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**Study of the Aeroelastic Response of two-degrees-of-
freedom Airfoils to Atmospheric Turbulence**

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STUDY OF THE AEROELASTIC RESPONSE
OF TWO-DEGREES-OF-FREEDOM
AIRFOILS TO ATMOSPHERIC TURBULENCE

Bachelor's Final Thesis

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Abstract

Abstract

The present work studies the aeroelastic response of a two degrees-of-freedom model for an airfoil, due to the effect of atmospheric turbulence. The study of the aeroelastic response to turbulence is of great importance, in order to ensure the structural integrity of wings. This project covers the obtention of the standard deviations of the degrees-of-freedom of the airfoil and all intermediate results due to atmospheric turbulence, aiming to determine to which extent this phenomena can damage the structure. Moreover, the obtention of these results are secured using computational tools. Therein, the results of the present study show that the coupling of the modes induces an attenuated response of the airfoil.

Resumen

El presente trabajo estudia la respuesta aeroelástica de un modelo de dos grados de libertad para un perfil aerodinámico, debido al efecto de la turbulencia atmosférica. El estudio de la respuesta aeroelástica a la turbulencia es de gran importancia, para asegurar la integridad estructural de las alas. Este proyecto abarca la obtención de las desviaciones estándar de los grados de libertad del perfil aerodinámico y todos los resultados intermedios debidos a la turbulencia atmosférica, con el objetivo de determinar en qué medida este fenómeno puede dañar la estructura. Además, la obtención de estos resultados se consigue utilizando herramientas computacionales. Así, los resultados del presente estudio muestran que el acoplamiento de los modos induce una respuesta atenuada del perfil aerodinámico.

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Chapter 1

Introduction

Turbulent [adjective] [1]:

1. in which there is a lot of sudden change, trouble, argument and sometimes violence;
2. (of air or water) changing direction suddenly and violently;
3. (of people) noisy and/or difficult to control.

The word turbulent has multiple definitions, whereas some of them refer to different environments or contexts, all of them aim to depict the same concept: disruption, sudden change and unexpected consequences.

Steadiness is one of the characteristics that is mostly pursued by humankind. Buildings, bridges, vehicles and other constructions developed by humankind seek to stand the test of time. Nevertheless, it is clear that the unsteadiness present in nature is fiercely opposed to this intention. Natural phenomena, of all kinds, fight an incessant battle against our tendency to steadiness. The reader may think of great disasters such as those associated with earthquakes, tsunamis, hurricanes... Whereas it is obvious that these phenomena have had its toll on numerous products of engineering, the unsteadiness of nature is aligned with the dynamism found in the everyday life. Any minimal disturbance of a system entails an unsteady motion.

1.1 Aeroelastic response to atmospheric turbulence

Aircrafts, which constitute one of the most complex transportation methods developed by humankind, are also subjected to unsteady and unpredictable motions: atmospheric turbulence.

1.1.1 Evolution of the study of turbulent fluids

The concept of turbulent fluids was introduced in the 16th century by Leonardo da Vinci, referring to a flow condition where the fluid becomes non-linearly unstable and generates vortices. The different works of Euler, D'Alembert, Barré, Stokes or Navier led to the obtention of the Navier-Stokes Equations, which fully describe the evolution of the flow for both laminar and turbulent regimes [2].

It was Reynolds' decomposal of the velocity field into a mean flow and a fluctuating flow that powered the study of turbulence as it is nowadays understood [3]. Moreover, the averaging method introduced by Reynolds marked a turning point for the study of turbulent flows. Whereas it was an advance in the study of turbulence, it introduced

a problem: the turbulence closure. This occurred as a result of the application of the averaging method to the Navier-Stokes equations, where there are more unknown variables than equations [4]. In addition, Boussinesq introduced the concept of eddy-viscosity, which establishes that turbulent fluctuations are dissipative on the mean flow.

A study of turbulent diffusion led Taylor to introducing the idea of a correlation function in 1921. During 1935-1936, in a series of publications, Taylor laid down the foundation of the statistical theory of turbulence [5].

In the 1920s, Prandtl and von Karman introduced semi-empirical turbulence theories, with the most notable being the mixing length theory, which proved to be particularly effective [5].

According to Kolmogorov's theory established in the 1940s, turbulent motions encompass a broad spectrum of scales, extending from a larger macroscale where energy is injected to a smaller microscale where energy is dissipated due to viscosity. In accordance with this theory, he was able to develop a model for turbulence that established a precedent for the actual formulations used to model turbulence [5], which will be explained in Chapter 2.

1.1.2 Aeroelasticity

Meanwhile unsteadiness is constantly threatening the prevalence of structures, the flexibility of these structures retains their failure. All structures are flexible, i.e. the capacity to deform when subjected to loads, up to a greater or lesser extent. Flexibility is especially relevant for wings, as it confers them with the capacity of enduring the aerodynamic loads which appear during flight. The study of the interaction of the aircraft's structure and the aerodynamic loads is called Aeroelasticity [6].

