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ORIGINAL PAPER

Spectral transformations and second kind polynomials associated with a hermitian linear functional

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Abstract
The aim of this paper is to analyze the relation between the Christoffel transformations in the framework of hermitian linear functionals and the associated polynomials of the second kind through of their associated Carathéodory formal power series. In particular, some results are given in the way of characterization of linear spectral transformations using Christoffel and Geronimus transformations. This gives a new example of rational spectral transformation that has not yet been studied in the literature.

Keywords Szegö orthogonal polynomials · Spectral transformations · Associated polynomials of second kind · Carathéodory formal power series

Mathematics Subject Classification Primary 42C05 · 33C50

1 Introduction

Let $\mathbf{u}$ be a complex-valued linear functional defined on the linear space of polynomials $\mathbb{P}$ with complex coefficients and real variable i.e.

$$\mathbf{u} : \mathbb{P} \to \mathbb{C}, \quad p(x) \to \langle \mathbf{u}, p(x) \rangle .$$

We denote the n-th moment of $\mathbf{u}$ by $\mathbf{u}_n := \langle \mathbf{u}, x^n \rangle$, $n = 0, 1, \ldots$. The linear functional $\mathbf{u}$ is said to be quasi-definite (resp. positive-definite) if every leading principal submatrix of the Hankel matrix $H = (\mathbf{u}_{i+j})_{i,j=0}^{\infty}$ is nonsingular (resp. positive-definite) (see [11]). Under these conditions, there exists a sequence of monic polynomials $(P_n)_{n\geq 0}$ satisfying $\deg P_n = n$ and $\langle \mathbf{u}, P_n(x) P_m(x) \rangle = K_n \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker delta and $K_n \neq 0$ (see [11,

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15]). The sequence $(P_n)_{n\geq 0}$ is said to be *the sequence of monic orthogonal polynomials* (SMOP) with respect to $\mathbf{u}$.

Given a quasi-definite linear functional $\mathbf{u}$ and if $(P_n)_{n\geq 0}$ is its corresponding SMOP, then there exist two sequences of complex numbers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$, with $a_n \neq 0$, such that

$$\begin{aligned} x\,P_n(x) &= P_{n+1}(x) + b_n\,P_n(x) + a_n\,P_{n-1}(x), \quad n \geq 0, \\ P_{-1}(x) &= 0, \qquad P_0(x) = 1. \end{aligned} \tag{1}$$

Conversely, from Favard's Theorem (see [11, 15]) if $(P_n)_{n\geq 0}$ is a sequence of monic polynomials generated by a three-term recurrence relation as in (1) with $a_n \neq 0$, $n \geq 1$, then there exists a unique linear functional $\mathbf{u}$ such that $(P_n)_{n\geq 0}$ is its SMOP. Another way to write the recurrence relation (1) is in matrix form. Indeed, if $\mathbf{P} = (P_0, P_1, \cdots)^\top$, where $A^\top$ denotes the transposed of a matrix A, then $x\mathbf{P} = J\mathbf{P}$ where J is the semi-infinite matrix

$$J = \begin{pmatrix} b_0 & 1 & & \\ a_1 & b_1 & 1 & \\ & a_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

which is known in the literature as a monic Jacobi matrix (see [11, 15]).

Definition 1 Let $(P_n)_{n\geq 0}$ be the SMOP with respect to the linear functional $\mathbf{u}$ satisfying the three-term recurrence relation (1). Then the sequence of associated polynomials of the k-th kind, $(P_n^{(k)})_{n\geq 0}$, is defined as the sequence of monic polynomials satisfying the recurrence relation

$$\begin{aligned} x P_n^{(k)}(x) &= P_{n+1}^{(k)}(x) + b_{n+k} P_n^{(k)}(x) + a_{n+k} P_{n-1}^{(k)}(x), \quad n \geq 0, \\ P_{-1}^{(k)}(x) &= 0, \quad P_0^{(k)}(x) = 1. \end{aligned}$$

According to Favard's Theorem, there exists a quasi-definite linear functional $\mathbf{u}^{(k)}$, called the k-th associated transformation of $\mathbf{u}$, such that $(P_n^{(k)})_{n\geq 0}$ is its corresponding SMOP.

The perturbations of quasi-definite functionals have been extensively studied in the literature (see for example [1, 33, 34] and references therein), being of particular importance the canonical transformations of Christoffel and Geronimus (Table 1). These two transformations are also known in the literature as canonical discrete Darboux transformations due to the existing relation between their corresponding Jacobi matrices and LU and UL factorizations [1, 33].

The necessary and sufficient conditions for the quasi-definiteness of the linear functionals $\widetilde{\mathbf{u}}$ and $\widehat{\mathbf{u}}$ appear in [15], among others.

A (general) Christoffel transformation $\widetilde{\mathbf{u}}$ of a linear functional $\mathbf{u}$ is a superposition of canonical Christoffel transformations, i. e., there exists a polynomial $p(x)$ such that $\widetilde{\mathbf{u}} = p(x)\mathbf{u}$. A (general) Geronimus transformation $\widehat{\mathbf{u}}$ of a linear functional $\mathbf{u}$ is a superposition

Table 1 Darboux transformations

$\mathbf{u}$ $J - cI = LU$	$\widetilde{\mathbf{u}} = (x-c)\mathbf{u}$ $\widetilde{J} - cI = UL$	Darboux (Christoffel) transformation without parameter
$\mathbf{u}$ $J - cI = UL$	$\mathbf{u} = (x-c)\widehat{\mathbf{u}}$ $\widehat{J} - cI = LU$	Darboux (Geronimus) transformation with parameter

of canonical Geronimus transformations, i.e., there exists a polynomial $q(x)$ such that $\mathbf{u} = q(x)\widehat{\mathbf{u}}$.

Given a quasi-definite linear functional $\mathbf{u}$ with moments $(\mathbf{u}_n)_{n\geq 0}$, the formal series

$$\mathbf{S}_{\mathbf{u}}(z) =: \sum_{n=0}^{\infty} \frac{\mathbf{u}_n}{z^{n+1}}$$

is said to be the *Stieltjes function* associated with $\mathbf{u}$. Due to the connection between the Stieltjes function and the coefficients of the three-term recurrence relation (1) through of continued fractions, the Stieltjes function is a very useful tool to study properties of quasi-definite linear functionals.

Definition 2 ([34]) Let $\widetilde{\mathbf{u}}$ be a quasi-definite linear functional and $\widetilde{\mathbf{S}}(z)$ its Stieltjes function. $\widetilde{\mathbf{u}}$ is said to be a *rational spectral transform of* $\mathbf{u}$ if there exist polynomials $A(z)$, $B(z)$, $C(z)$ and $D(z)$ such that

$$\widetilde{\mathbf{S}}(z) = \frac{A(z)\mathbf{S}_{\mathbf{u}}(z) + B(z)}{C(z)\mathbf{S}_{\mathbf{u}}(z) + D(z)}, \quad A(z)D(z) - B(z)C(z) \neq 0.$$

The above mapping between two linear functionals is said to be a *rational spectral transformation.* In particular, if $C(z) \equiv 0$, then $\widetilde{\mathbf{u}}$ is a *linear spectral transformed* of the linear functional $\mathbf{u}$. In such a case, the mapping between two linear functionals is said to be a *linear spectral transformation.*

If J is a Jacobi matrix, we can to add k new rows and columns to J. In such a way the original Jacobi matrix is shifted downward. We can remove the first k rows and columns of J, so that the original Jacobi matrix is shifted upward. In both cases, we obtain Jacobi matrices defining new linear functionals. In the first case, the functional obtained is known as the *anti-associated functional of order k* [30]. In the second case, the functional obtained is the k-th associated functional (see Definition 1). These two transformations are examples of rational spectral transformations. Together with Christoffel and Geronimus transformations they are relevant in the literature due to the following characterizations of linear (resp. rational) spectral transformations given by Zhedanov in [34].

Theorem 1 (Zhedanov, [34]) *Every linear spectral transformation is a superposition of the Christoffel and Geronimus transformations.*

Theorem 2 (Zhedanov, [34]) *Every rational spectral transformation is a superposition of Christoffel, Geronimus, associated, and anti-associated transformations.*

Taking into account the above, in [16] and [14], respectively, the following two problems were studied:

Problem 1 (Fig. 1) Let $\mathbf{u}$ be a quasi-definite functional, and let $\widetilde{\mathbf{u}} = (x-c)\mathbf{u}$ and $(x-c)\widehat{\mathbf{u}} = \mathbf{u}$ be a canonical Christoffel and Geronimus transformation of $\mathbf{u}$, respectively. Deduce the relations between $\mathbf{u}^{(1)}$ and $\widetilde{\mathbf{u}}^{(1)}$ (resp. $\widehat{\mathbf{u}}^{(1)}$) by using the LU and UL factorization of the monic Jacobi matrix associated with $\mathbf{u}$, as well as explicit algebraic relations between their corresponding SMOPs.

Problem 2 (Fig. 2) Let $\mathbf{u}$ be a quasi-definite functional and let $\widetilde{\mathbf{u}} = (x-c)\mathbf{u}$ and $(x-c)\widehat{\mathbf{u}} = \mathbf{u}$ be a canonical Christoffel and Geronimus transformation of $\mathbf{u}$, respectively. What is the relation between $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{u}^{(1)}}$ (resp. $\widehat{\mathbf{u}^{(1)}}$)? There, $\widetilde{\mathbf{u}^{(1)}}$ (resp. $\widehat{\mathbf{u}^{(1)}}$) is the Christoffel transformation (resp. Geronimus) of $\mathbf{u}^{(1)}$.

$$\begin{tikzcd}
\mathbf{u} \arrow[r, "\text{Associated Trans.}"] \arrow[d, "\text{Darboux Trans.}"'] & \mathbf{u}^{(1)} \arrow[d, dashed, "?"] \\
\widetilde{\mathbf{u}} \arrow[r] & \widetilde{\mathbf{u}}^{(1)}
\end{tikzcd}$$

Fig. 1 Problem 1

$$\begin{tikzcd}
\mathbf{u} \arrow[r, "\text{Darboux Trans.}"] \arrow[d, "\text{Associated Trans}"'] & \widetilde{\mathbf{u}} \arrow[d, dashed, "?"] \\
\mathbf{u}^{(1)} \arrow[r] & \widetilde{\mathbf{u}^{(1)}}
\end{tikzcd}$$

Fig. 2 Problem 2

The aim of this manuscript is to analyze the analog of Problem 1 for Christoffel transformations in the framework of hermitian linear functionals and their corresponding Carathéodory functions (see Definition 7).

Here, we stress that the study of properties of families of orthogonal polynomials with respect to these hermitian linear functionals (and more precisely when they induce inner products) has intensively attracted the interest of researchers in recent years (see [31, 32] and references therein). In particular, perturbations of quasi-definite (and, more specifically, positive-definite) hermitian functionals have been studied widely in the literature (see [2, 3, 5–10, 12, 17–22], among others) with a special emphasis on the analytic and algebraic properties of the sequences of orthogonal polynomials with respect to perturbed functionals as well as the relation between their corresponding Toeplitz matrices.

Taking into account the goals of this manuscript, the presentation is organized as follows. In Sect. 2, we provide the basic facts about hermitian linear functionals that will be needed in the sequel. In Sect. 3 we describe the most common perturbations of hermitian linear functionals and the relation between their Carathéodory formal power series. In particular, some results are given in the way of characterization of linear spectral transformations using Christoffel and Geronimus transformations. In Sect. 4 we study the analogue of Problem 1 for a Christoffel transformation giving a representation of the new family of orthogonal polynomials as well as an expression for the corresponding Carathéodory formal power series. This result illustrates a new example of rational spectral transformation that has not yet been studied in the literature as far as we know. Finally, in Sect. 5 some concluding remarks are established.

