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Stability Analysis of an Inflated, Axially Extended, Residually Stressed Circular Cylindrical Tube

Andrey Melnikov[1], **Jose Merodio**[2]

[1]Department of Mathematics, Nazarbayev University, Kabanbay Batyr Ave. 53, Astana, 010000, Kazakhstan, Email: andrey.melnikov@nu.edu.kz

[2]Departamento de Matemática Aplicada a las TIC, ETS de Ingeniería de Sistemas Informáticos, Universidad Politécnica de Madrid, Madrid, 28031, Spain, Email: merodioj@gmail.com

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Corresponding author: J. Merodio (merodioj@gmail.com)

Abstract. Residual stress may have an important influence on the mechanical response of residually stressed materials. This paper is concerned with the effects of residual stress on the stability of inflated, axially extended, residually stressed circular cylindrical tube. To this end, the theory of small incremental deformations superimposed on a large underlying finite deformation is used. Asymmetric and axisymmetric types of bifurcation are considered. It is found that for residual stress parameter $\hat{\gamma}$ of the same sign the effect of the residual stress is different depending on the type of bifurcation. For example, for asymmetric bifurcations with mode number $m = 1$ and with positive $\hat{\gamma}$ inclusion of residual stress makes the tube more stable, on the other hand, for axisymmetric bifurcations inclusion of residual stress, corresponding to positive residual stress parameter $\hat{\gamma}$, leads to increase of instabilities. In all cases, residual stress with positive and negative residual stress parameter $\hat{\gamma}$ leads to a symmetric character of bifurcation curves.

Keywords: Nonlinear elasticity; Residual stresses; Incremental elastic deformations; Tube bifurcation.

1. Introduction

The problem of extension and inflation of a thick-walled elastic tube under internal pressure and external pressure was extensively studied in [1].

The aforementioned contribution does not take into consideration the possible existence of residual stresses, which can be important as we elaborate on it below. In bush mountings for the support of engines residual stresses are often introduced during the vulcanization process or in manufacturing (see [2, 3]). In this case the residual stresses may have a negative impact on the material performance. On the other hand, in soft biological tissues, specifically in aortas and the heart, residual stresses may have a positive impact on the mechanical performance of these tissues.

Bifurcation analysis of a thick-walled cylindrical shell [4] was given in the context of a biomechanical problem concerned with the development of aneurysms of an arterial wall. This physiological abnormality may lead to very dangerous and even fatal consequences resulting from arterial wall tearing. Motivated by the desire to avoid arterial wall tearing, the authors [4] find the conditions for the onset of instabilities in the arterial wall in patients with marfan syndrome. However, as it was mentioned earlier, residual stresses, while often presented in arterial walls, were not considered by [4] in their analysis.

In this paper the bifurcation analysis of an inflated, axially extended, residually stressed circular cylindrical tube is given for simple strain-energy function. Apart from mentioned applications in mechanical engineering and biomechanics, due to the generality of this analysis it may be relevant for other applications as well, specifically for structures having a circular cylindrical geometry.

It is well known that a tube made of a rubber material under symmetrical load after passing a certain critical (bifurcation) point of deformation may take a final configuration which will deviate from perfect circular cylindrical geometry. Relevant experimental data and theoretical analysis can be found in [5, 6] and [7, 1], respectively. However, the bifurcation analysis which takes into account the presence of residual stress, to authors knowledge, is limited in literature. We mention recent work [8], where the stain-energy function based on invariant I_5 was used. The results obtained in [8] are different from those presented here for the strain-energy function based on I_6 invariant. See other references in [8] for the analytical bifurcation analysis. A numerical approach was used for a similar problem studied in [9].

In order to determine when deviations from the perfect cylindrical configuration are possible we use the theory of small incremental deformations superimposed on an underlying finite deformation.

This paper has the following structure. In Section 2, we summarize the most important definitions of the theory of elasticity relevant to the considered problem. The deformation from the reference to the current configuration is described in terms of respective polar cylindrical coordinates. Residual stress and equilibrium equations are introduced in Section 3, certain assumptions pertinent to the problem about residual stress tensor are made which simplify and reduce residual stress tensor to circumferential and radial components. In Section 4, for incompressible material strain energy is introduced generally as a function of nine invariants which account for deformation, residual stress and coupling between them. A general expression of Cauchy stress is given, and by evaluating it in the reference configuration, certain restrictions associated with residual stress for strain energy function are derived. In Section 5, invariants are presented in terms of principal stretches and residual stress components for the specified problem of extension and inflation of the tube. Connections for stress differences and derivatives of the strain energy function with respect to the principal stretches are provided. Specified form of the equilibrium equation is presented and integrated to find a general formula for pressure difference P at the boundaries of the tube. Along with formula for P, expressions for axial load N and reduced axial load F are given generally in terms of integrals as well. In Section 6, a simple form of strain-energy function and specific functions for components of residual stress are introduced. In Section 7, we summarize the equations governing incremental deformations superimposed on a deformed configuration. A general expression for elastic tensor accounting for residual stress is also provided. Then, in Section 8, we present a bifurcation analysis for the residually stressed elastic tube. First, general asymmetric bifurcations are analyzed. In this case the components of displacements due to superimposed deformation are dependent on the axial location along the tube, the radius and the angle in cylindrical polar coordinates. Second, axisymmetric bifurcations are considered. These are the configurations of the tube for which cross-sections remain circular (i.e. there is no dependence on the angle), but with the radius being depended on the axial location. The details of non-dimensionalization of the governing equations and respective boundary conditions are also presented. In Section 9, we discuss numerical results for asymmetric and axisymmetric types of bifurcations. The results are compared with cases where residual stress is not present.

For each of the asymmetric and axisymmetric bifurcations numerical results are based on the usage of the MATLAB code.

2. Kinematics and geometry

Let us consider an unstressed and unstrained continuum body in the *reference* configuration $\mathcal{B}_r$. A material point in this configuration is labelled by its position vector $\mathbf{X}$. The corresponding position vector is denoted by $\mathbf{x}$ in the deformed (or current) configuration $\mathcal{B}$, and the transformation from $\mathcal{B}_r$ to $\mathcal{B}$ is written $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, where the vector function $\boldsymbol{\chi}$ is referred to as the *deformation* (attention is confined to quasi-static deformations here). The deformation gradient tensor, denoted $\mathbf{F}$, is defined by

$$\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}) \tag{1}$$

where Grad is the gradient operator defined with respect to variable $\mathbf{X}$ in the reference configuration $\mathcal{B}_r$. Other important deformation tensors are right and left Cauchy-Green deformation tensors, denoted by $\mathbf{C}$ and $\mathbf{B}$, respectively, are defined by the formulas

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2 \tag{2}$$

where $^{\mathrm{T}}$ signifies the transpose of a second-order tensor, $\mathbf{U}$ and $\mathbf{V}$, respectively, are the right and left stretch tensors, which are positive definite and symmetric and come from the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, $\mathbf{R}$ being a proper orthogonal tensor. For a homogeneous incompressible nonlinearly isotropic elastic solid, the elastic stored energy (defined per unit volume) depends on only two invariants, which are the principal invariants of $\mathbf{C}$ (equivalently of $\mathbf{B}$), defined by

$$I_1 = \mathrm{tr}\,(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}\,(\mathbf{C}^2)] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \tag{3}$$

where $\lambda_i > 0$, $i \in \{1, 2, 3\}$ are the principal stretches, i.e. the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$. The incompressibility of the continuum body results in the constraint which may be written as

$$\det\mathbf{F} = 1 \quad \text{or} \quad \lambda_1\lambda_2\lambda_3 = 1, \tag{4}$$

equivalently in terms of components of $\mathbf{F}$ or in terms of the principal stretches, respectively.

2.1 Extension and inflation of the tube

We now consider a circular cylindrical tube, which, in terms of cylindrical polar coordinates (R, Θ, Z), is defined by the inequalities

$$A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad 0 \leqslant Z \leqslant L \tag{5}$$

in the reference configuration $\mathcal{B}_r$, where A and B are the internal and external radii and L in the length of the tube. In the reference configuration the position vector $\mathbf{X}$ of a point of tube is given by

$$\mathbf{X} = R\mathbf{E}_R + Z\mathbf{E}_Z, \tag{6}$$

where $\mathbf{E}_R$ and $\mathbf{E}_Z$ are the unit basis vectors associated with radial and axial directions, R and Z, respectively. We also denote by $\mathbf{E}_\Theta$ the corresponding unit vector associated with circumferential (azimuthal) direction.

Provided that the deformation, experienced by the tube, preserves its cylindrical circular shape, in the deformed configuration each material point will be located at the place, given by the position vector $\mathbf{x}$

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z, \tag{7}$$

where we make use of cylindrical polar coordinates (r, θ, z) in the current configuration $\mathcal{B}$, which are associated with unit basic vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$, respectively. The volume-preserving deformation consisting of axial extension, radial inflations is defined by

$$r = \sqrt{a^2 + \lambda_z^{-1}(R^2 - A^2)}, \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{8}$$

where λ_z is the (uniform) axial stretch of cylinder. The current deformation geometry of the tube defined by

$$a \leqslant r \leqslant b, \quad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant z \leqslant l = \lambda_z L. \tag{9}$$

For this deformation, the deformation gradient can be expressed as

$$\mathbf{F} = \lambda_r\mathbf{e}_r \otimes \mathbf{E}_R + \lambda_\theta\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_z\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{10}$$

where λ_r, λ_θ and λ_z are the principal stretches in the radial, azimuthal and axial directions. In particular, azimuthal stretch can be found as $\lambda_\theta = r/R$. In terms of principal stretches the incompressibility constraint (4) takes form

$$\lambda_r\lambda_\theta\lambda_z = 1. \tag{11}$$

The right and left Cauchy-Green deformation tensors (2) are calculated as

$$\begin{aligned} \mathbf{C} &= \lambda_r^2\mathbf{E}_R \otimes \mathbf{E}_R + \lambda_\theta^2\mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \lambda_z^2\mathbf{E}_Z \otimes \mathbf{E}_Z, \\ \mathbf{B} &= \lambda_r^2\mathbf{e}_r \otimes \mathbf{e}_r + \lambda_\theta^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_z^2\mathbf{e}_z \otimes \mathbf{e}_z. \end{aligned} \tag{12}$$

3. Equilibrium and residual stress

Throughout this paper, we assume that no body forces and no intrinsic couple stresses are present. Therefore, the Cauchy stress tensor $\boldsymbol{\sigma}$ (symmetric) and the nominal stress $\mathbf{T}$ satisfy the equilibrium equations

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}, \quad \operatorname{Div}\mathbf{T} = \mathbf{0}, \tag{13}$$

respectively, where div and Div are the divergence operators with respect to $\mathbf{x} \in \mathcal{B}$ and $\mathbf{X} \in \mathcal{B}_r$, respectively, and are connected by

$$\boldsymbol{\sigma} = \mathbf{F}\mathbf{T}\,. \tag{14}$$

If the traction is specified on all or part of the boundary we write the traction boundary condition as

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_{\mathrm{A}} \quad \text{on } \partial\mathcal{B}_{\mathrm{r}}, \tag{15}$$

where $\mathbf{t}_{\mathrm{A}}$ is the applied traction per unit area of $\partial\mathcal{B}_{\mathrm{r}}$ and $\mathbf{N}$ is the unit outward normal on $\partial\mathcal{B}_{\mathrm{r}}$.