Aircrafts can be subjected to unsteady aerodynamics during any phase of the flight. These pose an exceptional risk for aircrafts which can concur in structural damage. Tragically, throughout history several fatal accidents have led to the study and discovery of phenomena derived from unsteady aerodynamics. One of these phenomena is atmospheric turbulence. Adverse meteorological conditions is the main cause for incidents in aviation [7]. For example, on November 2001, a American Airlines flight from New York to Santo Domingo, shortly after take-off encountered turbulence caused by the wake of a preceding Boeing 747 aircraft. The turbulence was more severe than anticipated and resulted in the aircraft experiencing excessive yawing and rolling motions. The aircraft's tail eventually separated due to the extreme aerodynamic forces it experienced. This structural failure led to a loss of control, and the aircraft crashed [7]. Accidents of this kind have fueled the study of the loads induced by turbulence, with the aim of preventing structural damaging of the aircraft.

From an engineering standpoint, atmospheric turbulence is of great importance, as it allows the enhancement of the structural strength and fatigue, controllability and passenger and crew safety [2].

1.2 Objectives of the work

Whereas the study of turbulence itself is of great interest, one of the most important things in aeronautical engineering is the impact that turbulence can have on the structure of the aircraft. The analysis, from an aeroelastic point of view, of the response of the two degrees-of-freedom airfoil subjected to atmospheric turbulence, requires the completion of the following objectives:

- The obtention of the incremental pressure coefficient caused by a harmonic gust, to which an airfoil is subjected when encountered with the atmospheric turbulence.
- The obtention of the aerodynamic forces to which an airfoil is subjected when encountered with turbulence and their integration into a two-dimensional model for the airfoil.
- The obtention of the transfer matrix which relates the intensity of the atmospheric turbulence and the response of the system.
- The obtention of the standard deviations of the degrees-of-freedom of the response.
- The study the variation of the response due to the coupling of the degrees-of-freedom.
- The study of the fluctuations of the responses both for the coupled and decoupled systems when mass-related parameters vary is of great concern for the present study.
- The comparison of the response to atmospheric turbulence between lighter and heavier aircrafts.

The completion of these objectives will result in an in-depth understanding of the effects of the coupling of the modes in a two degrees-of-freedom model for an airfoil, when subjected to atmospheric turbulence. Most of the existing studies on the response to atmospheric turbulence consider only one degree-of-freedom for the airfoil. Therein, the present work contributes to the comprehension of the phenomena by providing an insight into the coupling influence on the response.

1.3 Outline of the project

In this section an outline of the main points will be presented. This work is divided in seven Chapters. These include:

- Chapter 2 explains the main concepts regarding atmospheric turbulence, the characterization and the main hypotheses of atmospheric turbulence. Moreover, it covers the treatment of random signals and the main atmospheric turbulence models.
- In Chapter 3 the structural model for the two degrees-of-freedom airfoil is presented, explaining the kinetic, elastic and dissipative energies associated to the airfoil. Finally, it includes the assembly of the equations of motion.
- In Chapter 4, the aerodynamic model for the airfoil when it is encountered with atmospheric turbulence is presented. Firstly, the hypotheses for the aerodynamic model are presented. In addition, this Chapter covers the obtention of the unsteady loads associated to the airfoil when subjected to atmospheric turbulence. Finally, the aerodynamic model assembly is outlined.
- In Chapter 5, the complete equations of motion are obtained. This Chapter presents the procedure followed to obtain the response of the airfoil both in the frequency and the time domain. Moreover, the decoupled model and the rigid body motion of the airfoil are explained.

- Chapter 6 presents the test cases on which the turbulence response will be studied. In addition, Chapter 6 presents all the results obtained for the present project, which include: the incremental pressure coefficient produced by a harmonic motion, $\Delta\tilde{C}_{p1}^G(\hat{x}, k)$; the dimensionless aerodynamic forces induced by a harmonic motion, $\tilde{Q}_h(k)$ and $\tilde{Q}_\alpha(k)$; the transfer acceleration functions for the bending and torsional motions, $|H_h|^2$ and $|H_\alpha|^2$; the Power Spectral Densities of the response, PSD_h and PSD_α ; and the standard deviations of both of the bending and torsional degrees-of-freedom.
- Chapter 7 explains the conclusions obtained for the present work. In addition, the future lines of work for the present project are presented.

Chapter 2

Atmospheric turbulence

2.1 Introduction to atmospheric turbulence

Throughout history a major concern, regarding the design of aircrafts, has been maintaining its structural integrity, through all phases of the flight. There are multiple types of loads that can damage severely the aircraft's structure. A well known phenomena that has hindered aircrafts since the beginning is atmospheric turbulence.

Atmospheric turbulence is the irregular random motion of the air caused by fluctuations in pressure, speed and temperature of the fluid. This phenomenon has an important impact on the aircraft on a structural level. Its severity depends on multiple parameters. Among these are those associated with the aircraft, such as: flight speed, the aircraft's mass range, its structural strength, etc. However, it is necessary to consider that atmospheric turbulence, as a consequence of its random nature, has a wide spectra of varieties when tracing back to its cause and severity. In this Chapter, the main types of turbulence, the hypotheses associated with it and the main existing atmospheric turbulence models will be presented.