2 Background

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ denote the open disk on the complex plane and the unit circle, respectively. Let $\mathbb{L} = \text{span}\{z^k\}_{k\in\mathbb{Z}}$ be the linear space of *Laurent* *polynomials with complex coefficients*. For $f(z) = \sum a_k z^k \in \mathbb{L}$ we define $f_*(z) = \overline{f}(z^{-1})$, where we understand by $\overline{f}(z) = \sum \overline{a_k} z^k$ and for $p \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$, we define the *reverse polynomial of* p as $p^*(z) = z^n p_*(z)$.

Definition 3 A polynomial $p(z)$ is said to be palindromic if $p(z) = p^*(z)$ and antipalindromic if $p(z) = -p^*(z)$.

The polynomial $p(z)$ will be called *self-reciprocal* if it is either palindromic or antipalindromic. From the above definition, if $p(z) = \sum_{k=0}^{m} a_k z^k$ is a self-reciprocal polynomial and z_0 is a zero of $p(z)$, then $1/\overline{z_0}$ is also a zero of $p(z)$. In addition, the coefficients are related by $a_k = \overline{a}_{m-k}$, $k = 0, \ldots, m$ if p is palindromic and $a_k = -\overline{a}_{m-k}$, $k = 0, \ldots, m$ if p is antipalindromic.

Let $L(z)$ be a Laurent polynomial such that $L(z) = L_*(z)$ (resp. $L(z) = -L_*(z)$), then $L(z)$ is said to be *hermitian (resp. anti-hermitian)*. In this case there exists a polynomial $r \in \mathbb{P}$ such that $L = r + r_*$ (resp. $L = r - r_*$). We will refer to $\deg r$ as the degree of L. Another characterization of a hermitian (resp. anti-hermitian) polynomial is the following: If $A(z) = z^{\deg r} L(z)$ with $A(z)$ a polynomial of $\deg A = 2\deg r$, then L is hermitian (resp. anti-hermitian) if and only if A is palindromic (resp. antipalindromic). As a consequence of the properties of the zeros of self-reciprocal polynomials, we have the following result.

Proposition 1 *If L is a hermitian polynomial (resp. anti-hermitian) of degree* $\deg r$, *then there exists $L_1, \ldots, L_{\deg r}$ hermitian (resp. anti-hermitian) Laurent polynomials, each one of degree one such that*

$$L = L_1 L_2 \cdots L_{\deg r}.$$

Let $\mathscr{L}$ be a linear functional defined on $\mathbb{L}$. Denote by $\mathcal{B}_{\mathscr{L}}(p(z), q(z)) := \mathscr{L}(p(z)\overline{q}(1/z))$ a sesquilinear form with $p, q \in \mathbb{P}$.

Definition 4 Consider the matrix $T = (c_{i-j})_{i,j=0}^{\infty}$, where $c_{i-j} := \mathscr{L}(z^{i-j}) = \mathcal{B}_{\mathscr{L}}(z^i, z^j)$, and let T_n be the leading principal submatrix of order n. Then

(i) If $c_{i-j} = \overline{c_{j-i}}$, $\mathcal{B}_{\mathscr{L}}$ is said to be hermitian sesquilinear form and $\mathscr{L}$ is said to be a hermitian linear functional in $\mathbb{P}$.
(ii) Taking into account that the function $\Phi(p) =: \mathcal{B}_{\mathscr{L}}(p, p)$ is a quadratic form, $\mathcal{B}_{\mathscr{L}}$ is said to be quasi-definite if $\Phi(p) \neq 0$ for all $p \in \mathbb{P}$. The above is equivalent to $\det T_n \neq 0$, for every non-negative integer n.
(iii) If $\mathcal{B}_{\mathscr{L}}(p, p) > 0$, for every $p \in \mathbb{P}$, $p(z) \neq 0$, or, equivalently, $\det T_n > 0$, for every non-negative integer n, then $\mathcal{B}_{\mathscr{L}}$ is said to be positive-definite.

Observe that if $\mathscr{L}$ is hermitian, then $\mathcal{B}_{\mathscr{L}}(p, q) = \overline{\mathcal{B}_{\mathscr{L}}(q, p)}$ for all $p, q \in \mathbb{P}$. If $\mathcal{B}_{\mathscr{L}}$ is a quasi-definite hermitian sesquilinear form, then there exists a sequence of monic polynomials, $(P_n)_{n\geq 0}$, orthogonal with respect to $\mathcal{B}_{\mathscr{L}}$, i.e., $\deg P_n = n$ and $\mathcal{B}_{\mathscr{L}}(P_k, P_n) = \mathbf{k}_n \delta_{n,k}, \mathbf{k}_n \neq 0$, for all $n \in \mathbb{N}$, where $\delta_{k,n}$ is the Kronecker delta (see [28]).

Theorem 3 (Szegő Recursion) *If $\mathcal{B}_{\mathscr{L}}$ is a quasi-definite hermitian sesquilinear form, then its corresponding sequence of monic orthogonal polynomials $(P_n)_{n\geq 0}$ satisfies the recurrence relations* (*see* [28])

$$\begin{aligned} &P_{n+1}(z) = zP_n(z) + P_{n+1}(0)P_n^*(z), && n \geq 0, \quad \textit{Forward relation,} \\ &P_{n+1}(z) = (1 - |P_{n+1}(0)|^2)zP_n(z) + P_{n+1}(0)P_{n+1}^*(z), && n \geq 0, \quad \textit{Backward relation.} \end{aligned} \tag{2}$$

Remark 1 The values $(P_n(0))_{n\geq 1}$ are known in the literature as Verblunsky coefficients.

As a consequence of Theorem 3

$$\mathbf{k}_{n+1} = (1 - |P_{n+1}(0)|^2)\mathbf{k}_n \tag{3}$$

and

$$P^*_{n+1}(z) = P^*_n(z) + \overline{P_{n+1}(0)}zP_n(z). \tag{4}$$

The reverse polynomials satisfy a property of orthogonality that can be summarized as follows (see [31]). If P_n is orthogonal to $\{1, z, \dots, z^{n-1}\}$ with respect to $\mathcal{B}_{\mathscr{L}}$ and $\mathcal{B}_{\mathscr{L}}(P_n(z), z^n) = \mathbf{k}_n \neq 0$, then $P^*_n(z)$ is orthogonal with respect to $\mathcal{B}_{\mathscr{L}}$ to $\{z, z^2 \dots, z^n\}$, i. e., to $z\mathbb{P}_{n-1}$ and

$$\mathcal{B}_{\mathscr{L}}(P^*_n(z), 1) = \mathbf{k}_n.$$

Theorem 4 (Verblunsky's Theorem [13]) *Assume* $(\alpha_n)_{n=1}^{\infty}$ *is a sequence of complex numbers such that* $|\alpha_n| \neq 1$ *for* $n = 1, 2, \dots$. *Suppose that a polynomial sequence* $(P_n))_{n\geq 0}$ *satisfies the Szegő recursion*

$$P_{n+1}(z) = zP_n(z) + \alpha_{n+1}P^*_n(z), n \geq 0, \quad P_0(z) = 1.$$

Then there exists a unique quasi-definite and hermitian functional $\mathscr{L}$ *such that* $(P_n)_{n\geq 0}$ *is orthogonal with respect* $\mathcal{B}_{\mathscr{L}}$. *Moreover, if* $|\alpha_n| < 1$ *for* $n = 1, 2, \dots,$ *then the hermitian functional* $\mathscr{L}$ *is positive-definite and there exists a unique non-trivial probability measure* μ *supported on* $\partial\mathbb{D}$ *such that* $(P_n)_{n\geq 0}$ *is orthogonal with respect to* μ.

The multiplication operator with respect to $(P_n(z))_{n\geq 0}$ is represented in a matrix form by

$$z\mathbf{P}(z) = \mathbf{H}_P\mathbf{P}(z),$$

where $\mathbf{P}(z) = [P_0(z), P_1(z) \cdots, P_n(z), \cdots]^{\top}$ and $\mathbf{H}_P$ is a lower Hessenberg matrix whose entries are

$$(\mathbf{H_P})_{n,j} = \begin{cases} 1, & j = n+1, \\ -\dfrac{\mathbf{k}_n}{\mathbf{k}_j}P_{n+1}(0)\overline{P_j(0)}, & j \leq n, \\ 0, & j > n+1. \end{cases}$$

Moreover, $\mathbf{H_P}\mathbf{D_P}\mathbf{H_P}^* = \mathbf{D_P}$, where $\mathbf{D_P} = \operatorname{diag}(\mathbf{k}_0, \mathbf{k}_1, \dots)$.

Definition 5 Let $\mathcal{B}_{\mathscr{L}}$ be a quasi-definite hermitian sesquilinear form. If $(P_n(z))_{n\geq 0}$ is its sequence of monic orthogonal polynomials, then we define the n-th kernel polynomial as

$$K_n(z, y) = \sum_{j=0}^{n} \frac{\overline{P_j(y)}P_j(z)}{\mathbf{k}_j},$$

where $\mathbf{k}_j = \mathcal{B}_{\mathscr{L}}(P_j, P_j)$.

The Kernel polynomials satisfy the following properties

Proposition 2 ([31]) *If* $K_n(z, y)$ *is the* n*-th Kernel polynomial associated with* $(P_n(z))_{n\geq 0}$, *then*

(i) *Christoffel–Darboux formula*

$$K_n(z,y)=\frac{\overline{P^*_{n+1}(y)}P^*_{n+1}(z)-\overline{P_{n+1}(y)}P_{n+1}(z)}{\mathbf{k}_{n+1}(1-z\overline{y})},\quad n\geq 0. \tag{5}$$

(ii) *In particular, if* $|y|=1$ *then*

$$K_n(y,y)=\frac{\overline{P_{n+1}(y)}(P_{n+1})'(y)-\overline{P^*_{n+1}(y)}(P^*_{n+1})'(y)}{\mathbf{k}_{n+1}},\quad n\geq 0.$$

(iii) *Reproducing property. For every polynomial* q *with* $\deg q\leq n$

$$\mathcal{B}_{\mathscr{L}}(K_n(z,\alpha),q(z))=\overline{q(\alpha)}\quad\textit{and}\quad\mathcal{B}_{\mathscr{L}}(q(z),K_n(z,\alpha))=q(\alpha).$$

For an arbitrary polynomial $p(z)$, $\deg p=n$, and $\mathscr{L}$ a hermitian linear functional, the polynomial of degree n defined by

$$\frac{1}{c_0}\mathscr{L}_y\left((z+y)\frac{p(z)-p(y)}{z-y}\right),$$

is said to be the associated polynomial of second kind of $p(z)$ with respect to $\mathscr{L}$. In particular, it is well known that the associated polynomials of the second kind denoted by $(Q_n)_{n\geq 0}$ corresponding to the orthogonal family $(P_n)_{n\geq 0}$ satisfy the following recurrence relations (see [28])

$$Q_{n+1}(z)=zQ_n(z)-P_{n+1}(0)Q^*_n(z),\quad Q_0(z)=1, \tag{6}$$

$$Q^*_{n+1}(z)=Q^*_n(z)-\overline{P_{n+1}(0)}zQ_n(z), \tag{7}$$

i.e., they satisfy the same recurrence relation as the family $(P_n)_{n\geq 0}$ (2) with $P_{n+1}(0)$ replaced by $-P_{n+1}(0)$. Furthermore, an important relation between $P_n(z)$ and $Q_n(z)$ can be found in [23, 28]

$$P_n(z)Q^*_n(z)+Q_n(z)P^*_n(z)=\text{const.}\,z^n\quad \text{const.}\in\mathbb{R}\setminus\{0\}.$$