We now assume that the reference configuration $\mathcal{B}_r$ is *residually* stressed, with the residual stress tensor denoted by $\boldsymbol{\tau}$. In this configuration, $\mathbf{T} = \boldsymbol{\sigma} = \boldsymbol{\tau}$, i.e. there is no difference between measures of stress since the deformation is measured from $\mathcal{B}_r$.

The residual stress $\boldsymbol{\tau}$ may associated with some prior material processing, plastic deformation or manufacturing process and it assumed to be known. It arises in the absence of body forces and surface tractions on the boundary $\partial\mathcal{B}_r$ of the material body $\mathcal{B}_r$. It is also assumed that it is not accompanied by intrinsic couple stresses, so that it is symmetric ($\boldsymbol{\tau}^{\mathrm{T}} = \boldsymbol{\tau}$) and therefore the rotational balance equations are satisfied in $\mathcal{B}_r$ (not shown here) along with the equilibrium equation

$$\operatorname{Div}\boldsymbol{\tau} = \mathbf{0}. \tag{16}$$

Due to the nature of residual stress there are no surface tractions, therefore, $\boldsymbol{\tau}$ must satisfy the boundary condition

$$\boldsymbol{\tau}\mathbf{N} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{r}}. \tag{17}$$

Note that $\boldsymbol{\tau}$ is a residual stress is defined according to Hoger [10] and is different from other types of initial stress, which may associated with surface tractions. It is important to note that residual stresses are necessarily non-uniform and geometry dependent, and therefore the elastic response of a residually stressed material body is inhomogeneous.

For the considered circular cylindrical geometry, we assume that only diagonal components of residual stress τ_{RR}, $\tau_{\Theta\Theta}$, τ_{ZZ} are present, i.e. there is no residual shear stress, which is also compatible with the boundary condition (17). The Z component of the equilibrium eq. (16) implies that τ_{ZZ} may be assumed to be constant. Furthermore, for consistency with boundary condition (17) we obtain that $\tau_{ZZ} \equiv 0$. The remaining components, τ_{RR} and $\tau_{\Theta\Theta}$, are assumed to be dependent only on R, and therefore the component of the equilibrium eq. (16) is the following radial equation which needs to be satisfied non-trivially

$$\frac{d\tau_{RR}}{dR} + \frac{1}{R}(\tau_{RR} - \tau_{\Theta\Theta}) = 0. \tag{18}$$

According to (17) equilibrium eq. (18) is appended by

$$\tau_{RR} = 0 \quad \text{on} \quad R = A, B. \tag{19}$$

For known expression of τ_{RR}, $\tau_{\Theta\Theta}$ can be obtained from eq. (18) as $d(R\tau_{RR})/dR$.

4. Constitutive equations

For a residually stressed elastic solid, the strain energy is a function of the deformation gradient $\mathbf{F}$ and residual stress $\boldsymbol{\tau}$, therefore we write the strain energy function as $W(\mathbf{F}, \boldsymbol{\tau})$ per unit volume, remembering that by objectively, W depends on $\mathbf{F}$ through the right Cauchy-Green tensor $\mathbf{C}$ defined in expression (2). For residually stressed incompressible elastic body the Cauchy and nominal stress tensors $\boldsymbol{\sigma}$ and $\mathbf{T}$ are obtained by

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}, \boldsymbol{\tau}) - p\mathbf{I}, \quad \mathbf{T} = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \tag{20}$$

where p is a Lagrange multiplier associated with the incompressibility constraint $(4)_1$ and $\mathbf{I}$ is the identity tensor in the current configuration $\mathcal{B}$.

We note that $W(\mathbf{F}, \boldsymbol{\tau})$ is automatically objective since $\boldsymbol{\tau}$ is unaffected by rotation in the deformed configuration $\mathcal{B}$ and W depends on $\mathbf{F}$ only through $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$.

If the material has preferred directions in $\mathcal{B}_r$ associated with $\boldsymbol{\tau}$ (its eigenvectors), then the elastic properties of the material relative to $\mathcal{B}_r$ are no longer isotropic, i.e. they are anisotropic.

In the reference configuration where deformation gradient tensor is an identity expression (20) reduces to

$$\boldsymbol{\tau} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}, \boldsymbol{\tau}) - p^{(r)}\mathbf{I}, \tag{21}$$

where $p^{(r)}$ is value of p in $\mathcal{B}_r$. Equation (21) imposes some restrictions on W and $\boldsymbol{\tau}$, which will be given at the end of the next subsection.

4.1 Invariant formulation

Following [12] for an incompressible material, we adopt that W depends on nine invariants of $\mathbf{C}, \boldsymbol{\tau}$ and their combinations. Therefore, invariants defined in terms of $\mathbf{C}$,

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}\left[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2\right], \tag{22}$$

which are basically standard invariants used for isotropic material. The absence of the third invariant is due to the incompressibility constraint resulting in $I_3 = \det\mathbf{C} = 1$. Similarly for residual stress tensor $\boldsymbol{\tau}$ we can define

$$I_4 = \{I_{41}, I_{42}, I_{43}\} \equiv \left\{\operatorname{tr}\boldsymbol{\tau}, \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{\tau})^2 - \operatorname{tr}(\boldsymbol{\tau}^2), \det\boldsymbol{\tau}\right]\right\}, \tag{23}$$

which are collectively denoted I_4. Of course, here incompressibility constraint has no effect on invariant $I_{43} = \det\boldsymbol{\tau}$ related to residual stress. The set of independent invariants is completed by the invariants which include coupling of $\mathbf{C}$ and $\boldsymbol{\tau}$

$$I_5 = \mathrm{tr}(\boldsymbol{\tau}\mathbf{C}), \quad I_6 = \mathrm{tr}(\boldsymbol{\tau}\mathbf{C}^2), \quad I_7 = \mathrm{tr}(\boldsymbol{\tau}^2\mathbf{C}), \quad I_8 = \mathrm{tr}(\boldsymbol{\tau}^2\mathbf{C}^2). \tag{24}$$

Therefore, W is taken as a function of the above nine invariants. We use the notation $W_i \equiv \partial W/\partial I_i$, where $i = 1, 2, 4, 5, 6, 7, 8$ for concise writing. By evaluation of derivatives $\partial I_i/\partial\mathbf{F}$, $i = 1, 2, 4, 5, 6, 7, 8$, the Cauchy stress tensor $(20)_1$ then expands out as

$$\begin{aligned}\boldsymbol{\sigma} =& 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + 2W_5\boldsymbol{\Sigma} + 2W_6(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}) \\ &+ 2W_7\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma} + 2W_8(\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}) - p\mathbf{I},\end{aligned} \tag{25}$$

where we have introduced the new Eulerian tensor $\boldsymbol{\Sigma}$, defined by $\mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}}$, which is push a forward of $\boldsymbol{\tau}$ from $\mathcal{B}_r$ to $\mathcal{B}$. In the reference configuration $\mathcal{B}_r$ where deformation gradient is an identity, i.e. $\mathbf{F} = \mathbf{I}$, the set of invariants reduces to

$$I_1 = I_2 = 3, \quad I_5 = I_6 = \mathrm{tr}\boldsymbol{\tau}, \quad I_7 = I_8 = \mathrm{tr}(\boldsymbol{\tau}^2). \tag{26}$$

We obtain the specialized expression of (21) by evaluating (25) in the reference configuration $\mathcal{B}_r$

$$\boldsymbol{\tau} = (2W_1 + 4W_2 - p^{(r)})\mathbf{I} + 2(W_5 + 2W_6)\boldsymbol{\tau} + 2(W_7 + 2W_8)\boldsymbol{\tau}^2, \tag{27}$$

where all $W_i, i \in \{1, 2, 4, 5, 6, 7, 8\}$, are evaluated for the invariants given by (26). The previous relation (27) implies that the following residual stress-dependent restrictions must be satisfied for the strain-energy function in $\mathcal{B}_r$

$$2W_1 + 4W_2 - p^{(r)} = 0, \quad 2(W_5 + 2W_6) = 1, \quad W_7 + 2W_8 = 0. \tag{28}$$

5. Application to specific deformation: extension and inflation

In terms of principal stretches and residual stress components, τ_{RR} and $\tau_{\Theta\Theta}$, from $(12)_1$ for the considered deformation we obtain invariants

$$\begin{aligned} I_1 &= \lambda_r^2 + \lambda_\theta^2 + \lambda_z^2, \quad I_2 = \lambda_\theta^2\lambda_z^2 + \lambda_r^2\lambda_z^2 + \lambda_r^2\lambda_\theta^2, \\ I_{41} &= \tau_{RR} + \tau_{\Theta\Theta}, \quad I_{42} = \tau_{RR}\tau_{\Theta\Theta}, \quad I_{43} = 0, \\ I_5 &= \lambda_r^2\tau_{RR} + \lambda_\theta^2\tau_{\Theta\Theta}, \quad I_6 = \lambda_r^4\tau_{RR} + \lambda_\theta^4\tau_{\Theta\Theta}, \\ I_7 &= \lambda_r^2\tau_{RR}^2 + \lambda_\theta^2\tau_{\Theta\Theta}^2, \quad I_8 = \lambda_r^4\tau_{RR}^2 + \lambda_\theta^4\tau_{\Theta\Theta}^2. \end{aligned} \tag{29}$$

Taking into consideration incompressibility condition (11) we observe that invariants (29) depend on two independent strain variables, and we take them as λ_θ, λ_z, together with τ_{RR} and $\tau_{\Theta\Theta}$, the third variable λ_r being given by $\lambda_r = \lambda_\theta^{-1}\lambda_z^{-1}$. This allows us to write the strain energy as a function of these variables, specifically as $\bar{W}(\lambda_\theta, \lambda_z, \tau_{RR}, \tau_{\Theta\Theta})$, which we connect to $W(I_1, I_2, I_4, I_5, I_6, I_7, I_8)$ by writing

$$\bar{W}(\lambda_\theta, \lambda_z, \tau_{RR}, \tau_{\Theta\Theta}) = W(I_1, I_2, I_4, I_5, I_6, I_7, I_8), \tag{30}$$

where invariants $I_1, I_2, I_4, I_5, I_6, I_7, I_8$ are given by (29). Then, using (30) and the expression for Cauchy stress (25), we obtain connections

$$\begin{aligned}\sigma_{\theta\theta} - \sigma_{rr} &= \lambda_\theta \frac{\partial\bar{W}}{\partial\lambda_\theta}, \\ \sigma_{zz} - \sigma_{rr} &= \lambda_z \frac{\partial\bar{W}}{\partial\lambda_z}\end{aligned} \tag{31}$$

with $\sigma_{r\theta} = \sigma_{rz} = \sigma_{\theta z} = 0$.