According to the physical mechanisms causing the motion of air masses, the main types of atmospheric turbulence are [8]:

1. **Mechanical Turbulence:** It is caused by friction between the air and the ground, especially irregular terrain and man-made obstacles, cause eddies and therefore turbulence in the lower levels. The intensity of this eddy motion depends on the strength of the surface wind, the nature of the surface and the stability of the air.
2. **Thermal (Convective) Turbulence:** It is caused by an uneven heating of the earth's surface due to the terrains' (rocks, ground, sea, etc.) different diffusivity, which is the ability of a substance to transmit energy, normally thermal. Isolated convective currents are therefore set in motion with warm air rising and cooler air descending, which are responsible for bumpy conditions as an airplane flies in and out of them. Turbulence extends from the base to the top of the convection layer, with smooth conditions found above. In the presence of clouds turbulence extends from the surface to cloud tops.
3. **Frontal Turbulence:** The lifting of the warm air by the sloping frontal surface and friction between the two opposing air masses produce turbulence in the frontal zone. This turbulence is most marked when the warm air is moist and unstable and will be extremely severe if thunderstorms develop. Turbulence is more commonly associated with cold fronts but can be present, to a lesser degree, in a warm front as well.

4. **Wind Shear Turbulence:** Wind shear is the change in wind direction and/or wind speed over a specific horizontal or vertical distance. Atmospheric conditions where wind shear exists include: areas of temperature inversions (it is normally found at a flight level surrounded by a warm air mass upward and a cold air mass downward), along troughs and lows (variation of the ground level i.e, valleys and mountains), and around jet stream (clear air turbulence, normally at high altitudes). When the change in wind speed and direction is pronounced, quite severe turbulence can be expected.

Moreover, the motion of air masses is defined by several parameters: speed, length or scale of the turbulence, differences in pressure, etc. These parameters have a direct impact on the response of the aircraft: acceleration, vertical displacement, uneasiness felt by passengers, etc. In order to distinguish the intensity of the turbulence and, consequently, its severity, a qualitative classification can be found [9]:

1. **Light Turbulence:** It momentarily causes slight changes in altitude and/or attitude or a slight bumpiness. Occupants of the airplane may feel a slight strain against their seat belts.
2. **Moderate Turbulence:** It is similar to light turbulence but somewhat more intense. There is, however, no loss of control of the airplane. Occupants will feel a definite strain against their seat belts and unsecured objects will be dislodged.
3. **Severe Turbulence:** It causes large and abrupt changes in altitude and/or attitude and, usually, large variations in indicated airspeed. The airplane may momentarily be out of control. Occupants of the airplane will be forced violently against their seat belts.
4. **Extreme Turbulence:** It results in the aircraft being tossed violently about and impossible to control. It may cause structural damage.

Although a qualitative approach is necessary for a preliminary classification, the concrete loads, acceleration and displacements caused by turbulence shall be calculated. The determination of these variables allows the design of aircrafts capable of enduring the aerodynamic and structural forces and moments to which they will be subjected, if encountered with turbulence.

2.2 Characterization of atmospheric turbulence

In order to determine the aeroelastic response of aircrafts to atmospheric turbulence, this phenomena shall be modeled. As previously explained, turbulence is a random motion which cannot be completely determined through space and time. If airspeed, pressure and temperature tests were to be carried out multiple times, in the exact same conditions, different values would be obtained. Therefore, it can be considered as a stochastic motion [10], which cannot be expressed through an analytical expression.

Over time turbulence has been studied following different approaches, but mainly two theories have proved to obtain significant results: empirical and statistical theories.

- Empirical theories are focused on the distribution of the mean speed and mean pressure, making assumptions in order to satisfy the equations of motion of the mean flow. These assumptions involve one or more empirical constants. Normally, the type of assumption is selected relying in the hypotheses regarding the character

of the fluctuations of speed and pressure. Nonetheless, the final theory rests on the assumption (and its associated constants), not on the hypotheses.

- In the statistical theories, frequency distribution and mean values of the pressure and of the components of the speed fluctuations play a principal role. More concretely, the relation between the statistical properties of the fluctuations and the mean motion are studied.

Depending on the nature of the study at stake, one of these two approaches to atmospheric turbulence must be used. For the selection of one of these two models, a detailed analysis of the model should first be carried out in order to establish which hypotheses are acceptable. It is important to select the Atmospheric Turbulence model so that the assumptions of the turbulence model are in agreement with those of the research.

2.2.1 Main hypotheses

For the purpose of turbulence modelling, it is vital to understand what lies behind this phenomena. Several studies and tests [8] on atmospheric turbulence have allowed the characterization of turbulence. These have concluded that the following hypothesis are applicable to atmospheric turbulence, simplifying its study:

1. **Stationarity and homogeneity:** These characteristics associated to a random process specify an invariance in its statistical characteristics, with respect to time and space. In the case of turbulence, this means that its statistical values are to remain constant with respect to the positioning in the turbulence, independently of the moment in time when the sampling interval is taken. Even though, the intensity of the turbulence depends on the meteorological conditions, which are known to evolve, this is not relevant for the purpose of the study. The results of different measurements of turbulence, laid out in Houbolt's [8] work on turbulence, show that it is acceptable to assume the stationarity and homogeneity of atmospheric turbulence.
2. **Isotropic:** Isotropy, when referred to a phenomena, is defined as the lack of variation in a magnitude according to the direction of measurement. The isotropy of atmospheric turbulence means that there is no mean rate of transfer of momentum across shearing surfaces [8], [11]. This translates to the lack of cross turbulent velocity between longitudinal, lateral and vertical turbulent velocities.
3. **Gaussian:** A random signal is considered to be Gaussian when its fluctuations have a normal probability distribution. Several references [8], [12], [8] on turbulence illustrate its Gaussian nature. Furthermore, Houbolt, in his study [8], exposes the cumulative frequency distribution, which aligns perfectly with a normal distribution, as a result it is reasonable to assume that turbulence is Gaussian. Hence, the system's response will be treated as Gaussian as well.
4. **Harmonic Motion:** Due to the nature of atmospheric turbulence (it fluctuates similarly upwards and downwards) its mean value is considered to be zero. As a consequence, it is reasonable (and it has been proven) to assume harmonic motion and thus it will be implemented, for the purposes of this study.