Proposition 3 (Mixed Christoffel–Darboux formula) *Let* $(P_n)_{n\geq 0}$ *be the monic orthogonal polynomials with respect to* $\mathcal{B}_{\mathscr{L}}$ *and* $(Q_n)_{n\geq 0}$ *the family of its associated polynomials of second kind. If we define*

$$\mathcal{K}_n(z,y)=\sum_{j=0}^{n}\frac{\overline{P_j(y)}Q_j(z)}{\mathbf{k}_j},$$

then

$$\mathcal{K}_n(z,y)=\frac{2/\mathbf{k}_0-\overline{P^*_{n+1}(y)}Q^*_{n+1}(z)-\overline{P_{n+1}(y)}Q_{n+1}(z)}{\mathbf{k}_{n+1}(1-z\overline{y})}.$$

Proof From (2), (4), (6), (7) and (3) we get

$$\frac{-\overline{P^*_{n+1}(y)}Q^*_{n+1}(z)-\overline{P_{n+1}(y)}Q_{n+1}(z)}{1-z\overline{y}}=\frac{\mathbf{k}_{n+1}}{\mathbf{k}_n}\left(\frac{-z\overline{y}\,\overline{P^*_n(y)}Q^*_n(z)-\overline{P_n(y)}Q_n(z)}{1-z\overline{y}}\right)$$

$$=\frac{\mathbf{k}_{n+1}}{\mathbf{k}_n}\left(\frac{-\overline{P^*_n(y)}Q^*_n(z)-\overline{P_n(y)}Q_n(z)}{1-z\overline{y}}\right)+\frac{\mathbf{k}_{n+1}}{\mathbf{k}_n}\overline{P_n(y)}Q_n(z),$$

and taking into account the initial conditions $P_0(y)=Q_0(z)=1$, we get the result. □

Let $\mathscr{L}$ be a hermitian linear functional and $\mathcal{B}_{\mathscr{L}}$ its corresponding sesquilinear form. If $\mathcal{B}_{\mathscr{L}}$ is quasi-definite, then we introduce the formal power series

$$\mathbf{F}(z) = c_0 + 2\sum_{n=1}^{\infty} c_{-n} z^n, \tag{8}$$

where $c_{-n} = \mathscr{L}(z^{-n})$. If $\mathcal{B}_{\mathscr{L}}$ is positive-definite, then $\mathbf{F}$ is said to be the *Carathéodory function* associated with $\mathscr{L}$. In such a case, $\mathbf{F}(z)$ is an analytic function in $\mathbb{D}$ and $\mathfrak{Re}(\mathbf{F}) > 0$ [29, 31]. Moreover,

Theorem 5 ([29, Lemma 2.1]) *Let* $\mathbf{F}$ *be analytic in* $|z| < 1$ *and suppose that* $\mathbf{F}$ *has simple poles at* $z_k \in \mathbb{D}$, $k = 1, \ldots, n$ *with* $\lim_{z\to z_k} F(z)(z-z_k) = \gamma_k$ *and* $\gamma_k/z_k \in \mathbb{R}$*. Furthermore, assume that the non-tangential boundary values* $\lim_{z\to e^{i\varphi}} \mathfrak{Re}(F(z) - \sum_{k=1}^{n} \gamma_k/(z-z_k))$ *exist a.e. on* $[0, 2\pi]$ *and are* L_p*-integrable on* $[0, 2\pi]$*, where* $p \in (1, \infty)$*. Then*

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\mu(\varphi)$$

with

$$d\mu(\varphi) = \left(\mathfrak{Re}\mathbf{F}(e^{i\varphi}) - a\right) d\varphi - \sum \frac{\gamma_k \pi}{z_k}\delta(e^{i\varphi} - z_k),$$

where

$$\mathfrak{Re}\mathbf{F}(e^{i\varphi}) = \lim_{z\to e^{i\varphi}} \mathfrak{Re}\mathbf{F}(z) \quad \textit{and} \quad a = \sum_{k=1}^{n} \frac{\gamma_k}{2z_k}$$

and $\delta(\cdot - z_k)$ *denotes the Dirac delta at* z_k*.*

As a natural extension, we call to $\mathbf{F}$ *Carathéodory formal power series* when the linear functional is quasi-definite.

Definition 6 (Jones, Njåstad, Thron [25]) Let $\{\delta_n\}_{n\geq 0}$ be a sequence of complex numbers such that $\delta_0 \neq 0$ and $|\delta_n| \neq 1$ for all $n = 1, 2 \ldots$. Then the hermitian Perron–Carathéodory continued fraction (HPC-fraction) is defined by

$$C = \delta_0 - \frac{2\delta_0}{1} \underset{+}{} \frac{1}{\overline{\delta_1}} \underset{+}{} \frac{(1-|\delta_1|^2)z}{\delta_1} \underset{+}{} \frac{1}{\overline{\delta_2}} \underset{+}{} \frac{(1-|\delta_2|^2)z}{\delta_2} \underset{+}{} \cdots \tag{9}$$

If we denote by B_n the numerator and A_n the denominator polynomials of the HPC fractions which are defined by the difference equations

$$B_{-1} = 1, \qquad B_0 = \delta_0, \qquad B_1 = -\delta_0, \tag{10}$$

$$A_{-1} = 0, \qquad A_0 = 1, \qquad A_1 = 1, \tag{11}$$

and

$$\begin{pmatrix} B_{2n} \\ A_{2n} \end{pmatrix} = \overline{\delta_n} z \begin{pmatrix} B_{2n-1} \\ A_{2n-1} \end{pmatrix} + \begin{pmatrix} B_{2n-2} \\ A_{2n-2} \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} B_{2n+1} \\ A_{2n+1} \end{pmatrix} = \delta_n \begin{pmatrix} B_{2n} \\ A_{2n} \end{pmatrix} + (1+|\delta_n|^2) z \begin{pmatrix} B_{2n-1} \\ A_{2n-1} \end{pmatrix}, \tag{13}$$

then $C_n = B_n/A_n$. is the n-th approximant of the HPC-fraction. Moreover, given a HPC-fraction as in (9) there exists a pair (L_0, L_∞) of formal power series

$$L_0(z) = c_0^{(0)} + 2\sum_{n=1}^{\infty} c_{-n} z^n, \quad L_\infty(z) = -c_0^{(\infty)} + 2\sum_{n=1}^{\infty} c_n z^{-n}$$

such that B_{2n}/A_{2n}, and B_{2n+1}/A_{2n+1}, are the weak (n, n) two-point Padé approximants for (L_0, L_∞) of orders $(n+1, n)$ and $(n, n+1)$, respectively, where for n, r, t non-negative integers and B and A polynomials of degree at most n, B/A is called the weak (n, n) two-point Padé approximant of order (r, t) for (L_0, L_∞) if (see [24, 26])

$$A(z)L_0(z) - B(z) = \mathcal{O}(z^r), \quad A(z)L_\infty(z) - B(z) = \mathcal{O}(z^{n-t}), \quad r + t \geq 2n + 1.$$

In particular, if for a quasi-definite hermitian linear functional $\mathscr{L}$ with monic orthogonal polynomials $(P_n)_{n\geq 0}$ and associated second kind polynomials $(Q_n)_{n\geq 0}$, we take $\delta_0 = c_0 = 1$ and $\delta_n = P_n(0)$, $n \geq 1$ in (9), then the numerator and denominator polynomials defined by (10), (11), (12) and (13) are (see [25])

$$\begin{aligned} A_{2n+1}(z) &= P_n(z), & A_{2n}(z) &= P_n^*(z), \\ B_{2n+1}(z) &= Q_n^*(z), & B_{2n}(z) &= -Q_n(z). \end{aligned}$$

Moreover, if $\mathbf{F}(z)$ is the Carathéodory formal power series for $\mathscr{L}$, then when $z \to 0$ (see [28, Theorem 2.1.(a)]) we get

(i) $P_n(z)\mathbf{F}(z) + Q_n(z) = \mathcal{O}(z^n)$,
(ii) $P_n^*(z)\mathbf{F}(z) - Q_n^*(z) = \mathcal{O}(z^{n+1})$.

When $\mathscr{L}$ is positive-definite we have that

$$F(z) = \lim_{n\to\infty} \frac{Q_n^*(z)}{P_n^*(z)}$$

uniformly on compact subsets of $\mathbb{D}$ (see [25, 29, 31]).

We conclude this section by stating the following important result.

Theorem 6 (Peherstorfer [29]) *Let $(P_n)_{n\geq 0}$ be a sequence of orthogonal polynomials with respect to a positive definite linear functional with Carathéodory formal power series,* $\mathbf{F}$. *Furthermore, let A, B, C, D be self-reciprocal polynomials of exact degree μ which satisfy the following conditions*

$$\frac{A(0)}{A^*(0)} = -\frac{B(0)}{B^*(0)} = \frac{D(0)}{D^*(0)} = -\frac{C(0)}{C^*(0)}, \tag{14}$$

$$(AD - BC)(z) = \mathcal{O}(z^\mu) \tag{15}$$

and

$$A(z) - B(z)\mathbf{F}(z) = \mathcal{O}(z^\lambda) \quad \textit{where} \quad 0 \leq \lambda \leq \mu.$$

The following statements hold for $n = 0, 1, 2\ldots$

(a)

$$\widetilde{P}_{n+\mu-\lambda}(z) = \frac{A(z)P_{n+\lambda}(z) + B(z)Q_{n+\lambda}(z)}{z^\lambda}$$

and

$$\widetilde{Q}_{n+\mu-\lambda}(z) = \frac{D(z)Q_{n+\lambda}(z) + C(z)P_{n+\lambda}(z)}{z^{\lambda}}$$

are both polynomials of exact degree $n + \mu - \lambda$ *with leading coefficient independent of* n.

(b) *The polynomial* $\widetilde{P}_{n+\mu-\lambda}(z)$ *is orthogonal with respect to a hermitian linear functional with Carathéodory function, given by*

$$\widetilde{\mathbf{F}}(z) = \frac{-C(z) + D(z)\mathbf{F}(z)}{A(z) - B(z)\mathbf{F}(z)}. \tag{16}$$

(c) *The polynomials* $\widetilde{P}_{n+\mu-\lambda}(z)$ *satisfy the recurrence relation*

$$\widetilde{P}_{n+\mu-\lambda+1}(z) = z\widetilde{P}_{n+\mu-\lambda}(z) + b_{n+\mu-\lambda}(\widetilde{P}_{n+\mu-\lambda})^*(z)$$

with

$$b_{n+\mu-\lambda} = \widetilde{P}_{n+\mu-\lambda+1}(0) = \frac{A^*(0)}{A(0)}P_{n+\lambda+1}(0), \quad n \geq 0. \tag{17}$$

Remark 2 It is necessary to clarify that in [28] the authors assume that the series $\mathbf{F}(z)$ defined in (8) converges on $|z| < \rho$ for some $\rho > 0$. They used this to provide the expression

$$\mathbf{F}(z) = \mathscr{L}_y\left(\frac{y+z}{y-z}\right).$$

However, the proof of [28, Theorem 2.1 (a)] only uses the fact that $\mathbf{F}(z)$ is a (formal) series. This is important because since the proof of [29, Theorem 2.2 (a)-(c)] (Theorem 6 in this document) is fully supported on [28, Theorem 2.1 (a)], the result is still valid when $\mathbf{F}(z)$ is taken as a formal series.

3 Perturbation of hermitian functionals

In this Section we will summarize the most important perturbations of hermitian linear functionals as well as of Carathéodory formal power series. With this in mind, we first provide the following definition. Hereinafter, if $\mathcal{B}_{\mathscr{L}}$ is quasi-definite, then we will assume without loss of generality that $c_0 = 1$.

