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Stability of a pair of co-rotating vortices

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The linear stability of a pair of co-rotating vortex filaments is studied with a view toward clarifying the behavior of these pairs in the shear layer. The configuration is found to be stable within the long wave approximation.

We study the stability of the motion of a co-rotating pair of parallel vortex filaments. Such pairs seem to arise in the large structure of shear layer flows[1] and have been associated[2] with the growth mechanism of the layer. Bradshaw[3] was the first to observe a "helical" instability in this pairing process, and, since then, some evidence has arisen linking this instability with the initial production of three dimensionality in the flow.[4]

The main problem in using vortex filaments to model actual flows is that, in the limit of zero core radius, any kink in the lines induces on itself an infinite velocity that instantly breaks up the configuration. Crow[5] computed the stability on the contrarotating vortices in the wake of an aircraft and, to avoid the self-induced infinity, introduced a cutoff distance such that the effects of all points closer than that distance are disregarded in computing the induced velocity. Even if this takes care of the infinity, the results depend on the cutoff length. Later, Widnall *et al.*[6] showed that this model arises as the long wave limit of a more exact theory, and gave an expression for the cutoff length as a function of the core diameter and structure. We use this simple model to investigate the stability of the co-rotating pair. The limitations introduced by the long wave approximation are that the vortex core radius be small compared with both the line separation and the wavelength of the instability. These seem to be reasonable approximations for the physical case.

We introduce dimensionless variables by normalizing all lengths with the distance between filaments, L, and the times with $\pi L^2/\Gamma$, where Γ is the circulation around each vortex. In these variables the fundamental motion of the pair consists of a rotation around a common axis, with unit angular velocity in the same sense as the circulation. To study the stability of this motion we perturb the vortices and study their behavior in time. The induced velocities are computed using a linearized version of the Biot-Savart law, which is enough for our purpose, including, of course, an appropriate cutoff length, and we only examine harmonic perturbations, since any other function can be expanded in its Fourier components.

It is convenient to use a different system of coordinates for the perturbations associated with each vortex filament. On each vortex we use a Cartesian, rotating system, with unit vectors $\mathbf{e}_1$ in the radial direction pointing outward, $\mathbf{e}_3$, parallel to the filament in the direction of the angular velocity vector of the pair, and $\mathbf{e}_2$ forming a right-handed system with these two (see Fig. 1). We also define a Lagrangian space coordinate z along the filaments. This coordinate labels the fluid particles in such a way that it is convected with the velocity of the fluid.

With this notation we can write the perturbations as $\mathbf{c}_j e^{ikz}$, with $j=1, 2$ standing for the two filaments. The induced velocities can be taken directly from Crow[5] and for the perturbation equation give

$$\frac{dc_{j1}}{dt}\,\mathbf{e}_1+\frac{dc_{j2}}{dt}\,\mathbf{e}_2=\{-[c_{j1}+M(k)c_{i1}]\mathbf{e}_2+\tfrac{1}{2}\mathbf{e}_3 \times[\mathbf{c}_j+Q(k)\,\mathbf{c}_i]\}-\tfrac{1}{2}\Omega(\mathbf{e}_3\times\mathbf{c}_j)-(\mathbf{e}_3\times\mathbf{c}_j)\,. \qquad (1)$$

In this equation it is assumed that

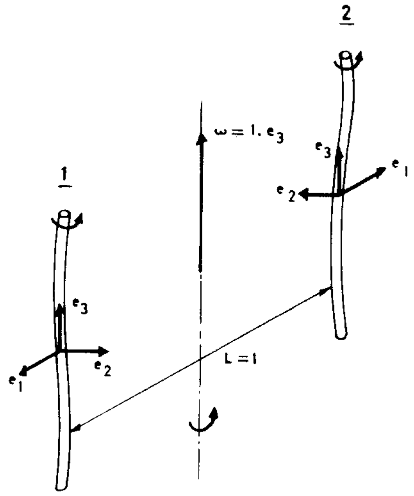

FIG. 1. Dimensionless coordinate systems for the perturbations.