Moreover, in this project, turbulence will only be considered to induce vertical speed along the vertical axis, in the aircraft's body axis system, in the z_b axis [12], in accordance

to hypothesis No. 2. The one-dimensional assumption constraints the instantaneous vertical or lateral gust velocities to be the same at all points in planes normal to the direction of aeroplane travel, meaning that the turbulence will only be studied, when acting in a concrete plane.

2.3 Statistical treatment of random signals

Considering $x(t)$ as a random excitation. The n^{th} order mean of $x(t)$ is defined as the integral shown in Eq. (2.1):

$$\langle x \rangle^n = \lim_{x \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)^n dt, \quad (2.1)$$

where T is the period and the interval $2T$ is wide enough.

As previously explained, the turbulence's first order mean is null. Therefore, the statistical parameter that will provide further information regarding the signal will be second order mean shown in Eq. (2.2):

$$\langle x \rangle^2 = \lim_{x \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)^2 dt. \quad (2.2)$$

The parameter shown in Eq. (2.2) is the squared-standard-deviation, which represents an estimate of the energy contained in the signal.

Another similar parameter that will be used in the analysis will be the Power Spectral Density (PSD), which is the measure of signal's power content in the frequency domain.

2.3.1 Power Spectral Density

The Power Spectral Density (PSD) or power spectrum enables the representation of the distribution of the signal's frequency components, which allows a faster understanding and interpretation than the complex Discrete Fourier Transform (DFT). It represents the portion of the total signal power contributed by each frequency component of a signal.

The PSD is extracted directly from the DFT, as the mean squared amplitude of each frequency component, averaged over the n samples in the record. The PSD varies with frequency. Thus, each element is a measure of the signal power contributed by each frequency interval. On the other hand, information such as the signal's offset is lost when using the PSD instead of the DFT.

The use of the PSD in the present study is due to its statistical application. The PSD allows the calculation of the squared standard deviation, as shown in Eq. (2.3):

$$\langle x \rangle^2 = \sigma_x^2 = \int_{-\infty}^{\infty} PSD(\omega) d\omega, \quad (2.3)$$

where ω is the frequency of the motion.

2.4 Atmospheric turbulence models

For the purpose of determining the systems' response to atmospheric turbulence, models approximating the signal caused by turbulence have been developed over time. Although all of them have contributed to the deeper understanding of turbulence, only those which

consider the hypotheses laid, earlier in this chapter, will be presented. Those include the Dryden and the Von Kármán models.

For what this study entails, the Power Spectral Density of the turbulence is the most relevant parameter. Shown below are both the Dryden and the Von Kármán turbulence intensity models for turbulence,

- Von Karman: $PSD(\frac{\omega L}{U_\infty}) = \sigma_w^2 \frac{L}{\pi U_\infty} \frac{1 + \frac{8}{3} \left(1.339 \frac{\omega L}{U_\infty}\right)^2}{\left[1 + \left(1.339 \frac{\omega L}{U_\infty}\right)^2\right]^{11/6}},$
- Dryden: $PSD(\frac{\omega L}{U_\infty}) = \sigma_w^2 \frac{L}{\pi U_\infty} \frac{1 + 3 \left(\frac{\omega L}{U_\infty}\right)^2}{\left[1 + \left(\frac{\omega L}{U_\infty}\right)^2\right]^2},$

where σ_w is the standard deviation of the vertical component of turbulent fluctuation speed, L is the turbulence length scale and U_∞ is the flight speed.

These models, even though based on the hypothesis previously mentioned, both have empirically derived coefficients to match flight test data. Even though von Kármán's model is more accurate and precise in the high frequencies range, for practical implementations, Dryden's model is often used due to the simplicity in reconstructing a time series from the PSD. Thus, the Dryden model will be used in this study.

