Computational and Applied Mathematics (2024) 43:81
https://doi.org/10.1007/s40314-023-02581-7

Numerical approximation of the boundary control for the wave equation in a square domain with a spectral collocation method

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Received: 11 April 2023 / Revised: 27 November 2023 / Accepted: 18 December 2023 /
Published online: 6 February 2024

Abstract

We propose a spectral collocation method to approximate the exact boundary control of the wave equation in a square domain. The idea is to introduce a suitable approximate control problem that we solve in the finite-dimensional space of polynomials of degree $N \in \mathbb{N}$ in space. We prove that we can choose a sequence of controls f^N associated with the approximate control problem in such a way that they converge, as $N \to \infty$, to a control of the continuous wave equation. Unlike other numerical approximations tried in the literature, this one does not require regularization techniques and can be easily adapted to other equations and systems where the controllability of the continuous model is known. The method is illustrated with several examples in 1-d and 2-d in a square domain. We also give numerical evidence of the highly accurate approximation inherent to spectral methods.

Keywords Numerical approximation · Controllability · Spectral collocation method · Wave equation

Mathematics Subject Classification 65N35 · 65K10 · 49M25 · 93B05

Communicated by José R. Fernánde.

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1 Introduction

Consider the wave equation on a square domain $\Omega = (-1,1)^d \subset \mathbb{R}^d$ ($d = 1, 2$) with a control f acting on one part of the boundary $\Gamma \subset \partial\Omega$ for some time $t \in (0, T)$

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = (0,T) \times \Omega \\ u = f & \text{on } (0,T) \times \Gamma \\ u = 0 & \text{on } (0,T) \times \partial\Omega \backslash \Gamma \\ u(0,x) = u^0(x), \;\; u_t(0,x) = u^1(x) & \text{in } \Omega. \end{cases} \tag{1}$$

Given any $f \in L^2((0,T) \times \Gamma)$ and some initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, problem (1) has a unique solution $(u, u_t) \in C([0,T], L^2(\Omega) \times H^{-1}(\Omega))$. It is also well known that, if $T > T_0$ with T_0 sufficiently large and $\Gamma \subset \partial\Omega$ satisfies some geometric conditions (see Bardos et al. 1992; Lions 1988), for any initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a control $f \in L^2((0,T) \times \Gamma)$, such that the solution of (1) $u \in C([0,T]; H_0^2) \cap C^1([0,T]; L^2)$ can be driven to any final target. We assume without loss of generality that this final target is the equilibrium, that is

$$u(T,x) = u_t(T,x) = 0, \;\; x \in \Omega. \tag{2}$$

In particular, this is true in dimension $d = 1$ when Γ is one extreme and $T_0 = 4$, and in dimension $d = 2$ when Γ is the union of two consecutive sides and $T_0 = 4\sqrt{2}$ (see Lions 1988). It is important to note that the control, when it exists, is not unique in general. Among the set of controls, a natural choice is to consider the minimal L^2-norm control which is usually unique.

In this work, we focus on the numerical approximation of these boundary controls f. This problem has been extensively studied in the last decades with different numerical methods. In particular, it is well known that a discretization of system (1) with finite elements or finite difference schemes reduces the problem to a finite-dimensional control problem which is not uniformly controllable with respect to the discretization parameter. This means that the discrete control problem does not provide a bounded sequence of controls, and therefore, it is not possible to use this strategy to approximate the continuous control. This was first observed in Glowinski et al. (1990) where the authors considered a finite-dimensional version of the Hilbert Uniqueness Method (HUM) introduced in Lions (1988). Since then, several cures have been proposed to recover convergence approximations of the controls as bigrid algorithms, Tychonoff regularization, filtering, mixed finite elements, etc. (see for instance, Castro 1999; Castro et al. 2008; Ervedoza and Zuazua 2012; Glowinski et al. 2008 and the review paper Zuazua 2005). We also mention more recent approaches where controls are obtained by minimizing a cost function that penalizes both the control and the state (Cindea et al. 2013) or those based on a space-time formulation that does not require regularizations (Burman et al. 2021).

Here, we propose a new numerical approach based on the spectral collocation method. For the background and details on this method as well as on other classes of spectral methods (Galerkin, collocation,…), we refer the reader to Gottlieb and Lustman (1983), Canuto et al. (1988), and Bernardi and Maday (1997). These methods have been extensively used in the past 30 years, especially for the numerical simulation of fluid dynamical problems (e.g., Canuto et al. 1988). According to this approach, the numerical solution is regarded as a smooth global polynomial of degree N (typically quite large), and satisfies the equilibrium equations point-wise at a family of collocation points that are the nodes of a high precision Gauss–Lobatto integration formula. This process can be regarded as a generalized Galerkin method: the distinguishing feature with respect to the conventional finite-element method is that trial (as well as test functions are global (rather than piecewise) polynomials of high degree. Two important consequences are derived from this fact. The first is that the spectral method is potentially extremely accurate. Indeed, for problems with smooth data, the order of convergence of the numerical solution is much higher than that achievable by finite-element approximations. To some extent, this feature still holds even for problems with low smoothness solution, such as those arising in fracture mechanics, or whenever loads are concentrated on a small part of the boundary. The second consequence of the global character of the test functions is that the spectral matrices are severely ill-conditioned and preconditioning techniques must be implemented for large-scale problems.

The use of spectral methods to approximate the control of the wave equation has been previously investigated by Boulmezaoud and Urquiza (2007) where, instead of collocation, a Galerkin spectral method is considered. The proposed approximation is not uniformly controllable. However, a bounded sequence of controls is obtained when trying to control the projection of the solution in a suitable low frequencies' space, similar as the result obtained for finite differences in Infante and Zuazua (1999). From a practical point of view, this is not satisfactory, since it requires to know an accurate representation of the eigenfunctions associated with the discrete problem, something which is not available in general.

The novelty here is that we are able to prove the uniform controllability, and therefore a convergent sequence of controls as $N \to \infty$, by adding an extra discrete boundary control that vanishes as $N \to \infty$. This provides an accurate approximation of the continuous control. The result relies on two key aspects: a uniform observability inequality for the associated discrete adjoint system and a detailed spectral analysis of the discrete low frequencies. The first property allows us to obtain the uniform boundedness of discrete controls, while the second one is used to obtain the convergence of the discrete control to the continuous one.

The method we present here to obtain the uniform observability inequality is new and considers, instead of the discrete collocation system, the equivalent continuous error equation associated with the polynomial approximation (see Gottlieb and Lustman 1983). This error equation is the same wave equation but with a nonhomogeneous second-hand term, known as the error term. Therefore, the observability inequality can be derived using the same techniques as in the continuous model and we only have to estimate this extra error term. This is an important advantage of the method, since it can be easily extended to more general equations (elasticity, fluid dynamics, etc.) and higher dimensions, as long as we consider rectangular domains.

The second important advantage of the method is in the convergence rate of the approximation. Here, we only prove that convergence holds, but the numerical experiments illustrate that one recovers the high accuracy expected by a spectral method, even when nonsmooth data are considered. For example, in experiment 2, we estimated rates of convergence for the L^2-norm of the order $N^{-1.5}$ and $N^{-0.4}$ for two chosen Lipschitz and discontinuous initial data respectively.

To clarify the exposition, we present detailed proofs in the one-dimensional case and the main results for the square domain. As we mentioned before, the proofs can be easily adapted by separation of variables due to the particular geometry of the domain considered here and the polynomial approximation. Of course, our approach is not suitable to tackle generalization to more general domains.

In this paper, we focus on the space discretization of the control problem. A complete numerical approximation would also require a suitable time discretization and this could obviously affect to the convergence of the fully discrete controls (both in space in time). This is an interesting question that we do not address here. Let us simply mention that we did not observe any singular phenomenon associated to the time discretization in the numerical experiments below.

The rest of the paper is divided in four more sections. Section 2 is devoted to state and prove the main results for the 1-d wave equation. Section 3 states the main results for the 2-d case. In Sect. 4, we present some numerical examples in 1-d and in 2-d in a square domain. Finally, the Appendix contains the main spectral results required for the analysis of the convergence.

2 The 1-d wave equation

In this section, we focus on the 1-d wave equation. Here, $\Omega = (-1, 1)$ and we assume that the control f acts at the right extreme $x = 1$. System (1) reads

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (t, x) \in \ (0, T) \times \Omega \\ u(t, 1) = f(t) & \text{in } t \in \ (0, T) \\ u(t, -1) = 0 & \text{in } t \in \ (0, T) \\ u(0, x) = u^0, \quad u_t(0, x) = u^1 & \text{in } x \in \ \Omega, \end{cases} \tag{3}$$

where $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ is the given initial data.

Assume that $T > 4$. We are interested in approximating one of the controls $f \in L^2(0, T)$ for which (2) is satisfied. More precisely, we follow HUM (see Lions 1988) and approximate the control that has minimal L^2-norm with respect to a suitable weighted norm, with a smooth weight $\eta(t)$ that is compactly supported bump function in $t \in (0, T)$. This ensures that the control itself is compactly supported and avoids possible singularities at times $t = 0, T$. To find a numerical approximation of this control f in (3), we proceed as follow: first we introduce a discrete version of the control problem (3), depending on a discrete parameter $N \longrightarrow \infty$. Then, we prove that this system is controllable for all N with three different controls, $f^N, g_R^N, g_L^N \in L^2(0, T)$, that we can choose in such way that

$$f^N \underset{N\to\infty}{\longrightarrow} f, \; g_R^N \underset{N\to\infty}{\longrightarrow} 0, \; g_L^N \underset{N\to\infty}{\longrightarrow} 0,$$

where f is a control of (3). Therefore, f^N is a numerical approximation of a continuous control f, while g_R^N, g_L^N can be understood as artificial controls which are only necessary to obtain f^N.

2.1 Approximation by the spectral collocation method

In this section, we introduce some notation and state the main results of the paper.

Let N be a natural number and consider $C = \{x_i, \; 0 \le i \le N\}$ the Legendre–Gauss–Lobatto (LGL) nodes in Ω that are the roots of

$$(1 - x^2)\frac{\mathrm{d}}{\mathrm{d}x} L^N(x),$$

where $L^k(t)$ is the k-th Legendre polynomial in $(-1, 1)$ (e.g., Szego 1939). We divide $C = C^\Omega \cup C^{Di}$ into interior and boundary nodes, that is

$$C^\Omega = C \cap \Omega = \{x_i, i \in I_\Omega\}$$
$$C^{Di} = \{-1, 1\} = \{x_i, i \in I_{Di}\},$$

where $I_\Omega,\ I_{Di}$ are the sets of indexes corresponding to the interior and boundary collocation nodes, respectively, and we denote $I = I_\Omega \cup I_{Di}$.

Let $\mathbb{P}_N(\Omega)$ be the space of continuous functions in $\bar{\Omega}$ which are polynomials of degree less than or equal to N and let $\mathbb{P}_N^{Di}(\Omega)$ be the subspace of $\mathbb{P}_N(\Omega)$ of those functions vanishing on $x = \{-1, 1\}$. We define the following discrete inner product that approximates that of $L^2(\Omega)$:

$$(w, z)_N = \sum_{i \in I} (wz)(t, x_i)\omega_i, \quad 0 \le t \le T. \tag{4}$$

Here, ω_i is the discrete weight associated with the one-dimensional LGL quadrature formula (e.g., Canuto et al. 1988, Chapter 2). Owing to the exactness of the integration LGL formula, we have

$$(w, z)_N = \int_\Omega wz \,\mathrm{d}x \ \text{ for all } w, z \text{ such that } wz \in \mathbb{P}_{2N-1}(\Omega). \tag{5}$$

The symbol $\|\cdot\|_N$ denotes the discrete norm which is defined as $\|z\|_N^2 = (z, z)_N$.

We recall that the discrete norm $\|\cdot\|_N$ is uniformly equivalent to the L^2-norm $|\cdot|_{L^2}$ in $\mathbb{P}_N(\Omega)$ (Canuto et al. 1988, Chapter 9). In other words, there exist two positive constants $C_1 = 1$, and $C_2 = 2 + \dfrac{1}{N}$, such that

$$C_1\, |p|_{L^2}^2 \le \|p\|_N^2 \le C_2\, |p|_{L^2}^2, \quad \forall p \in \mathbb{P}_N(\Omega). \tag{6}$$

We denote by Ψ_i the Lagrange polynomials which is 1 at x_i and 0 at all the other collocation nodes. For the commonly used Gauss–Lobatto points, one has

$$\Psi_i(x) = \frac{1}{N(N+1)L_N(x_i)} \frac{(1-x^2)L_N(x)}{x - x_i}. \tag{7}$$

Observe that $\{\Psi_i,\ i \in I\}$ constitutes a basis in $\mathbb{P}_N^{Di}(\Omega)$.