Definition 7 Let $\mathbf{F}(z)$ be a Carathéodory formal power series, A Carathéodory formal power series $\widetilde{\mathbf{F}}(z)$ is said to be a rational spectral transformation of $\mathbf{F}(z)$ if

$$\widetilde{\mathbf{F}}(z) = \frac{a(z)\mathbf{F}(z) + b(z)}{c(z) + d(z)\mathbf{F}(z)}, \tag{18}$$

where $a(z), b(z), c(z)$ and $d(z)$ are polynomials such that $a(z)c(z) - b(z)d(z) \neq 0$. In particular, if $d(z) = 0$, $\widetilde{\mathbf{F}}(z)$ is said to be a linear spectral transformation of $\mathbf{F}(z)$.

First, we point out some illustrative examples of linear spectral transformations that can be found in the literature.

3.1 Linear spectral transformations

3.1.1 Perturbation of the Toeplitz matrix

Let $\mathscr{L}$ be a hermitian linear functional with moments $(c_n)_{n\in\mathbb{Z}}$ and Carathéodory formal power series $\mathbf{F}(z)$. Consider a new hermitian functional $\widehat{\mathscr{L}}$ with moments $(\widehat{c}_n)_{n\in\mathbb{Z}}$ defined from a perturbation of the moments of $\mathscr{L}$ as follows: For $s, t \in \mathbb{N}$ with $s < t$ and fixed complex numbers $m_j,\ j = s, \dots, t$, let

$$\widehat{c}_j = c_j + \overline{m}_j, \quad \widehat{c}_{-j} = c_{-j} + m_j, \quad j = s, \dots, t,$$

and $\widehat{c}_k = c_k$, otherwise. These transformations were studied in [8] providing conditions for the quasi-definite character of $\widehat{\mathscr{L}}$ as well as the characterization of orthogonal polynomials associated with the corresponding sesquilinear form. Notice that if $\widehat{\mathbf{F}}$ is the Carathéodory formal power series corresponding to $\widehat{\mathscr{L}}$, then it can be written as a linear transformation of $\mathbf{F}$

$$\widehat{\mathbf{F}}(z) = \mathbf{F}(z) + 2\sum_{j=s}^{t} m_j z^j.$$

This is the simplest linear spectral transformation.

3.1.2 Christoffel transformation

Let $\mathscr{L}$ be a hermitian linear functional and $\mathcal{B}_{\mathscr{L}}$ its associated sesquilinear form. Consider the perturbation of $\mathcal{B}_{\mathscr{L}}$ defined by

$$\mathcal{B}_1(p(z), q(z)) := \mathcal{B}_{\mathscr{L}}(p(z), (z-\alpha)q(z)), \quad \alpha \in \mathbb{C}, \quad \alpha \neq 0.$$

Taking into account that $\mathcal{B}_1$ is non-hermitian, under certain conditions there exist two sequences of monic polynomials $(L_n)_{n\geq 0}$, $(R_n)_{n\geq 0}$ biorthogonal with respect to $\mathcal{B}_1$, i.e., for all $n \geq 0$, $\deg L_n(z) = \deg R_n(z) = n$ and

$$\begin{aligned} &\mathcal{B}_1(L_n(z), z^k) = 0, \quad 0 \leq k \leq n-1, && \mathcal{B}_1(L_n(z), z^n) \neq 0, \\ &\mathcal{B}_1(z^k, R_n(z)) = 0, \quad 0 \leq k \leq n-1, && \mathcal{B}_1(z^n, R_n(z)) \neq 0. \end{aligned}$$

In this case, $L_n(z)$ (resp. $R_n(z)$) is said to be left (resp. right) orthogonal with respect to $\mathcal{B}_1$.

Theorem 7 ([12]) *Suppose that $P_n(\alpha) \neq 0$ for every $n \geq 1$. Then*

(i) *The sequence*

$$R_n(z) = \frac{1}{(z-\alpha)}\left(P_{n+1}(z) - \frac{P_{n+1}(\alpha)}{P_n(\alpha)} P_n(z)\right)$$

is right orthogonal with respect to $\mathcal{B}_1$. Besides $\mathcal{B}_1(z^n, R_n) = -\overline{\frac{P_{n+1}(\alpha)}{P_n(\alpha)}}\mathbf{k}_n$.

(ii) *The sequence*

$$L_n(z) = \frac{\mathbf{k}_n}{\overline{P_n(\alpha)}} K_n(z, \alpha) \tag{19}$$

is left orthogonal with respect $\mathcal{B}_1$. In addition, $\mathcal{B}_1(L_n(z), z^n) = -\frac{\overline{P_{n+1}(\alpha)}}{\overline{P_n(\alpha)}}\mathbf{k}_n$.

Theorem 8 *Let* $L_n^{(1)}(z)$ *be the polynomial of degree n defined by*

$$L_n^{(1)}(z)=\frac{1}{(\overline{c_1}-\overline{\alpha})}\mathcal{B}_{1,y}\left((z+y)\frac{L_n(z)-L_n(y)}{(z-y)},1\right),\qquad n\geq 1. \tag{20}$$

Then

$$(\overline{c_1}-\overline{\alpha})zL_n^{(1)}(z)=(z\overline{c_1}+1)L_n(z)+\frac{\mathbf{k}_n}{\overline{P_n(\alpha)}}\left((1-\overline{\alpha}z)\mathcal{K}_n(z,\alpha)-\frac{2}{\mathbf{k}_{n+1}}\right).$$

Proof From (5) and (19) we get

$$\left(\frac{1}{z}-\overline{\alpha}\right)L_n(z)=\frac{\mathbf{k}_n}{\mathbf{k}_{n+1}\overline{P_n(\alpha)}}\,\frac{\overline{P^*_{n+1}(\alpha)}P^*_{n+1}(z)-\overline{P_{n+1}(\alpha)}P_{n+1}(z)}{z}.$$

If $\mathscr{L}$ is the linear functional associated with $\mathcal{B}$, then

$$\begin{aligned}
&\mathscr{L}_y\left(\frac{(z+y)}{(z-y)}\left[\left(\frac{1}{z}-\overline{\alpha}\right)L_n(z)-\left(\frac{1}{y}-\overline{\alpha}\right)L_n(y)\right]\right)\\
&\quad=\mathcal{B}_y\left(\frac{(z+y)}{(z-y)}\left(L_n(z)-L_n(y)\right),(y-\alpha)\right)+\mathscr{L}_y\left(\frac{(z+y)}{(z-y)}\left(\frac{1}{z}-\frac{1}{y}\right)L_n(z)\right)\\
&\quad=(\overline{c_1}-\overline{\alpha})L_n^{(1)}(z)-(z\overline{c_1}+1)\frac{L_n(z)}{z}.
\end{aligned}\tag{21}$$

On the other hand, for $n\geq 1$

$$\begin{aligned}
&\mathscr{L}_y\left(\frac{(z+y)}{(z-y)}\left[\overline{P^*_{n+1}(\alpha)}\left(\frac{P^*_{n+1}(z)}{z}-\frac{P^*_{n+1}(y)}{y}\right)\right]\right)\\
&\quad=\frac{\overline{P^*_{n+1}(\alpha)}}{z}\mathscr{L}\left(\frac{(z+y)}{(z-y)}\left(P^*_{n+1}(z)-P^*_{n+1}(y)\right)\right)-\frac{\overline{P^*_{n+1}(\alpha)}}{z}\mathcal{B}(P^*_{n+1}(y)(z+y),y)\\
&\quad=\frac{\overline{P^*_{n+1}(\alpha)}}{zP_{n+2}(0)}\left(Q_{n+2}(z)-zQ_{n+1}(z)\right)=-\frac{\overline{P^*_{n+1}(\alpha)}}{z}Q^*_{n+1}(z).
\end{aligned}\tag{22}$$

In a similar way we get

$$\begin{aligned}
&\mathscr{L}_y\left(\frac{(z+y)}{(z-y)}\left[\overline{P_{n+1}(\alpha)}\left(\frac{P_{n+1}(z)}{z}-\frac{P_{n+1}(y)}{y}\right)\right]\right)\\
&\quad=\frac{\overline{P_{n+1}(\alpha)}}{z}Q_{n+1}(z)-\frac{\overline{P_{n+1}(\alpha)}}{z}\mathcal{B}_y\left((z+y)P_{n+1}(y),y\right)=\frac{\overline{P_{n+1}(\alpha)}}{z}Q_{n+1}(z).
\end{aligned}\tag{23}$$

Thus, from (21), (22) and (23)

$$\begin{aligned}
&(\overline{c_1}-\overline{\alpha})L_n^{(1)}(z)\\
&\quad=(z\overline{c_1}+1)\frac{L_n(z)}{z}+\frac{\mathbf{k}_n}{\mathbf{k}_{n+1}\overline{P_n(\alpha)}}\left(-\frac{\overline{P_{n+1}(\alpha)}}{z}Q_{n+1}(z)-\frac{\overline{P^*_{n+1}(\alpha)}}{z}Q^*_{n+1}(z)\right)\\
&\quad=(z\overline{c_1}+1)\frac{L_n(z)}{z}+\frac{\mathbf{k}_n}{z\overline{P_n(\alpha)}}\left((1-\overline{\alpha}z)\mathcal{K}_n(z,\alpha)-\frac{2}{\mathbf{k}_{n+1}}\right).
\end{aligned}\tag{24}$$

□

Now let us consider the sesquilinear form $\mathcal{B}_2$ such that

$$\mathcal{B}_2(p(z),q(z)):=\mathcal{B}_{\mathscr{L}}((z-\alpha)p(z),(z-\alpha)q(z)),\qquad \alpha\in\mathbb{C}\setminus\{0\}.$$

Taking into account that $\mathcal{B}_{\mathscr{L}}$ is hermitian, it is not difficult to check that $\mathcal{B}_2$ is also hermitian. Moreover,

Theorem 9 ([12]) *The sesquilinear functional $\mathcal{B}_2$ is quasi-definite if and only if $K_n(\alpha,\alpha) \neq 0$, for every $n \geq 0$.*

Assume that $K_n(\alpha,\alpha) \neq 0$, for every $n \geq 0$ and let $(\widetilde{P}_n)_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to $\mathcal{B}_2$. Since

$$\mathcal{B}_2(p,q) = \mathcal{B}_1((z-\alpha)p,q) = \mathcal{B}_{\mathscr{L}}((z-\alpha)p,(z-\alpha)q),$$

then we have the following connection formula.