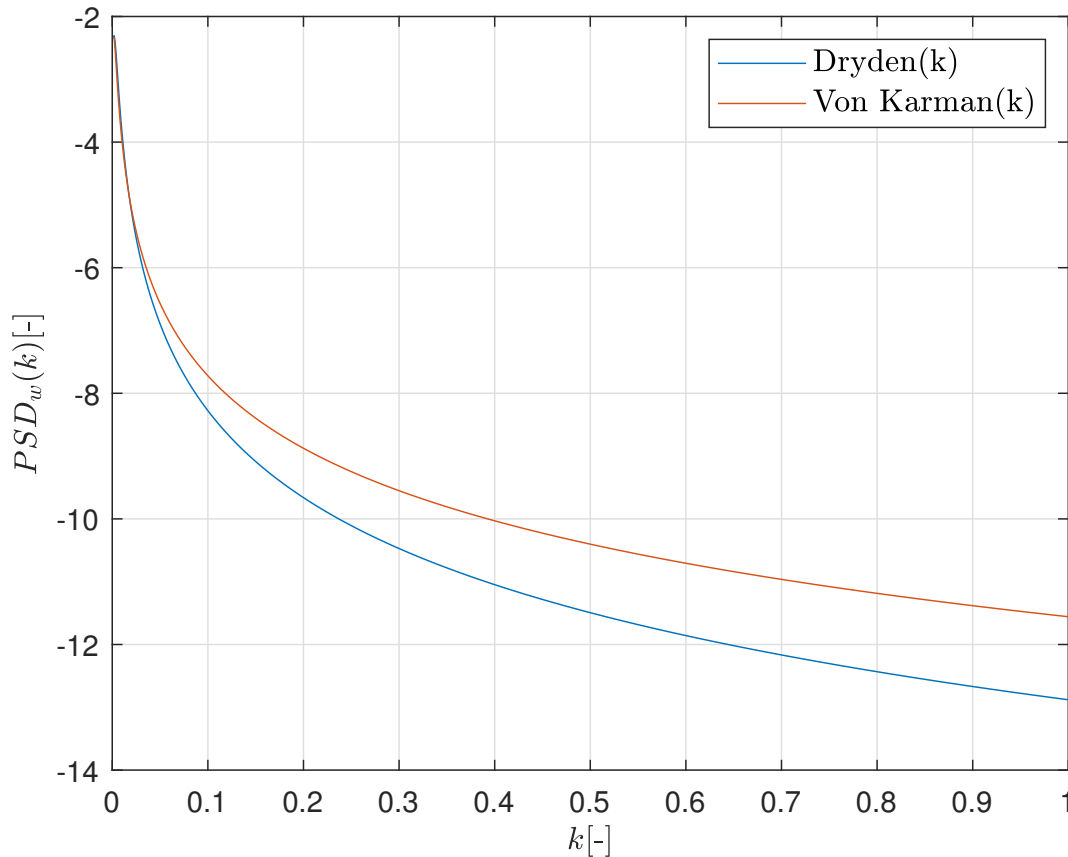


Figure 2.1: PSD_{wG} vs k

Figure 2.1 shows the logarithmic representation of both turbulence models, Dryden and Von Kármán, evaluated for the frequency range subjected to study in the present work. It can be seen that for low reduced frequencies both models match. Nonetheless, when

the reduced frequency increases it is seen that the Dryden model decreases more rapidly, in comparison to the Von Kármán model.

For the design of aircrafts, both EASA (European Agency for Aviation Safety) and FAA (Federal Aviation Administration) have adopted the von Kármán model because it includes an additional term that accounts for turbulence anisotropy [13]. Nonetheless, typically researchers employ the Dryden model in their projects due to the simplicity of its implementation.

Chapter 3

Structural model

An aeroelastic approach to the atmospheric turbulence problem implies an in depth analysis of the structural behaviour of the airfoil, which is considered to be an elastic body [14]. It is particularly important to consider the interaction between the deformation of the body and the forces exerted by the fluid on the body.

Whereas most physical systems are continuous, they can be discretized. This refers to the approximation of a system where its inertial proprieties are considered to be focalized in one or more exact coordinates. This technique allows the simplification of a complex continuous system with few or no analytical solutions, by a multiple degrees-of-freedom (the independent coordinates that allow the determination of the position of a body) system, with the aim of obtaining a precise and easier-to-obtain solution.

For example, as shown in Figure 3.1 an aircraft can be approximated by a multiple-degrees-of-freedom system, where further precision can be obtained by increasing the number of degrees-of-freedom.

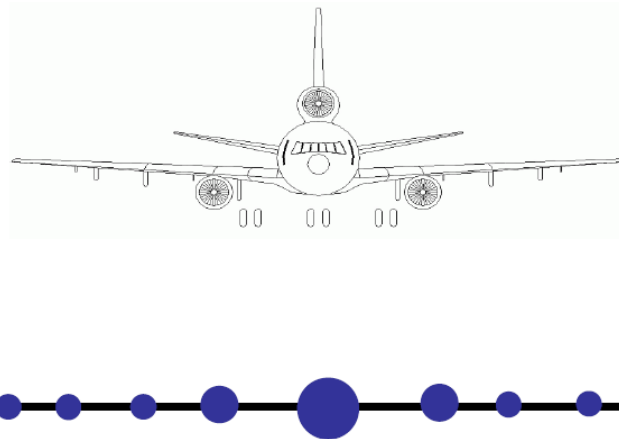


Figure 3.1: Aircraft approximated by a multiple degrees-of-freedom system

This type of approximation to a system is often used when carrying out a dynamic study. The role of mechanical vibration analysis is to use mathematical tools for modeling and predicting potential dynamic problems and solutions, which are usually not obvious in preliminary engineering problems. This project will be addressed through a vibrational approach. Thus, the response to turbulence of the pertinent airfoil shall be calculated using a static reference position. This chapter focuses on the description of the structural model (equations of motion and main characteristics) for the airfoil.

3.1 Airfoil parameters

As any mechanical system, the equations of motion can be obtained from energy considerations, through the Equations of Lagrange, shown in Eq. (3.1). In order to proceed with Lagrange's method, it is necessary to determine the generalized coordinates (the independent coordinates which allow the complete description of the system). The number of generalized coordinates is equal to the number of degrees-of-freedom.