Now, we introduce the following discrete control problem: Given $u^{0,N},\ u^{1,N} \in \mathbb{P}_N^{Di}(\Omega)$ and $T > 0$, find $f^N,\ g_R^N,\ g_L^N \in L^2(0, T)$, such that the solution $u^N \in C^2(0, T; \mathbb{P}_N(\Omega))$ of system:

$$\begin{cases} (u_{tt}^N - u_{xx}^N)(t, x_i) = g_L^N(t)G_L^N(x_i) + g_R^N(t)G_R^N(x_i) & \text{in } (t, x_i) \in (0, T) \times C^\Omega \\ u^N(t, 1) = f^N(t) & \text{in } t \in\ (0, T) \\ u^N(t, -1) = 0 & \text{in } t \in\ (0, T) \\ u^N(0, x_i) = u^{0,N}(x_i), u_t^N(0, x_i) = u^{1,N}(x_i) & \text{for}\ \ x_i \in C^\Omega, \end{cases} \tag{8}$$

satisfies

$$u^N(T, x_i) = u_t^N(T, x_i) = 0,\ \ x_i \in C^\Omega. \tag{9}$$

Here, $G_L^N, G_R^N \in \mathbb{P}_{N-1}(\Omega)$ are defined by

$$\begin{cases} G_L^N(x_i) = \left(\dfrac{h_{xx}^L}{\sqrt{\omega_0}} - \dfrac{\Psi_{0,x}}{\sqrt{\omega_0}\omega_0}\right)(x_i), \quad G_R^N(x_i) = \left(\dfrac{h_{xx}^R}{\sqrt{\omega_N}} + \dfrac{\Psi_{N,x}}{\sqrt{\omega_N}\omega_N}\right)(x_i) \\ h^L,\ h^R \in \mathbb{P}_N^{Di}(\Omega) \\ h^L(x_i) = \dfrac{1 - x_i}{2},\ \ h^R(x_i) = \dfrac{1 + x_i}{2},\ \ x_i \in C^\Omega. \end{cases} \tag{10}$$

Note that $h^L(x) \neq \dfrac{1-x}{2}$, since $h^L \in \mathbb{P}_N^{Di}(\Omega)$. Something similar can be said about $h^R(x)$.

The main results in this paper are the following:

Theorem 1 *Given $T > 4(2 + N^{-1})$ and $(u^{0,N},\ u^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$, there exist controls $f^N,\ g_L^N,\ g_R^N \in L^2(0, T)$, such that the solution u^N of (8) satisfies (9).*

Theorem 2 *Given $(u^0, u^1) \in L^2 \times H^{-1}$, there exists a sequence $(u^{0,N}, u^{1,N}) \in (\mathbb{P}_N^{Di}(\Omega))^2$, such that*

$$(u^{0,N}, u^{1,N}) \to (u^0, u^1) \quad in \ \ L^2 \times H^{-1}, \quad as \ \ N \to \infty.$$

Furthermore, for any $T > 4(2+N^{-1})$, we can choose the controls $f^N,\ g_L^N,\ g_R^N \in L^2(0, T)$, such that the solution u^N of (8) satisfies (9) and

$$f^N \to f, \quad g_R^N \to 0, \quad g_L^N \to 0, \quad as\ N \to \infty, in L^2(0, T),$$

where f is a control of the continuous wave equation (3).

When u^0 (resp. u^1) is a continuous functions, we can just take $u^{0,N} \in \mathbb{P}_N^{Di}$, such that $u^{0,N}(x_i) = u^0(x_i)$ (resp. $u^{1,N}(x_i) = u^1(x_i)$).

Remark 1 Note that the control time T in Theorem 2 is basically two times the time required in the continuous problem. This is due to the constant C_2 in (6) and probably not optimal, as we illustrate in the experiments below.

2.2 Existence of discrete controls: proof of Theorem 1

In this section, we prove Theorem 1. We first introduce a variational characterization of discrete controls (8) and then prove that a particular discrete control can be obtained as the minimizer of a convex quadratic functional defined on a polynomial space. Finally, we prove the coerciveness of the functional that guarantees the existence of minimizers.

Let us introduce the following bi-linear form in $\mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$:

$$\left\langle(\phi^{0,N}, \phi^{1,N}), (u^{0,N}, u^{1,N})\right\rangle_N = (u^{1,N}, \phi^{0,N})_N - (u^{0,N}, \phi^{1,N})_N. \tag{11}$$

Lemma 1 *Assume that $T > 0$, and consider some initial data $(u^{0,N}, u^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$. Any controls f^N, g_R^N, g_L^N that make the solution of the discrete system (8) satisfy (9) are solutions of*

$$\begin{aligned} &\int_0^T (\phi_x^N(t,1) - \omega_N \phi_{xx}^N(t,1)) f^N(t)\mathrm{d}t + \int_0^T \sqrt{\omega_N}\phi_{xx}^N(t,1) g_R^N(t)\mathrm{d}t \\ &\quad + \int_0^T \sqrt{\omega_0}\phi_{xx}^N(t,-1) g_L^N(t)\mathrm{d}t - \langle(\phi^N(0,\cdot), \phi_t^N(0,\cdot)), (u^{0,N}, u^{1,N})\rangle_N = 0, \end{aligned} \tag{12}$$

for all $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$, where $(\phi^N, \phi_t^N) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$ is the solution of the following collocation backwards wave equation:

$$\begin{cases} (\phi_{tt}^N - \phi_{xx}^N)(t, x_i) = 0 & in\ (t, x_i) \in (0, T) \times C^\Omega \\ \phi^N(t, 1) = \phi^N(t, -1) = 0 & in\ t \in (0, T) \\ \phi^N(T, x_i) = \phi^{0,N}(x_i), \quad \phi_t^N(T, x_i) = \phi^{1,N}(x_i) & at\ x_i \in C^\Omega. \end{cases} \tag{13}$$

Proof Multiplying the equation of $u^N(t, x_i)$ in (8) by $\omega_i \phi^N(t, x_i)$ and adding in $i \in I$, one obtains

$$\int_0^T (u_{tt}^N - u_{xx}^N, \phi^N)_N \mathrm{d}t = \int_0^T g_L^N \, (G_L^N, \phi^N)_N \mathrm{d}t + \int_0^T g_R^N \, (G_R^N, \phi^N)_N \mathrm{d}t. \tag{14}$$

We first simplify the left-hand side. Integrating by parts in time, space using (5), since the resulting integrand is also a polynomial of degree $2N-2$. Taking into account that f^N, g_L^N and g_R^N are controls, we have

$$\begin{aligned} 0 &= \int_0^T (u_{tt}^N - u_{xx}^N, \phi^N)_N \mathrm{d}t = \int_0^T (u^N, \phi_{tt}^N - \phi_{xx}^N)_N \mathrm{d}t \\ &\quad + \int_0^T f^N(t)\phi_x^N(t,1)\mathrm{d}t - \Big\langle (\phi^N(0,.), \phi_t^N(0,.)), (u^{0,N}, u^{1,N}) \Big\rangle_N \\ &= \int_0^T f^N(t)(\phi_x^N - \omega_N \phi_{xx}^N)(t,1)\mathrm{d}t - \Big\langle (\phi^N(0,.), \phi_t^N(0,.)), (u^{0,N}, u^{1,N}) \Big\rangle_N . \end{aligned} \tag{15}$$

The last equality is a consequence of the first equation in (13). Note that an extra term appears in the right-hand side of this expression coming from the fact that the first equation in (13) is only true for the interior nodes, while the discrete scalar product involves also the boundary nodes.

For the right-hand side in (14), we use again formula (5) and the fact that $G_R^N \phi^N$ is a polynomial of degree $2N-1$

$$\begin{aligned} (G_R^N, \phi^N)_N &= \frac{1}{\sqrt{\omega_N}} \left(h_{xx}^R + \frac{\Psi_{N,x}}{\omega_N}, \phi^N \right)_N = \frac{1}{\sqrt{\omega_N}} \int_{-1}^1 \left(h_{xx}^R + \frac{\Psi_{N,x}}{\omega_N} \right) \phi^N \mathrm{d}x \\ &= \frac{1}{\sqrt{\omega_N}} \left(\int_{-1}^1 h^R \phi_{xx}^N \mathrm{d}x - \int_{-1}^1 \frac{\Psi_N}{\omega_N} \phi_x^N \mathrm{d}x \right) \\ &= \frac{1}{\sqrt{\omega_N}} \left(\sum_{i \in I_\Omega} (h^R \phi_{xx}^N)(t, x_i)\omega_i - \phi_x^N(t,1) \right). \end{aligned} \tag{16}$$

To simplify the first term in the right-hand side, we observe that $h^R(x_i) = (1 + x_i)/2$ at the interior nodes, i.e., $i \in I_\Omega$. Then

$$\sum_{i \in I_\Omega} (h^R \phi_{xx}^N)(t, x_i)\omega_i = \int_{-1}^1 \frac{1+x}{2} \phi_{xx}^N \, \mathrm{d}x - \omega_N \phi_{xx}^N(t,1) = \phi_x^N(t,1) - \omega_N \phi_{xx}^N(t,1). \tag{17}$$

From (16)–(17), we easily obtain

$$(G_R^N, \phi^N)_N = -\sqrt{\omega_N} \phi_{xx}^N(t,1). \tag{18}$$

Combining (14), (15) and (18) we easily find (12). □

According to HUM, one possibility to construct controls f^N, g_R^N, g_L^N that satisfy the variational condition (9) is as minimizers of the following cost functional $J^N : \mathbb{P}_N^{Di}(\Omega) \times$

$\mathbb{P}_N^{Di}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} J^N(\phi^{0,N}, \phi^{1,N}) &= \frac{1}{2}\int_0^T \eta(t)\left|\phi_x^N(t,1) - \omega_N \phi_{xx}^N(t,1)\right|^2 \mathrm{d}t \\ &+ \frac{1}{2}\int_0^T \eta(t)\omega_N \left|\phi_{xx}^N(t,1)\right|^2 \mathrm{d}t + \frac{1}{2}\int_0^T \eta(t)\omega_0 \left|\phi_{xx}^N(t,-1)\right|^2 \mathrm{d}t \\ &- \left\langle(\phi^N(0,\cdot), \phi_t^N(0,\cdot)), (u^{0,N}, u^{1,N})\right\rangle_N, \end{aligned} \tag{19}$$

where (ϕ^N, ϕ_t^N) is the solution of (13) with final data $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$. The function $\eta(t)$ is a prescribed smooth function in $[0, T]$ introduced to guarantee that the controls vanish in a neighborhood of $t = 0, T$. Thus, we consider $\delta > 0$ a small number and $0 \le \eta(t) \le 1$, such that

$$\eta(t) = \begin{cases} 1 & \text{in } [2\delta, T - 2\delta] \\ 0 & \text{in } [0, \delta] \cup [T - \delta, T]. \end{cases} \tag{20}$$

Note that both η and J^N will depend on this parameter δ, but this is not relevant in the rest of the analysis and we will not make explicit this dependence in the notation.

Theorem 3 *Assume $(u^{0,N}, u^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$ and that $(\hat{\phi}^{0,N}, \hat{\phi}^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$ is a minimizer of J^N. If $\hat{\phi^N}$ is the corresponding solution of (13) with final data $(\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$, then*

$$\begin{aligned} &f^N(t) = \eta(t)(\hat{\phi_x^N}(t,1) - \omega_N \hat{\phi_{xx}^N}(t,1)) \\ &g_R^N(t) = \eta(t)\sqrt{\omega_N}\hat{\phi_{xx}^N}(t,1), \qquad g_L^N(t) = \eta(t)\sqrt{\omega_0}\hat{\phi_{xx}^N}(t,-1), \end{aligned} \tag{21}$$

are controls, such that the solution of (8) satisfies (9).

Proof If J^N achieves its minimum at $(\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$, its Gateaux derivative in the direction $(\phi^{0,N}, \phi^{1,N})$ must vanish, that is

$$\begin{aligned} 0 = &\int_0^T \eta(t)(\hat{\phi_x^N}(t,1) - \omega_N \hat{\phi_{xx}^N}(t,1))(\phi_x^N(t,1) - \omega_N \phi_{xx}^N(t,1))\mathrm{d}t \\ &+ \int_0^T \eta(t)\omega_N \hat{\phi_{xx}^N}(t,1)\phi_{xx}^N(t,1)\mathrm{d}t + \int_0^T \eta(t)\omega_0 \hat{\phi_{xx}^N}(t,-1)\phi_{xx}^N(t,-1)\mathrm{d}t \\ &- \left\langle(\phi^N(0,\cdot), \phi_t^N(0,\cdot)), (u^{0,N}, u^{1,N})\right\rangle_N, \end{aligned}$$

for all $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$. From Lemma 1, it follows that (21) are controls for which (8) holds. □

The functional J^N is clearly continuous and convex, so that the existence of a minimizer [and therefore a discrete control for system (8)] is guaranteed as soon as we prove its coercivity. This is a consequence of the following lemma. Note that this also concludes the proof of Theorem 1.

Lemma 2 *Given $T > 4(2 + N^{-1})$, there exists a constant $C > 0$, independent of N, such that the solutions of system (13) satisfy*

$$\begin{aligned} C\left\|(\phi_x^{0,N}, \phi^{1,N})\right\|_{N\times N}^2 \le &\int_0^T \left|\phi_x^N(t,1) - \omega_N \phi_{xx}^N(t,1)\right|^2 \mathrm{d}t \\ &+ \omega_N \int_0^T \left|\phi_{xx}^N(t,1)\right|^2 \mathrm{d}t + \omega_0 \int_0^T \left|\phi_{xx}^N(t,-1)\right|^2 \mathrm{d}t, \end{aligned} \tag{22}$$

for any final data $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$ *and* $\eta(t)$ *is defined as in* (20).

