbf{F}_0^T] \begin{bmatrix} \lambda_p \\ U \end{bmatrix} \right) \\ \begin{bmatrix} \psi^T & -\mathbf{B} \\ \mathbf{0}^T & \mathbf{F}_1^T \end{bmatrix} \begin{bmatrix} \lambda_p \\ U \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \\ \lambda_p \geq \mathbf{0} \end{aligned} \quad (2.128)$$

since $(\mathbf{dr})^T \lambda_p \geq \mathbf{F}^T \mathbf{U} = (\mathbf{F}_0^T + \gamma \mathbf{F}_1^T) \mathbf{U}$ therefore, with $\mathbf{F}_1^T \mathbf{U} = 1$, it results in $\gamma = \min \left((\mathbf{dr})^T \lambda_p - \mathbf{F}_0^T \mathbf{U} \right)$. The equations corresponding to the constraints remain the same as in the previous problem.

2.2.2.4 Limit Thrusts

Up to this point, we have approached, from the perspective of the limit analysis, problems where the geometric conditions — and with them the support conditions — are prescribed, and what is sought is to determine the limit loads compatible with these conditions, or the corresponding load factors.

A recurring problem in masonry or stonework structures (in arches) is to bound the maximum and minimum admissible values for some of the forces in the supports: the limit thrust problem. In this case, since thrust becomes a relevant static variable, it is essential to consider in the model its associated

variable: the support displacement. Thus the end joint(s) are no longer fixed, but must be connected to a fictional “voussoir” of zero thickness, which allows representing in its nodal parameters —forces and displacements— that are the resultants and the corresponding movements for the collapses associated with their maximum or minimum values.

In these cases, moreover, the resultant —the thrust— has several components of which only one in particular is of interest: consider, for example, the problem of thrust from a Gothic vault on the flying buttress: neither its more or less eccentric position in the connecting voussoir nor its greater or lesser inclination is usually relevant: what is basically of interest is the admissible range in the horizontal thrusts, given the geometry and the other loads on the flying buttress.

We can formulate this type of problem without much difficulty by establishing that the initial loads $\mathbf{F}_I$ are composed of, in addition to certain fixed loads $\mathbf{F}_0$, two additional components $\mathbf{F}_1$ and $\mathbf{F}_\rho$, where $\mathbf{F}_1$ represents the limit loads we seek to determine, and $\mathbf{F}_\rho$ values associated with these limit loads, such that being the initial load $\mathbf{F}_I = \mathbf{F}_0 + \mathbf{F}_1 + \mathbf{F}_\rho$, the failure load is $\mathbf{F} = \mathbf{F}_0 + \Gamma \mathbf{F}_1 + \mathcal{P}^T \mathbf{F}_\rho$. In this last expression, the factor Γ is the unknown value that interests us to express as a limit (limit thrust), while the factor(s) $\mathcal{P}$ only interest us insofar as they are necessary to describe the solution, but are not determinant. In the example of the flying buttress for determining the maximum admissible horizontal thrust exerted by the vault, in a plane model, $\mathbf{F}_0$ would be the loads on all voussoirs except the fictional or extreme voussoir that connects with the vault,

$\mathbf{F}_1$ would only provide a unit value for the horizontal “load” on the extreme voussoir, such that Γ can be that maximum thrust sought, and $\mathbf{F}_\rho$ would contain columns to express each of the other unit values associated with that thrust, namely, for the “load” tangent to the face of the voussoir in contact with the vault and for the “moment” on that face, associated with the eccentricity of the thrust, etc...

Thus formulated, the problem from the static perspective would be to obtain the maximum possible Γ , along with the values of the coefficients $\mathcal{P}$ corresponding to each of the unit values of the forces associated with the thrust, while complying with the conditions of strength and equilibrium.

$$\begin{aligned} \max_{\mathbf{f}, \Gamma, \mathcal{P}} & \left(\Gamma \equiv [\mathbf{0}^T \quad 1 \quad \mathbf{0}^T] \begin{bmatrix} \mathbf{f} \\ \Gamma \\ \mathcal{P} \end{bmatrix} \right) \\ \begin{bmatrix} \psi & \mathbf{0} & \mathbf{0} \\ -\mathbf{H} & \mathbf{F}_1 & \mathbf{F}_\rho \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \Gamma \\ \mathcal{P} \end{bmatrix} & \leq \begin{bmatrix} d\mathbf{r} \\ -\mathbf{F}_0 \end{bmatrix} \\ \Gamma & \geq 0 \end{aligned} \quad (2.129)$$

where again the restrictions are nothing more than the equations of strength and equilibrium.

For the second problem, of determining the minimum thrust, it would be sufficient to include in $\mathbf{F}_0$ a thrust greater than the maximum, and in $\mathbf{F}_1$ a “negative unit thrust” capable of subtracting from that contained in $\mathbf{F}_0$, maximizing that subtraction with the same previous expression.

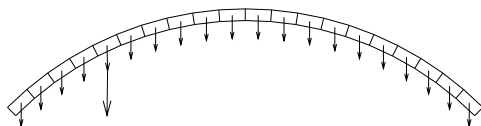


Figure 2.32: Arch with eccentric load: determining its limit value

If we now consider the dual kinematic problem of the previous one, we will have, according to what was seen in previous sections, the following formulation:

$$\begin{aligned}
 \min_{\lambda_p, U} & \left(\gamma(\lambda_p) = \begin{bmatrix} (dr)^T & -F_0^T \end{bmatrix} \begin{bmatrix} \lambda_p \\ U \end{bmatrix} \right) \\
 & \begin{bmatrix} \psi^T & -B \\ \mathbf{0}^T & F_1^T \\ \mathbf{0}^T & F_\rho^T \end{bmatrix} \begin{bmatrix} \lambda_p \\ U \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \\ \mathbf{0} \end{bmatrix} \\
 & \lambda_p \geq \mathbf{0}
 \end{aligned} \tag{2.130}$$

In this case, we are using the fact that $(dr^T \lambda) \geq F^T U$, with $U^T F = U^T (F_0 + \gamma F_1 + \rho^T F_\rho)$ such that $(dr)^T \lambda - F_0^T U \geq (\gamma F_1 + \rho^T F_\rho)^T U$, and where finally $F_1^T U = 1$ and $F_\rho^T U = \mathbf{0}$, resulting therefore that the minimization of the proposed product provides us with the sought value for γ .

2.2.3 Application Example: Arch of Rigid Blocks

Figure 2.32 presents an arch subjected to its own weight plus an eccentric load for which stability conditions are to be analyzed. The arch has been modeled with twenty voussoirs and twenty-one joints (the nineteen between the voussoirs plus the two connecting joints of the arch with the support) and, since global degrees of freedom are considered as the movements of the voussoirs, in case there is no sliding, it will necessarily result that the end joints of the arch do not move, they become immovable, even though they could “deform”. That is, this is a fixed support model. If the possibility of movement of one of the supports were to be considered, we would be considering an additional voussoir at this point, perhaps infinitely thin, in which the loads would precisely be the forces applied to the joint from the exterior through the support.

The arch is considered to be of circular layout, with a radius of 14.14 m and a span of 20 m, made with 20 voussoirs of 0.5 m depth, without tensile strength, and maximum compressive strength of 500 kN, subjected to a load of 10 kN per block plus a point load of 20 kN on the fifth voussoir.

It is desired to determine:

- the failure load factor for the total of the considered loads
- the failure load factor for amplification of only the point load
- the geometry of the loads that results worst for the arch stability
- the load factor for these worst loads and,
- the stress trajectories (corresponding to the funicular in graphic

calculation) and the collapse geometries of the three considered failure cases

2.2.3.1 Available Formulation

Summarizing the contributions from the theoretical sections, we have the following equations:

- EQUilibrium: $-\mathbf{H}\mathbf{f} + \mathbf{F} = \mathbf{0}$, with $\mathbf{F} = \mathbf{F}_0 + \Gamma\mathbf{F}_1$;
- COMpatibility: $-\mathbf{B}\mathbf{u} + \mathbf{U} = \mathbf{0}$, with $\mathbf{B} = \mathbf{H}^T$;
- resistance (MATERial): $\psi\mathbf{f} \leq d\mathbf{r}$.

where the following also hold —due to associated plasticity—the

- flow rule: $\psi^T\boldsymbol{\lambda} = \mathbf{u}$, and
- collapse condition: $(\psi\mathbf{f} - d\mathbf{r})^T\boldsymbol{\lambda} = \mathbf{0}$.

Finally, the static theorem provides the inequality between the energy loss of the loads and the plastic dissipation:

- $\mathbf{F}^T\mathbf{U} \leq \mathbf{f}^T\mathbf{u} = (d\mathbf{r})^T\boldsymbol{\lambda}$.

We can determine what would be the *central load* (optimal), that is, the one that provides the situation furthest possible from failure, associated with the stresses corresponding to the maximum possible *slack* through the following linear programming problem:

$$\begin{aligned} & \max_{z, \mathbf{f}_c} z \\ & z\mathbf{1} + \psi\mathbf{f}_c \leq d\mathbf{r} \\ & z \geq 0 \end{aligned}$$

and deducing said central load from these stresses through equilibrium, and thereby determining the “worst” load direction as the difference between the applied load and the central load — expressions in which we can also distinguish between permanent and variable loads, etc—

$$\begin{aligned} \mathbf{F}_c &= \mathbf{H} \mathbf{f}_c \\ \mathbf{F}_a &= \mathbf{F}_I - \mathbf{F}_c = \mathbf{F}_p + \mathbf{F}_v - \mathbf{F}_c \\ \mathbf{F} &= \Gamma \mathbf{F}_a + \mathbf{F}_c \end{aligned}$$

With all this, we have seen how solutions to the static problems can be approached (1.76) and its kinematic dual (2.128), and now it’s about materializing these solutions for the proposed problem.

For this, the different data matrices of the two problems must be constructed, although it is useful to consider the transposed format for the second problem¹³, since in that case the coefficient matrices for the constraints are identical in both problems.

The organization of data can be done efficiently in any programming language, but we will present here how a spreadsheet could be organized for the task, as it is very visual, and spreadsheet programs usually have the SIMPLEX solution pre-programmed.

The sheet, schematized in figure 2.33, allows obtaining the central load vector, as well as the solution to the static and kinematic problems.

Let’s see how it would be used.

13. in the way that form 1.72 is the transpose of 1.71.

We have on one hand the fixed data, namely, all columns of ones or zeros, the matrices ψ , $-\mathbf{H}$, $\mathbf{d}$, $\mathbf{r}$ (both allow fixing the column $\mathbf{r}_l = \mathbf{d}\mathbf{r}$), as well as positions to break down the possible load laws, permanent $\mathbf{F}_p$, variable $\mathbf{F}_v \dots$ or to place partial results such as central loads $\mathbf{F}_c$ and additional $\mathbf{F}_a$. These columns will allow breaking down the initial loads $\mathbf{F}_I$ of the problem between the constants $\mathbf{F}_0$ and those that will be altered by the load factor Γ , $\mathbf{F}_1$, or by the factors $\mathcal{P}$ that are linked to the loads we have called associated with the variable load, loads $\mathbf{F}_\rho$. The fixed loads $-\mathbf{F}_0$ are those inserted in the column that serves both as limit values and coefficients for calculating γ . The loads to be introduced in the data columns that will be affected by the load factors in any of the problems to be addressed are therefore $\mathbf{F}_1$ and $\mathbf{F}_\rho$, where applicable.

We have in turn the decision variables $z, \mathbf{f}_c$ for the central loads (the product $\begin{bmatrix} \mathbf{1} & \psi \end{bmatrix} \begin{bmatrix} z \\ \mathbf{f}_c \end{bmatrix}$ is stored in $\mathbf{r}_c$ for comparison with $\mathbf{r}_l$) and the solutions obtained with simplex will allow calculating and fixing $\mathbf{F}_c = \mathbf{H}\mathbf{f}_c$ and $\mathbf{F}_a = \mathbf{F}_I - \mathbf{F}_c$, where $\mathbf{F}_I$ is the initial load of the considered problem.

We have the decision variables $\mathbf{f}$, γ (and perhaps $\mathcal{P}$) for the static problem, in which the result $\mathbf{r}_s$ will be compared with the capacity objectives $\mathbf{r}_l$.

Finally, we have the decision variables and coefficients, λ , $\mathbf{U}$, for computing the load factor γ , variables that when multiplied by the large matrix containing the resistance coefficients ψ , equilibrium $-\mathbf{H}$ and loads $\mathbf{F}_1, \mathbf{F}_\rho$, allow calculating a row of results to compare with the row of objectives of zeros and one at the bottom of the sheet.

The determination of the decision variable values that solve any of the problems to be posed is performed with SIMPLEX with a process similar to that described in the example from a previous chapter, [1.4.13.3](#), establishing the target cell, the search direction (maximum or minimum), the decision variables, and the relevant constraints.

2.2.3.2 Model and Solutions

A detailed development of the example and its possible solutions is available in Cervera [2010a](#), appendix C. A complete model with a more precise representation of the material limits can be found in the spreadsheet attached to this document (ArcoPlastico.ods).



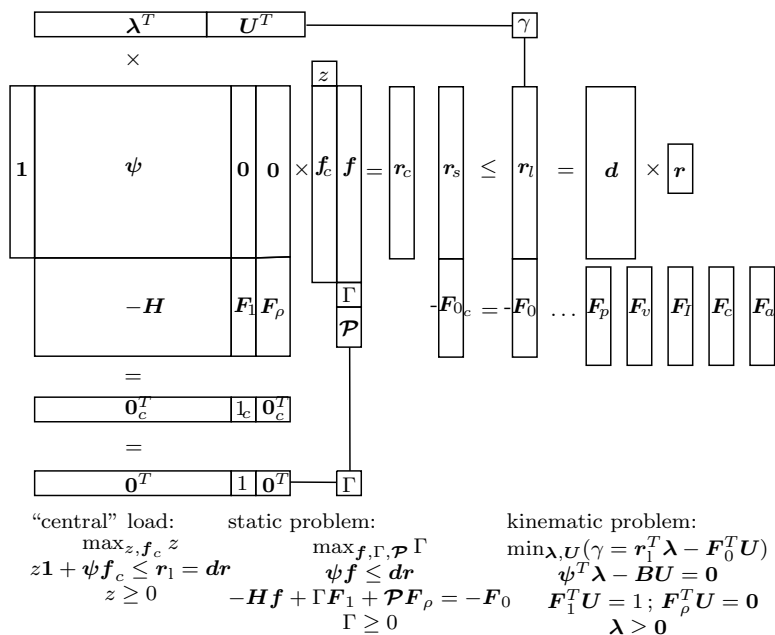


Figure 2.33: Spreadsheet scheme for the plastic solution. A single sheet for both static and kinematic solutions. The positions for the data are indicated (columns of ones or zeros, matrices ψ , $-\mathbf{H}$, $\mathbf{d}$, $\mathbf{r}$, permanent load laws $\mathbf{F}_p$ or variable, $\mathbf{F}_v \dots$ to construct $-\mathbf{F}_0$ and $\mathbf{F}_1$), for decision variables (z , $\mathbf{f}$, γ , $\mathcal{P}$ or λ , U), for results ($\mathbf{r}_c = [\mathbf{1} \ \psi] \times [z \ \mathbf{f}]$, $\mathbf{r}_l = \mathbf{d}\mathbf{r}$, $\mathbf{r}_s, \dots$) including the products from which the load factors Γ and γ result, etc; separating the values of the operations of the left and right parts of the inequalities to allow their comparison.

Chapter 3

Elastic Stability

This chapter presents and develops the concepts and methods required to address stability problems from an elastic or elastoplastic perspective. This analysis absolutely requires considering the equilibrium conditions in the deformed state of the piece or structure in question, and even in deformation situations perturbed from the possible equilibrium.

As shown in figure 3.1, it is only possible to distinguish between stable, neutral, or unstable equilibrium conditions of a system when we move away from the pure equilibrium situation, when we “perturb” it. If the potential energy of the system increases during the perturbation, the system will try to return to the initial equilibrium position and the system will be stable. If the total potential energy remains constant, the system will lose its initial position while remaining in equilibrium, which in

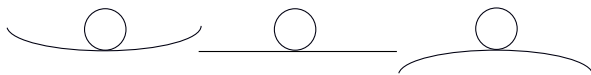


Figure 3.1: Types of equilibrium: stable, neutral (or “critical”) and unstable.

this case is neutral. And if the total potential energy decreases, the system will definitively lose balance, moving away from the initial position. The equilibrium is unstable.

The analysis can be done both by analyzing the potential energy and by reviewing the equilibrium conditions in the perturbed positions: since forces are the derivatives of energy with respect to displacement, the increasing or decreasing direction of the resultants in relation to the direction of perturbation allows characterizing the expected behavior. In this evaluation, the *critical situation* or neutral condition allows separating stable conditions from unstable ones.

3.1 Basic Concepts in Single Degree of Freedom Problems

In this section, we analyze systems whose perturbation derives from displacements governed by changes in a single degree of freedom. The simplicity of this model allows the introduction of the main concepts, which are generalized in subsequent sections.

3.1.1 Lateral Stability of Articulated Frames

We consider the case of a frame with columns and beams connected by hinged joints and subjected to vertical loads on the columns. It is considered that the initial geometry is perfect and, therefore, having sufficient strength in the columns, it can be considered in equilibrium. But if this geometry is perturbed by imposing a lateral displacement, the forces are no longer aligned with the columns and, therefore, the assembly becomes unbalanced: each of the vertical loads or reactions that compressed the columns can now be considered as composed of the sum of a horizontal component plus an oblique component aligned with the columns: the latter remains in equilibrium with the column compression, but now the horizontal components are no longer balanced, and require some mechanism to oppose them, that is, some rigid element that, when deformed, opposes contrary forces that counteract them.

3.1.1.1 Lateral Stability

In figure 3.2, we consider each column i subjected to a load P_i and perturb the initial equilibrium by considering a drift θ (or a lateral displacement v such that $v = \theta h$). In this figure, the stabilizing mechanism is a rigid horizontal bar, forced to extend or shorten when that lateral displacement is imposed. The contrast condition that allows us to identify the type of equilibrium is the relationship between the sum of the destabilizing forces, H , and the stabilizing capacity of the bar, R_h : the critical (neutral) condition in the drift is in the equality between the lateral

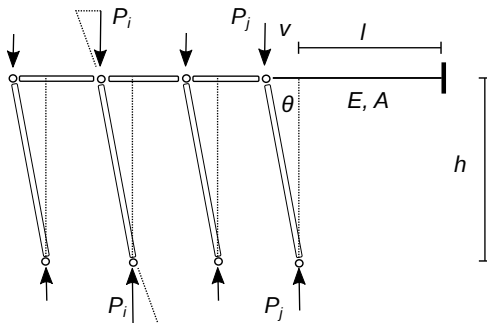


Figure 3.2: Frame stability: destabilizing and stabilizing effects.

load and the response. This situation will occur if all the loads to be stabilized reach a critical value P_k , which we could measure with a single load coefficient γ_k multiplied by the actually applied loads $\sum P_i$, such that their lateral effect equals the bar's response. That is:

$$\begin{aligned}
 H &= R_h \\
 \theta \sum (\gamma_k P_i) &= \theta P_k = \varepsilon EA = \frac{EA}{l} v \\
 \theta P_k &= EA \frac{h}{l} \theta = K_\theta \theta
 \end{aligned}$$

Ultimately, we arrive at the “critical” load, associated with the set of columns stabilized by the lateral bar, depends on the

stiffness properties of said lateral bar:

$$P_k = K_\theta = EA \frac{h}{l} \quad (3.1)$$

Thus, we have that the *critical load factor*, that is, *the factor that brings the loads to the critical limit* is

$$\gamma_k = \frac{P_k}{\sum P_i} = \frac{EA \frac{h}{l}}{\sum P_i} = \frac{K_\theta}{\sum P_i} \quad (3.2)$$

The critical load factor is therefore the quotient between the lateral stiffness against drift and the load that compresses the elements that drift, understanding drift as the angle of inclination produced.

For a load factor less than the critical, the structure will be stable: the drift will not occur because its initiation will set in the motion a response superior to the destabilizing load, returning the frame to its initial geometry. And conversely, for a higher load factor, the response will be less than the destabilizing effect and, therefore, being unstable equilibrium, the drift will increase until reaching the resistance limit of the bar, producing collapse.

3.1.1.2 Diagonal Bracing

In the case of the configuration in figure 3.3, the approach is the same although the stabilizing response differs: now the response is an oblique force, whose horizontal component provides the necessary stabilizing component, while the vertical component

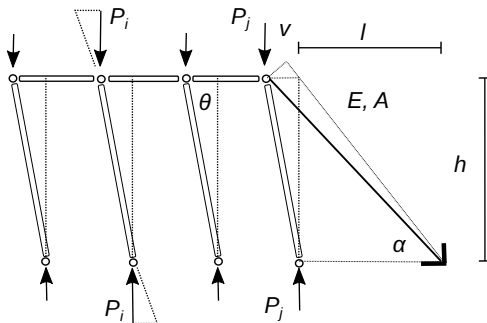


Figure 3.3: Frame bracing: destabilizing and stabilizing effects.

will imply increases in the total vertical loads. The equilibrium between the destabilizing lateral force and the diagonal response is now

$$H_a = \theta \left(\sum (\gamma_k P_i) + N_a \sin \alpha \right) = R_{h,a} = N_a \cos \alpha \quad (3.3)$$

where N_a is the axial force of the diagonal. This force depends on the drift and the geometry of the bracing triangle:

$$N_a = \varepsilon EA = \frac{v \cos \alpha}{h / \sin \alpha} EA$$

$$N_a = \theta \frac{\sin(2\alpha)}{2} EA$$

The critical condition is characterized by the condition:

$$\theta \sum (\gamma_k P_i) + \theta^2 \frac{\sin(2\alpha)}{2} EA \sin \alpha = \theta \frac{\sin(2\alpha)}{2} EA \cos \alpha$$

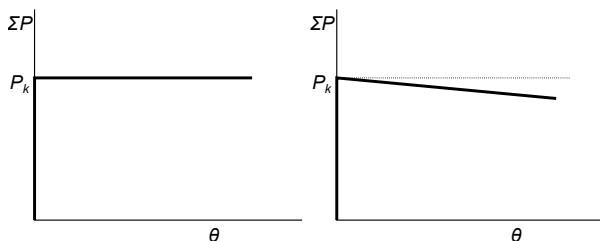


Figure 3.4: Types of postcritical behavior: both figures show differences in the evolution of the movement after the beginning of the collapse: with maintenance (as in the figure 3.2) or loss of lateral stiffness (figure 3.3) as the collapse progresses

This condition allows determining the critical load and the critical load factor of this configuration:

$$\begin{aligned} \gamma_k \sum P_i &= P_k = \frac{\sin(2\alpha) \cos \alpha}{2} EA \\ \gamma_k &= \frac{EA \frac{\sin(2\alpha) \cos \alpha}{2}}{\sum P_i} \end{aligned} \quad (3.4)$$

Again, we can see the critical load as a measure of the lateral stiffness provided by the stabilizing device: resistance is not required, the material's tensional response capacity is not present at all, because if there is sufficient stiffness, the perturbation does not initiate, since the response is greater than the lateral action and, therefore, the perturbation cannot progress.

Besides characterizing the equilibrium as stable, critical, or

unstable, depending on whether the sum of loads reaches or exceeds the critical condition, we can encounter different situations in the post-onset conditions in critical cases, in which the lateral stiffness, or stabilizing capacity might have been equal at the beginning for two different devices, but vary in the post-critical process, as reflected in figure 3.4, since in the case of bracing we could rewrite the expression 3.3 in the form $H = \theta \sum (\gamma_k P_i) = N_a (\cos \alpha - \theta \sin \alpha)$, with loss of capacity against the buckling effect as it advances.

3.1.2 Euler Buckling

The most visible local instability situation corresponds to the buckling of straight bars under compression, which often cause the bar failure to occur, not by crushing of any of its sections, but by uncontrolled lateral deformation. It has been historically one of the most studied problems both theoretically and experimentally. Euler's formulation is the basis of the current treatment of this problem. In what follows, we present the two ways of approaching it, namely, from the force equilibrium in the deformed situation, or from the consideration of the minimum potential energy condition.

3.1.2.1 Classical Second-Order Equilibrium Model

In the critical situation, the equilibrium condition is assumed for a certain lateral deformation of the bar given by the law $v(x)$ that expresses the displacement for each point with coordinate x of the original straight bar.

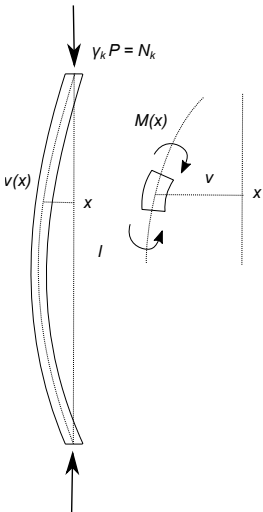


Figure 3.5: Euler Buckling: moments equilibrium in the deformed curve

The equilibrium between the bending moment derived from the compression eccentricity P brought up to its critical value $N_k = P_k = \gamma_k P$, and the bar's response derived from curvature, given its bending stiffness, can be expressed with the differential

equation

$$\begin{aligned}\gamma_k P v(x) &= N_k v = M(x) = EI c(x) = EI \frac{d^2 v}{dx^2} \\ N_k v &= EI v''\end{aligned}\tag{3.5}$$

The solution to the deformation v is the sinusoidal law that follows, where n indicates the number of half-waves between the two ends

$$v = v_{\max} \sin \frac{n\pi x}{l}$$

It is a law that, applied to the previous equation 3.5, allows expressing the response corresponding to the indifferent equilibrium and deducing from it the critical load value, which depends on the *mode* or number of half-waves of the deformation, and among which we select the smallest corresponding to the value of 1 for n :

$$\begin{aligned}N_k &= \frac{n^2 \pi^2 EI}{l^2} \\ N_{k,1} &= \frac{\pi^2 EI}{l^2} = \frac{\pi^2 EA i^2}{l^2} = \frac{\pi^2 EA}{\lambda^2}; \quad \lambda = \frac{l}{i}\end{aligned}\tag{3.6}$$

This is a critical condition, or indifferent equilibrium. For loads less than $N_{k,1}$, the response given by the section against curvature is greater than the moment of the load eccentricity and the piece returns to its straight geometry. For loads greater, the equilibrium becomes unstable and the piece deforms indefinitely until the structure's geometry is destroyed or until the moment breaks the piece in its central section.

Although the above is the classical formulation of the *Euler critical load*, it is easy to see that the critical condition is not so much the load as the stress, or better yet, the strain. Indeed, if we consider the strain corresponding to a centered compression in the section, it turns out that the critical condition is a strain that does not depend on the material considered, but only on the slenderness, on the piece's geometry

$$\sigma_{k,1} = \frac{\pi^2 E}{\lambda^2} \quad (3.7)$$

$$\varepsilon_{k,1} = \frac{\pi^2}{\lambda^2} \quad (3.8)$$

The above expression comes to point out the point at which it becomes easier for the piece to deform by bending than by shortening: for small axial deformations the piece shortens, but once the limit $\varepsilon_{k,1}$ is reached the piece adopts a different equilibrium configuration, *bifurcates* to a different deformed geometry, in bending, dependent only on the mechanical slenderness of the piece.

Considering the above, it is useful to identify a reference slenderness λ_0 for which the critical strain is identical to the strain at the yield limit: for slenderness less than this, the piece would fail by crushing, while for larger slenderness, the piece will fail by buckling.

$$\varepsilon_{y0} = \frac{\pi^2}{\lambda_0^2} \implies N_{k0} = N_u = A f_y \quad (3.9)$$

Such reference slenderness is used in codes to establish the *relative slenderness*, ratio between mechanical slenderness and ref-

erence slenderness, which is used in situations such as those in figure 3.7 for determining reduction coefficients of compressive strength due to buckling effect.

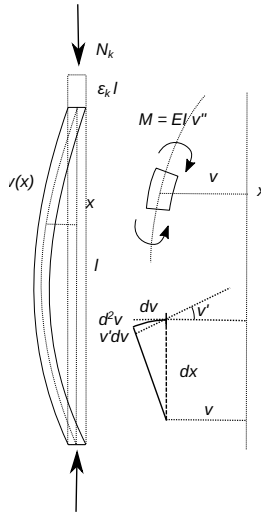


Figure 3.6: Euler Buckling: minimum in the total potential energy

3.1.2.2 Minimum Potential Energy

We can also approach the buckling problem from the potential energy perspective, for which we will seek the configuration that

makes it minimal. In this case, we must consider both the potential energy of the load, considering its possible descent due to the bar's deformation, and the potential deformation energy of the piece, integrating the deformation of all its sections. In figure 3.6 we see how the descent of a straight segment can be approximated by its rotation to the inclined position, so that we can measure the potential energy loss of the load in the resulting buckling curve along with the accumulated deformation energy in said curve:

$$W_P = - \int_0^l N_k \frac{(v')^2}{2} dx + \frac{1}{2} \int_0^l EI(v'')^2 dx = \mathcal{W}(v(x)) \quad (3.10)$$

The resulting expression is a functional $\mathcal{W}$ dependent on the deformation $v(x)$. To find its minimum, we consider variations of that function $v(x)$ in the form $v(x) + \xi u(x)$ with ξ a coefficient that we can make tend to 0 and $u(x)$ any arbitrary function, always satisfying the boundary conditions, which in this case implies that it is zero at the ends.

The minimum condition is that the functional gradient $\mathcal{W}$ be zero, or in other words, it is about finding the function v for which any infinitesimal variations¹ ξu suppose that $\mathcal{W}(v + \xi u) \geq \mathcal{W}(v)$.

Variational Calculus. Problems of the type above are the object of variational calculus.

Some basic concepts of this type of analysis are presented below.

1. ξ is used as a scale factor tending to zero, and u as a function that reflects how to alter v .

Functional. It is an operation that transforms functions $f(x)$ into scalars $\mathcal{F}(f)$ through integrals.

In a given domain X , the *scalar product* between functions: $\langle u(x), f(x) \rangle \equiv \int_X u(x)f(x)dx$, is a functional. Energy or work is a functional.

Gradients in Functionals, and Gâteaux Gradient. Just as the values of a variable in a function can differ between nearby points and we measure these differences (and differences per unit distance or gradient) using derivatives, we can consider variations or gradients in functionals depending on possible variations in the functions on which they are based.

There are many ways to express such gradients, or trends, in the functional depending on how the functions they depend on are altered. A classic form is the Gâteaux gradient, which can be expressed as follows, where $\mathcal{G}_u$ expresses the functional gradient $\mathcal{F}$ in the alterations of the function $f(x)$ that follow the direction of the arbitrary function $u(x)$

$$\mathcal{G}_u (\mathcal{F}(f(x))) = \lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} (\mathcal{F}(f(x) + \xi u(x))) \quad (3.11)$$

Condition of maximum or minimum in functionals: null gradient. The condition for maximum or minimum of a functional is that the gradient be null in all possible “directions” in which the functions it depends on can be conceived to be altered:

$$\mathcal{G}_u (\mathcal{F}(f(x))) = 0, \forall u. \quad (3.12)$$

This is a condition analogous to the maximum or minimum of a function: that its derivative be null in all directions of the variables on which said function depends.

Establishing the condition of minimum total potential energy. We have the functional for total potential energy, equation 3.10, and we equally have the condition for it to be a minimum, simply by combining 3.11 and 3.12, so that

$$\mathcal{G}_u(\mathcal{W}(v(x))) = \lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} (\mathcal{W}(v(x) + \xi u(x))) = 0, \quad \forall u.$$

This expression demands

$$\lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} \left(- \int_0^l N_k \frac{(v' + \xi u')^2}{2} dx + \frac{1}{2} \int_0^l EI(v'' + \xi u'')^2 dx \right) = 0$$

so that, developing the squares, deriving with respect to ξ and taking $\xi \rightarrow 0$ results in

$$\begin{aligned} - \int_0^l N_k u' v' dx + \int_0^l EI u'' v'' dx &= 0, \\ \int_0^l (-N_k v' - EI v''') u' dx, \quad \forall u \end{aligned}$$

from which results, finally, the necessary condition

$$N_k v' + EI v''' = 0 \tag{3.13}$$

equation which, as can be seen, refers back to 3.5.

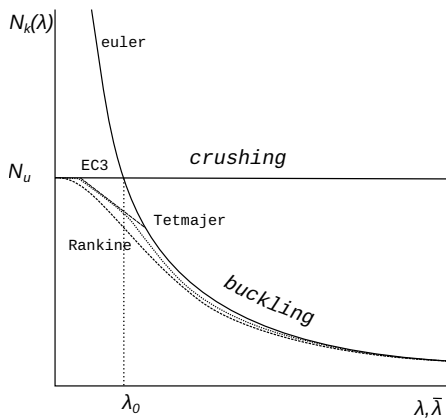


Figure 3.7: Resistance curves of compressed members. Possible load limits are indicated as a function of mechanical slenderness λ , according to crushing conditions, Euler buckling, as well as curves considering combinations of both effects: Rankine, Tetmajer, or Eurocode 3 cases. The reference slenderness $\lambda_0 = \pi/\sqrt{\varepsilon_y}$ is marked, see equation 3.9, which allows defining the relative slenderness $\bar{\lambda} = \lambda/\lambda_0$.

3.1.3 Limit states in compressed members

Based on what we have seen so far, for a compressed member we can consider two independent limits:

- ultimate crushing load: N_u
- Euler critical load: N_k

We could trace these limits as a function of slenderness in figures like 3.7, in order to select as available capacity the lowest value among the considered limits.

However, when contrasting the expressions with experiments, there is a limited adjustment under low slenderness conditions, close to λ_0 , and the basic reason is that the formulation has not yet taken into account the possibility of local plastification, nor the existence of uncertainties associated with imperfections, internal stress states derived from rolling processes, and so on.

There are many ways to approach the problem: historically we have the Tetmajer line which is merely a truncation in that problematic region, or the strategy provided by Rankine, based on classic expressions of combining phenomena, whether these are in “series” or “parallel”, with expressions of the type

$$f_c = \sqrt[n]{f_1^n + f_2^n}; \quad \frac{1}{f_c} = \sqrt[n]{\left(\frac{1}{f_a}\right)^n + \left(\frac{1}{f_2}\right)^n}. \quad (3.14)$$

In this case, these are parallel phenomena for which the simplest expression, adopted by Rankine, is

$$N_{\text{lim}} = \frac{1}{\frac{1}{N_u} + \frac{1}{N_k}} \quad (3.15)$$

The modern approach, reflected in codes such as Eurocode 3, is based on a development of the problem that involves considering, as opposed to the perfect geometries considered until now, the defects or imperfections in the compressed geometries.

3.2 Imperfections, *deflection amplification* and buckling factor

Up to this point, it has been considered that the geometry under compression is initially as exact as its theoretical model: perfectly vertical in braced frames, or perfectly straight in members. However, the existence of production or assembly defects prevents this from being possible in real situations, given the uncertainties in initial stress states, geometries or connection means, as well as in the actual on-site implementation processes. So the geometry of a problem like the one considered in section 3.1.1.1 is better represented, rather than with figure 3.2, with a figure that includes these initial drifts or misalignments with the vertical due to such imperfections, as in figure 3.8.

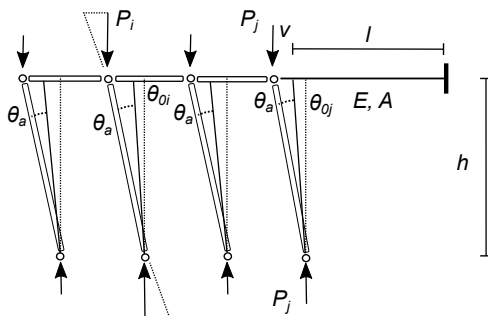


Figure 3.8: Stability of frame with initial imperfections: destabilizing and stabilizing effects.

3.2.1 Imperfection amplification

The model with imperfections assumes an initial geometry where each load P_i will rest on a column that suffers an initial drift θ_{oi} . The bar intended to stabilize the ensemble is mounted, without tension, in a position and initial length consistent with this imperfect geometry. Under these conditions, the application of loads would be producing a lateral thrust from the start due to its misalignment with the axial response of the columns, but since there is no initial deformation, there would be no response until a displacement occurs that extends the stabilizing bar until its deformation provides the response capable of stabilizing these lateral thrusts. Under these conditions, an increase in misalignment occurs, an additional drift θ_a , which affects all columns in the ensemble and which, in the case of sufficient stability, finally ensures the equality between the destabilizing action H and the stabilizing response R_H :

$$H = \sum \theta_i P_i = \sum (\theta_a + \theta_{oi}) P_i = \left(\theta_a + \frac{\sum \theta_{oi} P_i}{\sum P_i} \right) \sum P_i$$

$$R_H = EA\varepsilon = EA \frac{\theta_a h}{l} = P_k \theta_a \quad (3.16)$$

That is, the destabilizing horizontal force will depend on the total loads to be stabilized and the final average weighted drift, which includes the initial average misalignment, plus the additional one, common to all pieces, that occurs due to the amplification required to reach equilibrium. While the stabilizing action only has this additional deformation available to provide the necessary response. We see that the response is the product

of a stiffness by a deformation, precisely the stiffness that determines the critical load seen in section 3.1.1.1, equation 3.1. The equilibrium between action and response leads us to compare the effect of the total of loads on the total misalignment against the response given by stiffness in the added drift:

$$H = R_H \implies \bar{\theta} \sum P_i = (\theta_a + \bar{\theta}_o) \sum P_i = \theta_a P_k \quad (3.17)$$

Solving for the additional drift θ_a in the previous equation, we obtain the (average) final drift

$$\begin{aligned} \theta_a &= \bar{\theta}_0 \left(\frac{\sum P_i}{P_k - \sum P_i} \right) \\ \bar{\theta} = \bar{\theta}_0 + \theta_a &= \bar{\theta}_0 \left(\frac{1}{1 - \frac{\sum P_i}{P_k}} \right) = \bar{\theta}_0 \left(\frac{1}{1 - \frac{1}{\gamma_k}} \right); \quad \gamma_k = \frac{P_k}{\sum P_i} \end{aligned} \quad (3.18)$$

We see in this last expression that the final drift can be interpreted as an amplification of the initial weighted average drift, an amplification whose factor, always greater than 1, depends on the critical load factor seen in equation 3.2.

In this way, we see that the perfect situation, while not being suitable for describing the more realistic one, with imperfections, nevertheless provides a crucial parameter for describing the latter, since it is the relationship between the destabilizing load and the critical load that governs the drift amplification process: if both are equal, the drift becomes infinite.

Although not infinite, it may result that the amplified equilibrated drift requires deformations that, assuming excessive

stresses, are above what the structure can support. In the case of figure 3.8, this involves the requirement to add the verification of the bar's longitudinal deformation: is $\varepsilon = \theta_a h/l \leq \varepsilon_{yd}$?

Thus, the stability check will additionally require checking that the structure is capable of resisting the efforts in that final deformed situation. That is, a *second-order* analysis (of equilibrium and resistance in the deformed configuration) will generally be required, and the equilibrium of the original theoretical situation is not sufficient. If the factor $1/\gamma_k$ exceeds a small value (it is customary to consider 0.1 (or 10%) as a criterion), the regulations inexcusably require carrying out this type of analysis.

3.2.1.1 Lateral Resistance and P, Δ Effect

The second-order analysis of the lateral stability of frames requires checking the equilibrium situation against lateral forces considering the effects of deformation and lateral displacements. If in any of the problems in previous figures 3.2 or 3.8 we consider the existence of a horizontal action due to thrust, wind, or any other cause, the horizontal effect of the loads, the lateral action, in the original situation will be $H_{E0} = H + \bar{\theta}_0 \sum P$ where $\bar{\theta}_0$ is the weighted average collapse due to the imperfections.

However, the resistance check against the bar's resistive capacity, $H_R = A f_{yd}$, cannot be done considering that state, it must be done once the final drift situation is known: $H_E = H + \theta \sum P$ where θ becomes the definitive drift ratio. The typical non-linear process would involve iteratively calculating the lateral drift for the load H plus the effect of the previous step's

drift, until convergence is reached. We have seen before a possibility of shortcutting this iteration through equation 3.18 although now we must take into account that the initial drift to be amplified must include the addition from the lateral force H effect, that is $\bar{\theta}_0 = \theta_H + \bar{\theta}_{0i}$ if we call θ_{0i} the initial drift ratio due to imperfections. That is, the check will require

- determine the drift θ_H caused by the horizontal force H (which allows evaluating the lateral stiffness H/θ_H that, in turn, allows determining the critical load P_k since this is precisely that lateral stiffness, being indifferent to displacement in the expression $\theta P_k = \theta(H/\theta_H) = \theta\gamma_k \sum P_i$;
- add the average drift due to construction defects (or intentional inclination of the columns, if applicable);
- determine the final drift by the amplification of the original: $\theta = (\theta_H + \bar{\theta}_{0i})/(1 - 1/\gamma_k)$, with $\gamma_k = P_k/\sum P_i$, $\gamma_k = (H/\theta_H)/\sum P_i$;
- Check the resistive capacity for the total horizontal force effect of the loads and the associated displacement: $H_E \leq H_R$ or, in other words, check that $H + \theta \sum P \leq Af_{yd}$.

A method sometimes used in literature for this last check uses the stabilizing or destabilizing effect of each component in the overturning moment as the comparison element for the establishment of the stability condition. In that case, the verification equations change to the format $M_E = Hh \leq M_R$ (the effect moment of the displacement must be less than the moment given by the resistive capacity of the lateral stabilizing system, which, in the case considered here, takes the form $M_R = H_R h = hAf_{yd}$).

This will make that, the necessary strength verification for the stability in that problem:

$$H + \theta \sum P \leq A f_{yd},$$

can be rewritten in another form:

$$Hh \leq A f_{yd} h - \theta h \sum P = A f_{yd} h - \left(\sum P \right) \Delta \quad (3.19)$$

in which the verification would suggest a loss of capacity (loss in strength but also in stiffness) due to the displacement effect $P \Delta$. But this is more simply seen as the effect added to the horizontal action by the drift ratio in the final equilibrium situation.

The same can be done for the stiffness measure, which can be obtained as the ratio between the response to the load and the displacement, also expressible as the response being equal to the stiffness times the displacement. The determination would be transformed from the previous expression to the following form, for a given displacement, whether considering the displacement limit $\Delta_{\lim} = \epsilon_{yd} l$ or any other displacement Δ :

$$\begin{aligned} H_{\lim} h &= \left(\frac{A f_{yd} h}{\epsilon_{yd} l} - \sum P \right) \Delta_{\lim} = \left(\frac{EA h}{l} - \sum P \right) \Delta_{\lim}; \\ Hh &= \left(\frac{EA h}{l} - \sum P \right) \Delta \end{aligned} \quad (3.20)$$

This expression can be interpreted as if the load on the columns would reduce the stiffness that, associated with the lateral displacement Δ , acts in opposition to the destabilizing moment Hh exerted by the horizontal load H .

3.2.2 Buckling Factor.

We now return to the compressed straight bar, for which we analyze a possible initial imperfect, non-straight configuration. This analysis, whose first versions date back to Young (see Cervera Bravo et al. 2013), is at the base of the contemporary methods.

3.2.2.1 Initial and Equilibrium Geometries

An initial curved geometry $v_0(x)$, undeformed, with maximum eccentricity e_0 , is approximately assimilated to a sinusoid²:

$$v_0(x) = v_{0\max} \sin \frac{\pi x}{l} = e_0 \sin \frac{\pi x}{l}. \quad (3.21)$$

The final deformed curve is represented by the function $v(x)$, such that the deformation is the change from v_0 to v . The moment law, due to the applied load eccentricity N , is equated to the response by curvature change:

$$M = Nv = EI(v - v_0)''$$

from where, by derivation and considering the maximum eccentricity position, it is deduced

$$\begin{aligned} Ne &= \frac{\pi^2 EI}{l^2} (e - e_0) = N_k (e - e_0) \\ e &= e_0 \frac{1}{1 - \frac{N}{N_k}} = e_0 \frac{1}{1 - \frac{\varepsilon}{\varepsilon_k}} = e_0 \frac{1}{1 - \frac{1}{\gamma_k}} \end{aligned} \quad (3.22)$$

2. any other curve shape could be described by its Fourier series development; in such a case the first term of the development would be the chosen sinusoid and, in the deformation amplification analyzed subsequently, it is that first term that is basically responsible for the final geometry.

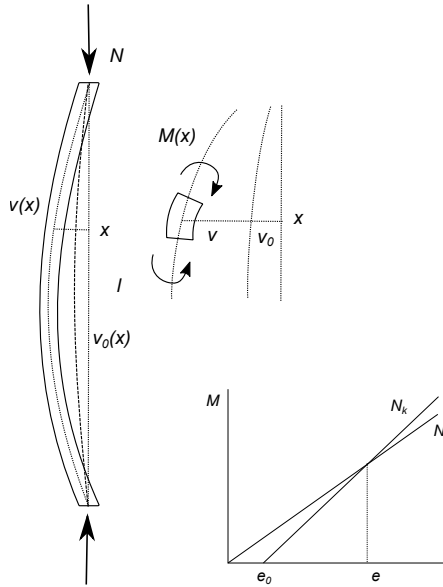


Figure 3.9: Deflection amplification under compression. The initial deformed v_0 with maximum eccentricity e_0 increases to the equilibrium deformation v . In this one, the moment effect of the load, $M = Ne$, is balanced with the piece's response, given its stiffness and the displacement due to the deformation, $N_k(e - e_0)$.

So the expression turns out to be similar to 3.18: the initial defect is amplified by reason of a deformation governed by the

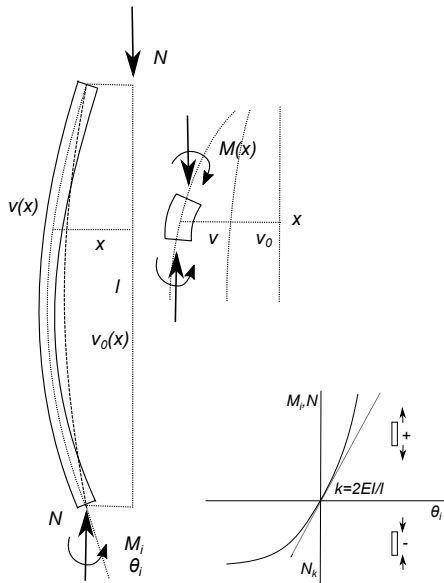


Figure 3.10: Deflection amplification under compression and stiffness change. Influence of the compression on the moment to rotation relationship at the column ends.

critical load factor, the ratio between the critical load and the applied load.

3.2.2.2 Stiffness Loss in a Compressed Support

We now consider the eccentric compression case, which therefore involves moments at the ends. We saw in equation 2.47 that the end stiffness in flexion, without axial effort, and for a symmetric deformation case, with moments at both ends was:

$$k_i = \frac{M_i}{\theta_i} = 2 \frac{EI}{l} \quad (3.23)$$

If we now consider a case like that in the figure 3.10, where there is also axial effort it will result that the stiffness varies: the relationship between M_i and θ_i is altered. We have seen that it reduces with the compression until becoming zero for the critical load, which means zero slope for the M, θ curve when N_k is reached. But analogously, said stiffness increases with traction, as it is easy to understand by seeing that the normal's eccentricity opposes the curvature —collaborates to reduce it— so the moment–rotation or moment–curvature law is not constant when varying N , as shown in said figure. We will study this effect later in detail in section 3.5.2.

3.2.2.3 Buckling Reduction Factor

We return to Figure 3.9. We now consider that it is feasible to limit the initial imperfection through manufacturing and installation controls associated with the current construction practices and regulations. We define this limitation as a fraction of the member length

$$e_0 \leq tl \quad (3.24)$$

where t is a tolerance term, typically around the value $1/350$.

Even when the compression is centered, due to the buckling flexural effect, we now consider the strength verification in the elastic range of the combined axial force and bending moment for the resulting eccentricity, e . The verification stress is:

$$\sigma_c = \frac{N}{A} + \frac{M}{W} = \frac{N}{A} \left(1 + e \frac{A}{W} \right) = \frac{N}{A} \left(1 + \frac{ev}{i^2} \right) \quad (3.25)$$

where we have used for W its value $I/v = I/(h/2) = Ai^2/v$, in which v is the distance to the center of gravity of the section from the farthest fiber, half the height in symmetric sections.

If we now deduce e from e_0 with 3.22, using as a reference for the compression state the average stress and strain of the member $\bar{\sigma} = N/A$, $\bar{\varepsilon} = N/EA$, we have the verification

$$\sigma_c = \bar{\sigma} \left(1 + e_0 \frac{1}{1 - \bar{\varepsilon}/\varepsilon_k} \frac{v}{i^2} \right) \leq \bar{\sigma} \left(1 + \frac{1}{1 - \bar{\varepsilon}/\varepsilon_k} \frac{tlv}{i^2} \right) \leq f_y$$

where ε_k is the Euler critical strain, equation 3.8.

If we now consider the relationship between the maximum possible average stress or strain in the member and its yield limit

$$\bar{\sigma} = f_y \chi = f_y \frac{1}{\omega}, \quad \bar{\varepsilon} = \varepsilon_y \chi = \frac{\varepsilon_y}{\omega}$$

having

$$\frac{f_y}{\omega} \left(1 + \frac{1}{1 - \varepsilon_y/(\omega\varepsilon_k)} \frac{tlv}{i^2} \right) \leq f_y$$

and if in addition to 3.8 we use the reference slenderness λ_0 from 3.9 we will have an expression from which a second-degree

equation in ω is derived, whose solution is

$$\omega = \frac{1}{\chi} \leq \phi + \sqrt{\phi^2 - \frac{\lambda^2}{\lambda_0^2}}, \quad \lambda_0 = \sqrt{\frac{\pi^2}{\varepsilon_y}}, \quad \phi = \frac{1}{2} \left(1 + \frac{tv\lambda}{i} + \frac{\lambda^2}{\lambda_0^2} \right) \quad (3.26)$$

In this expression, the term $\frac{tv\lambda}{i}$ can be expressed in relation to the slenderness λ_0 , using what is called the *imperfection factor*: $\hat{\alpha} = \frac{tv\lambda_0}{i}$, so that $\frac{tv\lambda}{i} = \hat{\alpha} \frac{\lambda}{\lambda_0}$. Thus the term ϕ of the previous equation takes a form that allows, for each material class (described by ε_y or by λ_0) and for each cross-sections family (the ratio v/i and the accepted tolerances, or the experimentally measured in that family, which can be summarized in the imperfection factor $\hat{\alpha}$) to determine the buckling reduction coefficient or buckling factor. In the following expressions are presented this form and the corrected form adopted by Eurocode 3 for steel structures after an extensive testing campaign carried out by the European Convention for Metal Construction (CECM) at the end of the 20th century.

$$\phi = \frac{1}{2} \left(1 + \hat{\alpha} \frac{\lambda}{\lambda_0} + \frac{\lambda^2}{\lambda_0^2} \right); \quad \phi = \frac{1}{2} (1 + \alpha (\bar{\lambda} - 0, 2) + \bar{\lambda}^2). \quad (3.27)$$

In such expressions, the relative slenderness $\bar{\lambda} = \lambda/\lambda_0$ can be expressed in many different ways considering 3.9:

$$\begin{aligned} \bar{\lambda} = \frac{\lambda}{\lambda_0} &= \lambda \sqrt{\frac{\varepsilon_y}{\pi^2}} = \sqrt{\frac{\varepsilon_y EA}{EA \pi^2}} \\ \bar{\lambda} &= \sqrt{\frac{Af_y}{N_k}} = \sqrt{\frac{N_u}{N_k}} = \sqrt{\frac{N_u}{N} \frac{N}{N_k}} \end{aligned}$$

The buckling factor ω , or its inverse, the buckling reduction coefficient χ can therefore be determined with the expression

$$\frac{1}{\chi} = \omega = \phi + \sqrt{\phi^2 - \bar{\lambda}^2}; \quad \phi = \frac{1}{2} (1 + \alpha (\bar{\lambda} - 0, 2) + \bar{\lambda}^2) \quad (3.28)$$

in which the coefficient α in the Eurocode is attributed to the different curves corresponding to each cross-section type (named “a₀, a, b, c, d”) through the values

$$\alpha([a_0; a; b; c; d]) = [0, 13; 0, 21; 0, 34; 0, 49; 0, 76].$$

It should be added that, if the (reduction) buckling factor is a function of the relative slenderness, $\omega = \omega(\bar{\lambda}, \alpha)$, there is also the inverse function, the relative slenderness as a function of the buckling factor, $\bar{\lambda} = \bar{\lambda}(\omega, \alpha)$, a function whose expression we include here to complete the analysis.

$$\bar{\lambda}(\omega, \alpha) = \frac{\sqrt{\omega^3 + \left(0, 2\alpha + \left(\frac{\alpha}{2}\right)^2 - 2\right) \omega^2 + (1 - 2\alpha)\omega - \frac{\alpha\omega}{2}}}{\omega - 1} \quad (3.29)$$

In Figure 3.11 the different Theoretical reduction values for straight perfect members, χ_T , corresponding to the lower value in the limit of crushing load or the Euler’s “critical” buckling load, are compared with the practical application values resulting from either the Rankine combination rules (χ_R and χ_{R_2}) seen above (section 3.1.3), or from the Eurocode formulation (χ_{CTE}), according to their respective expressions

$$\chi_T = \min \left(1, \frac{\varepsilon_k}{\varepsilon_y} \right) = \min \left(1, \frac{1}{\bar{\lambda}^2} \right) \quad (3.30)$$

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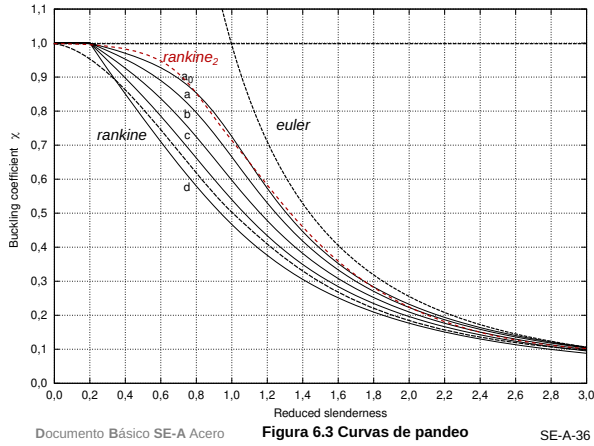


Figure 3.11: Buckling reduction curves: Theoretical values of $\chi(\bar{\lambda})$ plotted on a figure compared with the values given in the spanish code CTE,DB-SE-A, that was derived from Eurocode 3

$$\chi_R = \frac{1}{1 + \frac{1}{\varepsilon_k/\varepsilon_y}} = \frac{1}{1 + \bar{\lambda}^2} \quad (3.31)$$

$$\chi_{R_2} = \sqrt{\frac{1}{1 + \left(\frac{1}{\varepsilon_k/\varepsilon_y}\right)^2}} = \sqrt{\frac{1}{1 + \bar{\lambda}^4}} \quad (3.32)$$

$\chi_{CTE} = \chi(\bar{\lambda}, \alpha)$; see eq. 3.28.

$$\alpha([a_0, a, b, c, d]) = [0, 13; 0, 21; 0, 34; 0, 49; 0, 76]$$

3.2.2.4 Interpretations

From what has been seen so far, $\chi \leq 1$, or also $1/\chi = \omega \geq 1$. The elastic verification from which we started indicates that in a compressed member the average stress required to support the normal effort, σ_N , is not a good reference for verification, as it is distributed variably to face the moment derived from the resulting eccentricity e , therefore adding, for verification with the limit σ_y , a distribution with zero resultant and maximum value σ_M to this uniform distribution σ_N . We have therefore used the interpretation $\chi = \sigma_N/\sigma_y = \sigma_N/(\sigma_N + \sigma_M)$.

The eccentricity in that case, $e = M/N$, can be easily deduced:

$$e = \frac{M}{N} = \frac{W\sigma_M}{A\sigma_N} = \frac{W}{A} \left(\frac{\sigma_y - \sigma_N}{\sigma_N} \right) = \frac{W}{A} \left(\frac{1}{\chi} - 1 \right) \quad (3.33)$$

an expression in which W/A is around the value of $0,36 h$ in double T metal cross-sections. Since inertia and resistant modulus are respectively $I = Ai^2$, $W = I/v$, it also results

$$e = \frac{i^2}{v} (\omega - 1); \quad \frac{1}{\chi} = \omega = 1 + \frac{ev}{i^2}. \quad (3.34)$$

Three possible interpretations therefore result, collected in Figure 3.12. The first interpretation does not adequately account for elastoplastic situations, which is why the regulations have opted for the second. However, the third perspective should not be lost sight of: the buckling factor reflects the existence of an eccentricity. This third perspective is essential if the buckling effect involves joints, such as in a cantilever column: fixed

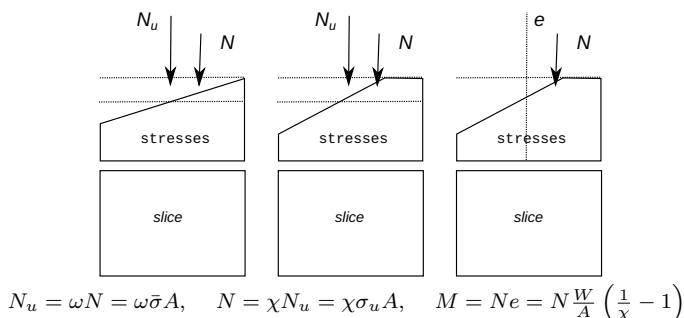


Figure 3.12: Buckling factor interpretations, as increase in verification stresses, reduction of resistance capacity, or eccentricity effect in compression.

at the base and free at the head. In this case, the resistance capacity to the combined effect of the axial force and the moment, described by that factor, is necessary both in the column and in its base joint. The eccentricity to be considered in the joint can be estimated from the properties and buckling factor obtained for the column.

3.3 Sizing in Buckling Problems

The determination of the buckling factor is immediate, as we have seen, when dealing with the problem of validate the state of a known member, but the design problem is to choose the valid member, so, with the approach seen so far, the design

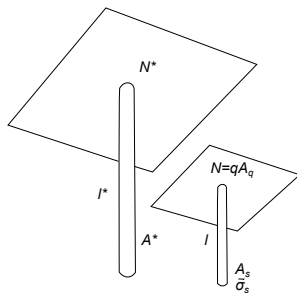


Figure 3.13: Proportionality in buckling problems: if the load per unit area is the same in both figures, a dimensional change that does not alter any proportion between them maintains the buckling factor intact

problem is not direct, it is necessarily iterative: one must try solution after solution until finding the appropriate one. This also makes it difficult to address broader reflections, as it deals only with solutions to particular cases.

However, we will see that the problem maintains proportionality conditions, which we will use to address the project more directly.

3.3.1 Proportionality in buckling

Figure 3.13 presents two different problems in which the geometry of the members, their sections and lengths, and the loaded areas change in dimension, maintaining all proportions. Under

these conditions, the buckling factor in both problems remains invariant.

Indeed, if we compute the magnitude l^2/N , with l being the member length and N the load, we have that

$$\frac{l^2}{N} = \frac{l^2}{A\bar{\sigma}} = \frac{l^2}{A\chi\sigma_y} = \frac{\mathcal{K}l^2}{i^2\chi\sigma_y} = \frac{\mathcal{K}}{\sigma_y} \frac{\lambda^2}{\chi} = \frac{\mathcal{K}}{\sigma_y} \omega\lambda^2 \quad (3.35)$$

in which we have used $\mathcal{K} = \frac{i^2}{A}$ which, in sections maintaining proportion, is a constant. Now, since for the function $1/\chi = \omega = \omega(\lambda, \alpha)$ there exists an inverse function $\lambda = \lambda_0\bar{\lambda}(\omega, \alpha)$ (see 3.29), it results that being the product $\omega\lambda^2$ invariant with the ratio l^2/N , the buckling factor itself does not vary with said ratio. We have

$$\frac{l^2}{N} = \frac{\mathcal{K}\lambda_0^2}{\sigma_y} \omega\bar{\lambda}(\omega, \alpha)^2 \implies \frac{l^2}{N} \frac{\sigma_y}{\mathcal{K}\lambda_0^2} = \omega\bar{\lambda}(\omega, \alpha)^2 \quad (3.36)$$

so that the buckling factor is actually a function of only two variables, one corresponding to the product $\frac{l^2}{N} \frac{\sigma_y}{\mathcal{K}\lambda_0^2}$, and the other, which is the imperfection factor α . Of the first, we are mainly interested in the problem terms, length l and load N . Since in the ω format this buckling factor has a value somewhat greater than unity, we can write its series expansion dependent on that first variable and keep the first term(s) of said expansion:

$$\omega = \omega\left(\frac{l^2}{N} \frac{\sigma_y}{\mathcal{K}\lambda_0^2}, \alpha\right) = 1 + ? \cdots = 1 + \beta \frac{l^2}{N} + \dots \quad (3.37)$$

To do this, we differentiate with respect to l^2/N both terms of

equation 3.36:

$$\frac{\partial}{\partial(l^2/N)} \left(\frac{l^2}{N} \frac{\sigma_y}{\mathcal{K}\lambda_0^2} \right) = \frac{\sigma_y}{\mathcal{K}\lambda_0^2} = \frac{\partial}{\partial(l^2/N)} (\omega \bar{\lambda}^2(\omega, \alpha)) \quad (3.38)$$

Using 3.29, considering equation 3.37 for the description of ω , and keeping only the first term of the expansion,

$$\beta \approx \frac{\partial}{\partial(l^2/N)} \omega = \frac{\partial}{\partial(l^2/N)} \left(1 + \beta \frac{l^2}{N} + \frac{1}{2} \beta_2 \left(\frac{l^2}{N} \right)^2 + \dots \right) \quad (3.39)$$

we can deduce the coefficient β (see Cervera Bravo et al. 2013), with the values in the following list

$$\beta(\alpha[a_0; a; b; c; d]) \approx [0.26; 0.31; 0.40; 0.50; 0.69] \frac{\sigma_y \varepsilon_y}{\pi^2 \mathcal{K}}. \quad (3.40)$$

It is important to note that β is a non-dimensional magnitude: it is a pressure whose product with l^2/N is dimensionless.

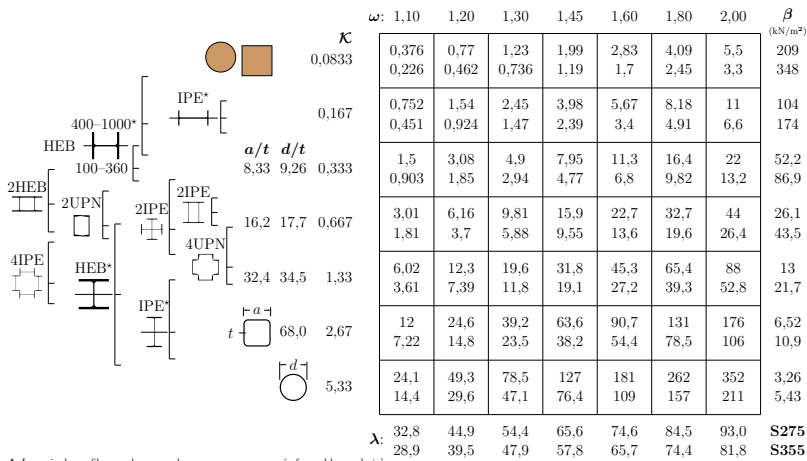
3.3.2 Estimating the buckling effect in design

With the previous results, we can now estimate the influence (the *weight*) of buckling in sizing. We use for this both the stress comparison expression and the eccentricity resulting from buckling:

For the first, we have

$$\sigma_c = \frac{N\omega}{A} = \frac{N}{A} \left(1 + \beta \frac{l^2}{N} + \dots \right) = \frac{1}{A} (N + \beta l^2 + \dots) \leq f_y \quad (3.41)$$

Mapa de pandeo para la curva 'c' del CTE. Casillas recuadradas: ℓ^2/N en m^2/MN para S275 y S355; β en kN/m^2 ; $\kappa = \ell^2/A$; $\lambda = \ell/i$.