The airfoil subjected to study is modeled as a 2-degrees-of-freedom system, as shown in Figure 3.2. The airfoil will have the typical properties of the 3/4 section of the selected wing, assuming a unitary wingspan [14].

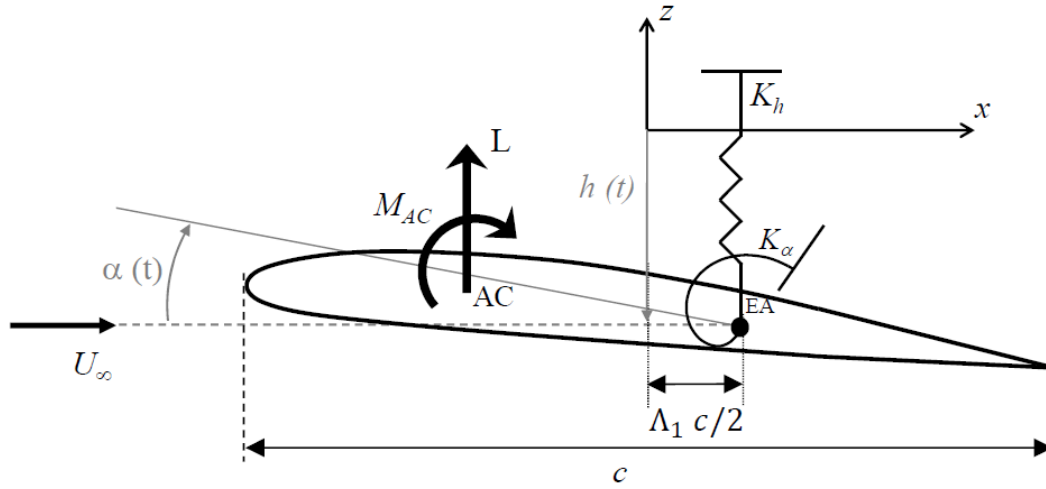


Figure 3.2: Aeroelastic model of the two-degrees-of-freedom airfoil

The degrees-of-freedom are the bending $h(t)$ and torsion $\alpha(t)$ modes, where these represent incremental variations relative to a static equilibrium position [14]. For the purposes of this study, the reference position shall be a leveled stationary flight condition (1g). Associated to both the bending and torsional modes are their inertial parameters shown in Table 3.1 along with other parameters inherent to the airfoil.

Table 3.1: Airfoil parameters definition

Parameter	Definition
Elastic Axis (EA)	The locus of flexural centers.
Flexural center	The point in a cross section, at which a shear force can be applied without producing a rotation of that section in its own plane.
c	Chord of the 3/4 airfoil of the wing. [m]
b	Wingspan of the aircraft. [m]
K_h and K_α	Equivalent structural stiffness of the 3/4 airfoil of the wing, for bending and torsion, respectively. [N/m] and [N/rad]
$\Lambda_1 \frac{c}{2}$	Position of the Elastic Axis (distance from geometrical center of the airfoil). [m]
ω_h	Bending natural frequency of the system. [rad/s]
ω_α	Torsion natural frequency of the system. [rad/s]
g_h	Dimensionless structural damping coefficient due to bending. [-]
g_α	Dimensionless structural damping coefficient due to torsion. [-]
M	Mass per wingspan unit of the airfoil [kg/m].
I_α	Inertial momentum per wingspan unit of the airfoil, in relation to the elastic axis [kg m].
S_α	Static inertia momentum per wingspan unit of the airfoil, in relation to the elastic axis [kg].

The Lagrange Equations are presented in Eq.(3.1), where T represents the kinetic energy, U the elastic energy and D the dissipation energy. Furthermore, $\tilde{p}_j$ represents the non-conservative loads on the model and q_j and $\dot{q}_j$ represent the degrees-of-freedom and its derivative, respectively:

$$\frac{\partial}{\partial t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + \frac{\partial U}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = \tilde{p}_j. \quad (3.1)$$

The provenance of these energies will be explained in the following subsections.

3.2 Kinetic energy

Kinetic energy is a property of a moving object or particle and depends not only on its motion but also on its mass. The kind of motion may be translation (or motion along a path from one place to another), rotation about an axis, vibration, or any combination of motions. The total kinetic energy of a body or a system is equal to the sum of the kinetic energies resulting from each type of motion.

The kinetic energy associated to a mass differential element of the airfoil (dm) is shown in Eq. (3.2). In this case, dm is the mass per unit chord and unit wingspan, where the number 1 represents the extrusion of such element along the unit wingspan [14].

$$dT = \frac{1}{2}(dm) \left[\dot{h} + \left(x - \Lambda_1 \frac{c}{2} \right) \dot{\alpha} \right]^2, \quad (3.2)$$

where x is the position in the chord where the energy is being evaluated. The integration along the chord yields:

$$T = \frac{1}{2}M\dot{h}^2 + \frac{1}{2}I_\alpha\dot{\alpha}^2 + S_\alpha\dot{h}\dot{\alpha}, \quad (3.3)$$

Eq. (3.3) shows the airfoil's kinetic energy, where inertial parameters are defined as:

- $M = \int_{-c/2}^{+c/2} dm$: which refers to the mass per unit of wingspan of the cross section.
- $I_\alpha = \int_{-c/2}^{+c/2} \left(x - \Lambda_1 \frac{c}{2} \right)^2 dm$: which is the inertia momentum per unit wingspan of the cross-section relative to the elastic axis.
- $S_\alpha = \int_{-c/2}^{+c/2} \left(x - \Lambda_1 \frac{c}{2} \right) dm$: which is the static moment per unit wingspan of the cross-section relative to the elastic axis. This parameter is particularly important in the 2-degrees-of-freedom model as it represents the structural coupling of the bending and torsional modes.