Proof We prove the following equivalent version of (22):

$$C\left\|(\phi_x^{0,N}, \phi^{1,N})\right\|_{N\times N}^2 \leq \int_0^T \left|\phi_x^N(t,1)\right|^2 \mathrm{d}t + \omega_N \int_0^T \left|\phi_{xx}^N(t,1)\right|^2 \mathrm{d}t + \omega_0 \int_0^T \left|\phi_{xx}^N(t,-1)\right|^2 \mathrm{d}t. \tag{23}$$

We first observe that the solutions of (13) solve the following equivalent system:

$$\begin{cases} \phi_{tt}^N - \phi_{xx}^N = -\phi_{xx}^N(t,-1)\Psi_0 - \phi_{xx}^N(t,1)\Psi_N & \text{in } (t,x) \in (0,T)\times\Omega \\ \phi^N(t,1) = \phi^N(t,-1) = 0 & \text{in } t \in (0,T) \\ \phi^N(T,x) = \phi^{0,N}, \phi_t^N(T,x) = \phi^{1,N} & \text{in } x \in \Omega. \end{cases} \tag{24}$$

In fact, this is easily seen by writing the polynomial $\phi_{tt}^N - \phi_{xx}^N$ in the Lagrangian basis and using system (13). Now, we try the classical multiplier technique to recover the observability inequality. Multiplying the equation in (24) by $(x+1)\phi_x^N$ and integrating by parts as for the continuous wave equation (see Lions 1988) if we set $X = \int_{-1}^1 (x+1)\phi_t^N\phi_x^N \mathrm{d}x\Big|_0^T$, one obtain

$$\frac{1}{2}\int_0^T\int_{-1}^1 (\left|\phi_t^N\right|^2 + \left|\phi_x^N\right|^2)\mathrm{d}x\mathrm{d}t \leq |X| + \int_0^T \left|\phi_x^N(t,1)\right|^2 \mathrm{d}t + \left|\int_0^T \phi_{xx}^N(t,-1)\int_{-1}^1 (x+1)\phi_x^N \Psi_0 \mathrm{d}x\mathrm{d}t\right| + \left|\int_0^T \phi_{xx}^N(t,1)\int_{-1}^1 (x+1)\phi_x^N \Psi_N \mathrm{d}x\mathrm{d}t\right|. \tag{25}$$

In the rest of this proof, we estimate each one of the terms in this expression.

We start with the left-hand side in (25). Define the discrete energy

$$E^N = \frac{1}{2}\left(\left\|\phi_t^N\right\|_N^2 + \left\|\phi_x^N\right\|_N^2\right). \tag{26}$$

It is easy to see that this energy is conserved, i.e., $E_t^N = E_0^N$ for all $t > 0$. We just multiply (13) by $\phi_t^N \omega_i$, add in $i \in I$ and integrate with respect time. This together with the norm equivalence in (6) gives

$$\frac{1}{2}\int_0^T\int_{-1}^1 \left(\left|\phi_t^N\right|^2 + \left|\phi_x^N\right|^2\right)\mathrm{d}x\mathrm{d}t \geq \frac{1}{C_2}\int_0^T E_t^N \mathrm{d}x\mathrm{d}t = \frac{T}{C_2}E_T^N.$$

We now estimate the terms in the right-hand side of (25). We start with X. Observe that

$$\left|\int_{-1}^1 (x+1)\phi_t^N\phi_x^N \mathrm{d}x\right| \leq 2\frac{1}{2}\left(\left\|\phi_t^N\right\|_{L^2}^2 + \left\|\phi_x^N\right\|_{L^2}^2\right) \leq \left(\left\|\phi_t^N\right\|_N^2 + \left\|\phi_x^N\right\|_N^2\right) = 2E_t^N = 2E_T^N. \tag{27}$$

Therefore, $|X| \leq 4E_T^N$. Let us turn to estimate the third term in the right-hand side in (25). Using Young's inequality, for any $\varepsilon > 0$, we can find a sufficient large constant $C_\varepsilon > 0$, such that

$$\left|\int_0^T \phi_{xx}^N(t,-1)\int_{-1}^1 (x+1)\phi_x^N \Psi_0 \mathrm{d}x\mathrm{d}t\right| \le \int_0^T \left|\phi_{xx}^N(t,-1)\right| |\Psi_0|_{L^2(\Omega)} \left|\phi_x^N\right|_{L^2(\Omega)} \mathrm{d}t$$
$$\le C_\varepsilon |\Psi_0|^2_{L^2(\Omega)} \left|\phi_{xx}^N(t,-1)\right|^2_{L^2(0,T)} + \varepsilon \left|\left|\phi_x^N\right|_{L^2(\Omega)}\right|^2_{L^2(0,T)}. \tag{28}$$

Taking into account the norm equivalence in (6), the conservation of the discrete energy proved above and the fact that, as $\Psi_0 \in \mathbb{P}_N(\Omega)$, $|\Psi_0|^2_{L^2(\Omega)} \le \|\Psi_0\|^2_N = \omega_0$, we obtain

$$\left|\int_0^T \phi_{xx}^N(t,-1)\int_{-1}^1 (x+1)\phi_x^N \Psi_0 \mathrm{d}x\mathrm{d}t\right| \le C_\varepsilon \omega_0 \left|\phi_{xx}^N(t,-1)\right|^2_{L^2(0,T)} + 2\varepsilon T E_T^N. \tag{29}$$

An analogous estimate holds for the last term in the right-hand side in (25).

It follows from (25), (29) and the fact that $\omega_N = \omega_0$:

$$\left(\frac{T}{C_2} - 4 - 4\varepsilon T\right) E_0^N \le \int_0^T \left|\phi_x^N(t,1)\right|^2 \mathrm{d}t + \omega_N C_\varepsilon \left|\phi_{xx}^N(t,1)\right|^2_{L^2(0,T)}$$
$$+ \omega_N C_\varepsilon \left|\phi_{xx}^N(t,-1)\right|^2_{L^2(0,T)}.$$

Then, inequality (23) holds as long as $\frac{T}{C_2} - 4\varepsilon T - 4 > 0$. As ε can be chosen arbitrarily small and $C_2 = 2 + 1/N$, we have the condition $T > 4(2 + N^{-1})$. □

Remark 2 The method we present here to obtain the uniform observability inequality relies on the continuous error equation (24), equivalent to (13). The observability inequality is derived using the same techniques as in the continuous model (see Lions 1988) and we only have to estimate the extra error term appearing in (24). Therefore, this can be easily adapted to other equations or systems and higher dimensions where the controllability of the continuous model is known.

2.3 Convergence of the discrete controls: proof of Theorem 2

We now prove the convergence result mentioned in Theorem 2. All along this section, we assume that the hypotheses of the theorem hold. To clarify the exposition, we proceed in three steps where we first define the sequence $(u^{0,N}, u^{1,N}) \in (\mathbb{P}_N^{Di}(\Omega))^2$ and prove the uniform boundedness of the associated sequence of controls, then characterize their weak limit, and finally prove the strong convergence, respectively.

Step 1: Uniform bound of the controls. We first state the following result that we prove in the Appendix below.

Lemma 3 *Given $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a sequence $(u^{0,N}, u^{1,N}) \in (\mathbb{P}_N^{Di}(\Omega))^2$, such that*

$$(u^{0,N}, u^{1,N}) \longrightarrow (u^0, u^1) \text{ in } L^2(\Omega) \times H^{-1}(\Omega). \tag{30}$$

On the other hand, for any $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$, we have

$$\left|\left\langle(\phi^{0,N}, \phi^{1,N}), (u^{0,N}, u^{1,N})\right\rangle_N\right| \le |(u^0, u^1)|_{L^2 \times H^{-1}} \|(\phi_x^{0,N}, \phi^{1,N})\|_{N\times N}. \tag{31}$$

Furthermore, if $(\phi^{0,N}, \phi^{1,N}) \to (\phi^0, \phi^1)$ *in* $H_0^1(\Omega) \times L^2(\Omega)$*, then*

$$\big\langle (\phi^{0,N}, \phi^{1,N}), (u^{0,N}, u^{1,N}) \big\rangle_N \to \big\langle (\phi^0, \phi^1), (u^0, u^1) \big\rangle, \tag{32}$$

where

$$\big\langle (\phi^0, \phi^1), (u^0, u^1) \big\rangle = \big\langle u^1, \phi^0 \mathrm{d}x \big\rangle_{H^{-1}, H_0^1} - \int_{-1}^{1} u^0 \phi^1 \mathrm{d}x, \tag{33}$$

and $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$ *is the duality product between* H^{-1} *and* H_0^1.

Remark 3 When u^0 and u^1 are continuous functions the polynomials $\tilde{u}^{0,N}, \tilde{u}^{1,N} \in \mathbb{P}_N^{Di}$ that coincides with u^0 and u^1 at the nodes $x_i \in C^\Omega$ give a discretization that satisfies

$$\tilde{u}^{0,N} \to u^0, \quad \tilde{u}^{1,N} \to u^1, \quad \text{in } L^2(-1, 1),$$

as $N \to \infty$. In this case, the result in Lemma 3 is still true when considering $(u^{0,N}, u^{1,N}) = (\tilde{u}^{0,N}, \tilde{u}^{1,N})$.

We choose $(u^{0,N}, u^{1,N}) \in (\mathbb{P}_N^{Di}(\Omega))^2$ as in this lemma. Note that, in particular, this sequence is uniformly bounded in $L^2 \times H^{-1}$ as $N \to \infty$ and the boundedness of the associated controls is a direct consequence of the following result.

Proposition 1 *Let* f^N, g_L^N, g_R^N *be the controls defined in* (21)*. Then, there exists a constant* $M > 0$*, independent of N, such that*

$$\int_0^T \left| f^N(t) \right|^2 \mathrm{d}t + \int_0^T \left| g_R^N(t) \right|^2 \mathrm{d}t + \int_0^T \left| g_L^N(t) \right|^2 \mathrm{d}t \le M \left| (u^0, u^1) \right|^2_{L^2 \times H^{-1}}. \tag{34}$$

Proof As $\hat{\phi}^N$ is the solution of the discrete adjoint system (13) associated with the the minimizer $(\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$ of J^N in $\mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$, we have in particular

$$J^N(\hat{\phi}^{0,N}, \hat{\phi}^{1,N}) \le J^N(0, 0) = 0. \tag{35}$$

Then, taking into account the conservation of the discrete energy, defined in (26), the approximation in (30), estimate (31), and the uniform observability inequality in Lemma 2, we obtain

$$\int_0^T \eta(t) \left| \hat{\phi}_x^N(t, 1)\, \mathrm{d}t - \omega_N \hat{\phi}_{xx}^N(t, 1) \right|^2 \mathrm{d}t + \int_0^T \eta(t) \omega_N \left| \hat{\phi}_{xx}^N(t, -1) \right|^2 \mathrm{d}t$$
$$+ \int_0^T \eta(t) \omega_N \int_0^T \left| \hat{\phi}_{xx}^N(t, 1) \right|^2 \mathrm{d}t \le C \left| (u^0, u^1) \right|^2_{L^2 \times H^{-1}},$$

which is equivalent to (34). □

Step 2: Weak convergence of the control f^N. Thanks to the bound (34) controls f^N, g_R^N, g_L^N are uniformly bounded in $L^2(0, T)$, and therefore, there exists a subsequence, still denoted by f^N, g_R^N, g_L^N, such that

$$f^N \underset{N \to \infty}{\rightharpoonup} h, \; g_R^N \underset{N \to \infty}{\rightharpoonup} h_R, \; g_L^N \underset{N \to \infty}{\rightharpoonup} h_L, \; \text{ weakly in } L^2(0, T). \tag{36}$$

Let us see that $h = f$ where f is the control with minimal L^2-weighted norm of system (3), with the weight function $\eta(t)$. In the next step, we show that $h_R = h_L = 0$. This control $f(t)$ can be characterized by the following two properties (see Castro 1999):

(P1) f satisfies the variational characterization of controls, that is

$$0=\int_0^T \phi_x(t,1)f(t)\mathrm{d}t-\big\langle(\phi(0,\cdot),\phi_t(0,\cdot)),(u^0,u^1)\big\rangle,\ \ \forall(\phi^0,\phi^1)\in H_0^1\times L^2, \tag{37}$$

where $\langle\cdot,\cdot\rangle$ is defined in (33) and ϕ is the solution of

$$\begin{cases}\phi_{tt}-\phi_{xx}=0 & \text{in } (t,x)\in(0,T)\times\Omega\\ \phi(t,1)=\phi(t,-1)=0 & \text{in } t\in(0,T)\\ \phi(T,x)=\phi^0,\quad \phi_t(T,x)=\phi^1 & \text{in } x\in\Omega.\end{cases} \tag{38}$$

(P2) $f(t)=\eta(t)\hat{\phi}_x(t,1)$ where $\hat{\phi}$ is a solution of adjoint continue problem (38) with final data $(\hat{\phi}^0,\hat{\phi}^1)$.

