A Marker for Studying the Turbulent Energy Cascade in Real Space

J.I. Cardesa and J. Jiménez

Abstract The equation for the kinetic energy based on the residual (sub-filter) velocity is written in several ways in which each term in the equation is Galilean invariant. A different expression for the inter-scale energy transfer term arises from each equation. The statistics of these different terms are studied, together with those of the subgrid-scale (SGS) dissipation. We report on the expression yielding the variance which best matches that of the SGS dissipation, and which is most negatively correlated with it. We argue why the term exhibiting these features is preferred over the others based on its physical meaning. Our study used direct numerical simulation data of homogeneous isotropic turbulence, and our conclusions were observed to be valid with both Gaussian and sharp spectral (low-pass) filters.

1 Introduction and Governing Equations

The subgrid-scale (SGS) dissipation is an example of a real-space quantity which means transfer of kinetic energy between resolved (large) and unresolved (small) scales. We define the large-scale velocity as a spacially low-pass filtered quantity as follows

$$\overline{u}_i\left(x_j\right) = \int u_i\left(x_j - r_j\right) G\left(r_j\right) dr_j, \tag{1}$$

so that the total velocity is decomposed into large- and small-scale components as

$$u_i = \overline{u}_i + u'_i. \tag{2}$$

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J. Peinke et al. (eds.), *Progress in Turbulence VI*,
Springer Proceedings in Physics 165, DOI 10.1007/978-3-319-29130-7_4

The evolution of the kinetic energy in the filtered velocity field is then given by

$$\left(\frac{\partial}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}\right) \frac{1}{2}\overline{u}_i\overline{u}_i = -\frac{\partial}{\partial x_j}\left(\overline{u}_j\overline{p} + \overline{u}_i\tau_{ij} - 2\nu\overline{u}_i\overline{S}_{ij}\right) - 2\nu\overline{S}_{ij}\overline{S}_{ij} + \overline{S}_{ij}\tau_{ij}, \quad (3)$$

where τ_{ij} is the SGS stress tensor $\overline{u_iu_j} - \overline{u}_i\overline{u}_j$ and $\overline{S}_{ij}\tau_{ij}$ is the SGS dissipation. Its statistics have been studied extensively [1, 3]. Here, we focus on an alternative equation, which is that of the kinetic energy contained in the residual scales. The reason behind this is that some terms within Eq. (3) are not, by themselves, Galilean invariant. Another reason is that $\overline{S}_{ij}\tau_{ij}$ contains the triple product $\overline{u}_i\overline{u}_j\overline{S}_{ij}$ which features interactions between large scales only. We are looking for an inter-scale transfer term arising uniquely from interactions between filtered and residual terms. Among the ways of writing the evolution equation for the kinetic energy of the residual (small-scale) velocities, we report on the following four:

$$\left(\partial_t + \overline{u}_j\partial_j\right)\frac{1}{2}u'_iu'_i = -\partial_j\left(u'_jp' - 2\nu u'_iS'_{ij}\right) - 2\nu S'_{ij}S'_{ij} \underbrace{+ u'_i\partial_j\left(\tau_{ij}\right) - u'_iu'_j\overline{S}_{ij} - u'_iu'_jS'_{ij}}_{T_A} \quad (4)$$

$$\left(\partial_t + \overline{u}_j\partial_j\right)\frac{1}{2}u'_iu'_i = -\partial_j\left(u'_jp' - 2\nu u'_iS'_{ij} - u'_i\tau_{ij}\right) - 2\nu S'_{ij}S'_{ij} \underbrace{- S'_{ij}\tau_{ij} - u'_iu'_j\overline{S}_{ij} - u'_iu'_jS'_{ij}}_{T_B} \quad (5)$$

$$\left(\partial_t + \overline{u}_j\partial_j\right)\frac{1}{2}u'_iu'_i = -\partial_j\left(u'_jp' - 2\nu u'_iS'_{ij} + \tfrac{1}{2}u'_iu'_iu'_j\right) - 2\nu S'_{ij}S'_{ij} \underbrace{+ u'_i\partial_j\left(\tau_{ij}\right) - u'_iu'_j\overline{S}_{ij}}_{T_C} \quad (6)$$

$$\left(\partial_t + \overline{u}_j\partial_j\right)\frac{1}{2}u'_iu'_i = -\partial_j\left(u'_jp' - 2\nu u'_iS'_{ij} - u'_i\tau_{ij} + \tfrac{1}{2}u'_iu'_iu'_j\right) - 2\nu S'_{ij}S'_{ij} \underbrace{- S'_{ij}\tau_{ij} - u'_iu'_j\overline{S}_{ij}}_{T_D}. \quad (7)$$