3.3 Elastic energy

Elastic potential energy is the energy resulting from applying a force to deform an elastic body. This energy is produced by forces considered conservative, which are those forces with the following property: the total work done in moving a particle between two points is independent of the path taken. Within the model at hand two types of conservative forces will be considered: elastic forces (opposed to the deformation of the airfoil) and the earth's gravitational action over the aircraft. Regarding the latter, as the height variation introduced by the atmospheric turbulence is negligible compared to the flight height, the potential energy variations associated with the earth's gravitational field are assumed to be zero [15].

Bodies have the capability of being deformed when subjected to an external force. In particular, wings have proved to be elastic bodies, structurally able to withstand the limit loads (maximum intended service loads) without permanent detrimental plastic deformation. Therefore, these will be modeled as elastic bodies with bending and torsional stiffness.

In order to determine the elastic energy of any body it is necessary to determine its stiffness. For the purpose of this study, only bending and torsional stiffnesses are to be considered for the model.

Being K_h and K_α the equivalent bending and torsional stiffnesses of the wing with respect to the elastic axis per unit span [14], the potential energy due to elastic deformation per unit wingspan will be as shown in Eq. (3.4):

$$U = \frac{1}{2}K_h h^2 + \frac{1}{2}K_\alpha \alpha^2. \quad (3.4)$$

Meanwhile the system is inertially coupled, elastically it is considered to be uncoupled. This decoupling can be guaranteed by the existence of the elastic axis.

The stiffness coefficients are often expressed as functions of the eigenfrequencies of the system assuming uncoupled modes. These are expressed as $K_h = M\omega_h^2$ and $K_\alpha = I_\alpha\omega_\alpha^2$ for the bending and torsional modes, respectively [14]. Therefore, Eq. (3.5) can be rewritten as:

$$U = \frac{1}{2}M\omega_h^2 h^2 + \frac{1}{2}I_\alpha\omega_\alpha^2 \alpha^2. \quad (3.5)$$

3.4 Dissipative energy

Damping is the removal of energy from a vibratory system. The energy lost is either transmitted away from the system by some mechanism of radiation or dissipated within the system. Commonly the decay of free oscillation or measurements are made of steady-state forced vibration in the vicinity of resonance [16].

Conservative systems are those dynamic systems in which the total energy (kinetic and potential energy) is conserved $W_T = T + U = Cte..$ On the contrary, dissipative systems are known to provide a damped response, which evolves temporally and tends continuously (when time tends to infinity) to an equilibrium position. Thermodynamically, there is no system without any kind of losses, which is why all existing systems can be considered to be dissipative.

These systems have dissipative or damping mechanisms such as relative motion of different parts of the system, internal damping mechanisms, fluidodynamic damping due to an external fluid exerting a force on the system, etc. The resisting damping forces are typically related to the velocity of the degrees of freedom, with the damping coefficients establishing the relationships between force and velocity. Therefore, multiple linear models for energy dissipation have been developed, with the aim of approximating the rate of energy loss with different types of parameters. Each one is best suited to a particular case. An overview of the main models will be presented and the one that best suits the airfoil model will be selected.

- **Proportional or Rayleigh Damping:** The dissipative force is a linear combination of the inertia and the stiffness coefficients. The important thing about this model is that it decouples the equations in the damping matrix, in other words, the modes of the same conservative system are those of the system with proportional damping. Mathematically it is the simplest model. However, from a physical point of view, it suffers from one important feature: for a damped system, phase shifts between the degrees of freedom of the modes can vary from 0 to 360°, whereas the determination of these shifts is not in the scope of this model. It only allows the determination of whether they are in phase or in counterphase.
- **Viscous Damping:** It is modeled with a term which is directly proportional to the velocity, and therefore proportional to the frequency of the motion in the case of harmonic motion. Its expression is $f_v = c_v \dot{q}$, where c_v is the viscous damping coefficient and $\dot{q}$ is the velocity. In a harmonic motion, the complex associated to the viscous force is $\tilde{f}_v = \tilde{f}_{v0} e^{i\omega t} = c_v \tilde{\dot{q}} = i\omega c_v \tilde{q}$. In this case, the harmonic force in the time domain is expressed as the real part of $\tilde{f}_v$.
- **Structural Damping:** It is modeled with a term independent of the frequency of the motion, proportional to the displacement but acting in quadrature with it (90

degree phase shift). In a one degree of freedom system, in the dynamics equation to model this type of damping it would be necessary to modify the dissipation term as

$$M\ddot{q} + \frac{c_s}{\omega}\dot{q} + Kq = F,$$

where g is the structural damping coefficient. When assuming harmonic motion, the final expression for the damping would be

$$\tilde{f}_s = igK\tilde{q}.$$

If the system consists of metallic elements and these are the only source of energy dissipation, the proportional dissipation model does not adequately represent the dissipation mechanism. The energy dissipated in such structures is independent of the excitation frequency. Moreover, for the purposes of this study, it is necessary to use a damping model that allows the determination of phase shift. Structural damping is mainly associated with internal and/or external frictions and is the one that normally appears in the deformation of aircraft-type structures [14]. Thus, the structural damping model will be used for this project.