5.1 Equilibrium and boundary load

Since σ_{zz} is uniform along the z axis and $\sigma_{\theta\theta}$ and σ_{rr} depend only on r (or equivalently R) with no shear stress $\sigma_{r\theta} = \sigma_{rz} = \sigma_{\theta z} = 0$, the equilibrium eq. $(13)_1$ reduces to one scalar equation, namely

$$r\frac{d}{dr}(\sigma_{rr}) + \sigma_{rr} - \sigma_{\theta\theta} = 0, \tag{32}$$

which can be integrated

$$\sigma_{rr}(b) - \sigma_{rr}(a) = \int_a^b (\sigma_{\theta\theta} - \sigma_{rr})\frac{dr}{r}, \tag{33}$$

where $\sigma_{rr}(a)$ and $\sigma_{rr}(b)$ are the values of σ_{rr} at the internal boundary surface $r = a$ and external boundary surface $r = b$ in the current configuration $\mathcal{B}$.

The situation in which the inner surface $r = a$ is subject to a pressure P_a on $r = a$ and the external surface $r = b$ is subject to a pressure P_b on $r = b$ leads to the boundary conditions

$$\sigma_{rr}(a) = -P_a \quad \text{and} \quad \sigma_{rr}(b) = -P_b. \tag{34}$$

Therefore, using previous boundary conditions (34) and connection $(31)_1$, eq. (33) can be rewritten as

$$P = P_a - P_b = \int_a^b \lambda_\theta \frac{\partial\bar{W}}{\partial\lambda_\theta}\frac{dr}{r}. \tag{35}$$

The axial load N on any cross section is given by

$$N = \int_a^b \int_0^{2\pi} \sigma_{zz} r \mathrm{d}r\mathrm{d}\theta = 2\pi \int_a^b \sigma_{zz} r\mathrm{d}r. \tag{36}$$

The usage of (32), the boundary values of σ_{rr} and (31), leads to an expression for the so-called *reduced axial load* F, which is defined as the total load N on the end of tube with closed ends reduced by the contributions P_a and P_b. This results in expression

$$F \equiv N - \pi a^2 P_a + \pi b^2 P_b = \pi \int_a^b \left(2\lambda_z \frac{\partial\bar{W}}{\partial\lambda_z} - \lambda_\theta \frac{\partial\bar{W}}{\partial\lambda_\theta}\right) r\mathrm{d}r. \tag{37}$$

6. A simple model accounting for residual stress

In order to proceed further and obtain numerical results we need to choose a model. To this end, we construct strain energy function using a basic neo-Hookean isotropic energy function with the term linear in I_6 accounting for the coupling of deformation and residual stress. Also we take into consideration restriction $(28)_2$. This leads to

$$W = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{4}(I_6 - \mathrm{tr}\boldsymbol{\tau}) \tag{38}$$

where $\mu(> 0)$ is a constant, which corresponds to the shear modulus in the reference configuration of a neo-Hookean (isotropic) material.

For the presented strain energy function (38), we obtain the Cauchy stress from (25)

$$\boldsymbol{\sigma} = \mu\mathbf{B} + \frac{1}{2}(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}) - p\mathbf{I}. \tag{39}$$

We note that the residual stress is accounted by the second term in (39) and we remind that $\boldsymbol{\Sigma} = \mathbf{F}\boldsymbol{\tau}\mathbf{F}^{\mathrm{T}}$.

The chosen model (38) must be also supplemented by the expression of the residual stress component τ_{RR} depending on R in the reference configuration. The boundary conditions (19) suggest that we can write

$$\tau_{RR} = \gamma(R - A)(R - B). \tag{40}$$

The other component, $\tau_{\Theta\Theta}$, can be easily obtained from (18) as

$$\tau_{\Theta\Theta} = \gamma[3R^2 - 2(A + B)R + AB], \tag{41}$$

where γ is constant which defines the strength of the residual stress. Depending on the sign of γ we obtain inequalities for τ_{RR}. Therefore, from (40) $\tau_{RR} < 0(> 0)$ for $\gamma > 0(< 0)$.

We write strain energy function in terms of principal stretches and components of residual stress

$$\bar{W} = \frac{1}{2}\mu\left(\lambda_\theta^2 + \lambda_z^2 + \lambda_\theta^{-2}\lambda_z^{-2} - 3\right) + \frac{1}{4}\left[(\lambda_\theta^{-4}\lambda_z^{-4} - 1)\tau_{RR} + (\lambda_\theta^4 - 1)\tau_{\theta\theta}\right].$$

This leads to the expressions of stress differences

$$\begin{aligned} \sigma_{\theta\theta} - \sigma_{rr} &= \lambda_\theta \bar{W}_{\lambda_\theta} = \mu(\lambda_\theta^2 - \lambda_\theta^{-2}\lambda_z^{-2}) + \lambda_\theta^4\tau_{\Theta\Theta} - \lambda_\theta^{-4}\lambda_z^{-4}\tau_{RR}, \\ \sigma_{zz} - \sigma_{rr} &= \lambda_z \bar{W}_{\lambda_z} = \mu(\lambda_z^2 - \lambda_\theta^{-2}\lambda_2^{-2}) - \lambda_\theta^{-4}\lambda_z^{-4}\tau_{RR}. \end{aligned} \tag{42}$$

7. Incremental formulation

In this section we summarize the equations governing incremental deformations superimposed on a current deformed configuration. A more detailed discussion of this theory can be found in [11].

7.1 Incremental equations and boundary conditions

We denote the increment of each variable by a superimposed dot. For example, increment $\dot{\mathbf{x}}$ can be viewed as a small displacement of the current position $\mathbf{x}$. Operations Grad and obtaining increments commute so that increment in deformation gradient can be written as $\dot{\mathbf{F}} = \mathrm{Grad}\,\dot{\mathbf{x}}$. The increment $\dot{\mathbf{T}}$ must satisfy the incremental governing equation

$$\mathrm{Div}\,\dot{\mathbf{T}} = \mathbf{0}. \tag{43}$$

For incompressible case ($J = 1$) the incremental form of the boundary condition (15) is

$$\dot{\mathbf{T}}^{\mathrm{T}}\mathbf{N} = \dot{\mathbf{t}}_{\mathrm{A}} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{r}}. \tag{44}$$

Now it is convenient to work in terms of the push-forward version of the increment in $\dot{\mathbf{T}}$ defined by (since $J = 1$)

$$\dot{\mathbf{T}}_0 = \mathbf{F}\dot{\mathbf{T}}. \tag{45}$$

Note that (45) can be considered as the incremental counterpart of (14), and also referred to as a quantity updated from the reference configuration $\mathcal{B}_{\mathrm{r}}$ to the deformed configuration $\mathcal{B}$, with updated quantity identified by a zero subscript. A detailed discussion of the concept of updating variables can be found in [11], and here we use this approach throughout the rest of the paper.

It can be shown that the governing eq. (43) is then updated to

$$\mathrm{div}\,\dot{\mathbf{T}}_0 = \mathbf{0}, \tag{46}$$

and the corresponding boundary condition is updated to

$$\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A0}} \quad \text{on} \quad \partial\mathcal{B}. \tag{47}$$

Incrementing $J = \det\mathbf{F}$, we obtain $\dot{J} = J\mathrm{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1})$, and for an incompressible material $\dot{J} = 0$, which leads to the incremental form of the incompressibility condition (4)

$$\mathrm{tr}\,\mathbf{L} \equiv \mathrm{div}\,\mathbf{u} = 0, \tag{48}$$

where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathrm{grad}\,\mathbf{u}$, $\mathbf{u}(= \dot{\mathbf{x}})$ being a function of $\mathbf{x}$.

Let us introduce orthogonal curvilinear coordinate system with $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ being unit basis vectors in this coordinate system. Then, in component form, eq. (46) is equivalent to the three scalar equations

$$\dot{T}_{0ji,j} + \dot{T}_{0ji}\mathbf{e}_k \cdot \mathbf{e}_{j,k} + \dot{T}_{0kj}\mathbf{e}_i \cdot \mathbf{e}_{j,k} = 0, \quad i = 1, 2, 3, \tag{49}$$

in which summation over repeated indices j and k from 1 to 3 is implied and the notation $_{,j}$ represents the derivative associated with the jth curvilinear coordinate, and is made explicit in Section 8 for cylindrical polar coordinates.

7.2 Incremental constitutive equations

Increment $\dot{\mathbf{T}}$ in the nominal stress is a result of increment $\dot{\mathbf{F}}$ in the deformation gradient, which leads to the following incremental form of the constitutive law. Incrementing constitutive eq. $(20)_2$ we obtain

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} - \dot{p}\mathbf{F}^{-1}, \tag{50}$$

where $\boldsymbol{\mathcal{A}}$, which is a fourth-order tensor, denotes elastic moduli associated with the strain energy W. The component form of it is written

$$\mathcal{A}_{\alpha i\beta j} = \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}, \tag{51}$$

which enjoy the symmetries

$$\mathcal{A}_{\alpha i\beta j} = \mathcal{A}_{\beta j\alpha i} \tag{52}$$

as a consequence of equality of mixed partial derivatives.

