



Invariants for the Lagrangian equivalence problem



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ABSTRACT

Let M be a connected smooth manifold, let $\text{Aut}(p)$ be the group automorphisms of the bundle $p: \mathbb{R} \times M \rightarrow \mathbb{R}$, and let $q: J^1(\mathbb{R}, M) \times \mathbb{R} \rightarrow J^1(\mathbb{R}, M)$ be the canonical projection. Invariant functions on $J^r(q)$ under the natural action of $\text{Aut}(p)$ are discussed in relationship with the Lagrangian equivalence problem. The second-order invariants are determined geometrically as well as some other higher-order invariants for $\dim M \geq 2$.

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1. Introduction and preliminaries

1.1. Statement of the problem

Let M be a connected smooth manifold of dimension m . An automorphism of the projection $p: \mathbb{R} \times M \rightarrow \mathbb{R}$, $p(x, y) = x$, is a pair $\phi \in \text{Diff } \mathbb{R}$, $\Phi \in \text{Diff}(\mathbb{R} \times M)$ making the following diagram commutative:

$$\begin{array}{ccc} \mathbb{R} \times M & \xrightarrow{\Phi} & \mathbb{R} \times M \\ \downarrow p & & \downarrow p \\ \mathbb{R} & \xrightarrow{\phi} & \mathbb{R} \end{array}$$

Let $\text{Aut}(p)$ be the group of automorphisms of p . The diffeomorphism ϕ is completely determined by Φ ; hence, $\text{Aut}(p)$ is a subgroup in $\text{Diff}(\mathbb{R} \times M)$. We denote by $J^r(\mathbb{R}, M)$ the bundle of jets of order r of (local) sections of the submersion p and by $\Phi^{(r)}: J^r(\mathbb{R}, M) \rightarrow J^r(\mathbb{R}, M)$ the r -th jet prolongation of Φ , i.e.,

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$$\Phi^{(r)}(j_x^r \sigma) = j_{\phi(x)}^r (\Phi \circ \sigma \circ \phi^{-1}), \quad \forall j_x^r \sigma \in J^r(\mathbb{R}, M).$$

Let $\mathcal{L}, \bar{\mathcal{L}}: J^1(\mathbb{R}, M) \simeq \mathbb{R} \times TM \rightarrow \mathbb{R}$ be two first-order (time dependent) Lagrangians on M . The Lagrangians densities $\mathcal{L}dx, \bar{\mathcal{L}}dx$ are said to be equivalent if there exists $\Phi \in \text{Aut}(p)$ such that, $(\Phi^{(1)})^*(\mathcal{L}dx) = \bar{\mathcal{L}}dx$, i.e., $\bar{\mathcal{L}}dx = (\mathcal{L} \circ \Phi^{(1)})\phi^*(dx)$, or even,

$$\bar{\mathcal{L}} = \frac{d\phi}{dx} (\mathcal{L} \circ \Phi^{(1)}). \tag{1}$$

The equivalence problem is one of the basic questions in the Calculus of Variations and has been dealt with in several works. The notion of equivalence itself has different interpretations. Equivalence up to an automorphism of p (also known as *fiber preserving* equivalence) is probably the most natural one and is the approach we follow in this article, although there are other possibilities, such as equivalence under the group $\text{Diff}(\mathbb{R} \times M)$ (point transformations) or even larger groups. The usual tool used to solve the equivalence problem in the literature has been the Cartan method. This procedure is quite algorithmic but the computations become early too complex. However, partial results have been obtained: Equivalence of quadratic Lagrangians in [8] or [1] (with respect to the Euler-Lagrange equations in the last case), equivalence of Lagrangian for $M = \mathbb{R}$ in [5] and [9], or the equivalence problems for field theory Lagrangians defined in the plane \mathbb{R}^2 and $M = \mathbb{R}$ in [4]. Our results are obtained for an arbitrary manifold M , agreeing with the previous results in the particular case $M = \mathbb{R}$. For that, we follow a different method connected with the notion of invariant and infinitesimal transformations. In this paper, we obtain the complete determination of the set of invariants up to order two. The second order invariant is also exploited from a geometrical point of view to give some higher order invariants in connection with the notion of metric invariant. The intrinsic-geometric interpretation of all these objects sheds light to the problem of determination of a complete set of invariants of arbitrary degree.

We now introduce the notion of invariant. We first note that sections of the projection

$$\begin{aligned} q: J^1(\mathbb{R}, M) \times \mathbb{R} &\rightarrow J^1(\mathbb{R}, M), \\ q(j_x^1 \sigma, \lambda) &= j_x^1 \sigma, \end{aligned}$$

correspond bijectively with functions in $C^\infty(J^1(\mathbb{R}, M))$, i.e., with first-order Lagrangians on the fibred manifold $p: \mathbb{R} \times M \rightarrow \mathbb{R}$. The section $s_{\mathcal{L}}: J^1(\mathbb{R}, M) \rightarrow J^1(\mathbb{R}, M) \times \mathbb{R}$ of q corresponding to $\mathcal{L} \in C^\infty(J^1(\mathbb{R}, M))$ is given by $s_{\mathcal{L}}(j_x^1 \sigma) = (j_x^1 \sigma, \mathcal{L}(j_x^1 \sigma))$. In what follows $s_{\mathcal{L}}$ and \mathcal{L} will be identified.

Let $\tilde{\Phi}^{(1)}: J^1(\mathbb{R}, M) \times \mathbb{R} \rightarrow J^1(\mathbb{R}, M) \times \mathbb{R}$ be the automorphism of q defined as follows:

$$\tilde{\Phi}^{(1)}(j_x^1 \sigma, \lambda) = (\Phi^{(1)}(j_x^1 \sigma), (\phi')^{-1} \lambda), \tag{2}$$

where $\phi' = d\phi/dx$. If (Ψ, ψ) is another automorphism of p , then

$$((\Phi \circ \Psi)^* dx)_x = \Psi^*(\Phi^* dx)_x = \psi^*(\phi' dx)_x = \phi' \psi'(dx)_x.$$

Hence

$$\begin{aligned} \tilde{\Phi}^{(1)}\left(\tilde{\Psi}^{(1)}(j_x^1 \sigma, \lambda)\right) &= \tilde{\Phi}^{(1)}(\Psi^{(1)}(j_x^1 \sigma), (\psi')^{-1} \lambda) \\ &= \left(\Phi^{(1)}\left(\Psi^{(1)}(j_x^1 \sigma)\right), (\phi')^{-1}(\psi')^{-1} \lambda\right) \\ &= \left((\Phi \circ \Psi)^{(1)}(j_x^1 \sigma), (\phi' \psi')^{-1} \lambda\right). \end{aligned}$$

In other words, $(\widetilde{\Phi \circ \Psi})^{(1)} = \tilde{\Phi}^{(1)} \circ \tilde{\Psi}^{(1)}$.

1.2. Differential invariants

Proposition 1.1. *With the preceding notations, $((\Phi^{-1})^{(1)})^*(\mathcal{L}dx) = \bar{\mathcal{L}}dx$ for $\Phi \in \text{Aut}(p)$ if and only if, $\tilde{\Phi}^{(1)} \circ s_{\mathcal{L}} \circ (\Phi^{(1)})^{-1} = s_{\bar{\mathcal{L}}}$. Accordingly, if the Lagrangian densities $\mathcal{L}dx$ and $\bar{\mathcal{L}}dx$ are equivalent, then every smooth function $I: J^r(q) \rightarrow \mathbb{R}$ which is invariant under the subgroup of $\text{Aut}(q)$ of automorphisms of the form $(\tilde{\Phi}^{(1)}, \Phi^{(1)})$, $\Phi \in \text{Aut}(p)$, takes the same value on the two r -jets of the sections $s_{\mathcal{L}}$ and $s_{\bar{\mathcal{L}}}$; namely,*

$$I\left(j_{j_x^1 \sigma}^r s_{\mathcal{L}}\right) = I\left(j_{j_x^1 \sigma}^r s_{\bar{\mathcal{L}}}\right), \quad \forall j_x^1 \sigma \in J^1(\mathbb{R}, M).$$

Therefore, invariant functions provide a method to distinguish non-equivalent Lagrangian densities.

Proof. According to (1), the equation $((\Phi^{-1})^{(1)})^*(\mathcal{L}dx) = \bar{\mathcal{L}}dx$ is equivalent to $\bar{\mathcal{L}} = (\phi')^{-1}(\mathcal{L} \circ (\Phi^{-1})^{(1)})$. We have

$$\begin{aligned} \left(\tilde{\Phi}^{(1)} \circ s_{\mathcal{L}} \circ (\Phi^{(1)})^{-1}\right)(j_x^1 \sigma) &= \left(\tilde{\Phi}^{(1)} \circ s_{\mathcal{L}}\right)\left((\Phi^{(1)})^{-1}(j_x^1 \sigma)\right) \\ &= \tilde{\Phi}^{(1)}\left(\left((\Phi^{(1)})^{-1}(j_x^1 \sigma), \mathcal{L}\left((\Phi^{-1})^{(1)}(j_x^1 \sigma)\right)\right)\right) \\ &= \left(j_x^1 \sigma, (\phi')^{-1}\left(\mathcal{L} \circ (\Phi^{-1})^{(1)}(j_x^1 \sigma)\right)\right) \\ &= s_{\bar{\mathcal{L}}}(j_x^1 \sigma). \quad \square \end{aligned}$$

The approach we follow for the determination of the invariant functions will be through the infinitesimal representation of projectable vector fields X of $\mathbb{R} \times M \rightarrow \mathbb{R}$ (resp. vector fields X that are vertical with respect to the projection $\mathbb{R} \times M \rightarrow \mathbb{R}$), that is, we will study the functions $I: J^r(q) \rightarrow \mathbb{R}$ such that $(\tilde{X}^{(1)})^{(r)}(I) = 0$, for $r \leq 2$, according to the notation of the next section. In principle, these functions will only be invariant under the connected component of the identity $\text{Aut}(p)_0$ (resp. $\text{Aut}^v(p)_0$ where $\text{Aut}^v(p)$ is the subgroup of vertical automorphisms, that is, automorphisms inducing the identity on \mathbb{R}), so that the number of independent invariant functions under the full group $\text{Aut}(p)$ (resp. $\text{Aut}^v(p)$) is less or equal to those obtained by that infinitesimal representation. However, we will conclude the main result by providing explicitly enough functions that are invariant under $\text{Aut}(p)$ (resp. $\text{Aut}^v(p)$).

1.3. Zero-order invariants

Let

$$X = u \frac{\partial}{\partial x} + v^\alpha \frac{\partial}{\partial y^\alpha}, \quad u \in C^\infty(\mathbb{R}), \quad v^\alpha \in C^\infty(\mathbb{R} \times M), \tag{3}$$

be the local expression of a p -projectable vector field on a fibred coordinate system (x, y^α) , $y^\alpha \in C^\infty(\mathbb{R} \times M)$, $1 \leq \alpha \leq m$, for the natural projection $p: \mathbb{R} \times M \rightarrow \mathbb{R}$, where we have used the Einstein summation convention. We have (cf. [2], [7, 2.1.1])

$$\begin{cases} X^{(1)} = u \frac{\partial}{\partial x} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_1^\alpha \frac{\partial}{\partial \dot{y}^\alpha}, \\ v_1^\alpha = \frac{\partial v^\alpha}{\partial x} - \frac{du}{dx} \dot{y}^\alpha + \frac{\partial v^\alpha}{\partial y^\beta} \dot{y}^\beta, \end{cases} \tag{4}$$

where $(x, y^\alpha, \dot{y}^\alpha)$ is the coordinate system on $J^1(\mathbb{R}, M)$ induced from (x, y^α) , $1 \leq \alpha \leq m$; i.e., $\dot{y}^\alpha(j_{x_0}^1 \sigma) = (d(y^\alpha \circ \sigma)/dx)(x_0)$.