*: La serie de perfiles puede comprobarse con una curva más favorable que la 'c'.

Figure 3.14: Buckling map for Eurocode 3 curve 'c'

in which we see that the pressure β applied to the square of the buckling length exerts an effect similar to what would add a load (a weight) βl^2 to the centered load.

For the buckling-derived eccentricity e_χ we have, according to expression 3.33

$$e_\chi = \frac{W}{A} (\omega - 1) \approx \frac{W}{A} \left(\beta \frac{l^2}{N} \right); M_\chi = N e_\chi \approx \frac{W}{A} \beta l^2 \quad (3.42)$$

As we can see, the design approximation becomes more immediate. Figure 3.14, from Cervera Bravo et al. 2013, presents the application for the case of Eurocode 3 curve c.

3.4 Local Elastic Stability Problems

The instability problem in compression is general: it occurs in every circumstance where a structure, member, section, or section element is subjected to compression and its geometry allows it to deviate from the line of pressure. Although we will not detail the formulation for each case, we present a small collection of figures of some possibilities to be considered, and whose verification expressions can be found in the codes.

3.4.1 In members: flexural-torsional, lateral buckling

We have seen flexural buckling, but the combination of flexure and torsion in member buckling is possible and relevant if misalignment between end forces and compressions stresses feeds rotations around the member axes, as seen in figure 3.15. In the first case, the variable section inclination induces that torsional moment. In the second case, the instability of the compressed chord makes it to deviate from the bending plane.

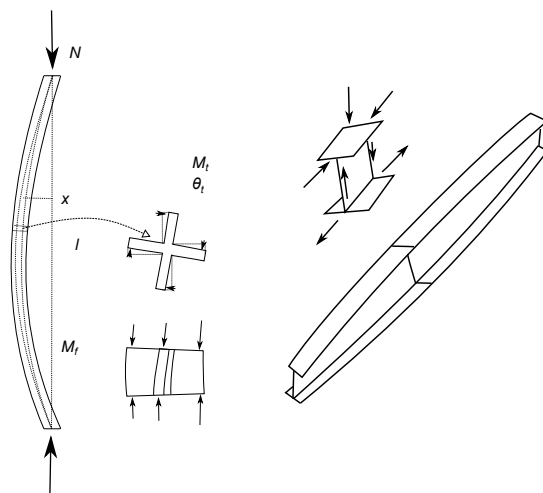


Figure 3.15: Buckling cases with torsion: Flexural-torsional and lateral buckling.

3.4.2 In plate elements: sidewall buckling...

The plates forming part of the section also have the possibility of deforming out of their plane due to compression stresses: this is the case of sidewall buckling. This can occur (figure 3.16) by load concentration, or by simple shear in slender plates: it is worth remembering that shear implies a state of tangential stresses that involve similar principal compression and tension stresses. Compression implies a reduction in bending stiffness,

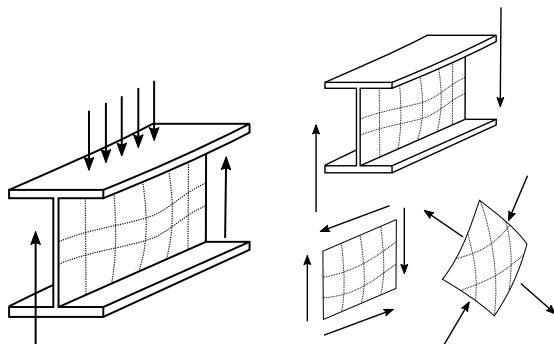


Figure 3.16: Buckling, by concentrated load, or by shear.

although tension implies increases. However, as figure 3.10 shows, the loss of stiffness in compression is greater than the gain in tension, and this therefore implies a net loss, with the consequence that the sidewall buckling phenomenon comes into play.

3.5 Global Buckling of Frames

Once been established the concepts and relationships relating to buckling of structural pieces or regions that can be represented by single-degree-of-freedom systems, we will now consider cases where this deformation cannot be adequately represented except by considering an appropriate (and appreciable) number of degrees of freedom. This is the problem of frames, although

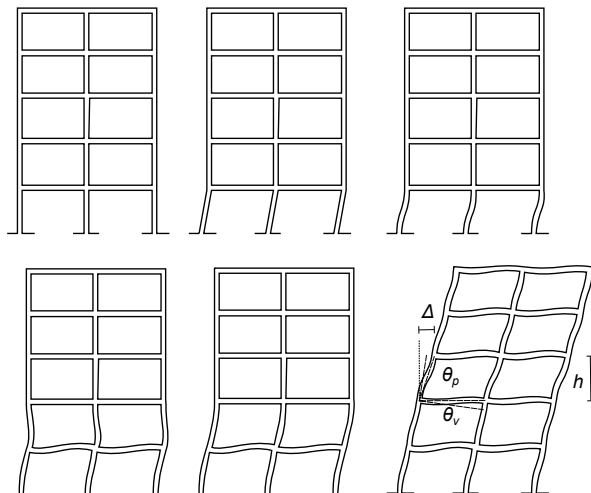


Figure 3.17: Drift of a rigid-jointed frame. Floor displacement induces rotations at the joints, therefore in beams, that adds to the drift ratio that would be derived from the sole displacement of the column head

it is also the case for all types of structures with a large number of members where loads must be balanced with stress paths involving compressions.

We approach the case of frames for greater clarity, although the techniques we will ultimately employ can be applied generally to structures of any geometry.

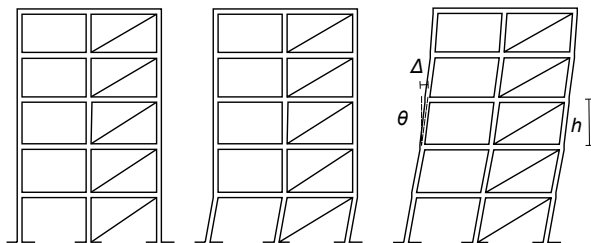


Figure 3.18: Drift of a braced articulated frame. Diagonal elongation is the fundamental variable as long as the frame does not reach significant heights.

3.5.1 Displacements and P-Delta Effect

In a first approximation, we characterize the displacement by a story drift (or drift ratio) represented by the angle $\theta = \Delta/h$ formed between the vertical and the line between joints of two successive stories after displacement.

If we look at the sequence of figures in 3.17, we verify that the total displacement is composed, in rigid frames, by:

- part from column displacement (column bending; θ_p)
- part from joint rotation (beam bending; θ_v)
- ... to which it could be added a part from joint deformation (by joint shear; θ_n , see figure 3.19)

However, in braced articulated frames, figure 3.18, this displacement is due

- fundamentally to diagonal elongation

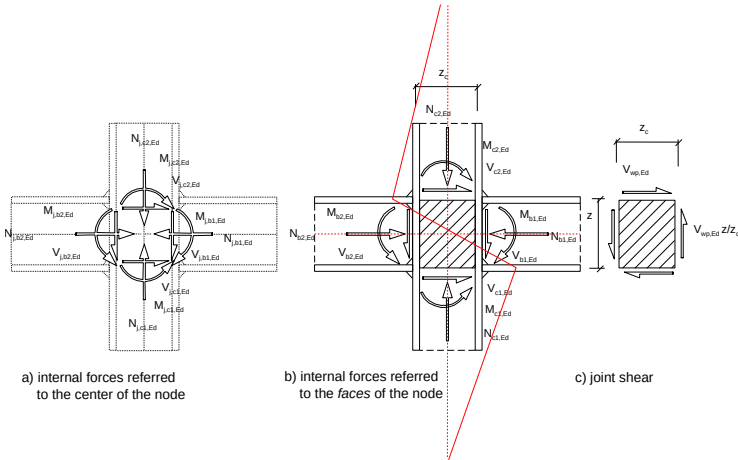


Figure 3.19: Joint Shear: Tangential stresses typically exceed those of adjacent members, since the moment jump between columns occurs over a very short distance, and the resulting distortion can therefore be relevant.

These displacements imply a misalignment of the vertical load effect with the inclined direction of column axes, and this supposes, as we saw in section 3.2.1.1, a horizontal imbalance that we can analyze in either of these ways:

- as an equivalent lateral load
- as a loss of stiffness

We will therefore reconsider the issue by analyzing the phe-

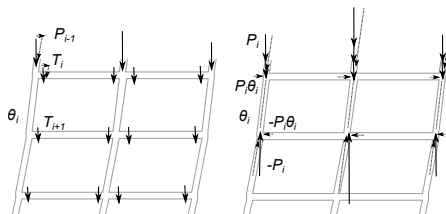


Figure 3.20: Equivalent loads in frame displacement. Loads above a certain region and shear forces from each story in the region are represented. Misalignment is analyzed below story i

nomenon story by story

3.5.1.1 Equivalent Lateral Load

Figure 3.20 shows the deformations, displacements, and loads above a certain height i of the structure (numbering stories from the top) and the compression and misalignment conditions of columns located below that story. For small displacement and drift values θ_i , the effect of the vertical loads can be understood as the summation of the horizontal $P_i \theta_i$ plus the oblique P_i aligned with the displaced column. The first-order analysis considers the effect of the compressions aligned with the vertical, thus ignoring the effect of what we call equivalent lateral load, $P_i \theta_i$. This load, as we see in the figure, is applied with the same value but with opposite directions at both ends of the column. If we consider the entire frame, for each story, the balance between columns above and below this implies an ad-

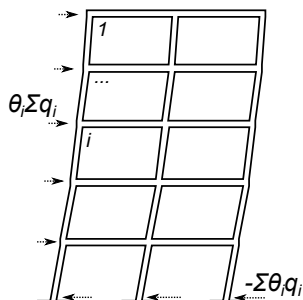


Figure 3.21: Equivalent loads applied by frame displacement. They represent in each story the product of the story load by the displacement, and in foundations the opposite of their sum.

ditional lateral load in each story equivalent to the story drift ratio multiplied by the beam shear or total load that is added to the column compressions supporting that story, figure 3.21. Additionally, it implies a total load in the foundation equal and opposite to the sum of all stories' lateral loads.

Thus, for each story of the frame, we have an effect similar to that considered in section 3.2.1, characterized by:

- the displaced vertical load $P = \sum q_i$
- the initial displacements due to imperfection or defects: θ_d
- the displacement due to the possible horizontal wind load W : θ_w

We will analyze the frame's horizontal equilibrium conditions.

3.5.1.2 Horizontal Equilibrium

Having the means to analyze the frame against horizontal wind loads, we can approximate an estimate of the lateral stiffness (load/drift ratio) of a story $K_{\theta,i}$ as the quotient between the wind shear in that story (sum of the wind loads including and above it) and its drift ratio due to this wind load:

$$K_{\theta,i} \approx \frac{W_i}{\theta_{W,i}} = \frac{\sum_1^i w_i}{\theta_{W,i}} \quad (3.43)$$

This is a first approximation where the interaction between the different stories is taken into account only indirectly, through the results of the analysis of the horizontal wind load hypothesis, an approximation that is more appropriate the more similar the wind load used in that analysis is to the combined load to be considered in the complete problem. But this approximation is capable of providing an important estimation.

We know that this lateral stiffness gives, for each story, the measure of the critical load because, for this load, the equilibrium is indifferent and for any displacement the equality between the response $K_{\theta}\theta$ and the action $P_k\theta$ occurs:

$$K_{\theta} = \frac{W}{\theta_W}, \quad \text{with} \quad K_{\theta}\theta = P_k\theta \implies P_k = K_{\theta}$$

We also have the initial lateral displacement before considering the compression effect in the columns:

$$\theta_0 = \theta_d + \theta_W$$

and therefore the final equilibrium condition involves, as we have already seen, an increase in the deformation to allow the response to reach the action:

$$P\theta = P(\theta_0 + \theta_{P\Delta}) = P_k\theta_{P\Delta} = P_k(\theta - \theta_0)$$

$$\theta = \theta_0 \frac{1}{1 - \frac{P}{P_k}} = (\theta_d + \theta_W) \left(\frac{1}{1 - \frac{\theta_W P}{W}} \right)$$

where we can use $\gamma_k = P_k/P$. Depending on the codes, the resulting expression takes various versions of the following:

$$\theta = (\theta_d + \theta_W) \left(\frac{1}{1 - r} \right); \quad r = \frac{P}{P_k} = \frac{1}{\gamma_k}$$

If $r < 0,1$ the horizontal effect $P\theta$ is less than $|1 - 1/0,9|$ times the total horizontal load, so this effect could be ignored. In the language of codes, this is a frame that can be assimilated to an *non-translational* one. However, the effect of the vertical loads on the lateral displacement must be considered if $r > 0,1$. It is then a frame classified as *translational*.

A linear approximation of this effect is given by the displacement amplification expression we just saw. This requires similar displacements in all stories (and therefore regularity in geometry, loads, sections...) but allows an approximate linear analysis considering as horizontal loads per story the sum of the wind load plus that derived from the displacement: $w_i + \theta q_i$, where θ is the drift ratio got from the preceding expression.

A more precise determination requires analyzing the assembly effect bar by bar in the frame, the only way to rigorously

analyze the interaction of all of them. It requires us to consider a model with more degrees of freedom, that will be assigned to all possible movement components for each one of the considered joints.

3.5.2 Material Stiffness and Geometric Stiffness

In the analysis of bar structures, we saw that the elastic approach used to include the interaction effects between pieces in the overall behavior of a structure led to a displacement formulation formalized in matrix format, linking displacements with loads through the assembly of local stiffnesses of each bar, represented in the stiffness matrices that connect displacements to end forces. On the other hand, in the analysis of the buckling phenomenon, we have seen that it can be interpreted as the alteration of the bar's stiffness (reduction where in compression), so we will now consider the problem from the perspective of this local stiffness and its possible subsequent assembly.

3.5.2.1 Pinned Bars

We first consider a model in which the bars—the columns—are pinned.

Figure 3.22 shows a bar subjected to axial effort where a node is laterally displaced relative to its former axis.

The equilibrium and compatibility equations in such a situation, described in first approximation in the deformed position

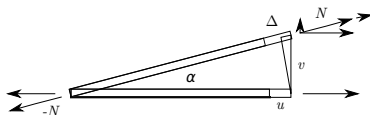


Figure 3.22: Pinned bar geometry under axial effort and lateral displacement relevant in stiffness

and if the angle derived from displacement is small, are:

$$\begin{bmatrix} f_H \\ f_V \end{bmatrix} = \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} N; \quad \Delta = l\epsilon = \begin{bmatrix} \cos \alpha & \sin \alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where one can express

$$\sin \alpha = \frac{v}{l + \Delta} = \frac{v}{l \left(1 + \frac{N}{EA}\right)}$$

even though the second term in the parentheses in the denominator, the unit elongation, is small in relation to the first and can be neglected.

If we now consider that the actual tension N of the bar, aligned with it in the deformed position, has two components parallel and perpendicular to the initial bar position, on which we represent the efforts, the perpendicular component will be

$$f_V \approx N \sin \alpha \approx N \frac{v}{l}$$

and therefore we would write the complete equilibrium in the format

$$\begin{bmatrix} f_H \\ f_V \end{bmatrix} = \left(\frac{EA}{l} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{N}{l} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u \\ v \end{bmatrix}$$

which we can symbolize as

$$\mathbf{f} = (\mathbf{k}_E + \mathbf{k}_G) \mathbf{u}$$

where the first term is the classic elastic stiffness, and the second term is called *geometric stiffness matrix* and accounts for the alteration derived from the effect of the axial load on displacement³.

If in the geometric matrix $\mathbf{k}_G$ we ignore the axial effects, and consider the lateral effects equal and opposite at both ends, we would have the following form:

$$\mathbf{f}_V = \mathbf{k}_G \mathbf{v} = \begin{bmatrix} f_{V,1} \\ f_{V,2} \end{bmatrix} = \frac{N}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (3.44)$$

This is a useful form in which the movements of both ends are expressed, although it is redundant compared to the previous one, which only considers the relative movement of one end with respect to the other.

If now we seek to estimate the *critical load*, that is, the axial load applied to a cantilever bar that makes the lateral response

3. It is also possible to find in the literature, depending on the formulation used, the expression

$$\begin{bmatrix} f_H \\ f_V \end{bmatrix} = \left(\frac{EA}{l} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{N}{l + \Delta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u \\ v \end{bmatrix}$$

For example, this would be the case if the procedure of section 2.1.1.7, equation 2.32, and shape functions of 2.24 are used, although with the local geometric matrix $\kappa_G = N = \bar{\sigma}A$, although the expression does not seem to have a reasonable basis and can be eliminated with a more rigorous formulation.

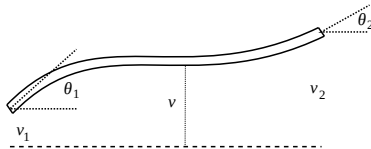


Figure 3.23: Bar geometry in flexure under axial effort and relevant transverse displacement (v) in stiffness

zero even when the displacement is not zero, we can approximate the lateral stiffness of the cantilever against transverse load as the sum of the stiffness of a flexural bar fixed at one end and the tensioned piece:

$$f_V = \left(\frac{3EI}{l^3} + \frac{N_k}{l} \right) v = 0$$

$$N_k = -\frac{3EI}{l^2}$$

The result is only approximate: the exact value from Euler's formulation is

$$N_k = -\frac{\pi^2}{4} \frac{EI}{l^2}$$

so an improvement in the approximation will require considering the flexural deformation in the tensioned model.

3.5.2.2 Bars in Flexion

Given the relative imprecision of assuming axial effects in isolation, we now consider the coupling between axial and flexural

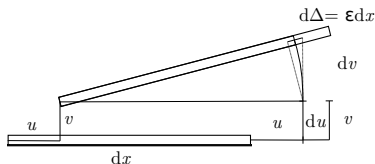


Figure 3.24: Unit Elongation for a Plane Displacement Deformation. Note the similarity in the displaced position between the right triangle with legs dv, dx and the orthogonal at the right end of the displaced segment, where the segment rotation is also represented.

efforts. We will first describe the lateral deformation based on the end node movements using shape functions and will consider only lateral displacement or rotation movements as relevant for the sought effects, having thus for the lateral displacement function⁴

$$v = \mathbf{N}\mathbf{u} = \mathbf{N}[v_1 \theta_1 v_2 \theta_2]^T \quad (3.45)$$

The elongations, including those added by this lateral deformation, can be measured, according to figure 3.24, with

$$\varepsilon = \frac{\partial u}{\partial x} + \left(\frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 \right)$$

4. Attention in the following to the notation difference between the shape functions $\mathbf{N}$, which will be those from equation 2.29, and the axial effort N of the bar.

We will have as end efforts $\mathbf{f} = [T_1 \ M_1 \ T_2 \ M_2]^T$ along with $[-N \ N]^T$.

We now measure the strain energy in the bar, considering both axial and flexural effects:

$$W = \int_0^l N \frac{\partial u}{\partial x} dx + \frac{1}{2} N \int_0^l \left(\frac{\partial v}{\partial x} \right)^2 dx$$

so that the geometric component, from flexure, is

$$\rightarrow W_{G,F} = \frac{1}{2} N \int_0^l \mathbf{u}^T \mathbf{N}'^T \mathbf{N}' \mathbf{u} dx$$

where parameters $\mathbf{u}$ can be extracted from the integral, and where N is the bar's compression and $\mathbf{N}$ are the shape functions used to construct $v(x) = \mathbf{N}\mathbf{u}$, shape functions as derived in 2.29.

If we now apply Castigliano's theorem, by which $f_i = \frac{\partial W_G}{\partial u_i}$, it will result for the component derived from the flexural effect

$$\mathbf{f}_G = N \left(\int_0^l \mathbf{N}'^T \mathbf{N}' dx \right) \mathbf{u} = \mathbf{k}_{G,F} \mathbf{u}$$

so that now the geometric stiffness matrix corresponding to lateral displacement and rotation parameters turns out to be

$$\mathbf{k}_{G,F} = \frac{N}{30l} \begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \quad (3.46)$$

This same result is obtained if the analysis is approached from the weak perspective of the equilibrium: internal work

equal to external work. In this case, for the same deformation expression from 3.45, we will have that the slopes can be determined with

$$\frac{\partial v}{\partial x} = v' = \mathbf{N}' \mathbf{u}$$

so that the transverse force component due to the rotation of the axial effort N will be

$$V = v' N$$

and therefore the equivalent end efforts $\mathbf{f} = [T_1 \ M_1 \ T_2 \ M_2]^T$ can be deduced from the internal and external work equality $W_i = W_e$

$$\bar{\mathbf{u}}^T \mathbf{f} = \int_0^l \bar{v}' V \, dx, \quad \forall \bar{\mathbf{u}}$$

expression that, written from shape functions, takes the form

$$\bar{\mathbf{u}}^T \mathbf{f} = \int_0^l \bar{\mathbf{u}}^T \mathbf{N}'^T N \mathbf{N}' \mathbf{u} \, dx, \quad \forall \bar{\mathbf{u}}$$

so that

$$\mathbf{f} = \int_0^l \mathbf{N}'^T N \mathbf{N}' \mathbf{u} \, dx$$

As the derivatives are now

$$\begin{aligned} N'_{\delta,i} &= \frac{6x^2}{l^3} - \frac{6x}{l^2} \\ N'_{\theta,i} &= \frac{3x^2}{l^2} - \frac{4x}{l} + 1 \\ N'_{\delta,d} &= \frac{6x}{l^2} - \frac{6x^2}{l^3} \\ N'_{\theta,d} &= \frac{3x^2}{l^2} - \frac{2x}{l} \end{aligned}$$

the definitive expression

$$\mathbf{f} = \mathbf{N} \left(\int_0^l \mathbf{N}'^T \mathbf{N}' \, dx \right) \mathbf{u} = \mathbf{k}_{G,F} \mathbf{u}$$

leads to the same matrix $\mathbf{k}_{G,F}$ from equation 3.46.

Such stiffness is positive in cases of positive efforts, of traction, but will result negative in cases of negative efforts, of compression: the stiffness is reduced by this effect.

3.5.3 Critical Load Factor, Eigenvalues

With the stiffness matrices that incorporate the lateral effects of axial loads, it is now simple to establish the general relationships between loads and displacements, although there is a nonlinearity problem: the geometric stiffness matrix depends on the internal stress state, which is not known while solving the problem. So the usual analysis procedures must consist of at

least two steps: the first in which the stress state is analyzed without considering geometric stiffness, and the second in which this geometric stiffness is estimated from the stresses of the first step. If it can be supposed that the added deformation by this modification (reduction) in stiffness can increase in the proportion to the load increase, this hypothesis allows the analysis of the possible critical collapse state and therefore of the critical load factor. We are talking about a linear buckling analysis or *linear buckling*.

However, in some cases, and certainly if the axial stresses change significantly in that second step, it is not possible to ensure that the geometry changes when the load increases can be represented by scaling the already estimated deformation and an iterative nonlinear analysis becomes essential that updates such stiffness at each deformation step and pursues the combined load and displacement evolution, an evolution that can involve displacement increases with load-bearing capacity loss. We are now talking about a nonlinear buckling analysis, and it will require iterative analysis methods capable of capturing this combined evolution (the arc length method, for example, see figure 2.12 in section 2.1.2.4). We will not advance further on this issue here, even though it can be extremely relevant in very slender compressed structures.

3.5.3.1 Global Stiffness under Axial Loads

Once we have a method to determine the (loss of) stiffness derived from the existence of axial stresses, we can rewrite the

global stiffness equations 1.24 in the form

$$\mathbf{F} = (\mathbf{K}_E + \mathbf{K}_G) \mathbf{U} \quad (3.47)$$

where $\mathbf{K}_G$ depends on the load, as we have seen, and where it is therefore possible to perform the analysis in two steps, with $\mathbf{F}_0$ and $\mathbf{K}_E$ in the first to determine the loads that allow determining $\mathbf{K}_G$, considering the load system capable of generating the most representative compression situation in the structure's life, and with $\mathbf{F}$ (with, at least, the $\mathbf{F}_0$ case again and stiffness $\mathbf{K}_E + \mathbf{K}_G$) for the analysis of any other hypotheses or load combinations of interest.

However, this analysis still does not inform us about the greater or lesser proximity to a critical situation, for which the determination of the (or the several) critical load factors is required, as we see next.

3.5.3.2 Critical Load Factor

The problem of determining the critical load, or the critical load factor, is linearized by assuming $\mathbf{F} = \gamma \mathbf{F}_0 + \mathbf{F}_l$ for an initial state, with $\gamma_0 = 1$, $\mathbf{F}_l = 0$, which allows determining a first state of displacements $\mathbf{U}_0$ and stresses $\mathbf{f}_0$ with which to construct $\mathbf{K}_{G,0} = \mathbf{K}_G(\mathbf{F}_0)$. For loads $\gamma_i \mathbf{F}_0$, geometric stiffness $\gamma_i \mathbf{K}_{G,0}$ is assumed.

In the following analysis, we seek to determine the load factor γ_k necessary to reach a situation where a deformation⁵ $\mathbf{U}_l$, which would respond to a perturbation $\mathbf{F}_l$, nonetheless results

5. Understood as a deformation involving misalignment in compressions

without response, that is, reaching a situation of indifferent equilibrium with indeterminate displacements. Or in mathematical format:

$$\mathbf{F}_l = (\mathbf{K}_E + \gamma_k \mathbf{K}_{G,0}) \mathbf{U}_l = 0, \mathbf{U}_l \neq \mathbf{0} \quad (3.48)$$

The problem is a generalized eigenvalue problem: it is the problem of determining the pairs of γ_k , $\mathbf{U}_l$ capable of ensuring the previous condition, and they correspond to situations where the matrix $(\mathbf{K}_E + \gamma_k \mathbf{K}_{G,0})$ becomes singular (with null determinant).

For each value of γ_k there will be a *mode* of displacement $\mathbf{U}_l$ linked to it, which will correspond to the geometry of the indifferent equilibrium corresponding to the critical load $\gamma_k \mathbf{F}_0$. This mode has no scale: any proportional deformation equally corresponds to this equilibrium form. If the load is less than the critical load and the geometry were initially perfect, there would be no deformation. If the load reaches the critical point, any deformation proportional to $\mathbf{U}_l$ is possible and, therefore, collapse is assured.

3.5.3.3 Instability as an Eigenvalue Problem

Problem 3.48 can be transformed into a standard eigenvalue problem if the inverse matrix of $\mathbf{K}_{G,0}$ exists. Now the problem can be formulated in the form

$$\begin{aligned} (\mathbf{K}_{G,0}^{-1} \mathbf{K}_E + \gamma_k \mathbf{I}) \mathbf{U}_l &= 0, \mathbf{U}_l \neq \mathbf{0} \\ (\mathbf{K}_{G,0}^{-1} \mathbf{K}_E) \mathbf{U}_l &= \gamma_k^* \mathbf{U}_l \end{aligned}$$

In the last expression, we use γ_k^* positive or negative depending on how the geometric stiffness $\mathbf{K}_{G,0}$ is being formulated. As we will see in the following section, it is possible to make explicit the stiffness reduction character of this.

As an application case of this eigenvalue problem, we can now address again the buckling problem of the cantilever support that we considered in section 3.5.2.1 using now the geometric stiffness of section 3.5.2.2:

$$\left(\begin{bmatrix} 12 & 6l \\ 6l & 4l^2 \end{bmatrix} \frac{EI}{l^3} + \frac{\gamma_k N}{30l} \begin{bmatrix} 36 & 3l \\ 3l & 4l^2 \end{bmatrix} \right) \begin{bmatrix} v_s \\ \theta_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.49)$$

Considering the solution corresponding to the smallest eigenvalue, we now obtain that $N_k = \gamma_k N = 2.486 \frac{EI}{l^2}$, that can be compared with the Euler solution, $\pi^2/4 \times EI/l^2$, that results in $2.467 EI/l^2$. Figure 3.25 shows the two buckling modes in the solution of the compressed cantilever problem formulated in 3.49.

3.5.3.4 Critical Load Factor as the Rayleigh Quotient

We will now reconsider the approach of equation 3.47, which considers both material and geometric stiffness.

It is interesting to make explicit the negative or reduction character of the geometric stiffness, so we rewrite it, similar to 3.48, and determine from this expression the scalar product that would measure the work in a displacement $\mathbf{U}_l$. This operation allows, in the case where $\mathbf{U}_l$ corresponds to a critical mode, to determine the value of the corresponding critical load factor:

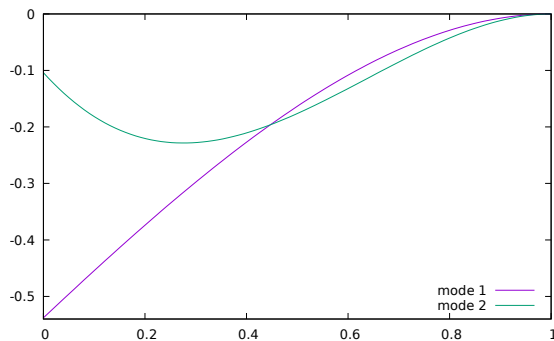


Figure 3.25: Buckling modes of compressed cantilever (horizontally), according to the stiffness obtained from the formulation in section 3.5.2.2, with the right node being the support.

$$\begin{aligned}
 \mathbf{F} &= (\mathbf{K}_E - \mathbf{K}_G) \mathbf{U} \\
 \mathbf{F}_l &= (\mathbf{K}_E - \gamma_k \mathbf{K}_{G,0}) \mathbf{U}_l = 0, \mathbf{U}_l \neq \mathbf{0} \\
 \mathbf{U}_l^T \mathbf{F}_l &= 0 = \mathbf{U}_l^T \mathbf{K}_E \mathbf{U}_l - \gamma_k \mathbf{U}_l^T \mathbf{K}_{G,0} \mathbf{U}_l \\
 \gamma_k &= \frac{\mathbf{U}_l^T \mathbf{K}_E \mathbf{U}_l}{\mathbf{U}_l^T \mathbf{K}_{G,0} \mathbf{U}_l}
 \end{aligned} \tag{3.50}$$

The last equation expresses the Rayleigh quotient and provides a procedure that allows to approximate γ_k by approximating the buckling mode $\mathbf{U}_l$.

Indeed, it is possible to iterate starting from an approximate form of $\mathbf{U}_l$, the $\mathbf{U}_{l,0}$, which would allow expressing $\mathbf{K}_E \mathbf{U}_{l,1} = \mathbf{F}_{G,0} = \gamma_0 \mathbf{K}_{G,0} \mathbf{U}_{l,0}$, which allows determining a better approximation $\mathbf{U}_{l,1}$, and so on.

3.6 Some Conclusions in Stability

As we have seen, the stability problem is linked to the need to analyze the conditions of equilibrium, compatibility, and material relationships in the deformed geometry of the structure, with its formulation in the initial geometry not being sufficiently valid even when linearity is maintained in material relationships: we have a situation of non-linearity derived from changes in geometry due to deformations: a geometric non-linearity. An even more complex problem will occur when this cause of non-linearity is added to the non-linearity in stress-strain relationships or when local deformations are appreciable in relation to the initial dimensions of the pieces.

3.6.1 Geometric Non-linearity by Large Displacements

This case of non-linearity that we have just studied implies that displacements are not small or negligible even though deformations continue to have linear relationships with stresses and remain small. In this case, we are talking about large displacements (as opposed to small displacements in linear analysis). In this case, it is possible that the non-linear buckling problem

can be linearized, or approximated linearly, considering that the transverse effect derived from the deformation is proportional to the product of compression by that transverse deformation or displacement: this presents us with two alternative ways to approach the problem:

- as one (or a set of) equivalent transverse load(s) that is added to the actually applied one and is measured by the difference between the loads balanced by the stress in the original geometry and those that would be balanced by that same stress in the deformed geometry (the difference between the vertical load and the one aligned with the column in the case of a tilted vertical column), a load that therefore increases lateral movements
- as a loss of lateral stiffness in compressed members, which therefore facilitates larger lateral displacements than would occur in the uncompressed case.

We are talking about linear buckling. Both strategies consider what we can call the $P\Delta$ effect, and applied to exact or perfect geometries, they allow us to determine the load factor that, when applied to the actual load, would lead to a situation of indifferent lateral equilibrium, or total stiffness loss (null determinant in the resulting stiffness matrix). This is the critical load factor, which can be approximated by techniques such as the Rayleigh quotient.

Real structures are not, however, perfect, so a better approximation of their behavior involves considering, foreseeing, possible *imperfections* (for which the deformed shapes corresponding to the buckling modes corresponding to the lowest

critical load factor would be chosen on a scale corresponding to the limit of tolerances admitted for these defects in construction processes) and performing the (non-linear) analysis for that geometry or, alternatively, estimating the lateral load that would correspond to the amplified deformation of these defects (where this amplification derives from the critical load factor, according to 3.18) and determining the resulting equilibrium condition, which could be considered safe if it does not exceed the amplified deformation assumed for the analysis, and the resulting efforts do not exceed the structural capability in any of the regions of the structure.

3.6.2 Geometric Non-linearity Adding Large Deformations

If the previous conditions are added to large deformations, it now results that the Cauchy tensors used for describing deformations or stresses are no longer valid and new formulations must be used, such as those provided by the Green-Lagrange tensors for deformation, and the second Piola-Kirchhoff tensor for stress.

This change will affect the Equilibrium and Compatibility relationships, which must be reformulated taking into account both geometries, the original and the deformed, considering the expressions of both tensors in them. It is a problem in elastic organic material structures like rubber, although relatively unusual in building structures.

Interested parties are referred to specialized texts.

3.7 Examples

3.7.1 Five-story Frame

We seek to analyze the stability state of the 5-story frame considered in section 2.1.3, figure 2.13.

That is, we maintain the same geometry, with heights: 3 m., total beam spans of 10 m and haunch length of 2 m. with facade columns modeled as hinged and rigid connection in the central column.

The frame is dimensioned with base beams formed by IPE 330, with haunch up to approximately 1.8 times the depth and quadruple moment of inertia, and HEB 180 columns in floors 1 and 2, HEB 200, 220 and 240 in floors 3, 4 and 5, counted from the roof.

The elastic matrix model is employed, for the 10 considered degrees of freedom, starting from the global and condensed stiffness matrices derived in said section.

3.7.1.1 Objectives: Analysis Planning

- Design loads and column normal forces (central and total) for load hypotheses 1: symmetric gravitational, 2: asymmetric, 3: (1) plus wind, 4: seismic (all with appropriate simultaneity and safety coefficients)
- modeling of local effects:
 - Horizontal shear forces per floor and inter-story and total deflection

- Euler critical loads and column buckling reduction coefficients
- modeling of global effects
 - $P\Delta$ effect per floor: equivalent horizontal load, floor lateral stiffness; and critical load and amplification factors.
 - Geometric stiffness matrices of central columns and frame
 - Minimum frame load factor (Rayleigh quotient or eigenvalue problem)
- Summary and evaluation of results

3.7.1.2 Development Details: Linear Analysis



We will develop the analysis with the aid of the spreadsheet “Portico5plantasEstabilidad.ods” available as an attached document in this pdf. The sheet is used with LibreOffice and has incorporated macros that should be activated when opening. These provide functions used in some calculations, such as the buckling factor according to Eurocode 3, etc⁶.

The sheet (ElastK.ENSAMB) has been developed detailing separately all calculations for each floor, even when the data are the same in all of them, in order to be able to use it with different data. Input fields are in green and calculated ones in various colors to identify different areas. Units are shown above

6. The function for determining eigenvalues and eigenvectors (*eigens* or *eigensw*) requires having installed and operational *r* (Linux) or *Rscript*, as it calls *R* through one of these procedures. Can be viewed in detail in Tools > Macros > Organize Macros > Basic and, expanding the reference to possible macros in the file, edit the *Module1*.

or next to the corresponding field. There are also cells with comments about their content or that of nearby cells identified by a red dot in the upper right corner and which are advisable to consult.

The first rows, up to row 20, provide the geometric and mechanical data of hot-rolled steel cross-sections that can be used in the frame.

Rows 21 to 32 provide the geometry and dimensioning of the frame, with heights and spans, the cross-sections chosen for columns and beams, and the light fraction representing the haunch.

Rows 33 to 38 calculate the static data of the haunch and the ratio of its maximum inertia to the base cross-section inertia.

Rows 39 to 63 determine the numerical coefficients of the stiffness of the fixed-end haunched beam relative to EI_0/l according to the procedure described in 2.1.3.1: columns B and C contain the stiffness of the constant section segment, columns D and E (the first two rows) that of the haunch segment, columns F, G, H, I assemble the stiffness of the complete piece incorporating both, and columns D and E (the second two rows) condense the stiffness of the complete piece by eliminating the parameters associated with the connection between the constant and haunched segments.

Rows 64 to 89 calculate the factors of perfect fixed-end moments (in black) and shears (in red) for uniformly distributed load on the beam, according to the procedure in 2.1.3.2: columns B and C contain the values corresponding to each segment, for constrained movements in the three nodes, column D the accumulated values at the end nodes, 1 and 3, column E (the first

two rows) the 'loads' at node 2, columns G, H, I, J, the inverse of the isolated beam stiffness matrix, column F the moments added to 1 and 3 when considering the displacements of loads at 2, and columns E and A (the lower two rows) the definitive moment and shear factors of the fixed-end haunched beam.

Rows 91 to 100 contain, in column A the stiffnesses relatives to the stiffness EI/l of the upper floor support, in columns B and C, the stiffness matrices of the supports, relatives to said stiffness, in column C the data of heights and spans, in columns E and F the matrices of the already calculated beams, but equally employing their values relatives to the stiffness of said support, column G, the fixed-end moments (relative to ql^2) and columns H, I, J the values for the hinged version of the beam (stiffness term, relative to the stiffness EI/l of the upper floor support, perfect fixed-end term, relative to ql^2 and shear term relative to ql at the central node).

Rows 103 to 113 contain the manual assembly of the frame stiffness matrix, and rows 114 to 124 its inverse that allows solving the problems for the 10 degrees of freedom considered, namely displacements and rotations of the five central nodes. The labeling clearly shows the chosen order. Again, the global matrices in terms of the scale EI/l of the upper floor support.

Rows 124 to 126 provide the values for the proposed loads, and rows 127 to 137 the load and displacement vectors for the different hypotheses considered, namely: purely vertical asymmetric load, wind horizontal load, and seismic load (the displacements both in their values dependent on the scale factor EI/l used, or in meters) Said rows also include the geometric and total load data necessary for the distribution of the total

seismic load among the different floors.

For the stability analysis discussed in section , the remaining rows of the spreadsheet will be used, as we see in the next section.

In the second sheet (ElastK_BTkB_comparar) the global matrix is constructed using the usual mathematical procedure ($\mathbf{K} = \mathbf{B}^T \mathbf{k} \mathbf{B}$) in order to compare the results of manual assembly and the matrix product result.

3.7.1.3 Stability Analysis

For the stability analysis, it is necessary to determine the normal forces in the central supports (for calculating stiffness loss due to compression, or 'geometric' stiffness) but also the effect on the lateral supports which, although hinged and not modeled in the previous analysis, provide the corresponding $P\Delta$ effect (or $P\theta$, equivalent lateral load) when deviating from the vertical. And as we have seen, this can alternatively be viewed as an equivalent lateral load or as a lateral (geometric) stiffness loss. Rows 141 to 148 are used to determine these compressions in the three hypotheses to be considered: maximum asymmetric load, maximum symmetric load, and gravitational load associated with the seismic hypothesis, with columns: A, for the load factor at the fixed end of the hunched beam, and C, D, E, plus G, H, I for the total accumulated load per support or floor in the three analyzed hypotheses.

Rows 150 to 157 evaluate the average and per-floor drift-ratio (inclination) derived from the elastic analysis. To facilitate the interpretation, the data in columns C, D, E is the inverse

of said deflection. Columns G, H, I determine the equivalent lateral load for the $P\Delta$ effect derived from these deflections.

Rows 159 to 165 provide in columns A and B an approximation of the floor lateral stiffness (ratio between floor shear, columns C, D, E and deflection, rows 153 to 157) and therefore the corresponding approximation to the critical load factor in columns G and H (see the interpretation in stiffness of the expression 3.2) and the coefficient of amplification of lateral deflections derived from expressions such as 3.18 or 3.22.

Rows 167 to 181 analyze the reduction of compression effectiveness or capacity in support members considered in isolation, according to several typical estimations, allowing comparison of their results, with row 182 emphasizing the safety of each approach in relation to the requirements of current regulations.

Rows 184 to 191 evaluate the local critical load factor, that is, the factor of compression increase for each support necessary to reach the Euler critical load, for the purpose of comparing with the corresponding values for the complete frame.

Rows 194 to 224 determine the geometric stiffness matrices of central and lateral supports, equations 3.46 and 3.44.

Rows 226 to 237 provide the manual assembly of the complete geometric matrix of the frame, which is automatically assembled ($\mathbf{K}_G = \mathbf{B}^T \mathbf{k}_G \mathbf{B}$) in rows 78 to 148 of the sheet named ElastK.BTkB.comparar, in which the matrix $\mathbf{B}$ shows the different effect of central columns (in combined flexion and compression) compared to lateral columns, only compressed.

In the first sheet (ElastK.ENSAMB), rows 239 to 260 evaluate the Rayleigh quotient, according to what is described in section 3.5.3.4, obtaining the critical load factors for any of the

lateral displacement hypotheses considered, logically different according to the lateral load acting in each case. In particular, this value is estimated for a sequence of iterations in the vertical load plus wind hypothesis.

Finally, and in case access to r or $Rscript$ is available as indicated in footnote 6 on page 287, the stability problem is formulated as an eigenvalue problem in rows 262 to 295 for the previously obtained elastic and geometric matrices.

Chapter 4

Fundamentals and Techniques of Project. Optimization.

In the preceding chapters, the foundations of the analysis have been established and the necessary techniques have been developed for analyzing structures formed by linear elements, whether bars with elastic or plastic behavior, or rigid linear elements in masonry structures. But the designer's task is not to analyze, but to propose. If only analysis techniques are available, the procedure is necessarily iterative: after proposing a solution, analysis tools will validate the suitability of the proposal. If the solution is not sufficiently good, due to the possibility of failure

or excessive resistant capacity and cost, the proposal must be reformulated and analyzed again.

In section 1.1.4 we already saw this duality, and both for the elastic approach, in 1.3.5, and for the plastic approach, in 1.4.8, we saw how to use the equations of each approach to venture solutions capable of solving the proposed problem with less iteration.

However, not all solutions equally valid in terms of safety are equivalent. Although every solution must obviously provide the same degree of safety, they are not all equal in elegance or cost. For each problem, we will seek the solution that, while maintaining the same structural requirements, involves the least use of resources. Therefore, the project is a problem that can be associated with optimization problems. This is what this chapter is about.

This chapter aims to provide a certain project theory that facilitates the designer's effectiveness in the necessary sequence of decision-making that every structural project involves.

To this end, possible alternatives are first analyzed, including the possibilities of using minimization tools, as well as the development of alternative approaches based on appropriate cost metrics. Having chosen the second path, and after simplifying these problems in simple flexural structures, the powerful metric of *structural volume* or *structure quantity* is analyzed and developed, which will be used to determine the relationships between the structure's geometry and its efficiency in the most common types.

4.1 Project Theory

4.1.1 Definitions

To establish solid foundations for a possible project theory, it is sensible to explicitly identify its objectives from the structural perspective. Therefore, it is appropriate to provide some definitions, even if they may seem unnecessary due to their obviousness.

Structural Qualities. Are the qualities of the building that ensure its survival.

Structural Requirements. Are the requirements whose fulfillment ensures that the necessary and sufficient structural qualities for survival are available.

4.1.2 Project Objectives

The objectives of the structural project are to ensure that structural requirements are effectively met, with sufficient reliability and efficiency for all load conditions likely to appear during its useful life:

- **Stability:** The structure is stable as a whole and in each of its parts.
- **Resistance:** All its elements have sufficient safety against failure.

- **Rigidity:** The structure's deformations under any load conditions are compatible with the intended use.
- **Durability:** The structure does not present alterations that could affect its safety in the short, medium, or long term.
- **Feasibility, Sustainability:** All this is achieved at an appropriate and reasonably low cost or impact.

The project theory would be, in this context, a tool to advance in meeting the objectives in the necessary process of reflection about the most appropriate structural form for the problem at hand: it would be about providing instruments that allow guidance in the best choice and validation of the form throughout the design process.

4.1.3 Premises: Conventionality versus Reality

Every project is largely an iterative process of decision, evaluation, and correction with increasing approximation and degree of definition in each iteration, from relatively imprecise and ambiguous situations to degrees of detail sufficiently high to allow the use of appropriate analysis tools for critical evaluation of what is designed. In this context, the use of simplified and perhaps conventional formulations is necessary. This is very clearly the case with actions, which cannot be estimated in all their variability and are therefore described through conventional models that can also be associated with the most conventional structural types.

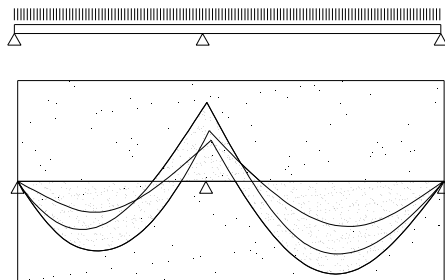


Figure 4.1: Conventionality of Actions: the uniformly distributed action is associated with conventional solutions of relatively constant section, for which the maximum values of the envelopes of the different foreseeable action situations (effort laws) can be adequately covered by the resistant capacity laws of the reference structural solution. In the graphs, some effort laws can be identified, their envelopes (which can be represented as areas or as pairs of limit lines), and the resistant capacities law of the element.

To show a simple example to visualize, in figure 4.1 a typical flexural solution is shown, which will be subjected to very varied load laws throughout its useful life, giving rise to different effort situations, and whose envelopes must in any case be smaller at every point than the corresponding resistant capacity of the element. For the purposes of analysis, this variability is replaced by a conventional uniform load whose value ensures with a certain reliability that the capacity chosen for the element is sufficient

for all future load situations.

So the probable loads and their different effort laws should not be confused with the envelopes of these laws and their approximation through the effort laws resulting from the conventional loads of codes, an approximation valid as long as it is less than the resistant capacity laws of the elements corresponding to the reference structural type: in the example of the figure, as long as the resistant capacity of the metal beam is constant and equal for moments of both signs, checking the worst point is sufficient, and the conventional action is adequate for the project. However, if we were considering a wire as a structure, that conventional action would no longer be adequate, as it does not represent the possibility of movement of that wire as the load position changes.

It is therefore essential in any simplification to consider its adequacy to the structural type implicit in the decision chain of the ongoing project, not confusing in this reflection the conventionality of the models with the reality of the structure intended to be designed.

4.1.4 Some Available Means for the Theory

4.1.4.1 Optimization

Insofar as our purpose is about determining forms and resistant capacities that can ultimately be described through parameters defining geometries and applicable material qualities, the possibility of using optimization methods for determining these parameters can be considered, so that the description of possible

appropriate solutions is done through some type of numerical or functional parameters, establishing some metric capable of expressing the goodness of the solution, and applying one of the multiple optimal determination tools to provide the sought solution.

Problems considered in this way have a certain range of input parameters, those that allow describing the solution, and a smaller range of output parameters that would respond to the coding or quantification of the required benefits that are the object of optimization (maximization or minimization). The modeling must be capable of determining what the output values would be from any input values, and the algorithms, in this case, try to determine the values of the input parameters that would optimize the output values.

Among the classic formulations used to model optimization problems, strategies based on variables representing possible numerical ranges for parameters have been widely used, but also strategies based on functions that would represent possible laws for the distribution of some of the sought geometric or material properties.

More recent formulations associated with the growing development of deep neural networks become less straightforward to describe. In this case, network training typically involves providing the network with a multiplicity of cases in which both input parameter values and output values resulting from those inputs are provided for each case, so that the parameter values associated with each neuron in the network (which are very numerous) ultimately adapt to the training history and subsequently allow, based on any other set of input values, determin-

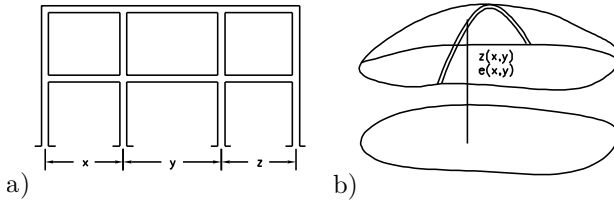


Figure 4.2: Optimization problem formulation: the goal is to determine the variables (a) or functions (b) that describe the optimal solution sought.

ing the most probable output values according to the received training model. In this case, optimization is oriented towards the process of adjusting the model to the training data, a model that will now be much more complex, by virtue of the high number of underlying parameters.

In any case, there are not yet models of this type capable of reproducing the decision, evaluation, and documentation processes that underlie any project (figure 4.3), although partial tools are available that can support specific steps in that process: modeling tools for defining the objects to be produced are evident, as are analysis tools for evaluating these models, etc.

In what follows, we point to the project support tools that have been employed throughout history, although this is an extensive line of study that would take us quite far (see Cervera 1982, Addis 2007, Kurrer 2018, Cervera Bravo 2022) and which would make no sense to develop here.

To trace the sources of the theoretical formulation adopted

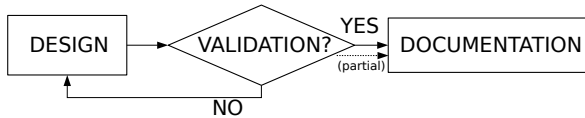


Figure 4.3: Basic project cycle schema: the documentation that guides the work process results from a decision (design or definition) and evaluation cycle in which decisions are progressively adopted and positively validated both individually and collectively within the technical and social context to which the project is destined.

and developed later, see also Cervera 1989, Cervera 1990 and Cervera Bravo and Vázquez Espí 2011.

4.1.4.2 Historical Precedents: Analysis versus Proportion Rules

Throughout history, constructed solutions have evolved from pure experimentation, where solutions capable of enduring over time would survive and therefore be replicated time and again, to current solutions based on verification through analysis.

In this process, proportion rules have endured since very distant times as a tool for extrapolating viable solutions. Indeed, as we will see later, in situations where self-weight is not relevant to resistance conditions, or in stability problems where there is no tensile strength capacity, the verification conditions we would apply today derive from proportion rules: if a solution is viable, similarly scaled solutions maintaining the initial proportions are

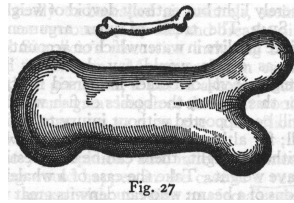


Figure 4.4: Human and Giant Bones according to Galileo

equally viable. This means that in this type of construction, proportion rules have been able to condense practical experience and have been successfully employed over time.

The application of the scientific method to the field of material and construction resistance has its foundational contribution in Galileo 1638, where reflection on the relationships between what he called *absolute resistance* (against axial actions) and *relative resistance* (against transverse actions) derives in a dimensionally correct theory about the resistance of bent pieces. Based on this theory, Galileo demonstrates the impossibility of giants' existence through the well-known *cube-square law*: weight effects grow with the cube of linear dimension alterations, while resistance capacities only grow with the square, so the relative growth of organs necessary for survival makes a proportional growth of humans to reach the size attributed to giants impossible (figure 4.4).

Thus, Galileo's theoretical critique would discredit proportion rules from a scientific perspective, opening a new scientific field that would soon be extensively explored from both theo-

retical and experimental approaches throughout the remainder of the 17th century and the entire 18th century, in which the tools of the newly developed mathematics for describing nature's laws, *analysis*, would make their way as a path for interpreting reality.

It is significant, however, that the theoretical discrediting of proportion rules did not render them obsolete in reality: they continued to be used throughout the 17th and 18th centuries, and only the social and industrial revolutions occurring at the turn of the 18th to 19th centuries would transform the landscape sufficiently to finally set them aside. Indeed, the industrial revolution brought changes in available materials, with old rules no longer applicable to new materials, along with the emergence of entirely new building types. On the other hand, the political alignments generated by the French Revolution meant that the new regime had to quickly train the necessary engineers (Cervera 1982), and that training, at the École Polytechnique, would be based on the entire scientific and experimental foundation developed throughout the 18th century, leading to the widespread use of analysis as a validation pathway for construction proposals, definitively abandoning the old rules.

But this does not imply that the strategy of addressing resistance or stability problems through a geometric lens is incorrect: what Galileo demonstrated was that proportion rules were not applicable in problems where self-weight is relevant, but not that it was impossible to establish geometric verification rules, of which, proportion rules can be part in certain cases, even if not applicable in others. In fact, Galileo's method is itself a purely geometric analysis method, even though its subsequent

development has made the initial geometric clarity lose its way by introducing physical magnitudes and constants of complex dimensionality. We will see next that it is possible to rewrite the analytical verification expressions we habitually use in geometric terms.

4.1.5 Method

What is sought is to rewrite the analytical expressions we use in structure validation based on the geometric parameters that allow characterizing the structures in question: indeed, in the initial design stages, geometric decisions are not only initial but also fundamental from the perspective of structure compatibility with other project constraints. Such that the geometric rules that can be established and that can provide information about structural form efficiency will be of great utility from the project's inception.

To do this, it is essential to restrict the variability of the objects considered and, therefore, resort to their distribution into classes or types that maintain geometric proximity. This will allow including in the formulation a set of geometric parameters that proves useful from the project perspective in said type.

The most productive strategy, initiated by Ricardo Aroca and developed within the Structural Design Seminar (SDE from their initials in spanish) at ETSAM, (see figure 4.5) is to establish types based on the proposed use, because indeed, we not only provide a reduction in geometric variability, but also that variability becomes restricted by predefined and invariable project components, completely independent of the structural

type decisions that might interest us.

Summarizing, therefore:

What we aim for: to classify and qualify structures based on few and powerful parameters. (It is worth noting the difference between *power* and *precision*: a parameter may be very imprecise in the way it details the decisions made, but very powerful in its implications: *size*, for example.)

Geometric Parameters to use: will preferably be those used in the crucial initial design phases, where the definition is still not very precise.

Comparison term: we are not only interested in safety or rigidity (effectiveness): all structures to be compared should be identical in performance. We are interested in efficiency, the ability to be effective with limited resources.

Efficiency measures: could consider human effort, resource use, environmental footprint, cost ...

Structural measure of efficiency: we are interested in a measure of the comparative efficiency of the structure, keeping all other factors equal, computable as far as possible using the variables specific to structural analysis.

So that:

1. We are going to develop and leverage efficiency concepts,
2. ... independent of material qualities,
3. We will make a comparative tour through types using these efficiency concepts,
4. ...to identify which are the relevant parameters and their effects.

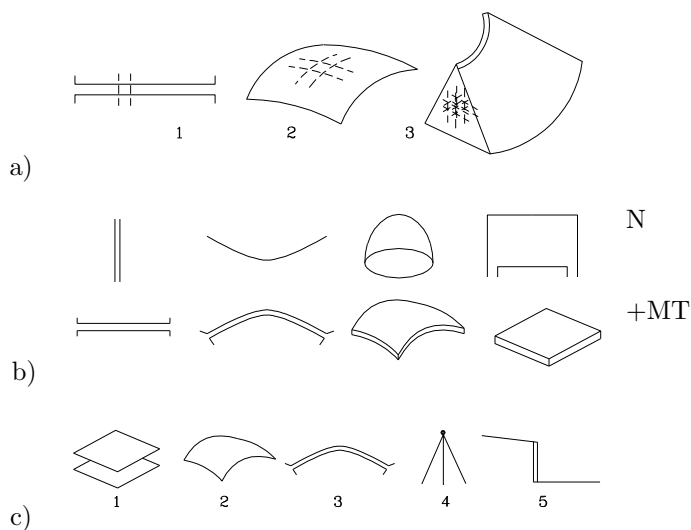


Figure 4.5: Three structural typification strategies: a), by dimensionality; b), by effort type and form; c), by use: 1: inhabiting parallel planes —*floor structures*—, 2: covering spaces —*roofs*—, 3: facilitating crossing from one point to another —*bridges*—, 4: fixing a point in space —*antennas*—, 5: containing masses —*deposits, dams, or retaining walls*—

4.2 Geometry and Efficiency in Beam Structures

As a first reflection, we will approach from a geometric perspective the analysis of structures of type c.1 in figure 4.5, that is the most common type in construction: floor structures, in which the basic geometric qualities respond to the configuration of parallel planes separated and relatively thin, typically configured by families of *beams* supported on each other and that, therefore, work basically in flexion.

In a simple case of material and geometry, which we will see is easily generalizable, we will successively analyze the way of approaching the analysis in geometric format, and the resulting geometric parameters and their incidence in structural costs. We will subsequently generalize this first analysis to materials, geometries, or more complex configurations, deriving from all of this the parameters that are relevant to characterize the efficiency of the solutions.

4.2.1 Classical Checks in Beam Structures

We will characterize the analysis of beam structures by exploring the beam of a simple case (see figure 4.6) of floor plus beam: a solution in which a floor subjected to a uniform load per square meter given by q is repeated indefinitely, and supported on beams with span l that are uniformly separated at distances s . These separations are equally the load span given the symmetry in the support conditions of the floors.

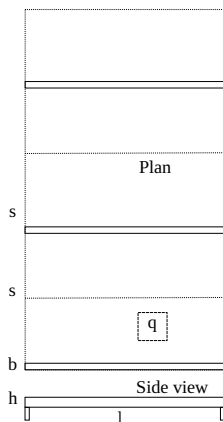


Figure 4.6: Beam solution repeated indefinitely, of floors on supported beam with rectangular section

4.2.1.1 Classic Format of Checks

The classical checks of strength and rigidity for the set of previous decisions on type, material, and geometry are simple, although such decisions imply the successive adoption of specific values for the general configuration and particular parameters. Let's see these decisions.

Type We consider a floor not excessively deformable for a total uniform load q , on supported beams with rectangular section. This establishes the general configuration (type of organization and section, deformation limits, etc.), in addi-

tion to the total load value q in which its characteristic value q_k can be distinguished (usual in service limit state checks), from its design value q_d used in ultimate limit state checks.

Material It will have symmetric behavior in tension and compression, being defined by its design resistance against flexural-derived stresses, normal f_{Md} , and shear f_{Vd} , and its elasticity modulus E .

Geometry This already implies more detailed decisions (or greater *precision*): the span l and beam separations s , as well as the dimensions of its section, depth $\times$ width: $h \times b$.

Although type and material decisions can be questioned at any time and have obvious implications in the solution's behavior, it is typical that in the validation process, and for the purposes of ensuring structural requirements, only the geometry characteristics are modified.

The usual checks are those of resistance to moment and shear, and rigidity, or deflection limitation. We see them in their coded format: (design) effect less than or equal to (design) capacity against said effect.

Shear:

$$V_{Ed} \leq V_{Rd} \equiv \frac{q_d s l}{2} \leq \frac{2}{3} b h f_{Vd} \quad (4.1)$$

Moment:

$$M_{Ed} \leq M_{Rd} \equiv \frac{q_d s l^2}{8} \leq \frac{b h^2}{6} f_{Md} \quad (4.2)$$

Deflection:

$$\delta \leq \delta_{\lim} \equiv \frac{5}{384} \frac{q_k s l^4}{EI} \leq \frac{l}{400} \quad (4.3)$$

These checks allow validating the adequacy of the adopted decisions or, otherwise, rejecting them, providing useful information to correct them.

4.2.1.2 Geometric Format of Checks

In its usual format, all parameters are considered indistinctly, regardless of the type of parameter and moment of decision in which it is adopted. And yet we have seen that typological or material selection decisions are usually less flexible. So it makes sense to rewrite the same previous check equations adopting a geometric format, in which these decisions are separated from each other.

If we do this by changing some term positions, taking the geometric ones to the first member and the rest to the second, now those same equations 4.1 to 4.3 can be rewritten in a different format of great interest.

Shear:

$$\frac{sl}{bh} \leq \frac{4}{3} \frac{f_{Vd}}{q_d} \equiv \lambda_s \lambda_l \leq k_\tau \quad (4.4)$$

Moment:

$$\frac{sl^2}{bh^2} \leq \frac{8}{6} \frac{f_{Md}}{q_d} \equiv \lambda_s \lambda_l^2 \leq k_\sigma \quad (4.5)$$

Deflection:

$$\frac{sl^3}{bh^3} \leq \frac{384}{24000} \frac{E}{q_k} \equiv \lambda_s \lambda_l^3 \leq k_\delta \quad (4.6)$$

It should be noted that in the previous expressions, the ratios between load and resistance parameters, or load and elasticity

modulus, are dimensionless ratios as both magnitudes are pressures in dimensional terms, so the resulting comparison terms (k_τ , k_σ , k_δ) are mere numbers derived from the decisions made about type and material, but completely independent of the details of the adopted geometry. And on the other hand, the ratios $s/b = \lambda_s$ (plant slenderness) and $l/h = \lambda_l$ (elevation slenderness) are easily identifiable proportions, and also dimensionless.

We see thereby that the usual checks, established in a way that facilitates taking into account purely geometric issues, are actually mere proportion checks: if the proportions given by rules 4.4 to 4.6 are met, the solution will be valid.

4.2.1.3 Proportion Rules and Verification Ranges

It is not only that the verifications 4.4 to 4.6 are proportion rules, but they also provide extremely interesting information: the application ranges for each of these verifications.

It is common that for a given structural solution to a certain problem, several independent verifications must be considered, among which one of them becomes dominant: in our current case, the beam could be dimensioned by deformation, although being redundant in terms of strength. That is, for certain solution ranges, the critical expression for validation can be one among all those considered: if the depth is very small, the stiffness verification will dominate.

The previous expressions allow us to get a comprehensive and much more precise idea of this situation. Let us imagine a solution for which the dominant expression is the deflection, so we are actually using the material below its admissible stresses

in order to keep the curvature controlled, resulting in a section that is more than sufficient to resist the moment. If we proceed with solutions that increase the section depth while maintaining the same deflection, the same curvatures, (for equal span this will equally reduce the slenderness λ_l), this will involve a process of increasing the used stresses, and therefore a reduction in the moment overstrength margin, until we reach the *limit slenderness* $\lambda_l = \lambda_{l(\text{limit})}$ for which the section continues at the deformation limit, but has reached the limit of normal flexural stresses.

In this condition, the two equations 4.5 and 4.6 are simultaneously satisfied in strict equality, so that by dividing the second by the first we will have

$$\lambda_{l(\text{limit})} = \frac{k_\delta}{k_\sigma} \quad (4.7)$$

In this particular case, and remembering the resulting values for these constants in these expressions, we can establish

$$\lambda_{l(\text{limit})} = \frac{6 \cdot 384}{8 \cdot 24000} \frac{E}{\frac{f_{Md}}{q_d/q_k}} = \frac{3}{250} \frac{1}{\varepsilon_s} \quad (4.8)$$

being the maximum service deformation $\varepsilon_s = (f_{Md}/\gamma_d)/E$.