In what follows, we see that h verifies these two properties. We start with the second one. By the boundedness of controls, the estimate (22) and the norm equivalence in (6), we deduce that $(\hat{\phi}^{0,N},\hat{\phi}^{1,N})$ is uniformly bounded in $H_0^1\times L^2$. Therefore, we can extract a subsequence, still denoted $(\hat{\phi}^{0,N},\hat{\phi}^{1,N})$, such that

$$(\hat{\phi}^{0,N},\hat{\phi}^{1,N})\rightharpoonup(\hat{\phi}^0,\hat{\phi}^1) \text{ weakly in } H_0^1(\Omega)\times L^2(\Omega). \tag{39}$$

Let $\hat{\phi}^N$ and $\hat{\phi}$ be the solutions of the discrete adjoint system (13) and the continuous one (38), associated with the final data $(\hat{\phi}^{0,N},\hat{\phi}^{1,N})$ and $(\hat{\phi}^0,\hat{\phi}^1)$, respectively. The following holds:

$$\hat{\phi}^N\rightharpoonup\hat{\phi},\ \text{ weakly-* in }\ L^\infty(0,T;H_0^1(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega)), \tag{40}$$

$$\hat{\phi}_x^N(t,1)-\omega_N\hat{\phi}_{xx}^N(t,1)\rightharpoonup\hat{\phi}_x(t,1),\ \text{ weakly in }\ L^2(0,T). \tag{41}$$

The convergence result (40) is easily deduced from the classical theory of spectral approximation in Canuto et al. (1988, Section 10.5). Concerning (41), we write system (13) in a weak form and take as test functions $\psi^N(x)=\frac{1+x}{2}\in\mathbb{P}_N$ and $l(t)\in C_0^1(0,T)$. Multiplying Eq. (13) by the weights w_i and $\psi^N(x_i)l(t)$, adding in $i\in I$ and integrating in time, we easily obtain the following identity:

$$\begin{aligned}0=&-\int_0^T(\hat{\phi}_{tt}^N(t,\cdot),\frac{1+x}{2})_N l(t)\mathrm{d}t+\int_0^T\int_{-1}^1\hat{\phi}_{xx}^N(t,x)\frac{1+x}{2}l(t)\mathrm{d}t\\ &-\int_0^T\omega_N\hat{\phi}_{xx}^N(t,1))l(t)\mathrm{d}t\\ =&\int_0^T\int_{-1}^1\hat{\phi}_t^N(t,x)\frac{1+x}{2}l_t(t)\mathrm{d}t-\frac{1}{2}\int_0^T\int_{-1}^1\hat{\phi}_x^N(t,x)l(t)\mathrm{d}t\mathrm{d}x\\ &+\int_0^T(\hat{\phi}_x^N(t,1)-\omega_N\hat{\phi}_{xx}^N(t,1))l(t)\mathrm{d}t,\ \ \forall l(t)\in C_0^1(0,T).\end{aligned} \tag{42}$$

Note that in the second term on the right-hand side, we have used the quadrature formula, since the integrand is a polynomial of degree $2N-2$, and this allowed us to integrate by parts in the x variable.

We can pass to the limit in (42) thanks to (36) and (40). Then, $\hat{\phi}$ verifies

$$0=\int_0^T\int_{-1}^1\hat{\phi}_t(x)\frac{1+x}{2}l_t(t)\mathrm{d}x\mathrm{d}t-\frac{1}{2}\int_0^T\int_{-1}^1\hat{\phi}_x(t,x)l(t)\mathrm{d}x\mathrm{d}t+\int_0^T\tilde{h}(t)l(t)\mathrm{d}t, \tag{43}$$

for all $l(t) \in C_0^1(0,T)$. Here, $\tilde{h}$ is the weak limit of $\hat{\phi}_x^N(t,1) - \omega_N \hat{\phi}_{xx}^N(t,1)$. On the other hand, as $\hat{\phi}$ is a solution of (38), it also verifies

$$0 = \int_0^T \int_{-1}^1 \hat{\phi}_t(t,x)\frac{1+x}{2} l_t(t)\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T \int_{-1}^1 \hat{\phi}_x(t,x) l(t)\mathrm{d}x\mathrm{d}t + \int_0^T \hat{\phi}_x(t,1) l(t)\mathrm{d}t, \tag{44}$$

for all $l(t) \in C_0^1(0,T)$. From (43) and (44), we finally deduce

$$\int_0^T \tilde{h}(t) l(t)\mathrm{d}t = \int_0^T \hat{\phi}_x(t,1) l(t)\mathrm{d}t, \ \forall\, l(t) \in C_0^1(0,T),$$

and then $\tilde{h}(t) = \hat{\phi}_x(t,1)$ with $\hat{\phi}$ the solution of (38). This finishes the proof of (41) and in particular that h satisfies property (P2) above.

Now, we check that the weak limit of $f^N(t)$, $h(t)$, also verifies the first property (P1) above. We need the following lemma that we prove in the appendix below:

Lemma 4 *Given $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a sequence $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{Di}(\Omega)$, such that*

$$(\phi^{0,N}, \phi^{1,N}) \longrightarrow (\phi^0, \phi^1), \ in\ H_0^1(\Omega) \times L^2(\Omega). \tag{45}$$

Furthermore, if ϕ^N is the solution of (13) with final data $(\phi^{0,N}, \phi^{1,N})$ and ϕ is the solution of (38) with final data (ϕ^0, ϕ^1) the following holds:

$$\phi^N \longrightarrow \phi, \ in\ C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \tag{46}$$

$$\phi_x^N(t,1) \longrightarrow \phi_x(t,1), \ \ in\ \ L^2(0,T), \tag{47}$$

$$\sqrt{w_N}\phi_{xx}^N(t,\pm 1) \longrightarrow 0, \ \ in\ \ L^2(0,T), \tag{48}$$

Given $(\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$, by Lemma 4, there exists a sequence $(\phi^{0,N}, \phi^{1,N}) \in \mathbb{P}_N^{Di}(\Omega) \times \mathbb{P}_N^{\partial\Omega}(\Omega)$, such that (45) holds. Furthermore, from formula (46), we deduce in particular that

$$(\phi^N(0,x), \phi_t^N(0,x)) \longrightarrow (\phi(0,x), \phi_t(0,x)) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega), \tag{49}$$

where ϕ^N is the solution of (13) with final data $(\phi^{0,N}, \phi^{1,N})$ and ϕ is the solution of (38) with final data (ϕ^0, ϕ^1). Now, passing to the limit, as $N \longrightarrow \infty$ in formula (12) and taking into account Lemmas 3 and 4, we obtain that h satisfies

$$\int_0^T \phi_x(t,1) h\mathrm{d}t - \left\langle (\phi(0,\cdot), \phi_t(0,\cdot)), (u^0, u^1)\right\rangle = 0, \tag{50}$$

$\forall (\phi^0, \phi^1) \in H_0^1(\Omega) \times L^2(\Omega)$. Therefore, h verifies property (P1) above.

Step 3: Strong convergence of the controls. From the lower semi-continuity of the norm with respect to the weak convergence, we have

$$|f|_{L^2}^2 + |h_R|_{L^2}^2 + |h_L|_{L^2}^2 \le \liminf_{N\to\infty} |f^N|_{L^2}^2 + |g_R^N|_{L^2}^2 + |g_L^N|_{L^2}^2. \tag{51}$$

On the other hand, if we consider formulas (12) and (37) with $(\phi^{0,N}, \phi^{1,N}) = (\hat{\phi}^{0,N}, \hat{\phi}^{1,N})$ and $(\phi^0, \phi^1) = (\hat{\phi}^0, \hat{\phi}^1)$, respectively, we obtain

$$
\begin{aligned}
0 = &\int_0^T \eta(t) \left| \hat{\phi}_x^N(t,1) - \omega_N \hat{\phi}_{xx}^N(t,1) \right|^2 \mathrm{d}t + \int_0^T \eta(t) \left| \sqrt{\omega_N} \hat{\phi}_{xx}^N(t,1) \right|^2 \mathrm{d}t \\
&+ \int_0^T \eta(t) \left| \sqrt{\omega_0} \hat{\phi}_{xx}^N(t,-1) \right|^2 \mathrm{d}t - \left\langle (\hat{\phi}^N(\cdot,0), \hat{\phi}_t^N(\cdot,0)), (u^{0,N}, u^{1,N}) \right\rangle_N,
\end{aligned} \tag{52}
$$

and

$$
\int_0^T \eta(t) \left| \hat{\phi}_x(t,1) \right|^2 \mathrm{d}t - \left\langle (\hat{\phi}(0,\cdot), \hat{\phi}_t(0,\cdot)), (u^0, u^1) \right\rangle = 0. \tag{53}
$$

The last term in (52) converges to the second term in (53), and therefore

$$
\begin{aligned}
&\int_0^T \eta(t) \left| \hat{\phi}_x^N(t,1) - \omega_N \hat{\phi}_{xx}^N(t,1) \right|^2 \mathrm{d}t + \int_0^T \eta(t) \left| \sqrt{\omega_N} \hat{\phi}_{xx}^N(t,1) \right|^2 \mathrm{d}t \\
&\quad + \int_0^T \eta(t) \left| \sqrt{\omega_0} \hat{\phi}_{xx}^N(t,-1) \right|^2 \mathrm{d}t \longrightarrow \int_0^T \eta(t) \left| \hat{\phi}_x(t,1) \right|^2 \mathrm{d}t \ in\ L^2(0,T),
\end{aligned} \tag{54}
$$

as $N \longrightarrow \infty$. Now, taking into account (51), (54) and the definition of the controls in (21), we deduce

$$
|f|_{L^2}^2 + |h_R|_{L^2}^2 + |h_L|_{L^2}^2 \le |f|_{L^2}^2,
$$

and therefore, $h_R = h_L = 0$. From (54), we also have

$$
|f^N|_{L^2} \to |f|_{L^2}, \quad |h_R^N|_{L^2}^2 \to 0, \quad |h_L^N|_{L^2}^2 \to 0, \tag{55}
$$

as $N \to \infty$. The strong convergence of the controls $f^N \underset{N\to\infty}{\longrightarrow} f$ in $L^2(0,T)$ is a consequence of the weak convergence stated in Step 2 above and the convergence of the norms stated in (55).

3 The 2-d wave equation

In this section, we show how to extend the numerical approximation of the control problem (1) to a square domain $\Omega = (-1,1)^2$. The results are easily extended by separation of variables, but the notation is a little cumbersome. Here, we only state the main result and sketch the proof of the uniform observability result.

Now, we assume that the control acts in the right and top sides $\Gamma = \Gamma_1 \cup \Gamma_2 = \{(1, x_2) \in \mathbb{R}^2 : -1 \le x_2 \le 1\} \cup \{(x_1, 1) \in \mathbb{R}^2 : -1 \le x_1 \le 1\}$. The rest of the boundary is then the left and down sides $\partial\Omega \backslash \Gamma = \Gamma_3 \cup \Gamma_4 = \{(-1, x_2) \in \mathbb{R}^2 : -1 \le x_2 \le 1\} \cup \{(x_1, -1) \in \mathbb{R}^2 : -1 \le x_1 \le 1\}$.

We also consider the discrete parameter $\mathbf{N} = (N_1, N_2) \in \mathbb{N} \times \mathbb{N}$ and the Legendre Gauss–Lobatto nodes in each variable $C = \{\mathbf{x_i} = (x_{1k}, x_{2m}), (0,0) \le \mathbf{i} = (k,m) \le (N_1, N_2)\}$. Note that C (and some other quantities defined below) depends on N, but we do not make explicit this dependence in the notation to simplify. We also set $C = C^{\Omega} \cup C^{\partial\Omega}$, such that $C^{\partial\Omega} = C^{\Gamma} \cup C^{\partial\Omega\backslash\Gamma}$ and $\mathbf{I} = \mathbf{I}_\Omega \cup \mathbf{I}_{\partial\Omega}$, such that $\mathbf{I}_{\partial\Omega} = \cup_{k=1}^4 \mathbf{I}_k$, where $\mathbf{I}_k$ is the set of indexes corresponding to the collocation nodes on the boundary Γ_k. Let $\mathbb{P}_\mathbf{N}(\Omega)$ be the space of continuous functions in $\bar{\Omega}$ which are polynomials of degree less than or equal to N_1

(respectively N_2) in the x_1-variable (respectively in the x_2-variable), and let $\mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$ be the subspace of $\mathbb{P}_{\mathbf{N}}(\Omega)$ of those functions vanishing on $\partial\Omega$.