Let us compute $\tilde{X}^{(1)}$. Let $z: J^1(\mathbb{R}, M) \times \mathbb{R} \rightarrow \mathbb{R}$ be the projection onto the second factor, i.e., $z(j_x^1 \sigma, \lambda) = \lambda$. The functions $(x, y^\alpha, \dot{y}^\alpha, z)$ are a system of coordinates on $J^1(\mathbb{R}, M) \times \mathbb{R}$. If X is the infinitesimal

generator of a pair of one-parameter groups $\phi_t \in \text{Diff } \mathbb{R}$, $\Phi_t \in \text{Diff}(\mathbb{R} \times M)$, then according to (2) we have $z \circ \tilde{\Phi}^{(1)} = z / \frac{d\phi_t}{dx}$. Hence

$$\tilde{X}^{(1)}(z) = z \frac{\partial}{\partial t} \Big|_{t=0} \left(\frac{d\Phi_t}{dx} \right)^{-1} = -z \frac{\partial}{\partial t} \Big|_{t=0} \left(\frac{d\Phi_t}{dx} \right) = -\frac{du}{dx} z,$$

and therefore $\tilde{X}^{(1)} = u \frac{\partial}{\partial x} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_1^\alpha \frac{\partial}{\partial \dot{y}^\alpha} - \frac{du}{dx} z \frac{\partial}{\partial z}$.

Let \mathcal{D}^0 be the involutive distribution on $J^1(\mathbb{R}, M) \times \mathbb{R}$ generated by the vector fields $\tilde{X}^{(1)}$, X being an arbitrary p -projectable vector field. The first integrals of \mathcal{D}^0 are the zero-order invariant functions. Taking the formulas (4) into account, we have

$$v_1^\alpha(j_x^1\sigma) = \frac{\partial v^\alpha}{\partial x}(\sigma(x)) + \frac{\partial(y^\beta \circ \sigma)}{\partial x}(x) \frac{\partial v^\alpha}{\partial y^\beta}(\sigma(x)) - \frac{\partial(y^\alpha \circ \sigma)}{\partial x}(x) \frac{du}{dx}(x),$$

and evaluating $\tilde{X}^{(1)}$ at a point $(j_x^1\sigma, \lambda)$,

$$\begin{aligned} \left(\tilde{X}^{(1)} \right)_{(j_x^1\sigma, \lambda)} &= u(x) \left(\frac{\partial}{\partial x} \right)_{(j_x^1\sigma, \lambda)} + v^\alpha(\sigma(x)) \left(\frac{\partial}{\partial y^\alpha} \right)_{(j_x^1\sigma, \lambda)} \\ &+ \frac{\partial v^\alpha}{\partial x}(\sigma(x)) \left(\frac{\partial}{\partial \dot{y}^\alpha} \right)_{(j_x^1\sigma, \lambda)} + \frac{\partial v^\alpha}{\partial y^\beta}(\sigma(x)) \frac{\partial(y^\beta \circ \sigma)}{\partial x}(x) \left(\frac{\partial}{\partial \dot{y}^\alpha} \right)_{(j_x^1\sigma, \lambda)} \\ &- \lambda \frac{du}{dx}(x) \left(\frac{\partial}{\partial z} \right)_{(j_x^1\sigma, \lambda)} - \frac{du}{dx}(x) \frac{\partial(y^\alpha \circ \sigma)}{\partial x}(x) \left(\frac{\partial}{\partial \dot{y}^\alpha} \right)_{(j_x^1\sigma, \lambda)} \\ &= u(x) \left(\frac{\partial}{\partial x} \right)_{(j_x^1\sigma, \lambda)} + v^\alpha(\sigma(x)) \left(\frac{\partial}{\partial y^\alpha} \right)_{(j_x^1\sigma, \lambda)} + \frac{\partial v^\alpha}{\partial x}(\sigma(x)) \left(\frac{\partial}{\partial \dot{y}^\alpha} \right)_{(j_x^1\sigma, \lambda)} \\ &+ \frac{\partial v^\alpha}{\partial y^\beta}(\sigma(x)) \left(\dot{y}^\beta \frac{\partial}{\partial \dot{y}^\alpha} \right)_{(j_x^1\sigma, \lambda)} - \frac{du}{dx}(x) \left(z \frac{\partial}{\partial z} + \dot{y}^\alpha \frac{\partial}{\partial \dot{y}^\alpha} \right)_{(j_x^1\sigma, \lambda)}. \end{aligned}$$

As the values $u(x)$, $(du/dx)(x)$, $v^\alpha(\sigma(x))$, $(\partial v^\alpha/\partial x)(\sigma(x))$, $(\partial v^\alpha/\partial y^\beta)(\sigma(x))$ are arbitrary, it follows that the vector fields $\partial/\partial x$, $\partial/\partial y^\alpha$, $\partial/\partial \dot{y}^\alpha$, $z\partial/\partial z$ are a local basis of the distribution \mathcal{D}^0 on the dense open subset $z \neq 0$; the generic rank of \mathcal{D}^0 thus coincides with the dimension of $J^1(\mathbb{R}, M) \times \mathbb{R}$; hence the only zero-order invariants are the constants.

1.4. Jet-prolongation formulas

Let $q: N \times \mathbb{R} \rightarrow N$ be the canonical projection onto the first factor, where N is a smooth manifold of dimension ν with local coordinates (t^1, \dots, t^ν) .

Let $z: N \times \mathbb{R} \rightarrow \mathbb{R}$ be the projection onto the second factor. The lift by infinitesimal contact transformation to the jet bundle $J^r(q)$ of a q -projectable vector field on $N \times \mathbb{R}$ with local expression

$$Y = \xi^a(t^1, \dots, t^\nu) \frac{\partial}{\partial t^a} + \eta(t^1, \dots, t^\nu, z) \frac{\partial}{\partial z},$$

is given by (cf. [2], [10]),

$$Y^{(r)} = \xi^a(t^1, \dots, t^\nu) \frac{\partial}{\partial t^a} + \eta(t^1, \dots, t^\nu, z) \frac{\partial}{\partial z} + \sum_{1 \leq |I| \leq r} \eta_I \frac{\partial}{\partial z_I}, \tag{5}$$

where $I = (i_1, \dots, i_\nu) \in \mathbb{N}^\nu$ is a multi-index of order $|I| = i_1 + \dots + i_\nu$, (t^a, z_I) , $1 \leq a \leq \nu$, $|I| \leq r$, is the coordinate system induced on $J^r(q)$, namely, $z_0 = z$, $z_I(j_\zeta^r \mathcal{L}) = (\partial^{|I|} \mathcal{L} / \partial t^I)(\zeta)$, $\mathcal{L} \in C^\infty(N)$, and the function η_I is defined as follows:

$$\eta_I = \mathbb{D}^I (\eta - \xi^a z_a) + \xi^a z_{I+(a)}, \tag{6}$$

with $(a) = (0, \dots, \overset{(a)}{1}, \dots, 0)$, $1 \leq a \leq \nu$, $\mathbb{D}_{t^a} = \frac{\partial}{\partial t^a} + \sum_{|I|=0}^\infty z_{I+(a)} \frac{\partial}{\partial z_I}$ denotes the total derivative with respect to the coordinate t^a and the operator \mathbb{D}^I is given by,

$$\mathbb{D}^I = (\mathbb{D}_{t^1})^{i_1} \circ \dots \circ (\mathbb{D}_{t^\nu})^{i_\nu}.$$

By using Leibnitz’s formula for the r -th derivative of a product, the formula (6) transforms into the following:

$$\eta_I = \mathbb{D}^I(\eta) - \sum_{J < I} \binom{I}{J} \frac{\partial^{|I-J|} \xi^a}{\partial t^{I-J}} z_{J+(a)}. \tag{7}$$

In the case we are dealing with, $N = J^1(\mathbb{R}, M)$, $\nu = 1 + 2m$, $Y = \tilde{X}^{(1)}$, and

$$\left. \begin{aligned} (t^a)_{a=1}^\nu &= (x, y^\alpha, \dot{y}^\alpha), \\ ((\xi^a)_{a=1}^\nu; \eta) &= (u, v^\alpha, v_1^\alpha; -\frac{du}{dx} z), \end{aligned} \right\} 1 \leq \alpha \leq m,$$

$u \in C^\infty(\mathbb{R})$, $v^\alpha \in C^\infty(\mathbb{R} \times M)$, and v_1^α is given in (4). Next, instead of the general formulas above, we use the following ones for $J^1(q)$ in our particular case: $(x, y^\alpha, \dot{y}^\alpha, z, z_x, z_{y^\alpha}, z_{\dot{y}^\alpha})$, $1 \leq \alpha \leq m$.

2. First-and second-order invariants

2.1. First order

First of all, let us compute $(\tilde{X}^{(1)})^{(1)}$. By applying (5) to $Y = \tilde{X}^{(1)}$ for $r = 1$, we have

$$\begin{aligned} (\tilde{X}^{(1)})^{(1)} &= u \frac{\partial}{\partial x} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_1^\alpha \frac{\partial}{\partial \dot{y}^\alpha} - \frac{du}{dx} z \frac{\partial}{\partial z} + A \frac{\partial}{\partial z_x} + B_\alpha \frac{\partial}{\partial z_{y^\alpha}} + C_\alpha \frac{\partial}{\partial z_{\dot{y}^\alpha}}, \\ A &= -\frac{d^2 u}{dx^2} z - 2 \frac{du}{dx} z_x - \frac{\partial v^\alpha}{\partial x} z_{y^\alpha} - \frac{\partial^2 v^\alpha}{\partial x^2} z_{\dot{y}^\alpha} + \frac{d^2 u}{dx^2} \dot{y}^\alpha z_{\dot{y}^\alpha} - \frac{\partial^2 v^\alpha}{\partial x \partial y^\beta} \dot{y}^\beta z_{\dot{y}^\alpha}, \end{aligned} \tag{8}$$

$$B_\alpha = -\frac{du}{dx} z_{y^\alpha} - \frac{\partial v^\beta}{\partial y^\alpha} z_{y^\beta} - \left(\frac{\partial^2 v^\beta}{\partial x \partial y^\alpha} + \frac{\partial^2 v^\beta}{\partial y^\alpha \partial y^\gamma} \dot{y}^\gamma \right) z_{\dot{y}^\beta}, \tag{9}$$

$$C_\alpha = -\frac{\partial v^\beta}{\partial y^\alpha} z_{\dot{y}^\beta}. \tag{10}$$