In that situation, it is usual for the section to be overabundant for shear. If we continue increasing the depth (reducing slenderness), always keeping the section at the admissible limit for moment, we can reduce the width more quickly (the resistant module depends on the depth squared and linearly on the width), which results in a reduction of the section area and

therefore of the cost. However, since that area is key in shear resistance, this reduction implies a reduction in the shear resistance margin, until we reach a point similar to the previous one, corresponding to what we call the *base slenderness*, for which the section is equally at the limit for both verifications. We call it base slenderness because, as we will see, a further reduction makes no sense: since the section area must be maintained to maintain shear resistance, from that depth there is no further possible cost reduction. Being in a situation of strict equality for the two equations 4.4 and 4.5, we can divide the second by the first obtaining

$$\lambda_{l(\text{base})} = \frac{k_\sigma}{k_\tau} \quad (4.9)$$

Thus we have references for the application ranges of the verifications, which we can complete at the high slenderness end with the following reflection.

If we increase slenderness while remaining in the stiffness verification conditions, that increase requires a reduction in the plant slenderness λ_s (an increase in width b if we do not modify the spacing s between beams), a reduction that also has a limit, corresponding to the case where the beam width, b , occupies the entire available space, s . At that point $\lambda_{s(\text{min})} = 1$ and therefore we reach a top value for the slenderness of what will now be a slab:

$$\lambda_{l(\text{top})} = \sqrt[3]{k_\delta} \quad (4.10)$$

Such that we can establish the set of ranges for the element's slenderness that determine which expression is dominant in the

verifications:

$$\begin{aligned}
 T : & \quad \lambda_l \leq \lambda_{l(\text{base})} = k_\sigma / k_\tau \\
 M : & \quad \lambda_{l(\text{base})} \leq \lambda_l \leq \lambda_{l(\text{limit})} = k_\delta / k_\sigma \\
 \delta : & \quad \lambda_{l(\text{limit})} \leq \lambda_l \leq \lambda_{l(\text{top})} = \sqrt[3]{k_\delta} \quad (4.11)
 \end{aligned}$$

Sometimes it happens that the moment verification does not apply under any circumstance, as the necessary capacity is covered, either by shear strength requirements or by the inertia needs required to limit deflection. This is often the case with sawn timber pieces in a doubly supported situation, where low tangential stress resistances are combined with appreciable deformability derived from support conditions. In this case the limit is between the application ranges of shear or stiffness verifications. The limit between both is, in that case:

$$\text{limit}(T \leftrightarrow \delta) = \lambda_{l(T\delta)} = \sqrt{k_\delta / k_\tau} \quad (4.12)$$

4.2.1.4 Unit Costs: Influence of Proportion Parameters

We have seen the explanatory value of the verification expressions when established in the proposed geometric format. But that explanatory value goes even further. To do this, we now consider the unit costs of any of the solutions, that is, the cost of the solution adopted per square meter of supported surface.

The total cost of one of the beams is proportional to its volume $V = l \times h \times b$, but we are interested in the unit cost per each unit of the total surface supported by said beam, which is

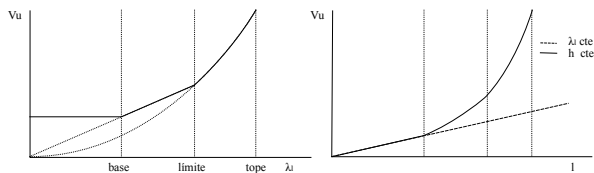


Figure 4.7: Unit costs for beam solutions, depending on the variation laws of spans and depths: **a**: variable slenderness and constant span, **b**: variable span and variable depth for constant slenderness (dashed) or constant depth (solid line).

$l \times s$. We are interested in C_u :

$$C_u \doteq V_u = \frac{lh b}{l s} = \frac{l}{\lambda_l \lambda_s} \quad (4.13)$$

If we now contrast this with the expressions in 4.11, considering that the dimensioning is done at the limit, that is, for the conditions of strict equality in the verification, the cost laws for each of the ranges in the dominant expression will result in:

$$\begin{aligned} T : \quad V_u &= \frac{l}{\lambda_l \lambda_s} = \frac{l}{k_\tau} \\ M : \quad V_u &= \frac{l \lambda_l}{\lambda_l^2 \lambda_s} = \frac{l \lambda_l}{k_\sigma} \\ \delta : \quad V_u &= \frac{l \lambda_l^2}{\lambda_l^3 \lambda_s} = \frac{l \lambda_l^2}{k_\delta} \end{aligned} \quad (4.14)$$

That is, if the dominant expression is shear, since resistance depends on the rectangular section area, regardless of how this

is resolved, the unit cost depends only on the span. However, for the conditions in which the moment is dominant, the cost depends linearly on the span and slenderness. Finally, if stiffness is the dominant condition, the cost depends linearly on the span and quadratically on slenderness. This can be graphically reflected, see figure 4.7, placing the range of required unit volumes, depending on the slenderness conditions under which decisions are established, between its minimum limit, to resist shear with a depth given by the base slenderness and the width required consequently, and a maximum limit, established to solve the problem with the minimum depth slab, defined in that case by deformation requirements, with the condition $\lambda_{l(\text{top})} = l/h_{\text{minimum slab}} = \sqrt[3]{k_{\delta}}$.

$$\frac{l}{k_{\tau}} \leq V_u \leq \frac{l}{\lambda_{l(\text{top})}} = h_{\text{minimum slab}} \quad (4.15)$$

4.2.2 Some Variants in Classical Verifications

One could criticize the previous simple analysis precisely for that: its simplicity, in that the most elementary case has been considered, of a doubly supported beam with a rectangular section.

It is evident that other support conditions are possible, which would exclusively affect the numbers involved in the verifications: the 1/8 factor in calculating the effect moment, or the 5/384 one in calculating the deflection, but this would only affect the concrete resulting values for the verification constants k_{τ} , k_{σ} , or k_{δ} , and their mutual relationships, without affecting

in any way the other reflections on the influence of proportion conditions on verifications or cost.

However, such criticisms can be extended to other more relevant variants in beam designs: the section may not be rectangular, or it could even be a more complex project condition, such as designing in a composite material.

We will therefore show how these reflections or part of them can be generalized through the study of reinforced concrete beam solutions, or their adaptation to hot-rolled steel cross-section series.

4.2.2.1 The Case of Reinforced Concrete

We will consider in the following the three basic verifications: the two for shear and moment resistance, and the one for sufficient stiffness, or deformation limit.

In the case of reinforced concrete solutions, we have two project components: the concrete section $h \times b$, which includes both the necessary compressed head part, which varies along the member, and the part required to separate this head from the tensioned steel reinforcement, also variable, and which in turn can contribute to shear resistance along with the contribution of the shear reinforcement. Both components are elements that contribute to the member's stiffness, in which, for greater complexity, it is necessary to consider the variable cracked situation, again varying along the member, in a variation that also depends on the criteria used to define the reinforcement ratio.

All these conditions might make one think about the impossibility of applying such simple rules as those described in the

preceding sections to single-material members. But we will see that the usual design conditions in reinforced concrete allow us to reduce this complexity to a large extent.

To do this, we will analyze the geometry of the concrete section, $h \times b$, its relationship with the steel reinforcement, and the verification conditions to be employed, seeking the minimum section appropriate to each problem.

Shear. Shear resistance requires two verifications:

- the capacity of the tensioned steel reinforcement which, before the 2021 structural code in Spain, could be considered working in collaboration with the concrete's tangential capacity, a collaboration that is ignored from that code onwards.
- the capacity of the section to accommodate the compressed strut in the concrete

The first verification has no implications regarding the rectangular section geometry: for the same reinforcement inclination, greater depth can simultaneously mean more stirrup length, with greater spacing, which maintains the required reinforcement volume. It is therefore the second verification that can delimit the necessary section.

The width of the compressed strut will define the maximum shear resistance capacity of the section

$$V_{Rd,c1} = f_{1cd} b_0 d \frac{\cot \alpha + \cot \theta}{1 + \cot \theta^2} \quad (4.16)$$

Bending moment. We consider that the beam section is established to ensure adequate resistant capacity to what we call *project moment* or *design moment*: M_D . This moment is a reference value for the entire member and represents a probable value for its sections, corresponding to the most common reinforcement ratio in the member, thus excluding the maximum values that may occur at negative moment peaks for which a punctual greater tensile reinforcement or even compressed reinforcement may be required. This moment is the isostatic one in the case of simply supported members, or a value between half and 60% of the isostatic moment in cases of members with continuity at both ends.

Facing this moment (*effect of actions*), the resistant *capacity* is established by making a decision about the compressed head depth, which will be a fraction γ of the effective depth d . γ is a mechanical ratio, since the resulting compression for this depth would be $\gamma b d f_{cd}$. For efficiency reasons, γ cannot exceed a value of 0.4 or 0.45 and, considering the need for local additional capacity at negative peaks, values of 0.35 are usual. In any case, it supposes a design decision that can be considered relatively independent of the others.

Once established, the moment resistant capacity we will use in the verification results

$$M_{Rd} = f_{cd} b \gamma d \left(1 - \frac{\gamma}{2}\right) d \quad (4.17)$$

Deflection. Considering the reflections in appendix B, we can consider that a good approximation of the cracked member's stiffness, the appropriate equivalent stiffness for its deformation

approximation is

$$EI \approx K \approx 0.5 E_s A_s h^2, \quad \text{with } A_s = \gamma b d \frac{f_{cd}}{f_{yd}} \quad (4.18)$$

so that the deformation limit verification (using $1/400$ of the span) takes the form

$$\frac{\delta}{l} \approx \frac{\alpha}{384} \frac{q_k s l^3}{K} \approx \frac{\alpha}{384} \frac{q_k s l^3}{0.5 E_s A_s h^2} \leq \frac{1}{400} \quad (4.19)$$

an expression in which α , between 1 and 5, will depend on the member's end conditions.

We can deduce the typical reinforcement area used for the moment effect selected as the design moment, $M_{Ed} = M_D$:

$$A_s = \frac{M_{Ed}}{z f_{yd}} = \frac{\beta q_d s l^2}{8} \frac{1}{\left(1 - \frac{\gamma}{2}\right) d f_{yd}} \quad (4.20)$$

and express the deformation verification in the form:

$$\frac{\delta}{l} \approx \frac{\alpha}{\beta} \frac{8}{192} \frac{q_k f_{yd}}{q_d E_s} \frac{l \left(1 - \frac{\gamma}{2}\right) d}{h^2} \leq \frac{1}{400} \quad (4.21)$$

where now, β is the fraction of the isostatic moment used to define the design moment.

In summary:

$$\frac{\delta}{l} \approx \frac{3}{100} \frac{\alpha}{\beta} \varepsilon_s \lambda \leq \frac{1}{400} \quad (4.22)$$

for being $f_{yd} q_k / q_d = f_s$, in service, $d \approx 0.8h$, and using values of $\gamma \in [0.25; 0.45]$, and the typical value $\gamma \approx 0.35$, which leads us to the expression:

$$\lambda_l \leq \frac{1}{12} \frac{\beta}{\alpha} \frac{1}{\varepsilon_s} = \lambda_{l(\text{limit})} \quad (4.23)$$

We see therefore that

- the resistant capacity of expression 4.16 is dimensionally similar to 4.1 since it depends on the section area
- the resistant capacity of expression 4.17 is dimensionally similar to 4.2 since it depends on the product of section area by depth
- the expression 4.19 is dimensionally similar to 4.3: the denominator is a stiffness, although measured from steel it can be transferred to the necessary concrete section, as done in 4.20 and 4.21,

Therefore, when reformulating the verification expressions to what we have called geometric format, identical forms to 4.4 to 4.6 will result, and where only the specific values that can be assigned to the geometric verification constants k_τ , k_σ and k_δ will vary.

We finally see that it is possible to establish the limit slenderness that delimits the verification ranges for moment or deflection, equation 4.23, which provides a tool that facilitates obtaining k_δ from k_σ , if we remember equation 4.7.

4.2.2.2 The Case of Hot-Rolled Steel

This case is less evident as the section decisions are reduced to choosing the cross-section series and determining the depth capable of meeting the requirements within the chosen series,

although a certain parallelism could be considered if we consider less efficient strategies such as placing several pieces in parallel.

The first of the decisions fixes the geometric relationships between different parameters (width, depth, web and flange thicknesses, and corner radii) so that from that point onward, the choices of depth and width are no longer independent, but are linked within the series.

Despite this, it is possible to see that part of the reflections remains useful. To do this, we can consider that, although the geometric qualities in each series are discrete, their relationships can be profitably approximated. In particular, the flexural resistance and sufficient stiffness checks continue to maintain a limit in the easily delimitable range of applicability: if for a particular slenderness both checks are strictly met, this is the limit slenderness. These checks are

$$M_{Ed} = \beta \frac{q_d s l^2}{8} \leq W_{x,pl} f_{yd} = M_{Rd}$$

$$\delta = \frac{\alpha q_k s l^4}{384 E I_x} \leq \frac{l}{400}$$

and being $W_{x,pl} = (W_{x,pl}/W_{x,el}) \times (I_x/(h/2))$, they can be rewritten as

$$\beta \frac{q_d s l^3}{8} \frac{h/2}{l} \leq \frac{W_{x,pl}}{W_{x,el}} I_x f_{yd}$$

$$\frac{\alpha q_k s l^3}{384} \leq E I_x \frac{1}{400}$$

so that at this limit, for which we would have $l/h = \lambda_{l(\text{limit})}$ and the equality conditions in both expressions, we can divide the

second equation by the first and obtain

$$\lambda_{l(\text{limit})} = \frac{3}{50} \frac{W_{x,el}}{W_{x,pl}} \frac{\beta}{\alpha} \frac{1}{\varepsilon_s} \quad (4.24)$$

The expression is similar to the one already established for reinforced concrete, see 4.23. For lower slenderness, the dominant expression will be the resistance one, while for higher slenderness, the stiffness check will dominate.

4.2.3 Problems Derived from Self-Weight

Up to this point, the effect of self-weight has not yet been considered in the checks, assuming, as is relatively usual, that it is no more than a small fraction of the total load, which is directly included through an approximate excess value.

But it is clear that, if the self-weight is a relevant part of the problem, this approximation is not sufficient, and therefore, in the following sections, we will address its formulation.

A first reflection forces us to slightly review expressions 4.4 to 4.6. Indeed, the load is no longer just the live load or the non structural dead load per supported surface unit (q_k or q_d depending on whether we consider its characteristic or design value) but it will be necessary to add the one derived from the self-weight. For the typology adopted there, the repercussion per unit surface would be the quotient between the total beam weight, $lbh\rho$, and the supported area, ls , where the specific weight of the material should be used in its characteristic ρ_k or design ρ_d value, as appropriate.

That is, instead of q_d we will have $q_d + \frac{lbh\rho_d}{l_s} = q_d \left(1 + \frac{\rho_d l}{q_d \lambda_l \lambda_s}\right)$. This is equivalent to using instead of q_d an amplified value: μq_d being $\mu = 1 + \frac{\rho_d l}{q_d \lambda_l \lambda_s}$ so that the geometric expressions 4.4 to 4.6 are transformed to the following:

$$T : \quad \frac{sl}{bh} \leq \frac{4}{3} \frac{f_{Vd}}{\mu q_d} \equiv \lambda_s \lambda_l \leq \frac{k_\tau}{\mu} \quad (4.25)$$

$$M : \quad \frac{sl^2}{bh^2} \leq \frac{8}{6} \frac{f_{Md}}{\mu q_d} \equiv \lambda_s \lambda_l^2 \leq \frac{k_\sigma}{\mu} \quad (4.26)$$

$$\delta : \quad \frac{sl^3}{bh^3} \leq \frac{384}{24000} \frac{E}{\mu_k q_k} \equiv \lambda_s \lambda_l^3 \leq \frac{k_\delta}{\mu_k} \quad (4.27)$$

And similarly, the limits that result for the ranges of dominant expressions are now

$$T \leftrightarrow M : \lambda_{l(\text{base})} = k_\sigma / k_\tau \quad (4.28)$$

$$M \leftrightarrow \delta : \lambda_{l(\text{limit})} = \frac{k_\delta \mu}{k_\sigma \mu_k} \quad (4.29)$$

$$\text{tope} : \lambda_{l(\text{tope})} = \sqrt[3]{k_\delta / \mu_k} \quad (4.30)$$

We see, therefore, that there is a modification in the values of the required verification constants, although we are still in proportion checks and where, moreover, the dominant ranges for dominant expressions remain easy to determine, some of them being identical or very close to those already obtained without considering the effects of self-weight.

But, insofar as the expressions are now dependent on the term μ that expresses the *load amplification* from the useful q

to the total μq , which includes the self-weight, how can we get an idea of this load amplification term?

To do this, we begin the reflection by going to the extreme case: the case in which the piece is capable of supporting only its own weight. Subsequently, we will analyze what happens when the piece has a margin of resistance to support additional useful or payloads to its weight.

4.2.3.1 When Everything is Self-Weight: The Limit Size

If the load is only the self-weight, we have that $\mu q = \frac{\rho l}{\lambda_s \lambda_l}$ and therefore the expressions 4.25 to 4.27 can be rewritten as

$$T : \quad \frac{sl}{bh} \leq \frac{4}{3} \frac{f_{Vd} \lambda_s \lambda_l}{\rho_d l} \equiv l \leq \frac{4}{3} \frac{f_{Vd}}{\rho_d} = L_\tau = k_\tau \frac{q_d}{\rho_d} \quad (4.31)$$

$$M : \quad \frac{sl^2}{bh^2} \leq \frac{8}{6} \frac{f_{Md} \lambda_s \lambda_l}{\rho_d l} \equiv l \leq \frac{8}{6 \lambda_l} \frac{f_{Md}}{\rho_d} = L_\sigma(\lambda_l) \quad (4.32)$$

$$\delta : \quad \frac{sl^3}{bh^3} \leq \frac{384}{24000} \frac{E \lambda_s \lambda_l}{\rho_k l} \equiv l \leq \frac{0,016}{\lambda_l^2} \frac{E}{\rho_k} = L_\delta(\lambda_l^2) \quad (4.33)$$

That is, these are now size checks: if the span l is less than or equal to a certain maximum size L , which in the cases of flexure or deflection checks depends linearly or quadratically on slenderness, then the structure is viable and will support its own weight. Larger sizes are unfeasible. It is worth noting now that the dimensionality of the expressions is given by the ratio between a pressure (the resisted stress or the modulus of elasticity) and a specific weight, from which a length results.

In particular, for the central expression, the one for resistance to normal stresses derived from bending, this magnitude, f_M/ρ or also σ/ρ is a magnitude we call *scope*, $\mathcal{A}$ (the symbol having been selected from the original spanish name for scope: “Alcance”), and is equivalent to the length of an arbitrary constant section cable hanging and at the limit of resistance to its own weight: because at this limit the expression $\sigma A = \rho A l$ becomes $\sigma_{\text{limm}} A = \rho A \mathcal{A}$, from where $\mathcal{A} = \sigma_{\text{limm}}/\rho$.

These are, therefore, clearly geometric checks, although now they are size checks, dependent on the slenderness of the piece and the various alternatives of the material’s scope.

4.2.3.2 The Inclusion of Self-Weight Effect in Checks

It is now straightforward to determine the load amplification factor μ required to incorporate the self-effect in the checks. Indeed, in each of the checks 4.31 to 4.33, if l is at the limit size, the margin in the comparison *pressure* (limit tension) used in the resistance expression is exhausted by the self-weight, such that a reduction in that value implies a reduction in the limit size and, similarly, a reduction in size implies a reduction in the pressure (tension) induced by the self-weight effect.

Considering the case of the flexure check, 4.32, the fraction of the resistant capacity $f = f_{Md}$ used to resist the self-weight, f_p/f , turns out to be proportional to the fraction that the piece size represents in relation to its maximum size, l/L . It is less than the total capacity f , leaving a margin of possible stresses $f_q = f - f_p$ available to resist payloads. And therefore, given

the proportionality between load and stresses

$$\frac{l}{L} = \frac{f_p}{f} = \frac{f - f_q}{f} = 1 - \frac{f_q}{f} = 1 - \frac{1}{\mu} \quad (4.34)$$

$$\mu = \frac{1}{1 - l/L} \quad (4.35)$$

In this expression, the ratio l/L can be identified as the key parameter for the influence of self-weight, and it is therefore useful to give it its own denomination: the *size ratio* or the *scale*.

The last expression for μ now allows us to use the check forms 4.25 to 4.27 and the limits to their application ranges, 4.28 to 4.30.

4.2.4 Basic Shape Parameters

If we now review the verification expressions, both for external load (equations 4.25, 4.26, and 4.27) and for self-weight only (equations 4.31, 4.32, and 4.33), also considering the corresponding load amplification factor (equation 4.35), as well as the reflections made on application ranges, costs, etc., we can identify the basic shape parameters from a structural perspective.

If we consider them in order of their importance (their role in the expressions, etc.), we have the following list, which is schematized in figure 4.8:

scheme: characterized by the values of $k_\tau, k_\sigma, k_\delta$, which are independently part of all expressions.

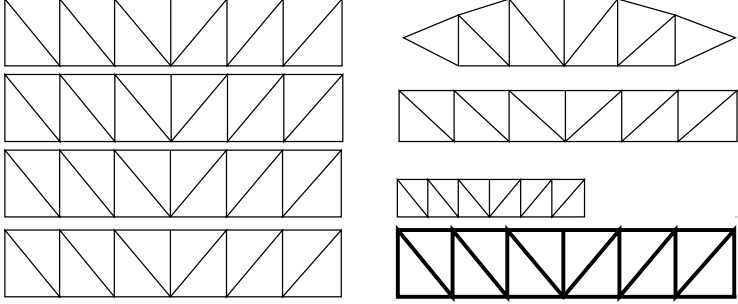


Figure 4.8: Shape parameters: scheme, proportion, size, and thickness. In each row in the figure, only one of these parameters changes.

proportion: defined by λ_l , which systematically appears and is also determinant in size expressions and thus in the impact of self-weight.

size: l , (or the size ratio, l/L), responsible for the effect of self-weight in the total load.

thickness: characterized by λ_s , which only appears in the resistance and stiffness verification expressions, without any influence on size expressions, and therefore does not intervene in the characterization of the self-weight effect.

In subsequent sections, we will check that this characterization can be generalized. But for this, we need more general ways to define shapes and their costs.

4.3 Structural Quantity and Structural Efficiency

In this section, we establish and explore the application of the concept of *structural quantity* or *stress volume*, a concept that will prove extremely useful for generalizing the ideas already advanced in the study of beam structures.

For this generalization, we will need to address arbitrary shapes for the analyzed structures and relate them to some magnitude capable of effectively measuring the consumption or cost invested in materializing such structures.

4.3.1 Structural Quantity

Throughout history, many magnitudes have been used to measure the investment in resources or materials necessary to set up a structure, such as weight, volume, or monetary cost. Currently, for sustainability reasons, it is common to consider greenhouse gas emissions or the possibilities and cost of reincorporating the employed products into a closed material cycle. Each of these magnitudes has advantages and disadvantages, but they all suffer from a fundamental initial problem: the need to consider a specific material for their determination. Without that material, these magnitudes cannot be computed. Therefore, it is interesting to consider an alternative magnitude that only considers the most elementary components of structural problems, namely, those that seek to generate forms capable of materializing the conditions of force equilibrium at a distance, loads, and

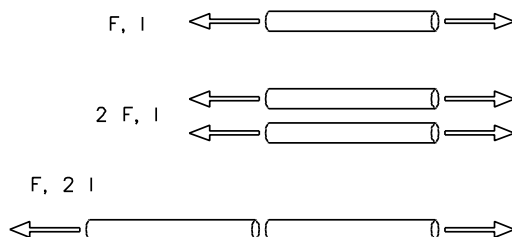


Figure 4.9: Concept of structural quantity: the problems in the two lower rows require double the investment of that required for the problem in the first row.

reactions.

Figure 4.9 shows three of these problems and exemplifies that structural solutions must employ resources both by the magnitudes of the forces to be balanced and by the distances between these. Such that a magnitude capable of measuring structural solutions should be linked exclusively to these two dimensions. The figure recalls Descartes' definition of *force* as the basis of his conception of statics—a concept equivalent to our current concept of work—when he affirmed that *the same force that can lift a weight, for example of 100 pounds to a height of two feet can also lift one of two hundred pounds to the height of one foot*. (Descartes 1638a and Descartes 1638c, cited in a

fascinating reflection by Duhem 1905–06, volume 1, pp. 339 et seq. although this formulation by Descartes is even earlier: Descartes 1637 and Descartes 1638b, as noted by Roux 2004. The work of Descartes, including his correspondence, is available in Tannery 1897..1913.) His definition, which would have given rise to a mechanics view based on work, did not prosper.

To narrow down the problem, we initially consider structures composed of bars subjected to axial efforts and define in these the *structural quantity* as the integral of the absolute value of efforts in the bars by their corresponding distances:

$$W = \int |N| dl \quad (4.36)$$

It is interesting to compare this magnitude with others very common in structure optimization problems:

$$\text{Material Volume: } \int dV = \int A dl = \int |N/\sigma| dl \quad (4.37)$$

$$\text{Stress Volume: } \int |\sigma| dV = \int |\sigma| A dl = \int |N| dl \quad (4.38)$$

$$\text{Strain Energy: } \sim \int \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = \int \varepsilon \sigma A dl = \int |N \varepsilon| dl \quad (4.39)$$

In the last expression, of energy or deformation volume¹ in which, to facilitate comparison, we have ignored the 1/2 factor of the elastic case, we consider the fact that in uniaxial states, stresses and strains have the same sign. It is equally interesting

1. *compliance* in Anglo-Saxon literature.

to see that the material quality used in this expression is one of the two key geometric qualities of the material, the maximum unit deformation in service, a quality that we have already identified as key in stability problems, see the reflections associated with expressions 3.8 and 3.9.

It can be seen that the structural quantity, or stress volume, is a magnitude independent of the material, and that is situated in an intermediate position between two of the most common magnitudes in structure optimization: the structure volume — or corresponding weight — or its strain energy. In the first, efforts are weighted by the inverse of stresses, while in the second they are weighted by deformations — proportional to stresses. In both cases, this weighting requires the adoption of a reference material, which is not the case in the stress volume quantity, as it directly results from the bar efforts and their lengths.

It is worth adding a consideration of great interest: in the case of structures dimensioned so that in all their sections the material is working at its maximum stress, the three magnitudes result in proportional values: if σ or ε adopt the same value in all sections, these values become the proportionality factor linking one magnitude to another.

All of the above forms part of the reasons that, as we will see, make structural quantity a key magnitude in measuring structural consumption and prove to be extremely productive.

We will therefore establish a precise framework of definitions and relationships for working with this magnitude.

Definition 4.1 (Maxwell’s Problem) ² *Defined as the prob-*

2. Denomination chosen in honor of the clear formulation established in

lem of adopting a structure for a system of forces (external to this) in equilibrium, and fixed in position, direction, and magnitude.

The forces are in equilibrium —of forces and moments— in space but this equilibrium requires a structure for its materialization. This set of forces includes both actions and reactions, without distinguishing between them.

Definition 4.2 (Maxwell’s Structure) *A structure that materializes the equilibrium of forces in a Maxwell’s problem through bars subjected to uniaxial effort, that is, under tension or compression.*

Definition 4.3 (Strict Structure) *A Maxwell’s structure in which all bars are subjected to the same stress, either in tension or compression, a stress that will correspond to the maximum admissible for the considered material.*

For the moment, we will consider materials for which the maximum stress in tension and compression are equal in absolute value, although later we will see materials with asymmetric behavior regarding these stress signs.

Under such conditions, we can reiterate the definition established in expression 4.36.

Definition 4.4 (Structural Quantity) *The integral of the absolute value of stresses extended to the entire volume of a structure composed of pieces working uniaxially. Equivalent to the*

Maxwell 1870, page 13 et seq.

sum, for all bars constituting the structure, of the product of the absolute value of the effort in each bar by its length.

4.3.1.1 Relationship with Other Magnitudes

Given that $dV = A ds$, we can relate, as we have seen, the quantity of structure with its volume and weight:

$$V = \int A ds = \int \frac{|N|}{|\sigma_s|} ds = \frac{W}{|\sigma_s|} \geq \frac{W}{\sigma} \quad (4.40)$$

$$W = \int |N| ds = \int A |\sigma_s| ds = \int |\sigma_s| dV \leq \mathcal{W} = \sigma V \quad (4.41)$$

$$P = \rho V = \rho \frac{\mathcal{W}}{\sigma} = \frac{\mathcal{W}}{\sigma/\rho} \quad (4.42)$$

where σ_s represent the working stress at each position s , and $|\sigma_s|$ the volume-weighted average of its absolute value, with σ being the verification stress of the material considered, and ρ its specific weight.

In the preceding expressions, we see the relationships between the magnitudes

P Weight of the structure (in force units)

V Volume of the structure (in volume units)

$\mathcal{W}$ Structural volume or structural capacity (in work units)

W Quantity of structure or stress volume (in work units)

In the expression 4.42 appears the second key geometric quality of materials, the ratio σ/ρ which, as we have already

seen in section 4.2.3.1, is a length we call the *scope* of the material, crucial in self-weight problems, as we saw there and as we will generalize in section 4.3.4.

4.3.1.2 Components of the Quantity of Structure

Breaking down the quantity of structure into its different components provides, as we will see, perspectives of great utility.

Tensile and Compressive Quantity. The first decomposition results from separately considering the tensile regions N^+ and the compressive regions N^- , thus separately considering the quantities of tensile structure W^+ or compressive structure W^- .

$$W^+ = \int |N^+| \, ds, \quad W^- = \int |N^-| \, ds, \quad (4.43)$$

$$W = \int |N| \, ds = \int_+ N^+ \, ds + \int_- |N^-| \, ds = W^+ + W^- \quad (4.44)$$

Horizontal and Vertical Quantity. Since gravitational loads are typically considered dominant loads, it is useful to distinguish between horizontal forces $F^=$ and vertical forces $F^\parallel$ (or orthogonal forces $F^\perp$ and parallel forces $F^\parallel$ to the dominant direction) and for similar reasons, distinguish between horizontal structural quantity $W^=$ and vertical structural quantity $W^\parallel$, using the parallelism that always exists between the directions

of efforts and bar lengths:

$$\begin{aligned}
 W &= \int |N| \, dl = \int |\mathbf{N} \cdot d\mathbf{l}| = \int (|N_x \, dx + N_y \, dy + N_z \, dz|), \\
 W &= \int |N_x| \, dx + \int |N_y| \, dy + \int |N_z| \, dz, \quad \text{where } \mathbf{N} \parallel d\mathbf{l}
 \end{aligned} \tag{4.45}$$

$$W^{\parallel} = \int |N_z| \, dz; \quad W^{\perp} = \int |N_x| \, dx + \int |N_y| \, dy \tag{4.46}$$

$$W = W^{\perp} + W^{\parallel} = W^{\perp} + W^{\parallel} \tag{4.47}$$

4.3.2 Structural Quantity Theorems

We can now formulate a set of theorems of the utmost relevance for characterizing the behavior of structures.

Theorem 4.1 (Maxwell's Theorem) *For every Maxwell structure solving the same Maxwell problem, the Maxwell number $M = \int N \, ds$ is a function of the applied forces and their points of application, independent of the structure's shape, being the same for all of them.*

Let's see the proof.

Let a space containing all structures that solve a problem, for fixed loads $\mathbf{F}_i$.

Let a uniform expansion with factor $(1 + e)$, around a point.

Let $\mathbf{e}_i$ be the displacement vector of each point i .

By the principle of virtual work and the consequent equality between the work of external forces and internal forces we will

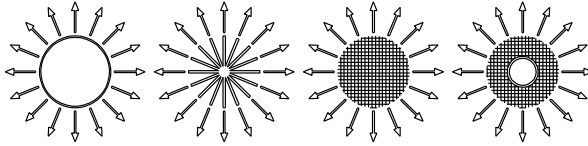


Figure 4.10: Identity between structures tensioned for being minimal

have

$$\begin{aligned} \sum \mathbf{F}_i \mathbf{e}_i &= \sum F_{xi} X_i \mathbf{e} + \sum F_{yi} Y_i \mathbf{e} + \sum F_{zi} Z_i \mathbf{e} = \\ U &= \int \sigma e \, dV = e \int \sigma \, dV = e \int N \frac{dV}{A} = e \int N \, ds \end{aligned}$$

so that

$$M = \int N \, ds = \sum F_{xi} X_i + \sum F_{yi} Y_i + \sum F_{zi} Z_i. \quad (4.48)$$

Corollaries of Maxwell's Theorem.

- A Maxwell structure that is only tensioned or only compressed is minimal, which allows considering identical all those that solve the same problem in that way. (figure 4.10).
- Minimizing the tensioned (or compressed) part of a structure implies minimizing the structure by having to maintain the difference between them.
- Structures can be classified by the value of M they solve (figure 4.12):

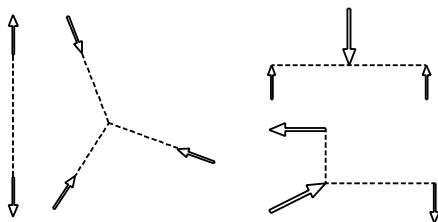


Figure 4.11: Structure types and Maxwell number

Tension $W \approx \geq M > 0$,

Compression $-W \approx \leq M < 0$,

Flexure $W \neq M \approx 0$.

Theorem 4.2 (Michell's Theorem) *A strict structure reaches the limit of economy —minimum structure— if the space in which it is situated can be subjected to a —virtual— deformation such that the unit elongations or shortenings increase equally in all pieces —with the same value, and in the sign of the original deformation—, and in value not less than the unit length change of any other element in the space.*

If the considered space extends to infinity in all directions, the volume of such a structure is minimum in relation to all possible designs, and otherwise minimum in relation to the designs that can be considered included in the same finite contour.

Let a space containing all structures that solve a problem, for fixed loads $\mathbf{F}_i$.

Let an arbitrary virtual deformation be established such that the absolute value of the principal deformations at every point in the space are less than a certain given value e . That is, in every direction of the space, the absolute value of the unit elongation or shortening μ is: $|\mu| \leq e$.

By the principle of virtual work, the variation in the deformation energy of each structure in said deformation δU is equal to the work done by external forces and therefore will be equal for all of them.

And therefore

$$\delta U = \int \mu \sigma \, dV = \int \mu \sigma A \, ds \leq \int |\mu| |N| \, ds \leq \int e |N| \, ds;$$

that is, $\delta U \leq e \int dW$. (4.49)

But it will occur that in the structures in which the Michell conditions are met, all the sign of the preceding expressions will be of equality, so these structures will have as deformation energy the product of their structure quantity by that unit deformation e and will therefore be minimal:

$$\delta U = \delta U_{e(\min)} = e \int dW_{e(\min)} \leq e \int dW \quad (4.50)$$

Corollary

Optimal structures can be characterized by a test deformation:

- Structures with stresses of equal sign: the test deformation will be an expansion or contraction of space: coincides with

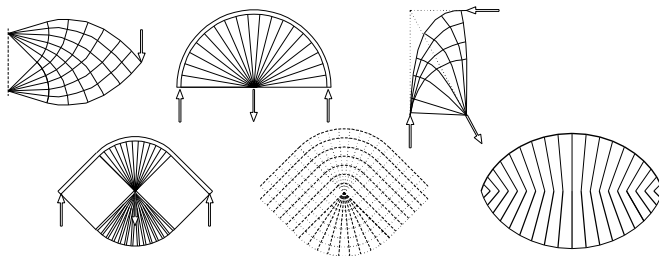


Figure 4.12: Michell Structures, Optimal or Quasi-Optimal in Bending. The bottom row shows the optimal design for point load and its test deformation, as well as a quasi-optimal approximation for uniform load.

the first corollary of Maxwell's theorem

- Structures formed by orthogonal bars before and after deformation: the test deformation is expansion and contraction in those directions

Theorem 4.3 (Rigidity Theorem) *Among all strict structures solving the same Maxwell problem, all those with equal structure quantity have equal deformation; the deformation increases with the structure quantity of the chosen scheme and, therefore, the minimum structure —the one with the least structure quantity— is also the most rigid.*

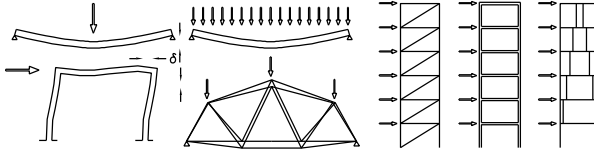


Figure 4.13: Weighted Deflection: one can measure the structure's deformation by the loss of potential energy of loads in the corresponding displacement.

Let a weighted measure of deflection be of the following form:

$$\delta_e = \sum \alpha_i \delta_i.$$

And use the loads themselves as weighting factors (see figure 4.13):

$$\delta_e = \sum P_i \delta_i$$

The measure of the resulting deformation is the loss of potential energy of the loads, equal to the strain energy plus the complementary energy of the structure (figure 4.14).

And therefore

$$\delta_e = U = \int \sigma \varepsilon \, dV = \int \sigma \varepsilon \frac{dW}{\sigma} = \int \varepsilon \, dW \quad (4.51)$$

Corollaries of the Rigidity Theorem.

- To minimize a structure, it is sufficient to seek the scheme that minimizes its deformation—with equal material utilization—

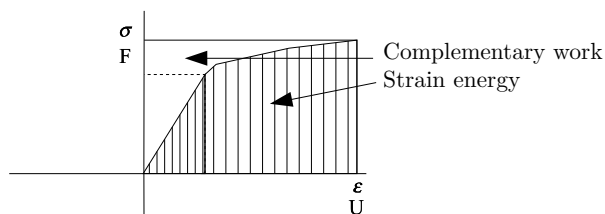


Figure 4.14: Strain and Complementary Energies



Figure 4.15: Equivalent Structures in Bending, because, if equal depth and same material's strains are used, the curvature is identical and therefore also the loss of potential energy of the loads.

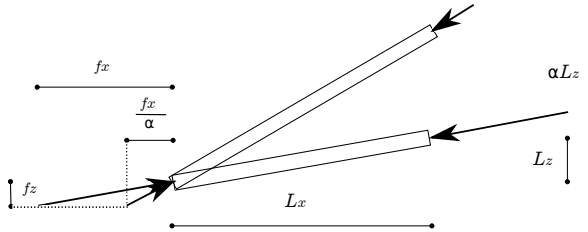


Figure 4.16: Effects of changing beam depth while maintaining the vertical component to balance the loads.

- Structures with equal loss of potential energy are equivalent. (see figure 4.15)
- If a strict structure shows a deformation α times greater than another established for the same loads, achieving in the first the same deformation as in the second will require accepting a consumption α^2 times the second, this due to being α times less efficient, and by needing to reduce stresses further by a factor of α to equalize the deformation.
- Structural materials in building construction are limited, except for strong oversizing, to unit deformations less than 2 per thousand, since usual slenderness and deflection limits prevent the exploitation of larger deformations³.

Theorem 4.4 (Aroca's Theorem, or Optimal Depth Theorem)

3. This statement can be demonstrated by reference to the resulting deformations in optimal or quasi-optimal structures in bending and the corresponding efficiency losses for the slenderness of typical types.

Starting from a structure subjected to vertical loads and reactions, if we consider all possible forms that can be obtained by maintaining the original topology and load positions, and alter the geometry through affine transformations that only affect the structure's depth, the optimal structure among these will be the one that satisfies the condition of equalizing the horizontal and the vertical structural quantities.

Let's examine what happens when changing the depth while ensuring equilibrium is maintained (see figure 4.16). We consider two phases:

Phase 0: Original dimensions, loads, and forces are maintained.

Phase 1: Changing depths (factor α) and horizontal forces (factor $1/\alpha$).

Checking Equilibrium and Geometric Compatibility

Phase 0.

The forces on the nodes are in equilibrium with the sum of efforts, with efforts parallel to the bar directions:

$$\begin{aligned} F_j^{\parallel} &= \sum f_i^{\parallel} \\ F_j^{\perp} &= \sum f_i^{\perp} \\ \frac{f_i^{\parallel}}{f_i^{\perp}} &= \frac{L_i^{\parallel}}{L_i^{\perp}} \end{aligned}$$

These efforts and lengths allow measuring the structural quantities

$$W^{\parallel}, W^{\perp}$$

Phase 1.

Transform the structure by multiplying the depth by a factor α , maintaining horizontal dimensions and reducing horizontal force and effort components by that factor. That is, for each variable V , we obtain the transformed $V(\alpha)$ as follows:

$$\begin{aligned} F_j^{\parallel}(\alpha) &= F_j^{\parallel}, & f_i^{\parallel}(\alpha) &= f_i^{\parallel} \\ L_i^{\parallel}(\alpha) &= \alpha L_i^{\parallel}, & L_i^{\perp}(\alpha) &= L_i^{\perp} \\ F_j^{\perp}(\alpha) &= \frac{1}{\alpha} F_j^{\perp}, & f_i^{\perp}(\alpha) &= \frac{1}{\alpha} f_i^{\perp} \end{aligned}$$

This maintains both equilibrium and alignment of efforts and bar directions:

$$\begin{aligned} F_j^{\parallel}(\alpha) &= \sum f_i^{\parallel}(\alpha) \\ F_j^{\perp}(\alpha) &= \sum f_i^{\perp}(\alpha) \\ \frac{f_i^{\parallel}(\alpha)}{f_i^{\perp}(\alpha)} &= \frac{L_i^{\parallel}(\alpha)}{L_i^{\perp}(\alpha)} \end{aligned}$$

Thus, the resulting structural quantities are:

$$W^{\parallel}(\alpha) = \alpha W^{\parallel}; \quad W^{\perp}(\alpha) = W^{\perp}/\alpha \quad (4.52)$$

Under these conditions, the product between both components remains constant, and the minimum will result from the minimum sum of two components with a constant product. The problem is analogous to the rectangle with minimum semi-perimeter



Figure 4.17: Structure with Curved Chords

for equal area, a problem whose solution is the square.

$$W^{\parallel} \cdot W^{\perp} = W^{\parallel}(\alpha) \cdot W^{\perp}(\alpha)$$

$$\min W = \min(W^{\parallel} + W^{\perp}) \implies W_{e(\min)}^{\parallel} = W_{e(\min)}^{\perp} \quad (4.53)$$

$$\min W = 2\sqrt{W^{\parallel} \cdot W^{\perp}} \quad (4.54)$$

Thus, the theorem is proven.

Corollaries of Aroca's Theorem

- The optimal slenderness can be obtained from the horizontal and vertical structural quantity values:

$$\begin{aligned} \alpha_o &= \frac{W_o^{\parallel}}{W^{\parallel}} = \frac{\sqrt{W^{\parallel} W^{\perp}}}{W^{\parallel}} = \sqrt{\frac{W^{\perp}}{W^{\parallel}}} \\ \lambda_o &= \frac{\lambda}{\alpha_o} = \lambda \sqrt{\frac{W^{\parallel}}{W^{\perp}}} \end{aligned} \quad (4.55)$$

- Flexural structures with curvature in both chords —and which occupy both sides of the line or plane connecting the supports— are approximately $\sqrt{2}$ times more efficient than those situated on only one side of this line (see figure 4.17)

The proof of the second statement is as follows:

Consider a tied arch on the horizontal, with the deck suspended by hangers, and assume it is at its optimal slenderness. We use $W^{+\parallel}$ as the unit of measure for structural quantity. By Maxwell's theorems (as it is a flexural structure with $M = 0$) and Aroca's theorem (as it is of optimal slenderness), the compressed and tensioned structural quantities are equal, as are the vertical and horizontal components. The tensioned quantity of arches and hangers corresponds to the compressed quantity of the arch, which can be split into horizontal and vertical parts, corresponding to the horizontal and vertical parts of the tie and hangers in tension. Such that

$$\begin{aligned} W^{+\parallel} &= W^{+=} = W^{-\parallel} = W^{-=} = 1 \\ W &= W^{+} + W^{-} = W^{+\parallel} + W^{+=} + W^{-\parallel} + W^{-=} = 4 \\ W^{\parallel} &= W^{=} = 2 \end{aligned} \tag{4.56}$$

We unfold, rotate, and reconnect the arch, causing the horizontal tie to disappear. The quantities are now $W_{*}^{=} = 1$, $W_{*}^{\parallel} = 2$, $W_{*} = W_{*}^{=} + W_{*}^{\parallel} = 3$, which are not optimal.

The optimum will be achieved with

$$\begin{aligned} W_o^{\parallel} &= W_o^{=} = \sqrt{W_{*}^{\parallel} W_{*}^{=} } = \sqrt{2} \\ W_o &= 2\sqrt{2} \\ \lambda_o &= \lambda_{*} \sqrt{\frac{W_{*}^{\parallel}}{W_{*}^{=} }} = \lambda_{*} \sqrt{\frac{2}{1}} = \lambda_{*} \frac{1}{\sqrt{2}} \end{aligned} \tag{4.57}$$

4.3.3 General Expression for the Structural Quantity

Based on what we have seen, the structural quantity is measured in work units, and its general expression can be characterized by the following properties

- It is proportional to the total load Q , since if we change the charge—and the corresponding reactions— without changing the geometry, the efforts grow proportionally.
- It is proportional to the basic dimension of the structure, its span l or its size. Because if we maintain the shape and proportions and the node-applied loads while changing the size, the bar lengths grow without alteration of the efforts.
- The expression that defines this work magnitude is, therefore, of the type $W = \nu Q l$

Suppose that for this form of expression, we consider cases with dominant bending. The optimal depth situation, according to Aroca's theorem is

$$W_o = W_o^- + W_o^{\parallel} = \nu_o Q l \quad \text{with } W_o^- = W_o^{\parallel}. \quad (4.58)$$

In this situation, the slenderness is optimal, λ_o , while for other depths we would have $\lambda = \alpha \lambda_o$, varying the depth with $\frac{1}{\alpha}$ (assuming $\alpha > 1$, the depth decreases).

Thus, the vertical amount of structure will vary—decrease—with $1/\alpha$ while the horizontal one will—increase—with α .

Therefore

$$\begin{aligned}
 W &= W^= + W^{\parallel} = \alpha W_o^= + \frac{1}{\alpha} W_o^{\parallel} \\
 W &= \frac{W_o}{2} \left(\alpha + \frac{1}{\alpha} \right) \\
 W &= \frac{W_o}{2} \left(\frac{\lambda}{\lambda_o} + \frac{\lambda_o}{\lambda} \right) \\
 W &= \frac{W_o}{2} \frac{\lambda}{\lambda_o} \left(1 + \frac{\lambda_o^2}{\lambda^2} \right). \tag{4.59}
 \end{aligned}$$

The last expression makes sense to the extent that the usual slenderness in bending is appreciably higher than the optimal solution, which makes the second term in the parentheses, $(\lambda_o/\lambda)^2$ generally very small relative to the first.

Now, being the expression for the optimal case of the form $W_o = \nu_o Q L$, considering possible slenderness changes we can write it as $W_o = \phi_o \lambda_o Q l$, and then we have as an expression for the structural quantity for any slenderness

$$\begin{aligned}
 W &= \frac{\phi_o \lambda_o}{2} Q l \frac{\lambda}{\lambda_o} \left(1 + \frac{\lambda_o^2}{\lambda^2} \right) = \frac{\phi_o}{2} \left(1 + \frac{\lambda_o^2}{\lambda^2} \right) \lambda Q l \\
 &= \phi_\lambda \lambda Q l \tag{4.60}
 \end{aligned}$$

In this expression, $\phi_\lambda = \phi_o (1 + (\lambda_o/\lambda)^2)/2$ is a function dependent on slenderness that varies between ϕ_o and $\phi_o/2$, but for usual slendernesses, high, can be approximated with $\gamma \approx \phi_\lambda \rightarrow \phi_\infty = \phi_o/2 = \nu_o/2\lambda_o$ such that the structural quantity is

represented in these cases as

$$W = \nu Ql \approx \gamma \lambda Ql \quad (4.61)$$

where $\nu = \phi_\lambda \lambda \approx \gamma \lambda$ is the *Michell number* of the structural geometry, and where $\gamma = \nu_o/2\lambda_o$ is a constant of the structural form not dependent on either size or slenderness, since it proceeds from fixed numbers determined from the optimal slenderness variant for that form.

That is, the structural quantity in bending problems is measured by the product of slenderness by total load and total span, affected by a coefficient of inverse effect to the structural form's efficiency.

If additionally the problem becomes dominated by deformation condition, by stiffness, by exceeding the slenderness λ over the configuration's limit slenderness, $\lambda_{e(\text{lim})}$, the necessary structural consumption will now be $W \times \lambda/\lambda_{e(\text{lim})}$ due to the need to reduce working stresses by the factor $\lambda/\lambda_{e(\text{lim})}$, so that costs will become dependent on the square of slenderness.

4.3.4 Structural Quantity and Self-Weight

We will consider strict structures for which the structural volume or capacity $\mathcal{W}$ is equal to the structural quantity W , and consider these made with material of specific weight ρ . Under these conditions, the structure's weight is

$$P = \frac{\rho}{\sigma} W; \quad \text{with } W = \gamma \lambda Ql \quad (4.62)$$

Insurmountable Size or Scope

We now consider the problem of the structure's scope or insurmountable size, that is the size L at which the structure supports only itself: it is strict for its own weight having reached the viable size limit. We consider as a valid approximation that the spatial distribution of this weight is similar to the one considered for the load Q in expression 4.62 due to its similarity to the distribution of loads applied in the structure's common use forms.

We will have

$$W = \gamma \lambda P L = \gamma \lambda W \frac{\rho}{\sigma} L$$

and solving for L

$$L = \frac{1}{\gamma \lambda} \frac{\sigma}{\rho} = \frac{1}{\nu} \frac{\sigma}{\rho} \quad (4.63)$$

We see that in this expression, that the crucial magnitude that provides the dimension is the *material scope*

$$\mathcal{A} = \frac{\sigma}{\rho} \quad (4.64)$$

and the other parameter determining the limit size is the inverse of the Michell number. With sizes l smaller than L we could employ proportionally lower stresses (capacities) in the material. Therefore, the fraction of available resistance employed in supporting its own weight is,

$$\chi = \frac{l}{L} = \frac{\sigma_p}{\sigma} = \frac{\nu l}{\sigma / \rho} \quad (4.65)$$

We call this quotient χ the *scale* or the *size ratio*, which represents the size relative to the insurmountable size of the structural type considered.

Load Amplification

We now assume the existence of loads additional to self-weight. The load will be $Q = P + R$, the self-weight plus additional load or live load, and we continue considering load and weight as homologous in their distribution over the structure.

Under these conditions

$$\frac{Q}{R} = \frac{\sigma}{\sigma - \sigma_p} = \frac{1}{1 - \frac{\sigma_p}{\sigma}} = \frac{1}{1 - \frac{l}{L}}$$

$$\frac{Q}{R} = \frac{1}{1 - \chi} \quad (4.66)$$

and therefore the total load Q , including self-weight, P , is the useful or payload R amplified by the factor $1/(1 - \chi)$. Here we can appreciate that the influence of size on the structure's weight is exclusively through the *size ratio* or *scale*, the relationship of the adopted size to the insurmountable size.

Structural Form Efficiency. A classic measure of structural efficiency has been the yield, as a ratio between the supported load (the payload) and the self-weight (the payload-to-weight

ratio):

$$r = \frac{R}{P} = \frac{Q - P}{P} = \frac{1 - \frac{P}{Q}}{\frac{P}{Q}} \quad (4.67)$$

$$r = \frac{R}{P} = \frac{1 - \chi}{\chi} = \frac{1}{\chi} - 1 \quad (4.68)$$

It is not a good measure, as we see it depends on the material employed and not only on the structure's geometry, since χ depends on the material's scope:

$$\frac{1}{\chi} = \frac{L}{l} = \frac{\sigma/\rho}{l} \frac{1}{\gamma\lambda}$$

A better measure for the efficiency is, therefore, the inverse of the Michell number, which involves both the geometric configuration of the structure and its slenderness, but has not any reference to the material:

$$\frac{1}{\gamma\lambda} = \frac{1}{\nu}. \quad (4.69)$$

Indeed, inconsistencies can be detected in the compared efficiency of structures according to their weights or *yields*. Let's see:

Efficiencies: Inverses of ν

Weights: P , dependent on material, σ/ρ , the payload R , the shape...

Yields: $r = R/P$

For different materials, the relationship between yields is not useful, but the relationship between efficiencies is.

$$e = \frac{1}{\nu} = \frac{1}{\chi} \frac{l}{\sigma/\rho} = (r+1) \frac{l}{\sigma/\rho} \quad (4.70)$$

$$\frac{e_1}{e_2} = \frac{1/\nu_1}{1/\nu_2} = \frac{(r_1+1)\mathcal{A}_2}{(r_2+1)\mathcal{A}_1}; \quad (4.71)$$

$$\frac{r_1+1}{r_2+1} = \frac{\mathcal{A}_1/\nu_1}{\mathcal{A}_2/\nu_2}; \quad \mathcal{A}_i = \frac{\sigma_i}{\rho_i} \quad (4.72)$$

If we know the weights of two solutions, P_1 , P_2 , and the payload R , which must be identical for both, the comparison of yields is immediate; the comparison of efficiencies from these weights requires considering the qualities of the materials used, which is also immediate, but the operation eliminates the dependence on those qualities on which yields depend. In such a way that discrepant conclusions can result: a better *yield* can be attributed to a worse structure simply by using a better material, a material of greater scope. Isolating the effectiveness of the form implies avoiding yield as an indicator.

4.3.5 Structure and Form: Structural Parameters of Form

Observing the expressions for the structural quantity and for the consideration of the self-weight, that is

$$W = \gamma\lambda Ql; \quad l \leq L = \frac{1}{\gamma\lambda} \frac{\sigma}{\rho} = \frac{1}{\gamma\lambda} \mathcal{A}; \quad \frac{Q}{R} = \frac{1}{1-\chi}; \quad \chi = \frac{l}{L}.$$

we observe the following elements that characterize the structure:

scheme defined by the coefficient γ ,

proportion or *slenderness*, characterized by λ ,

the Michell number, inversely linked to *efficiency*, is $\nu = \gamma\lambda$.

size or basic dimension of the problem, the span, l , less than the limit, *scope*, or insurmountable size L .

sizing associated with Q , but which turns out to be independent of L . It has no influence on the insurmountable size for self-weight. Changes in sizing involve changes in areas, which modify the weight and resistance in the same measure, such that they do not affect the fraction of resistance involved in resisting self-weight.

Thus, the list of parameters coincides with the one we obtained in section 4.2.4, albeit now in a much more general way, not restricted to lintel solutions.

4.4 Form and Cost in Flexural Roof Structures

We now analyze in more detail the solutions of cover structures: we will address the estimation of their costs—their structural quantity—based on the form parameters of possible solutions. These are flexural problems, characterized by null or close to zero Maxwell numbers, depending on the position of

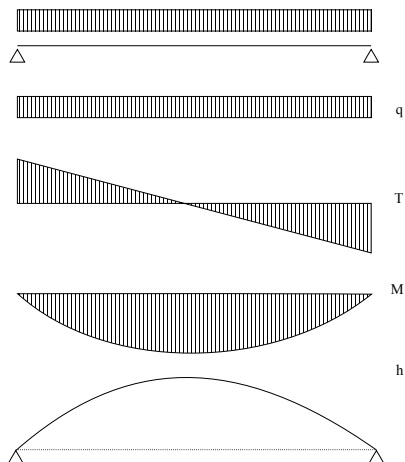


Figure 4.18: Flexural problem and basic functions for its representation: the problem of supporting a load between supports and the functions for the load, q , the shear, T and moment, M , or the antifunicular depth, h .

loads. Therefore, structures in which traction and compression components have comparable roles.

We address the computation of costs after considering the problem and solution types.

4.4.1 Problem and Geometries in Flexion

We consider the problem of transferring loads perpendicular to their line of action to a pair of supports located on the same line where the loads are situated. This is a flexural problem with a Maxwell number of zero, $M = 0$, given that all loads are orthogonal to their position vector if we place the origin on the line of supports (equation 4.48).

4.4.1.1 Problem and Typical Solutions

The flexural problem (figure 4.18) can be represented, regardless of how the structure materializes the equilibrium conditions:

- Through the moment graph (*beam* model),
- Through the funicular polygon (*arch* model).

The problem is characterized by the functions describing the load and the potential equilibriums of a slice: the functions for the load $q(x)$, the shear $T(x)$ and the moment $M(x)$, or through the funicular shape $h(x) = \frac{M(x)}{H(x)}$, where $H(x)$ is the horizontal component of the forces in the chords, functions that maintain the well-known equilibrium relationships in sections parallel to the loads: $q = \frac{\partial T}{\partial l}$, $T = \frac{\partial M}{\partial l}$; $H(x) = H = \text{constant} \implies h(x) = M(x)/H$.

In turn, this problem can be addressed with structures that respond to one of these three types (figure 4.19):

- Type 1: beam, characterized by a constant depth h , with the horizontal component in chords $H = M/h$ variable,
- Type 2: arch, characterized by a variable depth h , with $H = M/h$ constant by adopting the shape of the moment law M

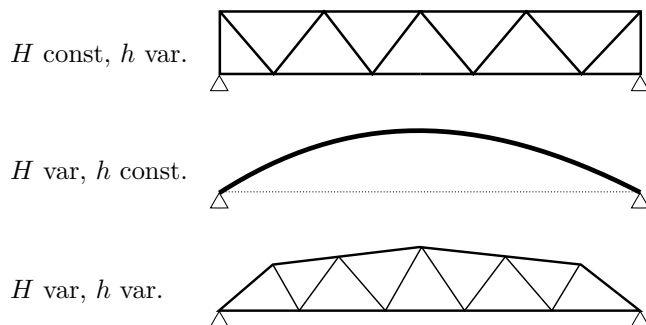


Figure 4.19: Flexural structure types: beam, arch or truss.

for h ,

- Type 3: truss, characterized by having both h and $H = M/h$ variable, while being necessary that their product equals the moment law M .

The following sections aim to visualize the costs of these solutions using the structural quantity and its properties.

4.4.1.2 Costs in Beams

We analyze the amount of structure required in chords and diagonals considering strict sizing, although allowing extension to non-strict sizing laws.

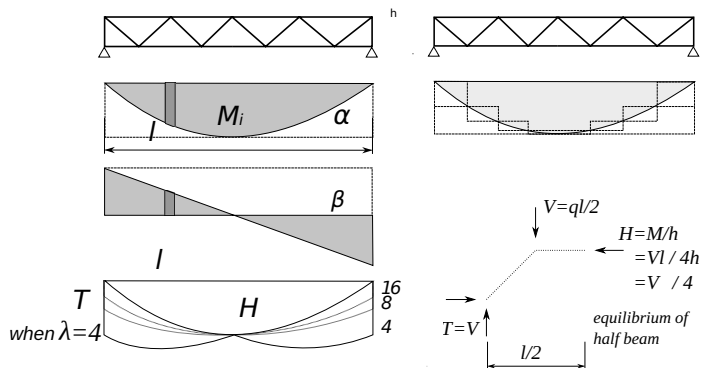


Figure 4.20: Cost estimation in beams. Represented by shaded areas. In the case of chords, this area represents the sum of both. The areas are displayed at the same scale in the lower part of the figure, depending on the slenderness λ . For this, the diagram on the right shows the relationship between the maximum force in chords H and the maximum shear, the reaction at the support.

Chords

The force in chords is on average $H = M/h$ since, depending on the triangulation geometry, its actual value in chords surrounds that value, above in one of them and below in the other, as shown in figure 4.20. So the structural quantity for each chord segment is twice the product of M/h by the segment length:

$$\begin{aligned} W_c &= 2 \int \frac{M}{h} dl = \frac{2}{h} \int M dl \\ &= \frac{2}{h} \int d(\breve{M}) = \frac{2}{h} \breve{M} \end{aligned} \quad (4.73)$$

$$= 2\alpha M_i \frac{l}{h} = 2\alpha M_i \lambda \quad (4.74)$$

$$= 2\alpha l H_{\max} \quad (4.75)$$

In the previous expressions, we have used the symbol $\breve{M}$ to express the area of the moment graph; α represents the fraction that this area represents with respect to the area of the rectangle $M_i l$, where M_i is the isostatic moment for the considered load. This allows considering non-strict sizing by using in the computation the area of the resistant capacities arranged in the beam instead of the effect moments area.

Diagonals

The forces in the diagonals, like their lengths, depend on their angle with the vertical, with the force $N = T/\cos\psi$ and the diagonal length of $h/\cos\psi$, for a beam length traversed in that segment of $h \tan\psi$.

In such a way that the structural quantity for each diagonal segment ds , corresponding to a segment dl of the beam length,

$$\int |N| ds = \int \frac{|T|}{\cos \psi} \frac{h/\cos \psi}{h \tan \psi} dl = \int \frac{|T|}{\sin \psi \cos \psi} dl = \int \frac{2|T|}{\sin 2\psi} dl \quad (4.76)$$

whose minimum value occurs when $\sin 2\psi_{d(\min)} = 1$, $\psi_{d(\min)} = 45^\circ$, which corresponds to the optimal inclination for diagonals. For other values or combinations of values depending on the diagonal family, we will have *penalties* to the cost —multiplication factors— measured by the ratio $\sin 2\psi_{d(\min)}/\sin 2\psi$ weighted to the fraction of the span corresponding to the inclination family ψ . In the particular case of $\psi = 90^\circ$ of Pratt beams or similar, their contribution to the cost, which is indeterminate in this last weighting, depends on the second family which determines the position and frequency of their use: it results in a local cost per vertical member of $|T|h$ for an effective segment given by $h \tan \psi_2$ and, therefore, they represent a contribution to the cost of $|T|/\tan \psi_2$ per unit length of the beam.

In this way, structural quantity in diagonals depends on their angles, penalizing the cost of the minimal possible, which is

$$W_D = 2\check{T} = 2\beta Tl = 2 \cdot 2 \int_0^{\frac{l}{2}} T dl = 4M_i = 8M_i\beta \quad (4.77)$$

Now β expresses the fraction of the rectangle $|T|l$ corresponding to the effectively disposed capacity and can, furthermore, add the factors corresponding to the penalties for non-optimal inclinations.

Total Cost

To compose both costs visually, it is useful to superimpose both moments and shear areas on a common scale, which can be done easily by comparing expressions 4.77 and 4.75, and putting T and H at the same scale for each slenderness λ , as shown in figure 4.20.

4.4.1.3 Costs in Arches

Now we analyze the tied arch, in which we place the loads on a horizontal deck suspended by hangers, figure 4.21. We successively analyze the structural quantity of each component.

Tie Rod

Its structural quantity, which is in tension and horizontal, will be

$$W_T = \int |N| dl = lH = W^{+=} \quad (4.78)$$

Hangers

Its structural quantity measures the amount necessary to raise the distributed load q to the height h of the corresponding arch segment:

$$W_P = \int_o^l h q dl = q \int h dl = q \check{h} \quad (4.79)$$

$$= \int h \frac{\partial T}{\partial l} dl = \int T \frac{\partial h}{\partial l} dl = \int T dz \quad (4.80)$$

$$= q \alpha h l = W^{+||} \quad (4.81)$$

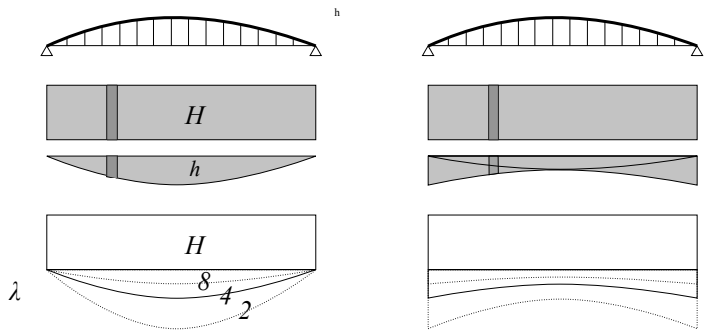


Figure 4.21: Estimation of costs in arches. Represented by the shaded areas, which are arranged at the same scale in the lower figure, depending on the slenderness λ . On the left the total vertical values are in a format suitable for comparing scales. On the right they are placed in the actual positions of such costs along l , considering the horizontal structural quantity components of the arch and the tie rod and the vertical component in hangers and arch.