The component form of equations (50) are then

$$\dot{T}_{\alpha i} = \mathcal{A}_{\alpha i\beta j}\dot{F}_{j\beta} + pF^{-1}_{\alpha k}\dot{F}_{k\beta}F^{-1}_{\beta i} - \dot{p}F^{-1}_{\alpha i}, \tag{53}$$

where $F^{-1}_{\alpha i}$ is defined as $(\mathbf{F}^{-1})_{\alpha i}$. The updated version of (50) is

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + p\mathbf{L} - \dot{p}\mathbf{I}, \tag{54}$$

and in component form the connections between the elastic moduli tensor (51) and its updated form is

$$\mathcal{A}_{0piqj} = F_{p\alpha}F_{q\beta}\mathcal{A}_{\alpha i\beta j}. \tag{55}$$

The symmetry (52) carries over to the updated version of the moduli.

In terms of invariants the updated elasticity tensor can be expanded in its component form as

$$\mathcal{A}_{0piqj} = \sum_{r\in\mathcal{I}} W_r F_{p\alpha}F_{q\beta}\frac{\partial^2 I_r}{\partial F_{i\alpha}\partial F_{j\beta}} + \sum_{r,s\in\mathcal{I}} W_{rs}F_{p\alpha}F_{q\beta}\frac{\partial I_r}{\partial F_{i\alpha}}\frac{\partial I_s}{\partial F_{j\beta}}, \tag{56}$$

where $W_{rs} = \partial^2 W/\partial I_r\partial I_s$ and $\mathcal{I}$ is again the index set $\{1,2,5,6,7,8\}$. For the specific model (39) we obtain elasticity tensor

$$\begin{aligned}\mathcal{A}_{0piqj} = \mu B_{pq}\delta_{ij} + \frac{1}{2}\left(\Sigma_{pq}B_{ij} + (\boldsymbol{\Sigma}\mathbf{B})_{pq}\delta_{ij}\right.\\ \left.+(\mathbf{B}\boldsymbol{\Sigma})_{pq}\delta_{ij} + \Sigma_{ij}B_{pq} + \Sigma_{pj}B_{iq} + \Sigma_{qi}B_{jp}\right).\end{aligned} \tag{57}$$

We also write here the useful connection

$$\mathcal{A}_{0jisk} - \mathcal{A}_{0ijsk} = (\sigma_{js} + p\delta_{js})\delta_{ik} - (\sigma_{is} + p\delta_{is})\delta_{jk}. \tag{58}$$

8. Bifurcation of a residually stressed circular cylinder

In this section, for consistency with the analysis in [1], which does not consider the effect of residual stress, we re-order the coordinates r, θ, z as θ, z, r, associated with the stretches $\lambda, \lambda_z, \lambda_r$, respectively. We also use the notation $\lambda_1, \lambda_2, \lambda_3$ for these stretches in the same order.

The unit basis vectors associated with the cylindrical polar coordinates θ, z, r are denoted $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and the derivatives $(\cdot)_{,k}$ in (49) denoted by subscripts with commas become $\partial/r\partial\theta$, $\partial/\partial z$, $\partial/\partial r$ for $k = 1, 2, 3$, respectively. For the cylindrical polar coordinates the only non-zero scalar products $\mathbf{e}_i \cdot \mathbf{e}_{j,k}$ in (49) are

$$\mathbf{e}_1 \cdot \mathbf{e}_{3,1} = -\mathbf{e}_3 \cdot \mathbf{e}_{1,1} = \frac{1}{r}. \tag{59}$$

The incremental displacement $\dot{\mathbf{x}} = \mathbf{u}$ is written

$$\mathbf{u} = v\mathbf{e}_1 + w\mathbf{e}_2 + u\mathbf{e}_3, \tag{60}$$

and the matrix of components of $\mathbf{L} = \operatorname{grad}\mathbf{u}$ with respect to the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$[L_{ij}] = \begin{bmatrix}(u + v_\theta)/r & v_z & v_r\\ w_\theta/r & w_z & w_r\\ (u_\theta - v)/r & u_z & u_r\end{bmatrix}, \tag{61}$$

where the subscripts θ, z, r without a preceding comma indicate the corresponding partial derivatives.

The incremental incompressibility condition (48) specializes to

$$(u + v_\theta)/r + w_z + u_r = 0. \tag{62}$$

In the next subsections we will consider asymmetrcic and axisymmetric bifurcations. Schematic representation of the configurations of the tube corresponding to asymmetric and axisymmetric bifurcations are given at Fig. 1.

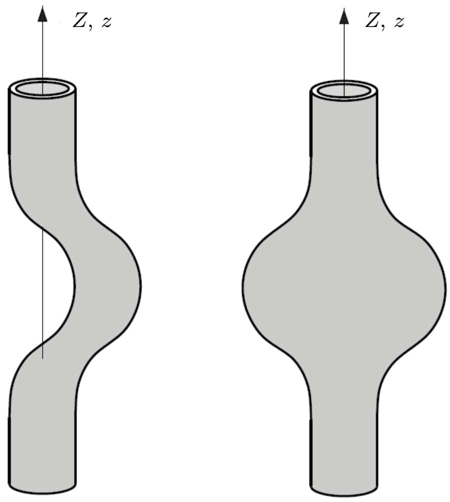

Fig. 1. Schematic representation of the configurations of the tube corresponding to asymmetric (left) and axisymmetric bifurcations (right).

8.1 Asymmetric bifurcations

Three scalar equations corresponding to $i = 1, 2, 3$ can be obtained from (49), using (59)

$$\dot{T}_{011,1} + \dot{T}_{021,2} + \dot{T}_{031,3} + \frac{1}{r}(\dot{T}_{031} + \dot{T}_{013}) = 0, \tag{63}$$

$$\dot{T}_{012,1} + \dot{T}_{022,2} + \dot{T}_{032,3} + \dot{T}_{032}/r = 0, \tag{64}$$

$$\dot{T}_{013,1} + \dot{T}_{023,2} + \dot{T}_{033,3} + \frac{1}{r}(\dot{T}_{033} - \dot{T}_{011}) = 0. \tag{65}$$

For the considered underlying cylindrical configuration the components of $\dot{\mathbf{T}}_0$ in the three above equations are given by

$$\dot{T}_{011} = \mathcal{A}_{01111}L_{11} + \mathcal{A}_{01122}L_{22} + \mathcal{A}_{01133}L_{33} + pL_{11} - \dot{p}, \tag{66}$$
$$\dot{T}_{022} = \mathcal{A}_{02211}L_{11} + \mathcal{A}_{02222}L_{22} + \mathcal{A}_{02233}L_{33} + pL_{22} - \dot{p}, \tag{67}$$
$$\dot{T}_{033} = \mathcal{A}_{03311}L_{11} + \mathcal{A}_{03322}L_{22} + \mathcal{A}_{03333}L_{33} + pL_{33} - \dot{p}, \tag{68}$$
$$\dot{T}_{012} = \mathcal{A}_{01212}L_{21} + \mathcal{A}_{01221}L_{12} + pL_{12}, \tag{69}$$
$$\dot{T}_{021} = \mathcal{A}_{02121}L_{12} + \mathcal{A}_{02112}L_{21} + pL_{21}, \tag{70}$$
$$\dot{T}_{013} = \mathcal{A}_{01313}L_{31} + \mathcal{A}_{01331}L_{13} + pL_{13}, \tag{71}$$
$$\dot{T}_{031} = \mathcal{A}_{03131}L_{13} + \mathcal{A}_{03113}L_{31} + pL_{31}, \tag{72}$$
$$\dot{T}_{023} = \mathcal{A}_{02323}L_{32} + \mathcal{A}_{02332}L_{23} + pL_{23}, \tag{73}$$
$$\dot{T}_{032} = \mathcal{A}_{03232}L_{23} + \mathcal{A}_{03223}L_{32} + pL_{32}, \tag{74}$$

where the components of the elastic moduli tensors $\boldsymbol{\mathcal{A}}_0$ are obtained from the general expression given by (57) for the specific underlying deformation.

Substitution of the expressions (66)–(74) into (63), (64) and (65) and use of the incompressibility condition (62) results in the expressions

$$\begin{aligned} \dot{p}_\theta &= (r\mathcal{A}'_{03131} + \mathcal{A}_{03131})(u_\theta + rv_r - v)/r + (\mathcal{A}_{01111} - \mathcal{A}_{01122} - \mathcal{A}_{02112})(u_\theta + v_{\theta\theta})/r \\ &+ \mathcal{A}_{02121}rv_{zz} + \mathcal{A}_{03131}rv_{rr} + (\mathcal{A}_{01133} - \mathcal{A}_{01122} - \mathcal{A}_{02112} + \mathcal{A}_{03113})u_{r\theta}, \end{aligned} \tag{75}$$

$$\begin{aligned} \dot{p}_z &= (r\mathcal{A}'_{03232} + \mathcal{A}_{03232})(u_z + \omega_r)/r + \mathcal{A}_{01212}(\omega_{\theta\theta} - ru_z)/r^2 + \mathcal{A}_{03232}\omega_{rr} \\ &+ (\mathcal{A}_{02222} - \mathcal{A}_{01221} - \mathcal{A}_{01122})\omega_{zz} + (\mathcal{A}_{02233} + \mathcal{A}_{03223} - \mathcal{A}_{01221} - \mathcal{A}_{01122})u_{rz}, \end{aligned} \tag{76}$$

$$\begin{aligned} \dot{p}_r &= \mathcal{A}_{01313}(u_{\theta\theta} - v_\theta)/r^2 + (r\mathcal{A}'_{01133} - r\mathcal{A}'_{02233} - \mathcal{A}_{01111} + \mathcal{A}_{01122} + \mathcal{A}_{03223})(v_\theta + u)/r^2 \\ &+ (\mathcal{A}_{01331} + \mathcal{A}_{01133} - \mathcal{A}_{03223} - \mathcal{A}_{02233})v_{r\theta}/r + (\mathcal{A}_{03333} - \mathcal{A}_{02233} - \mathcal{A}_{03223})u_{rr} \\ &+ \mathcal{A}_{02323}u_{zz} + (r\mathcal{A}'_{03333} + rp' - r\mathcal{A}'_{02233} + \mathcal{A}_{03333} - 2\mathcal{A}_{02233} + \mathcal{A}_{01122} - \mathcal{A}_{03223})u_r/r, \end{aligned} \tag{77}$$

respectively, where prime denotes differentiation with respect to r.