Hence

$$\begin{aligned} (\tilde{X}^{(1)})^{(1)} &= u \frac{\partial}{\partial x} + v^\alpha \frac{\partial}{\partial y^\alpha} \\ &\quad - \frac{du}{dx} \left(z \frac{\partial}{\partial z} + \dot{y}^\alpha \frac{\partial}{\partial \dot{y}^\alpha} + 2z_x \frac{\partial}{\partial z_x} + z_{y^\alpha} \frac{\partial}{\partial z_{y^\alpha}} \right) \\ &\quad + \frac{\partial v^\alpha}{\partial x} \left(\frac{\partial}{\partial \dot{y}^\alpha} - z_{y^\alpha} \frac{\partial}{\partial z_x} \right) + \frac{\partial v^\alpha}{\partial y^\beta} \left(\dot{y}^\beta \frac{\partial}{\partial \dot{y}^\alpha} - z_{y^\alpha} \frac{\partial}{\partial z_{y^\beta}} - z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{\dot{y}^\beta}} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{d^2 u}{dx^2} (-z + \dot{y}^\gamma z_{\dot{y}^\gamma}) \frac{\partial}{\partial z_x} - \frac{\partial^2 v^\alpha}{\partial x^2} z_{\dot{y}^\alpha} \frac{\partial}{\partial z_x} \\
& - \frac{\partial^2 v^\beta}{\partial x \partial y^\alpha} \left(\dot{y}^\alpha z_{\dot{y}^\beta} \frac{\partial}{\partial z_x} + z_{\dot{y}^\beta} \frac{\partial}{\partial z_{y^\alpha}} \right) \\
& - \sum_{\alpha \leq \beta} \frac{\partial^2 v^\gamma}{\partial y^\alpha \partial y^\beta} \frac{1}{1 + \delta_{\alpha\beta}} z_{\dot{y}^\gamma} \left(\dot{y}^\beta \frac{\partial}{\partial z_{y^\alpha}} + \dot{y}^\alpha \frac{\partial}{\partial z_{y^\beta}} \right),
\end{aligned}$$

and the distribution \mathcal{D}^1 generated by all the vector fields $(\tilde{X}^{(1)})^{(1)}$ on $J^1(q)$ is spanned by $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y^\alpha}$, and the following vector fields:

$$\begin{aligned}
\chi &= z \frac{\partial}{\partial z} + \dot{y}^\alpha \frac{\partial}{\partial \dot{y}^\alpha} + 2z_x \frac{\partial}{\partial z_x} + z_{y^\alpha} \frac{\partial}{\partial z_{y^\alpha}}, \\
\chi_\alpha &= \frac{\partial}{\partial \dot{y}^\alpha} - z_{y^\alpha} \frac{\partial}{\partial z_x}, \quad 1 \leq \alpha \leq m, \\
\chi_\alpha^\beta &= \dot{y}^\beta \frac{\partial}{\partial \dot{y}^\alpha} - z_{y^\alpha} \frac{\partial}{\partial z_{y^\beta}} - z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{\dot{y}^\beta}}, \quad \alpha, \beta = 1, \dots, m, \\
\chi' &= (-z + \dot{y}^\gamma z_{\dot{y}^\gamma}) \frac{\partial}{\partial z_x}, \\
\bar{\chi}_\alpha &= z_{\dot{y}^\alpha} \frac{\partial}{\partial z_x}, \quad 1 \leq \alpha \leq m, \\
\bar{\chi}_\beta^\alpha &= z_{\dot{y}^\beta} \left(\dot{y}^\alpha \frac{\partial}{\partial z_x} + \frac{\partial}{\partial z_{y^\alpha}} \right), \quad \alpha, \beta = 1, \dots, m, \\
\chi_{\alpha\beta}^\gamma &= \frac{1}{1 + \delta_{\alpha\beta}} z_{\dot{y}^\gamma} \left(\dot{y}^\beta \frac{\partial}{\partial z_{y^\alpha}} + \dot{y}^\alpha \frac{\partial}{\partial z_{y^\beta}} \right), \quad 1 \leq \alpha \leq \beta \leq m, 1 \leq \gamma \leq m.
\end{aligned}$$

Let us fix two indices α_0, β_0 . From the formulas above, on the dense open subset $U(\alpha_0, \beta_0) = \{z \neq 0, \dot{y}^{\beta_0} \neq 0, z_{\dot{y}^{\alpha_0}} \neq 0, z_{\dot{y}^{\beta_0}} \neq 0\} \subset J^1(q)$, we have

$$\begin{aligned}
\frac{\partial}{\partial z_x} &= \frac{1}{z_{\dot{y}^{\alpha_0}}} \bar{\chi}_{\alpha_0}, \\
\frac{\partial}{\partial \dot{y}^\alpha} &= \frac{1}{z_{\dot{y}^{\alpha_0}}} (z_{\dot{y}^{\alpha_0}} \chi_\alpha + z_{y^\alpha} \bar{\chi}_{\alpha_0}), \\
\frac{\partial}{\partial z_{y^\alpha}} &= \frac{1}{z_{\dot{y}^{\beta_0}}} (\bar{\chi}_{\beta_0}^\alpha - \dot{y}^\alpha \bar{\chi}_{\beta_0}), \\
\frac{\partial}{\partial z_{\dot{y}^\beta}} &= \frac{1}{z_{\dot{y}^{\alpha_0}} z_{\dot{y}^{\beta_0}}} \left[\dot{y}^\beta (z_{\dot{y}^{\beta_0}} \chi_{\alpha_0} + z_{y^{\alpha_0}} \bar{\chi}_{\beta_0}) - z_{y^{\alpha_0}} \bar{\chi}_{\beta_0}^\beta - z_{\dot{y}^{\beta_0}} \chi_{\alpha_0}^\beta + z_{y^{\alpha_0}} \dot{y}^\beta \bar{\chi}_{\beta_0} \right], \\
\frac{\partial}{\partial z} &= \frac{1}{z} (\chi - \dot{y}^\alpha \chi_\alpha) - \frac{1}{z z_{\dot{y}^{\beta_0}}} (2z_x \bar{\chi}_{\beta_0} + z_{y^\alpha} \bar{\chi}_{\beta_0}^\alpha),
\end{aligned}$$

thus proving that the rank of \mathcal{D}^1 on $U(\alpha_0, \beta_0)$ is equal to the dimension of the tangent space to $J^1(q)$ at each point. Hence the only first-order differential invariants are the constants.

2.2. The Hessian metric

Let $p^r: J^r(\mathbb{R}, M) \rightarrow \mathbb{R}$, $p^{r,r'}: J^r(\mathbb{R}, M) \rightarrow J^{r'}(\mathbb{R}, M)$, $r > r'$, be the canonical projections of the jet bundles of $p: \mathbb{R} \times M \rightarrow \mathbb{R}$.

The map $p^{10}: J^1(\mathbb{R}, M) \rightarrow J^0(\mathbb{R}, M) = \mathbb{R} \times M$ is an affine bundle modeled over the vector bundle $W = p^*T^*\mathbb{R} \otimes V(p) \cong \mathbb{R} \times TM$. Hence each fibre $F_{x,y} = (p^{10})^{-1}(x, y)$, $(x, y) \in \mathbb{R} \times M$, is an affine space modeled over T_yM . We set $\mathcal{L}^{x,y} = \mathcal{L}|_{F_{x,y}}$, $\forall \mathcal{L} \in C^\infty(J^1(\mathbb{R}, M))$.

Let A be a real affine space of finite dimension modeled over a real vector space V , endowed with its canonical C^∞ structure. Every vector $v \in V$ induces a vector field $\tilde{v} \in \mathfrak{X}(A)$ given by,

$$\tilde{v}_x(f) = \left. \frac{d}{dt} \right|_{t=0} f(x + tv), \quad \forall x \in A, \forall f \in C^\infty(A).$$

If v_1, \dots, v_n is a basis for V , then $\tilde{v}_1, \dots, \tilde{v}_n$ is a basis for the $C^\infty(A)$ -module $\mathfrak{X}(A)$.

There exists a unique linear connection D^A on A such that, $D^A\tilde{v} = 0$, $\forall v \in V$. This connection is symmetric and flat.

The image of a vector field $X \in \mathfrak{X}(M)$ by a diffeomorphism $\varphi: M \rightarrow M'$ is the vector field $\varphi \cdot X \in \mathfrak{X}(M')$ defined as follows: $(\varphi \cdot X)_{x'} = \varphi_*(X_{\varphi^{-1}(x')})$, $\forall x' \in M'$.

If D is a linear connection on M , then $\varphi \cdot D$ denotes the linear connection on M' defined by the following formula:

$$(\varphi \cdot D)_{X'Y'} = \varphi \cdot (D_{\varphi^{-1}X'}(\varphi^{-1} \cdot Y')), \quad \forall X', Y' \in \mathfrak{X}(M').$$

If ω is a 1-form on an affine space A , then $D^A\omega$ is the covariant tensor of degree 2 given by, $(D^A\omega)(X, Y) = (D_X^A\omega)(Y) = X(\omega(Y)) - \omega(D_X^AY)$, $\forall X, Y \in \mathfrak{X}(A)$.

Lemma 2.1. *With the previous notations and definitions, for every isomorphism of affine spaces $\alpha: A \rightarrow A'$ and every 1-form ω' on A' the following formulas hold:*

$$\alpha \cdot D^A = D^{A'}, \quad \alpha^*(D^{A'}\omega') = D^A(\alpha^*\omega').$$

Proof. Actually, from the very definition of an affine morphism there exists a linear isomorphism $\widetilde{\alpha}: V \rightarrow V'$ such that, $\alpha(v + a) = \widetilde{\alpha}(v) + \alpha(a)$, $\forall a \in A, \forall v \in V$, and we have $\widetilde{\alpha}(v) = \alpha \cdot \tilde{v}$, as follows from the next equalities:

$$\begin{aligned} (\alpha \cdot \tilde{v})_{x'} f' &= [\alpha_*(\tilde{v}_{\alpha^{-1}(x')})](f') \\ &= \tilde{v}_{\alpha^{-1}(x')}(f' \circ \alpha) \\ &= \lim_{t \rightarrow 0} \frac{(f' \circ \alpha)(\alpha^{-1}(x') + tv) - f'(x')}{t} \\ &= \lim_{t \rightarrow 0} \frac{f'(x' + t\widetilde{\alpha}(v)) - f'(x')}{t} \\ &= \widetilde{\alpha}(v)_{x'}(f'). \end{aligned}$$

By writing $u = \widetilde{\alpha}^{-1}(u')$, $v = \widetilde{\alpha}^{-1}(v')$, for all $u', v' \in V'$, we obtain

$$\begin{aligned} (\alpha \cdot D^A)_{u'}(\widetilde{v}') &= (\alpha \cdot D^A)_{\widetilde{\alpha}(u)}(\widetilde{\alpha}(v)) \\ &= (\alpha \cdot D^A)_{\alpha \cdot \tilde{u}}(\alpha \cdot \tilde{v}) \\ &= 0. \end{aligned}$$

If $X = \tilde{u}$, $Y = \tilde{v}$, then $(D^A\omega)(\tilde{u}, \tilde{v}) = \tilde{u}(\omega(\tilde{v}))$, by virtue of the definition of the connection D^A and we deduce

$$\begin{aligned} (\alpha^*(D^A\omega'))_a(\tilde{u}, \tilde{v}) &= (D^A\omega')_{\alpha(a)}(\alpha_*(\tilde{u}_a), \alpha_*(\tilde{v}_a)) \\ &= [(\alpha \cdot D^A)(\omega')]_{\alpha(a)}((\alpha \cdot \tilde{u})_{\alpha(a)}, (\alpha \cdot \tilde{v})_{\alpha(a)}) \\ &= [(\alpha \cdot D^A)(\omega')_{\alpha \cdot \tilde{u}}(\alpha \cdot \tilde{v})](\alpha(a)) \\ &= (\alpha \cdot \tilde{u})_{\alpha(a)}(\omega'(\alpha \cdot \tilde{v})) - \omega'(\alpha \cdot (D^A_{\tilde{u}}\tilde{v}))(\alpha(a)) \\ &= (\alpha \cdot \tilde{u})_{\alpha(a)}(\omega'(\alpha \cdot \tilde{v})) \\ &= \tilde{u}_a[\omega'(\alpha \cdot \tilde{v}) \circ \alpha] \\ &= \tilde{u}[\omega'(\alpha \cdot \tilde{v}) \circ \alpha](a) \\ &= \tilde{u}[(\alpha^*\omega')(\tilde{v})](a) \\ &= [D^A(\alpha^*\omega')]_a(\tilde{u}, \tilde{v}), \end{aligned}$$

thus allowing one to conclude the proof. \square

The Hessian metric of a function $\mathcal{L} \in C^\infty(J^1(\mathbb{R}, M))$ is the section of the vector bundle $\varkappa: S^2[V^*(p^{10})] \rightarrow J^1(\mathbb{R}, M)$ defined as follows (cf. [11, Definition 2.1]):

$$\text{Hess}_{j_x^1\sigma}(\mathcal{L}) = D^{F_{x,\sigma(x)}}(d\mathcal{L}^{x,\sigma(x)}). \tag{11}$$

In local coordinates, $\text{Hess}(\mathcal{L}) = \frac{\partial^2 \mathcal{L}}{\partial \dot{y}^\alpha \partial \dot{y}^\beta} d_{10}\dot{y}^\alpha \otimes d_{10}\dot{y}^\beta$, where $d_{10}f = df|_{V(p^{10})}$.