Expression 4.79 shows the structural quantity as proportional to the shape and area of the elevation. For expression 4.80, there have been used the relation between load and shear and the fact that, in the resulting partial integral, the values of h at the ends or of T at the center are null.

Arch

We obtain its structural quantity by combining the horizontal and vertical parts, and since in each section these components are H and T respectively, the result is

$$W_a = W^{-} + W^{-\parallel} = lH + \int T dz \quad (4.82)$$

$$= lH + l\alpha h q = l(H + 2R\alpha/\lambda) \quad (4.83)$$

in the last expression using the fact that $lq = 2R$ (where R is the reaction at each support) and being $h = l/\lambda$.

It should be noted the direct correspondence of the two components horizontal and vertical of the arch's structural quantity with the corresponding tension components of tie rod and hangers.

Total Cost

Remembering that when $\lambda = 4$ the reaction is equal to the maximum horizontal component in chords, $R = H$, we can again sum at the same scale both shaded regions corresponding to the two parts, horizontal and vertical, of the structural quantity to visualize the corresponding cost, as done in figure 4.21.

4.4.1.4 Costs in Trusses

Although the varied geometry of trusses seems to make it difficult, initially, to generalize the determination and visualization of costs in a manner similar to what has already been done for beams and arches, we can, however, quickly and simply approach these by comparing the geometry of the truss in question with the beam and arch of identical depth, of equal slenderness.

We will see this approximation in two cases, corresponding to two very common, but not too efficient forms, such as the triangular truss, two of whose variants are the *Spanish knife*, and the *saw tooth*. And for this we will use two figures, 4.22 and 4.23, in which we use the fact that at equal depth, the M graphs, for a given value of M_i in the beam, and of h in the arch, can be drawn at the same scale as $M_i = Hh$.

The Triangular Truss

In the case of the triangular truss, figure 4.22, we can use for the part corresponding to the chords the reference to the beam in the center of the piece, and the reference to the arch at the ends: at the ends the equilibrium at the start of both chords is indistinguishable from that of the arch, so the cost law will be that of a tangent arch, and therefore of double slenderness to the comparison arch. By being half the depth it would require a double horizontal component, which gives the cost evolution at the end. In contrast, in the center we have a cost similar to the center of the beam.

The interpolation between both values along the chords requires, with a linear law of depths, achieving a parabolic law of

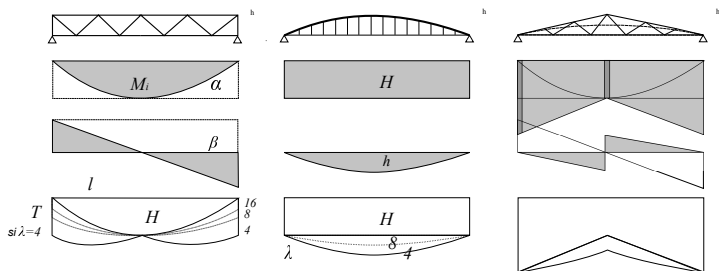


Figure 4.22: Estimation of costs in trusses. Represented, by reference to beams and arches, by the shaded areas, which are arranged at the same scale in the lower figure, depending on the slenderness λ

moments, which gives for this component, basically corresponding to the horizontal part of the cost, a linear law as well.

For the shear component, we have that at the end the shear is resolved by the inclination of the chord, which means subtracting from the capacity needed for shear the entire same value. But as the horizontal components decrease in chords to half the value at the support, with constant slope, the shear subtraction follows the same proportion, which results in excess in the center of the span: the result is that this truss works by carrying the load to the center, where it hangs from the vertex between chords, in a geometry of costs that tends to look more like the arch than the beam (see the grey areas in the figure).

The result in the sum of both components is clearly less

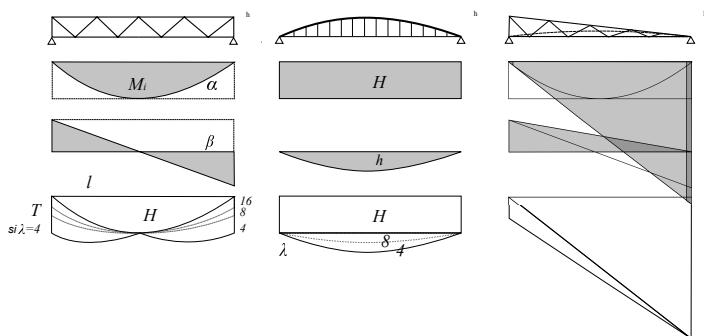


Figure 4.23: Cost Estimation in Trusses (Sawtooth Trusses). Represented, by reference to beams and arches, by shaded areas, which are arranged at the same scale in the lower figure, depending on the slenderness λ

effective than the beam or the arch given the poor angle at the supports of the piece.

The Sawtooth Truss

We perform the comparison in this way with the beam and the arch in figure 4.23.

In this case, the behavior of the chords at the left end will be similar to that of the beam, while at the right end it will resemble the tangent arch, which turns out to have a depth four times smaller than the studied piece. This provides two points for the unit costs at both ends, represented by values of 0 and $4H$

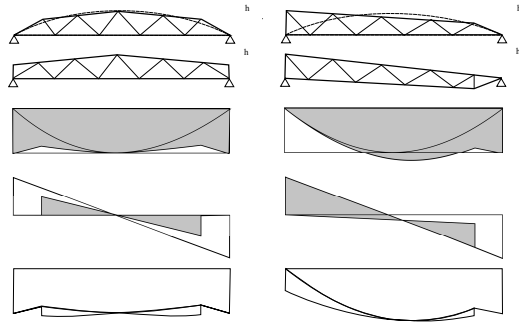


Figure 4.24: Improved Truss Forms and Approximate Resulting Costs

respectively, between which to interpolate the laws of H components responsible for ensuring moments M . A linear depth law for a parabolic moments law requires a linear interpolation, which provides the figure for that part of the cost law.

Now adding the shear part, which evolves between near identity with the beam at the left end, with the subtraction of the total at the right end, we have a law that must again add to the shear law a subtraction proportional to the horizontal component law due to the constant slope, which gives a triangular resulting law that carries loads from right to left.

As a result, it is a form whose extreme inefficiency becomes immediately visible.

Other Forms

The same technique used in previous forms can be used for other forms, although it becomes evident that any improvement will pass through improving the intersection angle between tensions and compressions at the supports, as suggested by a corollary of Michell's Theorem 4.2.

The subsequent increase of the angle of the last diagonal, which stops accompanying the upper chord or replaces it, represents an essential improvement and can be used in this second form for inclined roofs to accommodate water collection gutters.

The alternative involves altering the horizontality of the bottom chord to allow its inclined arrival at the support, bracing the diagonals against the corresponding change of direction of that chord, as suggested by a corollary of Aroca's theorem 4.4.

Figure 4.24 shows some of these variations.

4.4.1.5 Solutions for Circular Contour

The same computation method used for calculating the structural quantity in trusses in the previous section can be used to estimate that magnitude in more complex configurations, such as structures supported on an entire closed contour, like a circle. Figures in 4.25 show several of these possibilities and those in 4.26 the corresponding stress laws for a pair of these solutions, the second case for comparable rigidities and resistances in both directions.

If we measure the strict structural quantities in chords and

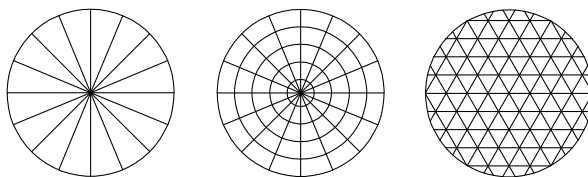


Figure 4.25: Beam Network in Circular Contour. Although the first solution is a set of radial trusses, the next two work radially and circumferentially coupled, in a behaviour closer to that of slabs

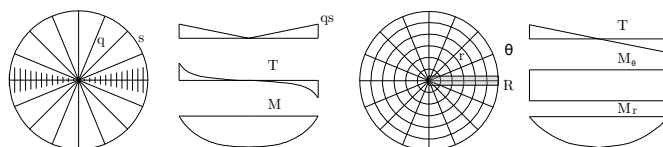


Figure 4.26: Load and Stress Laws in Beam Networks with Circular Contour, the second one drawn as the equivalent to the slab's laws

diagonals, we have in both cases the values (see Cervera 2001)

$$W_{Me} = \frac{1}{8}Ql\lambda; \quad (4.84)$$

$$W_{Te} = \frac{1}{3}Ql \quad (4.85)$$

for in the first of them we would have, with constant sizing, the magnitudes

$$W_M = \frac{1}{6}Ql\lambda;$$

$$W_T = Ql$$

easily determined from the maximum values in a circular sector, for the moment, $qsl^2/24$ and shear $qsl/4$, for arch s at the support, with savings on these quantities for a strict sizing of $1/4$ in chords and $2/3$ in the triangulation.

And in the second, for constant sizing, we would have, given the local laws of the figure for a rectangular element, integrated over the entire circle,

$$W_M = \frac{1}{6}Ql\lambda;$$

$$W_T = \frac{1}{2}Ql$$

now the savings on these quantities for strict sizing being $1/4$ in chords and $1/6$ in the triangulation.

But we see that this identity in the case of strict sizing is not surprising because, as demonstrated by a corollary of the Rigidity Theorem 4.3 (see figure 4.15), if the potential energy

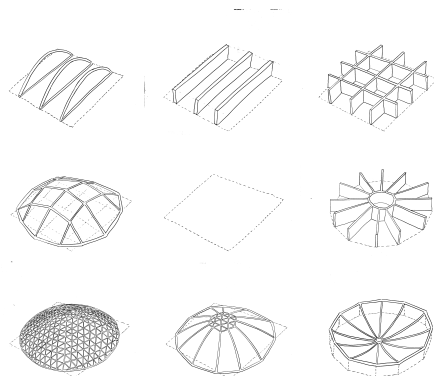


Figure 4.27: Comparable Roof Forms, to cover a space with a regular contour.

loss of the loads is equal, in strict solutions the structural quantity is also equal. And if the curvatures are identical as well as the shear distortions, the envelope of both deformed shapes is exactly the same.

This consideration will now allow us to comparatively traverse an entire set of structural types for roofs, apparently very different.

4.4.2 Equivalent Structures: Three-Dimensional Solutions

The relevant question is whether it is possible to establish some type of similarity or comparison in the very different forms employed in roof structures, schematized in figure 4.27: to what extent could some type of similarity be established between them?

The typological landscape presented below was first presented in Seville on July 8, 1992 (see Cervera Bravo 1993) as the result of a reflection on the possibilities of geometric transformation of structural forms without substantial change in costs that can be conceived, for the same degree of material utilization, if in addition

- loads, size, and slenderness are maintained,
- the corollaries of Maxwell's theorems 4.1 and the rigidity theorem 4.3 are utilized, which are summarized in figures 4.10 and 4.15,

That reflection and its drawings, those used here again, were the subject of a subsequent publication, (Cervera 2001) and have been a regular part of my Structural Projects courses at ETSAM. So here only a quick review of these ideas is made, which are based on the structural identities between the three different forms represented by the radial solution, the ring, or the mesh.

We start with the most elementary solution, the beam, figure 4.28, which can be unfolded as much as desired —the separation s between beams will only significantly influence the unit cost of the following family, since the structural quantity per unit area does not vary: if s is reduced there will be more beams, but

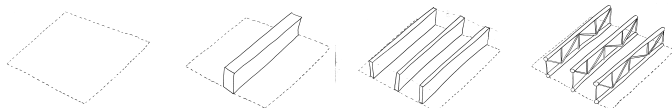


Figure 4.28: Beam

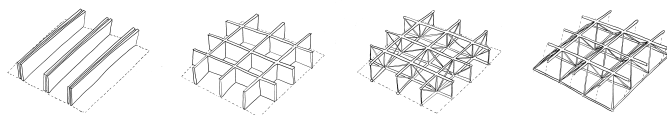


Figure 4.29: Grid

each one with less load— It is possible to lighten the solution by triangulation, which compared to well-designed solid web solutions is advantageous only from high spans, over 10, 12 m.

Unfolding the beam and rotating them to create the grid, figure 4.29, allows a very slight improvement in the structural quantity since the support on four sides reduces the descent near the added supports, in exchange for greater constructive complexity derived from the crossing of pieces, which can be reduced by vertically displacing them relative to each other. Therefore, in small-sized solutions, the beam will be advantageous compared to the advantage that the grid will provide in large-sized

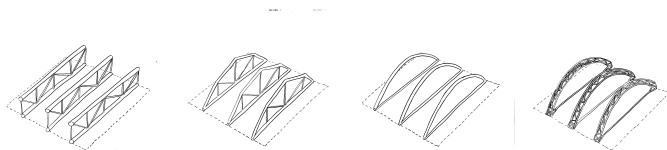


Figure 4.30: Arch

solutions. The usual triangulation is not done for isolated beams due to the multiplicity of nodes that result, but by displacing horizontally the two families of upper chords half a module relative to the lower ones, giving rise to the mesh of pyramids with a square base.

The alternative to the constant (or almost constant) depth of the beam is to tend towards the (anti)funicular form, passing through truss variants more or less close to the dominant moment diagram, giving rise to the arch, figure 4.30, which usually will need rigidity both because the compressed chord (buckling) and because it cannot adjust its form to all conceivable load laws.

Inverting the previous process, hanging the load instead of supporting it, leads to the catenary, 4.31, in which the need for bending rigidity of the loaded chord is still present, though now only due to load form variations. This rigidity is often provided through prestressing techniques, many times with an inverted catenary tensioning against the supporting component of the structural design, arch or catenary, occasionally with a rigid

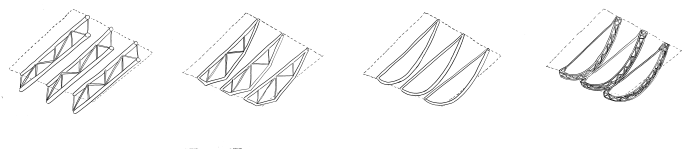


Figure 4.31: Catenary

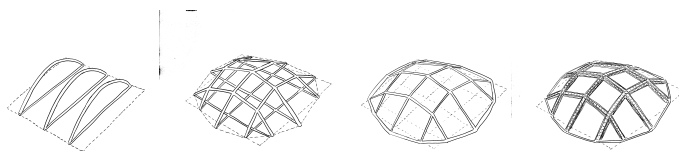


Figure 4.32: Crossed Arches

inverted arch parallel to the supporting catenary. The catenary generates, however, the problem of having to raise the supports when, in general, the ground supporting the foundation is at lower elevations than those of the roof.

An alternative to rotating the beams to generate the grid is to rotate the arches to generate crossed arches, figure 4.32, although now, if we want to make the elevations coincide at the crossings, it will be necessary to slightly modify the initially considered square plan to a polygonal form closer to the circumference. Now the mesh of tension ties is effectively replaced by a ring. The arches continue to be compressed chords with local

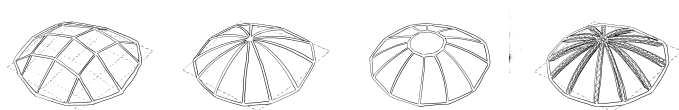


Figure 4.33: Radial Arches

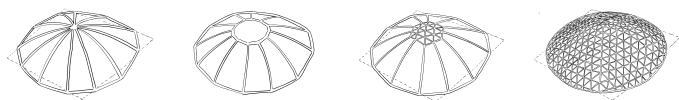


Figure 4.34: Dome

bending rigidity requirements, though somewhat lesser as load can be transferred in two directions.

The alternative to the mesh derived from the complete rotation of the main family is the radial solution, which in arches will result in radial arches, figure 4.33, well-known in traditional construction, and in which the keystone is usually resolved by a disc or a ring (perhaps leaving an oculus) to which all and each of the radii converge identically.

If the disc or ring of the keystone in the previous solution is replaced by a mesh and this is extended to the entire surface, the result is the dome, figure 4.34, which for small dimensions

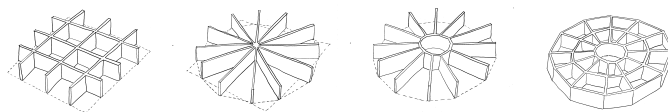


Figure 4.35: Drum

has important local rigidity given the double curvature, but for large dimensions again requires bending rigidity, which gives rise to double-layer solutions, etc.

The radial alternative to the mesh in beams leads to radial trusses, whose central rings for collecting compressed and tensile chords are usually transformed into drums, whose triangulated webs must transport the shear between trusses in cases of asymmetric loads. But this operation of annular transfer of loads and balances through the circumferential effect of the drum can be used in any other radii of the plan, giving rise to the type we call drum, figure 4.35, in which the material concentration that a radial truss network would demand in its center can be better diffused throughout the roof.

Finally, if in the radial solution of catenaries we use a ring as a substitute for the set of radial compressed chords, we have the form we call basket, figure 4.36, whose central tensioned ring can be of a considerable diameter such that the oculus growth allows it to cover only a fraction of the surface, such as the stands of a stadium, leaving the central arena free. The compressed ring

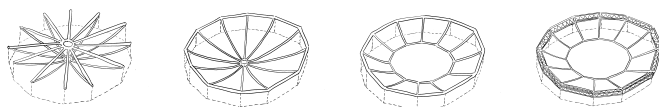


Figure 4.36: Basket

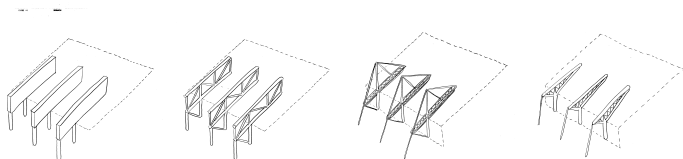


Figure 4.37: Cantilever

will require rigidity and, if this is appreciable, will allow resisting the plan flexions that may derive both from load asymmetries and possible deviations in plan from the circular form.

Opposite the families of solutions for the structure with supports on the contour would be the similar families that can be conceived for a support strategy in the center, with an entire typology of possible variants of compensated cantilevers forming mushrooms or parasols with all the variability already seen for the use of side supports. However, these solutions are less common as that central support is contradictory with much of the uses. If cantilever solutions are sought with only lateral sup-

port, the contribution of the structure necessary for the linking flexion of the cantilever becomes essential, figure 4.37, which reduces the range of possibilities by no longer having the support possibilities brought by the other sides or directions of the polygon.

Logically, these last cantilevered solutions on the sides of the polygon can be combined with span solutions in the center of the polygon, which gives rise to variants comparable to the beam with negative moments at support and positive in span, variants that are easily identifiable in many stadium roof solutions, etc., but which do not seem necessary to continue exploring here.

Chapter 5

Discretization Methods

In chapters 1 to 3 we have employed the technique of representing structural behavior based on the values and relationships of a set of discrete variables capable of describing forces and movements (or stresses and deformations) referred to a limited set of points in the structure, from which the set constituted by the continuum of this could be extrapolated or deduced: the variables $\mathbf{F}$, $\mathbf{U}$, $\mathbf{f}$, $\mathbf{u}$, and their relationships $\mathbf{K}$, $\mathbf{k}$, etc. . .

In bar structures, this strategy has allowed obtaining the algebraic relationships necessary to formulate and solve analysis or design problems, as defined in section 1.1.4, through the resources provided by the results of the elastic approach, 1.31, or

the plastic approach, 1.88.

In more complex geometries, based on structural material distributions across extensive surfaces or volumes, we must consider the existence of a continuum of infinite points over which the previous procedure is not immediately applicable. The mechanical, kinematic, and mutual relationships to be considered in the continuum must take into account that infinity. These are the field functions and equations of solid mechanics which, in a general form, adopt the format of differential equations, as summarized later in section 5.1.1, equations whose solution requires establishing both field conditions—in the interior of the considered continuum—and boundary conditions—on the surface or outer edge of said continuum—conditions that will respond to the type of continuum and problem considered.

However, if the difficulty in solving the differential problem is significant, usually due to geometric complexity and therefore in the form required for the functions that would describe the solutions, it is possible to establish methods that allow identifying variables localized in a finite number of points or regions, from which the relevant variables in the continuum and their relationships can be determined. This is what discretization methods consist of, whose description, section 5.1.2, and application to existing structural problems in building construction is the subject of this chapter.

Given that, among such methods, the Finite Element Method, FEM, has been generalized in the professional field, special attention is dedicated to this method in sections 5.2 to 6.4.

5.1 The Continuum Represented from Discrete Values

The objective of this section is to transfer the description of the structure's behavior from the point functions that would describe it in the continuum of its infinite points to a broad, but limited, finite set of variables.

To do this, we consider here that the structure's behavior can be described through the following functions and operators applied to the structural continuum:

displacements	$\boldsymbol{v}$	vector that defines the point's movement.
loads	φ	load density at the point, load per unit volume.
strains	$\boldsymbol{\epsilon}$	tensor describing the deformations in the environment of that point.
stresses	$\boldsymbol{\sigma}$	stress tensor in a differential element located around the point.
stiffness	$\boldsymbol{\kappa}$	relationship that allows determining, for the material located at the point, the stresses from the strains.

- operators:** $\partial(\cdot)$ Differential operator, which expresses the gradient, at the point considered, in the values of a certain variable, that to which the operator is applied. This characterizes the relationships between kinematic variables.
- $\partial^*(\cdot)$ Differential operator adjoint to the previous one (see 2.1.1.3) Characterizes the relationships between static variables. (5.1)

The field equations are those which, in each branch of the structural theory, relate the previous variables at all points of the structural continuum. We will see an extensive subset of these, although it is not the purpose of this text to justify them: they are taken as valid as detailed in the classic texts of the corresponding theories.

5.1.1 Field Equations in Solid Mechanics

The aim in this section is to generalize the equations of all theories in a common unified format

5.1.1.1 Classical Bar Theory

In this theory, where the properties and state of the bar are described through functions for the linear position x along the bar's directrix, the functions described in list 5.1 are, correspondingly: $v = w(x)$, the vertical displacement —deflection—

of the considered point, $\varphi = q(x)$, the vertical load density applied at said point, $\epsilon = c(x)$, the local curvature in the point's coordinates and $\sigma = M(x)$, the bending moment on the elementary slice at said coordinates. $\kappa = EI$ is the stiffness in the section, which in beam theory is the product of the modulus of elasticity by the moment of inertia, and which expresses the (material or *constitutive*) relationships between stress and strain in elastic bars: $M = EI c$.

Finally, the operator $\partial(\cdot)$ in bar theory is the operator $\partial(\cdot) = \partial = d^2/dx^2 = d^2(\cdot)/dx^2$, which establishes the relationships between deflection and curvature (compatibility). But it is identical to the operator $\partial^*(\cdot)$ that describes the relationships between local moment and load density (equilibrium), in the expressions $c = d^2w/dx^2$ and $q = d^2M/dx^2$. In this case $\partial(\cdot) = \partial^*(\cdot)$, a *self-adjoint* operator, as it is of second degree.

5.1.1.2 Elasticity Theory

Now the displacements $\mathbf{v}$ can be described by the column vector $[u \ v \ w]^T$, which describes the displacement in the three space directions, and the load density per unit volume at the point, as the column vector, also of three components, $\varphi = [p_x \ p_y \ p_z]^T$.

The stress tensor σ , is a symmetric three-dimensional tensor that can be represented by the matrix $\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$ which, being symmetric, equalizes the values around the diagonal, $\tau_{ij} = \tau_{ji}$, and where the convention $\sigma_i = \tau_{ii}$ is used. In the subscripts

of τ_{ij} , the first subscript is used to indicate the direction perpendicular to the elementary cube face being considered and the second subscript for the direction of the projection of the stress exerted on said face, such that the σ are the normal stresses and the τ are the shear stresses. In the matrix formats we will use later, it is usual to write the set of independent components of the stress tensor in column vector form, using the format $\sigma = [\sigma_x \ \sigma_y \ \sigma_z \ \tau_{yz} \ \tau_{zx} \ \tau_{xy}]^T$.

The relevant strain tensor is again three-dimensional, represented as $\begin{bmatrix} \epsilon_x & b_{xy} & b_{xz} \\ b_{yx} & \epsilon_y & b_{yz} \\ b_{zx} & b_{zy} & \epsilon_z \end{bmatrix}$, where $\epsilon_i = b_{ii}$ would be the displacement, relative to such of the point of interest, of another point located on axis i at a unit distance and measured in the same i direction and therefore measures the unit elongation in that direction, and where b_{ij} would be, for that same point on axis i relative to the same origin, the displacement in direction j derived from the deformation of the unit radius sphere. The angular distortion γ_{ij} between axes i and j is measured by the sum $b_{ij} + b_{ji}$, so being the tensor symmetric, $b_{ij} = b_{ji}$, we use $b_{ij} = \gamma_{ij}/2$. Given the rule of using energetically congruent static and kinematic variables in the matrix expressions we will employ, we will use the following column vector form: $\epsilon = [\epsilon_x \ \epsilon_y \ \epsilon_z \ \gamma_{yz} \ \gamma_{zx} \ \gamma_{xy}]^T$, since in the work density expression for a certain angular distortion we would have $\sigma_{ij}b_{ij} + \sigma_{ji}b_{ji} = \sigma_{ij}\gamma_{ij}$, by the symmetry of both tensors and the relationship between γ and b .

In this way, it is possible to formulate from these variables

the set of field equations from the theory of elasticity: the classic equations of equilibrium, compatibility, and material constitutive relations.

The equilibrium equations equate, for each direction of the three-dimensional space, the load density at the point to the difference, between opposite faces of the unit cube, of the stresses oriented in that direction:

$$\varphi = \boldsymbol{\partial}^* \boldsymbol{\sigma} \equiv \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial x} & 0 & 0 & 0 & -\frac{\partial}{\partial z} - \frac{\partial}{\partial y} \\ 0 & -\frac{\partial}{\partial y} & 0 & -\frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{\partial}{\partial z} - \frac{\partial}{\partial y} - \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \quad (5.2)$$

In the literature, it is common to find the symbolic expression $\boldsymbol{\partial}^{-T}$ for the adjoint $\boldsymbol{\partial}^*$, as suggested by the comparison with the following operator.

The local compatibility equations at the point, in which the differences between displacements of nearby points are equated to strains are

$$\boldsymbol{\epsilon} = \boldsymbol{\partial} \boldsymbol{v} \equiv \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad (5.3)$$

Finally, the equations that characterize the material behav-

ior, in its isotropic elastic version¹ are

$$\boldsymbol{\sigma} = \boldsymbol{\kappa} \boldsymbol{\epsilon}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} \quad (5.4)$$

equation that is usually deduced from the inverse relation (5.5), considering the elastic elongation effect in the direction of the tension and the simultaneous shortening by the Poisson effect in perpendicular directions, as well as the relation between longitudinal and transverse elastic moduli, $E/G = 2(1+\nu)$:

$$\boldsymbol{\epsilon} = \boldsymbol{\kappa}^{-1} \boldsymbol{\sigma}$$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \quad (5.5)$$

In building construction, it is uncommon to consider the triple state of stress-strain, typically resorting to plane states, either of stress, where the stresses on the z direction planes are

1. with identical behavior for all spatial directions

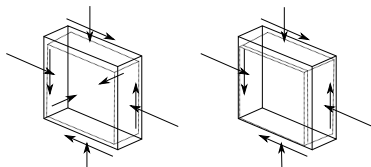


Figure 5.1: Plane States of Strain and Stress

considered negligible compared to the other two planes ($\sigma_z \approx 0$; $\tau_{zx} \approx 0$; $\tau_{zy} \approx 0$), or of strain, where complete constraint is considered for the z direction such that the corresponding strains are null ($\epsilon_z = \gamma_{zx} = \gamma_{zy} = 0$), figure 5.1. In these states, the variables are reduced by eliminating that direction from the set of stresses and strains considered, obtaining the following equations for elastic stress-strain relations (or their inverses):

Plane Strain State

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E/(1+\nu)}{(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (5.6)$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}(\epsilon_x + \epsilon_y) \neq 0$$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad (5.7)$$

In this case, the reduction is derived from the zeros in the strains, which involve the elimination of the corresponding rows and columns of expression 5.4 to generate 5.6, whose inverse provides 5.7.

Plane Stress State

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (5.8)$$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad (5.9)$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \neq 0$$

These equations are obtained by considering the stresses on the z face as null and thus eliminating the corresponding rows and columns in expression 5.5 to obtain 5.9, whose inverse is 5.8.

The expression 5.8 is written in matrix format, which in condensed form is represented as $\boldsymbol{\sigma} = \boldsymbol{\kappa}\boldsymbol{\epsilon}$, but it is also usual in the literature to use Einstein's format, $\sigma_i = \kappa_{ij}\epsilon_j$, which implies an implicit summation of all products with equal indices in cases of repeated indices.

An alternative to this formulation is the tensor-based approach: since both stress and strain are second-order tensors, the elastic stiffness that relates them is a fourth-order tensor,

and in this case the notation changes. The relevant tensors in the plane stress case are

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}; \begin{bmatrix} \epsilon_x & b_{xy} \\ b_{yx} & \epsilon_y \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix}; \quad \epsilon_{xy} = b_{xy} = \frac{\gamma_{xy}}{2}$$

and the elastic equation takes the form

$$\sigma_{ij} = \kappa_{ijkl} \epsilon_{kl} \quad (5.10)$$

where we can interpret κ_{ijkl} , with four dimensions, in the following way, where indices i, j select the block and k, l select the term within that block:

$$\kappa_{ijkl} = \kappa_{ij(kl)} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & 0 \\ 0 & \nu \\ 0 & \frac{1-\nu}{2} \\ \frac{1-\nu}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1-\nu}{2} \\ \frac{1-\nu}{2} & 0 \\ \nu & 0 \\ 0 & 1 \end{pmatrix} \quad (5.11)$$

5.1.1.3 Theory of Thin Plates in Bending

In this theory, the equilibrium equations of the shear differences with load density, and the equilibrium of the moment differences with that provided by the shear forces provide a combined equi-

librium equation for moments and load density:

$$\begin{aligned}
 \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q &= 0 \\
 \frac{\partial m_x}{\partial x} + \frac{\partial m_{yx}}{\partial y} + V_x &= 0 \\
 \frac{\partial m_{xy}}{\partial x} + \frac{\partial m_y}{\partial y} + V_y &= 0 \\
 \frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} + 2 \frac{\partial^2 m_{xy}}{\partial x \partial y} &= q
 \end{aligned} \tag{5.12}$$

And on the other hand, the compatibility relations that relate differences in deflections with plate inclinations and differences in these inclinations with curvatures allow linking curvatures with deflections:

$$\begin{aligned}
 c_x &= \frac{\partial \theta_x}{\partial x}; \quad c_y = \frac{\partial \theta_y}{\partial y}; \quad c_{xy} = c_{yx} = \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}; \\
 \theta_x &= \frac{\partial w}{\partial x}; \quad \theta_y = \frac{\partial w}{\partial y} \\
 c_x &= \frac{\partial^2 w}{\partial x^2}; \quad c_y = \frac{\partial^2 w}{\partial y^2}; \quad c_{xy} = 2 \frac{\partial^2 w}{\partial x \partial y}
 \end{aligned} \tag{5.13}$$

Finally the moments in the plate can be obtained by integrating the stresses corresponding to the plane stress states of each plane parallel to the mid-plane as the plate thickness is traversed and, since these stresses are elastically derived from the deformations of these planes, which can be deduced from the curvatures, the plate moment equations result dependent on the plate curvatures.

In matrix format we have, therefore, the same three families of equations:

Equilibrium of loads-shear-moments²

$$q = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y^2} & 2\frac{\partial^2}{\partial x\partial y} \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix}; \quad \varphi = \partial^T \sigma \quad (5.14)$$

Compatibility of deflections-rotations-curvatures

$$\begin{bmatrix} c_x \\ c_y \\ c_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ 2\frac{\partial^2}{\partial x\partial y} \end{bmatrix} w; \quad \epsilon = \partial v \quad (5.15)$$

Stiffness

$$\begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} c_x \\ c_y \\ c_{xy} \end{bmatrix}; \quad \sigma = \kappa \epsilon \quad (5.16)$$

where $\frac{Eh^3}{12(1-\nu^2)} = D$ is called the *plate constant*, a term equivalent to the section stiffness EI in beam theory. If in equation 5.16 we apply the differential operator from 5.14 to both members, substitute the result of the first according to 5.14, and substitute the curvatures of the second according to 5.15, that is, if we combine the three groups of equations, we obtain the plate differential equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}; \quad \partial^T(\kappa \partial v) = \varphi \quad (5.17)$$

2. The ∂^* operator, adjoint to the ∂ , is, in this case, $\partial^* = \partial^T$.

5.1.1.4 Boundary Conditions

Solving the differential problem requires, in addition to the general equations we have deduced, the imposition of boundary conditions on the volume, without which the solution would be indeterminate. Although these conditions can be varied, two major groups are generally considered:

“Essential” Conditions, Dirichlet: Kinematic, CDK, on support

$$\mathbf{v} = \mathbf{v}_{\Gamma_s}; \quad \mathbf{x} \in \Gamma_s \quad (5.18)$$

They establish or fix the displacements in the support region, or region of imposed movements, of the surface, Γ_s , and are generally straightforward to formulate.

“Natural” Conditions, Neumann: Static, CNS at free points

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\Gamma_l}; \quad \mathbf{x} \in \Gamma_l \quad (5.19)$$

They establish the loads applied in the free region of the surface, that part which does not have its movement directly constrained, Γ_l . Since the *response* loads depend on stress states and these on the unknown displacements, these are conditions more difficult to formulate in analytical approaches. A classic example is the trihedral of equilibrium at the surface in three-dimensional elasticity problems, figure 5.2 which indicates the global axes and the surface triangle dA , with its normal versor (vector) $\mathbf{n}$, subjected to a surface load density $\boldsymbol{\varphi}_{\Gamma}$. The (vector)

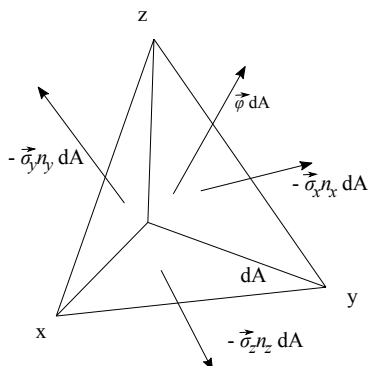


Figure 5.2: Equilibrium Trihedral at the Surface. The three hidden faces (the trihedral) are inside the volume Ω , while the visible face is on the outer free surface Γ_l of that volume. The tensions and surfaces of the four faces are shown, where those of the trihedral are a projection of the triangle on the surface

equilibrium implies

$$\begin{aligned}
 -\sigma_x \alpha_{nx} dA - \sigma_y \alpha_{ny} dA - \sigma_z \alpha_{nz} dA + \varphi_\Gamma dA &= 0 \\
 -\sigma_x n_x - \sigma_y n_y - \sigma_z n_z + \varphi_\Gamma &= 0 \\
 \sigma^T \mathbf{n}_\Gamma &= \varphi_\Gamma
 \end{aligned} \tag{5.20}$$

which can be expressed as³:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zx} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \varphi_{\Gamma x} \\ \varphi_{\Gamma y} \\ \varphi_{\Gamma z} \end{bmatrix}; \boldsymbol{\sigma}^T \mathbf{n} = \boldsymbol{\varphi}_{\Gamma}. \quad (5.21)$$

5.1.1.5 Field Equations and Differential Equation in All Elastic Theories.

In all cases, we have encountered the set of field equations in three groups

$$\boldsymbol{\varphi} = \boldsymbol{\partial}^* \boldsymbol{\sigma} \quad (\text{equ}) \quad (5.22)$$

$$\boldsymbol{\epsilon} = \boldsymbol{\partial} \mathbf{v} \quad (\text{com}) \quad (5.23)$$

$$\boldsymbol{\sigma} = \boldsymbol{\kappa} \boldsymbol{\epsilon} \quad (\text{mat}) \quad (5.24)$$

And if, as we have done in the case of plates, we combine these three expressions, we obtain as the *differential equation of the problem* the expression

$$\boldsymbol{\varphi} = \boldsymbol{\partial}^* (\boldsymbol{\kappa} \boldsymbol{\partial} \mathbf{v}) \quad (5.25)$$

which must be satisfied at all points of the volume, which we can denote by Ω . The first term of the equation is the applied charge density at each point $\mathbf{x}$ of the volume, which we can denote

3. in tensor format for $\boldsymbol{\sigma}$. If using the column vector format, $\boldsymbol{\sigma}^T = [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{yz} \quad \sigma_{zx} \quad \sigma_{xy}]$, and then the normal $\mathbf{n}$ is defined by $\mathbf{n}^T = \begin{bmatrix} n_x & 0 & 0 & 0 & n_z & n_y \\ 0 & n_y & 0 & n_z & 0 & n_x \\ 0 & 0 & n_z & n_y & n_x & 0 \end{bmatrix}$.

with φ_Ω , while the second becomes a function that determines the charge density capable of being balanced by a differential expression of the displacement $\mathbf{v}$, an expression written in the form of a *response* to said displacement, $\varphi(\Theta(\mathbf{v}))$, which would lead us, as in the case of the plate, to expressions in the form

$$\varphi(\Theta(\mathbf{v})) - \varphi_\Omega = 0, \quad \mathbf{x} \in \Omega \quad (5.26)$$

Which means that a certain operator applied to the displacements must equal the applied load, and where the operator combines the three groups of equations of the problem.

In addition to the differential equation 5.25, boundary conditions must be established, equations 5.18 and 5.19. And with this, we have the *strong* form of the problem. Since it is a form that is generally difficult or very difficult to solve, although it is useful to understand the characteristics of the problem, it is not directly employed in its solution.

5.1.1.6 Weak Form of Field Equations

To formulate it, we directly resort to the principle of virtual work, which affirms the equality between external and internal work performed by a stress state $\boldsymbol{\sigma}$ balanced with loads φ in the movement derived from adding any compatible deformation and displacement state $\bar{\boldsymbol{\epsilon}}, \bar{\mathbf{v}}$.

In the loads, we consider all applied to the body, which include both those applied to the interior of the volume, perhaps gravitational, φ_Ω , measured per unit of volume, as well as those applied on the free surface, φ_{Γ_l} , measured per unit of surface.

The nullity or complete constraint of displacement in the support regions of the surface φ_{Γ_s} is considered.

In such a way that this principle allows formulating the equilibrium problem in the following alternative form:

$$\int_{\Omega} \bar{\epsilon}^T \sigma \, d\Omega = \int_{\Omega} \bar{v}^T \varphi_{\Omega} \, d\Omega + \int_{\Gamma} \bar{v}^T \varphi_{\Gamma} \, d\Gamma, \\ \forall ([\bar{v}, \bar{\epsilon}], [\varphi_{\Omega}, \varphi_{\Gamma}, \sigma]) \text{ COM, EQU} \quad (5.27)$$

It is called *weak form* in contrast to the *strong form* contained in the expression 5.22, used in the combination that leads to the differential problem 5.25, insofar as the type of functions with which solutions could be tested in the differential expression requires a higher degree of continuity⁴ than that required by the functions with which solutions could be tested in this latter integral expression, 5.27. The state to be determined, σ , linked to the applied loads φ , must be able to satisfy this expression for any discernible compatible displacement state $\bar{v}, \bar{\epsilon}$.

The identity between both approaches can be demonstrated, starting from the *strong form* 5.25 using the Green–Gauss Theorem, which allows the integration by parts of the expression now used with the usual sign criterion in classic elasticity texts

$$\partial^* (\kappa \partial v) + \varphi = 0 \quad (5.28)$$

an expression in which, multiplying by an arbitrary displacement field $\bar{v}$, the nullity must be maintained at every point and,

4. associated with the possibility of jumps in functions that imply possible indeterminate or infinite values in some of their derivatives.

therefore, the nullity in the integral over the entire domain,

$$\int_{\Omega} \bar{\mathbf{v}}^T (\partial^* (\boldsymbol{\kappa} \partial \mathbf{v}) + \boldsymbol{\varphi}) \, d\Omega = 0 \quad (5.29)$$

now being able to proceed with the breakdown of terms and their integration by parts

$$\begin{aligned} & \int_{\Omega} \bar{\mathbf{v}}^T \partial^* (\boldsymbol{\kappa} \partial \mathbf{v}) + \int_{\Omega} \bar{\mathbf{v}}^T \boldsymbol{\varphi}_{\Omega} \, d\Omega = \\ & \int_{\Gamma} \bar{\mathbf{v}}^T \mathbf{n}^T \boldsymbol{\sigma} \, d\Gamma - \int_{\Omega} (\partial \bar{\mathbf{v}})^T (\boldsymbol{\kappa} \partial \mathbf{v}) \, d\Omega + \int_{\Omega} \bar{\mathbf{v}}^T \boldsymbol{\varphi}_{\Omega} \, d\Omega = \\ & \int_{\Gamma} \bar{\mathbf{v}}^T \boldsymbol{\varphi}_{\Gamma} \, d\Gamma + \int_{\Omega} \bar{\mathbf{v}}^T \boldsymbol{\varphi}_{\Omega} \, d\Omega - \int_{\Omega} \bar{\boldsymbol{\epsilon}}^T \boldsymbol{\sigma} \, d\Omega = W_e - W_i = 0, \\ & \quad \forall ([\bar{\mathbf{v}}, \bar{\boldsymbol{\epsilon}}], [\boldsymbol{\varphi}_{\Gamma}, \boldsymbol{\varphi}, \boldsymbol{\sigma}]) \text{ COM, EQU} \end{aligned} \quad (5.30)$$

5.1.2 Discretization Techniques

Once we have a model capable of representing the elastic equilibrium conditions in the continuum, its solution would require solving the corresponding differential problem, a task that is only possible in very few occasions. Therefore, the usual strategy is to transform the problem of determining the solution functions into a problem in which these functions can be approximated by a list of discrete values attributed to a large, but finite, set of variables, where the problem is reduced to determining the values for these variables that provide the best possible approximation to the functions sought in the differential problem.

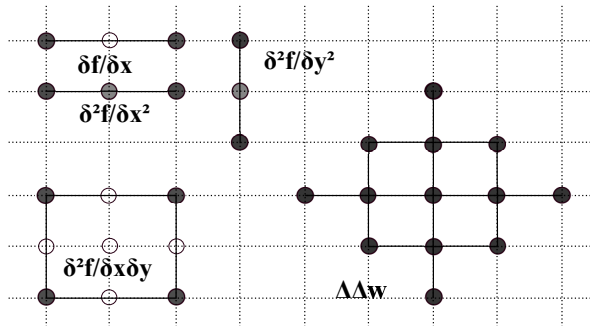


Figure 5.3: Finite Differences. Nearby points affected by the approximation to different differential operators at a point.

It is, therefore, the transformation of the continuum problem into a problem of values for discrete variables. Let's examine some of the appropriate strategies for this.

5.1.2.1 Finite Differences

In this method, the solution to the field problem in its *strong* formulation is addressed by discretizing the function and its derivatives. To do this, a regular grid of points is drawn over which the values of the field variables of said formulation (the descents, for example in the case of plates, equation 5.17) are sought to be determined. The differential expressions are approximated with the differences between the values of these variables at nearby

points, figure 5.3:

$$\begin{aligned}\frac{\partial w}{\partial x} &\approx \frac{w_{1,0} - w_{-1,0}}{2\Delta_x}; \quad \frac{\partial w}{\partial y} \approx \frac{w_{0,1} - w_{0,-1}}{2\Delta_y}; \\ \frac{\partial^2 w}{\partial x^2} &\approx \frac{\frac{w_{1,0} - w_{0,0}}{\Delta_x} - \frac{w_{0,0} - w_{-1,0}}{\Delta_x}}{\Delta_x} \\ \frac{\partial^2 w}{\partial x \partial y} &\approx \frac{w_{1,1} + w_{-1,-1} - w_{1,-1} - w_{-1,1}}{4\Delta_x \Delta_y}\end{aligned}$$

In this way, the problems become a set of linear equations in the values of the variables, which allows obtaining these and, therefore, the sought approximation by interpolation.

5.1.2.2 Nodal Approximations: From Points to Regions

The most current methods, by finite elements (finite element method, FEM), boundary elements (boundary element method, BEM), or meshless methods, establish a different approximation, which we will see in detail for the finite element method in section 5.2. In this approximation, we similarly seek to interpolate or approximate the solution functions based on the values of certain parameters, usually, but not exclusively, linked to the kinematic component of the problem, established at privileged points of the domain: the nodes. In the case of finite elements, the functions are interpolated locally in each element based on the values of the nodes in contact with the element, while con-

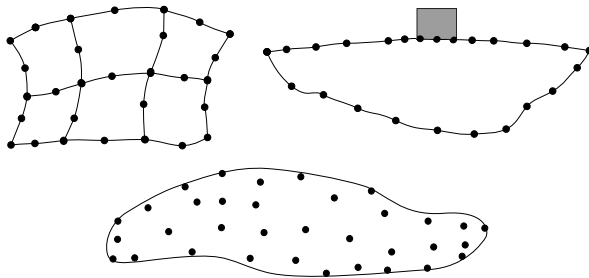


Figure 5.4: Nodal Approximations: Finite Elements (FEM), Boundary Elements (BEM), Particle Methods (meshless).

tinuity is ensured through nodes shared between contacting elements. In the case of boundary elements, the values of the variables on the domain boundary and their influence on the function of interest at any point inside the domain are considered. And in the case of meshless methods, the approximation is made through a more diffuse interpolation that determines the value of the function of interest at any point based on the existing variables of nearby nodes and their degree of proximity.

The formulations for these methods consist of rewriting the field equations, generally in their weak format, based on the established nodal variables, which allows transforming the differential problem into a numerical problem with a high number of variables, generally linear, or linearizable in steps as seen in section 2.1.2.4, problems susceptible to being solved using some of the numerical procedures typical of linear algebra.

Although all the cited methods have given rise to highly useful tools and remain current in some fields, we treat here only the finite element method, since it is almost universally employed in the software tools typically used in professional studies dedicated to structural engineering.

5.2 Base Equations of the Finite Element Method

A general description of the fundamentals of the Finite Element Method (FEM) and its main equations is developed below, without delving too deeply into the details, which will be analyzed in Chapter 6.

5.2.1 Field Equations Interpolation by Elements

Let us consider the equilibrium condition for a mechanical state (stress-load) σ, φ linked to a kinematic state (strain-displacement) ϵ, v that is compatible and satisfies the three groups of equations of elastic analysis,

$$\partial^* \sigma = \varphi; \quad \epsilon = \partial v; \quad \sigma = \kappa \epsilon \quad (5.31)$$

and consider the *weak* formulation by imagining compatible kinematic states (often qualified as *virtual*), $\bar{\epsilon}, \bar{v}$, which therefore verify

$$\bar{\epsilon} = \partial \bar{v} \quad (5.32)$$

This formulation, considering loads applied only on the boundary, ensures the equality between internal and external work of the balanced state in the compatible movement, such that

$$\int_{\Omega} \bar{\epsilon}^T \sigma \, d\Omega = \int_{\Gamma} \bar{v}^T \varphi \, d\Gamma, \quad \forall (\bar{\epsilon}, \bar{v} | \bar{\epsilon} = \partial \bar{v}) \quad (5.33)$$

What the methods provide is, at every point $\mathbf{x}$ of the domain Ω , a linear interpolation of the functions of this formulation based on the initially unknown values of a large number of parameters $\mathbf{a}$, linked to the regions of the domain. Different formats are possible depending on which functions are interpolated:

classic displacements $\mathbf{v}$ are interpolated

hybrid stress states σ are also interpolated

Pre-established and therefore known functions are used for the interpolations, dependent on the method and type of domain, functions that allow writing, depending on the form of approximation

$$\begin{aligned} \mathbf{v}(\mathbf{x}) &\approx \mathbf{N}(\mathbf{x})\mathbf{a}; \quad \epsilon(\mathbf{x}) \approx \partial \mathbf{N}(\mathbf{x})\mathbf{a} \\ \sigma(\mathbf{x}) &\approx \mathbf{M}(\mathbf{x})\mathbf{b}; \quad \epsilon(\mathbf{x}) \approx \kappa^{-1} \mathbf{M}(\mathbf{x})\mathbf{b} \end{aligned}$$

We will follow the development of the classic approximation. In it, the approximations to the field functions are

$$\mathbf{v} \approx \mathbf{N}\mathbf{a}; \quad \epsilon \approx \partial \mathbf{N}\mathbf{a}; \quad \sigma \approx \kappa \partial \mathbf{N}\mathbf{a} \quad (5.34)$$

These approximations can be used for both the equilibrium state whose solution is sought, and for all the compatible virtual strain

states. Therefore, the weak form can be expressed approximately:

$$\begin{aligned} \int_{\Omega} \bar{\epsilon}^T \boldsymbol{\sigma} d\Omega &= \int_{\Gamma} \bar{\mathbf{v}}^T \boldsymbol{\varphi} d\Gamma \approx \\ \int_{\Omega} \bar{\mathbf{a}}^T ((\partial \mathbf{N})^T \boldsymbol{\kappa} \partial \mathbf{N}) \mathbf{a} d\Omega &= \int_{\Gamma} \bar{\mathbf{a}}^T \mathbf{N}^T \boldsymbol{\varphi} d\Gamma \end{aligned} \quad (5.35)$$

and since the parameters of both the real and virtual states are numeric values independent of the functions, it results in

$$\begin{aligned} \bar{\mathbf{a}}^T \left(\int_{\Omega} (\partial \mathbf{N})^T \boldsymbol{\kappa} \partial \mathbf{N} d\Omega \right) \mathbf{a} &= \bar{\mathbf{a}}^T \int_{\Gamma} \mathbf{N}^T \boldsymbol{\varphi} d\Gamma, \quad \forall \bar{\mathbf{a}} \\ \left(\int_{\Omega} (\partial \mathbf{N})^T \boldsymbol{\kappa} \partial \mathbf{N} d\Omega \right) \mathbf{a} &= \int_{\Gamma} \mathbf{N}^T \boldsymbol{\varphi} d\Gamma \end{aligned} \quad (5.36)$$

where the integrals in both members can be performed independently of the unknown value of the approximation parameters $\mathbf{a}$. From this results a linear system of equations in which, if the parameters $\mathbf{a}$ are interpreted as displacements, the form becomes the already known stiffness expression: loads equal to stiffness times displacements, 1.24:

$$\mathbf{K} \mathbf{a} = \mathbf{F}; \quad \mathbf{a} \rightarrow \mathbf{U}; \quad \mathbf{F} = \mathbf{K} \mathbf{U} \quad (5.37)$$

What is characteristic of the Finite Element Method, in which the structure is decomposed into a list of connected elements, is that the expression 5.36 that allows determination of the coefficients of matrix $\mathbf{K}$ and vector $\mathbf{F}$, is calculated by replacing the integrals in the domain or boundary with the sum

of such integrals over the set of all elements that compose the structure:

$$\int_{\Omega} \phi \, d\Omega = \sum_e \int_{\Omega_e} \phi \, d\Omega; \quad \int_{\Gamma} \psi \, d\Gamma = \sum_e \int_{\Gamma_e} \psi \, d\Gamma \quad (5.38)$$

To ensure identity between the complete integral and the element-wise sum, it is necessary to ensure sufficient continuity in the involved functions to avoid loss of content when crossing the element boundaries: it must be ensured that these integrals become zero through boundaries. We will see the implications of this requirement later.

5.2.2 Free and Support Points: Reactions and Constraints

We saw in section 1.3.2 that the inclusion of all possible parameters in both free and supported regions leads to an expression, equation 1.28, that is not invertible as it allows movements without load, such as rigid displacements of the structure:

$$\begin{bmatrix} \mathbf{F}_l \\ \mathbf{F}_s \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ll} & \mathbf{K}_{ls} \\ \mathbf{K}_{sl} & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{U}_l \\ \mathbf{U}_s \end{bmatrix} \quad (5.39)$$

Such that it is the reduction that allows considering known the movements of the supported nodes, based on the constraints imposed by the support, which enables solvable expressions:

$$\mathbf{F}_l - \mathbf{K}_{ls}\mathbf{U}_s = \mathbf{F}_{l.eq} = \mathbf{K}_{ll}\mathbf{U}_l \quad (5.40)$$

$$\mathbf{R}_s = \mathbf{F}_s = \mathbf{K}_{sl}\mathbf{U}_l + \mathbf{K}_{ss}\mathbf{U}_s \quad (5.41)$$

The first expression would allow the determination of free displacements $\mathbf{U}_l$ by establishing the conjugate loads with these degrees of freedom, $\mathbf{F}_l$, along with the values of the forced or nullified displacements at the supported points, $\mathbf{U}_s$. If they are null, the relationship between loads and displacements at the free nodes is immediate: $\mathbf{F}_l = \mathbf{K}_{ll}\mathbf{U}_l$.

If they are non-null, such movements at the support will be interpreted as the equivalent loads that would be necessary to impose on the structure to make them compatible with zero movements of the free points: consider the equation 5.40 applied to the structure with no other loads than those necessary to impose the non-null pre-fixed values in $\mathbf{U}_s$ and null values in $\mathbf{U}_l$.

The second of these expressions, already seen in 1.29, allows the determination of reactions, which are nothing more than the forces applied at the support points necessary to maintain the equilibrium state with the loads applied to the model that are compatible with the displacements pre-fixed at the support.

In computer programs, the construction of the complete stiffness matrix in expression 1.28 is straightforward, but the reduction 5.40 is not as simple in cases where the movements $\mathbf{U}_s$ at the support are non-null. Therefore, occasionally the following trick procedure is applied: a very high number Ξ is placed on the pivots⁵ of $\mathbf{K}_{ss}$. In the force vector, fictitious forces $\mathbf{F}_s = \Xi\mathbf{U}_{s,c}$ are placed where $U_{s,c}$ is the known or prescribed value for the

5. the terms on the main diagonal of the matrix. It should be noted that altering any pivot eliminates the linear dependence of that row of the stiffness matrix with respect to the others, which, by doing so on all constrained rows, eliminates the singularity of the matrix.

corresponding U_s . Therefore, in the approximation, the corresponding equation to solve for a certain degree of freedom s will be

$$F_s = \Xi U_{s,c} = \sum_{i \neq s} K_{si} U_i + \Xi U_s$$

where the coefficients of the terms different from s would be small relative to Ξ , thus obtaining $U_s \approx U_{s,c}$.

5.2.3 Other Forms of Constraint

It is possible to consider additional forms of constraint, or the imposition of restrictions on the model's displacements. For example, in floor structure models based on frames, it is possible to represent only the frame bars. In this case, the existence of slabs — of rigid diaphragms between different frames — implies that the horizontal displacements of all nodes located on the same floor must be linked to each other, as they all derive from the displacement and rotation of the floor.

There may be other cases, such as size reductions of the model by imposing regularity conditions (turbine wings), etc. That is, some or all displacements $\mathbf{U}$ will depend, through imposed compatibility, on a smaller number of global displacements $\mathbf{U}_g$: $\mathbf{U} = \mathbf{B}\mathbf{U}_g$.

In general, this implies that there are equations constraining the displacements, in the form

$$\mathbf{C}\mathbf{U} = \mathbf{0}$$

where the list of displacements $\mathbf{U}$ could be separated between

the *free* $\mathbf{U}_l$ and the remaining restricted or constrained $\mathbf{U}_r$ having

$$\begin{bmatrix} \mathbf{C}_l & \mathbf{C}_r \end{bmatrix} \begin{bmatrix} \mathbf{U}_l \\ \mathbf{U}_r \end{bmatrix} = [\mathbf{0}]$$

$$\mathbf{U}_r = -\mathbf{C}_r^{-1} \mathbf{C}_l \mathbf{U}_l \quad (5.42)$$

making it possible to substitute the degrees of freedom $\mathbf{U}$ by $\mathbf{U}_l$, and, considering the duality 1.2.5, the corresponding conjugate loads and their matrix relationship with displacements:

$$\mathbf{U} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{C}_r^{-1} \mathbf{C}_l \end{bmatrix} \mathbf{U}_l = \mathbf{T} \mathbf{U}_l \quad (5.43)$$

$$\mathbf{F}_l = \mathbf{T}^T \mathbf{F}; \quad \mathbf{K}_l = \mathbf{T}^T \mathbf{K} \mathbf{T}. \quad (5.44)$$

5.2.4 Discretization of Element Behavior

Based on what has been seen in the preceding sections, we can now proceed to develop the properties of each of the elements into which we decompose the structure, properties that will be summarized in a limited, discrete set of variables. To do this the following must be done:

- to analyze the format of the equations at the element level
- to identify the interpolation functions to be used
- to construct the load and stiffness expressions of the element that will subsequently be assembled into the complete structure model

5.2.4.1 Equations in the Element

Nodes are considered in the element where degrees of freedom (or possible displacements) $\mathbf{u}_e$ are established, to which will correspond the conjugate forces on the element $\mathbf{f}_e$ that are exerted from the other elements in contact or as boundary loads on the element nodes that are on the exterior surface.

By the principle of virtual work, considering a compatible strain and displacement law $\bar{\epsilon}$, $\bar{\mathbf{v}}$, for which the corresponding nodal movements are $\bar{\mathbf{u}}_e$, we will have

$$\int_{\Gamma_e} \bar{\mathbf{v}}^T \boldsymbol{\varphi} d\Gamma + \bar{\mathbf{u}}_e^T \mathbf{f}_e = \int_{\Omega_e} \bar{\boldsymbol{\epsilon}}^T \boldsymbol{\sigma} d\Omega \quad (5.45)$$

Now, we are interpolating $\bar{\mathbf{v}}$ in the form $\bar{\mathbf{v}} \approx \mathbf{N}\bar{\mathbf{a}}$, so that

$$\bar{\mathbf{a}}^T \int_{\Gamma_e} \mathbf{N}^T \boldsymbol{\varphi} d\Gamma + \bar{\mathbf{u}}_e^T \mathbf{f}_e = \bar{\mathbf{a}}^T \int_{\Omega_e} (\partial \mathbf{N})^T \boldsymbol{\sigma} d\Omega$$

and if we choose as interpolation parameters $\mathbf{a}$ the same degrees of freedom $\mathbf{u}_e$, it will result $\bar{\mathbf{u}}_e = \bar{\mathbf{a}}$. Considering that the previous expression must be true for any virtual movement and that $\boldsymbol{\sigma} \approx \boldsymbol{\kappa} \partial \mathbf{N} \mathbf{u}_e$, it finally results

$$\int_{\Gamma_e} \mathbf{N}^T \boldsymbol{\varphi} d\Gamma + \mathbf{f}_e = \int_{\Omega_e} (\partial \mathbf{N})^T (\boldsymbol{\kappa} \partial \mathbf{N}) d\Omega \mathbf{u}_e \quad (5.46)$$

where we have the load vector for the loaded surface regions of the element

$$\mathbf{f}_{E,e} = \int_{\Gamma_e} \mathbf{N}^T \boldsymbol{\varphi} d\Gamma \quad (5.47)$$

and the elastic equilibrium condition in the element

$$\begin{aligned} \mathbf{f}_{E,e} + \mathbf{f}_e &= \mathbf{k}_e \mathbf{u}_e; \quad \mathbf{k}_e = \int_{\Omega_e} \mathbf{b}^T \boldsymbol{\kappa} \mathbf{b} d\Omega, \quad \text{with} \\ \mathbf{b} &= \partial \mathbf{N} = \mathbf{N}'; \quad \mathbf{N} = [\mathbf{N}_1(\Omega_e), \mathbf{N}_2(\Omega_e), \dots]. \end{aligned} \quad (5.48)$$

an expression in which the sum of the external load $\mathbf{f}_{E,e}$ and the internal efforts exerted by the rest of the structure on the element $\mathbf{f}_e$ are in the elastic equilibrium defined by the strains induced by the nodal displacements $\mathbf{u}_e$. The total forces on the element result equal to the stiffness of this by its nodal displacements.

If we define, therefore, the set of nodal functions $\mathbf{N}_i(\Omega_e)$ associated with each of the nodes i of the element such that they allow interpolating the movements of any point within the element from the movements $\mathbf{u}_{e,i}$ of such nodes, it will be possible to determine the external loads and local stiffness in the element, necessary to assemble the global structure, in the manner already studied for bar structures, in section 2.1.2.2.

5.2.4.2 Nodal Functions (or Shape Functions)

The strategy for establishing the interpolation functions is none other than attributing to each element and each node nodal or shape functions whose value is unity at the coordinates of said node, and zero at the coordinates of the other nodes of the element. Indeed, if this is the case, since the interpolation parameters are the movements of the nodes, the interpolation will assign to these such movements $\mathbf{u}_e$, attributing to the rest

of the points of the element movements compatible with these:

$$\mathbf{v}(\Omega_e) = \mathbf{N}(\Omega_e)\mathbf{u}_e = [\mathbf{N}_1(\Omega_e), \mathbf{N}_2(\Omega_e), \dots] \cdot [\mathbf{u}_1, \mathbf{u}_2, \dots]^T \quad (5.49)$$

$$\begin{aligned} \mathbf{u}_i = \mathbf{v}(\mathbf{x}_i) &= [\mathbf{0}, \dots, \mathbf{I}, \dots] \cdot [\mathbf{u}_1, \dots, \mathbf{u}_i, \dots]^T, \\ \text{being } \mathbf{N}_i(\mathbf{x}_i) &= 1; \mathbf{N}_i(\mathbf{x}_j) = 0, i \neq j. \end{aligned} \quad (5.50)$$

Additional continuity conditions will be seen later according to what was pointed out in section 5.2.1.

5.2.4.3 Getting the External Loads and Element Stiffness

The expression of the loads applied to the surface region of the structure that coincides with an element e is, equation 5.47:

$$\mathbf{f}_{E,e} = \int_{\Gamma_e} \mathbf{N}^T \boldsymbol{\varphi} d\Gamma = \begin{bmatrix} \int_{\Gamma_e} \mathbf{N}_1 \boldsymbol{\varphi} d\Gamma \\ \int_{\Gamma_e} \mathbf{N}_2 \boldsymbol{\varphi} d\Gamma \\ \vdots \end{bmatrix} \quad (5.51)$$

Similarly, from expression 5.48, the construction of the element stiffness can be derived:

$$\mathbf{k}_e = \int_{\Omega_e} \begin{bmatrix} \mathbf{N}'_1 \\ \mathbf{N}'_2 \\ \vdots \end{bmatrix} \boldsymbol{\kappa} [\mathbf{N}'_1 \quad \mathbf{N}'_2 \quad \dots] d\Omega = \begin{bmatrix} \int_{\Omega_e} \mathbf{N}'_1 \boldsymbol{\kappa} \mathbf{N}'_1 d\Omega & \int_{\Omega_e} \mathbf{N}'_1 \boldsymbol{\kappa} \mathbf{N}'_2 d\Omega & \dots \\ \int_{\Omega_e} \mathbf{N}'_2 \boldsymbol{\kappa} \mathbf{N}'_1 d\Omega & \int_{\Omega_e} \mathbf{N}'_2 \boldsymbol{\kappa} \mathbf{N}'_2 d\Omega & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad (5.52)$$

The assembly will attribute node i of the element to Node I in the structure ($i \rightarrow I$) according to the connectivity established in the model, such that each load or stiffness component of the element will be accumulated in the assembly process in the corresponding component of the structure, considering if necessary the required axis changes, in the case that their positions or alignments do not coincide, in accordance with what is described in 2.1.2.2.