We recall that the tube is loaded by different internal and external pressures, P_a and P_b. Therefore, the boundary condition (47) is specialized to

$$\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A}0} = \begin{cases} P_a\mathbf{L}^{\mathrm{T}}\mathbf{n} - \dot{P}_a\mathbf{n} & \text{on } r = a \\ P_b\mathbf{L}^{\mathrm{T}}\mathbf{n} - \dot{P}_b\mathbf{n} & \text{on } r = b \end{cases} \tag{78}$$

where $\dot{P}_a$ and $\dot{P}_b$ are prescribed constant increments in P_a and P_b, which we set here to zero on the boundaries $r = a$, $r = b$. In terms of components this gives

$$\dot{T}_{031} = \begin{cases} P_aL_{31} & \text{on } r = a \\ P_bL_{31} & \text{on } r = b, \end{cases} \quad \dot{T}_{032} = \begin{cases} P_aL_{32} & \text{on } r = a \\ P_bL_{32} & \text{on } r = b, \end{cases} \quad \dot{T}_{033} = \begin{cases} P_aL_{33} & \text{on } r = a \\ P_bL_{33} & \text{on } r = b. \end{cases} \tag{79}$$

It is reasonable to assume that $\mathcal{A}_{03131} \neq 0$, $\mathcal{A}_{03232} \neq 0$. Therefore, by using (72), (58) and the boundary conditions (79), the first of these can be written as

$$L_{13} + L_{31} = 0 \quad \text{on} \quad r = a, b, \tag{80}$$

using (74) the second of these can be written as

$$L_{23} + L_{32} = 0 \quad \text{on} \quad r = a, b, \tag{81}$$

while on use of (68) the third can be rewritten as

$$\mathcal{A}_{01133} L_{11} + \mathcal{A}_{02233} L_{22} + (\mathcal{A}_{03333} + \mathcal{A}_{03131} - \mathcal{A}_{01331}) L_{33} - \dot{p} = 0 \quad \text{on } r = a, b. \tag{82}$$

We now write displacements in the form

$$u = f(r) \cos m\theta \sin \alpha z, \tag{83}$$
$$v = g(r) \sin m\theta \sin \alpha z, \tag{84}$$
$$\omega = h(r) \cos m\theta \cos \alpha z, \tag{85}$$
$$\dot{p} = k(r) \cos m\theta \sin \alpha z, \tag{86}$$

where integers $m \geqslant 0$ and $\alpha \geqslant 0$. From the incompressibility condition (62) we obtain

$$h(r) = \frac{r f'(r) + f(r) + g(r) m}{\alpha r}. \tag{87}$$

Substitution of expressions (83)–(86) into (75)–(77) and elimination of $h(r)$ by means of (87) leads to the governing equations

$$\begin{aligned} &(r\mathcal{A}'_{03131} + \mathcal{A}_{03131} + \mathcal{A}_{01111} - \mathcal{A}_{01122} - \mathcal{A}_{02112}) m f(r) \\ &+ (\mathcal{A}_{01133} - \mathcal{A}_{01122} - \mathcal{A}_{02112} + \mathcal{A}_{03113}) r m f'(r) + [r\mathcal{A}^{*\prime}_{03131} + \mathcal{A}_{03131} \\ &+ m^2 (\mathcal{A}_{01111} - \mathcal{A}_{01122} - \mathcal{A}_{02112}) + \alpha^2 r^2 \mathcal{A}_{02121}] g(r) - (r\mathcal{A}^{*\prime}_{03131} + \mathcal{A}_{03131}) r g'(r) \\ &- r^2 g''(r) \mathcal{A}_{03131} - m r k(r) = 0, \end{aligned} \tag{88}$$

$$\begin{aligned} &[r\mathcal{A}'_{03232} - \mathcal{A}_{03232} + m^2 \mathcal{A}_{01212} - \alpha^2 r^2 (r\mathcal{A}'_{03232} + \mathcal{A}_{03232} - \mathcal{A}_{01212} - \mathcal{A}_{02222} + \mathcal{A}_{01221} + \mathcal{A}_{01122})] f(r) \\ &- [r\mathcal{A}'_{03232} - \mathcal{A}_{03232} - m^2 \mathcal{A}_{01212} - \alpha^2 r^2 (\mathcal{A}_{02222} - \mathcal{A}_{02233} - \mathcal{A}_{03223})] r f'(r) \\ &- (r\mathcal{A}'_{03232} + 2\mathcal{A}_{03232}) r^2 f''(r) - \mathcal{A}_{03232} r^3 f'''(r) \\ &+ [r\mathcal{A}'_{03232} - \mathcal{A}_{03232} + m^2 \mathcal{A}_{01212} + \alpha^2 r^2 (\mathcal{A}_{02222} - \mathcal{A}_{01221} - \mathcal{A}_{01122})] m g(r) \\ &- (r\mathcal{A}'_{03232} - \mathcal{A}_{03232}) m r g'(r) - \mathcal{A}_{03232} m r^2 g''(r) + \alpha^2 r^3 k(r) = 0, \end{aligned} \tag{89}$$

$$\begin{aligned} &(r\mathcal{A}'_{01133} - r\mathcal{A}'_{02233} - \mathcal{A}_{01111} + \mathcal{A}_{01122} + \mathcal{A}_{03223} - \alpha^2 r^2 \mathcal{A}_{02323} - m^2 \mathcal{A}_{01313}) f(r) + \\ &+ (r\mathcal{A}'_{03333} + r p' - r\mathcal{A}'_{02233} + \mathcal{A}_{03333} - 2\mathcal{A}_{02233} + \mathcal{A}_{01122} - \mathcal{A}_{03223}) r f'(r) \\ &+ (\mathcal{A}_{03333} - \mathcal{A}_{02233} - \mathcal{A}_{03223}) r^2 f''(r) \\ &+ (r\mathcal{A}'_{01133} - r\mathcal{A}'_{02233} - \mathcal{A}_{01111} + \mathcal{A}_{01122} + \mathcal{A}_{03223} - \mathcal{A}_{01313}) m g(r) \\ &+ (\mathcal{A}_{01331} + \mathcal{A}_{01133} - \mathcal{A}_{03223} - \mathcal{A}_{02233}) m r g'(r) - r^2 k'(r) = 0. \end{aligned} \tag{90}$$

Using (83)–(87) boundary condition (80) becomes

$$r g'(r) - f(r) m - g(r) = 0 \quad \text{on} \quad r = a, b, \tag{91}$$

using the previous expression boundary condition (81) can be rewritten as

$$r^2 f''(r) + r f'(r) + (\alpha^2 r^2 + m^2 - 1) f(r) = 0 \quad \text{on} \quad r = a, b, \tag{92}$$

and boundary condition (82) becomes

$$\begin{aligned} [f(r) + m g(r)](\mathcal{A}_{01133} - \mathcal{A}_{02233}) \quad &+ \quad r f'(r)(\mathcal{A}_{03333} + \mathcal{A}_{03131} - \mathcal{A}_{01331} - \mathcal{A}_{02233}) \\ &- \quad r k(r) = 0 \quad \text{on} \quad r = a, b. \end{aligned} \tag{93}$$

In order to tackle this problem numerically we need to rearrange governing equations (88)–(90) and boundary conditions (91)–(93). To this end, let us introduce new variables

$$y_1 = f(r), \quad y_2 = f'(r), \quad y_3 = f''(r), \quad y_4 = g(r), \quad y_5 = g'(r), \quad y_6 = k(r). \tag{94}$$

Therefore, governing equations (88)–(90) can be rewritten as a system of 6 first-order ordinary differential equations:

$$
\begin{aligned}
&y_1' = y_2, \quad y_2' = y_3, \quad y_4' = y_5,\\
&\mathcal{A}_{03232}r^3y_3'(r) + \mathcal{A}_{03232}mr^2y_5'(r) = [r\mathcal{A}_{03232}' - \mathcal{A}_{03232} + m^2\mathcal{A}_{01212}\\
&-\alpha^2r^2(r\mathcal{A}_{03232}' + \mathcal{A}_{03232} - \mathcal{A}_{01212} - \mathcal{A}_{02222} + \mathcal{A}_{01221} + \mathcal{A}_{01122})]y_1(r)\\
&-[r\mathcal{A}_{03232}' - \mathcal{A}_{03232} - m^2\mathcal{A}_{01212} - \alpha^2r^2(\mathcal{A}_{02222} - \mathcal{A}_{02233} - \mathcal{A}_{03223})]ry_2(r)\\
&-(r\mathcal{A}_{03232}' + 2\mathcal{A}_{03232})r^2y_3(r)\\
&+[r\mathcal{A}_{03232}' - \mathcal{A}_{03232} + m^2\mathcal{A}_{01212} + \alpha^2r^2(\mathcal{A}_{02222} - \mathcal{A}_{01221} - \mathcal{A}_{01122})]my_4(r)\\
&-(r\mathcal{A}_{03232}' - \mathcal{A}_{03232})mry_5(r) + \alpha^2r^3y_6(r),\\
&\mathcal{A}_{03131}r^2y_5'(r) = (r\mathcal{A}_{03131}' + \mathcal{A}_{03131} + \mathcal{A}_{01111} - \mathcal{A}_{01122} - \mathcal{A}_{02112})my_1(r)\\
&+(\mathcal{A}_{01133} - \mathcal{A}_{01122} - \mathcal{A}_{02112} + \mathcal{A}_{03113})rmy_2(r) + [r\mathcal{A}_{03131}' + \mathcal{A}_{03131}\\
&+m^2(\mathcal{A}_{01111} - \mathcal{A}_{01122} - \mathcal{A}_{02112}) + \alpha^2r^2\mathcal{A}_{02121}]y_4(r) - (r\mathcal{A}_{03131}' + \mathcal{A}_{03131})ry_5(r)\\
&-mry_6(r),\\
&r^2y_6'(r) = (r\mathcal{A}_{01133}' - r\mathcal{A}_{02233}' - \mathcal{A}_{01111} + \mathcal{A}_{01122} + \mathcal{A}_{03223} - \alpha^2r^2\mathcal{A}_{02323} - m^2\mathcal{A}_{01313})y_1(r) +\\
&+(r\mathcal{A}_{03333}' + rp' - r\mathcal{A}_{02233}' + \mathcal{A}_{03333} - 2\mathcal{A}_{02233} + \mathcal{A}_{01122} - \mathcal{A}_{03223})ry_2(r)\\
&+(\mathcal{A}_{03333} - \mathcal{A}_{02233} - \mathcal{A}_{03223})r^2y_3(r)\\
&+(r\mathcal{A}_{01133}' - r\mathcal{A}_{02233}' - \mathcal{A}_{01111} + \mathcal{A}_{01122} + \mathcal{A}_{03223} - \mathcal{A}_{01313})my_4(r)\\
&+(\mathcal{A}_{01331} + \mathcal{A}_{01133} - \mathcal{A}_{03223} - \mathcal{A}_{02233})mry_5(r).
\end{aligned} \tag{95}
$$