For every $\Phi \in \text{Diff}(\mathbb{R} \times M)$, the diffeomorphism $\Phi^{(1)}: J^1(\mathbb{R}, M) \rightarrow J^1(\mathbb{R}, M)$ transforms $V(p^{10})$ into itself because of the commutativity of the following diagram:

$$\begin{array}{ccc} J^1(\mathbb{R}, M) & \xrightarrow{\Phi^{(1)}} & J^1(\mathbb{R}, M) \\ \downarrow p^{10} & & \downarrow p^{10} \\ \mathbb{R} \times M & \xrightarrow{\Phi} & \mathbb{R} \times M \end{array}$$

Hence for every $\mathcal{L} \in C^\infty(J^1(\mathbb{R}, M))$ the inverse image of the Hessian metric $(\Phi^{(1)})^*\text{Hess}(\mathcal{L})$ is another section of $S^2[V^*(p^{10})] \rightarrow J^1(\mathbb{R}, M)$.

Proposition 2.2. *For every $\Phi \in \text{Aut}(p)$ and every $\mathcal{L} \in C^\infty(J^1(\mathbb{R}, M))$, we have*

$$(\Phi^{(1)})^*\text{Hess}(\mathcal{L}) = \text{Hess}(\mathcal{L} \circ \Phi^{(1)}).$$

Proof. If $\Phi(x, y) = (\varphi(x), \Psi(x, y))$, $\forall (x, y) \in \mathbb{R} \times M$, then, taking account of the fact that the affine bundle $J^1(\mathbb{R}, M) \rightarrow \mathbb{R} \times M$ is modeled over the vector bundle $\mathbb{R} \times TM \rightarrow \mathbb{R} \times M$ it follows that $\Phi^{(1)}: J^1(\mathbb{R}, M) \rightarrow J^1(\mathbb{R}, M)$ is an affine morphism whose associated linear morphism $\vec{\Phi}^{(1)}: \mathbb{R} \times TM \rightarrow \mathbb{R} \times TM$ is given by, $\vec{\Phi}^{(1)}(x, v) = (x, (\Psi_x)_*v)$, $\forall v \in T_yM$, and the statement is a consequence of Lemma 2.1, as

$$\begin{aligned} (\Phi^{(1)})^*\text{Hess}(\mathcal{L}) &= (\Phi^{(1)})^*D^{F_{x,\sigma(x)}}(d\mathcal{L}^{x,\sigma(x)}) \\ &= D^{F_{x,\sigma(x)}}((\Phi^{(1)})^*d\mathcal{L}^{x,\sigma(x)}) \end{aligned}$$

$$\begin{aligned} &= D^{F_{x,\sigma(x)}} d \left((\Phi^{(1)})^* \mathcal{L}^{x,\sigma(x)} \right) \\ &= D^{F_{x,\sigma(x)}} d \left(\mathcal{L}^{x,\sigma(x)} \circ \Phi^{(1)} \right) \\ &= \text{Hess}(\mathcal{L} \circ \Phi^{(1)}). \quad \square \end{aligned}$$

2.3. Second order

2.3.1. The basic invariant

Let $O^2 \subset J^2(q)$ be the dense open subset of elements $j_{j_{x_0}^1 \sigma_0}^2(\mathcal{L})$ for which the Hessian metric $\text{Hess}_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}$ is non-singular. In coordinates, O^2 is defined by the inequation $\det(z_{j^\alpha j^\beta})_{\alpha,\beta=1}^m \neq 0$. Hence, for every $j_{j_{x_0}^1 \sigma_0}^2(\mathcal{L}) \in O^2$ the linear mapping

$$\begin{aligned} \text{Hess}_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}^\flat: V_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}(p^{10}) &\rightarrow V_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}^*(p^{10}), \\ \text{Hess}_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}^\flat(X)(Y) &= \text{Hess}_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}(X, Y), \end{aligned}$$

is an isomorphism, the inverse of which is denoted by

$$\text{Hess}_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}^\sharp: V_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}^*(p^{10}) \rightarrow V_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}(p^{10}).$$

A contravariant metric in $S^2 V_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}(p^{10})$ is then defined as follows:

$$\sharp \text{Hess}_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}(w_1, w_2) = \text{Hess}_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}(\text{Hess}_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}^\sharp(w_1), \text{Hess}_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}^\sharp(w_2)),$$

for all $w_1, w_2 \in V_{j_{j_{x_0}^1 \sigma_0}^1(\mathcal{L})}^*(p^{10})$.

Proposition 2.3. *With the same notations as above, let $V: O^2 \rightarrow \mathbb{R}$ be the function defined by, $V(j_{j_x^1 \sigma}^2 \mathcal{L}) = \sharp \text{Hess}_{j_x^1 \sigma}(\mathcal{L})(d_{10} \mathcal{L}, d_{10} \mathcal{L})$.*

For every $\Phi \in \text{Aut}(p)$ and all $\mathcal{L} \in C^\infty(J^1(\mathbb{R}, M))$, the following formula holds:

$$V \left\{ ((\tilde{\Phi}^{-1})^{(1)})^{(2)}(j_{\Phi^{(1)}(j_x^1 \sigma)}^2 \mathcal{L}) \right\} = ((\phi)')^{-1} V(j_{\Phi^{(1)}(j_x^1 \sigma)}^2 \mathcal{L}).$$

Therefore, the function V is invariant under the natural action on $O^2 \subset J^2(q)$ of the normal subgroup $\text{Aut}^v(p) \subset \text{Aut}(p)$ of vertical automorphisms of p . Moreover, if O'^2 is the dense open subset of 2-jets $j_{j_x^1 \sigma}^2(\mathcal{L}) \in O^2$ such that $\mathcal{L}(j_x^1 \sigma) \neq 0$, then the function $I: O'^2 \rightarrow \mathbb{R}$ defined by,

$$I(j_{j_x^1 \sigma}^2 \mathcal{L}) = \frac{V(j_{j_x^1 \sigma}^2 \mathcal{L})}{\mathcal{L}(j_x^1 \sigma)}$$

is invariant under the full group $\text{Aut}(p)$ of automorphisms of p .

Proof. If $a = j_x^1 \sigma$, $a' = \Phi^{(1)}(j_x^1 \sigma)$, then

$$\begin{aligned} V \left\{ ((\tilde{\Phi}^{-1})^{(1)})^{(2)}(j_{a'}^2 s_{\mathcal{L}}) \right\} &= V \left\{ j_a^2 \left((\tilde{\Phi}^{-1}) \circ s_{\mathcal{L}} \circ \Phi^{(1)} \right) \right\} \\ &= V \left\{ j_a^2 s_{\bar{\mathcal{L}}} \right\} \end{aligned}$$

with $\bar{\mathcal{L}} = \Phi'^{-1}(\mathcal{L} \circ \Phi^{(1)})$ and hence

$$V \left\{ ((\tilde{\Phi}^{-1})^{(1)})^{(2)}(j_{a'}^2 s_{\mathcal{L}}) \right\} = \text{Hess}_a(\bar{\mathcal{L}})(\text{Hess}_a(\bar{\mathcal{L}})^\sharp(d_{10}\bar{\mathcal{L}}), \text{Hess}_a(\bar{\mathcal{L}})^\sharp(d_{10}\bar{\mathcal{L}})).$$

We first note that the following formulas hold:

$$\begin{aligned} \text{Hess}_a(\bar{\mathcal{L}}) &= (\phi')^{-1} \text{Hess}_a(\mathcal{L} \circ \Phi^{(1)}), \\ \text{Hess}_a(\bar{\mathcal{L}})^\sharp &= \phi' \text{Hess}_a(\mathcal{L} \circ \Phi^{(1)})^\sharp, \\ d_{10}\bar{\mathcal{L}} &= \phi^{-1} d_{10}(\mathcal{L} \circ \Phi^{(1)}), \end{aligned}$$

as $(\phi')^{-1}$ does not depend on vertical variables. Then, from the bilinearity of the Hessian we have

$$\begin{aligned} V \left\{ ((\tilde{\Phi}^{-1})^{(1)})^{(2)}(j_{a'}^2 s_{\mathcal{L}}) \right\} &= \\ (\phi')^{-1} \text{Hess}_a(\mathcal{L} \circ \Phi^{(1)})(\text{Hess}_a(\mathcal{L} \circ \Phi^{(1)})^\sharp(d_{10}(\mathcal{L} \circ \Phi^{(1)}), \text{Hess}_a(\mathcal{L} \circ \Phi^{(1)})^\sharp(d_{10}(\mathcal{L} \circ \Phi^{(1)}))) &= \end{aligned} \tag{12}$$

Moreover, from the covariance of the Hessian, $\text{Hess}(\mathcal{L} \circ \Phi^{(1)}) = (\Phi^{(1)})^* \text{Hess } \mathcal{L}$ (see Proposition 2.2), we conclude that its sharp operator is also covariant, namely,

$$\text{Hess}_a(\mathcal{L} \circ \Phi^{(1)})^\sharp = ((\Phi^{-1})^{(1)})_* \circ \text{Hess}_{a'}(\mathcal{L})^\sharp \circ ((\Phi^{-1})^{(1)})^*.$$

In addition, $d_{10}(\mathcal{L} \circ \Phi^{(1)}) = (\Phi^{(1)})^* d_{10}\mathcal{L}$ as $(\Phi^{(1)})_*$ transforms vertical vectors into vertical vectors. From these two facts and (12) we have

$$V \left\{ ((\tilde{\Phi}^{-1})^{(1)})^{(2)}(j_{a'}^2 s_{\mathcal{L}}) \right\} = (\phi')^{-1} \text{Hess}_{a'}(\mathcal{L})(\text{Hess}_{a'}(\mathcal{L})^\sharp(d_{10}\mathcal{L}), \text{Hess}_{a'}(\mathcal{L})^\sharp(d_{10}\mathcal{L})),$$

and we obtain the first formula in the statement. \square

2.3.2. The generic rank of \mathcal{D}^2 computed

The coordinate system induced by (x, y^α) on $J^2(q)$ is

$$\begin{aligned} &x, z, z_x, z_{xx}, \\ &y^\alpha, \dot{y}^\alpha, z_{y^\alpha}, z_{\dot{y}^\alpha}, z_{xy^\alpha}, z_{x\dot{y}^\alpha}, & 1 \leq \alpha \leq m, \\ &z_{y^\alpha y^\beta}, z_{\dot{y}^\alpha \dot{y}^\beta}, & 1 \leq \alpha \leq \beta \leq m, \\ &z_{y^\alpha \dot{y}^\beta}, & \alpha, \beta = 1, \dots, m. \end{aligned}$$

Hence, $\dim J^2(q) = 2m^2 + 7m + 4$.