5.3 Numerical Procedures

We have seen up to this point that it is feasible to construct the weak expression of the field equations through integral expressions (5.36) susceptible to being addressed by separate elements, (5.38, 5.51, 5.52) and that allow transforming the field problem into a system of linear equations (5.37) in nodal displacements that are used as parameters for interpolating the solution.

The problem now will be, therefore, to establish the way to address the integration phases necessary for determining the numerical coefficients of these equations. The usual strategy is to undertake integration through numerical procedures that allow an acceptable approximation of these coefficients.

5.3.1 Numerical Integration

The most common procedures for integrating the functions resulting from the weak formulation are analyzed below. The resulting continuous functions can generally be considered as polynomial functions, either because they arise from products of several functions if, as is usual, polynomial functions are used for shape functions, or because the resulting functions can be sufficiently approximated by series expansions. In this way, we only need to consider the problem of numerical integration, or quadrature, of polynomial functions.

5.3.1.1 Integration of Polynomial Functions

Given a polynomial function, its indefinite integral is immediate

$$f(x) = \sum_{i=0}^n a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots$$
$$\int f(x) dx = C + a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \dots \quad (5.53)$$

The numerical approximation to its definite form, by methods such as Simpson's etc., is also simple if values of the function are

available at a sufficient set of points in the interval of interest

$$\begin{aligned} \int_a^b f(x)dx &= a_0(x_b - x_a) + a_1 \frac{x_b^2 - x_a^2}{2} + a_2 \frac{x_b^3 - x_a^3}{3} + \dots \\ &\approx \sum_j W_j f(x_j), \quad x_j \in [x_a, x_b]. \end{aligned} \quad (5.54)$$

In the simple trapezoidal rule, for example, the points x_j are placed equidistantly over the interval and the weights W_j compute this equidistance for the inner points and their half for the endpoints.

It is of interest the same expression for the case where the interval is arranged symmetrically with respect to the origin, given that now the expression reduces the number of coefficients to be considered for a certain polynomial degree, and even more so when the endpoints have coordinates of unity:

$$\int_{-1}^1 f(x)dx = 2a_0 + 2\frac{a_2}{3} + 2\frac{a_4}{5} + \dots \quad i \in \text{even} \quad (5.55)$$

5.3.1.2 Gauss Integration

Gauss integration or quadrature, figure 5.5, seeks to establish, for the interval $[-1, 1]$, the best possible combination, in approximating the integral, of coordinates for determining the function value and its respective weights, considering the simplicity of expression 5.55 to approximate that version of the summation 5.54. To this end, and depending on the degree of the polynomial to be integrated, the function in 5.53 will require coefficients a_i up to the desired degree. A number of (symmetric)

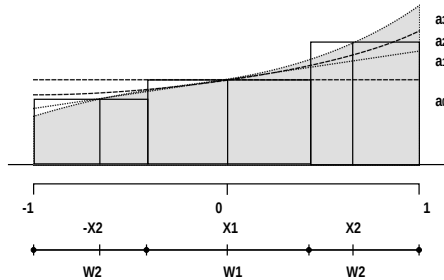


Figure 5.5: Gauss Integration

integration points is chosen, which implies the same number of terms in the expression of 5.54, and allows considering the same number of even coefficients. As the sum must equal the expression of these same coefficients and their multipliers in 5.55, this allows determining the corresponding weights and coordinates.

A (symmetric) point allows exact integration of linear expressions:

$$W_1 f(x_1) = 2a_0 \implies W_1 = 2, x_1 = 0 \quad (5.56)$$

Two (symmetric) points facilitate exact integration of polynomials up to the third degree:

$$\begin{aligned} W_1 f(x_1) + W_1 f(-x_1) &= W_1 2(a_0 + a_2 x_1^2) = \\ 2a_0 + 2\frac{a_2}{3} &\implies W_1 = 1; x_1 = \pm\sqrt{\frac{1}{3}} \end{aligned} \quad (5.57)$$

With three (symmetric) points, integration is exact for polynomials up to the fifth degree:

$$\begin{aligned}
 W_1 f(x_1) + W_2 (f(x_2) + f(-x_2)) &= \\
 W_1 a_0 + W_2 2(a_0 + a_2 x_2^2 + a_4 x_2^4) &= \\
 2a_0 + 2\frac{a_2}{3} + 2\frac{a_4}{5}; \text{ with} & \\
 W_1 + 2W_2 = 2; 2W_2 x_2^2 = \frac{2}{3}; 2W_2 x_2^4 = \frac{2}{5} \implies & \\
 x_1 = 0; x_2 = \pm\sqrt{\frac{3}{5}}; W_1 = \frac{8}{9}, W_2 = \frac{5}{9} & \quad (5.58)
 \end{aligned}$$

Classic texts related to the FEM include Gauss points and weights tables, which allow exact integration of polynomials up to degree $2n-1$, with n being the number of Gauss points used. So the integrals adopt the form

$$\int_{-1}^1 f(x) dx \approx \sum_j W_j f(x_j), \quad x_j, W_j \in \text{Gauss Table.} \quad (5.59)$$

5.3.2 Nonlinear Problems

We have described a means of transforming the elastic differential problem into a solvable system of linear equations, solvable through algebraic resources present in all scientific software libraries. However, many problems of interest are nonlinear, either because the material behavior is nonlinear, or due to inadequacy in the geometric approximation employed, which presupposes small deformations and displacements, allowing acceptance that the slight alteration of geometry does not influence

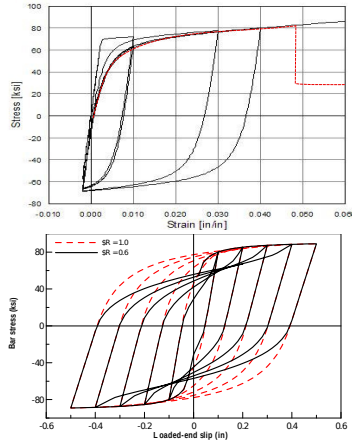


Figure 5.6: Material Nonlinearity, with hysteresis cycles. Local cases in metals (upper image) or bolted bars with the possibility of sliding (lower image).

behavior. These are two sources of nonlinearity in behavior, material and geometric, which can nevertheless be of relevant importance in the behavior of some structures.

Below we briefly note some particularities of these types of problems.

5.3.2.1 Material Nonlinearity

In situations where the material behaves nonlinearly, the usual problem is not only the loss of proportionality between stresses and strains: the material's *elasticity* could be maintained if that nonlinearity were the entire behavior, in the sense that the deformation energy could be completely recovered in cycles with loading and unloading phases where the path in the stress-strain graph would coincide in both phases without energy loss. However, the more common behavior involves non-coincident paths, where unloading begins with graph slopes similar to those at the start of loading, such that the entire cycle graph shows an energy loss in the phenomenon called *hysteresis*, which is often described as the superposition of two deformation components in the process, the elastic part, recoverable, and the plastic part, with energy loss through plastification and progressive material damage, as represented in figure 5.6, damage that even could progressively affect the geometry of the stress-strain curve in successive cycles.

Thus, in these types of problems, the goal is not only to describe the nonlinearity in the relationships between stress and strain, but to incorporate the description of some type of damage parameter and its influence on the curve geometry of the loading and unloading cycle.

The problems to be represented therefore include those of the evolution in stress-strain relationships dependent on previous history, which can include phenomena of increase or reduction of the elastic limit (the increase case, for example in cold-drawn steel hardening phenomena) of stiffness and/or strength losses,

and the characterization of the corresponding hysteretic cycles.

These are phenomena that will affect the so-called *constitutive* relations or relations derived from the constitution of the material.

Despite its importance, a detailed study of this set of problems is beyond the scope of this text.

5.3.2.2 Geometric Nonlinearity

As studied in chapter 3, another important source of nonlinear behaviors is that derived from the consideration on the final equilibrium conditions of the influence of the structure's deformation or displacement: a problem would remain linear in cases of linear elasticity of materials if, and only if, it is assumed that the deformed geometry is sufficiently close to the original geometry so as not to influence the equilibrium conditions, which are formulated over the original geometry. In the contrary case, that is, if it is necessary to consider the formulation of the equilibrium conditions over the deformed geometry, we are in problems of geometric nonlinearity, whether due to significant (large) displacements even if local deformations are small as can be buckling cases, or due to large deformations, corresponding to the behavior of highly deformable materials, such as rubber.

Addressing the problems of large displacements, as we did in chapter 3, involves considering the variation in stiffness derived from axial loading and its formalization through the concatenation of the following procedures

- analysis and solution of the eigenvalue problem and the corresponding buckling modes
- consideration of imperfections within the framework of manufacturing tolerances, selected proportional to those of the buckling modes (particularly the first) and
- incremental nonlinear analysis starting from this imperfection up to the equilibrium situation corresponding to the required design load.

It is worth recalling the formulations developed in section 3.5.3 that

- allow the determination of the critical load factor, 3.5.3.2,
- formalizing the generalized eigenvalue problem for a given load state and
 - addressing it as an eigenvalue problem, 3.5.3.3,
 - or alternatively through its iterative approximation using the Rayleigh quotient, 3.5.3.4.

For the case of large deformations, the formulation with Cauchy models for stress and strain states are incorrect, and one must handle the Green-Lagrange tensors for deformation and the second Piola-Kirchhoff tensor for stress. These are again methods beyond the scope of this text.

5.3.2.3 Solution Methods

The numerical solution of nonlinear problems requires using iterative procedures in which the use of a double iteration is common.

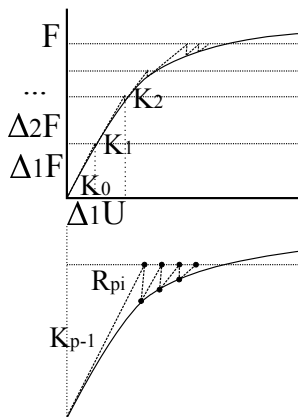


Figure 5.7: Newton–Raphson and Modified Newton–Raphson Procedures

In each of the steps of an incremental process at a certain time⁶ it must be determined the response $\mathbf{F}_{Ri}(\mathbf{U}_i)$ for the corresponding state i of loads and behavior. Starting with an initial state in step 0 with zero displacement, it is usual to consider for each step the load corresponding to an increasing factor: $\lambda_i \mathbf{F} = \mathbf{F}_i = \sum_i \Delta_i \mathbf{F}$, with $\lambda_0 = 0, \lambda_i \in (0, 1]$. For each load step the displacement increment $\Delta \mathbf{U}_i = \mathbf{U}_{i+1} - \mathbf{U}_i$ necessary to balance the difference between the load and the previous response is determined through the approximation $\lambda_{i+1} \mathbf{F} - \mathbf{F}_{Ri}(\mathbf{U}_i) =$

6. a fictitious time: this is not a dynamic analysis in which the consideration of inertial forces would be essential.

$K_i \Delta U_i$. It is usual to iterate at each load step to improve convergence. Thus arise the most widespread methods, schematized in figure 5.7.

Newton–Raphson The following recurrence is adopted for successive p load steps, and for each iteration i within them

- For displacement U_{pi} , the response is F_{Rpi} , with an unbalanced “residual” $R_{pi} = \lambda_p F - F_{Rpi} = \lambda_p F - F(U_{pi})$. Therefore, the “load” R_{pi} must be balanced. If the residual is sufficiently small, the process stops for that load step and moves to the next step.
- In order to reestablish the equilibrium, the stiffness to be used is that corresponding to the displacement reached up to that moment $K_{pi} = K(U_{pi})$
- The required displacement increment is estimated through the approximation $R_{pi} = K_{pi} \Delta U_{pi}$. The displacement reached will therefore be $U_{p,i+1} = U_{pi} + \Delta U_{pi}$

Modified Newton–Raphson The “stiffness” K_0 is maintained throughout the process (or for a given number of steps or load increments, . . . after which it is recalculated for a new group of load steps)

5.3.3 Computer Program Organization

In view of the above, it is feasible to propose the organization of a computer program capable of undertaking the modeling of elastic problems, their numerical representation and solution.

For this, a modular organization is adopted so that each module is independent of the content or processes of the others. Each module operates on memory regions, and only input or output modules communicate that memory with the exterior. The fundamental modules are outlined below.

Data Inputs Procedures for the identification of Materials, the Elements' selection (their shape functions $\mathbf{N} \dots$, Gauss Points and Weights. . .), the Structural Mesh layout (coordinates-connectivity), and the supports.

General Methods Modules for the determination, isolated and on-demand, of each group of numerical values:

stiffness: For each element, and for each Gauss point: material κ , generates the matrices $\mathbf{b} = \partial \mathbf{N}$, and integrates $\mathbf{k}$; assembles $\mathbf{K}$

load: For each element, and for each Gauss point, generates and integrates the vector $\mathbf{f}_E$; assembles $\mathbf{F}$;

solve: Solves $\mathbf{F} = \mathbf{K}\mathbf{U}$ (or rather $\Delta \mathbf{F} = \mathbf{K}\Delta \mathbf{U}$ in the more general case of incremental procedures)

displace: Updates $\mathbf{U} = \mathbf{U}_0 + \Delta \mathbf{U}$

stresses: computes $\mathbf{u} = \mathbf{B}\mathbf{U}$, $\mathbf{f} = \mathbf{k}\mathbf{u}$, $\epsilon = \mathbf{b}\mathbf{u}$, $\sigma = \kappa\epsilon$

reactions: $\varphi_{\Gamma,s} = \sigma n_{\Gamma,s}$

residuals: with $\mathbf{f}_e = \mathbf{k}(\mathbf{u})\mathbf{u}$ determines the unbalanced $\Delta \mathbf{F} = \mathbf{F} - \mathbf{B}^T \mathbf{f}_e$

Outputs On demand: any of the variables or groups or combinations of these in memory in any region of the structure.

Internal States (Memory)

- Global: coordinates, displacements, (loads and load step), (topology)
- Local: Elements: properties, loads, connection, strains, stresses, degradation or damage

Global Methods: Usual Order

- stiffness, load, solve, displace, reactions, residuals

Element Methods: By Global Call

- stiffness, load, stresses, residuals

General Analysis Scheme: Until residual=0, displace(solve(previous residual+ Δ load))

Chapter 6

The Finite Element Method

This chapter develops in more detail the method described in section 5.2, presenting both an overview of method usage, as employed in most available programs, as well as some of the appropriate techniques for the most common element types, with particular emphasis on how such elements are formulated and the resources available to ensure acceptable precision and error control. The intention is that this vision facilitates an informed and critical use of available tools, whether commercial or open-source.

Basic sources for the development of this section are Zienkiewicz 1977, Zienkiewicz 1980, Bathe 1982, Clough and Penzien 1975

and Wilson 2002.

6.1 Bases of Use of the Method

The method is based on a description of the static and kinematic behavior of the structure, no matter how complex, through the assembly of a number, generally very high, of elements that individually turn out to be relatively simple. The modeling and analysis of the structure therefore supposes a process that integrates, *(a)* the description of the structure and its boundary conditions (loads and supports) through the description of the elements that conform it and their connections, *(b)* the simulation of the model's behavior through the determination of movements and deformations suitable for fulfilling the three usual groups of equations in structural analysis (equilibrium, compatibility, and material constitutive relations), and *(c)* the detailed exploration of the values of the resulting variables in all points of the model to determine its viability, safety, or efficiency.

We have, therefore, three phases in the analysis, which we will now describe in more detail.

6.1.1 Phases of Analysis

The three phases, alluded to in figure 6.1 and which include the core of the analysis, that is, the determination of the model's state for the anticipated loads and support conditions, can be identified as:

- modeling (preprocessing)

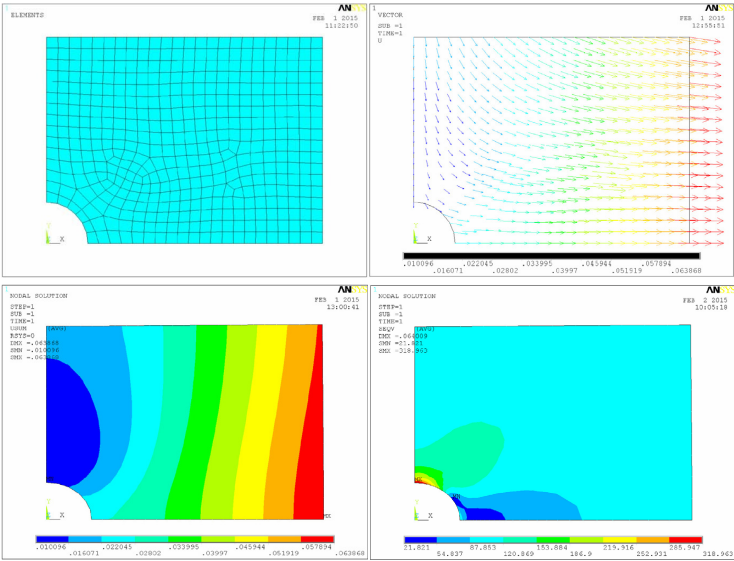


Figure 6.1: Phases of Analysis in the Finite Element Method: *modeling*, *analysis* and *evaluation*. Here (1) is the meshing to represent a plate with a hole under tension, with simplification by double symmetry, (2) the determination of displacements, and (3, 4) the interpretation of results, in this case the magnitude of such displacements and the comparison stress according to the Von-Mises model, the latter determined from the analysis on a new mesh, more refined at the hole's contour than the one in the first figure.

- analysis (processing)
- evaluation (postprocessing)

6.1.1.1 Modeling, or Preprocessing

In this phase, the objective is to construct a model that adequately represents the structure under study by decomposing it into relatively simple and finite-sized elements appropriately linked to each other. The goal is to ensure (1) that, locally, in each element, both the geometry and the formulation of the laws linking the static and kinematic variables of its behavior are simple and sufficiently close to those of a theory appropriate to the type of behavior expected in that region of the structure and, (2) that the representation of the connection between such elements and with the boundary (loads and supports) ensures appropriate movement identity and equilibrium in contact regions.

It therefore implies decisions regarding the type of behavior to be described: dimensionality (2D or 3D), relevant types of deformations and stresses (axial or bending), possible simplifications (by symmetries or other considerations)...

In this phase, therefore, the key decision is about the types of elements to be used in the model, their distribution throughout the structure's volume, and their typical size and shape in each region of this (meshing), the connectivity between these and with the boundary... It is usual for meshing to be performed automatically in large regions based on parameters established at the limits of such regions.

In choosing the type of element and its typical dimensions, it is necessary to consider the shape of the functions implemented in these and their possible adaptation to the expected response forms in specific regions, both within each element and in the transitions between elements.

It is equally of extreme importance, in cases of using various types of elements, to consider the appropriate connection conditions to ensure continuity in displacement and effort fields.

Element Type Selection

Depending on the geometry and behavior of the piece or region to be represented, the modeling will involve using one of the types provided by the program in question and, therefore, the qualities capable of being represented by that type of element. Denominations used in the Sap program are used here.

- linear region
 - for purely axial behavior: *truss*
 - if also combined with bending: *frame*
- surface region
 - for membrane (axial) behavior: *plane*
 - including bending behavior: *shell*
- volumetric region
 - 3D elastic element: *solid*

Elements are defined geometrically by reference of the nodes of the generic definition for the element type selected to specific nodes (coordinates) of the geometry of the structure being modeled, and in their material qualities by reference to typical

materials and specific values of such material properties, as well as by values of typical geometric properties of the selected element (section properties, etc., often by reference to groups of properties defined independently of specific elements and common to many of them)

Geometry and Materials Description

It is important to be clearly aware of the multiplicity of references that can be used: the structure's geometry will be defined in some sort of global axes, which can be defined as Cartesian, or polar or spherical in appropriate cases, and even specific regions of the general geometry may be better defined in axes with a local or particular orientation.

The elements, in turn, maintain local axes suitable to the formulation, so that any information provided or requested must be done with a clear definition of which reference axes are pertinent.

Regarding materials, depending on the type of analysis to be performed, their elastic or plastic properties, densities or specific weights, damping properties, or hysteretic properties, etc., must be considered and represented.

Boundary Conditions: Loads and Supports

It is customary in software to consider three differentiated groups of concepts

- load patterns,
- load cases,
- load case combinations.

A load pattern is a set of loads or movement restrictions coming from a common source that allows grouping them all in a single reference (for example, *structural mass*, or *north wind*). Each of the load patterns will involve establishing one or several of the two possible types of boundary conditions:

- loads, whether considering the masses of the elements, or specifically applied to these or to structure nodes.
- movement restrictions in certain structure nodes, either as complete constraints (supports, anchoring...) or as imposed displacements, or as links or forced combinations of some of these

In relevant cases, temperature conditions can be added: it is usual to consider default temperatures of 20 degrees Celsius, so that forced dilation or contraction effects—which are sometimes used to simulate prestressing effects—can be modeled by establishing different temperatures and the corresponding thermal expansion coefficients in the materials used.

Load cases establish groupings of patterns corresponding to some possible action hypothesis on the structure, and the way in which each pattern is applied. For example, establishing a load case corresponding to a certain dynamic analysis against wind might include the constant application in time of the structure's mass (pattern 1) and a certain characteristic mass supported on this (pattern 2) as well as the structure's ground anchoring conditions (pattern 3), along with the dynamic application in time of a certain scale of a lateral wind load pattern (pattern 4). Other possible load cases would correspond to other directions or scales of wind action, to variants for the application of

gravitational load, etc.

Load case combinations correspond to explorations of some form of combined effect of several of these cases. Such combinations can be of various types, depending on the desired result:

- sums, if the goal is to determine the effect of their superposition
- envelopes, if the intention is to establish the extreme values that result in each region of the structure
- “expected” values if what is sought is to determine the probable combination values (and perhaps other statistical properties) of the combination of probabilistic cases.

6.1.1.2 Analysis, or Process

This phase seeks to establish the state or sequences of states of the structural model through appropriate numerical methods.

The analysis process will construct, from the model information and the analyst’s intentions, the three usual groups of equations in structural theories, and solves the unknowns—usually displacements—that can be determined with these equations. In the description of the analysis to be performed, various approaches can be distinguished, depending on whether it is a static or dynamic analysis (in which inertial and damping effects will be considered...), a linear or non-linear analysis (and the type of non-linearity to be considered), as well as, in very particular cases such as the hybrid approximations described in section 5.2.1, the type of algorithm to consider in the analysis, insofar as the resulting fundamental matrix, usually the stiffness

matrix, the matrix that condenses the coefficients of all problem equations, may be non-symmetric, or for other reasons.

The result of the process is, therefore, the determination of the state variables corresponding to the unknowns in the employed formulation, usually displacements, for all and each of the load cases established in the modeling and for the required type of analysis.

Type and Analysis Algorithm

Three major groups can be considered in the types of analysis that can be addressed (and which must be declared to the program)

- Static: without consideration of inertia and damping effects:
 - linear (elastic)
 - incremental non-linear (or *pushover*) where part of the load increases in successive steps
 - “geometric” non-linear, with consideration of axial effects on stiffness in successive load steps
- Dynamic
 - analysis in the time domain, which can be done
 - * through modal response analysis and superposition procedures. Either through the eigenvalues–eigenvectors analysis got from the classical dynamic expression $(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{\Phi} = \mathbf{0}$, or from a sequence of the so-called *Ritz vectors*, particularly those *load dependent*¹,

1. the first one being proportional to the static deformation, an the following orthogonal to the preceding ones and to the mass matrix, see Wilson

- * through analyses that perform step-by-step time response integration
- analysis in the frequency domain seeking either the stationary response to a specific excitation, or the spectral response to a certain frequency range.
- Stability or global buckling analysis
 - *linear* approximation of global buckling through the eigenvalue problem of $(\mathbf{K} - \lambda \mathbf{K}_G(\mathbf{F}))\Phi = \mathbf{0}$.

For the required analysis, the program selects and activates the necessary algorithms and will generate the corresponding result data bases to be used later in post-processing.

Considerations on Non-linearity in Response

In most programs, non-linear properties in the response can be established, either through: non-linearity in material properties, via elastic stress limits or explicit stress-strain laws; non-linearity of specific components, such as moment-rotation laws in hinges or laws defining $P\Delta$ effects in cables; or non-linearity of the complete model, of the general geometry, due to global $P\Delta$ effects or the possibility of considering large displacements, that is, equilibrium conditions in the deformed geometries of each load step.

Considerations on Damping Properties in Dynamic Analysis

Sources of damping in dynamic behavior can be of two types:

2002, chapter 14.

viscous, dependent on interparticle velocities, and hysteretic, dependent on displacement history. Classic dynamic formulations easily incorporate the former through a matrix $\mathbf{C}$ of relations between responsive forces and velocities corresponding to each degree of freedom, and with much greater difficulty the latter, through a time-varying matrix $\mathbf{D}$ of relations between forces corresponding to energy dissipation that displacements entail for each degree of freedom and those same displacements.

For viscous damping, the matrix $\mathbf{C}$ is usually constructed, in the absence of a more appropriate and efficient theory, as a combination of mass and stiffness matrices, either by load case ($\mathbf{C} = c_k \mathbf{K} + c_m \mathbf{M}$), by material ($c_i = c_k \mathbf{k}_i + c_m \mathbf{m}_i$), or by modal combination. This relevant issue requires a dynamic justification that cannot be addressed in this text, but which nonetheless implies adopting specific decisions in declaring the type of analysis to be performed by the program.

6.1.1.3 Evaluation, or Post-processing

In this phase, the information data bases constructed in previous phases are exploited to characterize the structure's response and determine its safety or to provide the information that would facilitate the incorporation of modifications capable of improving the proposal.

Based on this information, the program user can trace on demand the values of other unresolved state variables, but deducible from the previous information, such as resulting deformation and stress fields, etc.

With this information, the states in each region of the struc-

ture can be compared with the admissible limit values for the desired safety degree, etc.

A very important part of this final phase is evaluating the results' validity. In this evaluation, it is essential to verify that the expected equilibriums are met: although the system adequately resolves the equilibriums for the provided data, there could be errors in the definition of loads or constraints resulting in responses alien to the intended objective due to model errors. Therefore, in this basic preliminary evaluation, an extremely necessary practice is to check that **the reactions obtained**

- correspond to and equal the sum of applied loads
- agree with the anticipated support conditions.

In post-processing

- plots of field (state) functions or their envelopes can be obtained, whether displacements, load case effort fields, or envelopes of load combination groups
- integral values for given regions can also be obtained (forces in sections of the structure itself, for example) through the definition of the cut and the sought integral expression. This allows checking the plausibility of the response by comparison with simpler models.

By type of analysis, it will be possible to determine

- in modal dynamic analysis: oscillation frequencies and modes, their participation factors, as well as the corresponding effective modal mass factors, where usually it will suffice to count with modes that accumulate an effective modal mass between 90% and 95%.

- in spectral response analysis, the response acceleration spectra for each mode, the corresponding modal amplitudes, and responses combined by standard methods, SRSS, (Square Root of Sum of Squares) or CQC (Complete Quadratic Combination)
- in static non-linear analysis, displacement-deformation-effort-load laws are determined. Attention must be paid in this type of analysis to the method for the incremental process control, whether by load or displacement increments, with the former being unstable at the load limit.
- the most sophisticated and costly analyses are those that solve dynamic processes as temporal series of states (“time history analysis” or THA), usually by direct integration, although not exclusively: FLA or “Fast Linear Analysis” is an alternative in programs like SAP. In these cases, given the abundance of information that can be generated, it is important to consider the required information, so that prior to the analysis, appropriate elements for tracking can be established (which in some programs are called *recorders*).

The evaluation of the results’ precision and margin of error is, finally, what will allow validating the obtained results as sufficiently representative of the modeled structure’s response. Due to its importance, the next section will be dedicated to this aspect.

6.1.2 Error Control: Methodology

In this section, we analyze the possible sources of error, whose control is necessary to ensure the utility of the analysis results. We examine the continuity requirements of field functions within the elements that divide the structure and at their interfaces, and the ways to quantify and address potential errors or refine the analysis results.

6.1.2.1 Sources of Error

Errors can arise from various causes, not always resulting from inaccuracies in modeling. We can group them into the following categories and causes:

- *Discretization*: considering continuity criteria or the precision of the functions employed,
- *Computation of integral expressions*: effects of numerical integration,
- *Geometry approximation*: discretization of shape and modeling of links and boundary conditions,
- *Numerical errors in solution obtainment*: associated with matrix conditioning, truncation, or rounding,
- *Material Constitutive Equations*: defects in parameter values or in their relationships in non-isotropic situations...

We will examine these in greater detail below.

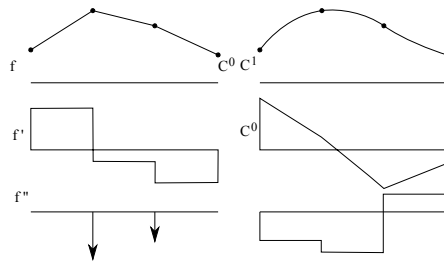


Figure 6.2: Continuity of the displacement field: within the element, through the polynomials chosen for shape functions; at the interface between them, if the displacement depends only on shared nodes. Derivatives have lower continuity orders. Terms resulting in non-zero values in integrals across the interface between one segment (element) and the next must be avoided.

6.1.2.2 Discretization, Functions Employed, and Integral Computation

Since the technique employed in the method involves replacing the continuous geometry with a discrete mesh of elements, the first cause of potential errors lies in the possibility that such substitution may not be adequate because the functions used and the resulting integrals from the formulation are unable to sufficiently approximate those expected in the differential solution of the continuous model, either within the elements or at their interfaces.

Thus, it is necessary to verify:

- The required continuity criteria:
 - This implies ensuring sufficient differentiability and integrability in the functions employed, such that the expressions used make sense: 5.47, 5.51, 5.48, 5.52. Therefore, a sufficient degree of continuity must be ensured for shape functions and their derivatives (see figure 6.2).
 - At the interface between elements, the identity in displacement fields and their derivatives must be ensured when viewed from any of the elements in contact, to guarantee the possibility of substituting domain integrals with their sum across elements, equation 5.38, which requires such integrals to be null when crossing said interface, and, insofar as the equations include a differential operator on these fields, implies excluding the possibility of singularities in these derivatives at the interface.
- The precision achieved through approximation:
 - where proximity of the approximated function to the sought function can be improved, either by reducing mesh size or increasing the order of functions employed. These are the “ h ” (representative element size) and “ p ” (polynomial degree of functions used in the element) methods
 - this improvement can be automated through adaptive meshing and coupling of differentiated meshes based on automatic detection of gradients in state functions.

Continuity and Differentiability and Integrability of Functions. Conformity

If the relevant differential operator is of order n , the modeling requires shape functions of at least the same degree to not vanish within the elements and, simultaneously, to ensure continuity at the interface of order $n - 1$: if functions are linear because the differential operator is first-order ($n = 1$), continuity C^0 must be maintained across the interface, or in the case of second-degree functions for an operator of that order ($n = 2$), continuity C^1 would be required—continuity in derivatives—so that the relevant differential term computed across the interface, even if potentially discontinuous, does not become singular, with the aim of ensuring the integral across said interface is null (see figure 6.2).

Elements are called compatible or *conforming* if they ensure sufficient continuity in displacement fields and their derivatives, guaranteeing adequate integrability of the resulting functions in the formulation (equations 5.38, 5.48...).

Elements whose formulations do not meet these continuity requirements are called *non-conforming*. Even these can remain acceptable if the lack of continuity disappears in regions of similar behavior, as a mesh refinement process would reduce the difference between proximate elements and, consequently, the discontinuity source of error, see the patch test, page 446.

Capacity to Reproduce Basic Movements and Deformations Shape functions represent the influence of each node's movement at every point of the element's continuum. And they must be capable of representing all types of basic movements without problems.

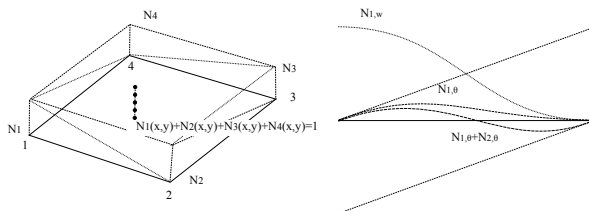


Figure 6.3: Sum of shape functions in the element: since they must be capable of representing undeformed movements, their sum must add up to one throughout the element. In cases of higher interpolation orders, functions associated with displacements and rotations are linked.

Let's examine the case of undeformed movements, displacements or rotations.

The displacement $\mathbf{u}_{\mathbf{x}}$ at a point $\mathbf{x}$ is expressed as $\mathbf{u}_{\mathbf{x}} = \sum \mathbf{N}_i(\mathbf{x})\mathbf{u}_i = \mathbf{N}(\mathbf{x})\mathbf{u}$, where i is an index to some node of the element and $\mathbf{u}_i$ is its displacement, with as many components as the dimensions of the represented space, and $\mathbf{N}_i(\mathbf{x})$ contains the shape functions associated with that node for all the components of the displacement. The matrix $\mathbf{N}$ is a block matrix, one block per node i , where each row associates a displacement component of point $\mathbf{x}$ with the complete node displacement (for the three-dimensional case and for any of the three dimensions d of the space in which $\mathbf{u}_i$ is defined, $u_{\mathbf{x},d} = \mathbf{N}_{id}\mathbf{u}_{id} = N_{i,d1}(\mathbf{x})u_{i,1} + N_{i,d2}(\mathbf{x})u_{i,2} + N_{i,d3}(\mathbf{x})u_{i,3}$). Such that if the displacement is $\boldsymbol{\nu}_R$, for each node i it must result that $\mathbf{u}_i = \boldsymbol{\nu}_R = \mathbf{I}\boldsymbol{\nu}_R$ and therefore the column of nodal displacements of the element, $\mathbf{u} = \mathbf{1}\boldsymbol{\nu}_R$

where $\mathbf{1}$ is a vertical block matrix, with one block per node being each block the identity matrix $\mathbf{I}$. In this way, as $\mathbf{u}_x = \mathbf{N}(\mathbf{x})\mathbf{u} = \mathbf{N}(\mathbf{x})\mathbf{1}\nu_R = \nu_R$ necessarily $\mathbf{N}(\mathbf{x})\mathbf{1} = \mathbf{I}$, which means that for each dimension d it must be $\sum_i N_{i,dd}(x) = 1$; $\sum_i N_{i,de}(x) = 0$, $d \neq e$, that is, the shape functions of all nodes must sum to unity at every point of the domain for displacement components coincident in direction or spatial dimension, and must cancel their sum for non-coincident components .

A rotation without deformation requires identifying the vector product $\boldsymbol{\theta} \times \mathbf{x}$ (which expresses such possible rotation) with the transformation $\sum \mathbf{N}_i(\boldsymbol{\theta} \times \mathbf{x}_i)$. This requires ensuring symmetry in the functions for all space dimensions. It is usual, due to the two previous conditions, to use isoparametric elements in which separate interpolation is applied for each of the space dimensions using the same number of parameters for each direction, which means $N_{i,de} = 0$, if $d \neq e$, and $N_{i,dd} \neq 0$, $\sum_i N_{i,dd} = 1$.

If, in addition to using displacements as nodal parameters, higher interpolation orders are used (thus requiring more parameters), adding rotations as is often the case with beams and slabs, achieving undeformed movement requires linking the functions associated with these movements: those linked to displacements $\mathbf{N}_w$ with those linked to rotations $\mathbf{N}_\theta$. As seen in figure 6.3, displacement without end rotations necessarily associates both group of movements when considering the deformed geometry relative to the line connecting both nodes: it must be $N_{1,\theta} + N_{2,\theta} = N_{1,w}/l + t(x)$ such that the second derivative (∂) of $t(x)$ is null.

In summary:

- an undeformed displacement $\boldsymbol{\nu}_R$ requires that $\mathbf{u}(\mathbf{x}) = \boldsymbol{\nu}_R$, $\forall \mathbf{x} \in \Omega$ and therefore $\mathbf{N}(\mathbf{x})(\mathbf{1}) = \mathbf{I}$, or equivalently, $\sum_i N_i(\mathbf{x}) = 1$.
- a rotation without deformation requires symmetry for shape functions in all space directions.
- in higher-order elements—which handle rotations at nodes as parameters in addition to displacements—the corresponding functions are necessarily linked.

Ability to Reproduce Constant Strain States

The need to make possible the representation of basic deformations includes the ability to adequately describe constant strain states, states to which elements naturally tend as the mesh is refined, and whose representation is therefore essential

- at least when the element dimension tends to zero
- to facilitate convergence, preferably in every finite-sized element
- and for every possible constant deformation

This capability allows relaxing the continuity conditions considered in section 6.1.2, limiting them to a more weak, less strict formulation, known as the *patch test*.

Patch Test: The necessary condition for convergence, even with non-conforming elements, is that the model allows representing constant strain states in large regions, constructed by many contiguous elements, and that, under this condition, the

corresponding patch maintains the required continuity conditions.

Various tests can be proposed to validate this criterion:

- A a linear displacement field is prescribed and the resulting forces –loads– are checked (they must be null in the interior of the patch, since if the strain is constant, internal stresses are also constant and therefore the applied balance at interior nodes is null)
- B considering that the displacement field is known, the displacement of some nodes is fixed, the displacements of the others are calculated using the model, and it is verified that they coincide with the movements corresponding to that field
- C forces in equilibrium on the boundary and the minimum constraint corresponding to rigid body movement are prescribed, and it is verified that the state of equilibrium and internal strain is maintained.

Degree and Completeness of Polynomials Approximating Field Functions

The approximation method implies the use of integration procedures that implicitly assume that fields are adequately represented by polynomial functions, while they can be approximated through Taylor series expansions. In these expansions, the basic terms of the expansion are those of lower order:

$$\begin{aligned}
f(x) &= f(x_0) + \frac{\partial f}{\partial x_{x_0}} (x - x_0) + \frac{\partial^2 f}{\partial x^2_{x_0}} \frac{(x - x_0)^2}{2!} + \dots \\
&\quad + \frac{\partial^n f}{\partial x^n_{x_0}} \frac{(x - x_0)^n}{n!} + O(x - x_0)^{n+1} \\
f(x) &\approx \sum_{i=0}^n \frac{h^i}{i!} \frac{\partial^i f}{\partial x^i_{x_0}} + O(h^{n+1}) = \sum_{i=0}^n \frac{1}{i!} \left(h \frac{\partial}{\partial x} \right)^i f_{x_0} + O(h^{n+1}); \\
&\hspace{25em} h = x - x_0, \\
f(x, y) &\approx \sum_{i=0}^n \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f_{x_0, y_0} + O((h + k)^{n+1}); \\
&\hspace{25em} h = x - x_0, k = y - y_0.
\end{aligned} \tag{6.1}$$

Such that the Taylor expansion is exact for polynomials of the considered degree, n , but for the rest it provides an error whose order depends on the element size raised to the power $n + 1$

$$e_n = f(x) - f_{T_n}(x) = O(h^{n+1}) \tag{6.2}$$

Thus, convergence requires (Bathe 1982) that the following condition be met:

$$\lim_{n \rightarrow \infty} \sup \sqrt[n]{\frac{h^n f^n}{n!}} < 1 \tag{6.3}$$

The construction of the basic functions therefore involves choosing the lowest-order polynomial terms which, for example in a 2D space, are located in the diamonds that can be drawn

centered on the height of Pascal's triangle, which we see in 6.4:

$$\begin{array}{ccccccccccc}
 & & & & 1 & & & & & & \\
 & & & & \xi & & \eta & & & & \\
 & & \xi^2 & & \xi\eta & & \eta^2 & & & & \\
 & \xi^3 & & \xi^2\eta & & \xi\eta^2 & & \eta^3 & & & \\
 & \xi^4 & & \xi^3\eta & & \xi^2\eta^2 & & \xi\eta^3 & & \eta^4 & \\
 \xi^5 & & \xi^4\eta & & \xi^3\eta^2 & & \xi^2\eta^3 & & \xi\eta^4 & & \eta^5 \\
 \xi^6 & \xi^5\eta & \xi^4\eta^2 & \xi^3\eta^3 & \xi^2\eta^4 & \xi\eta^5 & \eta^6 & & & &
 \end{array} \quad (6.4)$$

Computation of Integrals in the Elements

To the possible sources of error associated with the choice of shape functions described earlier are now added those from the integration forms used in the construction of load distribution models or stiffnesses derived from expressions 5.36 which, as was seen, are reduced to the form 5.37 once the integrals are performed, by elements, as indicated in 5.38.

Integration is usually performed numerically (by Gaussian integration, 5.3.1.2) so that the accuracy of the *quadrature* depends on the degree of the polynomials corresponding to the function to be integrated and the number of Gauss points chosen for each direction of the continuum to be represented: it will only be exact by Gauss if at least n points are used per direction to integrate polynomials of degree $2n - 1$. With fewer points, interpolation errors will result.

Consider the numeric format of the integral, as will be seen in section 6.1.3.2:

$\sum_i \Omega_i |J_i| \mathbf{b}_i^T \boldsymbol{\kappa}_i \mathbf{b}_i$ with $\mathbf{b} = \partial_{\mathbf{x}} \mathbf{N} = \frac{\mathbf{G}}{|J|} \partial_{\boldsymbol{\xi}} \mathbf{N}$. (6.22) where the sum is made over the value, weighted by Ω_i , of the expres-

sion at the Gauss points, i , being $\mathbf{N}$ the shape functions in the canonical or regular geometry of the element, $\partial_{\mathbf{x}}$ the relevant differential operator², $\mathbf{G}$ the matrix that projects such shape functions onto the topology and actual geometry of the element, $\mathbf{J}$ the Jacobian of the transformation from one geometry to another, and κ the matrix that describes locally the stress-strain relationships, all of them with the corresponding values for each Gauss point considered, i .

The degree of the polynomial to be integrated will be $\text{degree}(\mathbf{b})^2 * \text{degree}(\kappa) * \text{degree}(|\mathbf{J}|)$, but $\text{degree}(\mathbf{b}) = \text{degree}(\mathbf{G}) * \text{degree}(1/|\mathbf{J}|) * \text{degree}(\partial_{\xi} \mathbf{N})$ so that it becomes increasingly higher and further from the element's order as higher order is employed in its description. Thus, better “accuracy” is achieved in lower-order elements, but this will require smaller mesh dimensions to adequately represent variations in the functions, additionally resulting in lower continuity orders at the interface between elements.

As we will see in the section dedicated to bending elements, 6.3, problems in describing the coupling between shear and moment effects sometimes lead to formulations with excessive stiffness (see shear locking on page 505) which need to be corrected by providing a certain relaxation through reduced integration modes (fewer Gauss points, etc.) to ensure the necessary flexibility, but thereby accentuating potential problems in the numerical difference with the results that could be indicated as exact according to the precise form of the field equations.

2. according to local expressions 5.22, 5.23 and 5.24.

An Acceptable Mathematical Regularity is Therefore Required

Whatever the formulation, the non-singularity (spurious) of the element matrix and geometric invariance must be ensured, in the sense of insensitivity to coordinate axis changes and symmetry in the formulation with respect to different spatial directions.

The singularity of the element matrix, that is, a null determinant, implies a null response (null eigenvalue) with a non-null eigenvector, that is, it supposes being able to describe a movement without strain energy: $\mathbf{f} = \mathbf{k}\mathbf{u}$, $\{\mathbf{f} = 0, \mathbf{u} \neq 0\} \implies \mathbf{f} = \lambda\mathbf{u}$, with $\lambda = 0$; and $(\mathbf{k} - \lambda\mathbf{I})\mathbf{u} = 0$. From the element's point of view, it must be considered that movements without deformation must be possible, but there should be no additional sources of singularity that could imply element deformations without strain energy.

6.1.2.3 Numerical Errors in Solving the Equation System

The resulting equation system in each analysis step, in the linear format of 5.37 may have additional problems due to the limited mode of numerical representation in the computer. We review its possible sources, causes of numerical imprecisions, which can be grouped in

- poor conditioning of the matrix
- truncation errors
- rounding errors

Imprecision may arise from problems in the problem formulation that lead to a poor numerical conditioning of the fundamental expression $\mathbf{F} = \mathbf{K}\mathbf{U}$. This poor conditioning is reflected in the fact that small variations in $\mathbf{F}$ or $\mathbf{K}$ result in large variations in $\mathbf{U}$.

We can measure this degree of conditioning, $\text{cond}(\mathbf{K})$ (*condition number*), by the ratio between the error scale in $\mathbf{U}$ relative to the error scale in $\mathbf{F}$:

$$\text{cond}(\mathbf{K}) = \frac{\|e(\mathbf{U})\|}{\|\mathbf{U}\|} \div \frac{\|e(\mathbf{F})\|}{\|\mathbf{F}\|} \quad (6.5)$$

so that it is a number related to the eigenvalues of $\mathbf{K}$, which can be measured with the ratio between the largest and smallest of such eigenvalues:

$$\text{cond}(\mathbf{K}) = \frac{\lambda_{\max}(\mathbf{K})}{\lambda_{\min}(\mathbf{K})}. \quad (6.6)$$

This indicator of $\mathbf{K}$ allows predicting the number of incorrect digits in the result values' mantissa: $g_i = \log_{10} \text{cond}(\mathbf{K})$ and, as can be well understood, this problem is associated with formulations that describe situations of very reduced stiffness for some possible deformation modes.

But problems can also arise from the limited precision in the numerical representation used in the computer, either by

- rounding error (in the last digit)
- truncation error (in sums of numbers a and b that are very different, where they result from subtractions of equal or nearly equal numbers to the larger one)

The use of high precisions must be ensured: better than using *float* will be to employ *double* precision or higher. And perhaps it is relevant to pay attention to the relative scale of the different terms in the stiffness matrix $\mathbf{K}$: when dealing with terms of very different magnitudes (different units) at different scales, one might expect different precision in different types of results.

6.1.2.4 Other Modeling Errors

In general, the possibility of errors in modeling reality should be considered, with some of the sources being the following:

Errors in Material Constitutive Equations These can occur due to errors in parameter values, or by errors in the relationships between these parameters in situations without isotropy or homogeneity.

Considering linear behaviors in isotropic cases, for plane states, the Poisson's ratio is relevant. In non-isotropic cases, both Poisson's and Young's moduli are important. In dynamic cases, it is important to pay careful attention to an adequate estimation of damping.

In describing non-linear behaviors, care must be taken in describing the evolution of variables according to their loading and unloading history, with the corresponding evaluation of damage evolution (and accumulation), which requires an adequate representation of hysteresis cycles.

Errors arising from an inadequate treatment of these parameters can be of great importance.

Geometry Approximation The adequacy in representing the structural form, connections between elements, and any existing boundary conditions must be considered. This is a relatively frequent source of error, more probable the more automatic the program's method of use.

Some possible geometric modeling errors can be considered:

- that the representation is only approximate (due to interpolation error in shape). Not usually critical in building construction, although it is in aeronautics or in situations where the model serves for numerical control manufacturing.
- elimination of key elements in behavior. This is the case of partitions and staircases in dynamic analyses, whose elimination substantially alters the natural oscillation periods in initial or undamaged phases.
- incorrect geometry in element connections or load application, such as facade load positions in relation to structural elements.
- inadequate representation of clearances and initial states of the structure, or of its state evolution according to construction processes.

And it is of particular importance to pay correct attention to boundary conditions

- unfortunately, it is too frequent to introduce non-existent constraints ...
 - hinges without displacement possibility at truss support starts
 - complete rotational constraints in piece ends without such

constraint degree

- ... or poorly defined (inclined or elastic support in reality but not in the model ...)
- it is relevant, to validate a model, to consider the sensitivity of its behavior to changes in support conditions.

Therefore, as a validation tool of interest, it would be worth pointing out the utility of comparing results obtained from different modelings of the same problem.

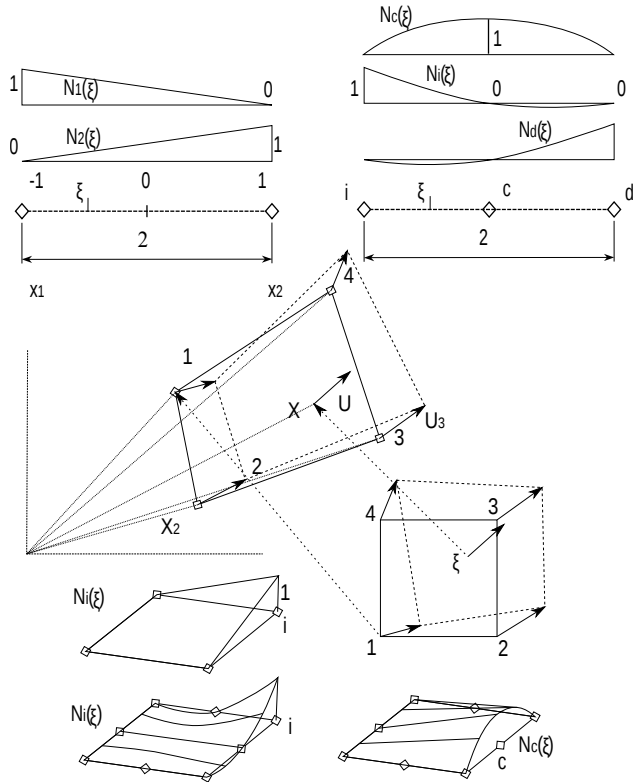


Figure 6.4: Shape Functions and Interpolation. Linear or quadratic functions in linear (top) or surface (bottom) elements. Displacements (in the center) are interpolated in the *natural* coordinate space from nodal values, and translated to corresponding points in the *real* coordinate space. The $\mathbf{x}$ corresponding to ξ is determined by interpolating from nodal coordinates.

6.1.3 Shape Functions and Properties of the Element

With all the considerations from the preceding sections, we can now explore alternatives for defining interpolation functions and the corresponding construction of element properties, and we will do this initially, due to its greater simplicity, by considering linear elements (bars) for axial stress and strain states or surface elements in plane states. In section 6.3 we will develop the element models for pieces subjected to bending or bending-compression states more extensively.

Thinking about Gauss integration, section 5.3.1.2, it is customary to systematically consider two coordinate systems: one that facilitates the identification of Gauss points, which we call the *natural* or canonical system, with coordinates in intervals $[-1, 1]$ for all directions of the considered space, that is, one for axial cases: ξ , two for surface cases: $\boldsymbol{\xi} = [\xi, \eta], \dots$ and another *real* system, with coordinates of values in *real* space units, x in axial cases, $\boldsymbol{x} = [x, y]$ in surface cases, such that the geometry of the *real* space is constructed from the *natural* coordinate system by interpolation from these, in analogy to how movements within the element are interpolated from node movements, with these interpolations being transferred from the *natural* coordinate space to the *real* coordinate space.

Given the conditions seen for the functions, that is, having $N_i(\boldsymbol{x}_i) = 1; N_i(\boldsymbol{x}_j) = 0, i \neq j$, with i, j being node indices, and given the criterion of choosing the lowest-order polynomial terms in the Pascal triangle, the appropriate polynomial functions associated with each node, depending on the number of

nodes per element, are like those in figure 6.4 for the *natural* coordinate space: the top figures represent linear or quadratic functions for a linear element with two or three nodes, those at the bottom for a surface element with four or eight nodes, and in the central figures, the right one shows the movement interpolation in the *natural* space and its transfer to the *real* space. The left central figure shows the node displacements and the resulting real deformation. It also shows the interpolation of the element's *real* geometry from the real coordinates of the nodes, an interpolation that allows establishing the correspondence between both spaces, from ξ to $\mathbf{x}$.

6.1.3.1 Typology and Expressions for Shape Functions

It would be possible to establish shape functions directly in the *real* coordinate space through polynomial functions chosen respecting the criteria described in sections 5.2.4.2 and 6.1.2.2, as will be done later in the example in section 6.2, subsection 6.2.4, which shows how to obtain the coefficients of the chosen monomials for these functions in the Pascal triangle, expression 6.4, formalized for coordinates in real space $\mathbf{x}$. However, it is much simpler to establish these functions in the *natural* coordinate space and transfer them to the *real* coordinate space through a geometric transformation that projects the first into the second, a very simple transformation to perform if we accept that the *real* geometry is interpolated from the *natural* one, ensuring that the projection places the nodes at the real coordinates, which can be done using the same shape functions used to interpolate displacements in the element from the node displacements.

Let's see some possible function types.

Lagrangian Functions

Built on the basis of Lagrange polynomials. For one dimension

$$N_i(\xi) = l_i^n(\xi) = \prod_{j=1(j \neq i)}^n \left(\frac{\xi - \xi_j}{\xi_i - \xi_j} \right) \quad (6.7)$$

The product of the $n - 1$ values constitutes a continuous function, which will be equal to unity for the coordinates ξ_i of node i to which the function refers, as each of the multiplicands will be, and will be zero at all other nodes, as some of these multiplicands will be zero, thus meeting the required conditions: $N_i(\mathbf{x}_i) = 1$; $N_i(\mathbf{x}_j) = 0$, $i \neq j$.

For more than one dimension, it would suffice to change the differences that appear in each quotient by a norm of the coordinates represented in that quotient: the distance, for example, or the product of those same polynomials applied separately to each space direction:

$$N_i(\xi, \eta) = N_i(\xi)N_i(\eta) \quad (6.8)$$

Serendipitous Functions

If we consider the *natural* system with coordinates in the interval $[-1, 1]$ for each space direction, it is possible to establish functions of lesser complexity quite easily, simply by taking advantage of the special positions of the nodes: for one dimension, the functions would be, without further complication, the following:

linear, with two nodes.

$$N_i = \frac{1 + \xi_i \xi}{2} \quad (6.9)$$

quadratic, with three nodes.

$$\begin{aligned} N_c &= 1 - \xi^2 \\ N_{i(i \neq c)} &= \frac{1 + \xi_i \xi}{2} - \frac{1 - \xi^2}{2} \end{aligned} \quad (6.10)$$

where c is the central node.

For the case of more dimensions, the strategy proposed earlier for Lagrangian functions, 6.8, can be used, which works well for elements with only nodes at the vertices, but requires for quadratic versions, and at the center nodes of the edges, considering products of the quadratic function for the direction that traverses the edge, with another parabolic function of zero value only at the opposite edge for orthogonal directions, in order to avoid zero values at the center of the element (at the coordinate origin) that would represent a spurious excess of stiffness.

Other formulations are possible, such as the use of Bézier functions, which are particularly useful for geometry interpolation in situations linked to computer assisted manufacturing (CAM) processes, as well as considering nodes not situated on the edges: for example, a central node in surface elements would allow avoiding the warning in the preceding paragraph. But the simplicity of the previous serendipitous formulation will facilitate the following exposition, so we will leave it here.

6.1.3.2 Formulation of Element Properties with Isoparametric Functions

Once the interpolation method is adopted, the task now is to construct the properties of each isolated element, properties that will subsequently be assembled into the structure using the procedures we saw in 2.1.2.2.

We will explore the case of surface elements for plane states, for which we consider the two coordinate systems

$$\mathbf{x} = [x \ y]^T; \quad \boldsymbol{\xi} = [\xi \ \eta]^T$$

The appropriate differential operator $\boldsymbol{\partial}$ for the problem is the one that relates displacements to strains at each point of the element continuum, see 5.3 applied to two dimensions, although now these functions will be interpolated from nodal values:

$$\boldsymbol{\epsilon} = \boldsymbol{\partial} \mathbf{v} \approx \boldsymbol{\partial} \mathbf{N} \mathbf{u} = \mathbf{b} \mathbf{u}; \quad \boldsymbol{\partial}_{\mathbf{x}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (6.11)$$

We use the subscript in $\boldsymbol{\partial}_{\mathbf{x}}$ to make explicit that it is an operator in the element's real coordinate space.

However, this operator cannot be applied directly as it is to the shape functions since these are defined in the natural coordinate space, so we must consider the geometry transformation. This transformation allows us to establish, for any function, the following differential relations:

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}; \quad \nabla_{\boldsymbol{\xi}} f = \mathbf{J} \nabla_{\mathbf{x}} f \quad (6.12)$$

where $\mathbf{J}$ is the Jacobian of the transformation of geometry from coordinates $\boldsymbol{\xi}$ to $\mathbf{x}$ (or of $\mathbf{x}$ as a function of $\boldsymbol{\xi}$).

The inverse transformation is evident:

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix}; \quad \nabla_{\mathbf{x}} f = \mathbf{J}^{-1} \nabla_{\boldsymbol{\xi}} f \quad (6.13)$$

so that the operator, in real coordinate space, established from differentials in natural coordinates, is:

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix}; \quad \det \mathbf{J} = |\mathbf{J}| \quad (6.14)$$

an expression that we can condense in matrix format:

$$\nabla_{\mathbf{x}} = \frac{\mathbf{G}_J}{|\mathbf{J}|} \nabla_{\boldsymbol{\xi}} \quad (6.15)$$

This expression now allows us to determine the form that the operator of the expression 6.11 must take in order to apply it to functions defined in the natural space. We adopt the format

$$\partial_{\mathbf{x}} = \frac{\mathbf{G}}{|\mathbf{J}|} \partial_{\boldsymbol{\xi}} \quad (6.16)$$

or, in more detail,

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \mathbf{J}_{22} & -\mathbf{J}_{12} & 0 & 0 \\ 0 & 0 & -\mathbf{J}_{21} & \mathbf{J}_{11} \\ -\mathbf{J}_{21} & \mathbf{J}_{11} & \mathbf{J}_{22} & -\mathbf{J}_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} & 0 \\ \frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial}{\partial \xi} \\ 0 & \frac{\partial}{\partial \eta} \end{bmatrix} \quad (6.17)$$

where the final operator ∂_ξ has been chosen in such a way that its application to the displacement vector $\mathbf{v} = [u \ v]^T$ allows complete the list of partial derivatives $[\frac{\partial u}{\partial \xi} \ \frac{\partial u}{\partial \eta} \ \frac{\partial v}{\partial \xi} \ \frac{\partial v}{\partial \eta}]^T$. Therefore,

$$\partial_x = \frac{G}{|J|} \partial_\xi; \quad \partial_x \mathbf{N} = \mathbf{b} = \frac{G}{|J|} \partial_\xi \mathbf{N} \quad (6.18)$$

where the format of the shape functions, for a node-wise ordering of the displacements u, v in the directions x, y , is:

$$\mathbf{N}(\boldsymbol{\xi}) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \quad (6.19)$$

where $N_i = N_i(\xi, \eta), i \in [1, 2, 3, 4]$ are functions chosen according to the preceding sections.

Integration in the Element

To proceed with the integration in the element, we must consider that since the functions are defined in the natural coordinate space $\boldsymbol{\xi}$, we must take into account how the entire area is traversed in real coordinates $\mathbf{x}$, for which the figure 6.5 can help us.

In this figure, we see that the real differential area can be interpreted as the area covered by the transformation to real space of the area delimited by the vectors $d\xi, d\eta$ in the natural space:

$$d\xi \rightarrow \vec{t}(d\xi) = \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{bmatrix} d\xi; \quad d\eta \rightarrow \vec{t}(d\eta) = \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \end{bmatrix} d\eta$$

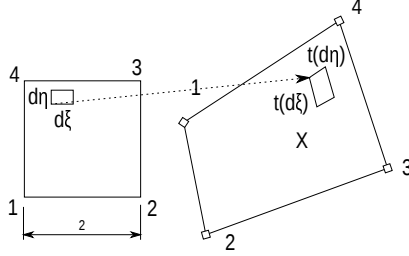


Figure 6.5: Transformation of Differential Area

in such a way that in real space the differential area can be measured by traversing the natural space in the form

$$dA = ||\vec{t}(d\xi) \times \vec{t}(d\eta)|| = |\mathbf{J}|d\xi d\eta \quad (6.20)$$

In this way, the element stiffness, which according to 5.48 is the integral

$$\mathbf{k}_e = \int_{\Omega} (\partial \mathbf{N})^T \boldsymbol{\kappa} \partial \mathbf{N} d\Omega = \int_{\Omega} \mathbf{b}^T \boldsymbol{\kappa} \mathbf{b} d\Omega \quad (6.21)$$

will adopt, when performed numerically by the Gauss points method, the following form:

$$\mathbf{k}_e = \sum_{i \in \text{Gauss}} \Omega_{\xi_i} |\mathbf{J}_i| \left(\frac{\mathbf{G}_i}{|\mathbf{J}_i|} \partial_{\xi} \mathbf{N}(\xi_i) \right)^T \boldsymbol{\kappa}_i \left(\frac{\mathbf{G}_i}{|\mathbf{J}_i|} \partial_{\xi} \mathbf{N}(\xi_i) \right) \quad (6.22)$$

Once the local stiffness of each element is obtained, its assembly in the complete structure, the identification and application of

boundary conditions—loads and supports— and the obtaining of nodal displacements as a solution to the resulting matrix problem, as well as the subsequent evaluation of these results are steps common to what has already been established in previous sections.

In the following sections, we use the described methodology for the study of appropriate element types for axial stress states, or bending.

6.2 Finite Elements in Axial Problems

In this section, we address the formulation of axial stress problems, whether linear, of bars in tension or compression along the piece's directrix, or surface, by membrane stresses contained in the surface plane. We carry out the entire process through a case study.

The example is approached from alternative formulation perspectives, with the aim of providing a comprehensive panoramic view of the entire formulation, which considers

1. the modes of discretization of the fundamental equations of the problem,
2. the construction of shape functions (or interpolation functions),
3. the formulation of boundary conditions and
4. the solution and interpretation of the results obtained and their quality.

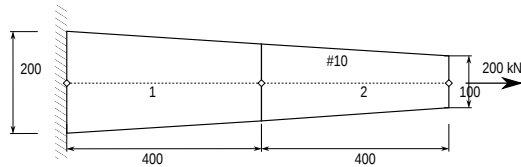


Figure 6.6: Case Study: Uniaxial or Linear Model

In a later section, problems requiring modeling of bending states will be addressed, which are generally more complex than those considered here.

The approach to the problem corresponds to the following scheme: after considering the available theoretical elements, that is, the appropriate equations and operators for the problem, the discretization parameters and shape functions are defined, exploring the required qualities and available types (polynomials, Lagrange, isoparametric...). With all of this, approximate solutions are addressed, either through bar models or plane models, to finally conclude with the comparison of the solutions obtained.

6.2.1 Equations and Parameters of the Problem; Linear Model

This is a simple traction problem on a variable cross-section plate, where the data corresponds to global considerations: the geometry, an applied load with a given resultant value, although not the detail of its application, and a support condition that

assumes none of the points on the supported boundary are capable of moving or changing orientation.

The choice of such a trivial model easily illustrates the virtues and limitations of the method and the various modeling alternatives available.

As the goal is to define the stress state at every point of the plate, the appropriate model would be plane stress for any of its points. A simpler and more immediate alternative representation, which is the one adopted in this section, is the linear model of variable section that requires some hypothesis about the distribution of stress in each section, with the simplest being the hypothesis of identical stress at all points of the same section. Therefore, the relevant equations are as follows, using two columns, the first for the generic description and the second for the details in the example case, considering the linear model.