Proposition 4 *Let $(P_n)_{n\geq 0}$, $(\widetilde{P}_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ be the sequences of (left) orthogonal polynomials with respect to $\mathcal{B}_{\mathscr{L}}$, $\mathcal{B}_2$ and $\mathcal{B}_1$, respectively. Then*

$$(z-\alpha)\widetilde{P}_n(z) = P_{n+1}(z) - \frac{P_{n+1}(\alpha)}{K_n(\alpha,\alpha)}K_n(z,\alpha), \quad n \geq 0, \tag{25}$$

$$(z-\alpha)\widetilde{P}_n(z) = L_{n+1}(z) - \frac{L_{n+1}(\alpha)}{L_n(\alpha)}L_n(z), \quad n \geq 0. \tag{26}$$

Proof The proof of (25) can be found in [12]. To prove (26) notice that the polynomials $(L_k(z))_{k=0}^{n+1}$ constitute a basis of $\mathbb{P}_{n+1}$. Thus, there exist complex coefficients $(\alpha_{n,k})_{k=0}^{n}$ such that

$$(z-\alpha)\widetilde{P}_n(z) = L_{n+1}(z) + \sum_{k=0}^{n}\alpha_{n,k}L_k(z).$$

The above yields

$$\mathcal{B}_2(\widetilde{P}_n(z), R_k(z)) = \mathcal{B}_1(L_{n+1}(z), R_m(z)) + \sum_{k=0}^{n+1}\alpha_{n,k}\mathcal{B}_1(L_k(z), R_m(z)).$$

Taking into account the orthogonality relations, we obtain the result. □

Corollary 1 *The Verblunsky coefficients of the original and perturbed sesquilinear form $\mathcal{B}_{\mathscr{L}}$ and $\mathcal{B}_2$, respectively, are related by*

$$\widetilde{P}_n(0) = -\frac{1}{\alpha}\left[\frac{K_{n+1}(\alpha,\alpha)}{K_n(\alpha,\alpha)}P_{n+1}(0) - \frac{P_{n+1}(\alpha)\overline{P^*_{n+1}(\alpha)}}{K_n(\alpha,\alpha)\mathbf{k}_{n+1}}\right], \quad |\alpha| \neq 1,$$

$$\widetilde{P}_n(0) = -\overline{\alpha}\left[\frac{K_{n+1}(\alpha,\alpha)}{K_n(\alpha,\alpha)}P_{n+1}(0) - \overline{\alpha}^{n+1}\frac{P^2_{n+1}(\alpha)}{K_n(\alpha,\alpha)\mathbf{k}_{n+1}}\right], \quad |\alpha| = 1.$$

Let $\widetilde{\mathbf{F}}(z)$ be the Carathéodory formal power series associated with $\mathcal{B}_2$ (Christoffel transformation). From the definition, it is not difficult to check that

$$\widetilde{\mathbf{F}}(z) = \frac{a(z)\mathbf{F}(z) + b(z)}{d(z)}, \tag{27}$$

where

$$\begin{aligned} a(z) &= (1-\overline{\alpha}z)(z-\alpha), \\ b(z) &= -\overline{\alpha}z^2 + 2i\mathfrak{Im}(\alpha\overline{c_1})z + \alpha, \end{aligned}$$

$$d(z) = z. \tag{28}$$

Observe that $\widetilde{\mathbf{F}}(z) = L(z)\mathbf{F}(z) + R(z)$, where $L(z) = a(z)/d(z)$ and $R(z) = b(z)/d(z)$ are hermitian and anti-hermitian Laurent polynomials, respectively.

Theorem 10 *Let $\mathbf{F}(z)$ be the Carathéodory formal power series (8) associated with a hermitian linear functional. Let*

$$\Omega_1 = \{\widetilde{\mathbf{F}}(z) : \widetilde{\mathbf{F}}(z) = L(z)\mathbf{F}(z) + R(z),\ L \textit{ hermitian pol.},\ R \textit{ anti-hermitian pol.}\},$$
$$\Omega_2 = \{\widetilde{\mathbf{F}}(z) : \widetilde{\mathbf{F}}(z) \textit{ Carathéodory formal power series associated with a hermitian functional}\}.$$

Then every $\widetilde{\mathbf{F}}(z) \in \Omega_1 \cap \Omega_2$ can be written as a superposition of Christoffel transformations.

Proof Suppose first that

$$\widetilde{\mathbf{F}}(z) = L(z)\mathbf{F}(z) + R(z) \tag{29}$$

with $R(z) = \lambda z + d - \overline{\lambda}/z$, $\mathfrak{Re}(d) = 0$, an anti-hermitian polynomial of degree one. Since $\widetilde{\mathbf{F}}(z) \in \Omega_2$, then it can be written as

$$\widetilde{\mathbf{F}}(z) = \mu_0 + 2\sum_{n=1}^{\infty} \mu_{-n} z^n,$$

where μ_n, $n = 0, 1, \ldots$ are the moments of a linear functional satisfying $\overline{\mu_{-n}} = \mu_n$. In order to preserve the hermitian structure of the linear functional associated with $\widetilde{\mathbf{F}}$, $L(z)$ should be a Laurent polynomial of degree one. With this in mind, let us assume that $L(z) = a_1 z + \gamma + a_2/z$ and from (29)

$$\widetilde{\mathbf{F}}(z) = \frac{a_2 - \overline{\lambda}}{z} + (\gamma + 2a_2 c_{-1} + d) + 2\left(\gamma c_{-1} + a_2 c_{-2} + \frac{a_1 + \lambda}{2}\right) z$$
$$+2\sum_{k=2}^{\infty} \left(a_1 c_{-(k-1)} + \gamma c_{-k} + a_2 c_{-(k+1)}\right) z^k,$$

thus

(i) $a_2 - \overline{\lambda} = 0$.
(ii) $\mu_0 = \gamma + 2a_2 c_{-1} + d \in \mathbb{R}$.
(iii) $\mu_{-1} = \gamma c_{-1} + a_2 c_{-2} + \dfrac{a_1 + \lambda}{2}$.
(iv) For $k \geq 2$, $\mu_{-k} = a_1 c_{-(k-1)} + \gamma c_{-k} + a_2 c_{-(k+1)}$.

From (i) $a_2 = \overline{\lambda}$. Now, for all $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, define

$$\widetilde{\mu}_k = a_1 c_{k+1} + \gamma c_k + a_2 c_{k-1}.$$

We want $\mu_k = \widetilde{\mu}_k$, for $k = \pm 2, \pm 3 \cdots$. It is clear from (iv) that for $k \leq -2$, $\mu_k = \widetilde{\mu}_k$. Now taking into account that by hypothesis $\mu_k = \overline{\mu_{-k}} = \overline{a_1} c_{k-1} + \overline{\gamma} c_k + \overline{a_2} c_{k+1}$, then $\mu_k = \widetilde{\mu}_k$ for $k \geq 2$ if and only if $\gamma \in \mathbb{R}$ and $a_1 = \overline{a_2} = \lambda$. Therefore, if (29) holds, then

$$L(z) = \lambda z + \gamma + \frac{\overline{\lambda}}{z}$$

which is a hermitian polynomial of degree one. Observe that condition (ii) implies $d = -2\mathfrak{Im}(\overline{\lambda} c_{-1})i$. Thus $\widetilde{\mathbf{F}}(z)$ coincides with a Christoffel transformation of $\mathbf{F}(z)$. In general, if

$\widetilde{\mathbf{F}}(z) = L(z)\mathbf{F}(z) + R(z) \in \Omega_1 \cap \Omega_2$ with $\deg(R) = m$, then from Proposition 1 we can factorize the polynomial $R = R_1 \cdots R_m$, where the polynomials R_k, $k = 1, \ldots, m$, are anti-hermitian polynomials of degree one. Thus, iterating the above procedure, we get the result. □

3.1.3 Geronimus transformation

Consider now the inverse problem of the previous section. More precisely, given a hermitian sesquilinear form $\mathcal{B}_{\mathscr{L}}$ defined from a hermitian linear functional $\mathscr{L}$, we seek a hermitian sesquilinear form $\widehat{\mathcal{B}_2}$ also defined from a hermitian linear functional $\widehat{\mathscr{L}_2}$ and such that

$$\mathcal{B}_{\mathscr{L}}(p(z), q(z)) := \widehat{\mathcal{B}_2}((z-\alpha)p(z), (z-\alpha)q(z)), \qquad \alpha \in \mathbb{C} \setminus \{0\}. \tag{30}$$

Let us consider the Laurent polynomial $L(z) = (z-\alpha)(1/z - \overline{\alpha})$, with $\alpha \neq 0, \alpha \in \mathbb{C} \setminus \partial\mathbb{D}$. Then (30) can be rewritten as

$$\widehat{\mathscr{L}_2}(p(z)\overline{q}(1/z)L(z)) = \mathscr{L}(p(z)\overline{q}(1/z)).$$

As a consequence $\widehat{\mathscr{L}_2}$ (resp. $\widehat{\mathcal{B}_2}$) is not unique, Furthermore, if $(\hat{c}_n)_{n\in\mathbb{Z}}$ are the moments of $\widehat{\mathscr{L}_2}$, then

$$c_n = -\overline{\alpha}\hat{c}_{n+1} + (1 + |\alpha|)^2)\hat{c}_n - \alpha\hat{c}_{n-1}, \tag{31}$$

and therefore $1 = (1 + |\alpha|)^2)\hat{c}_0 - 2\,\mathfrak{Re}(\alpha\overline{\hat{c}_1})$. This implies that $\widehat{\mathscr{L}_2}$ is completely defined up to the values of $\hat{c}_0 \in \mathbb{R}$ and $\hat{c}_1 \in \mathbb{C}$. Moreover, if $\widehat{\mathcal{B}_0}$ is a particular solution of (30), then every solution can be written as

$$\widehat{\mathcal{B}_2}(p(z),q(z)) = \widehat{\mathcal{B}_0}(p(z), q(z)) + Mp(\alpha)\,\overline{q}(1/\alpha) + \overline{M}\,p(1/\overline{\alpha})\,\overline{q(\alpha)}, \quad 0 < |\alpha| \neq 1.$$

In [4] it was shown that every sequence of orthogonal polynomials $(\widehat{P}_n)_{n\geq 0}$ with respect to $\widehat{\mathcal{B}_2}$, that is a solution of (30), can be written as

$$\widehat{P}_{n+1}(z) = (z + x_n)P_n(z) + y_n P_n^*(z),$$

where $x_n, y_n \in \mathbb{C}$ satisfy

$$\widehat{P}_{n+1}(0) = \frac{P_n(0) - y_{n-1}}{\overline{x}_{n-1}}, \quad \begin{pmatrix} 1 & \overline{P_n(0)} \\ P_n(0) & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n-1} + \overline{y}_{n-1}\widehat{P}_{n+1}(0) \\ \widehat{P}_{n+1}(0) \end{pmatrix}.$$

Taking $\widehat{P}_1(0)$ and x_0 as free parameters, an algorithm can be provided to construct the family $(\widehat{P}_n)_{n\geq 0}$ (see [4, Subsection 3.2.1])

Let $\widehat{\mathbf{F}}(z)$ be the Carathéodory formal power series associated with $\widehat{\mathcal{B}_2}$. Then from (31) (see [27]) we get

$$\mathbf{F}(z) = (z-\alpha)(1/z - \overline{\alpha})\widehat{\mathbf{F}}(z) + \left(\frac{\alpha}{z} - \overline{\alpha}z\right)\hat{c}_0 + \alpha\hat{c}_{-1} - \overline{\alpha}\hat{c}_1.$$

Thus $\widehat{\mathbf{F}}(z)$ is a linear spectral transformation of $\mathbf{F}(z)$

$$\widehat{\mathbf{F}}(z) = \frac{\hat{a}(z)\mathbf{F}(z) + \hat{b}(z)}{\hat{d}(z)}, \tag{32}$$

where

$$\hat{a}(z) = z,$$

$$\begin{aligned}\hat{b}(z) &= \overline{\alpha}\hat{c}_0 z^2 + 2i\,\mathfrak{Im}(\overline{\alpha}\hat{c}_1)z - \alpha\hat{c}_0,\\ \hat{d}(z) &= (1-\overline{\alpha}z)(z-\alpha).\end{aligned} \tag{33}$$

After some algebraic manipulations on (32), we get

$$\widehat{\mathbf{F}}(z) = \frac{\hat{a}(z)}{\hat{d}(z)}\mathbf{F}(z) + M\frac{\alpha+z}{\alpha-z} + \overline{M}\frac{1+\overline{\alpha}z}{1-\overline{\alpha}z}, \quad \text{with} \quad M = \frac{1}{2}\frac{2(\hat{c}_0 - \alpha\hat{c}_{-1}) - 1}{1-|\alpha|^2}.$$

Corollary 2 *Let* $\mathbf{F}(z)$ *be a Carathéodory formal power series corresponding to a hermitian linear functional. Let* $\widehat{\mathbf{F}}_\alpha(z)$ *and* $\widehat{\mathbf{F}}_{\alpha,M}(z)$ *be the Carathéodory formal power series corresponding to the Christoffel* (27) *and Geronimus* (32) *transformation, respectively, with parameter* α*. Then*

(i) $\widehat{\mathbf{F}}_\alpha(z) \circ \widehat{\mathbf{F}}_{\alpha,M}(z) = \mathbf{F}(z)$ *(Identity transformation).*

(ii) $\widehat{\mathbf{F}}_{\alpha,M}(z) \circ \widetilde{\mathbf{F}}_\alpha(z) = \mathbf{F}(z) + \dfrac{R(z)}{L(z)}$*. This transformation is known in the literature as Uvarov transformation* (*see* [27]).