Now, we introduce the following discrete control problem: Given $u^{0,\mathbf{N}}, u^{1,\mathbf{N}} \in \mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$ and $T > 0$, find $f^{\mathbf{N}} \in L^2(0, T; \Gamma)$, $g_k^{\mathbf{N}} \in L^2(0, T; \Gamma_k)$, $k = 1, \ldots, 4$, such that the solution $u^{\mathbf{N}} \in C^\infty(L^2(0, T); \mathbb{P}_{\mathbf{N}}(\Omega))$ of the following system:

$$\begin{cases} \left(u_{tt}^{\mathbf{N}} - \Delta u^{\mathbf{N}}\right)(t, \mathbf{x_i}) = \sum_{k=1}^{4} g_k^{\mathbf{N}}(t, \mathbf{x_i}) G_k^{\mathbf{N}}(\mathbf{x_i}) & \text{in } (t, \mathbf{x_i}) \in (0, T) \times C^\Omega \\ u^{\mathbf{N}}(t, \mathbf{x_i}) = f^{\mathbf{N}}(t, \mathbf{x_i}) & \text{in } (t, \mathbf{x_i}) \in (0, T) \times C^\Gamma \\ u^{\mathbf{N}}(t, \mathbf{x_i}) = 0 & \text{in } (t, \mathbf{x_i}) \in (0, T) \times C^{\partial\Omega\setminus\Gamma} \\ u^{\mathbf{N}}(0, \mathbf{x_i}) = u^{0,\mathbf{N}}(\mathbf{x_i}), \quad u_t^{\mathbf{N}}(0, \mathbf{x_i}) = u^{1,\mathbf{N}}(\mathbf{x_i}) & \text{in } \mathbf{x_i} \in C^\Omega, \end{cases} \tag{56}$$

satisfies

$$u^{\mathbf{N}}(T, \mathbf{x_i}) = u_t^{\mathbf{N}}(T, \mathbf{x_i}) = 0, \ \ \mathbf{x_i} \in C^\Omega. \tag{57}$$

Here, $G_k^{\mathbf{N}}, k = 1, .., 4$ are defined as in the 1-d case. For example, for the left and right boundaries, Γ_3 and Γ_1, the functions $G_3^{\mathbf{N}}$ and $G_1^{\mathbf{N}}$ depend only on the x_1 variable and coincide with $G^L(x_1)$ and $G^R(x_1)$ defined in (10) associated with the polynomials of degree N_1 in x_1. Analogously, for the bottom and top boundaries, Γ_4 and Γ_2, the functions $G_4^{\mathbf{N}}$ and $G_2^{\mathbf{N}}$ depend only on the x_2 variable and coincide again with $G^L(x_2)$ and $G^R(x_2)$ defined in (10), this time associated with polynomials of degree N_2 in x_2.

We now state the main results of existence and convergence for discrete control.

Theorem 4 *Given $T > 4\sqrt{2}(2+N_1^{-1})(2+N_2^{-1})$ and $(u^{0,\mathbf{N}}, \ u^{1,\mathbf{N}}) \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^2$, there exist controls $f^{\mathbf{N}} \in \ L^2(0, T; \Gamma)$, $\ g_k^{\mathbf{N}} \in \ L^2(0, T; \Gamma_k)$, $k = 1, \ldots, 4$, such that the solution $u^{\mathbf{N}}$ of (56) satisfies (57).*

Theorem 5 *Given $(u^0, u^1) \in L^2 \times H^{-1}$, there exists a sequence $(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \in (\mathbb{P}_{\mathbf{N}}^{Di}(\Omega))^2$, such that*

$$(u^{0,\mathbf{N}}, u^{1,\mathbf{N}}) \to (u^0, u^1) \ \ \text{in } L^2 \times H^{-1}.$$

Furthermore, for any $T > 4\sqrt{2}(2 + N_1^{-1})(2 + N_2^{-1})$, we can choose the controls $f^{\mathbf{N}}$, $g_k^{\mathbf{N}}$, $k = 1, \ldots, 4$, such that the solution $u^{\mathbf{N}}$ of (56) satisfies (57) and

$$f^{\mathbf{N}} \longrightarrow f, \quad \text{in } \ L^2(0, T; \Gamma), \quad g_k^{\mathbf{N}} \longrightarrow 0, \quad \text{in } \ L^2(0, T; \Gamma_k). \tag{58}$$

The proofs of these two results follow closely the one-dimensional case. They are based on a suitable variational characterization of the controls and the uniform observability inequality for the corresponding adjoint system

$$\begin{cases} (\phi_{tt}^{\mathbf{N}} - \Delta\phi^{\mathbf{N}})(t, \mathbf{x_i}) = 0 & \text{in } (t, \mathbf{x_i}) \in (0, T) \times C^\Omega \\ \phi^{\mathbf{N}}(t, \mathbf{x_i}) = 0 & \text{in } (t, \mathbf{x_i}) \in (0, T) \times C^{\partial\Omega} \\ \phi^{\mathbf{N}}(T, \mathbf{x_i}) = \phi^{0,\mathbf{N}}, \phi_t^{\mathbf{N}}(T, \mathbf{x_i}) = \phi^{1,\mathbf{N}} & \text{in } \mathbf{x_i} \in C^\Omega. \end{cases} \tag{59}$$

Lemma 5 *Given $T > 4\sqrt{2}(2+N_1^{-1})(2+N_2^{-1})$, there exists a constant $C > 0$, independent of $\mathbf{N}$, such that the solution of system (59) satisfies*

$$\begin{aligned} C \left\| (\nabla\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \right\|_{\mathbf{N}\times\mathbf{N}}^2 \leq & \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_1\cup\mathbf{I}_2} \left(\frac{\partial\phi^{\mathbf{N}}}{\partial\nu} - \omega_{\mathbf{N}}^{\xi_1} \frac{\partial^2\phi^{\mathbf{N}}}{\partial^2\nu} \right)^2 (t, \mathbf{x_i}) \omega_{\mathbf{i}}^{\xi_2} \mathrm{d}t \\ & + \int_0^T \sum_{\mathbf{i}\in\mathbf{I}_{\partial\Omega}} \omega_{\mathbf{N}}^{\xi_1} \left(\frac{\partial^2\phi^{\mathbf{N}}}{\partial^2\nu} \right)^2 (t, \mathbf{x_i}) \omega_{\mathbf{i}}^{\xi_2} \mathrm{d}t, \end{aligned} \tag{60}$$

for any final data $(\phi^{0,\mathbf{N}}, \phi^{1,\mathbf{N}}) \in \mathbb{P}_{\mathbf{N}}^{Di}(\Omega) \times \mathbb{P}_{\mathbf{N}}^{Di}(\Omega)$. *Here,* $\omega_{\mathbf{N}}^{\xi_1}$ *and* $\omega_{\mathbf{i}}^{\xi_2}$ *are defined by*

$$\begin{cases} \omega_{\mathbf{N}}^{\xi_1} = \omega_{N_1}^{x_1}, \quad \omega_{\mathbf{i}}^{\xi_2} = \omega_m^{x_2} \quad \text{if } \mathbf{i} \in \mathbf{I}_1 \cup \mathbf{I}_3, \\ \omega_{\mathbf{N}}^{\xi_1} = \omega_{N_2}^{x_2}, \quad \omega_{\mathbf{i}}^{\xi_2} = \omega_k^{x_1} \quad \text{if } \mathbf{i} \in \mathbf{I}_2 \cup \mathbf{I}_4, \end{cases} \tag{61}$$

and $\omega_k^{x_1}, \omega_m^{x_2}$ *are the discrete weights associated with the one-dimensional LGL quadrature formula in each one of the variables (e.g.,* Canuto et al. 1988, Chapter 2).

As in the one-dimensional case, the proof of Lemma 5 is based on the associated error equation, equivalent to (59), and given by

$$\begin{cases} \phi_{tt}^{\mathbf{N}} - \Delta\phi^{\mathbf{N}} = -\sum_{\mathbf{i}\in I_{\partial\Omega}} \dfrac{\partial^2\phi^{\mathbf{N}}}{\partial^2\nu}(t, \mathbf{x_i})\Psi_{i_1}^{x_1}\Psi_{i_2}^{x_2} & \text{in } Q = (0, T) \times \Omega \\ \phi^{\mathbf{N}} = 0 & \text{on } (0, T) \times \partial\Omega \\ \phi^{\mathbf{N}}(T, x) = \phi^{0,\mathbf{N}}, \phi_t^{\mathbf{N}}(T, x) = \phi^{1,\mathbf{N}} & \text{in } \Omega, \end{cases} \tag{62}$$

where $\mathbf{i} = (i_1, i_2)$ and $\Psi_k^{x_1}(x_1)$ (respectively $\Psi_m^{x_2}(x_2)$) are the Lagrange polynomial which is 1 at x_{1k} (respectively at x_{2m}) and 0 at all the other collocation points. For this error equation, we can apply the classical multipliers technique (see Lions 1988). The extra terms coming from the right-hand side in (62) are estimated following the same idea in 1-d case. The analysis is straightforward and it does not introduce new difficulties.

4 Numerical experiments

In this section, we illustrate the results in this paper approximating the boundary control both for the 1-d and 2-d wave equation in the square. For the time discretization, we use a classical Newmark method with parameters $\gamma = 1/2$, $\beta = 1/4$, and time step $dt = 10^{-2}$. With this choice, the time scheme is second order accurate (see Raviart 1983).

Experiment 1: we first consider the one-dimensional wave equation with two different types of initial position and velocity. The first one corresponds to a smooth bump that moves to the left-hand side and it is controlled from the right extreme. It is given by $u^0(x) = e^{-10x^2}$, $u^1(x) = -20xe^{-10x^2}$. The second one corresponds to a Lipschitz continuous initial data $u^0(x) = 1 - |x|$, $u^1(x) = 0$. We take as final time $T = 4.4$. Note that the time control is only slightly greater than the minimal control time for the continuous wave equation ($T = 4$) and lower than the time given by the discrete control problem in Theorems 2 and 3. The good approximation obtained in this case provides numerical evidence that the control time in the mentioned theorems is probably not optimal.

In Tables 1 and 2, we show the behavior of the norm of the controls when the degree of polynomials N grows. As stated in Theorem 3, we observe that the boundary control remains bounded, while the two artificial controls included in the system (g_L^N and g_R^N) vanish as N grows. The boundary controls are plotted in Fig. 1.

Experiment 2: Here, we illustrate the rate of convergence of the discrete control to the limit one with two different types of initial data: the first one is given by $u^0(x) = 1-|x|, u^1(x) = 0$ (Lipschitz) and the other considers $u^0(x) = 2(x + 1)1_{(-1,0)}$, $u^1(x) = 0$ (discontinuous). The exact control is known in the latter and it is given by $f(t) = -(t - 2)\ 1_{(1,3)}(t)$ (see Burman et al. 2023). For the first one, we compare the L^2-norm of the difference between the discrete control when $N = 200$ (that we take as continuous control) and the discrete control as N grows. For the second one, we compare directly the approximation with the exact control. The approximate controls are plotted in Fig. 1 (center and right plots).

Table 1 Norm of the null controls when $u^0(x) = e^{-10x^2}$, $u^1(x) = -20xe^{-10x^2}$ as N grows

N	$\|f^N\|_{L^2}$	$\|g_R^N\|_{L^2}$	$\|g_L^N\|_{L^2}$
20	5.6×10^{-1}	2×10^{-3}	2×10^{-3}
50	5.6×10^{-1}	1×10^{-4}	1×10^{-4}
100	5.6×10^{-1}	1×10^{-6}	1×10^{-6}

Table 2 Norm of the null controls as N grows for $u^0(x) = 1 - |x|$, $u^1(x) = 0$

N	$\|f^N\|_{L^2}$	$\|g_R^N\|_{L^2}$	$\|g_L^N\|_{L^2}$
20	5.8×10^{-1}	3×10^{-4}	3×10^{-4}
50	5.8×10^{-1}	4×10^{-4}	4×10^{-4}
100	5.8×10^{-1}	1×10^{-4}	1×10^{-4}
200	5.8×10^{-1}	1×10^{-6}	1×10^{-6}

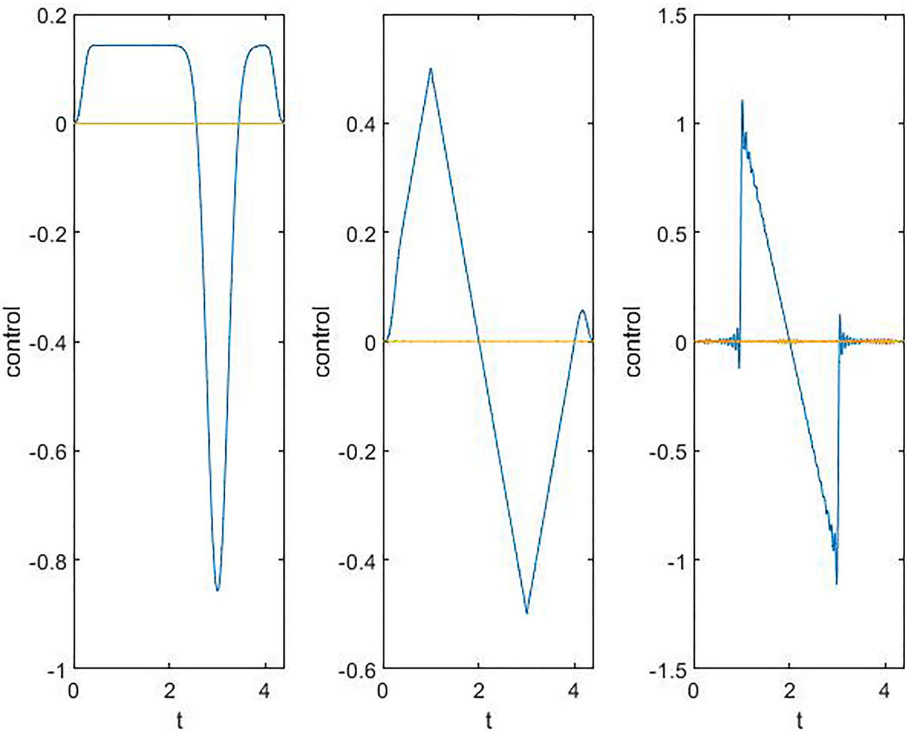

Fig. 1 Discrete controls f^{100} for different initial data: $u^0(x) = e^{-10x^2}$, $u^1(x) = -20xe^{-10x^2}$ (left one), $u^0(x) = 1 - |x|$, $u^1(x) = 0$ (center one) and $u^0(x) = 2(x+1)1_{(-1,0)}$, $u^1(x) = 0$ (right one)

In Tables 3 and 4, we show the error between the discrete control and the limit one. Comparing the error associated to values of $N \in [10, 100]$, we can give a rough estimate of the rate of convergence. More precisely, we observe that $\left|f^N - f\right|_{L^2} \sim N^{-\alpha}$ where α is estimated by the slope of the graphics relating $-\log_{10}\left|f^N - f\right|_{L^2}$ with $\log_{10} N$ (see Fig. 2).