Theorem 2.4. *On a dense open subset $O^2 \subset O^2 \subset J^2(q)$, where O^2 is the set of 2-jets whose Hessian metric is non-singular, the rank of the distribution \mathcal{D}^2 generated by all the vector fields of the form $(\tilde{X}^{(1)})^{(2)}$, X being an arbitrary p -projectable vector field on $\mathbb{R} \times M$, is $2m^2 + 7m + 3 = (m + 3)(2m + 1)$. Consequently, the invariant I defined in Proposition 2.3 is a basis for the invariants of second order.*

Proof. We first compute

$$\begin{aligned} (\tilde{X}^{(1)})^{(2)} &= u \frac{\partial}{\partial x} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_1^\alpha \frac{\partial}{\partial \dot{y}^\alpha} - \frac{du}{dx} z \frac{\partial}{\partial z} + A \frac{\partial}{\partial z_x} + B_\alpha \frac{\partial}{\partial z_{y^\alpha}} \\ &+ C_\alpha \frac{\partial}{\partial z_{\dot{y}^\alpha}} + D_{xx} \frac{\partial}{\partial z_{xx}} + E_{xy^\alpha} \frac{\partial}{\partial z_{xy^\alpha}} + F_{x\dot{y}^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\alpha}} \end{aligned}$$

$$+ \sum_{\alpha \leq \beta} G_{y^\alpha y^\beta} \frac{\partial}{\partial z_{y^\alpha y^\beta}} + H_{y^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{y^\alpha \dot{y}^\beta}} + \sum_{\alpha \leq \beta} K_{\dot{y}^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{\dot{y}^\alpha \dot{y}^\beta}},$$

where A , B_α , and C_α are given by the formulas (8), (9), and (10), respectively, and the coefficients D_{xx} , E_{xy^α} , $F_{x\dot{y}^\alpha}$, $G_{y^\alpha y^\beta}$, $H_{y^\alpha \dot{y}^\beta}$, $K_{\dot{y}^\alpha \dot{y}^\beta}$ are to be determined by using the formulas (5), and (7). We obtain

$$\begin{aligned} D_{xx} = & -3 \frac{du}{dx} z_{xx} - 2 \frac{\partial v^\alpha}{\partial x} z_{xy^\alpha} - 3 \frac{d^2 u}{dx^2} z_x + 2 \frac{d^2 u}{dx^2} \dot{y}^\alpha z_{x\dot{y}^\alpha} \\ & - \frac{\partial^2 v^\alpha}{\partial x^2} z_{y^\alpha} - 2 \frac{\partial^2 v^\alpha}{\partial x^2} z_{x\dot{y}^\alpha} - 2 \frac{\partial^2 v^\alpha}{\partial x \partial y^\beta} \dot{y}^\beta z_{x\dot{y}^\alpha} \\ & - \frac{d^3 u}{dx^3} z + \frac{d^3 u}{dx^3} \dot{y}^\alpha z_{\dot{y}^\alpha} - \frac{\partial^3 v^\alpha}{\partial x^3} z_{y^\alpha} - \frac{\partial^3 v^\alpha}{\partial x^2 \partial y^\beta} \dot{y}^\beta z_{\dot{y}^\alpha}, \end{aligned} \tag{13}$$

$$\begin{aligned} E_{xy^\alpha} = & -2 \frac{du}{dx} z_{xy^\alpha} - \frac{\partial v^\beta}{\partial x} z_{y^\alpha y^\beta} - \frac{\partial v^\beta}{\partial y^\alpha} z_{xy^\beta} \\ & - \frac{d^2 u}{dx^2} z_{y^\alpha} + \frac{d^2 u}{dx^2} \dot{y}^\beta z_{y^\alpha \dot{y}^\beta} - \frac{\partial^2 v^\beta}{\partial x^2} z_{y^\alpha \dot{y}^\beta} \\ & - \frac{\partial^2 v^\beta}{\partial x \partial y^\alpha} z_{y^\beta} - \frac{\partial^2 v^\beta}{\partial x \partial y^\alpha} z_{x\dot{y}^\beta} - \frac{\partial^2 v^\beta}{\partial x \partial y^\gamma} \dot{y}^\gamma z_{y^\alpha \dot{y}^\beta} \\ & - \frac{\partial^2 v^\beta}{\partial y^\alpha \partial y^\gamma} \dot{y}^\gamma z_{x\dot{y}^\beta} - \frac{\partial^3 v^\beta}{\partial x^2 \partial y^\alpha} z_{\dot{y}^\beta} - \frac{\partial^3 v^\beta}{\partial x \partial y^\alpha \partial y^\gamma} \dot{y}^\gamma z_{\dot{y}^\beta}, \end{aligned} \tag{14}$$

$$\begin{aligned} F_{x\dot{y}^\alpha} = & -\frac{du}{dx} z_{\dot{y}^\alpha x} - \frac{\partial v^\beta}{\partial x} z_{y^\beta \dot{y}^\alpha} - \frac{\partial v^\beta}{\partial y^\alpha} z_{x\dot{y}^\beta} \\ & + \dot{y}^\beta \frac{d^2 u}{dx^2} z_{\dot{y}^\alpha \dot{y}^\beta} - \frac{\partial^2 v^\beta}{\partial x^2} z_{\dot{y}^\alpha \dot{y}^\beta} - \dot{y}^\gamma \frac{\partial^2 v^\beta}{\partial x \partial y^\gamma} z_{\dot{y}^\alpha \dot{y}^\beta} - \frac{\partial^2 v^\beta}{\partial x \partial y^\alpha} z_{\dot{y}^\beta}, \end{aligned} \tag{15}$$

$$\begin{aligned} G_{y^\alpha y^\beta} = & -\frac{\partial^2 v^\sigma}{\partial y^\alpha \partial y^\beta} z_{y^\sigma} - \left(\frac{\partial^3 v^\sigma}{\partial x \partial y^\alpha \partial y^\beta} + \frac{\partial^3 v^\sigma}{\partial y^\alpha \partial y^\beta \partial y^\gamma} \dot{y}^\gamma \right) z_{\dot{y}^\sigma} \\ & - \frac{du}{dx} z_{y^\alpha y^\beta} - \frac{\partial v^\sigma}{\partial y^\beta} z_{y^\alpha y^\sigma} - \frac{\partial v^\sigma}{\partial y^\alpha} z_{y^\beta y^\sigma} \\ & - \left(\frac{\partial^2 v^\sigma}{\partial x \partial y^\beta} + \frac{\partial^2 v^\sigma}{\partial y^\beta \partial y^\gamma} \dot{y}^\gamma \right) z_{y^\alpha \dot{y}^\sigma} - \left(\frac{\partial^2 v^\sigma}{\partial x \partial y^\alpha} + \frac{\partial^2 v^\sigma}{\partial y^\alpha \partial y^\gamma} \dot{y}^\gamma \right) z_{y^\beta \dot{y}^\sigma}, \end{aligned} \tag{16}$$

$$H_{y^\alpha \dot{y}^\beta} = -\frac{\partial^2 v^\sigma}{\partial y^\alpha \partial y^\beta} z_{\dot{y}^\sigma} - \frac{\partial v^\sigma}{\partial y^\beta} z_{y^\alpha \dot{y}^\sigma} - \frac{\partial v^\sigma}{\partial y^\alpha} z_{y^\sigma \dot{y}^\beta} - \left(\frac{\partial^2 v^\sigma}{\partial x \partial y^\alpha} + \frac{\partial^2 v^\sigma}{\partial y^\alpha \partial y^\gamma} \dot{y}^\gamma \right) z_{\dot{y}^\beta \dot{y}^\sigma}, \tag{17}$$

$$K_{\dot{y}^\alpha \dot{y}^\beta} = \frac{du}{dx} z_{\dot{y}^\alpha \dot{y}^\beta} - \frac{\partial v^\sigma}{\partial y^\beta} z_{\dot{y}^\alpha \dot{y}^\sigma} - \frac{\partial v^\sigma}{\partial y^\alpha} z_{\dot{y}^\beta \dot{y}^\sigma}. \tag{18}$$

Hence, taking the formula for $(\tilde{X}^{(1)})^{(1)}$, and the expressions (13), (14), (15), (16), (17), and (18) for D_{xx} , E_{xy^α} , $F_{x\dot{y}^\alpha}$, $G_{y^\alpha y^\beta}$, $H_{y^\alpha \dot{y}^\beta}$, $K_{\dot{y}^\alpha \dot{y}^\beta}$ into account, we have

$$\begin{aligned} (\tilde{X}^{(1)})^{(2)} = & u \frac{\partial}{\partial x} + v^\alpha \frac{\partial}{\partial y^\alpha} + \frac{du}{dx} \chi_1^1 + \frac{\partial v^\alpha}{\partial x} \bar{\chi}_\alpha^1 + \frac{\partial v^\alpha}{\partial y^\beta} \tilde{\chi}_\alpha^\beta + \frac{d^2 u}{dx^2} \chi_1^{11} \\ & + \frac{\partial^2 v^\alpha}{\partial x^2} \bar{\chi}_\alpha^{11} + \frac{\partial^2 v^\alpha}{\partial x \partial y^\beta} \tilde{\chi}_\alpha^{\beta 1} + \sum_{\alpha \leq \beta} \frac{\partial^2 v^\sigma}{\partial y^\alpha \partial y^\beta} \hat{\chi}_\sigma^{\alpha \leq \beta} + \frac{d^3 u}{dx^3} \chi_1^{111} \\ & + \frac{\partial^3 v^\alpha}{\partial x^3} \bar{\chi}_\alpha^{111} + \frac{\partial^3 v^\beta}{\partial x^2 \partial y^\alpha} \tilde{\chi}_\beta^{11\alpha} + \sum_{\alpha \leq \beta} \frac{\partial^3 v^\sigma}{\partial x \partial y^\alpha \partial y^\beta} \chi_\sigma^{1, \alpha \leq \beta} \end{aligned} \tag{19}$$

$$+ \sum_{\alpha \leq \beta \leq \gamma} \frac{\partial^3 v^\sigma}{\partial y^\alpha \partial y^\beta \partial y^\gamma} \chi_\sigma^{\alpha \leq \beta \leq \gamma},$$

where

$$\begin{aligned} \chi_1^1 = & -z \frac{\partial}{\partial z} - \dot{y}^\alpha \frac{\partial}{\partial \dot{y}^\alpha} - 2z_x \frac{\partial}{\partial z_x} - z_{y^\alpha} \frac{\partial}{\partial z_{y^\alpha}} - 3z_{xx} \frac{\partial}{\partial z_{xx}} - 2z_{xy^\alpha} \frac{\partial}{\partial z_{xy^\alpha}} \\ & - z_{x\dot{y}^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\alpha}} - \sum_{\alpha \leq \beta} z_{y^\alpha y^\beta} \frac{\partial}{\partial z_{y^\alpha y^\beta}} + \sum_{\alpha \leq \beta} z_{\dot{y}^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{\dot{y}^\alpha \dot{y}^\beta}}, \end{aligned} \quad (20)$$

$$\bar{\chi}_\alpha^1 = \frac{\partial}{\partial \dot{y}^\alpha} - z_{y^\alpha} \frac{\partial}{\partial z_x} - 2z_{xy^\alpha} \frac{\partial}{\partial z_{xx}} - z_{y^\alpha y^\beta} \frac{\partial}{\partial z_{xy^\beta}} - z_{y^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{x\dot{y}^\beta}}, \quad (21)$$