Theorem 11 *Let* $\mathbf{F}(z)$ *be the Carathéodory formal power series* (8) *associated with a hermitian linear functional. Let*

$$\Omega_3 = \left\{\widetilde{\mathbf{F}}(z) : \widetilde{\mathbf{F}}(z) = \frac{L(z)\mathbf{F}(z) + R(z)}{W(z)},\ L,\ W,\ \textit{hermitian pol.},\ R \textit{ an anti-hermitian pol.}\right\}.$$

Then every $\widetilde{\mathbf{F}}(z) \in \Omega_1 \cap \Omega_3$ *can be written as a superposition of Christoffel and Geronimus transformations.*

Proof Let

$$\widetilde{\mathbf{F}}(z) = \frac{L(z)\mathbf{F}(z) + R(z)}{W(z)},$$

with L, W, hermitian polynomials, R anti-hermitian polynomial and such that $\widetilde{\mathbf{F}}(z) \in \Omega_1$. Suppose that $\deg(W) = m$. Then from Proposition 1, $W = W_1 \cdots W_m$, where each W_k, $k = 1, \ldots, m$, is a hermitian polynomial of degree one. Applying a Christoffel transformation (27) to $\widetilde{\mathbf{F}}(z)$ with $a(z) = zW_1(z)$ we get

$$\widetilde{\mathbf{F}}_1(z) = \frac{L(z)\mathbf{F}(z) + R(z)}{W_2(z)\cdots W_m(z)} + R_1(z),$$

where R_1 is an anti-hermitian polynomial. Since the Christoffel transformation preserves the structure of Carathéodory formal power series, then $\widetilde{\mathbf{F}}_1(z) \in \Omega_1$. Iterating the above process for each polynomial W_k, $k = 2, \ldots, m$,

$$\widetilde{\mathbf{F}}_{m-1}(z) = L(z)\mathbf{F}(z) + R(z) + R_{m-1}(z), \tag{34}$$

where $R_{m-1}(z)$ is an anti-hermitian polynomial. But the transformation (34) is reduced to the case already considered in Theorem 10. Hence, $\widetilde{\mathbf{F}}_{m-1}(z)$ is obtained from $\mathbf{F}(z)$ by means of a composition of Christoffel transformations. On the other hand, the function $\widetilde{\mathbf{F}}(z)$, by construction, is obtained from $\widetilde{\mathbf{F}}_{m-1}(z)$ by means of a composition of Geronimus transformations taking into account the fact that a canonical Geronimus transformation is the right inverse of a canonical Christoffel transformation (Corollary 2 (i)). Thus, $\widetilde{\mathbf{F}}(z)$ is obtained from $\mathbf{F}(z)$ by means of a composition of Christoffel and Geronimus transformation. □

3.2 Rational spectral transformations

3.2.1 Aleksandrov transformations

Definition 8 ([31]) If λ is a complex number such that $|\lambda| = 1$ and $\mathbf{F}(z)$ is a Carathéodory formal power series, then the Aleksandrov transformation of $\mathbf{F}(z)$ is defined by

$$\mathbf{F}^{\lambda}(z) = \frac{(\lambda + 1)\mathbf{F}(z) + \lambda - 1}{(\lambda - 1)\mathbf{F}(z) + \lambda + 1}.$$

This is an example of rational spectral transformation.

Theorem 12 ([31]) *Let $\mathscr{L}$ be a quasi-definite hermitian functional with Carathéodory formal power series $\mathbf{F}(z)$ and let $(P_n)_{n\geq 0}$ be the corresponding sequence of monic orthogonal polynomials. Then, for $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, there exists a sequence of monic orthogonal polynomials $(P_n^{\lambda})_{n\geq 0}$ associated with the Aleksandrov transformation satisfying the recurrence relation*

$$\begin{aligned} P_{n+1}^{\lambda}(z) &= zP_n^{\lambda}(z) + \lambda P_{n+1}(0)(P_n^{\lambda})^*(z), \\ P_{n+1}^{\lambda}(z) &= (1 - |P_{n+1}(0)|^2)zP_n^{\lambda}(z) + \lambda P_{n+1}(0)(P_{n+1}^{\lambda})^*(z). \end{aligned}$$

Moreover

$$\begin{aligned} P_n^{\lambda}(z) &= \frac{1}{2}(1 + \lambda)P_n(z) + \frac{1}{2}(1 - \lambda)Q_n(z), \\ (P_n^{\lambda})^*(z) &= \frac{1}{2}(1 + \overline{\lambda})P_n^*(z) + \frac{1}{2}(1 - \overline{\lambda})Q_n^*(z), \end{aligned}$$

where $Q_n(z)$ is the associated polynomial of second kind of $P_n(z)$.

Theorem 13 ([28]) *Let $(P_n)_{n\geq 0}$ be the monic orthogonal polynomials with respect to $\mathcal{B}_{\mathscr{L}}$, with Carathéodory formal power series $\mathbf{F}(z)$. If $(Q_n)_{n\geq 0}$ is the family of the second kind polynomials, then the Carathéodory formal power series $\mathbf{F}_Q(z)$ associated with the polynomials $(Q_n)_{n\geq 0}$ is*

$$\mathbf{F}_Q(z) = \frac{1}{\mathbf{F}(z)}. \tag{35}$$

Note that the result is a direct consequence of the Aleksandrov transformation with $\lambda = -1$.

3.2.2 Associated and anti-associated polynomials

Definition 9 Suppose that $(P_n)_{n\geq 0}$ satisfies the recurrence relation (2). Then for $n \geq 0$ we define the associated polynomials of order N $(P_n^{(N)})_{n\geq 0}$ which are generated by

$$P_{n+1}^{(N)}(z) = zP_n^{(N)}(z) + P_{n+1+N}(0)(P_n^{(N)})^*(z), \quad P_0^{(N)}(z) = 1.$$

If $(Q_n^{(N)})_{n\geq 0}$ is the sequence of associated polynomials of second kind corresponding to $(P_n^{(N)}(z))_{n\geq 0}$, as natural extension of [29], we get

$$\begin{aligned} P_n^{(N)}(z) &= \frac{\left(Q_N(z) + Q_N^*(z)\right) P_{n+N}(z) + \left(P_N^*(z) - P_N(z)\right) Q_{n+N}(z)}{z^N}, \\ Q_n^{(N)}(z) &= \frac{\left(P_N(z) + P_N^*(z)\right) Q_{n+N}(z) + \left(Q_N^*(z) - Q_N(z)\right) P_{n+N}(z)}{z^N}. \end{aligned}$$

Moreover, the Carathéodory formal series $\mathbf{F}^{(N)}$ associated with $(P_n^{(N)})_{n\geq 0}$, can be expressed as a rational spectral transformation of $\mathbf{F}$ as follows

$$\mathbf{F}^{(N)}(z)=\frac{\big(Q_N(z)-Q_N^*(z)\big)+\big(P_N(z)+P_N^*(z)\big)\,\mathbf{F}(z)}{\big(Q_N(z)+Q_N^*(z)\big)+\big(P_N(z)-P_N^*(z)\big)\,\mathbf{F}(z)}.$$

Definition 10 Suppose that $(P_n)_{n\geq 0}$ satisfies the recurrence relation (2) and let α_i, $i=0,\ldots,N-1$, be complex numbers such that $|\alpha_i|\neq 1$. Then for $n\geq 0$ we define the anti-associated polynomials $(P_n^{(-N)}(z))_{n\geq 0}$ of order N which are generated by

$$P_{n+1}^{(-N)}(z)=zP_n^{(-N)}(z)+\gamma_n(P_n^{(-N)})^*(z),\quad P_0^{(-N)}(z)=1,$$

where

$$\gamma_n=\begin{cases}\alpha_n, & n=0,\ldots,N-1,\\ P_{n+1-N}(0), & n\geq N.\end{cases}$$

Notice that this transformation is not unique since it depends on the choice of α_n, $n=0,\ldots,N-1$.

Proposition 5 *Let $M(z)$ be the polynomial defined by*

$$M(z)=zP_{N-1}^{(-N)}(z)+\alpha_{N-1}(P_{N-1}^{(-N)})^*(z),$$

and $R(z)$ its associated polynomial of second kind. Then

$$P_{N+k}^{(-N)}(z)=\frac{1}{2}(M(z)+M^*(z))P_k(z)+\frac{1}{2}(M(z)-M^*(z))Q_k(z), \tag{36}$$

$$Q_{N+k}^{(-N)}(z)=\frac{1}{2}(R(z)+R^*(z))Q_k(z)+\frac{1}{2}(R(z)-R^*(z))P_k(z). \tag{37}$$

Furthermore, in the positive definite case we get

$$\mathbf{F}^{(-N)}(z)=\frac{(R^*(z)+R(z))\mathbf{F}(z)+(R^*(z)-R(z))}{(M^*(z)-M(z))\mathbf{F}(z)+(M^*(z)+M(z))}. \tag{38}$$

Proof If in Theorem 6 we take

$$A(z)=\frac{1}{2}(M(z)+M^*(z)),\qquad B(z)=\frac{1}{2}(M(z)-M^*(z)),$$

$$C(z)=\frac{1}{2}(R(z)-R^*(z)),\qquad D(z)=\frac{1}{2}(R(z)+R^*(z)),$$

then A, B, C, D are self-reciprocal polynomials of degree N satisfying (14) and (15). Moreover $A(z)-B(z)\mathbf{F}(z)=\mathcal{O}(z^0)$. Thus taking $\mu=N$, $\lambda=0$ in Theorem 6 we have that the polynomial $\widetilde{P}_{N+k}(z)=A(z)P_k(z)+B(z)Q_k(z)$ is a polynomial of degree $N+k$. Notice that, in particular,

$$\begin{aligned}\widetilde{P}_{N+1}(z)&=\frac{1}{2}(M(z)+M^*(z))P_1(z)+\frac{1}{2}(M(z)-M^*(z))Q_1(z)\\&=zM(z)+P_1(0)M^*(z).\end{aligned}$$

Together with (17) this implies that $P_{n+N}^{(-N)}(z)=\widetilde{P}_{N+n}(z)$ for all $n=1,2\ldots,$ and (36) is proved. The proof of (37) is made in a similar way. Finally, (38) is a straightforward consequence of (16). ◻

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\substack{\text{Second.}\\ \text{Kind}}"] \arrow[d, "\substack{\text{Christoffel.}\\ \text{Transf.}}"'] & \mathcal{H} \arrow[d, dashed, "?"] \\
\mathcal{B}_2 \arrow[r] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
P_n \arrow[r, "\substack{\text{Second.}\\ \text{Kind}}"] \arrow[d, "\substack{\text{Christoffel.}\\ \text{Transf.}}"'] & Q_n \arrow[d, dashed, "?"] \\
\widetilde{P}_n \arrow[r] & Q_n^{[c]}
\end{tikzcd}$$

Fig. 3 Problem 3

In the positive definite case the following result holds.