There are additional ways of writing the right-hand side, but they lead to terms which are not, by themselves, Galilean invariant. These four equations give rise to four different inter-scale energy transfer terms outside of the divergence, which we label $T_{A\to D}$. They should be the small-scale counterparts of $\tau_{ij}\overline{S}_{ij}$. Our task will be to compare some simple statistics of these four resulting terms, with the aim of choosing the most suitable one for our needs.

2 Results and Conclusions

We computed the inter-scale transfer terms $T_{A\to D}$ from the direct numerical simulation of forced homogeneous isotropic turbulence found in the databases of the Johns Hopkins University [2]. Three 1024^3 velocity fields were used, separated by half a large-eddy turnover time between them. The Re_λ reported is 433. Entire fields of the three velocity components were downloaded and Fourier transformed, in order to filter and differentiate in wave-number space—thus taking full advantage of the simulation's spectral accuracy. We used a Gaussian and a sharp spectral filter. The filter widths were $[20, 28, 40, 57, 80, 113, 160, 226, 320, 453, 640, 905]\eta/\pi$, with η being the Kolmogorov length scale.

The dependence of the mean and standard deviation of terms $T_{A\to D}$ as well as that of $-\tau_{ij}\overline{S}_{ij}$ on the filter width is shown in Fig. 1a, c for the Gaussian filter, and in Fig. 1b, d for the sharp spectral filter. The mean values of $T_{A\to D}$ for a given filter type are equal to each other. This is consistent with the fact that they all differ by