$$\begin{aligned} \tilde{\chi}_\alpha^\beta = & \dot{y}^\beta \frac{\partial}{\partial \dot{y}^\alpha} - z_{y^\alpha} \frac{\partial}{\partial z_{y^\beta}} - z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{\dot{y}^\beta}} - z_{xy^\alpha} \frac{\partial}{\partial z_{xy^\beta}} - z_{x\dot{y}^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\beta}} \\ & - (1 + \delta_\sigma^\beta) z_{y^\sigma y^\alpha} \frac{\partial}{\partial z_{y^\sigma y^\beta}} - z_{y^\sigma \dot{y}^\alpha} \frac{\partial}{\partial z_{y^\sigma \dot{y}^\beta}} - z_{y^\alpha \dot{y}^\sigma} \frac{\partial}{\partial z_{y^\beta \dot{y}^\sigma}} \\ & - (1 + \delta_\sigma^\beta) z_{\dot{y}^\alpha \dot{y}^\sigma} \frac{\partial}{\partial z_{\dot{y}^\sigma \dot{y}^\beta}}, \end{aligned} \quad (22)$$

$$\begin{aligned} \chi_1^{11} = & (-z + \dot{y}^\gamma z_{\dot{y}^\gamma}) \frac{\partial}{\partial z_x} - 3z_x \frac{\partial}{\partial z_{xx}} + 2\dot{y}^\alpha z_{x\dot{y}^\alpha} \frac{\partial}{\partial z_{xx}} - z_{y^\alpha} \frac{\partial}{\partial z_{xy^\alpha}} \\ & + \dot{y}^\beta z_{y^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{xy^\alpha}} + \dot{y}^\beta z_{\dot{y}^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{x\dot{y}^\alpha}}, \end{aligned} \quad (23)$$

$$\bar{\chi}_\alpha^{11} = -z_{\dot{y}^\alpha} \frac{\partial}{\partial z_x} - z_{y^\alpha} \frac{\partial}{\partial z_{xx}} - 2z_{x\dot{y}^\alpha} \frac{\partial}{\partial z_{xx}} - z_{y^\beta \dot{y}^\alpha} \frac{\partial}{\partial z_{xy^\beta}} - z_{\dot{y}^\beta \dot{y}^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\beta}}, \quad (24)$$

$$\begin{aligned} \tilde{\chi}_\alpha^{\beta 1} = & -\dot{y}^\beta z_{\dot{y}^\alpha} \frac{\partial}{\partial z_x} - z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{y^\beta}} - 2\dot{y}^\beta z_{x\dot{y}^\alpha} \frac{\partial}{\partial z_{xx}} - z_{y^\alpha} \frac{\partial}{\partial z_{xy^\beta}} \\ & - z_{x\dot{y}^\alpha} \frac{\partial}{\partial z_{xy^\beta}} - \dot{y}^\beta z_{y^\sigma \dot{y}^\alpha} \frac{\partial}{\partial z_{xy^\sigma}} - \dot{y}^\beta z_{\dot{y}^\sigma \dot{y}^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\sigma}} - z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\beta}} \\ & - (1 + \delta_{\beta\gamma}) z_{y^\gamma \dot{y}^\alpha} \frac{\partial}{\partial z_{y^\gamma y^\beta}} - z_{\dot{y}^\sigma \dot{y}^\alpha} \frac{\partial}{\partial z_{y^\beta \dot{y}^\sigma}}, \end{aligned} \quad (25)$$

$$\begin{aligned} \hat{\chi}_\sigma^{\alpha \leq \beta} = & -\frac{1}{1 + \delta_{\alpha\beta}} z_{\dot{y}^\sigma} \left(\dot{y}^\beta \frac{\partial}{\partial z_{y^\alpha}} + \dot{y}^\alpha \frac{\partial}{\partial z_{y^\beta}} \right) \\ & - \frac{1}{1 + \delta_{\alpha\beta}} z_{x\dot{y}^\sigma} \left(\dot{y}^\beta \frac{\partial}{\partial z_{xy^\alpha}} + \dot{y}^\alpha \frac{\partial}{\partial z_{xy^\beta}} \right) - z_{y^\sigma} \frac{\partial}{\partial z_{y^\alpha y^\beta}} \\ & - \frac{1}{1 + \delta_{\alpha\beta}} \left((1 + \delta_{\beta\gamma}) \dot{y}^\alpha z_{y^\gamma \dot{y}^\sigma} \frac{\partial}{\partial z_{y^\gamma y^\beta}} + (1 + \delta_{\alpha\gamma}) \dot{y}^\beta z_{y^\gamma \dot{y}^\sigma} \frac{\partial}{\partial z_{y^\gamma y^\alpha}} \right) \\ & - \frac{1}{1 + \delta_{\alpha\beta}} z_{\dot{y}^\sigma} \left(\frac{\partial}{\partial z_{y^\alpha \dot{y}^\beta}} + \frac{\partial}{\partial z_{y^\beta \dot{y}^\alpha}} \right) \\ & - \frac{1}{1 + \delta_{\alpha\beta}} z_{\dot{y}^\gamma \dot{y}^\sigma} \left(\dot{y}^\beta \frac{\partial}{\partial z_{y^\alpha \dot{y}^\gamma}} + \dot{y}^\alpha \frac{\partial}{\partial z_{y^\beta \dot{y}^\gamma}} \right), \end{aligned} \quad (26)$$

$$\chi_1^{111} = (\dot{y}^\alpha z_{\dot{y}^\alpha} - z) \frac{\partial}{\partial z_{xx}}, \quad (27)$$

$$\bar{\chi}_\alpha^{111} = -z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{xx}}, \quad (28)$$

$$\tilde{\chi}_\beta^{11\alpha} = -\dot{y}^\alpha z_{\dot{y}^\beta} \frac{\partial}{\partial z_{xx}} - z_{\dot{y}^\beta} \frac{\partial}{\partial z_{xy^\alpha}}, \tag{29}$$

$$\chi_\sigma^{1,\alpha \leq \beta} = -\frac{z_{\dot{y}^\sigma}}{1 + \delta_{\alpha\beta}} \left(\dot{y}^\beta \frac{\partial}{\partial z_{xy^\alpha}} + \dot{y}^\alpha \frac{\partial}{\partial z_{xy^\beta}} \right) - z_{\dot{y}^\sigma} \frac{\partial}{\partial z_{y^\alpha y^\beta}}, \tag{30}$$

$$\chi_\sigma^{\alpha \leq \beta \leq \gamma} = -z_{\dot{y}^\sigma} \left(\dot{y}^\gamma \frac{\partial}{\partial z_{y^\alpha y^\beta}} + (1 - \delta_{\beta\gamma}) \dot{y}^\beta \frac{\partial}{\partial z_{y^\alpha y^\gamma}} + (1 - \delta_{\alpha\gamma}) \dot{y}^\alpha \frac{\partial}{\partial z_{y^\gamma y^\beta}} \right). \tag{31}$$

Therefore the vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y^\alpha}$, and (20)–(31) span the distribution \mathcal{D}^2 . Let us fix four indices $\alpha_0, \beta_0, \gamma_0, \sigma_0$. From (28) on the dense open subset $z_{\dot{y}^{\alpha_0}} \neq 0$, we have $\frac{\partial}{\partial z_{xx}} = -\frac{1}{z_{\dot{y}^{\alpha_0}}} \bar{\chi}_{\alpha_0}^{111}$. Replacing this expression into (29), on the dense open subset $z_{\dot{y}^{\alpha_0}} \neq 0, z_{\dot{y}^{\beta_0}} \neq 0$, we have $\frac{\partial}{\partial z_{xy^\alpha}} = \frac{\dot{y}^\alpha}{z_{\dot{y}^{\alpha_0}}} \bar{\chi}_{\alpha_0}^{111} - \frac{1}{z_{\dot{y}^{\beta_0}}} \tilde{\chi}_{\beta_0}^{11\alpha}$. From (31) we have $\frac{\partial}{\partial z_{y^\alpha y^\beta}} = -\frac{1}{\dot{y}^{\gamma_0} z_{\dot{y}^{\sigma_0}}} \chi_{\sigma_0}^{\alpha \leq \beta, \gamma_0}$ on the dense open subset $\dot{y}^{\gamma_0} \neq 0, z_{\dot{y}^{\sigma_0}} \neq 0$. Hence the distribution \mathcal{D}^2 is spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial z_{xx}}, \frac{\partial}{\partial z_{xy^\alpha}}, \frac{\partial}{\partial z_{y^\alpha y^\beta}}, \alpha \leq \beta$, and the following vector fields:

$$\begin{aligned} \chi_1^{11} &= -z \frac{\partial}{\partial z} - \dot{y}^\alpha \frac{\partial}{\partial \dot{y}^\alpha} - 2z_x \frac{\partial}{\partial z_x} - z_{y^\alpha} \frac{\partial}{\partial z_{y^\alpha}} - z_{xy^\alpha} \frac{\partial}{\partial z_{xy^\alpha}} \\ &\quad + \sum_{\alpha \leq \beta} z_{\dot{y}^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{\dot{y}^\alpha \dot{y}^\beta}}, \\ \bar{\chi}_\alpha^{11} &= \frac{\partial}{\partial \dot{y}^\alpha} - z_{y^\alpha} \frac{\partial}{\partial z_x} - z_{y^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{x\dot{y}^\beta}}, \\ \tilde{\chi}_\alpha^{1\beta} &= \dot{y}^\beta \frac{\partial}{\partial \dot{y}^\alpha} - z_{y^\alpha} \frac{\partial}{\partial z_{y^\beta}} - z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{\dot{y}^\beta}} - z_{xy^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\beta}} \\ &\quad - z_{y^\sigma \dot{y}^\alpha} \frac{\partial}{\partial z_{y^\sigma \dot{y}^\beta}} - z_{y^\alpha \dot{y}^\sigma} \frac{\partial}{\partial z_{y^\beta \dot{y}^\sigma}} - (1 + \delta_\sigma^\beta) z_{\dot{y}^\alpha \dot{y}^\sigma} \frac{\partial}{\partial z_{\dot{y}^\sigma \dot{y}^\beta}}, \\ \chi_1^{111} &= (-z + \dot{y}^\gamma z_{\dot{y}^\gamma}) \frac{\partial}{\partial z_x} + \dot{y}^\beta z_{\dot{y}^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{x\dot{y}^\alpha}}, \end{aligned} \tag{32}$$

$$\bar{\chi}_\alpha^{111} = -z_{\dot{y}^\alpha} \frac{\partial}{\partial z_x} - z_{\dot{y}^\beta \dot{y}^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\beta}}, \tag{33}$$

$$\begin{aligned} \tilde{\chi}_\alpha^{1\beta 1} &= -\dot{y}^\beta z_{\dot{y}^\alpha} \frac{\partial}{\partial z_x} - z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{y^\beta}} - \dot{y}^\beta z_{\dot{y}^\sigma \dot{y}^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\sigma}} - z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{x\dot{y}^\beta}} \\ &\quad - z_{\dot{y}^\sigma \dot{y}^\alpha} \frac{\partial}{\partial z_{y^\beta \dot{y}^\sigma}}, \end{aligned}$$

$$\hat{\chi}_\beta^{1\alpha\gamma} = -\dot{y}^\gamma z_{\dot{y}^\beta} \frac{\partial}{\partial z_{y^\alpha}} - z_{\dot{y}^\beta} \frac{\partial}{\partial z_{y^\alpha \dot{y}^\gamma}} - \dot{y}^\gamma z_{\dot{y}^\sigma \dot{y}^\beta} \frac{\partial}{\partial z_{y^\alpha \dot{y}^\sigma}}.$$