Theorem 14 (Peherstofer [29]) *Let* $\mathbf{F}(z)$ *be a Carathéodory formal series and suppose that the self-reciprocal polynomials* A, B, C *and* D *satisfy* (14), (15) *and* (16) *with* $\lambda = 0$. *Furthermore, assume that* $\mathfrak{Re}(-A(0)/B(0)) > 0$ *as well as* $\gamma = (A(0) - B(0))/(D(0) - C(0))$. *Then*

$$\widetilde{\mathbf{F}}(z) = \gamma \frac{D(z)\mathbf{F}(z) - C(z)}{A(z) - B(z)\mathbf{F}(z)}$$

is a Carathéodory function if and only if all zeros of $A(z)$ *and* $B(z)$ *lie on* $\partial\mathbb{D}$, *are simple and interlace.*

4 Second kind polynomials and spectral transformations

In this section we will study a new rational spectral transformation involving a perturbation of hermitian linear functionals and second-kind polynomials.

Problem 3 (Fig. 3) Let $\mathscr{L}$ be a quasi-definite hermitian functional with associated sesquilinear form $\mathcal{B}_{\mathscr{L}}$. Let $\mathcal{B}_2(p, q) = \mathcal{B}_{\mathscr{L}}((z-\alpha)p, (z-\alpha)q)$, $\alpha \neq 0$, be a Christoffel transformation of $\mathcal{B}_{\mathscr{L}}$, $\mathcal{H}(\cdot, \cdot)$ the sesquilinear form such that the sequence of polynomials of second-kind $(Q_n)_{n\geq 0}$ is the corresponding sequence of orthogonal polynomials. We will denote by $(Q_n^{[c]})_{n\geq 0}$ the sequence of monic polynomials of the second kind with respect to $\mathcal{B}_2$ which are orthogonal with respect to the sesquilinear form $\mathcal{C}(\cdot, \cdot)$. What is the relation between $\mathcal{C}$ and $\mathcal{H}$ (or equivalently, between their Carathéodory formal power series)? What is the relation between their orthogonal polynomials?

To deal with the above problem, we state the following proposition.

Proposition 6 *Let* $(Q_n^{[c]})_{n\geq 0}$ *be the second kind polynomials associated with* $(\widetilde{P}_n)_{n\geq 0}$. *Then*

$$(z-\alpha) Q_n^{[c]}(z) = W_{n+1}(z) - \frac{L_{n+1}(\alpha)}{L_n(\alpha)} W_n(z), \quad n \geq 0, \tag{39}$$

where $W_n(z) = r(z)L_n^{(1)}(z) - q(z)L_n(z)$,

$$r(z) = \frac{(\overline{c_1} - \overline{\alpha})(z-\alpha)}{(1+|\alpha|^2) - 2\mathfrak{Re}(\overline{\alpha}c_1)}, \qquad q(z) = \frac{(\overline{c}_1 - \overline{\alpha})z + 1 - \overline{\alpha}c_1}{(1+|\alpha|^2) - 2\mathfrak{Re}(\overline{\alpha}c_1)},$$

and $L_n(z)$ *and* $L_n^{(1)}(z)$ *are defined in* (19) *and* (20), *respectively.*

Proof From (26), we have

$$\mathcal{B}_1\left(\frac{(z-\alpha)\widetilde{P}_n(z)-(y-\alpha)\widetilde{P}_n(y)}{z-y}(z+y),1\right)=(\overline{c}_1-\overline{\alpha})\left(L^{(1)}_{n+1}(z)-\frac{L_{n+1}(\alpha)}{L_n(\alpha)}L^{(1)}_n(z)\right).$$

On the other hand, the left-hand side of the previous expression is

$$\begin{aligned}&\mathcal{B}_1\left((z+y)\left[\frac{(z-\alpha)\widetilde{P}_n(z)-(y-\alpha)\widetilde{P}_n(z)+(y-\alpha)\widetilde{P}_n(z)-(y-\alpha)\widetilde{P}_n(y)}{z-y}\right],1\right)\\&=\widetilde{P}_n(z)\mathcal{B}_1(z+y,1)+\mathcal{B}_2\left(\frac{\widetilde{P}_n(z)-\widetilde{P}_n(y)}{z-y}(z+y),1\right)\\&=[(\overline{c}_1-\overline{\alpha})z+1-\overline{\alpha}c_1]\widetilde{P}_n(z)+\left[(1+|\alpha|^2)-2\mathfrak{Re}(\alpha\overline{c_1})\right]Q^{[c]}_n(z).\end{aligned}$$

This yields

$$\begin{aligned}&\left[(1+|\alpha|^2)-(\alpha\overline{c_1}+\overline{\alpha}c_1)\right]Q^{[c]}_n(z)\\&=(\overline{c}_1-\overline{\alpha})\left(L^{(1)}_{n+1}(z)-\frac{L_{n+1}(\alpha)}{L_n(\alpha)}L^{(1)}_n(z)\right)-[(\overline{c}_1-\overline{\alpha})z+1-\overline{\alpha}c_1]\widetilde{P}_n(z).\end{aligned}$$

Multiplying both sides of the above equation by $(z-\alpha)$ and taking into account (26), the statement follows. □

Theorem 15 *Let define*

$$A(z)=\overline{\alpha}z^2+2i\,\mathfrak{Im}(\overline{\alpha}c_1)z-\alpha,\quad \textit{and}\quad B(z)=-\overline{\alpha}z^2+(1+|\alpha|^2)z-\alpha. \tag{40}$$

Then,

(i) *The polynomial*

$$\mathcal{R}_{n+1}(z)=\frac{A(z)P_{n+1}(z)+B(z)Q_{n+1}(z)}{z},\quad n\geq 0,$$

has exact degree $n+1$ and its leading coefficient is $B_2(1,1)$.

(ii) *The reverse polynomial of $\mathcal{R}_{n+1}(z)$ can be written as*

$$\mathcal{R}^*_{n+1}(z)=\frac{-A(z)P^*_{n+1}(z)+B(z)Q^*_{n+1}(z)}{z},\quad n\geq 0. \tag{41}$$

(iii) *The sequence $(\mathcal{R}_{n+1})_{n\geq 0}$ satisfies the recurrence relation*

$$\mathcal{R}_{n+1}(z)=z\mathcal{R}_n(z)-P_{n+1}(0)\mathcal{R}^*_n(z),\quad n\geq 0.$$

Proof Taking into account that $P_1(0)=-c_1$, then from (2) and (6) with $n=1$, we get

$$\mathcal{R}_1(z)=B_2(1,1)z+c_1\left[(1+|\alpha|^2)-2i\mathfrak{Im}(\overline{\alpha}c_1)\right]-2\alpha,$$

which is a polynomial of degree 1. Now assume that $R_{k+1}(z)$, is a polynomial of degree $k+1$ for all $k=0,1,\ldots,n$. Again from (2) and (6) and the fact that $Q^*_{n+1}(0)=P^*_{n+1}(0)=1$, we get

$$\mathcal{R}_{n+2}(z)=\mathcal{R}_{n+1}(z)+P_{n+2}(0)\left(\frac{A(z)P^*_{n+1}(z)-B(z)Q^*_{n+1}(z)}{z}\right), \tag{42}$$

is a polynomial of degree $n+2$ and, as consequence, (i) follows for all $n\geq 0$ from the induction process. The above analysis also shows that the leading coefficient of $\mathcal{R}_n(z)$, is $B_2(1,1)$ for all $n=1,2,\ldots$. To prove (ii), let us assume that $A(z)P_{n+1}(z)=\sum_{k=0}^{n+3}a_{n+1,k}z^k$

and $B(z)Q_{n+1}(z)=\sum_{k=0}^{n+3}b_{n+1,k}z^k$. Since $\deg \mathcal{R}_{n+1}(z)=n+1$, then $a_{n+1,0}=-b_{n+1,0}$, $a_{n+1,n+3}=-b_{n+1,n+3}$ and

$$\mathcal{R}^*_{n+1}(z)=\sum_{k=1}^{n+2}(\overline{a_{n+1,k}}+\overline{b_{n+1,k}})z^{n-k+2}.$$

On the other hand, since $A(z)=-A^*(z)$ and $B(z)=B^*(z)$

$$\begin{aligned}\frac{-A(z)P^*_{n+1}(z)+B(z)Q^*_{n+1}(z)}{z}&=\frac{A^*(z)P^*_{n+1}(z)+B^*(z)Q^*_{n+1}(z)}{z}\\&=\frac{(A(z)P_{n+1}(z))^*+(B(z)Q_{n+1}(z))^*}{z}\\&=z^{n+3}\sum_{k=0}^{n+3}\overline{a_{n+1,k}}z^{-k-1}+z^{n+3}\sum_{k=0}^{n+3}\overline{b_{n+1,k}}z^{-k-1}\\&=z^{n+3}\sum_{k=0}^{n+3}(\overline{a_{n+1,k}}+\overline{b_{n+1,k}})z^{n-k+2}.\end{aligned}$$

Thus, we get (41). Finally, (iii) is a direct consequence of (42) and the expression of $R^*_{n+1}(z)$. □

Notice that according to Theorems 4 and 15 (iii), $(\mathcal{R}_n)_{n\geq 0}$ is a sequence of orthogonal polynomials with respect to a hermitian linear functional defined on $\mathbb{L}$. Taking into account that we are interested in finding relations between Carathéodory formal power series, we state the following Theorem.

Theorem 16 *Let $A(z)$, $B(z)$ be defined in (40). If*

$$C(z)=\overline{\alpha}z^2+(1+|\alpha|^2)z+\alpha \quad \text{and} \quad D(z)=-\overline{\alpha}z^2+2i\,\mathfrak{Im}(\overline{\alpha}c_1)z+\alpha,$$

then

(i) *The second kind polynomials associated with $(\mathcal{R}_n)_{n\geq 0}$, denoted by $(\mathcal{T}_n)_{n\geq 0}$, satisfy*

$$\mathcal{T}_{n+1}(z)=\frac{D(z)Q_{n+1}(z)+C(z)P_{n+1}(z)}{z},\quad n\geq 0,$$

with exact degree $n+1$.

(ii) *The reverse polynomial of $\mathcal{T}_{n+1}(z)$ is*

$$\mathcal{T}^*_{n+1}(z)=-\frac{D(z)Q^*_{n+1}(z)-C(z)P^*_{n+1}(z)}{z}.$$

(iii) *The sequence $(\mathcal{T}_{n+1})_{n\geq 0}$ satisfies the recurrence relation*

$$\mathcal{T}_{n+1}(z)=z\mathcal{T}_n(z)+P_{n+1}(0)\mathcal{T}^*_n(z),\quad n\geq 0.$$

(iv) *The Carathéodory formal power series $\mathbf{F}_{\mathcal{R}}(z)$ (resp. $\mathbf{F}_{\mathcal{T}}(z)$) associated with $(\mathcal{R}_n)_{n\geq 0}$ (resp. $(\mathcal{T}_n)_{n\geq 0}$) is given by*

$$\mathbf{F}_{\mathcal{R}}(z)=\frac{-C(z)+D(z)\mathbf{F}(z)}{A(z)-B(z)\mathbf{F}(z)}\quad\left(\text{resp.}\quad \mathbf{F}_{\mathcal{T}}(z)=\frac{1}{\mathbf{F}_{\mathcal{R}}(z)}\right),$$

where $\mathbf{F}(z)$ is the Carathéodory formal power series associated with the original family $(P_n)_{n\geq 0}$

Proof Since the polynomials $A(z)$, $B(z)$, $C(z)$, $D(z)$ satisfy

$$\frac{A(0)}{A^*(0)} = -\frac{B(0)}{B^*(0)} = \frac{D(0)}{D^*(0)} = -\frac{C(0)}{C^*(0)},$$

as well as

$$A(z)D(z) - B(z)C(z) = -\left((1-|\alpha|^2)^2 + 4(\mathfrak{Im}(\overline{\alpha}c_1))\right)z^2,$$

then the statements follow from Theorem 2.2 in [29]. □

As we have mentioned earlier, $(\mathcal{R}_n)_{n\geq 0}$ is the sequence of orthogonal polynomials with respect to a hermitian linear functional on $\mathbb{L}$. We will denote it by $\mathscr{L}_R$.