Table 3 Convergence of the discrete control to the limit as N grows for $u^0(x) = 1 - |x|$, $u^1(x) = 0$

N	$\log(\|f^N - f^{200}\|_{L^2})$	$\log(\|g_R^N\|_{L^2})$	$\log(\|g_L^N\|_{L^2})$
10	−1.5	−2.5	−2.5
50	−2.5	−3.3	−3.3
100	−3.0	−3.8	−3.8

Table 4 Convergence of the discrete control to the limit as N grows for $u^0(x) = 2(x+1)1_{(-1,0)}$, $u^1(x) = 0$

N	$\log(\|f^N - f\|_{L^2})$	$\log(\|g_R^N\|_{L^2})$	$\log(\|g_L^N\|_{L^2})$
10	−0.4	−0.8	−0.8
50	−0.7	−1.5	−1.5
100	−0.8	−1.7	−1.7

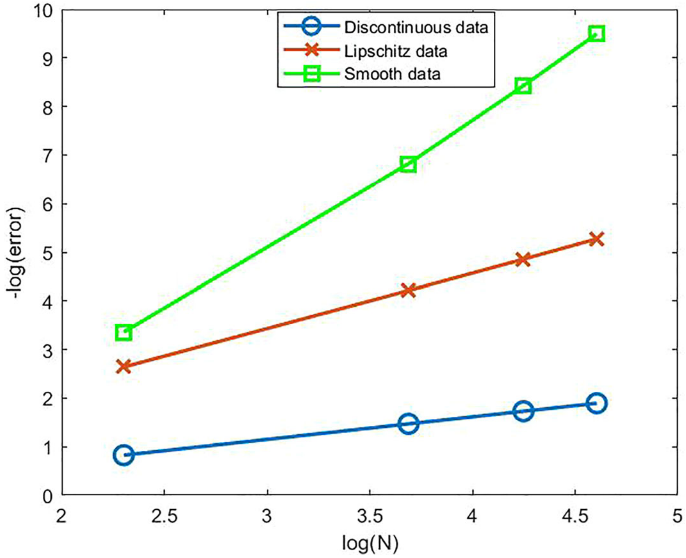

Fig. 2 The slope of each graphic gives an experimental estimate of the value α for which $|f^N - f|_{L^2} \sim N^{-\alpha}$. The three graphics correspond to the three initial data considered in experiments 1 and 2. We observe that α is greater for smoother data, as expected in spectral methods

It is interesting also to compare the results with those obtained in Burman et al. (2023) for the space-time finite-element method. While the error in the latter depends linearly on the size of the finite-element mesh, for the spectral method, the dependence is a power of the polynomial degree N. This power depends on the smoothness of the initial data. Of course, there are other important parameters involved as the complexity of the programming or the time cost that are much more difficult to assess.

Experiment 3: Now, we consider a two-dimensional square domain $(-1, 1)^2$. The control acts on the two sides $\{1\} \times (-1, 1) \cup (-1, 1) \times \{1\}$ in the time interval $t \in (0, 4.4)$, with step $dt = 10^{-2}$. We consider the degree of polynomial is $\mathbf{N} = (80, 80)$ in the x_1, x_2-variable, respectively. The initial position and velocity given by a bump function $u^0 = e^{-10x_1^2}e^{-10x_2^2}$ and $u^1 = (-20x_1e^{-10x_1^2})(-20x_2e^{-10x_2^2})$. As in 1-d, the time control is lower than the time given by the discrete control problem in Theorems 4 and 5, and also lower than the control

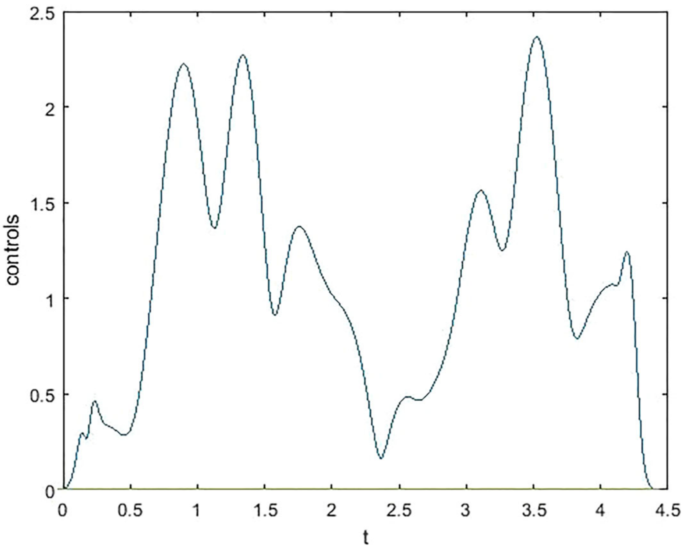

Fig. 3 The behavior of control $f^{\mathbf{N}}$ during the time for $\mathbf{N} = (80, 80)$

Table 5 Norm of the controls for $u^0(x) = e^{-10(x_1^2+x_2^2)}$, $u^1(x) = (20^2 x_1 x_2)\, e^{-10(x_1^2+x_2^2)}$ as $\mathbf{N}$ grows

$\mathbf{N}$	$\lvert f^{\mathbf{N}}\rvert_{L^2}$	$\lvert g_1^{\mathbf{N}}\rvert_{L^2}$	$\lvert g_2^{\mathbf{N}}\rvert_{L^2}$	$\lvert g_3^{\mathbf{N}}\rvert_{L^2}$	$\lvert g_4^{\mathbf{N}}\rvert_{L^2}$
(20, 20)	7.2×10^{-1}	8.5×10^{-3}	8.2×10^{-3}	8.5×10^{-3}	8×10^{-3}
(50, 50)	7.2×10^{-1}	7.4×10^{-4}	7.4×10^{-4}	7.4×10^{-4}	7.4×10^{-4}
(80, 80)	7.2×10^{-1}	6.4×10^{-5}	6.4×10^{-5}	6.4×10^{-5}	6.4×10^{-5}

time for the continuous wave equation ($T = 4\sqrt{2}$) (Lions 1988). However, the initial data are almost compactly supported in the disc $|\mathbf{x}| < 1$ inside the domain and this makes these special data controllable for the chosen time. In Fig. 3, we have drawn the behavior of the norm of control acting in the two sides of the square during the time, since the other controls are of the order 10^{-5}. As in 1-d in Table 5, we show the behavior of the norm for the controls when the degree of polynomials $\mathbf{N}$ grows. As stated in Theorem 5, we observe that the boundary control remains bounded, while the four artificial controls included in the system ($g_k^{\mathbf{N}}$, $k = 1, \ldots, 4$) vanish as $\mathbf{N}$ grows.

Acknowledgements The authors were supported by grant PID2021-124195NB-C31 from the Spanish government (MICINN). The first author also thanks the support of the Algerian government for the scholarship offered to finance the PhD at the Polytechnic University of Madrid.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

5 Appendix

In this section, we give a proof of Lemmas 3 and 4. Both proofs rely on a careful spectral analysis that we address first. Let $(\lambda_k, \varphi_k)_{k\in\mathbb{N}}$ be the eigenvalues and the eigenfunctions associated to the Laplace equation

$$\begin{cases} \varphi_{xx} + \lambda\varphi = 0, & \text{in } x \in \Omega = (-1, 1) \\ \varphi(1) = \varphi(-1) = 0. \end{cases} \tag{63}$$

The eigenvalues are simple and can be computed explicitly ($\lambda_k = (k\pi/2)^2$, $k \in \mathbb{N}$), while the associated eigenfunctions $\{\varphi_k\}_{k\in N}$ constitute an orthogonal basis in $L^2(-1, 1)$.

Associated to the collocation numerical approximation of the wave equation, we introduce the following discrete eigenvalue problem:

$$\begin{cases} \varphi^N_{xx}(x_i) + \lambda^N \varphi^N(x_i) = 0, & \text{at } \ x_i \in C^{\Omega} \\ \varphi^N(-1) = \varphi^N(1) = 0. \end{cases} \tag{64}$$

It is known that this eigenvalue problem admits $N - 1$ eigenvalues which are simple and real numbers (see Vandeven 1990; Weideman and Trefethen 1988). We assume that they are written in increasing order, i.e., $0 < \lambda^N_1 < \lambda^N_2 < \cdots < \lambda^N_{N-1}$. The associated eigenfunctions $\{\varphi^N_k\}_{k=1}^{N-1}$ constitute an orthogonal basis in $\mathbb{P}^{Di}_N$ with the discrete scalar product $(\cdot, \cdot)_N$.

From now on, we assume that both φ^N_k and φ_k are normalized in the L^2-norm.

The following Theorem states the spectral approximation results that we need.

Theorem 6 *Let $m \geq 2$, $\alpha \in (0, 2/\pi)$ and $r(N) = \alpha N^{\frac{1}{8}}$. Then, for any $k \in \{1, \ldots, r(N)\}$, there exists constants C_α, C, independent of N, such that the following estimates hold:*

$$\left| \sqrt{\lambda^N_k} - \sqrt{\lambda_k} \right| \leq C_\alpha p(\alpha)^{N^{2/3}}, \ 0 < p(\alpha) < 1, \tag{65}$$

$$\left| \varphi^N_k - \varphi_k \right|_{L^2} \leq C N^{-3/4}, \tag{66}$$

$$\left| \varphi^N_k - \varphi_k \right|^2_{H^1_0} \leq C \left(p(\alpha)^{N^{2/3}} + N^{-1} \right), \tag{67}$$

$$\left| \sqrt{w^N_0} \varphi^N_{k,xx}(\pm 1) \right| \leq C N^{-1/2}, \tag{68}$$

$$\left| \varphi^N_{k,x}(1) - \varphi_{k,x}(1) \right|_{L^2} \leq C \left(p(\alpha)^{N^{2/3}} + N^{-1} \right). \tag{69}$$

Proof **Step 1: Estimate** (65). This is a direct consequence of the following estimate proved in Vandeven (1990):

$$\left| \lambda^N_k - \lambda_k \right| \leq C_\alpha p(\alpha)^{N^{2/3}}. \tag{70}$$

Step 2: Estimate (66). We follow the idea in Raviart (1983, Lemma 6.4-3) where a related result is obtained for the Galerkin approximation. Let us introduce the variational characterizations of both (63) and (64)

$$a(\varphi, v) = \lambda(\varphi, v), \quad \forall v \in H^1_0(\Omega), \tag{71}$$

where $a(\cdot,\cdot)$ is the bi-linear form defined by $a(u,v) = (u_x, v_x), \forall u, v \in H_0^1$, and $(\cdot,\cdot)$ denotes the scalar product in $L^2(-1,1)$, and

$$a(\varphi^N, v^N) = \lambda^N (\varphi^N, v^N)_N, \qquad \forall v^N \in \mathbb{P}_N^{Di}(\Omega). \tag{72}$$

Note that the bi-linear form $a(\cdot,\cdot)$ is the same both in the continuous and the discrete formulation. In fact, (72) is deduced from (64) multiplying the equations by $\omega_i^N v^N(x_i)$ and adding in $i \in I$, that is

$$\begin{aligned} 0 &= \sum_{i\in I} \varphi_{xx}^N(x_i)\omega_i^N v^N(x_i) + \sum_{i\in I} \lambda^N \varphi^N(x_i)\omega_i^N v^N(x_i) = \int_{-1}^1 \varphi_{xx}^N v^N \, \mathrm{d}x + \lambda^N(\varphi^N, v^N)_N \\ &= -\int_{-1}^1 \varphi_x^N v_x^N \, \mathrm{d}x + \lambda^N(\varphi^N, v^N)_N = -a(\varphi^N, v^N) + \lambda^N(\varphi^N, v^N)_N. \end{aligned}$$

The main difference with the case treated in Raviart (1983) is that here the right-hand side in the variational characterization (72) makes appear the discrete scalar product $(\cdot,\cdot)_N$, instead of the L^2 one, and this introduces some technical details. Let us define

$$\varrho_{k,N} = \max_{i\in I_\Omega, i\neq k} \frac{\lambda_k}{\left|\lambda_i^N - \lambda_k\right|}, \quad I_\Omega = \{1, \dots, N-1\}, \tag{73}$$

and the orthogonal projection $\Pi_N \varphi_k \in \mathbb{P}_N^{Di}$ characterized by

$$a(\Pi_N \varphi_k - \varphi_k, v^N) = 0, \qquad \forall v^N \in \mathbb{P}_N^{Di}(\Omega). \tag{74}$$