From (33), we have $\frac{\partial}{\partial z_x} = -\frac{1}{z_{\dot{y}^{\alpha_0}}} \bar{\chi}_{\alpha_0}^{111} - \frac{z_{\dot{y}^\beta \dot{y}^{\alpha_0}}}{z_{\dot{y}^{\alpha_0}}} \frac{\partial}{\partial z_{x\dot{y}^\beta}}$ on the dense open subset $z_{\dot{y}^{\alpha_0}} \neq 0$. Replacing the previous formula into (32) and letting free the index α_0 , we obtain

$$C_\beta^\alpha \frac{\partial}{\partial z_{x\dot{y}^\beta}} = v^\alpha, \quad 1 \leq \alpha \leq m, \tag{34}$$

where $C_\beta^\alpha = (z - \dot{y}^\gamma z_{\dot{y}^\gamma}) z_{\dot{y}^\beta \dot{y}^\alpha} + \dot{y}^\gamma z_{\dot{y}^\alpha} z_{\dot{y}^\gamma \dot{y}^\beta}$, $v^\alpha = (z - \dot{y}^\gamma z_{\dot{y}^\gamma}) \bar{\chi}_\alpha^{111} - z_{\dot{y}^\alpha} \chi_1^{111}$. On the dense open subset O'^2 defined by

$$0 \neq \det (C_\beta^\alpha)_{\alpha, \beta=1}^m = z (z - \dot{y}^\gamma z_{\dot{y}^\gamma})^{m-1} \det (z_{\dot{y}^\alpha \dot{y}^\beta})_{\alpha, \beta=1}^m,$$

we can solve (34) for $\frac{\partial}{\partial z_{x\dot{y}\beta}}$, thus proving that $\frac{\partial}{\partial z_{x\dot{y}\beta}}, \frac{\partial}{\partial z_x} \in \mathcal{D}^2|_{O^2}$. Hence the distribution \mathcal{D}^2 is spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z_x}, \frac{\partial}{\partial z_{xy^\alpha}}, \frac{\partial}{\partial z_{x\dot{y}\beta}}, \frac{\partial}{\partial z_{y^\alpha\dot{y}\beta}}, \alpha \leq \beta$, and the following vector fields:

$$\begin{aligned} \chi_1''^1 &= -z \frac{\partial}{\partial z} - \dot{y}^\alpha \frac{\partial}{\partial \dot{y}^\alpha} - 2z_x \frac{\partial}{\partial z_x} - z_{y^\alpha} \frac{\partial}{\partial z_{y^\alpha}} + \sum_{\alpha \leq \beta} z_{\dot{y}^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{\dot{y}^\alpha \dot{y}^\beta}}, \\ \bar{\chi}_\alpha''^1 &= \frac{\partial}{\partial \dot{y}^\alpha}, \end{aligned} \tag{35}$$

$$\begin{aligned} \tilde{\chi}_\alpha''^\beta &= \dot{y}^\beta \frac{\partial}{\partial \dot{y}^\alpha} - z_{y^\alpha} \frac{\partial}{\partial z_{y^\beta}} - z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{\dot{y}^\beta}} \\ &\quad - z_{y^\sigma \dot{y}^\alpha} \frac{\partial}{\partial z_{y^\sigma \dot{y}^\beta}} - z_{y^\alpha \dot{y}^\sigma} \frac{\partial}{\partial z_{y^\beta \dot{y}^\sigma}} - (1 + \delta_\sigma^\beta) z_{\dot{y}^\alpha \dot{y}^\sigma} \frac{\partial}{\partial z_{\dot{y}^\sigma \dot{y}^\beta}}, \\ \tilde{\chi}_\alpha''^{\beta 1} &= -z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{y^\beta}} - z_{\dot{y}^\sigma \dot{y}^\alpha} \frac{\partial}{\partial z_{y^\beta \dot{y}^\sigma}}, \end{aligned} \tag{36}$$

$$\hat{\chi}_\beta''^{\alpha\gamma} = -\dot{y}^\gamma z_{\dot{y}^\beta} \frac{\partial}{\partial z_{y^\alpha}} - z_{\dot{y}^\beta} \frac{\partial}{\partial z_{y^\alpha \dot{y}^\gamma}} - \dot{y}^\gamma z_{\dot{y}^\sigma \dot{y}^\beta} \frac{\partial}{\partial z_{y^\alpha \dot{y}^\sigma}}. \tag{37}$$

From (35) we have $\frac{\partial}{\partial \dot{y}^\alpha} = \bar{\chi}_\alpha''^1$, and on the dense open subset $z_{\dot{y}^\alpha 0} \neq 0$, from (36) we have $\frac{\partial}{\partial z_{y^\beta}} = -\frac{1}{z_{\dot{y}^\alpha 0}} \tilde{\chi}_{\alpha 0}''^{\beta 1} - \frac{z_{\dot{y}^\sigma \dot{y}^\alpha 0}}{z_{\dot{y}^\alpha 0}} \frac{\partial}{\partial z_{y^\beta \dot{y}^\sigma}}$. Replacing the previous formula into (37), we obtain

$$\bar{C}_{\beta\sigma}^\gamma \frac{\partial}{\partial z_{y^\alpha \dot{y}^\sigma}} = \bar{v}_\beta^{\alpha\gamma}, \quad \alpha, \beta, \gamma = 1, \dots, m, \tag{38}$$

with $\bar{C}_{\beta\sigma}^\gamma = \dot{y}^\gamma z_{\dot{y}^\beta} \frac{z_{\dot{y}^\sigma \dot{y}^\alpha 0}}{z_{\dot{y}^\alpha 0}} - z_{\dot{y}^\beta} \delta_\sigma^\gamma - \dot{y}^\gamma z_{\dot{y}^\sigma \dot{y}^\beta}$, $\bar{v}_\beta^{\alpha\gamma} = \hat{\chi}_\beta''^{\alpha\gamma} - \frac{\dot{y}^\gamma z_{\dot{y}^\beta}}{z_{\dot{y}^\alpha 0}} \tilde{\chi}_{\alpha 0}''^{\beta 1}$. Letting $\alpha = \alpha_1, \beta = \beta_0$, the system (38) transforms into the following system of m equations and m unknowns: $\bar{C}_{\beta_0\sigma}^{\alpha_1\gamma} \frac{\partial}{\partial z_{y^{\alpha_1} \dot{y}^\sigma}} = \bar{v}_{\beta_0}^{\alpha_1\gamma}$, and, in particular, for $\beta_0 = \alpha_0$, we have

$$\begin{aligned} \bar{C}_{\alpha_0\sigma}^{\alpha_1\gamma} &= \dot{y}^\gamma z_{\dot{y}^{\alpha_0}} \frac{z_{\dot{y}^\sigma \dot{y}^{\alpha_0}}}{z_{\dot{y}^{\alpha_0}}} - z_{\dot{y}^{\alpha_0}} \delta_\sigma^\gamma - \dot{y}^\gamma z_{\dot{y}^\sigma \dot{y}^{\alpha_0}} = -z_{\dot{y}^{\alpha_0}} \delta_\sigma^\gamma, \\ \det(\bar{C}_{\alpha_0\sigma}^{\alpha_1\gamma})_{\gamma,\sigma=1}^m &= \det(-z_{\dot{y}^{\alpha_0}} \delta_\sigma^\gamma)_{\gamma,\sigma=1}^m = (-z_{\dot{y}^{\alpha_0}})^m \neq 0. \end{aligned}$$

Therefore, we can solve the equations (38) with respect to $\frac{\partial}{\partial z_{y^\alpha \dot{y}^\sigma}}$, thus concluding that the distribution \mathcal{D}^2 is spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \dot{y}^\alpha}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z_x}, \frac{\partial}{\partial z_{xy^\alpha}}, \frac{\partial}{\partial z_{xx}}, \frac{\partial}{\partial z_{xy^\alpha}}, \frac{\partial}{\partial z_{x\dot{y}\beta}}, \frac{\partial}{\partial z_{y^\alpha\dot{y}\beta}}, \alpha \leq \beta, \frac{\partial}{\partial z_{y^\alpha \dot{y}^\sigma}}$, and the following $m^2 + 1$ additional vector fields:

$$\begin{aligned} \zeta_1^1 &= -z \frac{\partial}{\partial z} + \sum_{\alpha \leq \beta} z_{\dot{y}^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{\dot{y}^\alpha \dot{y}^\beta}}, \\ \bar{\zeta}_\alpha^\beta &= -z_{\dot{y}^\alpha} \frac{\partial}{\partial z_{\dot{y}^\beta}} - (1 + \delta_\sigma^\beta) z_{\dot{y}^\alpha \dot{y}^\sigma} \frac{\partial}{\partial z_{\dot{y}^\sigma \dot{y}^\beta}}. \end{aligned}$$

For every $j_{j_x^1}^2 \mathcal{L} \in O'^2$ we thus have

$$\begin{aligned} \dim \mathcal{D}^2|_{j_{j_x^1}^2 \mathcal{L}} &= \frac{3}{2}m^2 + \frac{11}{2}m + 3 \\ &\quad + \text{rank} \left\{ \zeta_1^1|_{j_{j_x^1}^2 \mathcal{L}}, \bar{\zeta}_\alpha^\beta|_{j_{j_x^1}^2 \mathcal{L}} : \alpha, \beta = 1, \dots, m \right\}. \end{aligned}$$

Therefore, we only need to prove that the rank of the system $\zeta_1^1|_{j_{j_x^1}^2 \mathcal{L}}, \bar{\zeta}_\alpha^\beta|_{j_{j_x^1}^2 \mathcal{L}}, \alpha, \beta = 1, \dots, m$, is $\frac{1}{2}m(m+3)$. To do this, we choose coordinates $(y^\alpha)_{\alpha=1}^m$ adapted to the Hessian metric $\text{Hess}_{j_{x_0}^1 \sigma_0}(\mathcal{L})$, namely

$$z_{j^\alpha \dot{y}^\beta}(j_{j_x^1}^2(\mathcal{L})) = \varepsilon_\alpha \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, m,$$

$$\varepsilon_\alpha = \begin{cases} +1, & 1 \leq \alpha \leq m^+, \\ -1, & 1 + m^+ \leq \alpha \leq m, \end{cases}$$

the pair (m^+, m^-) , $m^- + m^+ = m$, being the signature of $\text{Hess}_{j_{x_0}^1 \sigma_0}(\mathcal{L})$. Hence

$$\zeta_1^1|_{j_{j_x^1}^2 \mathcal{L}} = -z \frac{\partial}{\partial z} + \varepsilon_\alpha \frac{\partial}{\partial z_{j^\alpha \dot{y}^\alpha}} \Big|_{j_{j_x^1}^2 \mathcal{L}},$$

$$\bar{\zeta}_\alpha^\alpha|_{j_{j_x^1}^2 \mathcal{L}} = -z_{j^\alpha \dot{y}^\alpha} \frac{\partial}{\partial z_{j^\alpha \dot{y}^\alpha}} - 2\varepsilon_\alpha \frac{\partial}{\partial z_{j^\alpha \dot{y}^\alpha}} \Big|_{j_{j_x^1}^2 \mathcal{L}},$$

$$\bar{\zeta}_\alpha^\beta|_{j_{j_x^1}^2 \mathcal{L}} = -z_{j^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{j^\alpha \dot{y}^\beta}} - \varepsilon_\alpha \frac{\partial}{\partial z_{j^\alpha \dot{y}^\beta}} \Big|_{j_{j_x^1}^2 \mathcal{L}}, \quad \alpha \neq \beta.$$