EQU	$\partial^* \sigma = \varphi$	$-\frac{dF}{dx} = p$
COM	$\partial v = \epsilon$	$\frac{du}{dx} = \epsilon$
MAT	$\kappa \epsilon = \sigma$	$EA \epsilon = F$

“Exact” Solution of the Problem

Considering the linear bar model, the displacement results from the sum (integral) of strains, such that $u = \int_0^x \epsilon \, dx = \frac{Fl}{EA_0} \int_0^x \frac{d(x/l)}{1-(1-\alpha)x/l}$; where $\alpha = \frac{A_1}{A_0}$ is used;

Therefore, the analytical solution would be

$$u(x) = \frac{Fl}{EA_0} \frac{\log(1-(1-\alpha)x/l)}{-(1-\alpha)}. \text{ At the end of the bar, } u(l) = 0.5545 \text{ mm}$$

6.2.2 Bar Discretization: Shape Functions

Now we consider the discretization of the bar for the same linear model.

We must take into account that we need to represent, from discrete (nodal) conditions, both the continuum of field functions (displacements, strains, and stresses) and the geometry itself (the section law in this case). And for this, we use the interpolation (or shape) functions that seem most appropriate.

We consider the double coordinate system, the *natural* system with coordinates in the interval $[-1, 1]$, ξ , and the *real* system, with coordinates of values in real space units, x , in such a way that the geometry of the *real* space is also constructed from the *natural* coordinate system by interpolation from these. Such that the geometry is interpolated with

$$\begin{aligned} x(\xi) &= N_1(\xi)x_1 + N_2(\xi)x_2 \\ A(\xi) &= N_1(\xi)A_1 + N_2(\xi)A_2 \end{aligned} \quad (6.23)$$

and the displacements with:

$$u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2 \quad (6.24)$$

in which we are using identical shape functions for both groups of interpolations. We could not do so, and use a different number and positions of parameters for geometry and mechanics, but it is quite common to use these *isoparametric* elements, which use the same number and meaning of parameters for both purposes. Figure 6.7 shows the two coordinate systems and the shape functions that have been adopted.

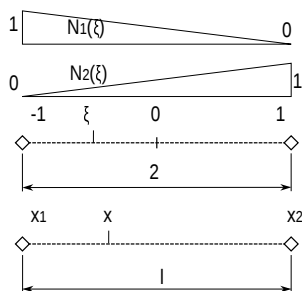


Figure 6.7: Bar Element and Shape Functions

The adopted isoparametric shape functions are very simple

$$\mathbf{N} = [N_1(\xi) \ N_2(\xi)]$$

$$N_1 = \frac{1 - \xi}{2}$$

$$N_2 = \frac{1 + \xi}{2}$$

The expression can be generalized in the form

$$N_i = \frac{1 + \xi_i \xi}{2} \quad (6.25)$$

Differential Operator

The differential operator that allows deriving strains from displacements, or its adjoint that derives distributed loads from stresses, is an operator in real coordinate space. For the one

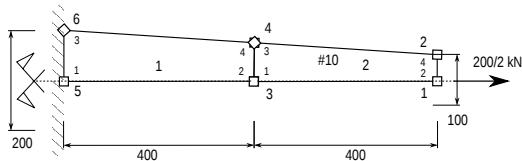


Figure 6.8: Case Study: Surface Model

that links kinematic variables:

$$\boldsymbol{\vartheta} = \frac{d}{dx}(\cdot) \quad (6.26)$$

However, since the expressions derived from the shape functions will end up being expressions in the natural coordinate space, it is necessary to shape it in these, which is feasible by considering the coordinate transformations between one and the other space:

$$\begin{aligned} \frac{df}{d\xi} &= \frac{df}{dx} \frac{dx}{d\xi}, \\ \frac{df}{dx} &= \left(\frac{dx}{d\xi} \right)^{-1} \frac{df}{d\xi} \end{aligned} \quad (6.27)$$

The relationship with the expression 6.18 can be seen if in addition to identifying $\boldsymbol{\vartheta}_x$ with d/dx we consider in this case the identities $\frac{1}{|J|} \equiv \left(\frac{dx}{d\xi} \right)^{-1}$, $\mathbf{G} \equiv 1$ and $\boldsymbol{\vartheta}_\xi \equiv \frac{d}{d\xi}$.

6.2.3 Equations and Parameters of the Problem; Surface Model

An alternative to the model conceived as a succession of linear pieces of variable cross-section is to consider the plate as a surface element of constant thickness in which the geometry is defined by the association of trapezoidal plane elements. In Figure 6.8 this alternative is represented, considering in the modeling the symmetry of the problem around the piece's directrix: therefore, only half of the section is modeled.

The field equations are now those corresponding to the plane stress state, so that in each unit of thickness we will have:

$$\begin{aligned}
 \text{EQU } \varphi &= \partial^* \sigma \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial y} \\ 0 & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \\
 \text{COM } \epsilon &= \partial v \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\
 \text{MAT } \sigma &= \kappa \epsilon \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}
 \end{aligned} \tag{6.28}$$

and for a different unit thickness, it suffices to consider the product of stress by the thickness.

Now we do not have an exact solution. To formulate this according to the expressions of elasticity theory, we would consider not only the internal conditions according to expressions like that defined in 5.25, but we would also need to establish the boundary conditions of the Dirichlet type, 5.18, on the clamped

edge: impossibility of movement, and of the Neumann type, 5.19, of stress equilibrium with the applied load, both on the edge where the distributed load is applied, and on the free edges above and below the plate where the load is zero. In the symmetric half-piece model, the conditions on the symmetry axis would require mixed conditions, where the vertical displacement would be prevented (Dirichlet) but not the horizontal one, requiring for this degree of freedom the condition that the load applied tangentially from the other half of the plate is zero (Neumann).

6.2.4 Discretization of the Plate: Shape Functions

We will first analyze the possibility of establishing the shape functions in the real coordinate system, x, y where x is the bar's directrix and y the orthogonal direction, at the plate's edge.

Considering the issues pointed out in sections 5.2.4.2 and 6.1.2.2, we decouple the movements in the two plane directions and use polynomial functions of four terms for each, as they will be associated with four nodes, with one parameter per node.

The organization of the set would be as follows:

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} N_1(x, y) & 0 & N_2(x, y) & 0 & N_3(x, y) & 0 & N_4(x, y) & 0 \\ 0 & N_1(x, y) & 0 & N_2(x, y) & 0 & N_3(x, y) & 0 & N_4(x, y) \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} \quad (6.29)$$

with $N_i(x_i, y_i) = 1$; $N_i(x_j, y_j) = 0$, when $i \neq j$, being i, j indexes to the nodes, of coordinates x_i, y_i , and being the parameters u_i, v_i their respective displacements.

Each of the functions will therefore have the polynomial form:

$N_i = a_{i1} + a_{i2}x + a_{i3}y + a_{i4}xy$, so that

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ xy \end{bmatrix}$$

$$\mathbf{N} = \mathbf{a}\mathbf{x} \quad (6.30)$$

where the matrix $\mathbf{a}$ containing the sixteen required numerical coefficients can be resolved if we establish the conditions previously mentioned of $N_i(x_i, y_i) = 1$; $N_i(x_j, y_j) = 0$, $i \neq j$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{bmatrix}$$

$$\mathbf{I} = \mathbf{a}\mathbf{x}_n; \quad \mathbf{a} = \mathbf{x}_n^{-1} \quad (6.31)$$

where, as can be seen, the matrix $\mathbf{x}_n$ groups the values of the appropriate polynomial terms to the real node coordinates, with $\mathbf{a}$ being its inverse.

6.2.5 Summary of Discretization Alternatives

In figure 6.9, four alternatives for modeling the piece can be seen, considering variants in the element type and the order of

polynomials used for their shape functions.

These are two bar models and two membrane models. The two bar models are models with two elements each, where the first attributes two nodes to each element, therefore with linear functions of two parameters, and the second with three nodes per element, and therefore elements that will require quadratic interpolation functions. The membrane models represent symmetry conditions and both use two elements for half the piece, the first with four nodes at the vertices of each element, and the second with eight nodes, by adding nodes at the center of each of the four edges.

The approximate displacements, represented in the natural coordinate space of each element, would be obtained as follows, from the displacements of the nodes of the considered element:

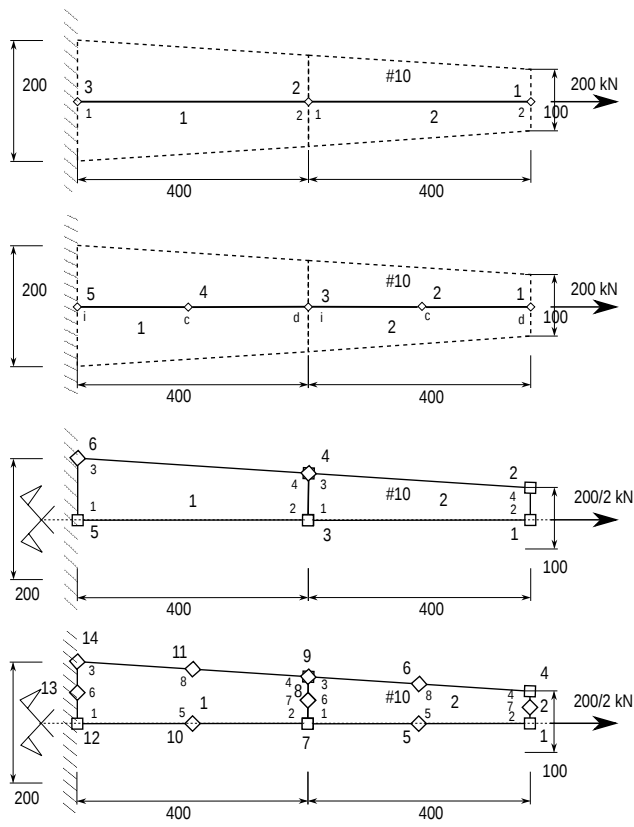


Figure 6.9: Discretization alternatives of the piece: two element models (linear or surface) and two strategies for their functions (linear or quadratic) The figures indicate in large the global numbering of nodes and elements (an arbitrary order does not alter the model), and in small the local (the order must strictly follow the adopted formulation).

a: linear bar

$$u(\xi) = u_1 N_1 + u_2 N_2 = u_1 \frac{1-\xi}{2} + u_2 \frac{1+\xi}{2}$$

b: quadratic bar

$$\begin{aligned} u(\xi) &= u_i N_i + u_c N_c + u_d N_d = \\ u_i \left(\frac{1-\xi}{2} - \frac{1-\xi^2}{2} \right) &+ u_c (1 - \xi^2) + u_d \left(\frac{1+\xi}{2} - \frac{1-\xi^2}{2} \right) \end{aligned}$$

c: linear membrane

$$\begin{aligned} \begin{bmatrix} u(\xi, \eta) \\ v(\xi, \eta) \end{bmatrix} &= \begin{bmatrix} u_1 N_1 + u_2 N_2 + u_3 N_3 + u_4 N_4 \\ v_1 N_1 + v_2 N_2 + v_3 N_3 + v_4 N_4 \end{bmatrix} = \\ \mathbf{v}(\xi) &= \mathbf{N} \mathbf{u} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix}; \\ N_1 &= \frac{(1-\xi)(1-\eta)}{4}, N_2 = \frac{(1+\xi)(1-\eta)}{4}, N_3 = \frac{(1-\xi)(1+\eta)}{4}, \\ N_4 &= \frac{(1+\xi)(1+\eta)}{4}. \quad N_i = \frac{(1+\xi_i \xi)(1+\eta_i \eta)}{4}. \end{aligned}$$

d: quadratic membrane

$$\begin{aligned} N_1 &= N_{--} = \frac{1}{4}(-\xi + \xi^2)(-\eta + \eta^2); & N_2 &= N_{+-} = \frac{1}{4}(\xi + \xi^2)(-\eta + \eta^2) \\ N_3 &= N_{-+} = \frac{1}{4}(-\xi + \xi^2)(\eta + \eta^2); & N_4 &= N_{++} = \frac{1}{4}(\xi + \xi^2)(\eta + \eta^2) \\ N_5 &= N_{0-} = \frac{1}{4}(1 - \xi^2)(1 - \eta^2); & N_6 &= N_{-0} = \frac{1}{4}(1 - \xi^2)^2(1 - \eta^2) \\ N_7 &= N_{+0} = \frac{1}{4}(1 + \xi)^2(1 - \eta^2); & N_8 &= N_{0+} = \frac{1}{4}(1 - \xi^2)(1 + \eta)^2 \end{aligned}$$

The approximation in this last model is again $\mathbf{v}(\xi) = \mathbf{N} \mathbf{u}$, where $\mathbf{u}$ will group the sixteen nodal displacements and $\mathbf{N}$ the thirty-two terms that, as in the previous case, reiterate the use

of the eight functions in the separate interpolation of movements along the x and y axes. It can be verified that the functions, chosen in the serendipitous manner, satisfy the condition $\sum_1^8 N_i(\xi, \eta) = 1$.

Also, it can be seen that, in this case, the monomials of the resulting polynomials are nine versus the eight nodes, by including the fourth-degree term, as can be seen in the Pascal triangle, 6.4, unlike case **c**, where the number of nodes and monomials coincide. This means that, while in case **c** the serendipitous functions chosen coincide with those resulting from the approximation seen in 6.2.4, this is not the case in case **d** where the $\xi^2\eta^2$ term would not be used by choosing the eight lowest-order monomials strictly necessary to match the coefficients with the eight conditions resulting from the values the function must adopt at each node.

The difference between the two families of shape functions is a term of the form $1/4(\xi+1)(\xi-1)(\eta+1)(\eta-1)$, which is added to the first four serendipitous functions, and subtracted from the last four, and which constrains the geometry of the resulting interpolation to some extent: the strategy adopted is forcing the functions of the nodes located at the vertices to become zero at all points of the zero-coordinate axes, and not just at nodes other than the vertex considered, thereby forcing the geometry of the central node functions, by being obliged that the sum of the eight functions is unity in the entire domain. This term is exactly one-fourth of the shape function that would correspond to a node at the center of the element if we were to choose a nine-node model, which would therefore free up that degree of freedom and eliminate the noted constraint. However, in this

case, if we considered that this central node is not loaded nor connected and condensed its degrees of freedom into the eight border nodes, we would return to the result provided by the strategy in 6.2.4. For the eight-node membrane models defined in page 490 for the example in the next section, you can see the difference in results from using both strategies, pages 493 ff.

Having established the discretization, we will now develop the process that allows obtaining the approximate solution.

6.2.6 Solutions and Evaluation

The procedure requires constructing the stiffness matrices of each element $\mathbf{k}_e$, using 6.22, assembling them in the complete model, according to 2.1.2.2, which requires associating the local load and displacement parameters of each element to the corresponding global model parameters, applying boundary conditions, support conditions, and loads, and solving the resulting system of equations $\mathbf{F} = \mathbf{KU}$.

We will now go through these steps for the two types of elements that have been considered: linear (bars) or surface (membrane) elements.

6.2.6.1 Bar Elements

We recover the available formulation and the expressions of the numerical integrals (5.59, 6.21, 6.18, 6.22) for two cases, a: element with linear functions, or b: with quadratic functions.

$$\partial_x N = \mathbf{b} = \frac{G}{|J|} \partial_\xi N; \quad \mathbf{k}_e = \sum \Omega_i |J_i| \mathbf{b}_i^T \boldsymbol{\kappa}_i \mathbf{b}_i \quad (6.32)$$

Considering that these are isoparametric elements, the geometry is interpolated with the same functions as the displacements, as collected on page 476. For both types of element, we have only one dimension, which transforms ξ into x so that in both cases

$$\mathbf{J} = \frac{dx}{d\xi}; \quad \frac{d\xi}{dx} = \frac{1}{\mathbf{J}};$$

and therefore in case a:

$$x = x_1 \frac{1 - \xi}{2} + x_2 \frac{1 + \xi}{2}; \quad \mathbf{J} = \frac{-x_1 + x_2}{2} = \frac{l}{2}$$

(the reader could determine $\mathbf{J}$ for case b: as an exercise)

On the other hand, $\mathbf{G}$ corresponds to the transformation between differential operators corresponding to the two coordinate spaces:

$$\frac{\partial(\cdot)}{\partial x} = \frac{\mathbf{G}}{|\mathbf{J}|} \frac{\partial(\cdot)}{\partial \xi}; \quad \text{thus now} \quad \frac{d\xi}{dx} = \frac{\mathbf{G}}{|\mathbf{J}|}; \quad \frac{dx}{d\xi} = \frac{|\mathbf{J}|}{\mathbf{G}};$$

and therefore, comparing with the preceding expressions

$$\begin{aligned} \text{a: } \mathbf{J} &= \frac{l}{2}; \quad \mathbf{G} = 1, \\ \text{b: } \mathbf{J} &= \frac{l}{2} + \xi(x_i + x_d - 2x_c); \quad \mathbf{G} = 1. \end{aligned} \quad (6.33)$$

Strains and Stiffness

In case a: we have 2 nodes and need 1 Gauss point: $\xi_g = 0$. Applying the preceding expressions

$$\begin{aligned}
\boldsymbol{\epsilon} &= \mathbf{b}\mathbf{u}; \quad \varepsilon = \frac{2}{l} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\mathbf{k}_e &= 2\frac{2}{l} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \frac{E(A_1 + A_2)}{2} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{EA_c}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned} \tag{6.34}$$

In case b: we have 3 nodes. 2 Gauss points will be sufficient:

$$\xi_g = \pm \sqrt{\frac{1}{3}}.$$

$$\varepsilon = \frac{2}{l} \begin{bmatrix} \xi - \frac{1}{2} & -2\xi & \xi + \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_i \\ u_c \\ u_d \end{bmatrix};$$

With 6.32 again:

$$\begin{aligned}
\partial_{\mathbf{x}} \mathbf{N} &= \mathbf{b} = \frac{\mathbf{G}}{|\mathbf{J}|} \partial_{\xi} \mathbf{N}; \quad \mathbf{k}_e = \sum \Omega_i |\mathbf{J}_i| \mathbf{b}_i^T \boldsymbol{\kappa}_i \mathbf{b}_i, \text{ with} \\
\mathbf{J} &= \frac{l}{2} + \xi(x_i + x_d - 2x_c) = \frac{l}{2}, \quad x_c = \frac{x_i + x_d}{2}; \quad \mathbf{G} = 1.
\end{aligned}$$

expressions that allow us to determine

$$\begin{aligned}
\mathbf{k}_e &= \frac{2}{l} \sum_{i \in \text{Gauss}} \begin{bmatrix} \xi_i - \frac{1}{2} \\ -2\xi_i \\ \xi_i + \frac{1}{2} \end{bmatrix} EA_i \begin{bmatrix} \xi_i - \frac{1}{2} & -2\xi_i & \xi_i + \frac{1}{2} \end{bmatrix} = \\
&= \frac{EA_c}{3l} \begin{bmatrix} 7 + 4\alpha & -(8 + 4\alpha) & 1 \\ -(8 + 4\alpha) & 16 & -(8 - 4\alpha) \\ 1 & -(8 - 4\alpha) & 7 - 4\alpha \end{bmatrix}, \\
&\text{with } A_c = \frac{A_i + A_d}{2}; \quad \alpha = \frac{A_i - A_d}{2A_c}. \tag{6.35}
\end{aligned}$$

We see in the last expression that the resulting bar matrix is 3x3 as three parameters have been used, although the bar will only be connected at its two ends, so the parameter associated with its central node is not relevant in the global model. Because of this, we can recall what we had already addressed in section 2.1.2.1, where we sought the condensed stiffness matrix for a piece in which we were not interested in isolating some of the degrees of freedom, which we condensed to obtain the equation 2.57.

Although this is a technique we have already used several times, we will review it again in a general way. The degrees of freedom are decomposed between the “internal”, i , and the “connected”, c . The local stiffness expressions are, in the block format corresponding to the ordering of the degrees of freedom

$$\begin{bmatrix} \mathbf{f}_c \\ \mathbf{f}_i \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{e,cc} & \mathbf{k}_{e,ci} \\ \mathbf{k}_{e,ic} & \mathbf{k}_{e,ii} \end{bmatrix} \begin{bmatrix} \mathbf{u}_c \\ \mathbf{u}_i \end{bmatrix}$$

where, without altering the mechanics of the piece, we are interested in maintaining visible only the relationships between the displacement and load parameters on the connected nodes. We can deduce

$$\begin{aligned} \mathbf{u}_i &= \mathbf{k}_{e,ii}^{-1}(\mathbf{f}_i - \mathbf{k}_{e,ic}\mathbf{u}_c); \\ \mathbf{f}_c &= \mathbf{k}_{e,cc}\mathbf{u}_c + \mathbf{k}_{e,ci}(\mathbf{k}_{e,ii}^{-1}(\mathbf{f}_i - \mathbf{k}_{e,ic}\mathbf{u}_c)) = \\ &(\mathbf{k}_{e,cc} - \mathbf{k}_{e,ci}\mathbf{k}_{e,ii}^{-1}\mathbf{k}_{e,ic})\mathbf{u}_c + \mathbf{k}_{e,ci}\mathbf{k}_{e,ii}^{-1}\mathbf{f}_i. \end{aligned}$$

and seeking to represent $\mathbf{f}_c - \mathbf{k}_{e,ci}\mathbf{k}_{e,ii}^{-1}\mathbf{f}_i = \mathbf{f}_c^* = \mathbf{k}_{e,c}\mathbf{u}_c$,

$$\mathbf{k}_{e,c} = \mathbf{k}_{e,cc} - \mathbf{k}_{e,ci}\mathbf{k}_{e,ii}^{-1}\mathbf{k}_{e,ic}$$

so in the case we are considering, for which the degree of freedom to be condensed is the central node's, we will have

$$\mathbf{k}_{e,c} = \frac{EA_c}{l} \begin{bmatrix} 1 - \frac{\alpha^2}{3} & -\left(1 - \frac{\alpha^2}{3}\right) \\ -\left(1 - \frac{\alpha^2}{3}\right) & 1 - \frac{\alpha^2}{3} \end{bmatrix} = \frac{EA_c^*}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (6.36)$$

with $A_c^* = A_c(1 - \alpha^2/3)$, $\alpha = (A_i - A_d)/A_c$.

We see, therefore, that we obtain appropriate bar stiffness expressions for both models, although in the quadratic model the equivalent area is somewhat smaller than the average obtained in the linear model.

6.2.6.2 Membrane Elements

In this case, although we study alternative models c: and d: from page 476, we will verify that the process is identical to the bar model cases: after analyzing the components of the formulation and the transformation between coordinate spaces, we will show the procedure for constructing the stiffness matrix, which we will examine in detail for the 4-node element. The summary would be as follows.

Element Matrix Construction Process

For the adopted isoparametric formulation, the following scheme can be considered:

- For each element e
 - the expressions for $\partial_{\xi} \mathbf{N}$ are known
 - the expressions for $\mathbf{J}(\mathbf{x}(\xi))$ and for $\mathbf{G}(\mathbf{J})$ are known

- the matrix $\mathbf{k}_e$ is initialized to $\mathbf{0}$
- for each Gauss point g
 - * Calculate $\mathbf{J}_g(\mathbf{x}(\boldsymbol{\xi}))$
 - * Construct $\mathbf{G}_g$ from the elements of $\mathbf{J}_g$
 - * Calculate $\mathbf{b}_g = \frac{\mathbf{G}_g}{|\mathbf{J}_g|} \partial_{\boldsymbol{\xi}} \mathbf{N}|_g$
 - * Determine $\boldsymbol{\kappa}_g$
 - * Calculate $\Delta \mathbf{k}_{eg} = \Omega_g |\mathbf{J}_g| \mathbf{b}_g^T \boldsymbol{\kappa}_g \mathbf{b}_g$
 - * Accumulate $\Delta \mathbf{k}_{eg}$ in $\mathbf{k}_e$
- assemble $\mathbf{k}_e$ into the global matrix $\mathbf{K}$

Membrane Elements, Four or Eight Nodes

We have the general expressions

$$\partial_{\mathbf{x}} \mathbf{N} = \mathbf{b} = \frac{\mathbf{G}}{|\mathbf{J}|} \partial_{\boldsymbol{\xi}} \mathbf{N}; \quad \mathbf{k}_e = \sum \Omega_i |\mathbf{J}_i| \mathbf{b}_i^T \boldsymbol{\kappa}_i \mathbf{b}_i \quad (6.32)$$

although now the geometric transformation is two-dimensional, so that, considering the differential operators for plane stress already seen in section 6.1.3.2, it results, both for model c: and model d:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} & 0 & 0 \\ 0 & 0 & -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \end{bmatrix} \quad (6.37)$$

Therefore, continuing with the four-node case c:, the transformation and its Jacobian correspond to the following expressions:

$$\begin{bmatrix} x(\xi, \eta) \\ y(\xi, \eta) \end{bmatrix} = \begin{bmatrix} N_1(\xi, \eta) & 0 & N_2(\xi, \eta) & 0 & N_3(\xi, \eta) & 0 & N_4(\xi, \eta) & 0 \\ 0 & N_1(\xi, \eta) & 0 & N_2(\xi, \eta) & 0 & N_3(\xi, \eta) & 0 & N_4(\xi, \eta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix},$$

that is

$$\begin{aligned} x &= \sum x_i \frac{(1 + \xi_i \xi)(1 + \eta_i \eta)}{4}; \quad y = \sum y_i \frac{(1 + \xi_i \xi)(1 + \eta_i \eta)}{4}; \\ \mathbf{J} &= \frac{1}{4} \begin{bmatrix} \sum x_i \xi_i (1 + \eta_i \eta) & \sum y_i \xi_i (1 + \eta_i \eta) \\ \sum x_i \eta_i (1 + \xi_i \xi) & \sum y_i \eta_i (1 + \xi_i \xi) \end{bmatrix} \end{aligned} \quad (6.38)$$

For the eight-node case d: we would have

$$\begin{aligned} x &= \sum x_i N_i(\xi, \eta); \quad y = \sum y_i N_i(\xi, \eta); \\ \mathbf{J}(\xi, \eta) &= \begin{bmatrix} \sum x_i \frac{\partial N_i(\xi, \eta)}{\partial \xi} & \sum y_i \frac{\partial N_i(\xi, \eta)}{\partial \xi} \\ \sum x_i \frac{\partial N_i(\xi, \eta)}{\partial \eta} & \sum y_i \frac{\partial N_i(\xi, \eta)}{\partial \eta} \end{bmatrix}, \end{aligned}$$

where i is an index for each of the eight nodes of the element, and N_i is each of the eight shape functions from page 476.

Gaussian numerical integration requires for model c: a single integration point, centered at $(\xi, \eta)_g = (0, 0)$, so that, at this point,

$$\mathbf{J}_0 = \begin{bmatrix} (-x_1 + x_2 - x_3 + x_4)/4 & (-y_1 + y_2 - y_3 + y_4)/4 \\ (-x_1 - x_2 + x_3 + x_4)/4 & (-y_1 - y_2 + y_3 + y_4)/4 \end{bmatrix} \quad (6.39)$$

while in model d: a combination of two points is needed for each direction, four points therefore, at the coordinates $\left(\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{1}{3}}\right)$, so that we would have the four $\mathbf{J}$ values corresponding to each of the Gauss points g

$$\mathbf{J}_g = \mathbf{J}(\xi_g, \eta_g) = \begin{bmatrix} \sum x_i \frac{\partial N_i(\xi, \eta)}{\partial \xi} & \sum y_i \frac{\partial N_i(\xi, \eta)}{\partial \xi} \\ \sum x_i \frac{\partial N_i(\xi, \eta)}{\partial \eta} & \sum y_i \frac{\partial N_i(\xi, \eta)}{\partial \eta} \end{bmatrix} \bigg|_g. \quad (6.40)$$

Four-Node Element Matrix

We will continue to exemplify case c:, a four-node plane element whose relevant field equations are 6.28.

The interpolation functions $\mathbf{N}(\boldsymbol{\xi})$ are those from section c: on page 476: $N_i = (1 + \xi_i \xi)(1 + \eta_i \eta)/4$.

The coordinate transformation $\mathbf{x}(\boldsymbol{\xi})$, $\mathbf{J}(\mathbf{x}(\boldsymbol{\xi}))$, and the operator transformation from 6.18, $\mathbf{G}(\mathbf{J})$, as collected in 6.17 and 6.37, correspond to the geometric transformation defined in 6.38, whose Jacobian we can develop:

$\mathbf{J} =$

$$\frac{1}{4} \begin{bmatrix} (x_2 - x_1)(1 - \eta) + (x_4 - x_3)(1 + \eta) & (y_2 - y_1)(1 - \eta) + (y_4 - y_3)(1 + \eta) \\ (x_3 - x_1)(1 - \xi) + (x_4 - x_2)(1 + \xi) & (y_3 - y_1)(1 - \xi) + (y_4 - y_2)(1 + \xi) \end{bmatrix}$$

We consider the expression 6.32 to construct the matrix.

We have seen that there is only one Gauss point: $(0, 0)$, for which the Jacobian value is that of 6.39, which we can apply to

the expression for $\partial_{\mathbf{x}} \mathbf{N}$, which takes the form

$$\begin{aligned} \partial_{\mathbf{x}} \mathbf{N} &= \frac{\mathbf{G}}{|\mathbf{J}|} \partial_{\xi} \mathbf{N}|_0 \\ &= \frac{\mathbf{G}}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial}{\partial \xi} & 0 \\ \frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial}{\partial \xi} \\ 0 & \frac{\partial}{\partial \eta} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \\ &= \frac{\mathbf{G}}{|\mathbf{J}|} \frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & 1-\eta & 0 & -(1+\eta) & 0 & 1+\eta & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & 1-\xi & 0 & 1+\xi & 0 \\ 0 & -(1-\eta) & 0 & 1-\eta & 0 & -(1+\eta) & 0 & 1+\eta \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & 1-\xi & 0 & 1+\xi \end{bmatrix} \end{aligned}$$

So, being $(\xi, \eta)_0 = (0, 0)$, it results

$$\begin{aligned} \mathbf{b}_0 &= \partial_{\mathbf{x}} \mathbf{N}|_0 = \frac{\mathbf{G}}{|\mathbf{J}|} \partial_{\xi} \mathbf{N}|_0 = \frac{\mathbf{G}_0}{|\mathbf{J}_0|} \partial_{\xi} \mathbf{N}|_0 = \frac{1}{4|\mathbf{J}_0|} \times \\ &\begin{bmatrix} J_{12} - J_{22} & 0 & J_{12} + J_{22} & 0 & -J_{12} - J_{22} & 0 & -J_{12} + J_{22} & 0 \\ J_{21} - J_{11} & J_{12} - J_{22} & -J_{21} - J_{11} & J_{12} + J_{22} & J_{21} + J_{11} & -J_{12} - J_{22} & -J_{21} + J_{11} & -J_{12} + J_{22} \end{bmatrix} \Big|_0 \\ &= \frac{1}{\alpha} \begin{bmatrix} y_3 - y_2 & 0 & y_1 - y_4 & 0 & y_4 - y_1 & 0 & y_2 - y_3 & 0 \\ 0 & x_2 - x_3 & y_3 - y_2 & x_4 - x_1 & y_1 - y_4 & x_1 - x_4 & y_4 - y_1 & x_3 - x_2 \end{bmatrix}; \\ &\alpha = -8|\mathbf{J}_0| = x_3 * y_4 - x_2 * y_4 - x_4 * y_3 + x_1 * y_3 + \\ &\quad x_4 * y_2 - x_1 * y_2 - x_3 * y_1 + x_2 * y_1. \end{aligned}$$

where the coordinates (x_i, y_i) will be the real coordinates of each considered element.

In the final stiffness calculation, $\mathbf{k}_e = \sum \Omega_g |\mathbf{J}_g| \mathbf{b}_g^T \boldsymbol{\kappa}_g \mathbf{b}_g$, we will use $\Omega_g = \Omega_1 = 4$ for a sum consisting of a single point, $g = 1$, with coordinates $(0, 0)$, which takes into account the weight of the entire element.

The local material stiffness matrix $\boldsymbol{\kappa}_g$ will be that of plane stress (6.28), in which stress (per unit thickness) can be consid-

ered, although in general it is better to consider the unit force stress $\times$ thickness:

$$\kappa = \frac{E t}{(1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

So by performing the complete process we would obtain the stiffness matrices for the two elements.

Solutions

Once the stiffness matrices are built, assembled, and the load terms are determined, the common sequence is applied to determine all solution parameters: $\mathbf{U} = \mathbf{K}^{-1}\mathbf{F}$, $\mathbf{u} = \mathbf{B}\mathbf{U}$, $\boldsymbol{\epsilon} = \mathbf{b}\mathbf{u} = \partial_x \mathbf{N}\mathbf{u}$, $\boldsymbol{\sigma} = \kappa\boldsymbol{\epsilon}$.

We will now develop the complete process using Maxima.

Using Maxima. Four-node Plane Element

It is possible to follow the previous sequence with Maxima, which we do for the 4-node element. The greatest difficulty could be in the application of the matrix operator ∂_{ξ} , but this can be solved with ease using the `matrix_element_mult` variable that establishes how matrix multiplication is interpreted: in the following list, matrix operator names are declared, and how the corresponding operators are applied is defined in that variable. The process is simple: creation of the matrix $\mathbf{N}=\mathbf{Nmat}$:

```
n1 : (1-xi)*(1-eta)/4;
n2 : (1+xi)*(1-eta)/4;
```

```

n3 : (1-xi)*(1+eta)/4;
n4 : (1+xi)*(1+eta)/4;
Nmat :
  matrix([n1,0,n2,0,n3,0,n4,0],[0,n1,0,n2,0,n3,0,n4]);

```

The actual geometry, starting from the element geometry:

```

xyr : Nmat . [x1,y1,x2,y2,x3,y3,x4,y4];

```

and for the Jacobian construction we create a matrix `nabla` with the operator names and an interpretation of the product of these operators in the matrix product in `matrix_element_mult`:

```

nabla : matrix([ddxi],[ddeta]);
matrix_element_mult :
  lambda([x,y],if x = ddx then diff(y,xi)
           elseif x = ddeta then diff(y,eta)
           else x * y);

```

so that the Jacobian, and its value at natural coordinates (0,0) are

```

Jac : nabla . transpose(xyr) ;
Jaco : subst([xi=0,eta=0],Jac);

```

The matrix $\mathbf{G}(\mathbf{J})$ will be

```

GJ : matrix([ Jac[2,2],-Jac[1,2], 0 , 0 ],
             [ 0 , 0 ,-Jac[2,1], Jac[1,1]],
             [-Jac[2,1],Jac[1,1],Jac[2,2],-Jac[1,2]]);

```

and the operator ∂_{ξ} and its application to the matrix $\mathbf{N}$:

```
dd : matrix([ddxi, 0],[ddeta,0],[0,ddxi],[0,ddeta]);
ddN : dd . Nmat;
```

so that the matrix $\mathbf{b}$ and its particularization at the Gauss point will be

```
matrix_element_mult : "*" ;
bmat : factor(1/(4*determinant(Jaco))*
             subst([xi=0,eta=0],4*GJ . ddN));
```

where we previously returned the original meaning to multiplication between terms in the matrix product.

The expression for the stiffness matrix is thus immediately obtained

```
kappa : t*e/(1-nu*2)*matrix([1 , nu , 0      ],
                             [nu, 1  , 0      ],
                             [0 , 0  ,(1-nu)/2]);
ke : factor(4*determinant(Jaco)*
            transpose(bmat) . kappa . bmat);
```

where, for the corresponding problem values,

```
vars: [nu=0.25,t= 10, e= 210000];
coor1: [x1=0,y1=0,x2=400,y2=0,x3=0,y3=100,x4=400,y4=75];
coor2: [x1=400,y1=0,x2=800,y2=0,x3=400,y3=75,x4=800,y4=50];
```

we will obtain, by substituting:

```
k1: subst(append(coor1,vars),ke);
k2: subst(append(coor2,vars),ke);
```

To build the complete matrix $\mathbf{K}$ we would assemble the degrees of freedom of nodes 1,2,3 and 4 of the first element into the global nodes 5,3,6 and 4, and those of the second element into nodes 3,1,4 and 2. On the other hand, nodes 5 and 6 have Dirichlet support conditions (zero displacement) so they will not be included in the final model, just as, by symmetry, the vertical displacements y of nodes 1 and 3. The remaining degrees of freedom have Neumann conditions (known external load) and will therefore be those considered in the assembly. The complete model $\mathbf{F} = \mathbf{K}\mathbf{U}$ would be, considering the coincidence of the real axis directions of the elements with the global axes, and ordering the degrees of freedom by global node and direction:

$$\begin{bmatrix} F_{x1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{x4} \\ F_{y4} \end{bmatrix} = \begin{bmatrix} k_{2,x2,x2} & k_{2,x2,x4} & k_{2,x2,y4} & k_{2,x2,x1} & k_{2,x2,x3} & k_{2,x2,y3} \\ k_{2,x4,x2} & k_{2,x4,x4} & k_{2,x4,y4} & k_{2,x4,x1} & k_{2,x4,x3} & k_{2,x4,y3} \\ k_{2,y4,x2} & k_{2,y4,x4} & k_{2,y4,y4} & k_{2,y4,x1} & k_{2,y4,x3} & k_{2,y4,y3} \\ k_{2,x1,x2} & k_{2,x1,x4} & k_{2,x1,y4} & k_{2,x1,x1} & k_{2,x1,x3} & k_{2,x1,y3} \\ k_{2,x3,x2} & k_{2,x3,x4} & k_{2,x3,y4} & k_{1,x2,x2} & k_{1,x2,x4} & k_{1,x2,y4} \\ k_{2,y3,x2} & k_{2,y3,x4} & k_{2,y3,y4} & k_{2,x3,x1} & k_{2,x3,x3} & k_{2,x3,y3} \\ k_{2,y3,x1} & k_{2,y3,x3} & k_{2,y3,y3} & k_{1,x4,x2} & k_{1,x4,x4} & k_{1,x4,y4} \\ k_{1,y4,x2} & k_{1,y4,x4} & k_{1,y4,y4} & k_{2,y3,x1} & k_{2,y3,x3} & k_{2,y3,y3} \\ k_{1,y4,x1} & k_{1,y4,x3} & k_{1,y4,y3} & k_{2,y3,x3} & k_{2,y3,y3} & k_{2,y3,y3} \end{bmatrix} \begin{bmatrix} U_{x1} \\ U_{x2} \\ U_{y2} \\ U_{x3} \\ U_{x4} \\ U_{y4} \end{bmatrix}$$

The load vector will be $\mathbf{F}^T = [50000 \ 50000 \ 0 \ 0 \ 0 \ 0]$,

Eight-node Plane Element

For the eight-node model, the serendipity shape functions indicated on page 476 can be used, although, as commented there, the choice of polynomial functions with the minimum possible degree provides less constrained results. In the attached file (matrixCuatroOchoNodos.wxm), a commented development is



included for Maxima, using both groups of functions, and whose results are used for the comparisons in the following section.

6.2.6.3 Comparative Results

Bar Models

In both models, the expressions to be used are similar, except for the equivalent area A^* employed in the elements:

$$\mathbf{F} = \mathbf{K}\mathbf{U} \equiv \begin{bmatrix} 200 \\ 0 \\ R_3 \end{bmatrix} = \frac{E}{l} \begin{bmatrix} \boxed{A_{c2}^*} & \boxed{-A_{c2}^*} & 0 \\ -A_{c2}^* & \boxed{A_{c2}^* + A_{c1}^*} & \boxed{-A_{c1}^*} \\ 0 & \boxed{-A_{c1}^*} & \boxed{A_{c1}^*} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix}$$

The results for the two-node model (linear displacement laws, therefore with constant strain and stress in the element) are:

$$\mathbf{U} = \begin{bmatrix} 0 \\ 0.2286 \\ 0.5486 \end{bmatrix} \text{ mm}; \mathbf{u} = \mathbf{B}\mathbf{U} = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ U_2 \\ U_2 \\ U_1 \end{bmatrix}; \boldsymbol{\sigma} = \boldsymbol{\kappa}\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} \approx \mathbf{b}\mathbf{u}; \sigma_i \approx E \frac{2}{l} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix}; \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \end{bmatrix} = \begin{bmatrix} 160.00 \\ 114.29 \end{bmatrix} \text{ N/mm}^2$$

The error between interpolated or smoothed stresses, $\hat{\sigma}$ (see section 6.4.1.2) and the constant element stresses, $\bar{\sigma}$, can be measured:

$$e = \frac{1}{E} \int_v (\hat{\sigma} - \bar{\sigma})^2 dv \approx \frac{1}{E} \left(\frac{2}{3} (114.29 - 91.44)^2 \times 200 \cdot 1750 + \frac{2}{3} (160.00 - 137.14)^2 \times 200 \cdot 1250 \right) = 1.044 J$$

The relative error measure turns out to be: $e/(E + e) = 1.87\%$.

The results for the three-node model (with quadratic displacement laws and therefore with strain and stress varying linearly in the element) are:

$$U = \begin{bmatrix} 0 \\ 0.2317 \\ 0.5538 \end{bmatrix} \text{ mm}; \mathbf{u} = \mathbf{BU} = \begin{bmatrix} u_{1i} \\ u_{1c} \\ u_{1d} \\ u_{2i} \\ u_{2c} \\ u_{2d} \end{bmatrix} = \begin{bmatrix} 0 \\ u_{1c} \\ U_2 \\ U_2 \\ u_{2c} \\ U_1 \end{bmatrix}; \boldsymbol{\sigma} = \boldsymbol{\kappa}\boldsymbol{\varepsilon} \approx \boldsymbol{\kappa}\mathbf{bu}$$

$$\boldsymbol{\sigma}_i \approx E \frac{2}{l} \begin{bmatrix} \xi_i - \frac{1}{2} & -2\xi_i & \xi_i + \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{ii} \\ u_{ic} \\ u_{id} \end{bmatrix}; \begin{bmatrix} \bar{\boldsymbol{\sigma}}_1^T \\ \bar{\boldsymbol{\sigma}}_2^T \end{bmatrix} = \begin{bmatrix} 174.4 & 147.8 \\ 129.2 & 102.5 \end{bmatrix},$$

with stress values at the two Gauss points of each element in N/mm^2 . The stress results are plotted in figure 6.10.

In the displacements, it is observed that the model with more degrees of freedom is slightly less rigid, although the differences in the smoothed stress laws are very limited.

It is also noted that the stress values at the Gauss points exactly match the analytical values, although this can be considered a consequence of the problem's simplicity: although the approximations at these points are generally better than in the rest of the element, they remain approximations, as observed in the quadratic element case.

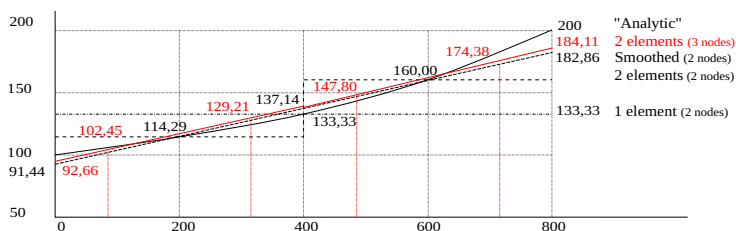


Figure 6.10: Stress values and laws for two bar models, linear (in black, considering one or two elements) and quadratic (in red, for two elements). For the linear model with two elements, smoothed stresses are plotted considering values from both. The curved law corresponds to the analytical solution of the tensile piece problem with linearly varying area.

Membrane Models

The 4-node membrane element provides displacements of 0.528 and 0.519 mm in the y-axis at the element's edge at the load application point, and 0.219 0.217 mm respectively in the middle section, with horizontal normal stresses of 160.0 and 114.3 N/mm² at the respective Gauss points, which coincides with the two-node bar model, although the model predicts, given the piece geometry, the existence of slight tangential stresses, of -5.0 and -3.57 N/mm² respectively at these points. The stress distribution in the elements is not exactly constant and is of limited quality.

For the 8-node element, it is relevant to consider which load vector should be used: the load is distributed stress over the

entire end of element 2, between nodes numbered 1, 2, and 4. Given the equidistance between these, it might be tempting to distribute the 100 kN with 25 for the corner nodes and 50 for the central node, but this is in contradiction with the rigorous application of the theory used: according to expression 5.47, the distributed load φ must be applied to each node according to the integral along the edge of its product with the node's shape function, which, when using parabolic functions, means attributing two-thirds of the total load to the central node and one-sixth to each corner node: 66.66 and 16.66 kN respectively. We call these consistent loads.

The displacements obtained for the end and mid sections of the piece are, from axis to edge, 0.539 0.522 0.535 and 0.225 0.214 0.222 mm with non-consistent loads and 0.533 0.523 0.528 and 0.223 0.215 0.219 with consistent loads in the serendipitous eight-node model with nine-monomial polynomial functions. For the model with eight-monomial polynomial functions, the displacements for consistent loads are 0.534 0.529 0.522 and 0.222 0.219 0.216 mm respectively. See the comments about the difference between the two models on pages 476 et seq.

The differences between models show the slight distortion effect that is caused by the incorrect application of loads, as well as the over-stiffening in the region of the non-existent ninth central node that is caused by the shape functions of nine monomials compared to those of only eight, although these differences are slight.

For this last model, with eight nodes and eight-monomial polynomial functions, with consistent load application, the stresses at the clamped and end nodes are, in the same order, 105.1 104.1

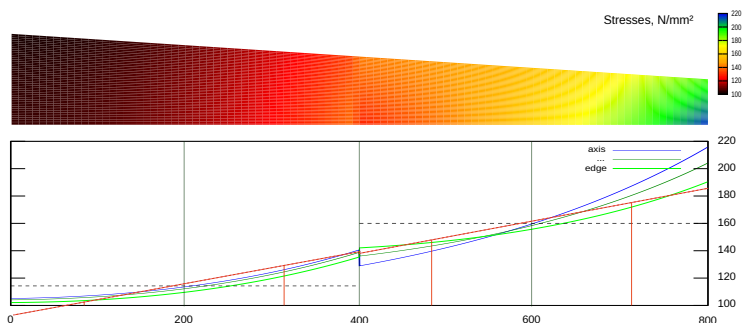


Figure 6.11: Horizontal normal stress distribution and diagrams in the symmetric half-plate model with two eight-node elements, minimum degree polynomial functions and load applied consistently. Bar model stress reference points from two and three node bars are also traced.

102.1 and 215.9 204.1 190.5 N/mm^2 and the resulting graphs are those in 6.11, which are compared with those of the bar elements.

6.3 Finite Elements in Bending Problems

The objective of this section is twofold: on the one hand, to provide tools for formulating elements capable of approximating solutions to bending problems, according to available theoret-

ical alternatives and their corresponding differential equations, and on the other hand to present and analyze the application fields of such formulations, considering the problems that arise from the coupling between the different functions describing displacements and the method chosen for its representation.

In this analysis, we traverse the modeling of beam and plate theories, starting with the simplest formulations, Navier and Kirchhoff theories respectively, which do not account for shear distortions, and extending them to those that do, Timoshenko and Reissner-Mindlin, to finally present some ideas about how to formulate shell elements that, in addition to coupling both axial and bending effects, must describe curved geometries.

6.3.1 Beam Flexure

Figure 6.12 shows, in addition to the static variables involved in the equilibrium of a slice between two nearby sections of a beam, two alternative ways of interpreting the slice deformation. The first, most common, presents the Navier model, in which the tangential deformation due to shear is neglected and, consequently, the sections remain orthogonal to the centerline deformation, such that the curvature c and section rotation θ are coupled, by identifying the slice rotation with the centerline rotation. In the second approximation, the slice undergoes distortions derived from the tangential stress, distortions that vary along the beam depth, but provide a distortion γ that decouples the slice rotation θ , measured from its initial position, from the centerline rotation φ . This second approximation corresponds to the Timoshenko model. While the first model is appropriate

Where I is the section's moment of inertia. The deformation function is thus the curvature c , to which the bending moment M is associated, with rotations and shear forces derived from these. The curvature derives directly from the displacement w .

Timoshenko For less slender beams, shear distortion is considered:

$$\begin{aligned} {}^T\theta &= \frac{\partial w}{\partial x} - \gamma; \quad \varepsilon_x(z) = z \frac{\partial \theta}{\partial x}; \quad c = \frac{\partial \theta}{\partial x} \neq \frac{\partial^2 w}{\partial x^2} = c + \frac{\partial \gamma}{\partial x}; \\ M &= EI c; \quad T = GA^* \gamma = GkA \gamma; \quad G = \frac{E}{2(1 + \nu)} \end{aligned} \quad (6.42)$$

Where $A^* = kA$ is the effective area against shear distortion. Now there are two functions to describe the deformation, γ and c , to which two stress functions are associated, T and M . Now the displacement w results from the combined effect of both deformation components.

6.3.1.1 Navier

Although known as the Navier model, this is a model for slender beams in flexure developed progressively by Leonhard Euler, Jacques Bernoulli, and Claude-Louis Henry Navier.

Its finite element approximation requires describing the vertical displacement w and ensuring continuity at the nodes for both this displacement and the rotations θ . Thus, the nodal parameters must be displacement and rotation to express the displacement function for the element based on these. For elements with two nodes, i, d , and two degrees of freedom per node,

w , θ , fourth-order polynomial functions –of third degree– will suffice, which we have already seen in 2.29. In real coordinates between 0 and l we have

$$w = \left(\frac{2x^3}{l^3} - \frac{3x^2}{l^2} + 1 \right) w_i + \left(\frac{x^3}{l^2} - \frac{2x^2}{l} + x \right) \theta_i + \left(-\frac{2x^3}{l^3} + \frac{3x^2}{l^2} \right) w_d + \left(\frac{x^3}{l^2} - \frac{x^2}{l} \right) \theta_d.$$

and the process of constructing the element stiffness matrix, equivalent loads, etc. has already been developed with algebraic integration in section 2.1.1.7.

Now we will consider using *natural coordinates*, as we have done with axial problems, to facilitate processes with numerical integration. We will have $x = x_0 + \frac{l}{2}(\xi + 1)$, $\frac{\partial w}{\partial x} = \frac{2}{l} \frac{\partial w}{\partial \xi}$.

The shape functions, for elements with coordinates ξ in the interval $[-1, 1]$ will be

$$N_{1,w} = \frac{1}{4}(2 - 3\xi + \xi^3); N_{1,\theta} = \frac{l}{8}(1 - \xi - \xi^2 + \xi^3) \\ N_{2,w} = \frac{1}{4}(2 + 3\xi - \xi^3); N_{2,\theta} = \frac{l}{8}(-1 - \xi + \xi^2 + \xi^3)$$

The procedure is identical to those already used, where the components of the process are summarized in

$$\partial_x N = \mathbf{b} = \frac{\mathbf{G}}{|\mathbf{J}|} \partial_\xi N; \mathbf{k}_e = \sum \Omega_i |\mathbf{J}_i| \mathbf{b}_i^T \boldsymbol{\kappa}_i \mathbf{b}_i; \quad (6.32)$$

and where we can specialize for the equations of the adopted

beam model:

$$\begin{aligned}
 J &= \frac{dx}{d\xi}; \quad \partial_x w = \frac{\partial^2 w}{\partial x^2} = \left(\frac{\partial \xi}{\partial x} \right)^2 \frac{\partial^2 w}{\partial \xi^2} = \left(\frac{\partial \xi}{\partial x} \right)^2 \partial_\xi w = \\
 &= \frac{\mathbf{G}}{|\mathbf{J}|} \partial_\xi w \implies G = \frac{1}{dx/d\xi} \\
 \partial_\xi N &= \frac{d^2 N}{d\xi^2} = N''; \quad J = \frac{l}{2}; \quad G = \frac{2}{l};
 \end{aligned} \tag{6.43}$$

so that, considering two Gauss points for integration, each with a weight Ω_i of 1, we will have

$$\mathbf{k}_e = \sum_{\xi=\pm\sqrt{1/3}} 1 \left(\frac{l}{2} \right)^{-3} \begin{bmatrix} N''_{1,w} \\ N''_{1,\theta} \\ N''_{2,w} \\ N''_{2,\theta} \end{bmatrix} EI \begin{bmatrix} N''_{1,w} & N''_{1,\theta} & N''_{2,w} & N''_{2,\theta} \end{bmatrix} \tag{6.44}$$

where $\begin{bmatrix} N''_{1,w} & N''_{1,\theta} & N''_{2,w} & N''_{2,\theta} \end{bmatrix} = \begin{bmatrix} \frac{6\xi}{4} & \frac{-2+6\xi}{8} & -\frac{6\xi}{4} & \frac{2+6\xi}{8} \end{bmatrix}$, so that

$$\mathbf{k}_e = \begin{bmatrix} \frac{12EI}{l^3} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{2EI}{l} & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix} \tag{6.45}$$

As we can see, these are the same flexural stiffness relations that we previously obtained by algebraic method, in 2.42 or 2.44.

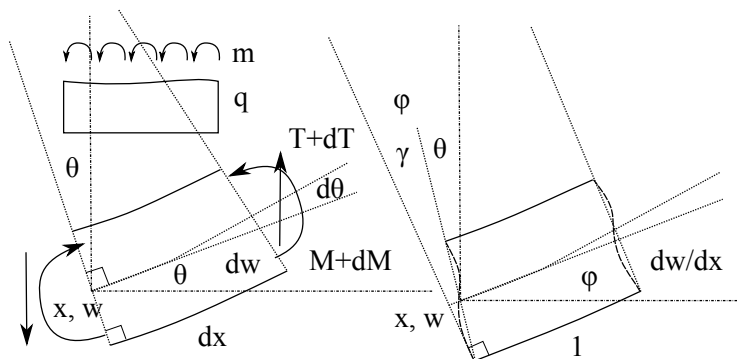


Figure 6.13: Timoshenko beam flexure. The deformation with shear distortion requires two movement parameters, and consequently also two load parameters.

6.3.1.2 Timoshenko

We have seen that the inclusion of distortion in non-slender beams decouples the centerline descent w from the section rotation, θ , equations 6.42. Thus, two movement parameters are needed and, by correspondence, two conjugate load parameters: the applied load parameters on the infinitesimal slice that would produce work in each of these movements. Therefore, a moment density m appears in figure 6.13 as an applied load, which is added to the load density q . The relevant equations are applied

through expressions that now take the following form:

$$\begin{aligned}
 \epsilon &= \partial v; \quad \varphi = \partial^* \sigma; \quad \sigma = \kappa \epsilon; \quad v \approx N a \\
 v &= \begin{bmatrix} w \\ \theta \end{bmatrix}; \quad \epsilon = \begin{bmatrix} c \\ \gamma \end{bmatrix}; \quad \sigma = \begin{bmatrix} M \\ T \end{bmatrix}; \quad \partial = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -1 \end{bmatrix} \\
 \kappa &= \begin{bmatrix} EI & 0 \\ 0 & GkA \end{bmatrix}; \quad \varphi = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -1 \end{bmatrix} \begin{bmatrix} M \\ T \end{bmatrix} = \begin{bmatrix} q \\ m \end{bmatrix}
 \end{aligned} \tag{6.46}$$

Shape Functions

We consider the case of two-node elements, located at both ends of the piece segment to be modeled, and consider nodal parameters as the displacements and rotations of both ends:

$$a = u = \begin{bmatrix} w_1 & \theta_1 & w_2 & \theta_2 \end{bmatrix}^T; \quad N = ?$$

Shape functions are selected considering displacements and rotations as decoupled, and since the differential operator is first-order, linear separated functions will suffice for both movements. Therefore, in natural coordinates, the already known functions are used:

$$N_i = (1 + \xi_i \xi)/2, \quad i \in [1, 2]; \quad \xi \in [\xi_1, \xi_2] = [-1, 1].$$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix};$$

Stiffness Matrix

The different parts of the problem are as always

$$\mathbf{b} = \partial \mathbf{N}; \quad \partial_x \mathbf{N} = \frac{\mathbf{G}}{|\mathbf{J}|} \partial_\xi \mathbf{N}; \quad \mathbf{k}_e = \sum \Omega_i |\mathbf{J}_i| \mathbf{b}_i^T \boldsymbol{\kappa}_i \mathbf{b}_i; \quad (6.32)$$

$$J = \frac{dx}{d\xi} = \frac{l}{2}; \quad \frac{\mathbf{G}}{|\mathbf{J}|} \partial_\xi = \begin{bmatrix} 0 & \frac{2}{l} \frac{\partial}{\partial \xi} \\ \frac{2}{l} \frac{\partial}{\partial \xi} & -1 \end{bmatrix};$$

and the strains can be approximated with

$$\begin{aligned} \begin{bmatrix} c \\ \gamma \end{bmatrix} &= \begin{bmatrix} c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} = \mathbf{b} \mathbf{u} = \partial \mathbf{N} \mathbf{u} = \\ &\left(\begin{bmatrix} 0 & \frac{2}{l} \frac{\partial}{\partial \xi} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{2}{l} \frac{\partial}{\partial \xi} & -1 \end{bmatrix} \right) \mathbf{N} \mathbf{u} = (\mathbf{b}_c + \mathbf{b}_\gamma) \mathbf{u}. \\ \mathbf{b}_c &= \begin{bmatrix} 0 & -\frac{1}{l} & 0 & \frac{1}{l} \\ 0 & 0 & 0 & 0 \end{bmatrix}; \\ \mathbf{b}_\gamma &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{l} & -N_1(\xi) & \frac{1}{l} & -N_2(\xi) \end{bmatrix}. \end{aligned}$$

so that the effects of curvature and distortion components can be decoupled in the stiffness, given that, $\boldsymbol{\kappa}$ being diagonal, it results $\mathbf{b}_c^T \boldsymbol{\kappa} \mathbf{b}_\gamma = \mathbf{0}$. Therefore, it can be done

$$\mathbf{k}_e = \mathbf{k}_{e,c} + \mathbf{k}_{e,\gamma} = \sum_{c,\gamma} \sum_i \Omega_i |\mathbf{J}_i| \mathbf{b}_i^T \boldsymbol{\kappa}_i \mathbf{b}_i \quad (6.47)$$

The component $\mathbf{k}_{e,c}$ can be integrated with one Gauss point. The component $\mathbf{k}_{e,\gamma}$, with quadratic terms, requires two. Re-

sulting in:

$$\begin{aligned} \mathbf{k}_{e,c} &= \frac{EI}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}; \\ \mathbf{k}_{e,\gamma} &= \frac{kAG}{l} \begin{bmatrix} 1 & \frac{l}{2} & -1 & \frac{l}{2} \\ \frac{l}{2} & \frac{l^2}{3} & -\frac{l}{2} & \frac{l^2}{6} \\ -1 & -\frac{l}{2} & 1 & -\frac{l}{2} \\ \frac{l}{2} & \frac{l^2}{6} & -\frac{l}{2} & \frac{l^2}{3} \end{bmatrix} \end{aligned} \quad (6.48)$$

We can now test the result in a simple problem: the approximation of the behavior of a cantilever beam, with load P on the right. We consider a single element and restrict the degrees of freedom to the free, loaded node.

$$\begin{aligned} \begin{bmatrix} R_1 \\ M_1 \\ P \\ 0 \end{bmatrix} &= \mathbf{k}_e \begin{bmatrix} 0 \\ 0 \\ w_2 \\ \theta_2 \end{bmatrix} \\ \begin{bmatrix} P \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{kAG}{l} & -\frac{kAG}{2} \\ -\frac{kAG}{2} & \frac{EI}{l} + \frac{kAGl}{3} \end{bmatrix} \begin{bmatrix} w_2 \\ \theta_2 \end{bmatrix}. \\ \mathbf{k}_e^{-1} &= \frac{l}{EI + kAGl^2/12} \begin{bmatrix} \frac{EI}{kAG} + \frac{l^2}{3} & \frac{l}{2} \\ \frac{l}{2} & 1 \end{bmatrix} = \\ &= \frac{1}{1 + \frac{kAGl^2}{12EI}} \begin{bmatrix} \frac{l}{kAG} + \frac{l^3}{3EI} & \frac{l^2}{2EI} \\ \frac{l^2}{2EI} & \frac{l}{EI} \end{bmatrix} \end{aligned} \quad (6.49)$$

Now we see an unexpected and undesired effect.

Shear Locking. If the coefficient $\alpha = \frac{kAGl^2}{12EI} = \frac{k\lambda^2}{24(1+\nu)}$ tends to 0, the flexibility tends to the theoretical beam theory, $w = Pl^3/3EI + (Pl/kAG)$, $\theta = Pl^2/2EI$, ... but if α tends to ∞ , or what is the same, if the slenderness of the piece $\lambda \rightarrow \infty$, the flexibility disappears [$\mathbf{k}_e^{-1} \rightarrow \mathbf{0}$]! or what is the same, an unexpected excess of stiffness occurs.

Comparing with Navier's beam. We would have

$$\begin{aligned} w_N &= \frac{Pl^3}{3EI}; w_T = \frac{1}{1+\alpha} \left(\frac{l^3}{3EI} + \frac{l}{kAG} \right) P; \\ \frac{w_T}{w_N} &= \frac{1}{1 + \frac{k\lambda^2}{24(1+\nu)}} \left(1 + \frac{6(1+\nu)}{k\lambda^2} \right) = \\ &\quad \frac{24(1+\nu)(6(1+\nu) + k\lambda^2)}{k\lambda^2(24(1+\nu) + k\lambda^2)} \end{aligned} \quad (6.50)$$

The ratio should tend to 1 if slenderness tends to ∞ . So there is an excess of stiffness that can be attributed to the shear-distortion component. To verify this, let's look at a normalized expression of local end efforts: $\bar{\mathbf{f}} = \frac{l^3}{3EI} \mathbf{f}$, efforts associated with end displacements $\mathbf{u}$ through stiffness $\mathbf{k}$, which we disaggregate and normalize in its two components:

$$\begin{aligned} \bar{\mathbf{f}} &= \frac{l^3}{3EI} \mathbf{f} = \frac{l^3}{3EI} \mathbf{k} \mathbf{u} = \frac{l^3}{3EI} (\mathbf{k}_c + \mathbf{k}_\gamma) \mathbf{u} = \\ &= \frac{l^3}{3EI} \left(\frac{3EI}{l^3} \bar{\mathbf{k}}_c + \frac{kAG}{l} \bar{\mathbf{k}}_\gamma \right) \mathbf{u} = (\bar{\mathbf{k}}_c + 4\alpha \bar{\mathbf{k}}_\gamma) \mathbf{u} \end{aligned}$$

We see that it is the shear component that contributes this excess. The usual response to this problem is to make this

component, $\bar{\mathbf{k}}_\gamma$ or $\mathbf{k}_\gamma$ singular. And the way to do it is by performing reduced integration: use one Gauss point instead of the two required for exact integration.

Now we would have

$$\begin{aligned}\mathbf{k}_\gamma &= 2\frac{l}{2} \begin{bmatrix} 0 & -\frac{1}{l} \\ 0 & -\frac{1}{2} \\ 0 & \frac{1}{l} \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} EI & 0 \\ 0 & kAG \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{l} & -\frac{1}{2} & \frac{1}{l} & -\frac{1}{2} \end{bmatrix}, \\ \mathbf{k}_\gamma &= \frac{kAG}{l} \begin{bmatrix} 1 & \frac{l}{2} & -1 & \frac{l}{2} \\ \frac{l}{2} & \frac{l^2}{4} & -\frac{l}{2} & \frac{l^2}{4} \\ -1 & -\frac{l}{2} & 1 & -\frac{l}{2} \\ \frac{l}{2} & \frac{l^2}{4} & -\frac{l}{2} & \frac{l^2}{4} \end{bmatrix}; \\ \mathbf{k}_e &= \mathbf{k}_{e,c} + \mathbf{k}_{e,\gamma}.\end{aligned}\tag{6.51}$$

It is left as an exercise for the reader to verify the consequences of this correction.