$$\mathbf{c}_j = c_{j1}\,\mathbf{e}_1 + c_{j2}\,\mathbf{e}_2 + c_{j3}\,\mathbf{e}_3, \tag{2}$$

and the coefficients are defined, from Crow, as

$$\begin{aligned} M(k) &= \tfrac{1}{2}k^2 K_2(k),\\ Q(k) &= k K_1(k) + k^2 K_0(k), \\ \Omega &= \tfrac{1}{2}k^2\,(1.059 - \ln k a_v), \end{aligned} \tag{3}$$

with the K's standing for the modified Bessel functions and a_v for the vortex core diameter.

In Eq. (1) the first parenthesis is associated with the induction of one line on the other and the second with the self-induction of the line on itself, while the last one takes into account the fact that the coordinate system rotates with the pair. It should be realized the (1) actually consists of two equations, one for each filament, corresponding to the two values of j. There are also two other equations for the evolution of the longitudinal components c_{j3}, but they decouple from (1) and introduce nothing new to the stability analysis.

To ascertain the behavior of the solutions of (1) it is convenient to consider the symmetric and antisymmetric parts of the perturbations separately. Consider the vectors

$$\mathbf{p} = \tfrac{1}{2}(\mathbf{c}_1 - \mathbf{c}_2),\ \mathbf{q} = \tfrac{1}{2}(\mathbf{c}_1 + \mathbf{c}_2). \tag{4}$$

The first one, p, represents the oscillations of the two lines as a whole, while q describes internal perturbations of the system, in the sense that the mean position of the lines is left unchanged by them. We can get independent equations for p and q by adding and substracting the two equations in (1). Writing the resulting equation for p in component form, we get

$$\begin{aligned} \frac{dp_1}{dt} &= \tfrac{1}{2}(Q+1+\Omega)\,p_2, \\ \frac{dp_2}{dt} &= [M-1-\tfrac{1}{2}(Q+1+\Omega)]\,p_1 . \end{aligned} \tag{5}$$

These constitute a linear system whose characteristic roots are given by

$$\lambda^2 = \tfrac{1}{2}(Q+1+\Omega)[M-1-\tfrac{1}{2}(Q+1+\Omega)]. \tag{6}$$

It is easy to see that the first parenthesis in (6) is always positive, while the second is negative, so that the roots are pure imaginary numbers, and this mode of the motion has neutral stability for all the wavelengths.

Now looking at the symmetric mode, q, we get another linear system of equations whose roots are given by

$$\lambda^2 = \tfrac{1}{2}(Q-1-\Omega)[M+1-\tfrac{1}{2}(Q-1-\Omega)]. \tag{7}$$

The only way for these roots to be real is for both factors to be positive and this gives as a stability criterium that

$$0 < \Omega < Q-1,$$

which is equivalent to

$$a_v > a_0(k) = \frac{2.883}{k}\,\exp\left[-\frac{2}{k^2}\,(Q-1)\right].$$

This boundary, as well as the corresponding values for ka_0 are given in Table I.

TABLE I. Stability limits for symmetric perturbations.

k	a_0	ka_0
0	4.23	0
0.1	4.18	0.42
0.2	4.06	0.81
0.5	3.59	1.79
1	2.75	2.75
2	1.64	3.29
5	0.62	3.10
10	0.29	2.94
∞	0	2.88

It will be remembered that the long wave approximation that we are using in this work implies that both a_v and ka_v should be much smaller than 1, so that the instability region given here is well out of the range of validity of the theory. We can, therefore, conclude that, within the long wave assumptions, the symmetrical mode is also stable.

We see from the previous analysis that the process in which two line vortices rotate around each other is stable to small perturbations, if we do not take into account the effect of other possible singularities external to the pair.

Even if we only analyze the stability to long wavelength perturbations, this seems to be the appropriate limit for the physical situation. We can, therefore, conclude that, to explain the instabilities observed in the pairing of two vortices in a shear layer, we have to take into account the global flow, and not only the local phenomena.

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