We split the third group above and choose new generators as follows:

$$\bar{\zeta}_\alpha^\beta|_{j_{j_x^1}^2 \mathcal{L}} = -z_{j^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{j^\alpha \dot{y}^\beta}} - \varepsilon_\alpha \frac{\partial}{\partial z_{j^\alpha \dot{y}^\beta}} \Big|_{j_{j_x^1}^2 \mathcal{L}}, \quad \alpha < \beta,$$

$$\hat{\zeta}_\alpha^\beta|_{j_{j_x^1}^2 \mathcal{L}} = \varepsilon_\beta \bar{\zeta}_\alpha^\beta - \varepsilon_\alpha \bar{\zeta}_\beta^\alpha = \varepsilon_\alpha z_{j^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{j^\alpha \dot{y}^\beta}} - \varepsilon_\beta z_{j^\alpha \dot{y}^\beta} \frac{\partial}{\partial z_{j^\alpha \dot{y}^\beta}}, \quad \beta < \alpha.$$

The system $\zeta_1^1|_{j_{j_x^1}^2 \mathcal{L}}, \bar{\zeta}_\alpha^\alpha|_{j_{j_x^1}^2 \mathcal{L}}, \hat{\zeta}_\alpha^1|_{j_{j_x^1}^2 \mathcal{L}}, 2 \leq \alpha \leq m, \bar{\zeta}_\alpha^\beta|_{j_{j_x^1}^2 \mathcal{L}}, 1 \leq \alpha < \beta \leq m$, is readily seen to be independent. As for the rest of the vectors, one obtains the relations, $\hat{\zeta}_\alpha^\beta|_{j_{j_x^1}^2 \mathcal{L}} = \frac{z_{j^\beta \dot{y}^\beta}}{z_{j^\alpha \dot{y}^\alpha}} \hat{\zeta}_\alpha^1|_{j_{j_x^1}^2 \mathcal{L}} - \frac{z_{j^\alpha \dot{y}^\alpha}}{z_{j^\beta \dot{y}^\beta}} \hat{\zeta}_\beta^1|_{j_{j_x^1}^2 \mathcal{L}}, 2 \leq \beta < \alpha \leq m$, so that the rank of the system $\zeta_1^1|_{j_{j_x^1}^2 \mathcal{L}}, \bar{\zeta}_\alpha^\beta|_{j_{j_x^1}^2 \mathcal{L}}, \alpha, \beta = 1, \dots, m$, is $\frac{1}{2}m(m+3)$. Therefore

$$\dim \mathcal{D}^2|_{j_{j_x^1}^2 \mathcal{L}} = \frac{3}{2}m^2 + \frac{11}{2}m + 3 + \frac{1}{2}m(m+3) = 2m^2 + 7m + 3. \quad \square$$

Example 2.5. For $M = \mathbb{R}^m$, we now show the expression of the invariant I when applied to quadratic Lagrangians

$$\mathcal{L} = a_{\alpha\beta} \dot{y}^\alpha \dot{y}^\beta + b_\alpha \dot{y}^\alpha + c,$$

where $a_{\alpha\beta}, b_\alpha, c \in C^\infty(\mathbb{R} \times M)$ and $a_{\alpha\beta} = a_{\beta\alpha}, \alpha, \beta = 1, \dots, m$. We consider that the matrix $A = (a_{\alpha\beta})_{\alpha, \beta=1}^m$ is non-singular and we use the notation $(a^{\alpha\beta})_{\alpha, \beta=1}^m$ for its inverse A^{-1} . The Hessian matrix of \mathcal{L} is $2A$ so that we have

$$I(j_{j_x^1}^2 \mathcal{L}) = \frac{\frac{1}{2} a^{\alpha\beta} (a_{\alpha\gamma} \dot{y}^\gamma + b_\alpha)(a_{\beta\rho} \dot{y}^\rho + b_\beta)}{\mathcal{L}} = \frac{\mathcal{L} + b_\alpha \dot{y}^\alpha + a^{\alpha\beta} b_\alpha b_\beta - c}{2\mathcal{L}}.$$

Hence, two Lagrangians \mathcal{L} and \mathcal{L}' define the same invariant if and only if

$$\frac{b_\alpha \dot{y}^\alpha + a^{\alpha\beta} b_\alpha b_\beta - c}{\mathcal{L}} = \frac{b'_\alpha \dot{y}^\alpha + a'^{\alpha\beta} b'_\alpha b'_\beta - c'}{\mathcal{L}'},$$

that is,

$$(b_\alpha \dot{y}^\alpha + a^{\alpha\beta} b_\alpha b_\beta - c) \mathcal{L}' = (b'_\alpha \dot{y}^\alpha + a'^{\alpha\beta} b'_\alpha b'_\beta - c') \mathcal{L}.$$

If the previous identity is not satisfied, the Lagrangians are not equivalent. As a particular example, for Lagrangians with no linear part (that is $b = b' = 0$), if $cA' \neq c'A$ then \mathcal{L} and \mathcal{L}' are not equivalent.

3. Metric invariants

The Hessian metric $\text{Hess}(\mathcal{L})$ of a function $\mathcal{L} \in C^\infty(J^1(\mathbb{R}, M))$ is the section of the bundle $\varkappa: S^2[V^*(p^{10})] \rightarrow J^1(\mathbb{R}, M)$ defined in the formula (11) of the subsection 2.2.

Let $\mathcal{M} \subset S^2[V^*(p^{10})]$ be the open subbundle whose fibre $\mathcal{M}_{j_x^1\sigma}$ over $j_x^1\sigma$ is the set of non-degenerate symmetric bilinear forms $V_{j_x^1\sigma}(p^{10}) \times V_{j_x^1\sigma}(p^{10}) \rightarrow \mathbb{R}$.

Every $\Phi \in \text{Aut}(p)$ induces an isomorphism $(\Phi^{(1)})_*: V_{j_x^1\sigma}(p^{10}) \rightarrow V_{j_x^1\sigma}(p^{10})$, as $p^{10} \circ \Phi^{(1)} = \Phi \circ p^{10}$. Hence Φ induces a diffeomorphism $\Phi_{\mathcal{M}}^{(1)}: \mathcal{M} \rightarrow \mathcal{M}$ defined by $\Phi_{\mathcal{M}}^{(1)}(g_{j_x^1\sigma}) = (\Phi^{(1)})^{-1} * (g_{j_x^1\sigma})$ so that $\varkappa \circ \Phi_{\mathcal{M}}^{(1)} = \Phi \circ \varkappa$.

A function $I \in C^\infty(J^r\mathcal{M})$ is said to be a metric invariant of order r (cf. [6, §2]) if the following property holds:

$$\begin{aligned} I(\Psi_{\mathcal{M}}^{(r)}(j_{j_x^1\sigma}^r g)) &= I(j_{j_x^1\sigma}^r g), \quad \forall \Psi \in \text{Diff } J^1(\mathbb{R}, M), \\ &\forall j_x^1\sigma \in J^1(\mathbb{R}, M), \quad \forall g \in \Gamma(\varkappa), \end{aligned}$$

where $\Psi_{\mathcal{M}}: \mathcal{M} \rightarrow \mathcal{M}$ is defined in the same way as $\Phi_{\mathcal{M}}^{(1)}$, and $\Psi_{\mathcal{M}}^{(r)}$ is its r -th jet prolongation.

In particular, if I is a metric invariant, then $I((\Phi_{\mathcal{M}}^{(1)})^{(r)}(j_{j_x^1\sigma}^r g)) = I(j_{j_x^1\sigma}^r g)$, $\forall \Phi \in \text{Aut}(p)$.

Proposition 3.1. *Every metric invariant of order r induces an invariant in the sense of Proposition 1.1 of order $r + 2$. Therefore, if $\dim M = m \geq 2$, then there exists $\text{Aut}(p)$ -invariant functions that cannot be obtained as derivatives of the second-order basic invariant I defined in Proposition 2.3, but all of these invariants are of order ≥ 3 .*

Proof. Let $O'^2 \subset J^2(q)$ be the dense open subset of elements $j_{j_{x_0}^1\sigma_0}^2(\mathcal{L})$ for which the Hessian metric $\text{Hess}_{j_{x_0}^1\sigma_0}(\mathcal{L})$ is non-singular and $\mathcal{L}(j_{x_0}^1\sigma_0) \neq 0$, as in §2.3.1.

Let $q^{kh}: J^k(q) \rightarrow J^h(q)$, $k \geq h$, be the canonical projection. For every $r \geq 2$, let O'^r be the dense open subset in $J^r(q)$ given by $O'^r = (q^{r2})^{-1}(O'^2)$ and for every $r \geq 0$ let $\Theta^r: J^{r+2}(O'^2) \rightarrow J^r(\mathcal{M})$ the fibred map defined as

$$\Theta^r(j_{j_x^1\sigma}^{r+2}\mathcal{L}) = j_{j_x^1\sigma}^r(\text{Hess}(\mathcal{L})),$$

which is $\text{Aut}(p)$ -equivariant with respect to the natural actions on these spaces by virtue of Proposition 2.2. Hence, every invariant function $I \in C^\infty(J^r\mathcal{M})$ induces an invariant function $I \circ \Theta^r$ of order $r + 2$.

Moreover, as is well known, the basic metric invariants are the scalar contractions of the successive covariant differentials of the curvature tensor of the corresponding Levi-Civita of a metric. Hence, for a general metric $g \in \Gamma(\varkappa)$, every metric invariant is of order ≥ 2 , but the curvature tensor of a Hessian metric

$$g = \sum_{h,i=1}^m g_{hi} dy^h \otimes dy^i, \quad g_{hi} = g_{ih} = \frac{\partial^2 \mathcal{L}}{\partial \dot{y}^h \partial \dot{y}^i},$$

depends on the third derivatives of \mathcal{L} (e.g., see [11, Proposition 2.3–(1)]), thus concluding. \square

Example 3.2. If $\dim M = m = 2$, then the basic metric invariant is the Gaussian curvature (cf. [3, formula (1.9)]):

$$\begin{aligned}
4 \left[\frac{\partial^2 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^1} \frac{\partial^2 \mathcal{L}}{\partial \dot{y}^2 \partial \dot{y}^2} - \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^2} \right)^2 \right]^2 K = & \\
- \frac{\partial^2 \mathcal{L}}{\partial \dot{y}^2 \partial \dot{y}^2} \left[\frac{\partial^3 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^1 \partial \dot{y}^1} \frac{\partial^3 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^2 \partial \dot{y}^2} - \left(\frac{\partial^3 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^1 \partial \dot{y}^2} \right)^2 \right] & \\
+ \frac{\partial^2 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^2} \left[\frac{\partial^3 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^1 \partial \dot{y}^1} \frac{\partial^3 \mathcal{L}}{\partial \dot{y}^2 \partial \dot{y}^2 \partial \dot{y}^2} - \frac{\partial^3 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^1 \partial \dot{y}^2} \frac{\partial^3 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^2 \partial \dot{y}^2} \right] & \\
- \frac{\partial^2 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^1} \left[\frac{\partial^3 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^1 \partial \dot{y}^2} \frac{\partial^3 \mathcal{L}}{\partial \dot{y}^2 \partial \dot{y}^2 \partial \dot{y}^2} - \left(\frac{\partial^3 \mathcal{L}}{\partial \dot{y}^1 \partial \dot{y}^2 \partial \dot{y}^2} \right)^2 \right], &
\end{aligned}$$

which is an invariant of third order.

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