Boundary conditions (91)–(93) become

$$ry_5(r) - y_1(r)m - y_4(r) = 0 \quad \text{on} \quad r = a, b, \quad \text{on} \quad r = a, b, \tag{96}$$

$$r^2y_3(r) + ry_2(r) + (\alpha^2r^2 + m^2 - 1)y_1(r) = 0 \quad \text{on} \quad r = a, b, \tag{97}$$

$$
\begin{aligned}
[y_1(r) + my_4(r)](\mathcal{A}_{01133} - \mathcal{A}_{02233}) \quad &+ \quad ry_2(r)(\mathcal{A}_{03333} + \mathcal{A}_{03131} - \mathcal{A}_{01331} - \mathcal{A}_{02233})\\
&- \quad ry_6(r) = 0 \quad \text{on} \quad r = a, b.
\end{aligned} \tag{98}
$$

To proceed further with the numerical solution we need to use incremental boundary conditions at the end of the tube. We assume that there are no radial and rotational displacements and the axial component of the increment in the nominal stress tensor is also zero. Therefore, we write

$$u = v = 0, \quad \dot{T}_{022} = 0 \quad \text{on} \quad z = 0, l. \tag{99}$$

From (83) and using the previous incremental boundary condition (99) we obtain

$$\alpha = \frac{\pi n}{\lambda_z L}, \quad n = 1, 2, 3... \tag{100}$$

The numerical solutions are obtained for the non-dimensionalized form of the system of governing ordinary differential equations (95) and corresponding boundary conditions (96)–(98). The essence of the numerical scheme and the details of the non-dimensionalization procedure are discussed in the next section detailing axisymmetric bifurcations.

8.2 Axisymmetric bifurcations

Axisymmetric bifurcations imply that $v = 0$ and u and w are independent of θ, therefore the components of the displacement gradient specialize to

$$[L_{ij}] = \begin{bmatrix} u/r & 0 & 0 \\ 0 & w_z & w_r \\ 0 & u_z & u_r \end{bmatrix}, \tag{101}$$

and the incompressibility condition (48) can be obtained as

$$u/r + w_z + u_r = 0. \tag{102}$$

For axisymmetric incremental deformations with $v = 0$ and no dependence on θ the component of the equilibrium eq. (49) for $i = 1$ is satisfied automatically. Equations for $i = 3, 2$ specialize, respectively, to

$$\dot{T}_{023,2} + \dot{T}_{033,3} + \frac{1}{r}(\dot{T}_{033} - \dot{T}_{011}) = 0, \tag{103}$$

$$\dot{T}_{022,2} + \dot{T}_{032,3} + \frac{1}{r}\dot{T}_{032} = 0, \tag{104}$$

where the components

$$
\begin{aligned}
\dot{T}_{023} &= \mathcal{A}_{02323}L_{32} + \mathcal{A}_{02332}L_{23} + pL_{23}, && (105)\\
\dot{T}_{032} &= \mathcal{A}_{03232}L_{23} + \mathcal{A}_{03223}L_{32} + pL_{32}, && (106)\\
\dot{T}_{011} &= \mathcal{A}_{01111}L_{11} + \mathcal{A}_{01122}L_{22} + \mathcal{A}_{01133}L_{33} + pL_{11} - \dot{p}, && (107)\\
\dot{T}_{022} &= \mathcal{A}_{02211}L_{11} + \mathcal{A}_{02222}L_{22} + \mathcal{A}_{02233}L_{33} + pL_{22} - \dot{p}, && (108)\\
\dot{T}_{033} &= \mathcal{A}_{03311}L_{11} + \mathcal{A}_{03322}L_{22} + \mathcal{A}_{03333}L_{33} + pL_{33} - \dot{p} && (109)
\end{aligned}
$$

are obtained by specializing $(54)_1$. Substitution of these into (103) and (104) and use of incremental incompressibility condition (102) leads to

$$\begin{aligned}\dot{p}_r &= (r\mathcal{A}'_{01133} - \mathcal{A}_{01111})u/r^2 + (r\mathcal{A}'_{03333} + rp' + \mathcal{A}_{03333})u_r/r + \mathcal{A}_{03333}u_{rr}\\ &+ \mathcal{A}_{02323}u_{zz} + (r\mathcal{A}'_{02233} + \mathcal{A}_{02233} - \mathcal{A}_{01122})w_z/r + (\mathcal{A}_{02233} + \mathcal{A}_{03223})w_{rz},\end{aligned} \tag{110}$$

$$\begin{aligned}\dot{p}_z &= \mathcal{A}_{03232}w_{rr} + (r\mathcal{A}'_{03232} + \mathcal{A}_{03232})w_r/r + \mathcal{A}_{02222}w_{zz} + (\mathcal{A}_{02233} + \mathcal{A}_{03223})u_{rz}\\ &+ (r\mathcal{A}'_{03223} + rp' + \mathcal{A}_{03223} + \mathcal{A}_{01122})u_z/r.\end{aligned} \tag{111}$$

We note that, alternatively, expressions (110) and (111) can be obtained directly from (77) and (76), recalling that for axisymmetric case $v = 0$ and there is no dependence on θ.

The traction boundary condition again has the form (78) but now specializes to

$$\dot{T}_{032} = \begin{cases} P_a L_{32} & \text{on } r = a\\ P_b L_{32} & \text{on } r = b,\end{cases} \qquad \dot{T}_{033} = \begin{cases} P_a L_{33} & \text{on } r = a\\ P_b L_{33} & \text{on } r = b.\end{cases} \tag{112}$$

Proceeding further, by using the components (106) and (109), we obtain, provided that $\mathcal{A}_{03232} \neq 0$

$$L_{23} + L_{32} = 0 \quad \text{on} \quad r = a, b. \tag{113}$$

The use of (109) and the incompressibility condition $L_{11} + L_{22} + L_{33} = 0$,

$$(\mathcal{A}_{03333} - \mathcal{A}_{02233} + p)L_{33} + (\mathcal{A}_{01133} - \mathcal{A}_{02233})L_{11} - \dot{p} = \begin{cases} P_a L_{33} & \text{on } r = a\\ P_b L_{33} & \text{on } r = b.\end{cases} \tag{114}$$

We seek for solutions of displacements in the form

$$u = f(r)\sin\alpha z, \tag{115}$$

$$\omega = h(r)\cos\alpha z. \tag{116}$$

We cross-differentiate expressions (110) and (111) with respect to z and r. This leads to elimination of second-order cross derivatives of $\dot{p}$. Furthermore, the use of incompressibility condition (102) allows us to eliminate $h(r)$ from the resulting expression. Thus, we obtain a single governing equation for $f(r)$

$$\begin{aligned}& r^4[\mathcal{A}_{03232}f''' + (r\mathcal{A}'_{03232} + 2\mathcal{A}_{03232})f''/r + (r\mathcal{A}'_{03232} - \mathcal{A}_{03232})f'/r^2 - (r\mathcal{A}'_{03232} - \mathcal{A}_{03232})f/r^3]'\\ &+ \alpha^2r^2[(2\mathcal{A}_{02233} + 2\mathcal{A}_{03223} - \mathcal{A}_{03333} - \mathcal{A}_{02222})r^2f''\\ &+ (2r\mathcal{A}'_{03223} + 2r\mathcal{A}'_{02233} - r\mathcal{A}'_{03333} - r\mathcal{A}'_{02222} - \mathcal{A}_{03333} - \mathcal{A}_{02222} + 2\mathcal{A}_{02233} + 2\mathcal{A}_{03223})rf'\\ &+ (r^2\mathcal{A}''_{03223} + r^2p'' + r\mathcal{A}'_{03223} + r\mathcal{A}'_{01122} - r\mathcal{A}'_{01133} - r\mathcal{A}'_{02222} + r\mathcal{A}'_{02233}\\ &+ \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{03223})f] + \alpha^4r^4\mathcal{A}_{02323}f = 0,\end{aligned} \tag{117}$$

and the corresponding two boundary conditions (113) and (114) as

$$r^2f'' + rf' + (\alpha^2r^2 - 1)f = 0 \quad \text{on} \quad r = a, b \tag{118}$$

and

$$\begin{aligned}& \mathcal{A}_{03232}r^3f''' + (r\mathcal{A}'_{03232} + 2\mathcal{A}_{03232})r^2f'' + (r\mathcal{A}'_{03232} - \mathcal{A}_{03232})rf'\\ &- (r\mathcal{A}'_{03232} - \mathcal{A}_{03232})f - \alpha^2r^2[(\mathcal{A}_{03333} + \mathcal{A}_{02222} - 2\mathcal{A}_{02233} - 2\mathcal{A}_{03223} + \mathcal{A}_{03232})rf'\\ &- (r\mathcal{A}'_{03232} + \mathcal{A}_{01122} - \mathcal{A}_{02222} + \mathcal{A}_{02233} - \mathcal{A}_{01133} + \mathcal{A}_{03223} + \mathcal{A}_{03131} - \mathcal{A}_{01313})f] = 0 \quad \text{on} \quad r = a, b.\end{aligned} \tag{119}$$

Boundary condition (119) was obtained by differentiating (114) with respect to z and then by substituting expression for $\dot{p}_z$ from (111) into the resulting expression.

8.2.1 Numerical solution

In order to obtain numerical results we use incremental boundary condition

$$u = 0 \quad \text{on} \quad z = 0, l. \tag{120}$$

Thus, radial displacements at the ends of the cylinder are not permitted and the increment $\dot{T}_{022}$ in the axial load is not present. Therefore, we obtain from (120) and (115)

$$\alpha = \frac{\pi n}{l} = \frac{\pi n}{\lambda_z L}, \tag{121}$$

where $n = 1, 2, 3, ...$ is the mode number. We observe from (121) that α may be changed either by mode number n or the length of the cylinder L. Therefore, it is convenient to fix $n = 1$ and to perform the analysis for different lengths of the cylinder, recognizing that the effect of increasing the mode number n can be equivalently substituted by a decrease in the value of L.