We now write $\Pi_N \varphi_k$ in the basis φ_i^N. If we normalize φ_i^N in such a way that $|\varphi_i^N|_{L^2} = 1$, then

$$\Pi_N \varphi_k = \sum_{i\in I_\Omega} \frac{(\Pi_N \varphi_k, \varphi_i^N)_N}{\|\varphi_i^N\|_N^2} \varphi_i^N. \tag{75}$$

If we denote

$$v_k^N = \frac{(\Pi_N \varphi_k, \varphi_k^N)_N}{\|\varphi_k^N\|_N^2} \varphi_k^N. \tag{76}$$

Then

$$\left|\Pi_N \varphi_k - v_k^N\right|_{L^2}^2 = \left| \sum_{i\in I_\Omega, i\neq k} \frac{(\Pi_N \varphi_k, \varphi_i^N)_N}{\|\varphi_i^N\|_N^2} \varphi_i^N \right|_{L^2}^2 \leq 2 \sum_{i\in I, i\neq k} \frac{\left|(\Pi_N \varphi_k, \varphi_i^N)_N\right|^2}{\|\varphi_i^N\|_N^2}. \tag{77}$$

From (72), (71), and (74), we easily obtain

$$(\lambda_i^N - \lambda_k)(\Pi_N \varphi_k, \varphi_i^N)_N = \lambda_k (\varphi_k - \Pi_N \varphi_k, \varphi_i^N)_N + \lambda_k \Big((\varphi_k, \varphi_i^N) - (\varphi_k, \varphi_i^N)_N \Big).$$

Therefore, using (73), we get

$$\left|(\Pi_N \varphi_k, \varphi_i^N)_N\right|^2 \leq 2\varrho_{k,N}^2 \left|(\varphi_k - \Pi_N \varphi_k, \varphi_i^N)_N\right|^2 + 2\varrho_{k,N}^2 \left|(\varphi_k, \varphi_i^N) - (\varphi_k, \varphi_i^N)_N\right|^2, \tag{78}$$

that we now substitute in (77). Taking into account the norm equivalence in (6), we easily deduce

$$\left|\Pi_N\varphi_k - v_k^N\right|_{L^2} \leq \sqrt{6}\varrho_{k,N}\left(\|\varphi_k - \Pi_N\varphi_k\|_N + \sum_{i\in I_\Omega, i\neq k}\left|(\varphi_k,\varphi_i^N) - (\varphi_k,\varphi_i^N)_N\right|\right). \tag{79}$$

The idea now is to replace the $\|\cdot\|_N$ norm in the right hand side by the $|\cdot|_{L^2}$ norm. In fact, these two norms are equivalent for polynomials in $\mathbb{P}_N$, so that we first replace φ_k by a polynomial. Let us define the interpolation $I_N : H_0^1 \longrightarrow \mathbb{P}_N^{Di}$ as follows:

$$\begin{cases} I_N\varphi_k \in \mathbb{P}_N^{Di}, \\ I_N\varphi_k(x_i) = \varphi_k(x_i),\ \forall i \in I. \end{cases} \tag{80}$$

Note that

$$\|\varphi_k - \Pi_N\varphi_k\|_N = \|I_N\varphi_k - \Pi_N\varphi_k\|_N \leq \sqrt{3}\,|I_N\varphi_k - \Pi_N\varphi_k|_{L^2}. \tag{81}$$

Therefore, substituting in (79) and then adding and subtracting φ_k, we obtain

$$\begin{aligned} \left|\Pi_N\varphi_k - v_k^N\right|_{L^2} \leq \sqrt{6}\varrho_{k,N}\Big(&\sqrt{3}\,|I_N\varphi_k - \varphi_k|_{L^2} + \sqrt{3}\,|\varphi_k - \Pi_N\varphi_k|_{L^2} \\ &+ \sum_{i\in I_\Omega, i\neq k}\left|(\varphi_k,\varphi_i^N) - (\varphi_k,\varphi_i^N)_N\right|\Big). \end{aligned} \tag{82}$$

Furthermore, (76) and $\left|\varphi_k^N\right|_{L^2} = |\varphi_k|_{L^2} = 1$ give

$$|\varphi_k|_{L^2} - \left|\varphi_k - v_k^N\right|_{L^2} \leq \left|v_k^N\right|_{L^2} \leq |\varphi_k|_{L^2} + \left|\varphi_k - v_k^N\right|_{L^2},$$

and

$$\left|\frac{\left|(\Pi_N\varphi_k,\varphi_k^N)_N\right|}{\|\varphi_k^N\|_N^2} - 1\right| \leq \left|\varphi_k - v_k^N\right|_{L^2}. \tag{83}$$

As it is always possible to choose the eigenfunctions φ_k^N, such that $(\Pi_N\varphi_k,\varphi_k^N)_N \geq 0$, we obtain

$$\left|v_k^N - \varphi_k^N\right|_{L^2} \leq |\varphi_k - \Pi_N\varphi_k|_{L^2} + \left|\Pi_N\varphi_k - v_k^N\right|_{L^2}. \tag{84}$$

Finally, from (82) and (84)

$$\begin{aligned} \left|\varphi_k^N - \varphi_k\right|_{L^2} \leq\ & 2(1+3\sqrt{3}\varrho_{k,N})\,|\varphi_k - \Pi_N\varphi_k|_{L^2} + 2(3\sqrt{2}\varrho_{k,N})\,|\varphi_k - I_N\varphi_k|_{L^2} \\ & + 2(\sqrt{6}\varrho_{k,N})\left(\sum_{i\in I_\Omega, i\neq k}\left|(\varphi_k,\varphi_i^N) - (\varphi_k,\varphi_i^N)_N\right|\right). \end{aligned}$$

We recall that, if $v \in H_0^m$ for some $m \geq 1$ and $v^N \in \mathbb{P}_N^{Di}$, then there exist constants $c, c_1 > 0$, such that (see Canuto et al. 1988, chapter (9))

$$|v - \Pi_N v|_{L^2} \leq cN^{-m}\,|v|_{H^m} \quad \text{and} \quad |v - I_N v|_{L^2} \leq c_1 N^{1/2-m}\,|v|_{H^m}. \tag{85}$$

On the other hand, there exists a constant $c_2 > 0$, such that [see Cividini et al. 1993, estimation (3.22)]

$$|(v, v^N)_N - (v, v^N)| \leq c_2 N^{-m} |v|_{H^m} |v^N|_{L^2}. \tag{86}$$

From the classical projection results for spectral methods in (85) and (86) when $m = 2$ and the fact that $|\varphi_k|_{H^2} \leq c\lambda_k \leq cN^{1/4}$ (since by hypotheses $k \leq cN^{1/8}$), we deduce that there exists a constant $C > 0$, such that

$$\left|\varphi_k^N - \varphi_k\right|_{L^2} \leq CN^{-3/4}, \quad k \leq r(N). \tag{87}$$

Step 3: Estimate (67). From (71), (72) and the norm equivalence in (6), we can write

$$a(\varphi_k^N - \varphi_k, \varphi_k^N - \varphi_k) \leq 3\lambda_k^N + \lambda_k - 2\lambda_k(\varphi_k, \varphi_k^N).$$

We also have

$$\left|\varphi_k^N - \varphi_k\right|_{L^2}^2 = 2(1 - (\varphi_k, \varphi_k^N)),$$

and then

$$a(\varphi_k^N - \varphi_k, \varphi_k^N - \varphi_k) \leq 3\left(\lambda_k^N - \lambda_K + \lambda_K \left|\varphi_k^N - \varphi_k\right|_{L^2}^2\right).$$

From the coercivity of the bi-linear form a in (71), (66) and (70), the fact that $\lambda_k \leq M_\alpha N^{\frac{1}{4}}$ we can deduce, there exists a constant $C_1 > 0$, such that

$$\left|\varphi_k^N - \varphi_k\right|_{H_0^1}^2 \leq C_1\left(p(\alpha)^{N^{2/3}} + N^{-1}\right).$$

Step 4: Estimate (68). It is enough to prove the estimate at $x = 1$, since the other one is similar. First, we observe that we can write the discrete eigenvalue problem (64) in the following equivalent form:

$$\begin{cases} \varphi_{k,xx}^N + \lambda_k^N \varphi_k^N = \varphi_{k,xx}^N(-1)\Psi_0^N + \varphi_{k,xx}^N(1)\Psi_N^N & \text{in } x \in \Omega \\ \varphi_k^N(1) = \varphi_k^N(-1) = 0, \end{cases} \tag{88}$$

where $\Psi_0^N \in \mathbb{P}_N(\Omega)$ (resp. Ψ_N^N) is the Lagrangian polynomial which is 1 at $x = -1$ (resp. $x = 1$) and 0 at the rest of quadrature points in $C^\Omega = C^{\Omega,N}$. At this point, we make explicit the dependence on N of the set of quadrature points by writing $C^{\Omega,N}$, since we consider different sets below.

It is easy to see that the eigenfunctions associated with (88) are either even or odd. We focus on the case of even eigenfunctions, since the other one is similar. In this case $\varphi_{k,xx}^N(-1) = \varphi_{k,xx}^N(1)$. Multiplying (88) by $\Psi_{N-1}^{N-1} \in \mathbb{P}_{N-1}(\Omega)$ the Lagrangian polynomial which is 1 at $x = 1$ and 0 at all other collocation points in $C^{\Omega,N-1}$, one has

$$\varphi_{k,xx}^N(1)\omega_{N-1}^{N-1} + \lambda_k^N \int_{-1}^1 \varphi_k^N \Psi_{N-1}^{N-1} \mathrm{d}x = \varphi_{k,xx}^N(1)\omega_N^N. \tag{89}$$

Note that in the first term on the left-hand side, we have used the quadrature formula with nodes in $C^{\Omega,N-1}$, which is exact for polynomials of degree $2(N-1)-1$. In fact, the term inside the integral is a polynomial of degree $2N-3$, and by hypotheses, Ψ_{N-1}^{N-1} is 1 at $x = 1$ and 0 at the other quadrature nodes. On the right-hand side, we have used the quadrature formula with nodes in $C^{\Omega,N}$ which is also exact, since the integrated is a polynomial of degree $2N-1$. Therefore

$$
\begin{aligned}
\left|\sqrt{\omega_N^N}\varphi_{k,xx}^N(1)\right| &= \left|\frac{\sqrt{\omega_N^N}\lambda_k^N \int_{-1}^1 \varphi_k^N \Psi_{N-1}^{N-1}\mathrm{d}x}{(w_{N-1}^{N-1}-w_N^N)}\right| \\
&= \left|\frac{\sqrt{\omega_N^N}\lambda_k^N \int_{-1}^1 (\varphi_k^N - P_{N-2}\varphi_k^N)\Psi_{N-1}^{N-1}\mathrm{d}x}{(\omega_{N-1}^{N-1}-\omega_N^N)}\right|,
\end{aligned}
\tag{90}
$$

where $P_{N-2}\varphi_k^N$ is the orthogonal projection of φ_k^N in $\mathbb{P}_{N-2}^{Di}(\Omega)$ with respect to the L^2-scalar product. The last equality comes from the fact that $\int_{-1}^1 P_{N-2}\varphi_k^N \Psi_{N-1}^{N-1}\mathrm{d}x = 0$ by the quadrature formula with nodes in $C^{\Omega,N-1}$.

Now, using Cauchy–Schwarz inequality in (90) and taking into account the norm equivalent in (6), as $\Psi_{N-1}^{N-1} \in \mathbb{P}_{N-1}(\Omega), \left|\Psi_{N-1}^{N-1}\right|_{L^2} \le \left\|\Psi_{N-1}^{N-1}\right\|_{N-1} = \sqrt{\omega_{N-1}^{N-1}}$ and $\omega_{N-1}^{N-1} = \dfrac{2}{(N-1)N}, \omega_N^N = \dfrac{2}{N(N+1)}$, from (85) and the fact that $\lambda_k^N \le M_\alpha N^{\frac{1}{4}}$ we obtain, there exists a constant $c_3 > 0$, such that

$$
\begin{aligned}
\left|\sqrt{\omega_N^N}\varphi_{k,xx}^N(1)\right| &\le \frac{\sqrt{\omega_N^N}\sqrt{\omega_{N-1}^{N-1}}\lambda_k^N \left|\varphi_k^N - P_{N-2}\varphi_k^N\right|_{L^2}}{\left|(\omega_{N-1}^{N-1}-\omega_N^N)\right|} \\
&\le c_3 N^{5/4}(N-2)^{-m}\left|\varphi_k^N\right|_{H^m}.
\end{aligned}
\tag{91}
$$

To estimate this last term, we use the following result, □

Lemma 6 *Assume that $m=2$, then there exists a constant $M_2 > 0$, such that*

$$
\left|\varphi_k^N\right|_{H^2}^2 \le M_2|\lambda_k^N|^2(1 + w_N^N\left|\varphi_{k,xx}^N(1)\right|^2). \tag{92}
$$

From Lemma 6, the fact that $\lambda_k^N \le M_\alpha N^{\frac{1}{4}}$ and (91) we easily deduce estimate (68).

We now prove Lemma 6.