6.3.2 Plate Bending

The plate formulation follows, in a two-dimensional geometry, a development similar to beam theory in one dimension: again the dilemma is raised between considering the deformations derived from shear stress distortion or not, according to their relative importance: in slender plates this effect is neglected as irrelevant: we have the Kirchhoff model, which assumes that the material line perpendicular to the plate plane remains perpendicular during movement, while the alternative model, Reissner-

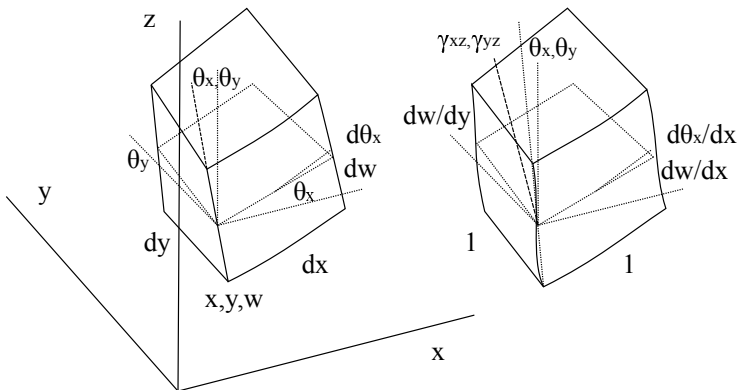


Figure 6.14: Plate Bending. The models consider or do not consider distortion, which implies the loss of perpendicularity to the mid-plane of the material line that was perpendicular before the movement.

Mindlin, assumes that this perpendicularity can be lost due to such distortions, as shown in figure 6.14.

We will, as in other cases, establish the bases of modeling, starting from the functions that represent the statics and kinematics of the problem.

6.3.2.1 Modeling of Plate Bending

The figure 6.15 identifies the different components of the equilibrium: as in all cases, it involves identifying the loads φ , the

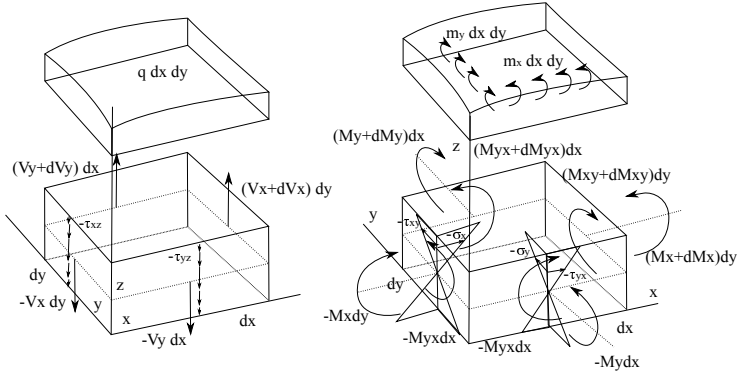


Figure 6.15: Plate Bending Statics: components of equilibrium with perpendicular load and internal stresses, between each other, and possibly with externally applied moments.

internal stresses σ , and the differential operator that links them in the equilibrium $\mathcal{D}^*$, without forgetting the necessary correspondence with the kinematic counterparts, displacements $\mathbf{u}$, strains ϵ , and the corresponding compatibility operator, $\mathcal{D}$.

We use as loads $\varphi = [q(x, y), m_x(x, y), m_y(x, y)]^T$, (where m_x, m_y are usually $= 0$, depending on the model).

For the identification of internal stresses, we employ the resultants of the stresses $\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}$ considering that the vertical normal stresses σ_z are approximately null, as are equally the stresses $\sigma_{x0}, \sigma_{y0}, \tau_{xy0}$, where the 0 indicates those corresponding to the mid-plane.

These resultants can be obtained with the expressions

$$\begin{aligned} M_x &= - \int_t z \sigma_x dz, \quad M_y = - \int_t z \sigma_y dz, \quad M_{xy} = - \int_t z \tau_{xy} dz; \\ V_x &= \int_t \tau_{xz} dz, \quad V_y = \int_t \tau_{yz} dz; \end{aligned} \quad (6.52)$$

with positive stresses being those in the increasing direction on the increasing face of the cuts, tractions in the case of normal stresses.

They are grouped in the different shear forces and moments: $\sigma = [V_x, V_y, M_x, M_y, M_{xy}]^T$, and the equilibrium conditions with external loads are

$$\begin{aligned} \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q &= 0; \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} + V_x + m_x &= 0 \\ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} + V_y + m_y &= 0 \end{aligned} \quad (6.53)$$

where the positive signs are those in the figure on the increasing faces.

The kinematic parameters of interest correspond to the displacement of the mid-surface points, where horizontal displacements can be assumed null, only requiring consideration of the vertical component ($u = v = 0, w \neq 0$). However, points at the depth of the cross-section distinct from this do have horizontal movements u, v that can be deduced from the deviations with respect to the original direction of the material line *normal* to the

mid-surface, composed of the rotations of said perpendicular θ_x , θ_y and the angular distortions γ_{xz} , γ_{yz} that reflect the departure of the material line from the direction of the line perpendicular to the mid-plane.

We now consider the strains; Those of the plane state of each plane parallel to the mid-plane, $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ can be deduced from the movements u, v in the plate thickness and the qualities of said plane state in each layer: $[\varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y)]$ as they are plane stress states. The angular distortions, γ_{xz}, γ_{yz} , are linked to the corresponding shear stresses, of which we use the vertical ones: τ_{xz}, τ_{yz} .

With this, we have tools to compare the two classic alternative models.

6.3.2.2 Kirchhoff versus Reissner–Mindlin

In the Kirchhoff model, similar to Navier's for beams, distortions are ignored, which prevents separately modeling the vertical shear stresses and therefore the shear forces, which must therefore be deduced from the moments. The expressions for the movement of the normal to the plate are, therefore:

$$\theta_x = \frac{\partial w}{\partial x}; \theta_y = \frac{\partial w}{\partial y}. \quad (6.54)$$

However, in the Reissner–Mindlin model, the rotation of the normal to the mid-plane of the plate is decoupled from the change of direction of the material line initially coincident with that normal:

$$\theta_x = \frac{\partial w}{\partial x} - \gamma_{xz}; \theta_y = \frac{\partial w}{\partial y} - \gamma_{yz}. \quad (6.55)$$

Both models are developed below.

Kirchhoff Model

For displacements $\mathbf{v}$, w is sufficient, which allows deducing θ_x, θ_y , from which in turn the strains are deduced:

$$\begin{aligned}\varepsilon_x &= -z \frac{\partial \theta_x}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}; \quad \varepsilon_y = -z \frac{\partial \theta_y}{\partial y} = -z \frac{\partial^2 w}{\partial y^2}; \\ \gamma_{xy} &= -z \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) = -z 2 \frac{\partial^2 w}{\partial x \partial y};\end{aligned}\quad (6.56)$$

where again signs are considered according to the directions of local strain on the increasing faces of the section.

These strains respond to plane stress states in the thickness, and as seen in the equations, they are deducible from three unique functions that can be interpreted as curvatures, and which constitute the necessary and sufficient group to describe the strains $\boldsymbol{\varepsilon}$:

$$c_x = \frac{\partial^2 w}{\partial x^2}, \quad c_y = \frac{\partial^2 w}{\partial y^2}, \quad c_{xy} = 2 \frac{\partial^2 w}{\partial x \partial y}. \quad (6.57)$$

From the strains, the stresses of the corresponding plane states can be derived, and the moments resulting from these stresses in the sections, moments that are sufficient as stress functions $\boldsymbol{\sigma}$, and are in concordance with the functions that describe the strains

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} c_x \\ c_y \\ c_{xy} \end{bmatrix} \quad (6.58)$$

Finally, the load $\varphi = q$ can be deduced by equilibrium from these and thus the relationships between load and displacement ($q \leftrightarrow w$) are completed

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = q \quad (6.59)$$

The set is, therefore:

$$\varphi = q; \mathbf{v} = w; \boldsymbol{\epsilon} = \begin{bmatrix} c_x \\ c_y \\ c_{xy} \end{bmatrix}; \boldsymbol{\sigma} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}; \boldsymbol{\partial} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ 2 \frac{\partial^2}{\partial x \partial y} \end{bmatrix};$$

$$\boldsymbol{\epsilon} = \boldsymbol{\partial} \mathbf{v}; \varphi = \boldsymbol{\partial}^* \boldsymbol{\sigma} = \boldsymbol{\partial}^T \boldsymbol{\sigma}; \boldsymbol{\sigma} = \boldsymbol{\kappa} \boldsymbol{\epsilon}, \quad (6.60)$$

with $\boldsymbol{\kappa}$ as stated in 6.58 for the moments–curvatures relationships.

The question now is to construct the approximation to the displacement functions: $\mathbf{v} = \mathbf{N} \mathbf{u}$; $\mathbf{N} = ?$; $\mathbf{u} = ?$. They must be chosen taking into account the strategy already repeatedly used, which involves imagining a partition of the structure into elements, and establishing the stiffness qualities of each element through the expressions:

$$\partial_x \mathbf{N} = \mathbf{b} = \frac{\mathbf{G}}{|\mathbf{J}|} \partial_\xi \mathbf{N}; \mathbf{k}_e = \sum \Omega_i |\mathbf{J}_i| \mathbf{b}_i^T \boldsymbol{\kappa}_i \mathbf{b}_i \quad (6.32)$$

The typical elements can be triangular, rectangular, or trapezoidal, with nodes at the vertices and perhaps on the sides, and with movement parameters at each node that facilitate the desired interpolation for $\mathbf{v} = w$. In general, and given that the

differential operator $\boldsymbol{\partial}$ to be applied is of second degree, third- or higher-degree functions will be needed, which requires several parameters per node. One evident parameter is its displacement, and for reasons of symmetry of behavior, the rotations around the two plane axes also become evident parameters. Less evident is the necessity of using additional parameters, which would imply adding cross derivatives. Later we will see that there are reasons why some models incorporate them in their formulation.

This results in

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \dots \\ \mathbf{u}_n \end{bmatrix}; \mathbf{u}_i = \begin{bmatrix} w_i \\ \theta_{xi} = \left(\frac{\partial w}{\partial x} \right)_i \\ \theta_{yi} = \left(\frac{\partial w}{\partial y} \right)_i \\ \Gamma_i = \left(\frac{\partial^2 w}{\partial x \partial y} \right)_i ? \end{bmatrix}; \quad (6.61)$$

From these, interpolation functions must be imagined that, meeting the conditions considered in section 6.1.2.2, adequately approximate the displacements in the model. Given the added geometric difficulties in this problem, we will exclusively consider a very simple model, the Melosh–Zienkiewicz–Cheung (MZC) model, as represented in figure 6.16.

It is a rectangular element where the coordinates and nodal parameters are

$$x = \xi a + x_0; y = \eta b + y_0$$

$$\mathbf{u}_i = [w_i, \theta_{xi}, \theta_{yi}]^T \quad (6.62)$$

where natural coordinates ξ, η between -1 and 1 have been used, and real coordinates given by the dimensions of the rectangle

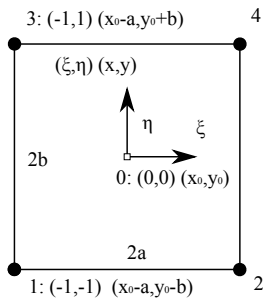


Figure 6.16: MZC Rectangular Plate Element.

sides, $2a$, $2b$, and using the following shape functions:

$$\begin{aligned}
 w &= \sum_{i=1}^4 \left(N_i w_i + \bar{N}_i \theta_{xi} + \bar{\bar{N}}_i \theta_{yi} \right) \\
 N_i &= \frac{1}{8} (1 + \xi_i \xi) (1 + \eta_i \eta) (2 + \xi_i \xi + \eta_i \eta - \xi^2 - \eta^2) \\
 \bar{N}_i &= \frac{a}{8} (\xi^2 - 1) (\xi + \xi_i) (1 + \eta \eta_i) \\
 \bar{\bar{N}}_i &= \frac{b}{8} (\eta^2 - 1) (\eta + \eta_i) (1 + \xi \xi_i)
 \end{aligned} \tag{6.63}$$

where the index refers to the node to which the function corresponds.

In the following expressions, the second index refers to the node whose coordinates are applied to the function, and the prime to the partial derivative to be applied to the function,

according to the direction expressed by the letter that follows it.

Given the linear and axis-separated relations between natural and real coordinates, it is easy to verify that the precise conditions are met:

$$\begin{aligned}
 N_i(i) &= N_{ii} = 1, N_{ik \neq i} = 0, \\
 N_{ij'x} &= \frac{\partial N_i}{\partial x_j} = 0, N_{ij'y} = 0, \\
 \bar{N}_{ij} &= 0, \bar{N}_{ij'y} = 0, \\
 \bar{N}_{ii'x} &= 1, \bar{N}_{ik'x} = 0, \\
 \bar{\bar{N}}_{ij} &= 0, \bar{\bar{N}}_{ij'x} = 0, \\
 \bar{\bar{N}}_{ii'y} &= 1, \bar{\bar{N}}_{ik'y} = 0.
 \end{aligned}$$

To which can be added both the condition $\sum_{i=1}^4 N_i = 1$, necessary to ensure rigid movement in the case of common displacement of all nodes without rotation, or the rigid rotation conditions, which we do not develop here.

We can now construct the element stiffness considering that, given the geometric relationships between both coordinate systems, we will have

$$\mathbf{J} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}; \quad \partial_{\mathbf{x}} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ 2 \frac{\partial^2}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} \frac{1}{a^2} \frac{\partial^2}{\partial \xi^2} \\ \frac{1}{b^2} \frac{\partial^2}{\partial \eta^2} \\ \frac{2}{ab} \frac{\partial^2}{\partial \xi \partial \eta} \end{bmatrix} \quad (6.64)$$

having as always

$$\mathbf{b} = \partial_{\mathbf{x}} \mathbf{N}; \quad \mathbf{k}_e = \sum \Omega_i |\mathbf{J}_i| \mathbf{b}_i^T \boldsymbol{\kappa}_i \mathbf{b}_i \quad (6.32)$$

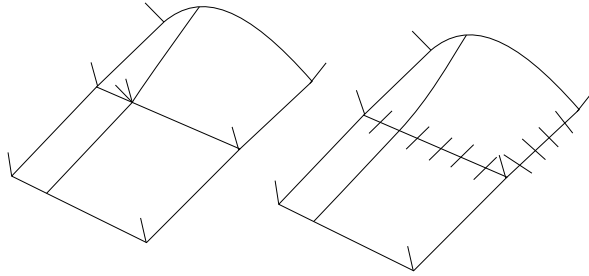


Figure 6.17: Non-conforming Elements: if the deformation of a border is not generated exclusively by the parameters of the nodes of that border, discontinuities are produced in the derivatives along the common border. Forcing continuity in these at the common border implies the non-existence or singularity of the cross derivatives at the nodes.

and where the sum would be made for $2 \times 2 = 4$ Gauss points.

It must be considered that, since twelve parameters per element are handled (three per node), the chosen functions select twelve monomials in the Pascal triangle that, to allow symmetry between the two plane directions, do not exactly coincide with a complete rhombus associated with the triangle's height. In the case of having used one more parameter, namely the cross derivatives, it would be sixteen terms, which can be chosen in different ways, although now it is possible to use that rhombus around the height. The possible advantage of using this second approximation becomes apparent if we consider Figure

6.17: the element considered is a non-conforming element, in which, since the deformation on the borders is generated from the movements and their gradients of all nodes, including those not on the considered border, discontinuity occurs in the slopes (in the rotations), with a consequent loss of deformation energy (moment by rotation) by not measuring this through the border. The alternative requires more parameters, but in turn implies the generation of singularities in the second derivatives, as $\partial^2 w / \partial x \partial y$ does not coincide with $\partial^2 w / \partial y \partial x$, as can be seen in that same figure.

However, this problem may not affect convergence if the patch test stated in section 6.1.2.2 is met, which is true in this model.

Reissner–Mindlin Model

In this case, corresponding to the second image in figure 6.12, the field functions are handled as follows:

$$\begin{aligned} \mathbf{v} &= [w \ \theta_x \ \theta_y]^T, \ \boldsymbol{\varphi} = [q \ m_x \ m_y]^T, \\ \boldsymbol{\epsilon} &= [c_x \ c_y \ c_{xy} \ \gamma_{xz} \ \gamma_{xy}]^T, \\ \boldsymbol{\sigma} &= [M_x \ M_y \ M_{xy} \ V_x \ V_y]^T. \end{aligned} \quad (6.65)$$

The strains in the depth of the cantilever are

$$\begin{aligned} \varepsilon_x &= -zc_x = -z \frac{\partial \theta_x}{\partial x}, \ \varepsilon_y = -zc_y = -z \frac{\partial \theta_y}{\partial y}, \\ \gamma_{xy} &= -zc_{xy} = -z \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right), \end{aligned} \quad (6.66)$$

The functional relationships are as follows:

The compatibility equations will be:

$$\begin{aligned} c_x &= \frac{\partial \theta_x}{\partial x}; \quad c_y = \frac{\partial \theta_y}{\partial y}; \\ \gamma_{xz} &= \frac{\partial w}{\partial x} - \theta_x, \quad \gamma_{yz} = \frac{\partial w}{\partial y} - \theta_y. \end{aligned} \quad (6.67)$$

The material constitution equations, for the isotropic case, are:

$$\begin{aligned} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} &= \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} c_x \\ c_y \\ c_{xy} \end{bmatrix} \\ \begin{bmatrix} V_x \\ V_y \end{bmatrix} &= ktG \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}; \quad \boldsymbol{\sigma} = \boldsymbol{\kappa} \boldsymbol{\epsilon}. \end{aligned} \quad (6.68)$$

The equilibrium equations are 6.53.

So that

$$\boldsymbol{\partial} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -1 & 0 \\ \frac{\partial}{\partial y} & 0 & -1 \end{bmatrix}; \quad \boldsymbol{\partial}^* = \begin{bmatrix} 0 & 0 & 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial y} & -1 & 0 \\ 0 & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 & -1 \end{bmatrix} \quad (6.69)$$

The nodal parameters can be established as

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \dots \\ \mathbf{u}_n \end{bmatrix}; \quad \mathbf{u}_i = \begin{bmatrix} w_i \\ \theta_{xi} = \left(\frac{\partial w}{\partial x} \right)_i \\ \theta_{yi} = \left(\frac{\partial w}{\partial y} \right)_i \end{bmatrix}; \quad (6.70)$$

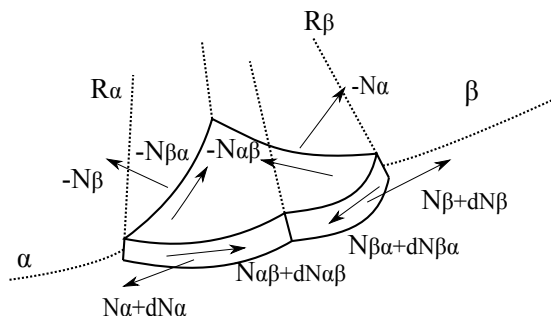


Figure 6.18: Local equilibrium in shells. Even when only membrane stresses are represented, the coupling between membrane and plate stresses becomes evident in the curved geometry.

and as $w_i, \theta_{xi}, \theta_{yi}$ and the associated fields are independent, C^0 functions are valid with the following approximation:

$$\begin{bmatrix} w \\ \theta_x \\ \theta_y \end{bmatrix} = \sum_{i=1}^n \begin{bmatrix} N_i w_i \\ N_i \theta_{xi} \\ N_i \theta_{yi} \end{bmatrix} = \mathbf{N} \mathbf{u}; \quad (6.71)$$

6.3.3 Shells

Shells are structural types, generally curved, in which neither the stresses contained in the tangent plane to the local surface (membrane stresses) nor the transversal stresses to said plane (plate stresses) can be ignored.

As shown in Figure 6.18, to the extent that both types of

stress contribute to the equilibrium against transverse loads, their coupling in the overall behavior is inevitable. And even in thin shells where the contribution of the bending component to the global equilibrium may be of small magnitude, such bending can be crucial in checking local strength in wide regions of the structure.

Therefore, to represent these issues in the modeling of a structure, it is necessary to consider the joint representation of both types of stresses.

Furthermore, it is necessary to enable the representation of curvature in the structure's geometry. To this end, a first approximation could be considered through a faceted or polyhedral description of said geometry, using small planar elements, adequately respecting the three-dimensional connection of these elements in space. This approach would also allow emulating the laminar behavior by superimposing membrane and plate elements connected in the same space, restricting deformation compatibility to that forced by the coincidence of nodal displacements.

A more elegant and coherent approach requires the creation of laminar element models capable of adequately representing both the geometry of the region occupied by the element and the combination of stresses and deformations in that geometry.

In the following sections, we consider, without fully developing them, relevant issues in the formulation of such type of elements.

6.3.3.1 Local Equilibrium Conditions in Shells

In the plane tangent to a point on the shell surface, one can consider the equilibrium of the resultant forces in the plane directions and the transverse direction. The curvature of the region implies that both plate stresses (bending stresses), capable of balancing normal loads as we have seen, and membrane stresses are involved in this equilibrium.

Let's examine the contribution to equilibrium in the normal direction to the surface from membrane or plate components, according to the nomenclature in Figure 6.18. To do this, we consider curved, orthogonal axes tangent to the surface in a differential region of dimensions $R_\alpha d\alpha \times R_\beta d\beta$, where R_i represents the curvature in the considered direction and di , $i \in (\alpha, \beta)$ represents the angle between the two considered cuts in that direction.

The membrane equilibrium would be, considering the effect of unit stresses N_i on lengths $R_i di$, for the angle change dj :

$$\begin{aligned}
 N_\alpha R_\beta d\beta d\alpha + N_\beta R_\alpha d\alpha d\beta + N_{\alpha\beta} R_\beta d\beta \frac{\partial^2 \beta}{\partial \alpha \partial \beta} d\alpha + \\
 N_{\beta\alpha} R_\alpha d\alpha \frac{\partial^2 \alpha}{\partial \alpha \partial \beta} d\beta + p R_\alpha d\alpha R_\beta d\beta = 0; \\
 \frac{N_\alpha}{R_\alpha} + \frac{N_\beta}{R_\beta} + 2 \frac{N_{\alpha\beta}}{R_{\alpha\beta}} + p = 0 \quad (6.72)
 \end{aligned}$$

in the final expression, using the equality $N_{\alpha\beta} = N_{\beta\alpha}$ and the definition $2/R_{\alpha\beta} = 1/R_\alpha \frac{\partial^2 \beta}{\partial \alpha \partial \beta} + 1/R_\beta \frac{\partial^2 \alpha}{\partial \alpha \partial \beta}$.

The contribution of the shear, for example, the one contained

in the α faces, would be

$$-V_\alpha R_\beta d\beta + \left(V_\alpha + \frac{\partial V_\alpha}{\partial \alpha} d\alpha \right) \left(R_\beta + \frac{\partial R_\beta}{\partial \alpha} d\alpha \right) d\beta \quad (6.73)$$

So that the normal surface equilibrium equation becomes

$$\frac{N_\alpha}{R_\alpha} + \frac{N_\beta}{R_\beta} + 2 \frac{N_{\alpha\beta}}{R_{\alpha\beta}} + \frac{1}{R_\alpha R_\beta} \frac{\partial(V_\alpha R_\beta)}{\partial \alpha} + \frac{1}{R_\alpha R_\beta} \frac{\partial(V_\beta R_\alpha)}{\partial \beta} + p = 0 \quad (6.74)$$

an expression that clearly shows the coupling between membrane and bending stresses.

Similarly, we will find these types of coupled expressions for the other five equilibrium conditions in the space of that differential element, although one of them, the rotational equilibrium of the tangent plane, results in the usual equality of tangential stresses measured in orthogonal cuts, $N_{\alpha\beta} = N_{\beta\alpha}$, if no moment is applied.

6.3.3.2 Parameters for a Laminar Element Model

For its formulation, it will be necessary to consider geometric issues, where the length elements are measured as the product of curvature radii by angle differentials: $dl_\alpha = R_\alpha d\alpha$, $dl_\beta = R_\beta d\beta$.

If we review the possible components of the different field functions, we would have for the different pairs:

Loads and displacements:

$$\begin{aligned} \varphi &= [p \ p_\alpha \ p_\beta \ m_\alpha \ m_\beta]^T, \\ \mathbf{v} &= [w \ u_\alpha \ v_\beta \ \theta_\alpha \ \theta_\beta]^T; \end{aligned} \quad (6.75)$$

Stresses and strains:

$$\begin{aligned}\boldsymbol{\sigma} &= [N_\alpha \ N_\beta \ N_{\alpha\beta} \ V_\alpha \ V_\beta \ M_\alpha \ M_\beta \ M_{\alpha\beta}], \\ \boldsymbol{\epsilon} &= [\varepsilon_\alpha \ \varepsilon_\beta \ \gamma_{\alpha\beta} \ \gamma_\alpha \ \gamma_\beta \ c_\alpha \ c_\beta \ c_{\alpha\beta}];\end{aligned}\tag{6.76}$$

The differential operator $\boldsymbol{\partial}$ required for the construction of the stiffness matrix would be the one corresponding to compatibility conditions and will have a dimension of 8×5 . The equilibrium conditions allow us to construct the adjoint operator, of which we will only show here the first row $\boldsymbol{\partial}_1^*$, corresponding to the first of the equilibrium conditions, which is the only one we have shown, but serves to demonstrate the complexity of the resulting formulation.

$$\boldsymbol{\partial}_1^* = \begin{bmatrix} \frac{1}{R_\alpha} & \frac{1}{R_\beta} & \frac{2}{R_{\alpha\beta}} & \left(\frac{1}{R_\alpha R_\beta} \frac{\partial R_\beta}{\partial \alpha} + \frac{1}{R_\alpha} \frac{\partial}{\partial \alpha} \right) & \left(\frac{1}{R_\alpha R_\beta} \frac{\partial R_\alpha}{\partial \beta} + \frac{1}{R_\beta} \frac{\partial}{\partial \beta} \right) & 0 & 0 & 0 \end{bmatrix};$$

Once $\boldsymbol{\partial}$ is constructed, we would again use the chain of expressions from 6.32 to construct the stiffness matrix that condenses the properties of the element.

6.3.3.3 Variants in Shell Element Formulations

From what has been reflected so far, several groups of issues can be deduced that require attention

Relevant Issues to Decide

- “planar” or curved
- metrics: angles and lengths (arcs, radii),
- thin (Kirchhoff) or thick (Mindlin–Reissner)
- triangular, rectangular, trapezoidal
- number of nodes per side
- anodal parameters or interior nodes
- locking elimination techniques
 - compression (volumetric, membrane...)
 - shear
- continuity improvements
- stress distribution at the edge
- geometry representation

Alternative Variants

- Adapted solid elements
- Isogeometric elements (NURBS)
- ...

From all this, a set of basic recommendations is derived before using any of the element types provided by the software to be used.

Recommendations

- Read the documentation of the used element
 - theoretical bases
 - field of application
 - convergence examples
- Test the element in simple and theoretically documented problems
 - with regular meshes
 - with irregular meshes
 - with variable refinement
- If user parameters exist, test their influence in simple problems and contrast the values provided by default in the program.

6.4 Convergence and Control of Results

Some of the potential problems associated with convergence and the appearance of errors were briefly discussed in section 6.1.2.

In this section, the possible strategies for validation and improvement of model results are analyzed in more detail.

6.4.1 Refinement, Extrapolation, and Stress Smoothing

Every model involves choosing a finite number of parameters to describe possible movements, a number that, even though it may

be very high, is inevitably finite, thus representing a restriction on the deformation modes that can be represented. Therefore, part of the possible model errors can be interpreted as errors forced by this restriction, which to some extent represents a forced over-stiffness of the model compared to the theoretical object it attempts to represent.

Thus, one way to reduce this source of error is to introduce more degrees of freedom in the representation, more parameters for describing possible displacement fields.

For this, two alternative strategies can be used:

- h strategy: mesh refinement
- p strategy: increasing the order of functions (polynomial degree)

In the first strategy, the aim is to reduce the typical mesh size h , thus increasing the number of elements and consequently the total number of model parameters. In the second strategy, while maintaining the typical size and number of elements, the aim is to increase the complexity of the movement that can be represented by increasing the degree p of the polynomials used in the interpolation functions of the element movement, an increment that will require more nodes per element and, as we have seen in section 5.3.1.2, more Gauss points to ensure the correct numerical realization of the integrals required by the method.

Of both possible strategies, the first is more common since it does not require element reformulations. And it is for this reason that the quality of meshing is a usual issue in error analysis and control.

While the global analysis results basically provide the model displacements from which local deformations and stresses are deduced element by element, an immediate consequence is the lower quality or precision of these derived field variables, in which differences can be detected in the values attributed to points in contact between different elements, particularly at vertices where several elements converge.

So one way to evaluate mesh quality is by comparing such values.

Since a more refined mesh provides a better description of the sought behavior, the expected error order between two different meshes is different. Therefore, the results of the more refined mesh, even without reaching the exact expected values, will deviate from the results of the less refined mesh in the correct direction, such that it is possible to consider to what extent both results can be combined to achieve an even better approximation. This reflection is at the origin of the following extrapolation.

6.4.1.1 Richardson Extrapolation

For a polynomial order p and mesh size h , the error between the exact value³ u_{ex} and the value provided by the model u_1 can be characterized in a magnitude order of $O(h_1^{p+1})$, where $p + 1$ accounts for the number of parameters used in the polynomial representation and h_1 represents the mesh size of the

3. The value that would correspond to the “exact” solution of the differential equations and boundary conditions of the problem.

first model. Similarly, the error order can be determined for the second model.

Therefore

$$\begin{aligned} 1 : \quad u_{\text{ex}} - u_1 &= O(h_1^{p+1}) \\ 2 : \quad u_{\text{ex}} - u_2 &= O(h_2^{p+1}) \approx O(h_1^{p+1}) \left(\frac{h_2}{h_1} \right)^{p+1} \end{aligned}$$

Eliminating $O(h_1^{p+1})$ between the two expressions we will have

$$\begin{aligned} u_{\text{ex}} \left(\left(\frac{h_1}{h_2} \right)^{p+1} - 1 \right) &\approx \left(\frac{h_1}{h_2} \right)^{p+1} u_2 - u_1 \\ u_{\text{ex}} &\approx \frac{\alpha u_2 - u_1}{\alpha - 1}; \quad \alpha = \left(\frac{h_1}{h_2} \right)^{p+1}. \end{aligned} \tag{6.77}$$

6.4.1.2 Stress Smoothing

Since deformations are deduced from displacement differentials, and stresses from those, displacement continuity does not ensure stress continuity which, as we have seen, cannot be expected to coincide in value at nodes shared by several elements. This is because nodal values are obtained by deploying in the element those obtained for integration points, points where the approximation is more exact than in the rest of the element. In such a way that a common action is to *smooth* their values to eliminate value jumps between elements. This can be done locally, bringing stress laws to make values coincide at nodes with the

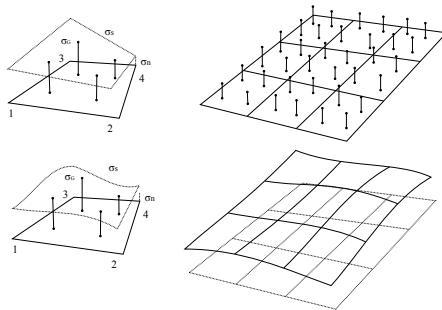


Figure 6.19: Stress Smoothing. Stresses are evaluated in each element from their values at Gauss points, and their lack of nodal coincidence at element contacts is corrected (smoothed), either locally or globally.

weighted average of those obtained at these in all elements sharing them. Also globally, interpolating values of an entire region from Gauss points of all elements in that region.

In this manner, one way for calibrating the error obtained in a solution can now be the difference between the value provided by each element's model and the value resulting from smoothing, with the error being greater the larger these differences are. But how to measure it?

6.4.2 Measures of “error”

The objective is to compare the “exact” fields $\mathbf{v}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\sigma}$, and approximated $\bar{\mathbf{v}}$, $\bar{\boldsymbol{\epsilon}}$, $\bar{\boldsymbol{\sigma}}$ of the problem variables and provide a measure of the approximation error. We have

$$\mathbf{v} \approx \bar{\mathbf{v}} = \mathbf{N}\mathbf{u}, \quad \boldsymbol{\epsilon} \approx \bar{\boldsymbol{\epsilon}} = \mathbf{b}\mathbf{u}, \quad \boldsymbol{\sigma} \approx \bar{\boldsymbol{\sigma}} = \boldsymbol{\kappa}\mathbf{b}\mathbf{u} \quad (6.78)$$

where $\mathbf{u}$ represents the model results in the obtained nodal displacement values.

In such a way that the fields describing the errors will be

$$\mathbf{e}_v = \mathbf{v} - \bar{\mathbf{v}}, \quad \mathbf{e}_\epsilon = \boldsymbol{\epsilon} - \bar{\boldsymbol{\epsilon}}, \quad \mathbf{e}_\sigma = \boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}} \quad (6.79)$$

From these, a global error measure can be conceived through an “energy norm” based either on strains or stresses, in accordance with the description contained in the model:

$$\begin{aligned} \|\mathbf{e}_\epsilon\| &= \sqrt{\int_{\Omega} \mathbf{e}_\epsilon^T \boldsymbol{\kappa} \mathbf{e}_\epsilon d\Omega} \\ \|\mathbf{e}_\sigma\| &= \sqrt{\int_{\Omega} \mathbf{e}_\sigma^T \boldsymbol{\kappa}^{-1} \mathbf{e}_\sigma d\Omega} \end{aligned} \quad (6.80)$$

The problem lies in that the exact values are unknown, so that the error measure is based on the difference between local strains (stresses) and smoothed strains (stresses). The usual measure is made with the smoothed stresses $\boldsymbol{\sigma}_s$ and the global error

measure seeks to be limited in the following way:

$$\begin{aligned} \|e_\sigma\| &\leq \eta \|W\|, \quad e_\sigma = \sigma_s - \bar{\sigma}; \\ \|W\| &= \sqrt{\int_{\Omega} \sigma_s^T \kappa^{-1} \sigma_s d\Omega} \end{aligned} \quad (6.81)$$

where η represents a tolerance limit in the global error.

Therefore, it can be considered as a descriptive error parameter the quotient

$$\xi_g = \frac{\|e_\sigma\|}{\eta \|W\|} \quad (6.82)$$

The problem with this representation is that the error is usually distributed in a non-uniform manner over the model, such that very significant local errors can occur even when the global average error is reduced.

Therefore, it is of great interest to measure the local and global error densities, so that not only a reduced error is of interest, but also a relatively uniform error density across the entire model.

A uniform distribution would require that the density in each element be similar and therefore similar to the global mean.

$$\begin{aligned} \|\text{Dens}(e_\sigma)\|^e &= \sqrt{\frac{1}{\Omega_e} \int_{\Omega_e} e_\sigma^T \kappa^{-1} e_\sigma d\Omega} = \frac{\|e_\sigma\|^e}{\sqrt{\Omega_e}} \\ \text{with the objective } \frac{\|e_\sigma\|^e}{\sqrt{\Omega_e}} &= \frac{\|e_\sigma\|}{\sqrt{\Omega}} \end{aligned}$$

We do not delve into more details here, which largely exceed the objectives of this text.

Chapter 7

Uncertainty Modeling: Reliability

This chapter presents a very brief introduction to statistical methods applied to structural safety evaluation, methods usually referred to under the label *reliability*.

The content of the chapter adapts to the structure of the final part of the course on “Fundamentals of Structural Theories and Their Numerical Calculation” to which the entire book refers, as noted in the presentation, although it is developed in more detail and with more bibliography in Cervera [2019](#).

The guiding thread consists of characterizing the relevant sources of uncertainty and their impact on determining the degree of reliability with which the safety of a specific structural

element can be affirmed, through an examination of an example studying the frame already seen in figure 2.22 in section 2.2.1.3, though for simplicity, only failure mechanisms by bending will be considered.

In this journey, the measure of the influence of different uncertainties in determining failure modes and parameters is analyzed, the random variables used for this measurement, and is provided an introduction to the probabilistic models used in characterizing structural safety conditions and their implementation in codes, through partial safety coefficients.

7.1 Sources and Factors of Uncertainty

Neither reality, nor the information obtained through its observation can be considered deterministic. Any measurement, essential for understanding or acting upon reality, is impregnated with uncertainties originating, in general, from two types of sources. The first type corresponds to the nature of reality itself, where the complex interaction of numerous variables prevents the exact and identical repetition of the same phenomenon. The second responds to human factors, either due to a poor understanding of the phenomenon to be measured and its defective modeling, or due to defects in measurement procedures or instruments.

Such that the uncertainty in describing real behavior through its theoretical modeling stems from differentiated factors:

- physical or mechanical uncertainties (the variability in time and space of the considered qualities)

- measurement uncertainties (due to errors or biases in measuring such qualities)
- statistical uncertainties themselves (when working with limited information, finite samples, etc...)
- uncertainties due to the modeling of reality itself (in our case, of structures), which entails limitations in the predictive capacities of this model.

Thus, a correct understanding of the real behavior of structures involves considering their probabilistic qualities, derived from the randomness of each of the variables considered when modeling them.

In the example case, for which the failure scenarios studied from a deterministic perspective with the plastic approach are those in figure 7.1, there is uncertainty not only in the actual geometry h, l , the actions F_1, F_2 , or in the values of the flexural resistances of the members (column and beam), M_p, M_v , but also that these latter can be different at each end depending on the execution conditions of the connections, or the (even small) variability in the resistant capacities of the cross-sections used... such that the characterization of the *failure modes* may require considering the potential correlation between these resistances in different regions of the structure, as expressed by the alternatives in figure 7.2.

So that, if we formulate the failure condition through some type of expression in which positive values indicate no failure and negative values indicate failure, the expressions for both alternatives in the plastic formulation could be the following for the three collapse mechanisms of the figure (lateral displace-

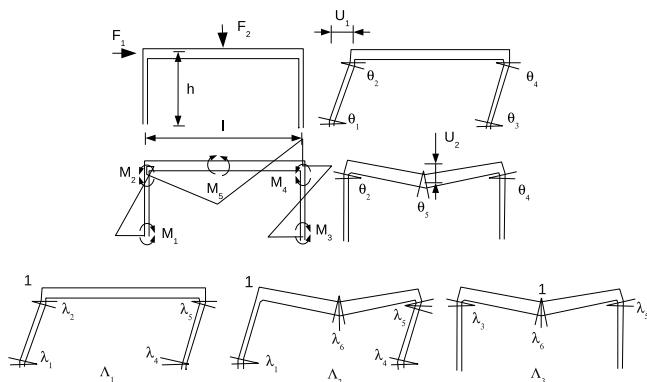


Figure 7.1: Components of statics and kinematics of failure in a simple frame, and the corresponding collapse mechanisms.

ment, combined, and deck failure):

identical resistance values

$$\begin{aligned}
 4M_p - F_1 h &< 0 \\
 (4M_p + 2M_v) - (F_1 h + F_2 l/2) &< 0 \\
 2(M_p + M_v) - F_2 l/2 &< 0
 \end{aligned} \tag{7.1}$$

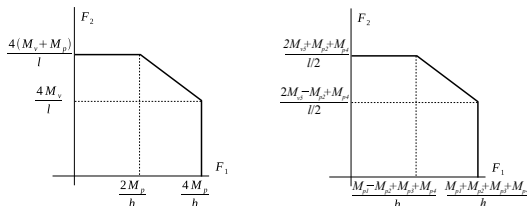


Figure 7.2: Limit loads in a simple frame, depending on whether the strength limits at the ends of different members are considered related or not

resistance values different but correlated

$$\begin{cases} 1 : Z_1 < 0 \\ 2 : Z_2 < 0 \\ 3 : Z_3 < 0, \text{ with} \end{cases}$$

$$\begin{aligned} Z_1 &= M_{p1} + M_{p2} + M_{p3} + M_{p4} - F_1 h \\ Z_2 &= M_{p1} + M_{p3} + 2M_{p4} + 2M_{p5} - F_1 h - F_2 l/2 \\ Z_3 &= M_{p2} + 2M_{v5} + M_{p4} - F_2 l/2 \end{aligned} \quad (7.2)$$

In the previous expressions, we would be considering random the different variables $M_i = W_i f_{y,i}$, F_j , h , l , in such a way that the event (or set of events) that characterizes the possibility of failure is expressed with the three-dimensional random variable $\mathbf{Z} = [Z_1 \ Z_2 \ Z_3]^T$ and the failure condition with the fulfillment of any of the conditions $\mathbf{Z} < \mathbf{0}$.

More generally, let us recall the formulations we used in plastic analyses, in which we established the limit conditions (or

limit state) through some *limit surface* or *failure surface*, formulations in which this surface is defined in spaces representing the verification variables, with expressions we have seen that adopt the form $\psi \mathbf{f} \leq \mathbf{dr}$, see 1.33, or 1.40, and which we can also write as $\mathbf{dr} - \psi \mathbf{f} \geq 0$. In these expressions, we established that the resistant capacity $\mathbf{dr}$ must be greater than the effect of loads, $\psi \mathbf{f}$, that is, the verification sought to ensure sufficient strength, in a format we could generalize:

$$\mathbf{G}(\mathbf{X}) \geq 0; \quad \mathbf{X} = (G_i, Q_j, f_k, T_l, \dots) \quad (7.3)$$

where the expression $\mathbf{G}$ depends on permanent loads G_i or variable loads Q_j , material strengths f_k , and the topology and geometry of sections, pieces and their connections T_l , etc.

These verifications can also be formulated in formats such as the following, where R represents the resistant capacity and E the effect of actions against such capacity:

$$\begin{aligned} R_i &\geq E_i; \quad R_i - E_i \geq 0 \\ \frac{R_i}{E_i} &\geq 1; \quad \log(R_i) - \log(E_i) \geq 0 \end{aligned} \quad (7.4)$$

In this framework, characterizing failure instead of strength involves establishing a variable for the event to be measured, Z , (where it can be $Z = R - E$) and formulating the condition that expresses failure, which is done with the expression $Z < 0$, which, translated into analysis terms, means considering that failure occurs when any of the conditions expressed in $\mathbf{dr} - \psi \mathbf{f} < 0$ are met. Therefore, the variable $\mathbf{Z}$ that expresses the

possible failure condition has the same number of dimensions as the verification equations, being $\mathbf{Z} = \mathbf{dr} - \psi \mathbf{f}$.

The deterministic formulation of failure can be expressed in the format

$$Z_i = R_i - E_i < 0 \quad (7.5)$$

7.1.1 Random Variables. Types and Distributions

We must consider any possible non-deterministic measure as a random variable. We will denote this variable as X , of which we can only obtain different estimations or measurements X_i for each “test” or “realization” case, in such a way that X represents the phenomenon, of which we only know a finite set of observations, X_i , which will include values, perhaps repeated, that are distributed in a certain range of values x . Now x represents ranges of possible values for the phenomenon we denote with X , but a value of x does not necessarily correspond to values observed in the set X_i . In such a way that, for precision in language, we must distinguish between the *frequency* with which a certain value (or set of values) X_i is observed in the total of observations, and the *probability* for a value x (or for values comprised in a range or interval of values $[x_a..x_b]$) of being observed. In the first case, we are counting cases in a sequence of experiments or observations; in the second, we are projecting future observation expectations.

7.1.1.1 Basic and Derived Variables

In our models, some variables will correspond to directly measurable properties: dimensions, weights, material strength tests, etc. We call these *basic* variables.

On the contrary, a second group of variables may correspond to properties defined from the former, to properties constructed through some theoretical model. These are *derived* variables. A simple example would be the strength properties of specific sections: while we might think of testing and measuring each of these sections, it is usual to construct their properties from the geometry and materials with which they are built.

In general, the qualities of interest (the failure modes of a structure) are defined by derived variables, so their uncertainty must be evaluated from the combination of uncertainties in basic variables and the model by which they are defined.

7.1.1.2 Variables by Their Effect in Relation to Failure Condition

An important difference in variables related to structural safety derives from their effect on failure: we will call *actions* those variables whose increment increases the probability of failure, and *resistances* those others whose increment results in a reduction of the failure probability.

This distinction is essential in determining the failure probability given the uncertainty of such variables, but we must be aware that this is a classification that seriously depends on the failure mode considered, such that the same source or variable

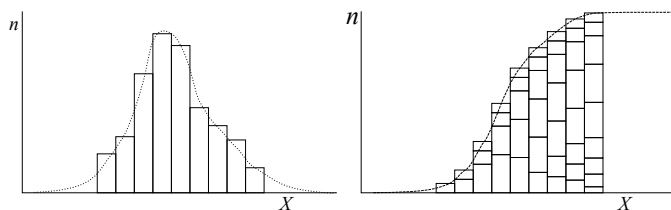


Figure 7.3: Histogram: number n of occurrences of observed values of X in the interval indicated on the abscissa. Distribution (cumulative): number n of occurrences of observed values of X in the interval indicated on the abscissa or previous intervals. Both figures include a “smooth” graph approximated to the qualities of the experimental graph.

may need to be considered as an action in some cases and as a resistance in others: an example is the case of loads on a canopy with double cantilever over a central column, where the load on the right side can be part of the resistance effect against the column’s bending, loaded on the left cantilever, while both loads are actions for the bending of any of the cantilever connections against the node.

7.1.1.3 Distributions of Random Variables

The repeated measurement of a random variable provides different values even when they may be close to each other. Histograms provide a graphical view of the distribution (figure 7.3 a): in a Cartesian graph, we represent on the x-axis a given set

of equal intervals of possible values for the measurement and on the y-axis the number of values obtained for the variable in each of these intervals. The sum of the histogram values will represent the number of measurements made and, if the histogram bars cover the interval width, the proportion of the area corresponding to each interval in relation to the area enclosed in the complete figure provides the proportion between the number of observations made in this interval with respect to the total number, or in other words, the frequency of observations corresponding to the interval.

Another interesting graph (figure 7.3 b) uses the same intervals on the x-axis but counts in the y-axis the accumulated number of values obtained both in that interval and in all previous ones: for a given variable, the graph will express the number of cases in which the variable has had values equal to or less than those of the interval. This graph ends at the maximum values of the variable with a horizontal line at the ordinate corresponding to the total number of observations. This second graph of the cumulative distribution is nothing more than the integral of the previous one.

If instead of measuring the number of cases we measure the frequency, the distribution function becomes a monotonically increasing function that can be smoothed while maintaining the overall distribution properties, and which is situated in the range of values $[0, 1]$, and the histogram is transformed into a derivative function of the distribution that, dimensionally, expresses the density of that distribution for each possible value x of the observed range, where, assuming that the distribution is stable over time, this probability density can be attributed to future

events.

So we have the following functions and relationships:

Probability Density Function:

$$f(x); \quad \Pr(a \leq X < b) = \int_a^b f(x) dx; \quad \int_{-\infty}^{\infty} f(x) dx = 1 \quad (7.6)$$

Distribution Function:

$$F(x) = \Pr(X < x) = \int_{-\infty}^x f(x) dx. \quad (7.7)$$

If we assume that the value of X is directly related to the survival capacity of the structure (for example, it expresses the load at which the structure collapses), the distribution function $F(x)$ can be called a *fragility* function or curve because it expresses the probability of failure (or exceeding a certain damage limit) of the structure when the load value given by x is reached.

Since structures that do not fail survive, we can define:

Survival Function:

$$S(x) = 1 - F(x) = \Pr(X \geq x) = \int_x^{\infty} f(x) dx. \quad (7.8)$$

The probability of failing of structures that survived is called the failure rate, hazard rate or

Hazard Function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{1 - \int_{-\infty}^x f(x) dx} \quad (7.9)$$

Main Distributions in Structural Reliability The figures contained in 7.4 show some of the most common distributions in structural safety, which are outlined below

Normal Distribution, also called *Gaussian Distribution*, or *Gauss Bell Curve*. This is the distribution of a random variable X with mean μ and variance σ^2 , representable as $X \sim \mathcal{N}(\mu, \sigma^2)$ and whose probability density function is the following:

$$f_{\mu, \sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad (7.10)$$

If the parameters μ, σ^2 are respectively 0 and 1, the distribution $\mathcal{N}(0, 1)$ for the corresponding variable Z is called standard normal distribution, whose probability density is therefore, if we denote the corresponding cumulative distribution with $\Phi(x)$:

$$f(x) = \frac{d}{dx}\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad (7.11)$$

$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \quad (7.12)$$

The normal distribution is symmetric, with mean, median, and mode being equal.

The cumulative distribution function corresponding to a normal distribution is, therefore

$$F_{\mu, \sigma^2}(x) = \Phi\left(\frac{x - \mu}{\sigma}\right) \quad (7.13)$$

or also

$$F_{\mu, \sigma^2}(x) = \frac{1}{2} \left(1 + \operatorname{erf}\left(\frac{x - \mu}{\sigma\sqrt{2}}\right) \right) \quad (7.14)$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the “error” function.

It is a very ubiquitous distribution, as it is the one to which any random variable tends to be configured as the sum of a set of independent random variables¹.

Lognormal Distribution. It involves the distribution of a random variable X such that the natural (or Napierian) logarithm of said variable has a normal distribution: that is, if X follows a lognormal distribution, $Y = \ln X$ has a normal distribution. And similarly, if Y follows a normal distribution, its exponential $X = e^Y$ has a lognormal distribution. This distribution is the one to which the result of the product of many independent random variables tends. The domain of the variable X is restricted to positive values: $0 \leq X \leq \infty$.

Weibull Distribution This is a distribution defined for positive values of the variable, which responds to the types of distributions called *extreme value*. These respond to the determination of extreme values, maximum or minimum, that are obtained for a variable when obtaining values in batches of given sizes, or corresponding to fixed time periods, such as the lowest resistances of batches of n test specimens, or the highest surface loads measured in a given area in a reference period of one year.

1. By the central limit theorem.

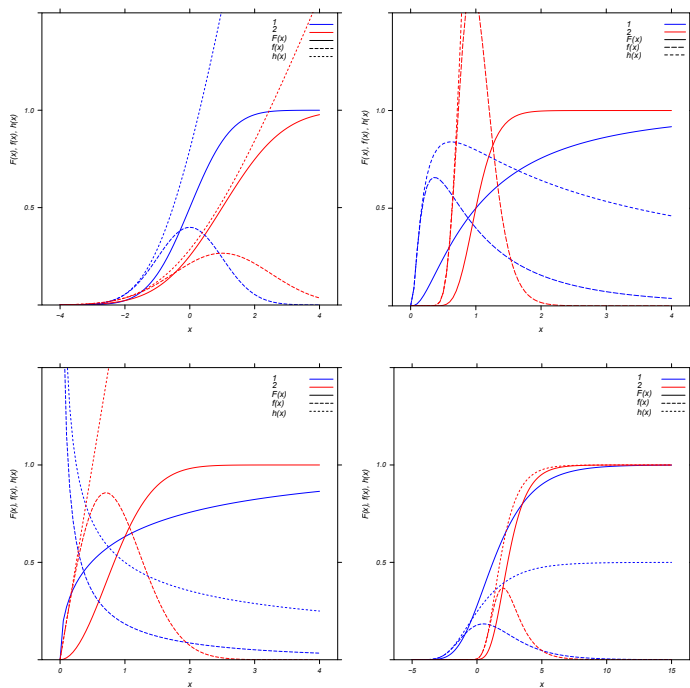


Figure 7.4: From left to right and top to bottom, normal, lognormal, Weibull, and Gumbel distributions. Represented for two cases in each figure are the distribution functions $F(x)$, probability density functions $f(x)$, and hazard functions $h(x)$.

Gumbel Distribution This distribution (generalized extreme value distribution, type-I) is used to model how the maximum values of a series of samples from some other distribution are distributed. It is thus related, like the preceding one, to extreme value theories.

It is used for characterizing the annual maximum values of a certain variable action measured in a long sample (25, 50, 100 years, ...). It is theoretically the distribution that should be used to obtain the *characteristic value* of a variable action. When this latter value is colloquially defined, it refers to the 95% fractile, that is, the value that only has a 5% probability of being exceeded in a given year. Or also, an action whose *return period* is $20 = 1/0.05$ years. It is important to note that the *return period* should not be considered a temporal indicator of a possible rhythm or cycle in the action in question, it is only the inverse of an annual probability.

7.1.1.4 Parameters or Basic Indicators of a Distribution

It is useful to summarize the properties of a distribution in representative values or indicators. The most usual ones and their symbolism are pointed out below.

Central Values:

Mean $\bar{X}$ and *expected value* $E(X)$ of the variable. For a discrete distribution, the mean is the sum of the values obtained in the different observations divided by the number of these:

$$\bar{X} \approx E(X) = \mu \quad (7.15)$$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i; \quad i \in [1, n], \quad (7.16)$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx \quad (7.17)$$

The form of expression 7.17 is identical to the distance of the center of gravity of the probability density function graph to the origin of abscissas. That is why this parameter is also called *first moment*—similar to the static moment used in mass geometry to determine the position of the center of gravity—

Median. The median of a variable X is defined as the value m for which the probability of obtaining lower values is identical to the probability of obtaining higher values. It is therefore the value for which the areas of the graphs enclosed under the probability density function before and after the vertical corresponding to the value m are identical.

$$F(m) = \Pr(X \leq m) = \Pr(X \geq m) = S(m) = \int_{-\infty}^m f(x) dx = \frac{1}{2} \quad (7.18)$$

The median is not as skewed—and distant from the value perceived as central—as the mean in strongly asymmetric distributions, such as income distribution in a population, and is always bounded when the mean may not be. It is therefore a more robust statistical parameter.

Mode. This is the most probable value M in a distribution, corresponding therefore to the value that maximizes the ordinate in the histogram, or in the probability density function.

$$M = x \left(x_M | f(x_M) = \max_x f(x) \right) \quad (7.19)$$

It is also usual to consider local maximum values, so that a distribution may have more than one local maximum in the probability density function: one case is that of bimodal distributions, with two modes, which will have two local maxima in $f(x)$, a situation often interpreted as the effect of superposing two different populations, each with its own distribution.

Dispersion Measures:

Mean Absolute Deviation. This is the mean of the error made when considering the mean $\mu = \bar{X}$ as the value of the variable when considering a given observation X_i : $|X_i - \mu|$ in absolute value. The mean absolute deviation will thus be

$$\frac{1}{n} \sum_{i=1}^n |X_i - \mu| \quad (7.20)$$

Deviations can also be considered with respect to the median, or the mode. As handling absolute values is mathematically laborious, it is more common to consider quadratic expressions to eliminate the negative terms of the previous sum. This leads us to the following parameter:

Variance. This is the mean of the squares of the differences

between the observed values and their mean:

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 \quad (7.21)$$

If n contains the entire population (all possible observations), the mean μ is correct, and the expression has no flaws. But if n is only a sample of the population, the estimation of the mean is only approximate, so the previous estimation is potentially incorrect. That is, the difference $X_i - \mu = X_i - \bar{X}$ would not be the error but only an estimate of this, which is usually called a residual. It is easy to see that if we have n observations, we can only estimate its variability through $n - 1$ differences between one observation and the others: with only one observation it is impossible to know if there will be any variability at all. So if we are dealing only with representative samples that do not cover the complete population, the variance is determined using the *Bessel correction*—or including the *Bessel factor*—through the expression:

$$\sigma^2 = \frac{n}{n-1} \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 \quad (7.22)$$

For a continuous distribution described by a probability distribution function $f(x)$, we will have, as we saw for the calculation of the mean, the following expression:

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \quad (7.23)$$

It should be noted that the expression is the *second moment*

which, in mass geometry, would be measuring the inertia with respect to the center of gravity.

This expression can be developed and also written as

$$\sigma^2 = \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2 \quad (7.24)$$

an expression very close to Steiner's theorem in mass geometry.

Standard deviation, or typical deviation. To have a measure of deviation that is dimensionally comparable to the metric of the variable, the square root of the variance is used, with identical considerations as in the variance, depending on the category of the sample considered.

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \mu)^2} \quad (7.25)$$

$$\sigma = \sqrt{\int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx} \quad (7.26)$$

The continuous expression is equivalent to determining, in mass geometry, the radius of gyration for the figure of the probability density function $f(x)$, whose mean would be located at its center of gravity².

Coefficient of variation (sometimes denoted by COV, for which there is no unanimous symbol: variants such as ρ , V , ... can be found). When the variable predominantly has values of the same sign, it may make statistical sense to consider the

2. and whose total area is known to be unity.

proportion between the deviation values relative to the mean values of the variable: this is the case, for example, of resistances in material samples, where a negative resistance would not make sense for a given test.

$$\rho = \frac{\sigma}{|\mu|} \quad (7.27)$$

Distributions with coefficients of variation greater than unity are called high-variance distributions.

7.1.1.5 Problems of two or more random variables.

Usual problems require considering different random variables: for example, actions and resistances in structural resistance problems.

Each of the variables X_i in the set $\mathbf{X}$ —here the index i refers to the list of possible variables—will have its own properties: their respective evolution spaces x_i , and their qualities in these (distribution, density, mean, etc.) exist separately from the consideration relative to the rest. But it is also important to consider their joint properties if, instead of analyzing their isolated variation, we analyze that of the set for events or experiments in which we measure all variables simultaneously, thus considering the joint distribution.

In the space containing all dimensions or variables of the problem considered, $\mathbf{x}$, with $\mathbf{X} \in \mathbf{x}$, we will have, in addition to the probability densities for each dimension x_i , those corre-

sponding to the joint space $\mathbf{x}$, or for its subspaces $\mathbf{x}^*$:

$$\Pr(X_i = x_i) = \sum_{\mathbf{x}^*} \Pr(X_i = x_i, \mathbf{X}^* = \mathbf{x}^*) \quad (7.28)$$

$$f(x_i) = \int_{x_i = \text{const.}} f(\mathbf{x}) d\mathbf{x}^* \quad (7.29)$$

where the integral is performed for all dimensions (subspaces, or variables) except the dimension or variable considered (the subspace indicated by $\mathbf{x}^*$, that is, all dimensions different from those indicated in i).

The probability distributions or densities of a subset of variables, measured independently of the rest, as in the preceding expression, are called in the literature marginal distribution or density functions. It can be seen that they are densities referring to different dimensions: in the case of two variables, the joint density is per unit area of the two-dimensional space, while the marginal density is linear, per unit length of the considered variable's dimension.

The probability distributions of a variable or subset of variables X_i , measured for predetermined values of the other variables $\mathbf{X}^*$ are called *conditional* probabilities, and are denoted with the formalism $\Pr(X_i|\mathbf{X}^*)$.

For these problems, in addition to individual parameters, joint parameters can be measured, of which we will point out the most important, starting from the case of two variables where $\mathbf{X} = [X, Y]$ and $\mathbf{x} = [x, y]$. Note that a row vector is used for the set: each column is a variable and separate rows can be used for successive measurements of each pair of values.

In this case, the joint probability density $f(\mathbf{x}) = f(x, y)$ and the marginal $f(x)$, $f(y)$ or conditional $f(x|y)$, $f(y|x)$, —where for notation brevity we represent all with the same $f()$ although as we have seen they refer to different spaces— are related by the expressions

$$\begin{aligned} f(x) &= \int_y f(x, y) dy = \int_y f(x|y)f(y) dy \\ f(y) &= \int_x f(x, y) dx = \int_x f(y|x)f(x) dx. \end{aligned} \quad (7.30)$$

Covariance. In a problem with two random variables X and Y , it is the expected value of the product $(X - \mu_x)(Y - \mu_y)$ for all events in which both variables are measured jointly: $[X_i, Y_i]$, $i \in (1 : n)$

$$\begin{aligned} \sigma_{xy} = \text{cov}(X, Y) &= E[(X - \mu_x)(Y - \mu_y)] = \\ &= \frac{1}{n} \sum_{i=1}^n (X_i - \mu_x)(Y_i - \mu_y) \end{aligned} \quad (7.31)$$

It can be seen that the variance σ_x^2 for one isolated variable is nothing more than an “auto-covariance” considering pairs of that same variable: $\sigma_x^2 = \sigma_{xx}$.

Covariance matrix. It is the matrix that groups the covariances corresponding to a list of random variables. In the case

of two variables (two dimensions):

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} E[(X_i - \mu_x)(X_i - \mu_x)] & E[(X_i - \mu_x)(Y_i - \mu_y)] \\ E[(Y_i - \mu_y)(X_i - \mu_x)] & E[(Y_i - \mu_y)(Y_i - \mu_y)] \end{bmatrix} \quad (7.32)$$

This can be generalized to spaces with multiple dimensions, where the term *variance* is also used in a multidimensional sense, as an extension of the one-dimensional variance concept, extending the row of variables used $\mathbf{X}$:

$$\Sigma = \text{var}(\mathbf{X}) = E[(\mathbf{X} - E[\mathbf{X}])^T(\mathbf{X} - E[\mathbf{X}])] = E[\mathbf{X}^T \mathbf{X} - 2\boldsymbol{\mu}^T \mathbf{X} + \boldsymbol{\mu}^T \boldsymbol{\mu}] = E[\mathbf{X}^T \mathbf{X}] - \boldsymbol{\mu}^T \boldsymbol{\mu}. \quad (7.33)$$

This is a symmetric matrix.

Correlation. Linear correlation, or Pearson correlation. Between two random variables X and Y , it is the quotient between their covariance and the product of their standard deviations:

$$\rho_{xy} = \frac{\text{cov}(X, Y)}{\sigma_x \sigma_y} = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x \sigma_y} \quad (7.34)$$

It is a value between -1 and 1, where the absolute value is an indicator of the degree of linear relationship between the two referenced variables.

Correlation Matrix. It is the (symmetric) matrix constituted

by the correlations of the variables considered in the set.

$$\text{cor}(\mathbf{X}) = \boldsymbol{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{bmatrix} \quad (7.35)$$

It is easy to see that the diagonal is formed by unit values.

The interest of covariances and correlation lies in the linear approximation of the values of one variable from those of another, in the case of non-zero correlation, as shown in Cervera 2019, section 2.4.2 et seq.:

$$Y_i = \bar{Y} + \rho_{xy} \sigma_y \frac{X_i - \bar{X}}{\sigma_x} + \varepsilon_i, \quad (7.36)$$

The expression determines the displacement or distance with respect to the mean in the dependent variable Y that must be added to said mean to explain Y from the distance in the independent variable X with respect to its own mean. In this expression, standard deviations should be interpreted as scale parameters or measurement unit parameters for such distances.

Operations with Random Variables The combination of the effects of two or more random variables through mathematical operations gives rise to derived variables, whose statistical parameters depend on those, and on the operation itself. We will see some cases.