We introduce the dimensionless variables and material constants defined by

$$\begin{aligned}&\hat{r} = r/A, \quad \hat{a} = a/A, \quad \hat{b} = b/A, \quad \hat{f}(\hat{r}) = \frac{f(r)}{A},\\ &\hat{g}(\hat{r}) = g(r)/A, \quad \hat{k}(\hat{r}) = k(r)/\mu, \quad \hat{p}(\hat{r}) = p(r)/\mu, \quad \hat{\alpha} = \alpha A,\\ &\hat{\gamma} = \gamma A^2/\mu, \quad \hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/\mu, \quad \hat{\boldsymbol{\tau}} = \boldsymbol{\tau}/\mu, \quad \hat{\boldsymbol{\mathcal{A}}}_0 = \boldsymbol{\mathcal{A}}_0/\mu.\end{aligned} \tag{122}$$

We note that nondimensional $\hat{g}(\hat{r})$ and $\hat{k}(\hat{r})$ were used in Section 8.1 for more general case of asymmetric bifurcations.

Also we define the nondimensional variables

$$\hat{y}_1(\hat{r}) = \hat{f}_\alpha(\hat{r}), \quad \hat{y}_2(\hat{r}) = \hat{f}'_\alpha(\hat{r}), \quad \hat{y}_3(\hat{r}) = \hat{f}''_\alpha(\hat{r}), \quad \hat{y}_4(\hat{r}) = \hat{f}'''_\alpha(\hat{r}) \tag{123}$$

so that we can rewrite governing eq. (117) as a system of first-order ordinary differential equations

$$\begin{aligned}
&\hat{y}'_1 = \hat{y}_2, \quad \hat{y}'_2 = \hat{y}_3, \quad \hat{y}'_3 = \hat{y}_4,\\
\hat{r}^4\hat{\mathcal{A}}_{03232}\hat{y}'_4 = &-[3(\hat{r}\hat{\mathcal{A}}'_{03232} - \hat{\mathcal{A}}_{03232}) - \hat{r}^2\hat{\mathcal{A}}''_{03232} + \hat{\alpha}^2\hat{r}^2(\hat{r}^2\hat{\mathcal{A}}''_{03223} + \hat{r}^2\hat{p}'' + \hat{r}\hat{\mathcal{A}}'_{03223} + \hat{r}\hat{\mathcal{A}}'_{01122}\\
&- \hat{r}\hat{\mathcal{A}}'_{01133} - \hat{r}\hat{\mathcal{A}}'_{02222} + \hat{r}\hat{\mathcal{A}}'_{02233} + \hat{\mathcal{A}}_{01111} + \hat{\mathcal{A}}_{02222} - 2\hat{\mathcal{A}}_{01122} - 2\hat{\mathcal{A}}_{03223}) + \hat{\alpha}^4\hat{r}^4\hat{\mathcal{A}}_{02323}]\hat{y}_1\\
&- [\hat{r}^3\hat{\mathcal{A}}''_{03232} - 3\hat{r}^2\hat{\mathcal{A}}'_{03232} + 3\hat{r}\hat{\mathcal{A}}_{03232} + \hat{\alpha}^2\hat{r}^3(2\hat{r}\hat{\mathcal{A}}'_{03223} + 2\hat{r}\hat{\mathcal{A}}'_{02233} - \hat{r}\hat{\mathcal{A}}'_{03333} - \hat{r}\hat{\mathcal{A}}'_{02222}\\
&- \hat{\mathcal{A}}_{03333} - \hat{\mathcal{A}}_{02222} + 2\hat{\mathcal{A}}_{02233} + 2\hat{\mathcal{A}}_{03223})]\hat{y}_2\\
&- [\hat{r}^3(3\hat{\mathcal{A}}'_{03232} + \hat{r}\hat{\mathcal{A}}''_{03232}) - 3\hat{r}^2\mathcal{A}_{03232} + \hat{\alpha}^2\hat{r}^4(2\hat{\mathcal{A}}_{02233} + 2\hat{\mathcal{A}}_{03223} - \hat{\mathcal{A}}_{03333} - \hat{\mathcal{A}}_{02222})]\hat{y}_3\\
&- (2\hat{r}^4\hat{\mathcal{A}}'_{03232} + 2\hat{r}^3\hat{\mathcal{A}}_{03232})\hat{y}_4
\end{aligned} \tag{124}$$

with corresponding boundary conditions, obtained from (118) and (119),

$$\hat{r}^2\hat{y}_3 + \hat{r}\hat{y}_2 + (\hat{\alpha}^2\hat{r}^2 - 1)\hat{y}_1 = 0, \quad \text{on} \quad r = \hat{a}, \hat{b}, \tag{125}$$

$$\begin{aligned}
&\hat{\mathcal{A}}_{03232}\hat{r}^3\hat{y}_4 + (\hat{r}\hat{\mathcal{A}}'_{03232} + 2\hat{\mathcal{A}}_{03232})\hat{r}^2\hat{y}_3 + (\hat{r}\hat{\mathcal{A}}'_{03232} - \hat{\mathcal{A}}_{03232})\hat{r}\hat{y}_2 - (\hat{r}\hat{\mathcal{A}}'_{03232} - \hat{\mathcal{A}}_{03232})\hat{y}_1\\
&- \hat{\alpha}^2\hat{r}^2[(\hat{\mathcal{A}}_{03333} + \hat{\mathcal{A}}_{02222} - 2\hat{\mathcal{A}}_{02233} - 2\hat{\mathcal{A}}_{03223} + \hat{\mathcal{A}}_{03232})\hat{r}\hat{y}_2 - (\hat{r}\hat{\mathcal{A}}'_{03232} + \hat{\mathcal{A}}_{03223} + \hat{\mathcal{A}}_{03131} - \hat{\mathcal{A}}_{01313}\\
&+ \hat{\mathcal{A}}_{01122} - \hat{\mathcal{A}}_{02222} + \hat{\mathcal{A}}_{02233} - \hat{\mathcal{A}}_{01133})\hat{y}_1] = 0, \quad \text{on} \quad r = \hat{a}, \hat{b}.
\end{aligned} \tag{126}$$

Expressions for the elastic moduli specialized for the energy function (38) are given by (57).

We write the initial values for the system (124) in the form

$$\hat{y}_i(\hat{a}) = \delta_{ik}, \quad i = 1, ..., 4, \tag{127}$$

where δ_{ik} is the Kronecker delta. Each k $(= 1, ..., 4)$ in (127) corresponds to the solution $\mathbf{y}^k$ of the system (124), the general solution of which can be written in the form

$$\hat{\mathbf{y}} = \sum_{k=1}^{4} c_k \hat{\mathbf{y}}^k, \tag{128}$$

where c_k are constants.

We require the solution (128) to satisfy boundary conditions (125)–(126). Substitution of (128) into (125)–(126) leads to a 4×4 determinant of coefficients of c_k, vanishing of which represents the bifurcation criterion for this problem.

9. Numerical results

9.1 Asymmetric bifurcation

In this section we consider the results for asymmetric bifurcations for most likely mode number $m = 1$ and moderate values of residual stress parameter $\hat{\gamma}$. Bifurcation curves are denoted by continuous lines with corresponding zero pressure curves denoted by dashed lines. We use red color for positive values of $\hat{\gamma}$ for the respective bifurcation curves and we use blue color for the negative values of $\hat{\gamma}$. Black color corresponds to the case without residual stress. We see from Fig. 2 that residual stress corresponding to negative value of parameter $\hat{\gamma}$ has destabilizing effect for asymmetric bifurcation: for the same values of λ_z the values of λ_a are lower than those required to achieve bifurcation without residual stress. On the other hand, residual stress corresponding to a positive residual stress parameter $\hat{\gamma}$ has a reverse effect and makes the asymmetric bifurcations achievable at values of λ_a higher than those values without residual stress. Thus, in this case residual stress has a stabilizing effect for asymmetric bifurcations.

Let us consider bifurcation curve corresponding to $\hat{\gamma} = 0.5$ with respect to regions of pressure with the same sign. We see that at Fig. 2 zero pressure curves divide bifurcation curve into three regions. Zero pressure curves were obtained from expression (53) for $\psi^* = 0$ in [12] using 'fimplicit' in MATLAB [13]. If we start moving upwards from the bottom of the figure, we first find that asymmetric bifurcation for $\hat{\gamma} = 0.5$ happens at negative pressure (at external pressure). Then moving forward, we get to the region of positive pressure between the two red dashed lines (here bifurcation becomes possible for internally pressurized tube). Moving forward along the third longest part of bifurcation curve, we again get into the area of negative pressure, and thus bifurcation here becomes possible for externally pressurized tube.

Now let us consider the bifurcation curve corresponding to $\hat{\gamma} = -0.5$. We can observe that the shown bifurcation curve is split into two parts by zero pressure curves. Again, for analysis of the result we go upwards along the blue bifurcation curve. The region below the intersection of blue zero pressure curve and bifurcation curve corresponds to negative pressure, and thus externally pressurized tube bifurcates into asymmetric regime in this region. If we move forward upwards along the bifurcation curve, we end up in the region of positive pressure, and thus internally pressurized tube can bifurcate into asymmetric regime for these values of λ_z and λ_a on bifurcation curve. At Fig. 3 we show the results for much longer tube $L/B = 50$. The obtained results look very similar to those shown at Fig. 2.

We also obtained bifurcation curves for $m = 2$ for moderate values of $\hat{\gamma} = -1, 0, 1$, but the effect of residual stress on these bifurcation curves is very small, almost negligible. Therefore, we do not show these results here.