Proof Multiply (88) by φ_k^N and integrating by parts, one has

$$
\int_{-1}^1 \left|\varphi_{k,x}^N\right|^2 \mathrm{d}x = \lambda_k^N \int_{-1}^1 \left|\varphi_k^N\right|^2 \mathrm{d}x - 2\varphi_{k,xx}^N(1)\int_{-1}^1 \Psi_N^N \varphi_k^N \mathrm{d}x.
$$

The last equality comes from the fact that $\int_{-1}^1 \Psi_0^N\varphi_k^N\mathrm{d}x = \int_{-1}^1 \Psi_N^N\varphi_k^N\mathrm{d}x$. Now, multiplying and dividing the second term on the right-hand side by $\sqrt{\omega_N^N}$, first using young's inequality and then Cauchy–Schwarz inequality, we obtain

$$
\left|\varphi_{k,x}^N\right|_{L^2}^2 \le \left(\lambda_k^N + \frac{|\Psi_N^N|_{L^2}^2}{\omega_N^N}\right)\left|\varphi_k^N\right|_{L^2}^2 + \omega_N^N\left|\varphi_{k,xx}^N(1)\right|^2. \tag{93}
$$

On the other hand, multiplying (88) by $\varphi_{k,xx}^N$ and integrating by parts, one obtains

$$
\left|\varphi_{k,xx}^N\right|_{L^2}^2 = \lambda_k^N\left|\varphi_{k,x}^N\right|_{L^2}^2 + 2\omega_N^N\left|\varphi_{k,xx}^N(1)\right|^2. \tag{94}
$$

Here, on the right-hand side, we have used the quadrature formula with nodes in C^N, since the integrated is a polynomial of degree $2N-2$ and by hypotheses Ψ_0^N (resp. Ψ_N^N) is 1 at $x=-1$ (resp. $x=1$) and 0 at the rest of quadrature points in C^N and the fact that φ_k^N is even. Finally, from (93), (94), and the normalization of the eigenfunctions $\left|\varphi_k^N\right|_{L^2}=1$, we easily obtain (92). □

Step 5: Estimate (69). Multiplying the equation of φ_k^N in (64) by $\frac{1+x_i}{2}\omega_i^N$ and adding in $i\in I$, one obtains

$$\begin{aligned}0&=\sum_{i\in I}\varphi_{k,xx}^N(x_i)\frac{1+x_i}{2}\omega_i^N+\lambda_k^N\sum_{i\in I}\varphi_k^N(x_i)\frac{1+x_i}{2}\omega_i^N-\varphi_{k,xx}^N(1)\omega_N^N\\&=\varphi_{k,x}^N(1)+\lambda_k^N\int_{-1}^1\varphi_k^N\frac{1+x}{2}\mathrm{d}x-\varphi_{k,xx}^N(1)\omega_N^N.\end{aligned}\tag{95}$$

Note that in the first and second terms on the right-hand side, we have used the quadrature formula. In particular, this allowed us to integrate by parts the first term.

Now, multiplying the equation in (63) by $\frac{1+x}{2}$ and integrating by parts, one has

$$\varphi_{k,x}(1)+\lambda_K\int_{-1}^1\varphi_k\frac{1+x}{2}\mathrm{d}x=0.\tag{96}$$

Combining (95) and (96), we obtain

$$\varphi_{k,x}^N(1)-\varphi_{k,x}(1)=\lambda_k\int_{-1}^1\varphi_k\frac{1+x}{2}\mathrm{d}x-\lambda_k^N\int_{-1}^1\varphi_k^N\frac{1+x}{2}\mathrm{d}x+\varphi_{k,xx}^N(1)w_N^N.$$

Here, the right-hand side is easily estimated using (66),(70) and the normalization of the eigenfunctions $\left|\varphi_k^N\right|_{L^2}=1$. This gives (69) and concludes the proof of the theorem. □

We now move to give a proof of Lemma 4 and Lemma 3.

Proof of Lemma 4. Define $\mu_k=sign(k)\sqrt{\lambda_{|k|}}$ and $\Phi_k=(\varphi_{|k|}/(i\mu_k),\varphi_{|k|})/\sqrt{2}$ for $k\in\mathbb{Z}^*=\mathbb{Z}\backslash\{0\}$. Note that $\{\Phi_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis in $H_0^1\times L^2$. Thus, given $(\phi^0,\phi^1)\in H_0^1\times L^2$, we can write

$$(\phi^0,\phi^1)=\sum_{k\in\mathbb{Z}^*}a_k\Phi_k,\quad |(\phi^0,\phi^1)|^2_{H_0^1\times L^2}=\sum_{k\in\mathbb{Z}^*}|a_k|^2<\infty,\tag{97}$$

for some Fourier coefficients $a_k\in\mathbb{C}$. Analogously, we define $\mu_k^N=sign(k)\sqrt{\lambda_{|k|}^N}$ and $\Phi_k^N=(\varphi_{|k|}^N/(i\mu_k^N),\varphi_{|k|}^N)/\sqrt{2}$ for $|k|\le N,\ k\neq 0$. Again, $\{\Phi_k^N\}_{|k|\le N}$ is an orthonormal basis of $H_0^1\times\mathbb{P}_N$ where the scalar product in $\mathbb{P}_N$ is the discrete inner product $(\cdot,\cdot)_N$. Let us consider

$$(\phi^{0,N},\phi^{1,N})=\sum_{|k|\le r(N)}a_k\Phi_k^N.\tag{98}$$

From the convergence results in Theorem 6, we have

$$\begin{aligned}|(\phi^0,\phi^1)-(\phi^{0,N},\phi^{1,N})|^2_{H_0^1\times L^2}\le&\sup_{|k|\le r(N)}\left(|\Phi_k-\Phi_k^N|^2_{H_0^1\times L^2}\right)\sum_{|k|\le r(N)}|a_k|^2\\&+\sum_{|k|>r(N)}|a_k|^2\to 0,\quad\text{as }N\to\infty.\end{aligned}\tag{99}$$

This concludes the proof of (45). Moreover, the solution of the continuous wave equation (38) is given by

$$(\phi(t,x), \phi_t(t,x)) = \sum_{k\in\mathbb{Z}^*} a_k e^{i\mu_k t}\Phi_k, \tag{100}$$

while the one associated to (13) with the initial data $(\phi^{0,N}, \phi^{1,N})$ is given by

$$(\phi^N(t,x), \phi_t^N(t,x)) = \sum_{|k|\le r(N)} a_k e^{i\mu_k^N t}\Phi_k^N. \tag{101}$$

Again, the uniform convergence of the low frequencies stated in Theorem 6 allows us to obtain (46)–(48).

Proof of Lemma 3. We follow the idea in the proof of Lemma 4. Let us define $\hat{\Phi}_k = (i\mu_k)\Phi_k$ for $k \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$ where Φ_k where introduced at the beginning of Lemma 4. Note that $\{\hat{\Phi}_k\}_{k\in\mathbb{Z}}$ is now an orthonormal basis in $L^2 \times H^{-1}$. Thus, given $(u^0, u^1) \in L^2 \times H^{-1}$, we can write

$$(u^0, u^1) = \sum_{k\in\mathbb{Z}^*} \hat{b}_k\hat{\Phi}_k, \quad |(u^0, u^1)|^2_{L^2\times H^{-1}} = \sum_{k\in\mathbb{Z}^*} |\hat{b}_k|^2 < \infty, \tag{102}$$

for some Fourier coefficients $\hat{b}_k \in \mathbb{C}$. Analogously, we define $\hat{\Phi}_k^N = i\mu_k^N\Phi_k^N$ for $|k| \le N,\ k \ne 0$. Let us consider

$$(u^{0,N}, u^{1,N}) = \sum_{|k|\le r(N)} \hat{b}_k\hat{\Phi}_k^N. \tag{103}$$

Note that if (u^0, u^1) are continuous functions, the sequences that we choose $(u^{0,N}, u^{1,N})$ are the polynomial which coincides with the value of (u^0, u^1) at the collocation points.

Arguing as in (99), the convergence result in (30) can be reduced to prove

$$\sup_{|k|\le r(N)} \left(|\hat{\Phi}_k - \hat{\Phi}_k^N|^2_{L^2\times H^{-1}}\right) \to 0, \quad \text{as } N \to \infty.$$

Note that

$$|\hat{\Phi}_k - \hat{\Phi}_k^N|^2_{L^2\times H^{-1}} = |\varphi_{|k|} - \varphi_{|k|}^N|^2_{L^2} + |i\mu_k\varphi_{|k|} - i\mu_k^N\varphi_{|k|}^N|^2_{H^{-1}}.$$

The first term here can be estimated uniformly for $|k| \le r(N)$ by Theorem 6 and converges to zero as $N \to \infty$. Concerning the second term, we use the fact that the eigenfunctions φ_k (resp. $\varphi_{|k|}^N$) satisfy (64) (resp. (88)) together with the isometry of the Laplacian between H_0^1 and H^{-1}. Therefore

$$|i\mu_k\varphi_{|k|} - i\mu_k^N\varphi_{|k|}^N|^2_{H^{-1}} = \left|\frac{\varphi_{|k|,xx}}{\mu_k} - \frac{\varphi_{|k|,xx}^N - \varphi_{|k|,xx}^N(-1)\Psi_0^N - \varphi_{|k|,xx}^N(1)\Psi_N^N}{\mu_k^N}\right|^2_{H^{-1}} \tag{104}$$

$$\le \left|\frac{\varphi_{|k|}}{\mu_k} - \frac{\varphi_{|k|}^N}{\mu_k^N}\right|^2_{H_0^1} + \frac{|\varphi_{|k|,xx}^N(-1)|^2}{\lambda_k^N}|\Psi_0^N|^2_{H^{-1}}$$

$$+\frac{|\varphi_{|k|,xx}^N(1)|^2}{\lambda_k^N}|\Psi_N^N|^2_{H^{-1}}, \tag{105}$$

that converges uniformly to zero for $|k| \le r(N)$ as a consequence of Theorem 6 and the uniform bound of $|\Psi_0^N|_{L^2}$ and $|\Psi_0^N|_{L^2}$.

We now prove (31), Observe that $\{\Phi_k^N\}_{|k|\le N}$ is orthonormal in $\mathbb{P}^N \times \mathbb{P}^N$ with the scalar product

$$\Big((v^{0,N}, v^{1,N}), (w^{0,N}, w^{1,N})\Big)_N^* = (v_x^{0,N}, w_x^{0,N})_N + (v^{1,N}, v^{1,N})_N,$$

whose associated norm is equivalent to the usual norm in $H_0^1 \times L^2$.

Therefore, if write any $(\phi^{0,N}, \phi^{1,N})$ as $\sum_{|k|\le N} a_k^N \Phi_k^N$ and by the orthogonality of the eigenfunctions φ_k^N with respect to the discrete scalar product $(\cdot,\cdot)_N$ and the duality product (11), we have

$$\begin{aligned}
&\Big|\big\langle(\varphi^{0,N}, \varphi^{1,N}), (u^{0,N}, u^{1,N})\big\rangle_N\Big| = \left|\left\langle \sum_{|k|\le r(N)} \hat{b}_k \hat{\Phi}_k^N, \sum_{|k|\le N} a_k^N \Phi_k^N \right\rangle_N\right| \\
&\le \left|\sum_{|k|\le r(N)} \hat{b}_k a_k^N\right| \le \left(\sum_{|k|\le r(N)} |\hat{b}_k|^2\right)^{1/2} \left(\sum_{|k|\le r(N)} |a_k^N|^2\right)^{1/2} \\
&\le |(u^0, u^1)|_{L^2\times H^{-1}} \|(\varphi_x^{0,N}, \varphi^{1,N})\|_{N\times N}.
\end{aligned}$$

Finally, we prove (32). We assume now that $(\phi^{0,N}, \phi^{1,N}) \to (\phi^0, \phi^1)$ in $H_0^1 \times L^2$ that we write as $(\phi^0, \phi^1) = \sum_{k\in\mathbb{Z}^*} a_k \Phi_k$. We have

$$\begin{aligned}
&\big\langle(\varphi^{0,N}, \varphi^{1,N}), (u^{0,N}, u^{1,N})\big\rangle_N = \left\langle \sum_{|k|\le r(N)} \hat{b}_k \hat{\Phi}_k^N, \sum_{|k|\le N} a_k^N \Phi_k^N \right\rangle_N, \\
&\big\langle(\varphi^0, \varphi^1), (u^0, u^1)\big\rangle = \left\langle \sum_{k\in\mathbb{Z}^*} \hat{b}_k \hat{\Phi}_k, \sum_{k\in\mathbb{Z}^*} a_k \Phi_k \right\rangle.
\end{aligned}$$

The convergence results in Theorem 6 allow to prove the estimate

$$|a_k^N - a_k| \le C N^{-1/4}, \quad |k| \le r(N),$$

and using the strong convergence $(\phi^{0,N}, \phi^{1,N}) \to (\phi^0, \phi^1)$

$$\left\langle \sum_{|k|\le r(N)} \hat{b}_k \hat{\Phi}_k^N, \sum_{|k|\le N} a_k^N \Phi_k^N \right\rangle_N \to \left\langle \sum_{k\in\mathbb{Z}^*} \hat{b}_k \hat{\Phi}_k, \sum_{k\in\mathbb{Z}^*} a_k \Phi_k \right\rangle.$$

This concludes the proof of (32).

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