Negative of a random variable:

$$\begin{aligned}
 Y &= -X \\
 \bar{Y} &= \frac{1}{n} \sum_{i=1}^n (-X_i) = -\bar{X} \\
 \sigma_y &= \sqrt{\sigma_y^2} = \sqrt{E[(Y - \bar{Y})^2]} = \\
 &\quad \sqrt{E[(-X + \bar{X})^2]} = \sigma_x
 \end{aligned} \tag{7.37}$$

Sum of two random variables:

$$Z = X + Y \tag{7.38}$$

$$\begin{aligned}
 \mu_z = \bar{Z} &= E[Z] = E[X + Y] = \\
 &= \frac{1}{n} \sum_{i=1}^n (X_i + Y_i) = \bar{X} + \bar{Y}
 \end{aligned} \tag{7.39}$$

$$\begin{aligned}
 \sigma_z^2 &= E[(Z - \bar{Z})^2] = E[Z^2 + \bar{Z}^2 - 2Z\bar{Z}] = \\
 &= E[(X + Y)^2 + (\bar{X} + \bar{Y})^2 - 2(X + Y)(\bar{X} + \bar{Y})] \\
 &= E[(X - \bar{X})^2 + (Y - \bar{Y})^2 + 2(X - \bar{X})(Y - \bar{Y})] \\
 &= \sigma_x^2 + \sigma_y^2 + 2\rho_{xy}\sigma_x\sigma_y
 \end{aligned} \tag{7.40}$$

In the case of uncorrelated variables, it will be $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$, but for a complete positive correlation, with $\rho_{xy} = 1$ we will have $\sigma_z^2 = (\sigma_x + \sigma_y)^2$.

Difference of two random variables: proceeding analogously

$$\begin{aligned} Z &= X - Y \\ \mu_z &= \bar{Z} = E[Z] = E[X - Y] = \bar{X} - \bar{Y} \\ \sigma_z^2 &= E[(Z - \bar{Z})^2] = \sigma_x^2 + \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y. \end{aligned} \quad (7.41)$$

Product of two random variables: Retrieving the expression 7.33 for two variables X and Y we have that

$$\begin{aligned} \text{cov}(X, Y) &= E[(X - E[X]) \cdot (Y - E[Y])], \\ &= E[X \cdot Y] - E[X] \cdot E[Y] \end{aligned} \quad (7.42)$$

so that we can establish

$$Z = X \cdot Y \quad (7.43)$$

$$E[Z] = \mu_Z = \mu_X\mu_Y + \text{cov}(X, Y) \quad (7.44)$$

Likewise, the variance can be determined with

$$\begin{aligned} \sigma_Z^2 &= E[(X \cdot Y)^2 - (E(X \cdot Y))^2] \\ &= E(X^2Y^2) - (\mu_X\mu_Y + \text{cov}(X, Y))^2 \\ &= \text{cov}(X^2, Y^2) + (\mu_X^2 + \sigma_X^2) \cdot (\mu_Y^2 + \sigma_Y^2) - \\ &\quad - (\mu_X\mu_Y + \text{cov}(X, Y))^2 \\ &= \text{cov}(X^2, Y^2) + (\mu_X^2 + \sigma_X^2) \cdot (\mu_Y^2 + \sigma_Y^2) - \mu_Z^2 \end{aligned} \quad (7.45)$$

To obtain σ_Z^2 it is therefore necessary to know moments higher than the second, required to determine $\text{cov}(X^2, Y^2)$, the covariance between the squared values of both variables.

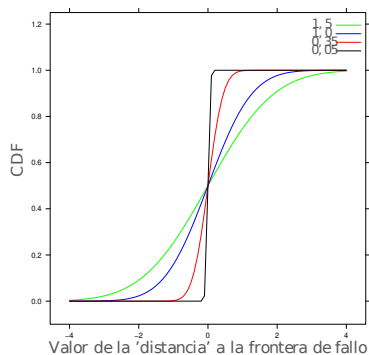


Figure 7.5: Failure probability function interpreted as a “boundary” between regions with or without failure condition (sign of F), where the distance to the boundary is measured in relevant units for a normal distribution of the failure condition $F_i = R_i - E_i$. Probabilistic variants are represented for the deviations $\sigma = [0.05; 0.35; 1.0; 1.5]$.

The deterministic version corresponds to $\sigma = 0$.

7.2 Security Models against Failure

In this section, we give a brief overview of security models. The classical models are deterministic, which consider the non-randomness of variables and models representing the different factors that influence determining whether the structure is safe or not, whether it will fail or not. To the extent that this determinism is not exact, in this model, we seek to distance the unfavorable effects from the favorable ones by means of safety factors or safety coefficients that reduce the estimation of the latter, in order to distance the possibility of failure. The models in admissible stresses are a classic form of these.

However, the different variability of different types of variables has long led to considering the need to adapt such factors to these types, giving rise in modern codes to partial safety factors, based on semi-probabilistic models for their calibration. These models operate like the deterministic ones, but with *design* values for the variables involved, which are established from accepted statistical values and modified by partial factors, which are calibrated to achieve an accepted and standardized probable failure rate.

Finally, probabilistic models operate with the random variability in models and variables seeking to ensure that the values adopted in the project provide a failure rate lower than that adopted in the regulations as acceptable for each type of project.

In what follows, we will study this in more detail.

7.2.1 Deterministic and Semiprobabilistic Safety Models

From a deterministic approach to structural behavior and failure models, a dual consideration is used:

problems can be characterized by several groups of variables:

- structural geometry, which includes dimensions, sections, connection forms, etc.
- qualities of the materials that make up this geometry: resistant capacities and deformability for different stress combinations, etc.
- actions on the structural geometry, loads or actions, of diverse qualities and origins: permanent or variable, gravitational or thermal, etc.

and theories of structures, material strength, etc. allow establishing failure models in which checks compare the effects E_i of actions Q_j across the entire structural geometry $\mathcal{E}$ with the resistant capacities R_i against such effects provided by this geometry and the qualities f_k of the materials arranged in it.

To accept the validity of such assumptions—or in other words, that their use allows confidence in the evaluation results (whether or not there will be structural failure)—the higher values among those observed or calculated are prescribed for actions, and for material resistant capacities, “admissible values” lower than those that effectively cause their rupture.

So the verifications adopt formats of the following type, which we assume for now are common in their application $\mathcal{P}_j$

by different professionals³:

$$R_i(\mathcal{E}, f_1, f_2, \dots, f_m) \geq E_i(\mathcal{E}, Q_1, Q_2, \dots, Q_n), \quad i \in [1..k] \quad (7.46)$$

where k is the number of failure conditions evaluated, which is assumed to bound or encompass all possible conditions for the problem considered.

These expressions can equally be written in the following alternative formats:

$$R_i(\mathcal{E}, f_1, f_2, \dots, f_m) - E_i(\mathcal{E}, Q_1, Q_2, \dots, Q_n) \geq 0 \quad (7.47)$$

$$\frac{R_i(\mathcal{E}, f_1, f_2, \dots, f_m)}{E_i(\mathcal{E}, Q_1, Q_2, \dots, Q_n)} \geq 1; \quad (7.48)$$

The failure condition i would thus be characterized by expressions such as⁴

$$F_i = R_i - E_i < 0 \quad (7.49)$$

From this approach, the failure condition can be characterized as a hypersurface, a *limit surface* in the space of possible values for the problem variables, a hypersurface that separates these possible value combinations into two regions: the region where, by fulfilling all the system's inequalities, the absence of failure can be assured, and the region of assured failure, separated by the surface for which some failure condition is met.

It should be noted that, if we consider an evolution in the values of a problem's variables approaching the failure surface,

3. Otherwise, each application $\mathcal{P}_j$ could involve a different set of verifications $R_{i,j} \geq E_{i,j}$, perhaps even with non-coinciding variables.

4. taking logarithms in the last one

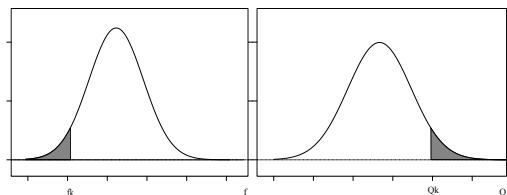


Figure 7.6: Selection of “biased” values on the safe side for resistances or actions: the nominal “characteristic” values.

crossing it, and moving away from it, we can imagine this evolution as a very particular failure probability distribution function: before reaching the boundary, the failure probability is 0, and becomes 1 after crossing the boundary. Figure 7.5 clearly expresses the difference between this deterministic approach and a probabilistic approach where the variable F_i indicating the failure event is random.

We can therefore see the need to choose values biased on the safe side as nominal values for resistances or actions (figure 7.6), but this alone is not sufficient, since the bias does not eliminate the probability of failure in the comparison, figure 7.7, so to the nominal bias must be added an increase in actions and a reduction in resistances that sufficiently reduces the probability of simultaneously finding low resistances with high actions that produce the undesired failure. This involves the use of design values, derived from nominal values affected by factors that expand or reduce the unfavorable or favorable effect considered in the relevant failure model, see figure 7.8.

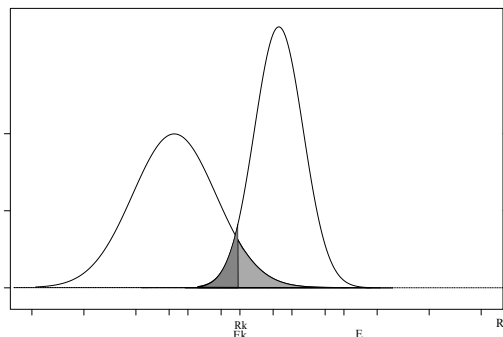


Figure 7.7: Comparison between nominal or characteristic values is unsafe.

7.2.1.1 Semiprobabilistic Safety Model

From a probabilistic perspective, the objective is to establish a sufficiently reduced failure probability. The semiprobabilistic perspective seeks to add to deterministic models safety factors that are adequate to ensure that same reduced failure probability.

That failure probability is, in the abstract, easy to determine: facing a specific type of failure i , if the probability functions of action effects and resistant capacities against these are known, the failure probability is

$$\Pr(R - E < 0) = \int_{-\infty}^{\infty} f_E(x) F_R(x) dx \quad (7.50)$$

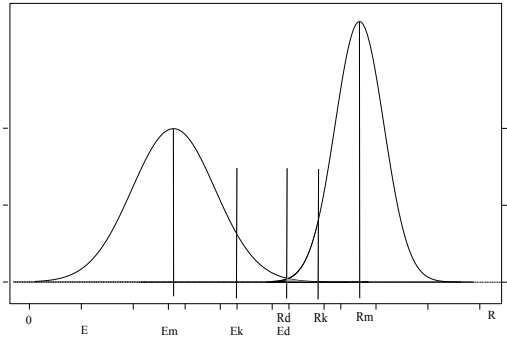


Figure 7.8: Partial safety factors, and design values for comparison, in a semiprobabilistic safety approach.

since $F_R(x) = \Pr(R < x)$ is the probability of having resistances less than x ; and being $f(x)$ the probability density function of having effects equivalent to x , the product $f_E(x) dx = \Pr(x < E < x+dx)$ is the expression of the probability of effects between x and $x+dx$, equal to or greater than x . So that the product of both evaluates the failure probability in the interval $[x, x+dx]$, and its integral the total failure probability.

Given that the objective is to ensure that this probability is sufficiently reduced and admissible, failure is established as a limit state that should not be reached with a determined prob-

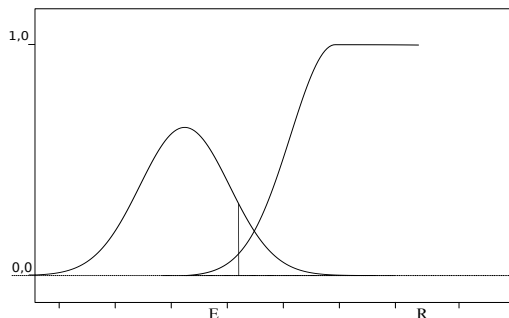


Figure 7.9: Calculation of failure probability. For any value x , $F_R(x)$ indicates the cumulative probability of resistances less than x and $f_E(x)$ the probability density of effects equal to that value x

ability degree.

$$\Pr(R - E < 0) \leq P_{\text{adm.}} \quad (7.51)$$

$$\int_{-\infty}^{\infty} f_E(x) F_R(x) dx \leq P_{\text{adm.}} \quad (7.52)$$

However, a failure causing personal injury is not the same as one involving only material damage, so that admissible probability will depend on the type of failure or limit state considered. Limit states are thus categorized as follows:

- Ultimate Limit State
- Service Limit State

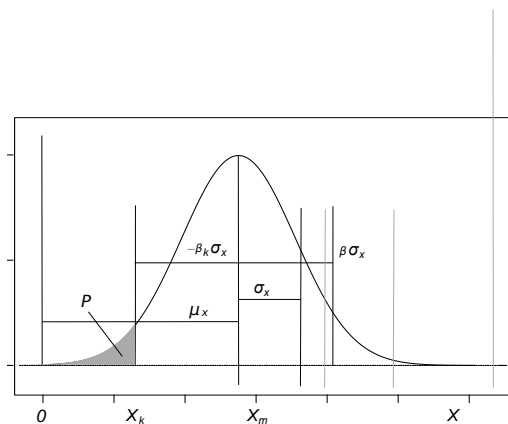


Figure 7.10: Probability density function of resistances, and significant points: mean $\mu = \bar{X} = X_m$, standard deviation σ , or reliability index β positive or negative $\beta\sigma$. The areas under the curve indicate the probability P of finding resistances in that value range of x . The characteristic value X_k is usually established so that unsafe values have a probability of less than 5% (correspond to the 5% fractile) or equivalently, reach the reliability index β_k corresponding to that probability.

- *Accidental Limit State...*

And for each of them, admissible probabilities are normalized through indices established in parallel with other possible considerations about safety or damage probability.

7.2.1.2 Types of Failure and Reliability Index

It is evident that the consequences of a failure depend on the degree of damage it may cause, such that different *reliability indices* can be assigned to the structure in relation to failures, according to whether their consequences are more or less serious. These indices would be nothing more than reference values, somehow linked to measures of the associated failure probability, with greater reliability (higher index) the lower the failure probability.

Given the multiplicity of statistical distributions seen before, and in order to handle the references to failure probabilities, or the corresponding indices at significant distribution points, the definitions are established on the normal, or Gaussian distribution model already seen, and whose most significant points or parameters we highlight in figure 7.10.

Mean μ , or Expected Value $E(X)$:

$$\bar{X} = \mu_X = \frac{1}{n} \sum_{i=1..n} X_i \quad (7.53)$$

$$\mu_x = \int x f(x) dx \quad (7.54)$$

Variance σ^2 and standard deviation σ :

$$\sigma_X^2 = \frac{1}{n-1} \sum_{i=1..n} (X_i - \bar{X})^2 \quad (7.55)$$

$$\sigma_X^2 = \int (x - \mu_x)^2 f(x) dx \quad (7.56)$$

(The divisor for standard deviation uses Bessel's correction, considering whether the estimation is made from a sample, $1/(n - 1)$, or from the entire population, $1/n$)

Coefficient of Variation V_X :

$$V_X = \frac{\sigma_X}{\mu_X} \quad (7.57)$$

The standardization of language leads to the use of the following definitions, described on page 544 and following:

Normal distribution with mean μ and variance σ^2 :

$$X \sim \mathcal{N}[\mu, \sigma^2]$$

$$F_X(x) = \Pr(X < x) = \Phi\left(\frac{x - \mu}{\sigma}\right); \quad \bar{X} = \mu_X = \mu, \sigma_X = \sigma \quad (7.58)$$

where Φ is the standard normal distribution function

Standard normal distribution: $X \sim \mathcal{N}[0, 1]$, $F_X(x) = \Phi(x)$.

Thus, the reliability index β becomes associated with a certain probability $P = \Phi(-\beta)$, as graphically represented in figure 7.10:

$$P = \Phi(-\beta) = \Pr(X < \mu_X - \beta\sigma_X) \quad (7.59)$$

7.2.1.3 What Failure Probability is Considered Admissible?

The decision is normative, contained in the codes, and established by comparison with other types of risk.

Smoking 10 cigarettes per day	1 in 200
All natural causes, 40 years of age	1 in 850
Any violence or poisoning	1 in 3300
Flu	1 in 5000
Road accident (in Europe)	1 in 8000
Leukemia	1 in 12500
Earthquake, living in Iran	1 in 23000
Team sports	1 in 25000
Domestic accident	1 in 26000
Work accident	1 in 43500
Floods, living in Bangladesh	1 in 50000
Radiation, nuclear industry workers	1 in 57000
Homicide, in Europe	1 in 100000
Floods, living in northern China	1 in 100000
Railway accident (in Europe)	1 in 500000
Earthquake, living in California	1 in 2000000
Death by lightning	1 in 10000000
Storm and strong winds, northern Europe	1 in 10000000

Table 7.1: Phenomenon considered and annual mortality probability P_a . British Medical Association 1987, table 10.1

A classic table in the literature is table 7.1. Starting from references like this, the codes establish the reliability indices β or acceptable failure probabilities depending on the period to be considered.

In the case of building construction, this risk estimation and attribution of acceptable probability depends on both the type of use to which the construction is destined (and the degree of occupancy throughout its life), as well as its expected lifetime, and the impact on damages to people or property corresponding to the type of failure considered.

To consider the expected lifetime of the building, a classic expression in statistics is useful, which relates the annual probability P_a of an event E (whose inverse is, by definition, the return period, T_R) with its probability in a given time (lifetime) T_v , assuming that the occurrence of each case of such event type is independent:

$$T_R = \frac{1}{P_a} = \frac{T_v}{-\ln(1 - P_{E,T_v})} \quad (7.60)$$

In relation to the type and severity of damages that can be expected for each possible failure, limit states have been established for normal use, distinguishing between those that can cause collapses and therefore affect the safety of people and the building itself (ultimate limit states that are imperative to avoid) and those that only cause material damage affecting the functionality, comfort, or appearance of the building (serviceability limit states). Implications are also considered for the requirements against failure in possible risk scenarios or accidental situations, such as fire (sometimes referred to as accidental limit

Class	limit values			
	1 year		50 years	
	β	Prob	β	Prob.
RC3	5,2	10^{-7}	4,3	$8,5 \cdot 10^{-6}$
RC2	4,7	$1,3 \cdot 10^{-6}$	3,8	$7,2 \cdot 10^{-5}$
RC1	4,2	$1,3 \cdot 10^{-5}$	3,3	$4,8 \cdot 10^{-4}$

Table 7.2: Risk class and reliability, or reference failure probability, required for ultimate limit states, according to Tables B.2 in Annex B of EN-1990 and in Appendix B of Annex 18 of the Spanish Structural Code of 2021. See also Appendix C of this.

states) due to their different probabilities compared to those of normal use, and their dependence on corresponding preventive and mitigation systems.

Finally, different reliability requirements are assigned to different types of constructions and the risk they entail by their mode of use.

All these requirements are ultimately measured through reliability indices, indices that can be related, through the standard normal distribution, to normalized failure probabilities for each case. It should be noted that the choice of this distribution is merely normative: the appropriate distributions for the failure phenomenon may be others, particularly extreme value distributions, so the reference failure probability computed with the normal distribution may not be a good measure of the probability in the actual case.

In structural regulations, constructions are classified into

classes according to their damage risk:

RC1 low in lives and cost

RC2 medium in lives or cost

RC3 high in lives or cost

and for each of these, the required reliability indices are established. An example for ultimate limit states is the table 7.2⁵.

The semi-probabilistic models establish the partial safety factors of the regulations seeking that their effect allows adjusting the results of the checks to the pre-fixed reliability indices, as illustrated in section 7.2.1.2.

7.2.2 Probabilistic Safety Models

Probabilistic models address the direct determination of failure probability, such that the checks, in which the complete random problem is handled, allow ensuring that the failure probability

5. Note that the value for RC2 implies a relationship between the so-called failure return period $T_R = 1/P_a$ and the building's life span T_V which is established between 14000 and 15000 for a service period of 50 years, being 10 times smaller for the RC1 case and 8 times larger for RC3. It is common to establish the required reliability for a one-year period and derive from this the necessary reliability for different periods. So that the reliability referred to one year turns out to be a fixed value by risk class. A different case is the table C2 of Annex C of CTE-DB-SE, which establishes the required reliability according to the consequences of failure and the required cost order to reduce its effects, although it fixes it for the entire life span of the structure, so that the required reliability for the one-year period will depend on the expected service time. The target value of T_R/T_V set in this case for average situations coincides with that established in EN1990 for RC2 at 50 years.

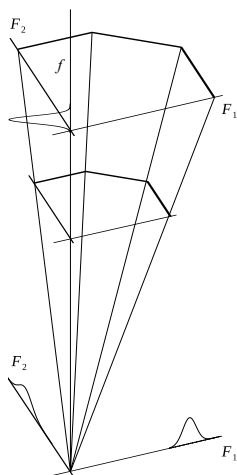


Figure 7.11: Failure in the space of random variables. Three of them are represented, F_1 , F_2 , and f , and their probability density functions, as well as the limit or boundary between failure and non-failure conditions. Figure 7.2 would express that boundary for defined values of these variables, although it can be interpreted as a “fuzzy” boundary for random variables, expressed on their median values, and considering the interpretation provided by figure 7.5.

established by the regulations is not exceeded in the project or, in other words, that the structure meets the required reliability.

To do this, the failure event and its probability are analyzed,

with the computation of failure probability being formally simple, although complex in practice. It suffices to integrate, for the space of values of the variables where failure occurs, the product of the probability density functions of these variables at each point in that space region by the “volume” element at that point.

For example, in the case of the frame in figure 7.1, for which the resistance limit figure would be one of those in 7.2, if we consider the distributions for all possible values of limit loads F_1 and F_2 , and for possible steel resistances f , as expressed in figure 7.11, we would have as failure event and probability the expressions

$$\begin{aligned} Z(f, F_1, F_2, \dots) &= R(f, \dots) - E(F_1, F_2, \dots) < 0 \\ \Pr(Z < 0) &< P_{\text{adm}}; \end{aligned} \quad (7.61)$$

where the probability computation would be

$$\Pr(Z < 0) = \iiint F_c(f, F_1, F_2) f(f) f(F_1) f(F_2) \, df \, dF_1 \, dF_2 \quad (7.62)$$

$$\begin{aligned} &= \iint f(F_1) f(F_2) \, dF_1 \, dF_2 \int F_c(f, F_1, F_2) f(f) \, df \\ &= \iint F_c(F_1, F_2) f(F_1) f(F_2) \, dF_1 \, dF_2 \end{aligned} \quad (7.63)$$

where $F_c \in [0, 1]$ indicates the collapse probability for the values of the variables it depends on at the point considered in the space containing all possible values for each variable (first

and second rows of the preceding list of expressions), or in its “projection” (in the third row) to the plane where the resistance distribution f is condensed through the “fuzzy” boundary indicated in figure 7.11. Note that in the preceding expressions, f as a function argument is resistance, but as a function name is probability density function, and F as an argument is load, although as a function name is the cumulative distribution function of failure probability. In this representation, F_c has values 0 or 1 in the two rows where it depends on the three random variables considered in the model ($F_c(f, F_1, F_2)$) and has values in the continuous range between 0 and 1 in the third row, where it only depends on load values ($F_c(F_1, F_2) = \int F_c(f, F_1, F_2) f(f) df$), having evaluated the failure probability conditioned to those specific values of F_1 and F_2 and derived from the uncertainty in resistance f .

The logic of the expression does not hide the difficulty of its practical obtention, for which indirect methods are often used, such as the Monte Carlo method.

7.3 Methods to Determine Reliability

From what has been seen so far, the reliability index β provides important information, at least qualitatively, about failure probability, information that is also quantitative if the relevant distribution is normal. So that determining this index becomes key.

We will assume henceforth that we have models to characterize structural failure (deterministic), that we are able to

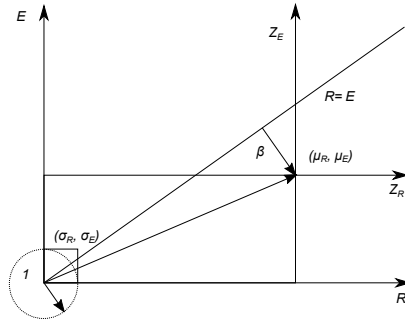


Figure 7.12: Reliability in the space of normalized variables: Geometric interpretation of the reliability index

establish the structure's life span and that, for that period, we are able to establish the relevant random variables and their distributions. Under these conditions, we seek to determine the reliability index and its sensitivity, that is the possible types and magnitudes of error in its estimation.

7.3.1 The Basic Reliability Problem and Its Generalization

We will describe the simplest reliability estimation strategy, usually referenced as First Order Reliability Method (FORM).

We consider a problem with only two variables E and R , effects of actions and resistance to these. We assume normal distributions for both, with parameters μ_E , σ_E and μ_R , σ_R re-

spectively. The failure condition will be expressed as

$$F = R - E < 0 \quad (7.64)$$

where the F distribution is also normal, with parameters

$$\mu_F = \mu_R - \mu_E \quad (7.65)$$

$$\sigma_F = \sqrt{\sigma_E^2 + \sigma_R^2 - 2\rho_{ER}\sigma_E\sigma_R} \quad (7.66)$$

where ρ_{ER} would be the correlation between load effects and resistive capacities. If we consider they are not correlated⁶, $\rho_{ER} = 0$ and therefore $\sigma_F = \sqrt{\sigma_E^2 + \sigma_R^2}$.

Recalling expressions 7.58 and 7.59

$$\begin{aligned} \Pr(F < x) &= \Phi\left(\frac{x - \mu_F}{\sigma_F}\right) \\ \Phi(-\beta) &= \Pr(F < 0) = \Phi\left(-\frac{\mu_F}{\sigma_F}\right). \end{aligned} \quad (7.67)$$

with

$$\beta = \frac{\mu_F}{\sigma_F} = \frac{\mu_R - \mu_E}{\sqrt{\sigma_R^2 + \sigma_E^2}} \quad (7.68)$$

We now consider variables E and R normalized to their transformed Z_E and Z_R , with standard normal distribution,

6. Attention must be drawn to the fact that in this case we may be ignoring situations where there is negative correlation: $\rho_{ER} = -1$. We would then have $\sigma_F^2 = (\sigma_E + \sigma_R)^2$. Think, for example, of low or very low control situations where it is possible to associate conditions of exceeding usual loads with low resistances in the executed construction.

where the transformation implies for both a translation and a scale change.

$$Z_E = \frac{E - \mu_E}{\sigma_E}; Z_R = \frac{R - \mu_R}{\sigma_R} \quad (7.69)$$

We can represent the boundary between non-failure and failure regions in the Z_R, Z_E graph (see figure 7.12) where the original values $X (R, \mu_R, \dots)$ on the abscissa axis and $Y (E, \mu_E, \dots)$ on the ordinate axis are at scales $X/\sigma_R, Y/\sigma_E$.

The vector from the initial origin $(R, E) = (0, 0)_o$ to the translated or normalized $(Z_R, Z_E) = (0, 0)_n$ is, in the normalized scale

$$\boldsymbol{\mu} = \left(\frac{\mu_R}{\sigma_R}, \frac{\mu_E}{\sigma_E} \right) \quad (7.70)$$

and in turn, the intersection points of the limit line $R = E$ on the abscissa $Z_E = 0$ and ordinate $Z_R = 0$ axes can be determined from 7.69, being respectively

$$\left(-\frac{\mu_R - \mu_E}{\sigma_R}, 0 \right); \left(0, \frac{\mu_R - \mu_E}{\sigma_E} \right). \quad (7.71)$$

Thus the direction of the limit line is determined and the unit vector (versor) perpendicular to it is

$$\left(\frac{\sigma_R}{\sqrt{\sigma_R^2 + \sigma_E^2}}, -\frac{\sigma_E}{\sqrt{\sigma_R^2 + \sigma_E^2}} \right) \quad (7.72)$$

If we project the vector $\boldsymbol{\mu}$ from 7.70 onto this normal to the limit line, we obtain

$$\frac{\mu_R - \mu_E}{\sqrt{\sigma_R^2 + \sigma_E^2}} = \beta \quad (7.73)$$

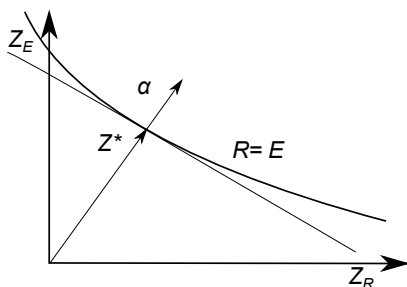


Figure 7.13: Hashofer-Lind Index: reliability in the space of normalized variables as distance from origin to “project point”

where the identification with β comes from the previously seen expression, 7.68.

Therefore, the reliability index can be interpreted as the distance β to the point of the mean of the variable that expresses the failure condition ($[Z_R, Z_E] = [0, 0]$) from the closest position of said failure condition (“project” point) in the space of *normalized variables*.

Considering now the figure in the space of normalized variables (figure 7.13), this same distance can be computed from the origin to the project point, Z^* , the closest point expressing any of the failure conditions. The index is valid for any conditions, both linear and non-linear, and for spaces with higher

dimensions:

$$\beta = |\mathbf{Z}^*| = \min_{\mathbf{Z} \in \text{Failure}} \sqrt{\sum_i^n Z_i^2} = \min_{\mathbf{Z} \in \text{Failure}} \sqrt{\mathbf{Z}^T \cdot \mathbf{Z}} \quad (7.74)$$

Expression 7.68 is thus of great utility for estimating reliability.

Sensitivity

It is now interesting to consider the sensitivity of this index to alterations in some of the variables Z_i , through the corresponding partial derivatives:

$$\alpha_i = \frac{\partial \beta}{\partial Z_i} \quad (7.75)$$

The sign of α_i indicates whether Z_i is an “action” or “resistance”, because changes of the same sign between Z_i and β (α_i positive) imply growth in reliability with the increase of the variable’s value, and therefore point to variables that act as resistances, as opposed to the case where the increase of the variable reduces reliability, indicating actions.

7.3.2 Partial Safety Factors

The use of safety factors has a long history, from the initial strategy of applying them only to resistances, through admissible stresses, passing through more refined approaches initiated by engineers like Rankine or Maillart to differentiate by origin type (loads and types in these or resistances) to their contemporary probabilistic strategy based on partial factors (Pugsley),

see Cervera Bravo 2022, such that in current codes they are maintained, as parallel paths for their definition, either through their experience over decades of use, or through statistical procedures as described here.

We now address the contemporary way of ensuring reliability for random events. The procedure requires setting and calibrating partial safety factors so that deterministic failure checks, in the form 7.5, and using design values $E_d = \gamma_E E$, $R_d = R/\gamma_R$ obtained from nominal biased values for basic variables (their characteristic values), with bias increased in the direction of safety by the corresponding partial factor, ensure the admissible reliability or failure probability set a priori.

In Cervera 2019, two approaches are analyzed, depending on the distribution considered, of which we summarize here, for simplicity, the second, following Văcăreanu, Aldea, and Lungu 2007, applicable to lognormal distributions that avoid the existence of negative values in the variables of interest (negative resistances?).

7.3.2.1 Models for Actions and Resistances

The definition of actions as random variables is a whole world, see sections 4.4 and 4.5 in Cervera 2019, but they can be simplified by grouping their effects E into lognormal distributions, with parameters μ_E and $V_E = \sigma_E/\mu_E$ with values for this last coefficient of variation

- ≈ 0.10 for permanent loads
- ≈ 0.30 for (the maxima of) variable loads

And using as nominal values the characteristic values, defined as those with a 95% probability of **not** being exceeded. It should be noted that with this definition the temporal variability of loads is not reflected, even though these are time-varying variables. It can be understood that the definitions adopted by the standards refer to the 50-year period which is the usual reference for standard structures.

Strengths have lower variability, and their effects can be approximately condensed in the resistant capacity R into lognormal distributions with parameters μ_R and $V_R = \sigma_R/\mu_R$, with this last coefficient of variation having values

- < 0.10 for concretes
- < 0.05 for steels

Now characteristic values are used, that is, those with a 5% probability of **not** being exceeded, and again the effects of temporal degradation are not considered here, referring to that same standard 50-year period.

The different bias in nominal values for actions or resistances can be observed, although in both cases only the 5% most unsafe values are left out.

7.3.2.2 Failure Condition

We consider the following variable to measure the failure condition and evaluate its probability:

$$Z = \ln \frac{R}{E} \quad (7.76)$$

with zero value when reaching the collapse limit, negative for collapse situations, and positive for situations where collapse does not occur. The mean and variation coefficient of Z are

$$\mu_Z = \mu_{\log \frac{R}{E}} \approx \log \frac{\mu_R}{\mu_E} \quad (7.77)$$

$$\sigma_Z = \sigma_{\log \frac{R}{E}} \approx V_{\frac{R}{E}} = \sqrt{V_R^2 + V_E^2} \quad (7.78)$$

where sufficiently valid approximations are used in cases of reduced variability (see Cervera 2019 section 2.3.2).

The failure probability, and the corresponding reliability, will be, according to 7.67:

$$\begin{aligned} \Pr(Z < 0) &= \Phi\left(-\frac{\mu_Z}{\sigma_Z}\right) = \Phi(-\beta) \\ \beta &= \frac{\log \frac{\mu_R}{\mu_E}}{\sqrt{V_R^2 + V_E^2}} \approx \frac{\log \frac{\mu_R}{\mu_E}}{\alpha_R V_R + \alpha_E V_E} \end{aligned} \quad (7.79)$$

where the values of α_R and α_E used in the approximation can be interpreted as the cosines of the angles of the projection of the sides V_R or V_E of a rectangle onto its diagonal $\sqrt{V_R^2 + V_E^2}$, values that can be approximated. It can thus be written

$$\begin{aligned} \frac{\mu_R}{\mu_E} &= e^{\beta(\alpha_R V_R + \alpha_E V_E)} = e^{\beta \alpha_R V_R} e^{\beta \alpha_E V_E} \\ \frac{\mu_R}{e^{\beta \alpha_R V_R}} &= \mu_E e^{\beta \alpha_E V_E} \end{aligned} \quad (7.80)$$

The last expression can be interpreted as that, given a certain reliability β , the mean values of resistance and action effects can

be compared by reducing the former and increasing the latter with factors dependent on both the considered reliability and the coefficients of variation of the corresponding distributions.

As the nominal values are not the mean values, but the characteristic values, biased to the probability of unsafe values of only 5%, the expression 7.80 can be rewritten:

$$\frac{R_k}{\gamma_R} = E_k \gamma_E;$$

$$\text{where } R_k = \mu_R e^{-\beta_{k,R} V_R}; E_k = \mu_E e^{\beta_{k,E} V_E} \quad (7.81)$$

where $\beta_{k,R}, \beta_{k,E}$ are the reliabilities associated with leaving out only that 5% of unsafe values (the factor by which the standard deviation must be shifted relative to the mean to achieve this), and from which the following safety factors can finally be deduced as necessary to achieve the target reliability β :

$$\gamma_R = e^{(\alpha_R \beta - \beta_{k,R}) V_R}; \quad \gamma_E = e^{(\alpha_E \beta - \beta_{k,E}) V_E} \quad (7.82)$$

7.4 Example: Frame Reliability

We now address the determination of reliability for a practical case: the frame in figure 7.1, for which we have seen the verification space of the failure condition in figure 7.11, corresponding to the deterministic limit conditions of figure 7.2.

We consider the failure conditions described in expressions 7.2, with random variables $M_i = W_i f_{y,i}$, where $i \in [1 \dots 5]$, F_j , $j \in [1, 2]$, h and l .

The failure probability described in 7.61 would be determined directly through expressions 7.62 or 7.63, but here we will do it indirectly by determining the reliability index corresponding to the distributions of the problem variables.

7.4.1 Basic Variables

We consider hereafter that the frame dimensions are deterministic variables, with $h = 4\text{m.}$, $l = 8\text{m.}$

We equally consider that the design conditions propose for the Beam and Column IPE 330 and HEB 180 cross-sections respectively, both of S275 steel.

As initial random variables we establish the following table, in which we have highlighted in bold, as reference values, the nominal values, whether the mean $\bar{X}$ or the characteristic X_k of each variable X , according to usual criteria and to codes or geometric property tables of the cross-sections, from which plastic modules are collected, and likewise their coefficient of variation V_x , according to tables such as those in Cervera 2019, along with the usual safety factors γ_X , in the case of F_2 considering that it contains part permanent load and part variable load.

From these, the remaining parameters have been determined, as well as corresponding design values X_d . Note the difference, in the case of loads, between nominal (characteristic) and mean values.

In the case of geometric qualities of the cross-sections, there is a usual inconsistency: mean values are handled, but the variability in these would imply considering characteristic values lower than the means and, even with unity safety factors, would

imply design values lower than said mean. And yet mean values are usually used in calculations, which is why these values are highlighted in color.

X	$\bar{X}$	V_X	σ_X	X_k	γ_X	X_d
F_1 kN	24.1	0.45	10.9	42	1.5	63
F_2 kN	76.2	0.35	26.7	120	1.4	168
$W_{1:4}$ cm ³	481.4	0.09	43.3	419.3	1.0	481.4
W_5 cm ³	804.3	0.05	40.2	743.2	1.0	804.3
$f_{1:5}$ MPA	310.8	0.07	21.8	275	1.05	261.9

In the following table, the characteristic and mean values corresponding to the design values of the plastic modules used are marked in red. Finally, the last two columns detail the values ultimately used to characterize the relevant design variables, with their means and variances suitable to the corresponding distributions.

X	X_k	$\bar{X}$	$\bar{X}$	σ_X^2
F_1 kN			24.1	118.8 k ² N ²
F_2 kN			76.2	712.9 k ² N ²
$W_{1:4}$ cm ³	481.4	565.1	481.4	1874.9 cm ⁶
W_5 cm ³	804.3	876.4	804.3	1616.0 cm ⁶
$f_{1:5}$ MPA			310.8	475.2 MPA ²

In the case of variables associated with the frame resistances, $[W_1, W_2, W_3, W_4, W_5, f_{y,1}, f_{y,2}, f_{y,3}, f_{y,4}, f_{y,5}]$, we are considering that these are independent variables, but they are not completely so. These are correlated variables, as the origins of the

pieces are probably from common manufacturers, such that if there is a rise or fall in quality in one of them, there will probably be the same in the others, etc. The correlation is higher between similar cross-sections and even more so between the ends of the same piece, even though both ends can be considered with different random qualities depending on the differentiated execution of both nodes.

With all these considerations, the correlation matrix is proposed as follows between the variables, as they have been ordered in the previous paragraph:

$$\begin{bmatrix} 1 & 0.7 & 0.75 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 1 & 0.6 & 0.75 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.75 & 0.6 & 1 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0.75 & 0.7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.95 & 0.85 & 0.75 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0.95 & 1 & 0.75 & 0.85 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0.85 & 0.75 & 1 & 0.95 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0.75 & 0.85 & 0.95 & 1 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0.7 & 0.7 & 0.7 & 1 \end{bmatrix}$$

7.4.2 Derived Variables

We now consider the variables corresponding to resistive capacities, $M_i = W_i f_i$.

We consider normal distributions of variables X and Y , with known parameters, and seek the parameters of the distribution of variables derived from these, $Z = Z(X, Y)$.

According to what was considered in section 7.1.1.5 for the combination of two variables X, Y with distributions

$$\begin{aligned} X &\sim \mathcal{N}(\bar{X}, \sigma_X^2); \\ Y &\sim \mathcal{N}(\bar{Y}, \sigma_Y^2); \end{aligned}$$

we will have the following distributions for their combinations:

Sum of two variables

$$\begin{aligned} Z &= X + Y \\ \bar{Z} &= \bar{X} + \bar{Y} \\ \sigma_Z^2 &= \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y = \sum \Sigma_{XY} = \sum \text{cov}(X, Y). \end{aligned}$$

Product of two variables

$$\begin{aligned} Z &= X \cdot Y \\ \bar{Z} &= \bar{X} \cdot \bar{Y} + \sigma_{XY} = \bar{X} \cdot \bar{Y} + \text{cov}(X, Y) \\ \sigma_Z^2 &= (\bar{X}^2 + \sigma_X^2)(\bar{Y}^2 + \sigma_Y^2) + \text{cov}(X^2, Y^2) - \bar{Z}^2 \end{aligned}$$

Accordingly, considering the preceding correlation matrix, we have the following statistics for the resistances $(M_1 \dots M_5)$, and their correlation matrix:

Distributions of $M_1 \dots M_5$:

$R :$	$M_1..M_4$	M_5
$\bar{R}$	175.61	272.37
σ_R	20.05	23.45

Correlations between $M_1 \dots M_5$:

1.000	0.794	0.787	0.656	0.349
0.794	1.000	0.656	0.787	0.349
0.787	0.656	1.000	0.794	0.349
0.656	0.787	0.794	1.000	0.349
0.349	0.349	0.349	0.349	1.000

For the computation of this last matrix, the covariance and its corresponding correlation have been evaluated for each pair M_i, M_j , where $M_i = W_i f_i$, as follows.

The variables derive from their respective products: $Z_i = X_i Y_i$, $Z_j = X_j Y_j$, whose means $\bar{Z}_i, \bar{Z}_j$ and variances $\sigma_{Z_i}^2, \sigma_{Z_j}^2$ are determined with the preceding expressions.

Covariance is the expected value of the product between the differences of each pair of measurements and their respective means. In the following computation, X will be the geometry W , and Y the resistance f , variables that are not correlated.

$$\begin{aligned}
 \sigma_{Z_i Z_j} &= \text{cov}(Z_i, Z_j) = E[(Z_i - \bar{Z}_i)(Z_j - \bar{Z}_j)] \\
 &= E[Z_i Z_j] - \bar{Z}_i \bar{Z}_j = E[(X_i Y_i X_j Y_j)] - \bar{X}_i \bar{Y}_i \bar{X}_j \bar{Y}_j \\
 &= E[(X_i X_j)(Y_i Y_j)] - \bar{X}_i \bar{X}_j \bar{Y}_i \bar{Y}_j = \sigma_{(X_i X_j)(Y_i Y_j)} \\
 &= E(X_i X_j) E(Y_i Y_j) - \bar{X}_i \bar{X}_j \bar{Y}_i \bar{Y}_j \\
 &= (\bar{X}_i \bar{X}_j + \text{cov}(X_i, X_j)) + (\bar{Y}_i \bar{Y}_j \text{cov}(Y_i, Y_j)) - \bar{X}_i \bar{X}_j \bar{Y}_i \bar{Y}_j \\
 &= \text{cov}(X_i, X_j) \bar{Y}_i \bar{Y}_j + \bar{X}_i \bar{X}_j \text{cov}(Y_i, Y_j) + \\
 &\quad \text{cov}(X_i, X_j) \text{cov}(Y_i, Y_j)
 \end{aligned}
 \tag{7.83}$$

Correlations can be determined from these.

$$\rho_{Z_i Z_j} = \frac{\sigma_{Z_i Z_j}}{\sigma_{Z_i} \sigma_{Z_j}} \tag{7.84}$$

7.4.3 Failure Probability

Considering all of the above, we can now determine the statistics for the three failure conditions proposed in 7.2:

	Z_1	Z_2	Z_3
$\bar{Z}$	605.91	846.03	591.34
σ_Z	84.23	133.39	165.47
β_Z	7.19	6.34	3.57
$P(Z < 0)$	3.15 E-13	1.13 E-10	1.76 E-4

The calculation shows that the most probable failure condition is the 3rd, the deck failure, both due to the closer proximity to zero in the mean of the difference between resistant capacity and load effect, and due to the greater variance in that difference. Against this failure mode the structure has a reliability of 3.57, a reliability that turns out to be somewhat lower than that established by regulations for buildings of risk class RC2 at 50 years, and corresponds to a failure probability slightly more than double the reference for that class and period, but which would amply meet the requirement for class RC1 at 50 years, or which could be valid for a ratio between the failure return period and the building’s life span of 5680 times versus the approximate value between 14000 and 15000 required for risk class RC2. See the note associated with table 7.2.

A deterministic calculation with the safety factors adopted above would use the verification $Z_3 = R - E = M_2 + 2M_5 +$

$M_4 - F_2 l / 2$, for which the design values (the characteristic values affected by partial factors) would provide for R and for E the respective values of 673 and 672 mkN, which suppose the validity of the project.

7.5 Conclusions

Although the chapter provides details that are not, in general, necessary for the design of reliable structures, a set of conclusions can nevertheless be extracted that are relevant, and which are summarized below.

7.5.1 On Variability, Probability, and Consequences of Failure

- The magnitudes handled in structural calculation present spatial and, where applicable, temporal variability, so for design, only a statistical description of their value or probable maximum is available.
- Absolute safety is therefore not possible; very low failure probabilities must be accepted (which are established in the codes)
- With an appropriate statistical description of these magnitudes, it is possible and admissible, though complex, to verify structures using statistical procedures that ensure that these failure probabilities are not exceeded.

- The accepted failure probability depends on the consequences associated with the type of failure: hence the classification of Limit States into:
 - Ultimate Limit States (ULS)
 - Serviceability Limit States (SLS) (reversible or irreversible)
 - Others (accidental limit states, or associated with damage degrees, etc...)

7.5.2 On Bias in Nominal Values and Partial Factors

- To reduce the possibility of failure when combining the probable values of these magnitudes, for each variable, biased values are adopted on the appropriate side in relation to failure: actions or resistances. These values (*characteristic*) are usually those that only exclude 5% of cases.
- In addition, to control and limit the failure probability to the desired value, for each variable these values are modified on the appropriate side by an additional safety factor, depending on the dispersion of the statistical distribution: these are the partial safety factors and the corresponding verification values, the *design values*.
- The quality of the variable in the considered verification (by its effect for or against failure) is the factor that defines the amplifying or reducing character of the adopted partial safety factor.

- The Codes *calibrate* these partial factors to the desired failure probability by simultaneously considering the uncertainties associated with:
 - The accepted statistical models for action and resistant capacity variables (and therefore the specifications for acceptance or rejection of the corresponding products)
 - The professional model: the models of behavior representation and failure verification expressions established or admitted in the code itself.
 - ... therefore it is not sensible to mix criteria from different codes.

7.5.3 On Simultaneity Coefficients and Other Terminology Derived from Semi-Probabilistic Methods

- When several independent actions are considered simultaneously, as the temporal records are not synchronized, the maximum of one does not occur when the other's maximum occurs, so the probable maximum of the sum is less than the sum of probable maximums, hence reduction coefficients can be introduced.
- If an element must support actions at several points without spatial synchronization, the same occurs, and the effect of the sum of several is not the sum of the effects of each, so reduction coefficients can be introduced.
- If several elements or points collaborate to jointly resist the

effect of the same cause, a resistance value closer to the mean value can likewise be adopted.

From all of this, then, terms such as the following groups are derived:

- mean and characteristic value,
- *favorable* or *unfavorable* effect,
- partial factors, amplification, reduction, design value,
- simultaneity and reduction conditions derived from area, number of heights or type of use, shared load, etc.

Appendix A

Limit State Resistance Planes

A.1 Three-Point Plane

The generic plane equation is $a_1x + b_1y + c_1z - d_1 = 0$, which can also be written as $xa_0 + yb_0 + zc_0 = 1$, where $a_0 = a_1/d_1 \dots$ so that the conditions can be established

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad (\text{A.1})$$

and therefore

$$\begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} y_1 z_2 + y_2 z_3 + y_3 z_1 - y_1 z_3 - y_2 z_1 - y_3 z_2 \\ z_3 x_1 + z_1 x_2 + z_2 x_3 - z_3 x_2 - z_1 x_3 - z_2 x_1 \\ x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_3 y_1 - x_3 y_2 \end{bmatrix}, \quad (\text{A.2})$$

$$\Delta = x_1 y_2 z_2 + y_1 z_2 x_3 + z_1 x_2 y_3 - x_1 y_3 z_2 - y_1 z_3 x_2 - z_1 x_3 y_2;$$

which can also be expressed in the canonical format

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = d \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}; d = \frac{1}{\sqrt{a_0^2 + b_0^2 + c_0^2}}, \quad (\text{A.3})$$

format in which $\mathbf{g} = [a \ b \ c]^T$ is the plane's unit vector, and where for any point $\mathbf{p} = [x \ y \ z]^T$, the substitution $\mathbf{g}^T \mathbf{p} - d = ax + by + cz - d = d_p$ measures the distance of the point to the plane, positive: from the *exterior*, or negative: from the *interior*. This is because d is the distance from the origin to the plane measured in the direction of $\mathbf{g}$ and the scalar product $\mathbf{g}^T \mathbf{p}$ is the projection of the position vector $\mathbf{p}$ onto said direction.

Once the limit plane expression is known, it is possible to determine the existence or not of collapse for a given stress state, or characterize the degree of safety in relation to the failure condition described by this.

The situation is safe if the resulting factor exceeds the reference factor for the problem in question, in this case the section subjected to biaxial bending and compression. We can also consider its inverse format:

$$\frac{1}{\gamma} = \frac{an + bm_y + cm_z}{d} \leq \frac{1}{\gamma_{\text{ref}}} \quad (\text{A.5})$$

But in any case, it is evident that in cases where the origin O is not centered in relation to the resistance limit surface, the ratio $\gamma/\gamma_{\text{ref}}$ may not be representative of the safety in situations of stress close to zero (or null stress situation), which leads us to consider the new factor.

A.2.2 Robust Safety Factor

In this case, the aim is to consider equally safe all cases of similar departure from failure conditions, cases given by a centered contraction of the limit surface, around the point F_c representing the stress state furthest from any imaginable failure conditions, in the same figure A.1. Now we will have

$$\gamma = \frac{d - d_c}{an + bm_y + cm_z - d_c} = \frac{|[AB] + [B3]|}{|[AB]|} = \frac{|[F_cB] + [B2]|}{|[F_cB]|}$$

$$\gamma \geq \gamma_{\text{ref}}; \quad d_c = an_c + bm_{yc} + cm_{zc} \quad (\text{A.6})$$

where now d_c is the distance of the *central* point given by F_c to the origin O measured in the orthogonal projection to the limit plane, with $an_c = N_c/Af_{yd}$, $bm_{yc} = M_{yc}/W_{ply}f_{yd}$, $bm_{zc} = M_{zc}/W_{plz}f_{yd}$.

So that the condition to be checked can be expressed in any of these other forms, which lead us in any case to the typical format of resistance equations in limit analysis:

$$\begin{aligned}\gamma_{\text{ref}}(an + bm_y + cm_z) &\leq d + (\gamma_{\text{ref}} - 1)d_c \\ an + bm_y + cm_z &\leq \frac{d}{\gamma_{\text{ref}}} + \left(1 - \frac{1}{\gamma_{\text{ref}}}\right)d_c \\ \boldsymbol{\psi} \mathbf{f} &\leq \mathbf{d} \mathbf{r}\end{aligned}\tag{A.7}$$

It is worth comparing the following version with equation A.5 to verify the difference between both criteria (although the expression is not particularly intuitive)

$$\frac{an + bm_y + cm_z}{d} \leq \frac{1}{\gamma_{\text{ref}}} + \left(1 - \frac{1}{\gamma_{\text{ref}}}\right) \frac{d_c}{d}\tag{A.8}$$

Appendix B

Equivalent Stiffness in Reinforced Concrete

B.1 Introduction and Warning

This appendix collects the information provided as informative sheet No. 6 in the Structural Project classes at ETSAM for estimating deflections in reinforced concrete. Although the first versions of this appendix date from the first decade of the 2000s, this version incorporates some minor subsequent revisions and, although it would be useful to update it to adapt to the alterations introduced by the Structural Code of 2021 (Annex 19, section 7.4), it is not essential to validate the core of the matter, which consists of

1. supporting the preference of establishing the approximation to the stiffness of bent bars based on steel reinforcement and depth, rather than based on corrections to the gross concrete section, and
2. obtaining the approximation to the deformed law of the cracked bar with variable reinforcement from the section decided by what we call the *design moment* or *design moment*.

B.2 Equivalent Stiffness

We first analyze how to estimate the stiffness of a section, subsequently extending the analysis to the entire member.

B.2.1 Section

We consider a section whose reinforcement A_s has been prescribed based on the anticipated moment of solicitation, M_d , and criteria that assign for the corresponding Ultimate Limit State (ULS) a defined depth, γd , to the concrete compression block. Under these conditions

$$\begin{aligned} A_s &= \frac{f_{cd}}{f_{sd}} \gamma b d \\ M_d &= f_{cd} \gamma \left(1 - \frac{\gamma}{2}\right) b d^2 \end{aligned} \tag{B.1}$$

The stiffness of the section for the use condition depends on its cracking situation. The conventional gross stiffness $K_b = E_c I_b$ does not consider the reinforcement and as an approximation is

excessively crude. More interesting is the cracked stiffness K_f , whose approximate expression as a function of steel area would simply be equal to the steel stiffness E_s by the reinforcement area A_s by the product of the compression-tension lever arm, by the tension depth—distance of the reinforcement to the neutral fiber—that is $K_f \approx E_s A_s \times 0.8h \times 0.6h \approx 0.5E_s A_s h^2$. Since the approximations to the lever arm and tension depth may seem crude, we will carefully review the problem.

Although we consider the compressed block rectangular in the Ultimate Limit State (ULS) analysis, in the Service Limit State (SLS) and in the analysis of stiffness implications, a triangular block with increasing stresses from the neutral line is more realistic, a compression block whose depth we now represent by χd . Under these conditions we have a markedly linear behavior in service where, by equalizing tensions and compressions, we will have

$$\begin{aligned} A_s &= \frac{1}{2} \frac{\varepsilon_c E_c}{\varepsilon_s E_s} \chi b d \\ M &= \frac{1}{2} \varepsilon_c E_c \chi \left(1 - \frac{\chi}{3}\right) b d^2 \end{aligned} \quad (\text{B.2})$$

with the additional condition that the deformed section is plane:

$$\frac{\varepsilon_c + \varepsilon_s}{d} = \frac{\varepsilon_c}{\chi d} \quad (\text{B.3})$$

Isolating χ in the first of equations B.2 and in B.3 and equating them, we will have

$$2 \frac{\varepsilon_s E_s}{\varepsilon_c E_c} \frac{A_s}{b d} = \frac{\varepsilon_c}{\varepsilon_c + \varepsilon_s} = \frac{1}{1 + \epsilon}$$

which can be written, using $\epsilon = \varepsilon_s/\varepsilon_c$

$$2\epsilon \frac{E_s}{E_c} \frac{As}{bd} = \frac{1}{1 + \epsilon}$$

a quadratic equation in ϵ whose explicit —positive— solution is

$$\epsilon = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{2E_c bd}{E_s A_s}}$$

in which we can substitute A_s which is known, according to [B.1](#):

$$\epsilon = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{2 E_c f_{sd}}{\gamma E_s f_{cd}}} \quad (\text{B.4})$$

As χ is deduced from [B.3](#) we can now obtain χ from γ and the materials used.

In this way, we know the potentially cracked region of concrete and can determine the section stiffness. The curvature in service is $c = (\varepsilon_c + \varepsilon_s)/d = \varepsilon_c(1 + \epsilon)/d$ for a moment M . Considering the expression [B.2](#) for M and recalling [B.1](#) for A_s , the stiffness can be expressed in either of the following ways,

$$\begin{aligned} K_f &= \frac{1}{2} \frac{\chi \left(1 - \frac{\chi}{3}\right)}{1 + \epsilon} E_c b d^3 = \psi_c E_c I_c \\ K_f &= \frac{1}{2} \frac{\chi \left(1 - \frac{\chi}{3}\right)}{1 + \epsilon} \frac{1}{\gamma} \frac{E_c f_{sd}}{E_s f_{cd}} E_s A_s d^2 = \psi_s E_s A_s h^2 \end{aligned} \quad (\text{B.5})$$

We see that the cracked stiffness can be expressed in terms of the concrete section or in terms of the steel section, through two

alternative coefficients ψ_c , ψ_s so that for quick and approximate evaluations in the project conception phase, the choice preferably will be the one corresponding to the least variability—or greatest stability—in said coefficient.

$$\begin{aligned}\psi_c &= 6\chi^2 \left(1 - \frac{\chi}{3}\right) \frac{d^2}{h^2} \\ \psi_s &= \frac{1}{2}\chi^2 \left(1 - \frac{\chi}{3}\right) \mathcal{N} \frac{d^2}{h^2}\end{aligned}\tag{B.6}$$

with

$$\begin{aligned}\chi &= \frac{1}{1 + \epsilon}; \\ \epsilon &= \frac{1}{2}(-1 + \sqrt{1 + 2\mathcal{N}}) \\ \mathcal{N} &= \frac{E_c f_{sd}}{\gamma E_s f_{cd}}\end{aligned}\tag{B.7}$$

The basic parameter is $\mathcal{N}$ which depends on general design decisions, namely, the selection of materials and safety criteria and the compressed depths corresponding to the reinforcements. The following tables clarify the panorama for usual safety criteria and a mechanical cover of one-tenth of the depth.

Values for ψ_c :

		γ		0.25		0.35		0.45	
BS	H	25	35	25	35	25	35	25	35
400		0.55	0.65	0.70	0.82	0.83	0.96		
500		0.47	0.55	0.60	0.70	0.72	0.83		

Values for ψ_s :

		γ		0.25		0.35		0.45	
BS	H	25	35	25	35	25	35	25	35
400		0.46	0.43	0.41	0.38	0.38	0.35		
500		0.48	0.46	0.44	0.41	0.41	0.38		

In the tables, the greater stability of ψ_s is verified.

It is usually established that the section stiffness is derived from a weighted average of the cracked and gross stiffness using the expression¹

$$K_e = \left(\frac{M_f}{M_a} \right)^3 K_b + \left(1 - \left(\frac{M_f}{M_a} \right)^3 \right) K_f \quad (\text{B.8})$$

dependent on the applied moments $M_a = M$ and section cracking $M_f = f_{ctm,fl} W_b$, a stiffness that is only slightly greater than the cracked stiffness in the analyzed cases, and on which we will not elaborate. In this way, the section stiffness will be the cracked stiffness, measured with

$$K_f = \psi_s E_s A_s h^2 \quad (\text{B.9})$$

B.2.2 Beam

It is understood that the stiffness of a reinforced concrete beam must integrate the varied stiffnesses of its sections. This is usually done through some type of integral—Branson—from which a weighted equivalent average can be deduced. The standards

1. The Structural Code, Annex 19, uses the exponent 2 in its expression 7.18 instead of the 3 in the classic formula.

accept simplifications in view of the usual reinforcement strategies, such as considering the section of dominant moment in statically determinate bars or, in cases of continuous bars with moment sign changes, considering the section of greatest moment for each region weighted by an approximation of the fraction of the span that the region occupies —conventionally considered as 25% of the span at support regions—.

We can characterize continuous beams with variable bending moment according to their proximity to canonical cases: fully fixed or fixed-supported. In both cases, the decision about section dimensions is adopted for an intermediate moment of the beam's moments—it is easy to consider the known as *plastic moment*— even though the reinforcement is carried out according to the analysis results. In such cases, we call *design moment* $M_{\mathcal{D}}$ the selected design value, or *design section* the one used to characterize the bar. The design decision involves a depth in the compressed block $\gamma_{\mathcal{D}}$ for this. The final reinforcements correspond to the design values at support(s) and span M_{ad} , M_{vd} , to which depths of the block γ_a , γ_v can be easily deduced

$$\begin{aligned}\gamma \left(1 - \frac{\gamma}{2}\right) &= \frac{M}{M_{\mathcal{D}}} \gamma_{\mathcal{D}} \left(1 - \frac{\gamma_{\mathcal{D}}}{2}\right) \\ \gamma &= 1 - \sqrt{1 - 2 \frac{M}{M_{\mathcal{D}}} \gamma_{\mathcal{D}} \left(1 - \frac{\gamma_{\mathcal{D}}}{2}\right)}\end{aligned}\quad (\text{B.10})$$

depth limited by $\gamma \leq 0,45$ after which compression reinforcement is added. Analogously, the resulting reinforcements for support and span will be those resulting from the support and span moments, and can be deduced from their relationship with

the one used for *design*, for the same section. That is, the depths, reinforcements, and support and span stiffnesses can be determined, and therefore the average stiffness of the bar, from the corresponding values to what we have called *design section*:

$$\begin{aligned}\alpha K_v + (1 - \alpha K_a) &= \alpha \psi_v E_s A_{sv} h^2 + (1 - \alpha) \psi_a E_s A_{sa} h^2 \\ K &= \psi_{eq} E_s A_{sD} h^2 \text{ with } \psi_{eq} = \alpha \psi_v \frac{A_{sv}}{A_{sD}} + (1 - \alpha) \psi_a \frac{A_{sa}}{A_{sD}}\end{aligned}\tag{B.11}$$

The following tables provide the coefficient ψ_{eq} corresponding to the equivalent average stiffness of the bar, a coefficient which, together with the reinforcement of the design section, characterizes said stiffness.

		BiFixed		FixedSupp.	
γ_D	BS H	25	35	25	35
0,25	400	0,45	0,42	0,45	0,42
	500	0,48	0,45	0,48	0,45
0,35	400	0,42	0,39	0,41	0,38
	500	0,45	0,42	0,44	0,41
0,45	400	0,39	0,36	0,38	0,35
	500	0,41	0,39	0,41	0,38

The previous results could be nuanced —increasing in value— if we consider that in bars with low depth —flat beams— we would need to account for greater stiffness in spans given the collaboration to positive moments of the compression layer, and considering that in beams with appreciable depth the cover is a smaller proportion than the one used for the tables. For all of

this, and for its simplicity, it seems reasonable to approximate the stiffness of the bar with the expression we had proposed—applied to the design section— of enormous interest for the phase corresponding to initial decisions

$$K_f = 0,5E_sA_s h^2 \quad (\text{B.12})$$

B.3 Beam Deflections

Once the average stiffness of the beam sections is established, it is elementary to address the limit state of deformation: the expression for the instantaneous deflection is simply

$$\delta = \frac{\alpha}{384} \frac{qsl^4}{K} \quad (\text{B.13})$$

for the load q per square meter applied to beams with span l at distances s and being α a coefficient that expresses the end conditions based on the end moments M_i , and M_d —negative—and the isostatic M_I :

$$\alpha = 5 + 3 \frac{M_i + M_d}{M_I} \quad (\text{B.14})$$

Sizing with strictly necessary steel

$$\delta = \frac{\alpha}{384} \frac{qsl^4}{0,5E_sA_s h^2} \quad \text{with } A_s \approx \frac{\beta qsl^2}{8} \frac{\gamma_f}{0,72hf_{sd}}$$

$$\frac{\delta}{l} \approx \frac{3}{100} \frac{\alpha}{\beta} \frac{f_{sd}}{\gamma_f E_s} \lambda = \frac{3}{100} \frac{\alpha}{\beta} \varepsilon_s \lambda \quad (\text{B.15})$$

an expression in which —to round off— a usual sizing criterion has been used $\gamma \approx 0,4d \implies z \approx 0,72h$ and being $\lambda = l/h$ the beam's slenderness, γ_f the partial safety coefficient for loads and $\beta = M_D/M_I$ the fraction between the moment used for the bar's design and the isostatic. Other reinforcement criteria would simultaneously alter the coefficients expressing ψ_{eq} and the depth, reducing the deflection for lower reinforcement ratios.

B.3.1 Active Deflection

The active deflection to be considered in housing use and considering standard construction methods is close to the instantaneous deflection for the total load, so we use the expression [B.15](#). The usually admitted approximate analysis implies discounting from the total accumulated deflection the instantaneous deflection of permanent loads plus the partitions' load, along with that part of the deferred loads corresponding to self-weight, partition wall and flooring, in the times elapsed since the formwork removal or the material's storage, respectively, until the partition finish.

B.4 Verification Conditions

The limit slenderness values that ensure that deformation limit conditions are not exceeded depend on these, and can be easily deduced from [B.15](#). For a limit of $\delta/l \leq 1/400$ and B400S steel, the following values are deduced —similar to those in EHE—

cantilever	6	double pinned	14
support–continuity	17	pinned–fixed	24
double continuity	22	double fixed	35

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