9.2 Axisymmetric bifurcation

At Fig. 4 and Fig. 5 we obtained new results for axisymmetric bifurcations for the residually stressed tube. Also, we reproduced axisymmetric bifurcation curves obtained in [1] without residual stress. We note that values of L/B should be divided by 2 in Fig. 3 in [1] for correct caption of the figure. Again, we can observe a symmetric picture with respect to the case without residual stress. But now unlike in the case for asymmetric bifurcations, residual stress corresponding to positive residual stress parameter $\hat{\gamma}$ has

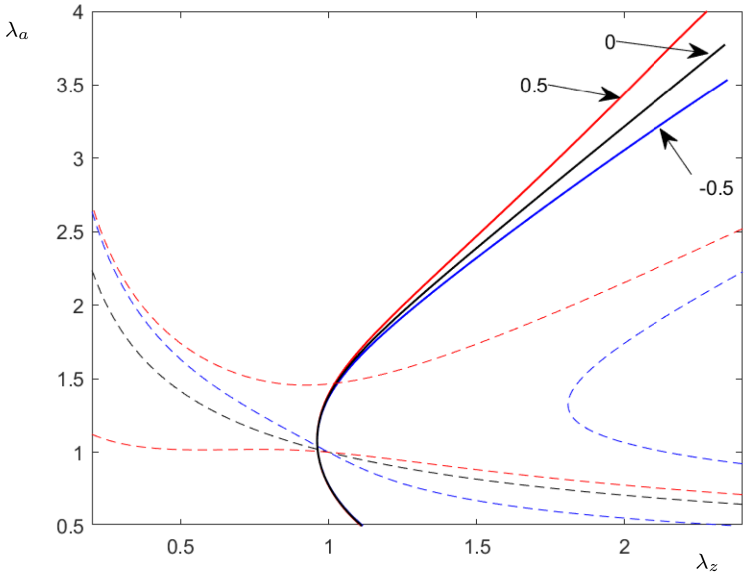

Fig. 2. Plots of the asymmetric bifurcation curves (continuous curves) for the augmented neo-Hookean elastic material (38) for $A/B = 0.8$, $L/B = 10$, $\hat{\gamma} = -0.5, 0, 0.5$, mode number $m = 1$. The values of $\hat{\gamma}$ are shown next to the relevant curves. Dashed lines are zero pressure curves corresponding to residual stress parameter $\hat{\gamma}$ and shown with the same color as respective bifurcation curve.

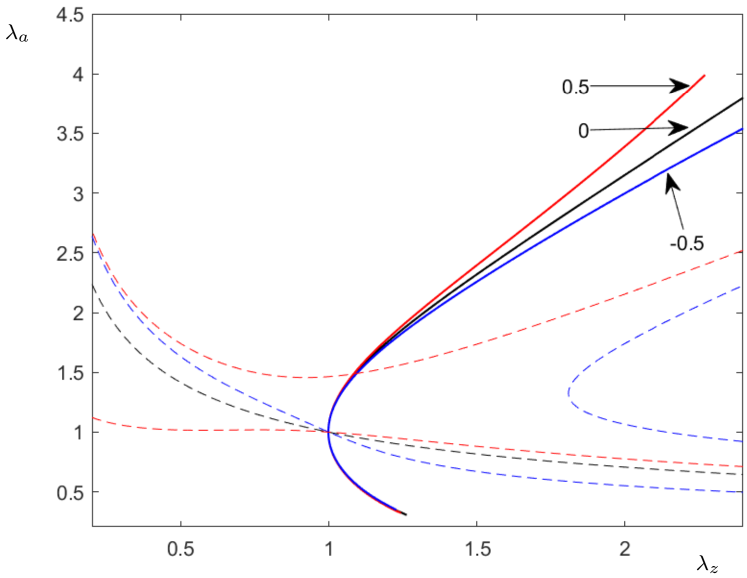

Fig. 3. Plots of the asymmetric bifurcation curves (continuous curves) for the augmented neo-Hookean elastic material (38) for $A/B = 0.8$, $L/B = 50$, $\hat{\gamma} = -0.5, 0, 0.5$, mode number $m = 1$. The values of $\hat{\gamma}$ are shown next to the relevant curves. Dashed lines are zero pressure curves corresponding to residual stress parameter $\hat{\gamma}$ and shown with the same color as respective bifurcation curve.

a destabilizing effect: for the same fixed values of λ_z, axisymmetric bifurcation can happen at lower values of λ_a as opposed to the case without residual stress. On the other hand, residual stress corresponding to negative residual stress parameter $\hat{\gamma}$ has a stabilizing effect: for the same fixed values of λ_z, axisymmetric bifurcations occur at larger values of λ_a, and thus it is more difficult to achieve these values of λ_a and switch into this bifurcation regime.

Now let us discuss for what pressure (internal or external) and for which values of λ_a and λ_z axisymmetric bifurcations become possible. At Fig. 4 for the case without residual stress, zero pressure curve divides the region of values of λ_a and λ_z into two areas. The upper-right region corresponds to the area of positive pressure, and thus internally pressurized tube can bifurcate here into axisymmetric regime. The lower-left area corresponds to the limited bifurcation values of λ_z and λ_a ($\lambda_z \approx 0.5, \lambda_a < 1.5$), where bifurcation is possible under external pressure.

For residually stressed tube the division of the area by zero pressure curves at Fig. 4 is more complicated. Let us first consider bifurcation curve corresponding to $\hat{\gamma} = -0.8$ at Fig. 4. The middle region between blue dashed zero pressure curves corresponds to the area of positive pressure and thus internally pressurized residually stressed tube can bifurcate into axisymmetric regime if the tube is deformed within the values of λ_a and λ_z, which are limited by these two zero pressure curves (blue dashed lines). Outside this region there are two zones of negative pressure on the left and right hand sides, and thus axisymmetric bifurcation is possible under external pressure.

Now we consider bifurcation of a residually stressed tube with parameter $\hat{\gamma} = 0.8$. The middle region between the two red dashed zero pressure curves correspond to the area of positive pressure and thus bifurcation is possible under internal pressure. The upper and lower regions correspond to the areas of negative pressure, where bifurcation of a residually stressed tube becomes possible under external pressure. For the upper region this corresponds to the ranges $0.6 \lessapprox \lambda_z \lessapprox 1.6$ and $1.7 \lessapprox \lambda_a \lessapprox 2.1$. The results for longer tube $L/B = 10$ are shown at Fig. 5. The character of the results is very similar to those results shown for shorter tube at Fig. 4. We note that bifurcation curves have higher maxima for longer tube, while zero pressure curves do not depend on the ratio L/B.

The results shown here are in agreement qualitatively with the ones obtained numerically in [9], [14], [15].

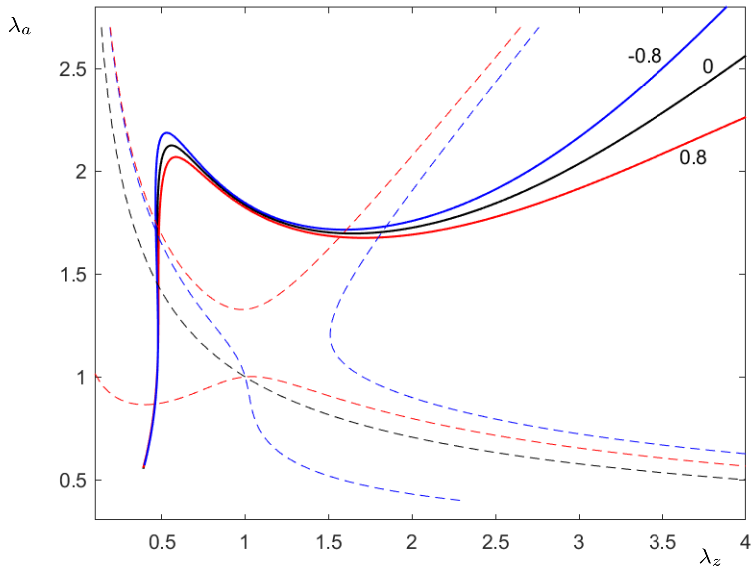

Fig. 4. Plots of the axisymmetric bifurcation curves (continuous curves) for the augmented neo-Hookean elastic material (38) for $A/B = 0.85$, $L/B = 5$, $\hat{\gamma} = -0.8, 0, 0.8$. The values of $\hat{\gamma}$ are shown next to the relevant curves. Dashed lines are zero pressure curves corresponding to residual stress parameter $\hat{\gamma}$ and shown with the same color as respective bifurcation curve.

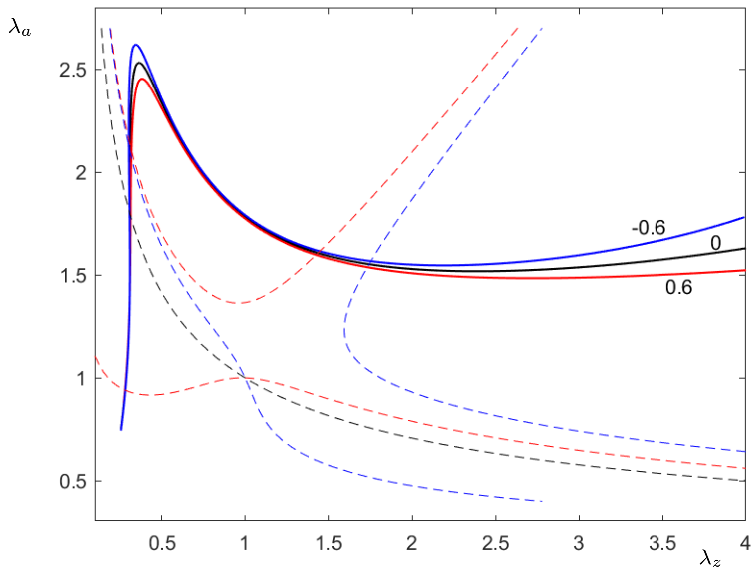

Fig. 5. Plots of the axisymmetric bifurcation curves (continuous curves) for the augmented neo-Hookean elastic material (38) for $A/B = 0.85$, $L/B = 10$, $\hat{\gamma} = -0.6, 0, 0.6$. The values of $\hat{\gamma}$ are shown next to the relevant curves. Dashed lines are zero pressure curves corresponding to residual stress parameter $\hat{\gamma}$ and shown with the same color as respective bifurcation curve.

10. Concluding remarks

In the present paper we obtained new results for axisymmetric and asymmetric bifurcations for the model (38), accounting for residual stress. The obtained results are different from those published in [8] for different models. The present findings suggest that the influence of residual stress, associated with positive and negative residual stress parameter, $\hat{\gamma}$, is different and the resulting bifurcation curves can be viewed as symmetric with respect to the bifurcation curves without residual stress. For asymmetric bifurcations with mode number $m = 1$ inclusion of residual stress with positive $\hat{\gamma}$ makes the tube more stable (this is shown at Fig. 2 and Fig. 3), on the other hand, for axisymmetric bifurcations inclusion of residual stress corresponding to positive residual stress parameter $\hat{\gamma}$ leads to increase in instabilities (see Fig. 4 and Fig. 5). For the negative values of $\hat{\gamma}$ the effect of residual stress is reversed for asymmetric and axisymmetric bifurcations.

Author Contributions

A. Melnikov: formal analysis, validation, writing—original draft, writing—review and editing.; J. Merodio: formal analysis, validation, writing—review and editing.

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Conflict of Interest

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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ORCID iD

Andrey Melnikov https://orcid.org/0000-0001-8639-0000

Jose Merodio https://orcid.org/0000-0001-5602-4659

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