Proposition 7 *Let $\mathscr{L}_R$ be the hermitian linear functional such that the polynomials $(\mathcal{R}_n)_{n\geq 0}$ are orthogonal and let $\mathcal{B}_R(\cdot,\cdot)$ be the sesquilinear form associated with $\mathscr{L}_R$. Then*

(i) *The kernel polynomial (Definition 5) $K_{n,\mathcal{R}}(x,y)$ associated with $\mathscr{L}_R$ can be written as*

$$K_{n,\mathcal{R}}(z,y) = \frac{\overline{\mathcal{R}^*_{n+1}(y)}\mathcal{R}^*_{n+1}(z) - \overline{\mathcal{R}_{n+1}(y)}\mathcal{R}_{n+1}(z)}{\mathbf{k}_{n+1,\mathcal{R}}(1-z\overline{y})},$$

where $\mathbf{k}_{n,\mathcal{R}} = \mathcal{B}_R(\mathcal{R}_n,\mathcal{R}_n)$.

(ii) *The polynomials $(W_n)_{n\geq 0}$ defined in Proposition 6 can be written as*

$$\begin{aligned} W_n(z) &= \frac{\mathbf{k}_n}{\delta\overline{\mathcal{R}_n(\alpha)}}\left(\frac{\overline{\mathcal{R}^*_{n+1}(\alpha)}\mathcal{R}^*_{n+1}(z) - \overline{\mathcal{R}_{n+1}(\alpha)}\mathcal{R}_{n+1}(z)}{\mathbf{k}_{n+1}(1-z\overline{\alpha})}\right) \\ &= \frac{\mathbf{k}_{n,\mathcal{R}}}{\overline{\mathcal{R}_n(\alpha)}}K_{n,\mathcal{R}}(z,\alpha), \end{aligned} \tag{43}$$

where $\delta := \left[(1+|\alpha|^2) - 2\mathfrak{Re}(\alpha\overline{c_1})\right] = B_2(1,1)$.

(iii) *For each $n \geq 0$*

$$\mathbf{k}_{n+1,\mathcal{R}} = (\delta)^{n+1}\frac{\mathbf{k}_{n+1}}{\mathbf{k}_0}\mathbf{k}_{0,\mathcal{R}}.$$

Proof Let $(\mathfrak{r}_n)_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to $\mathcal{B}_{\mathcal{R}}(\cdot,\cdot)$ and $\widetilde{\mathbf{k}}_{j,\mathcal{R}} = \mathcal{B}_{\mathcal{R}}(\mathfrak{r}_j(z),\mathfrak{r}_j(z))$. Taking into account that for each $n \geq 0$ the leading coefficient of $\mathcal{R}_n(z)$ is δ, we get

$$\begin{aligned} K_{n,\mathcal{R}}(z,y) &= \sum_{j=0}^{n}\frac{\mathfrak{r}_j(z)\overline{\mathfrak{r}_j(y)}}{\widetilde{\mathbf{k}}_{j,\mathcal{R}}} = \frac{\delta^2}{\delta^2}\frac{\overline{\mathfrak{r}^*_{n+1}(y)}\mathfrak{r}^*_{n+1}(z) - \overline{\mathfrak{r}_{n+1}(y)}\mathfrak{r}_{n+1}(z)}{\widetilde{\mathbf{k}}_{n+1}(1-z\overline{y})} \\ &= \frac{\overline{\mathcal{R}^*_{n+1}(y)}\mathcal{R}^*_{n+1}(z) - \overline{\mathcal{R}_{n+1}(y)}\mathcal{R}_{n+1}(z)}{\mathbf{k}_{n+1,\mathcal{R}}(1-z\overline{y})} = \sum_{j=0}^{n}\frac{\mathcal{R}_j(z)\overline{\mathcal{R}_j(y)}}{\mathbf{k}_{j,\mathcal{R}}}, \end{aligned}$$

and (i) is proved. Now, from the definition of $W_n(z)$, (19) and (24) we get

$$\begin{aligned} W_n(z) = {} & \frac{k_n}{\delta\, k_{n+1}\overline{P_n(\alpha)}}\left(\frac{\overline{\alpha}z^2 + 2i\mathfrak{Im}(\overline{\alpha}c_1)}{1-\overline{\alpha}z}\right)\left(\frac{\overline{P^*_{n+1}(\alpha)}P^*_{n+1}(z) - \overline{P_{n+1}(\alpha)}P_{n+1}(z)}{z}\right) \\ & + \frac{k_n}{\delta\, k_{n+1}\overline{P_n(\alpha)}}\frac{(z-\alpha)(1-\overline{\alpha}z)}{(1-\overline{\alpha}z)}\left(-\frac{\overline{P_{n+1}(\alpha)}}{z}Q_{n+1}(z) - \frac{\overline{P^*_{n+1}(\alpha)}}{z}Q^*_{n+1}(z)\right), \end{aligned}$$

and since

$$A(z)=\overline{\alpha}z^2+2i\Im\mathfrak{m}(\overline{\alpha}c_1)z-\alpha, \quad B(z)=(z-\alpha)(1-\overline{\alpha}z)$$

we get

$$W_n(z)=\frac{k_n}{\delta\, k_{n+1}\overline{P_n(\alpha)}}\left(\frac{-\overline{P_{n+1}^*(\alpha)}\mathcal{R}_{n+1}^*(z)-\overline{P_{n+1}(\alpha)}\mathcal{R}_{n+1}(z)}{\mathbf{k}_{n+1}(1-z\overline{\alpha})}\right). \tag{44}$$

Thus, the first part of (ii) is a straightforward consequence of

$$\mathcal{R}_{n+1}(\alpha)=\mathcal{B}(z+\alpha,z-\alpha)P_{n+1}(\alpha), \quad \mathcal{R}_{n+1}^*(\alpha)=-\mathcal{B}(z+\alpha,z-\alpha)P_{n+1}^*(\alpha).$$

Notice that (44) is equivalent to

$$W_n(z)=\left(\frac{1}{\delta}\frac{\mathbf{k}_n}{\mathbf{k}_{n+1}}\frac{\mathbf{k}_{n+1,\mathcal{R}}}{\mathbf{k}_{n,\mathcal{R}}}\right)\frac{\mathbf{k}_{n,\mathcal{R}}}{\overline{\mathcal{R}_n(\alpha)}}K_{n,\mathcal{R}}(z,\alpha),$$

and since $W_n(z)$ is a monic polynomial we get

$$\frac{1}{\delta}\frac{\mathbf{k}_n}{\mathbf{k}_{n+1}}\frac{\mathbf{k}_{n+1,\mathcal{R}}}{\mathbf{k}_{n,\mathcal{R}}}=1.$$

Thus (iii) is proved. □

Corollary 3 *The monic polynomials $(W_n)_{n\geq 0}$ are left orthogonal with respect to the sesquilinear form $\mathcal{B}_{\mathcal{R},1}$ defined by $\mathcal{B}_{\mathcal{R},1}(p(z),q(z)) := \mathcal{B}_{\mathcal{R}}(p(z),(z-\alpha)q(z))$. Furthermore, the sequence of second-kind polynomials $(Q_n^{[c]})_{n\geq 0}$ associated with $(\widetilde{P}_n)_{n\geq 0}$ is orthogonal with respect to the sesquilinear form $\mathcal{B}_{\mathcal{R},2}(p(z),q(z)) := \mathcal{B}_{\mathcal{R}}((z-\alpha)p(z),(z-\alpha)q(z))$. In other words, polynomials $(Q_n^{[c]})_{n\geq 0}$ are orthogonal with respect to a Christoffel transformation of $\mathcal{B}_{\mathcal{R}}$. Besides,*

$$(z-\alpha)Q_n^{[c]}(z)=\frac{1}{\delta}\mathcal{R}_{n+1}(z)-\frac{\mathcal{R}_{n+1}(\alpha)}{K_{n,R}(\alpha,\alpha)}K_{n,R}(z,\alpha).$$

Proof The result is a direct consequence of (43) and (39), respectively. □

Corollary 4 *Let $\mathbf{F}_Q(z)$ and $\mathbf{F}_{Q^{[c]}}(z)$ be the Carathéodory formal power series associated with the sesquilinear forms $\mathcal{H}$ and $\mathcal{C}$, respectively. Then*

$$\mathbf{F}_{Q^{[c]}}(z)=\frac{d(z)\mathbf{F}_Q(z)}{a(z)+b(z)\mathbf{F}_Q(z)},$$

where $a(z)$, $b(z)$, $c(z)$ and $d(z)$ were defined in (28)

Proof The proof is a direct consequence of (35) and (27). □

Problem 4 (Fig. 4) Let $\mathscr{L}$ be a quasi-definite hermitian functional and let $\mathcal{B}_{\mathscr{L}}$ be the associated sesquilinear form. Let $\widehat{\mathcal{B}}_2((z-\alpha)p,(z-\alpha)q) := \mathcal{B}_{\mathscr{L}}(p,q)$ be a Geronimus transformation of $\mathcal{B}_{\mathscr{L}}$, $\mathcal{G}(\cdot,\cdot)$ the sesquilinear form such that the polynomials of the second kind $(Q_n(z))_{n\geq 0}$ constitute the corresponding sequence of orthogonal polynomials, and let $(Q_n^{[g]})_{n\geq 0}$ be the sequence of polynomials of the second kind with respect to $\widehat{\mathcal{B}}_2$, which are orthogonal with respect to the sesquilinear form $\mathcal{G}(\cdot,\cdot)$.

What is the relation between $\mathcal{G}$ and $\mathcal{H}$ (or alternatively, between their Carathéodory formal power series)? On the other hand, what is the relation between their corresponding sequences of orthogonal polynomials?

What is the relation between the corresponding Hessenberg matrices?

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\text{Second. Kind}"] \arrow[d, "\text{Geronimus Transf.}"'] & \mathcal{H} \arrow[d, dashed, "?"] & P_n \arrow[r, "\text{Second. Kind}"] \arrow[d, "\text{Geronimus Transf.}"'] & Q_n \arrow[d, dashed, "?"] \\
\widehat{\mathcal{B}}_2 \arrow[r] & \mathcal{G} & \widehat{P}_n \arrow[r] & Q_n^{[g]}
\end{tikzcd}$$

Fig. 4 Problem 4

Corollary 5 *Let* $\mathbf{F}_Q(z)$ *and* $\mathbf{F}_{Q^{[g]}}(z)$ *be the Carathéodory formal power series associated with the sesquilinear forms* $\mathcal{H}$ *and* $\mathcal{G}$, *respectively. Then*

$$\mathbf{F}_{Q^{[g]}}(z) = \frac{\hat{d}(z)\mathbf{F}_Q(z)}{\hat{a}(z) + \hat{b}(z)\mathbf{F}_Q(z)},$$

where $\hat{a}(z)$, $\hat{b}(z)$, *and* $\hat{d}(z)$ *are given* (33).

5 Concluding remarks

In the same way as in the real case, a very interesting problem is to describe all the rational spectral transformations of Carathéodory formal power series. We note that Theorem 11 and Theorem 14 describe a subset of rational spectral transformations (18) that can be constructed under certain conditions; however, there is not a closed result that gives a characterization of this type of transformations. It is clear that the Christoffel and Geronimus transformations are not sufficient to generate all the linear spectral transformations. Thus, it is necessary to seek a new set of "basis" transformations that enable us to generate all linear spectral transformations, preserving the hermitian condition of the linear functional.

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