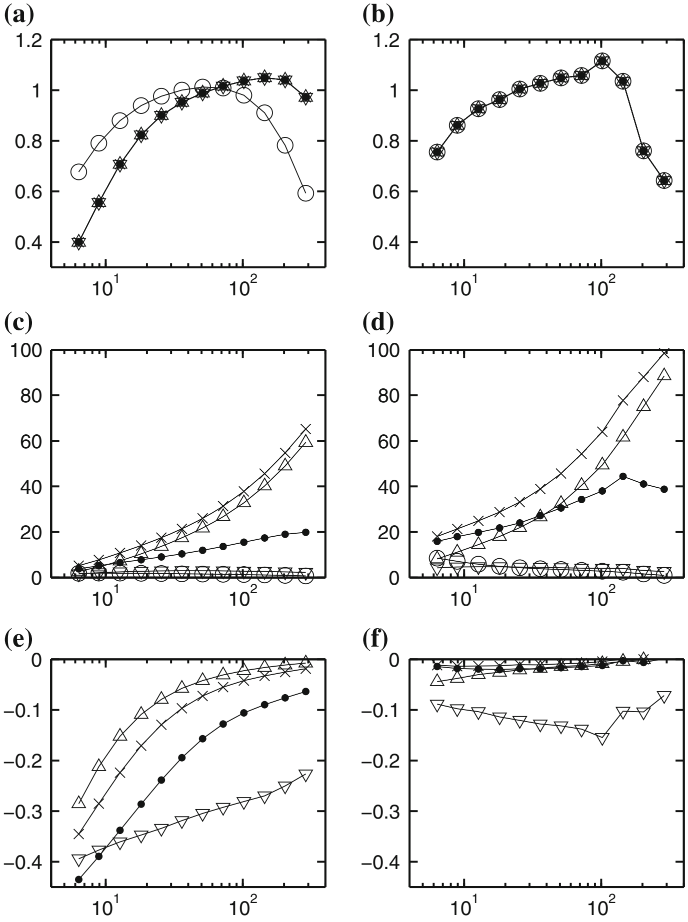

Fig. 1 Statistics of the transfer terms $T_{A\to D}$ obtained with a Gaussian filter (*left*) and a sharp spectral filter (*right*), as a function of filter width Δ. The horizontal axis is Δ/η on all figures. Symbol legend: $\triangle$ T_A, $\times$ T_B, $\triangledown$ T_C, $\bullet$ T_D, $\bigcirc$ $-\overline{S}_{ij}\tau_{ij}$. **a**, **b** Mean of the transfer terms normalised by the mean total viscous dissipation. **c**, **d** Standard deviation of the transfer terms normalised by the mean total viscous dissipation. **e**, **f** Spatial correlation between $\tau_{ij}\overline{S}_{ij}$ and $T_{A\to D}$

a divergence term which vanishes on average in homogeneous flows. However, the mean that the terms converge to changes depending on the filter type. For both filter types, it remains of the same order as the mean total viscous dissipation. A significant effect of the filter type is the fact that $\langle -\tau_{ij}\overline{S}_{ij}\rangle$ is different from the mean of terms $T_{A\rightarrow D}$ for the Gaussian filter. Yet these are all equal for the sharp spectral filter, which is a *projection* contrary to the Gaussian filter. From Fig. 1c, d we conclude that the standard deviation of T_C is the most similar to that of $\tau_{ij}\overline{S}_{ij}$, whereas those of T_A, T_B and T_D are larger. This lead us to investigate the spatial correlation between quantities $T_{A\rightarrow D}$ with $\tau_{ij}\overline{S}_{ij}$ found with the same filter width and type. It seemed reasonable to expect, at least on average, that there exists a negative correlation between $\tau_{ij}\overline{S}_{ij}$ and the quantities $T_{A\rightarrow D}$. Such a negative correlation would imply that when $\tau_{ij}\overline{S}_{ij}$ acts as a sink in the equation for the kinetic energy of the large-scale velocity field, terms $T_{A\rightarrow D}$ behave as sources in the kinetic energy of the small-scale velocity field—and vice-versa. These correlations are shown on Fig. 1e, f. It is clear from them that the most negative correlation is given by T_C, indicating that $\tau_{ij}\overline{S}_{ij}$ and T_C behave more like "communicating vessels" than the combination of $\tau_{ij}\overline{S}_{ij}$ with T_A, T_B or T_D.

From the results outlined so far, we are left with the conclusion that T_C features the best-behaved statistics in the sense that they are the most consistent with the observed statistics of $\tau_{ij}\overline{S}_{ij}$. Our conclusion so far is based solely on the data. We now try to get to the same conclusion by looking back at the equations where terms $T_{A\rightarrow D}$ appear. T_A and T_B are found to be the same as T_C and T_D with the additional $u'_i u'_j S'_{ij}$ product. This product can be written entirely as part of the divergence—using continuity, and so it contributes nothing to the mean but affects only the standard deviation. As seen on Fig. 1c, d, $u'_i u'_j S'_{ij}$ increases the standard deviation. Furthermore, this triple product is a coupling of small-scale quantities interacting with each other, and hence cannot convey the meaning of an energy transfer resulting from large- and small-scale interaction. So T_A and T_B could be discarded based on that criterion alone. In order to chose the best expression—for our purpose—between T_C and T_D, we see that the equation with T_D includes the term $u'_i\tau_{ij}$ inside the divergence. The latter term contains a coupling between filtered (large) and residual (small) quantities which is outside of T_D. Such coupling terms within the equation with T_C have been entirely confined to the inter-scale transfer term, so that all large- and small-scale interactions lie inside T_C. To summarise, all the products coupling large and small scales in Eq. (6) are inside T_C, while no additional term within T_C is the result of small-scale quantities interacting with each other. It is the only expression out of Eqs. (4)→(7) which achieves this, and for this reason we chose it as our preferred way of writing the evolution equation of the small-scale kinetic energy.

Acknowledgments This work was supported by the European Research Council Multiflow grant ERC-2010.AdG-20100224.

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