As can be extracted from its definition, structural damping is applied to ω frequency harmonic motions, while viscous damping can also be used in transients. On the other hand, the dissipated energy (or dissipation function) is also often used to model structural damping. The equation for this is $D_s = \frac{1}{2} \frac{gK}{\omega} \dot{q}^2$. Although this dissipation function is only valid for harmonic motion, this hypothesis is valid, due to the characteristics of atmospheric turbulence, described in the previous Chapter.

The dissipation function per unit span associated with a given structural damping is shown in Eq. (3.6):

$$D = \frac{1}{2} \frac{K_h}{\omega} g_h \dot{h}^2 + \frac{1}{2} \frac{K_\alpha}{\omega} g_\alpha \dot{\alpha}^2 = \frac{1}{2} M \frac{\omega_h^2}{\omega} g_h \dot{h}^2 + \frac{1}{2} I_\alpha \frac{\omega_\alpha^2}{\omega} g_\alpha \dot{\alpha}^2, \quad (3.6)$$

where g_h and g_α are the dimensionless structural damping coefficients due to bending and torsion, respectively and ω is the excitation frequency of the harmonic motion to which the structure is subjected.

3.5 Motion equations

Employing the Lagrangian Equations with the kinetic energy Eq. (3.3), the elastic deformation energy Eq. (3.5) and the dissipation function Eq. (3.6), developed in the previous sections, the expression shown in Eq. (3.7) is obtained in matrix form.

$$\begin{bmatrix} M & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} M \frac{\omega_h^2}{\omega} g_h & 0 \\ 0 & I_\alpha \frac{\omega_\alpha^2}{\omega} g_\alpha \end{bmatrix} \begin{Bmatrix} \dot{h} \\ \dot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} Q_h \\ Q_\alpha \end{Bmatrix}. \quad (3.7)$$

It is important to mention that Eq. (3.7) establishes the structural model that will be used hereafter for the calculation of the response of an airfoil to atmospheric turbulence. Moreover in Eq. (3.7), Q_h and Q_α represent the generalized aerodynamic forces (per unit span) generated as a consequence of the incremental non-stationary motion of the airfoil.

Generally, in aeroelasticity the aerodynamic forces usually depend linearly on the generalized coordinates, their first and second derivatives. Physically, the meaning of this dependence translates into:

- Terms proportional to the second derivatives ($\ddot{h}, \ddot{\alpha}$) correspond to virtual masses or aerodynamic inertia due to the acceleration effect of the fluid;
- Terms proportional to the first derivatives ($\dot{h}, \dot{\alpha}$) correspond to the aerodynamic damping which can be positive or negative, indicating that the fluid delivers or extracts energy from the system;
- The coefficients proportional to the generalized coordinates (h, α) correspond to the aerodynamic stiffness, i.e., the contributions of the fluid motion to the degrees of freedom of the solid.

Nevertheless, this will be discussed in greater depth in forthcoming chapters.

Chapter 4

Aerodynamic model

Aerodynamics is the branch of science which studies the forces and moments necessary to have a controlled and sustainable flight in air. It studies the effect of the velocity fields induced by the motion during flight.

Fluids can be characterized by their inherent quantities and properties that determine their behavior in terms of its motion, forces, energies, etc. The motion of a fluid is fully described by the Navier-Stokes equations shown in Eqs. (4.1) and (4.2).

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{u}) = 0, \quad (4.1)$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \nabla \vec{u} = \rho \vec{f}_m + \nabla \tau, \quad (4.2)$$

where ρ is the fluid's density, $\vec{u}$ refers to the three dimensional vector, where each element is the velocity in the x, y and z axis, respectively, $\vec{f}_m$ refers to the three dimensional vector containing the massic forces and τ refers to the shearing forces.

4.1 Main hypotheses for the aerodynamic model

In general, an experimentally based characterization of the fluid and the transfer of acceptable hypotheses is desirable. This method allows manipulation of equations which are simpler and tailored to a particular fluid. In order to do this, multiple dimensionless numbers have been established in order to characterize fluids. For the purposes of this study, air will be modeled as an ideal gas.

- **Strouhal number:** Strouhal number represents the ratio of local acceleration to convective acceleration. If $St \ll 1$, a process or a motion of a fluid is considered steady as local variations are considerably smaller than convective motions. Moreover, when $St \sim 1$, local variations have a similar effect on the motion of the fluid as convective variations. It is therefore necessary to consider both contributions for an accurate description of the fluid motion. This type of motion is called unsteady-state motion. The fluid under atmospheric turbulence conditions is treated as a unsteady fluid, where the Strouhal number is of unit order, $St \sim 1$.
- **Reynolds number:** It is the ratio of inertia force to the viscous force. It describes the predominance of inertia forces to the viscous forces occurring in the flow systems. The Reynolds number allows the characterization of the fluid in terms of the importance of the viscosity in the motion of the aforementioned: large Reynolds number indicate a predominance of convective terms, resulting in a laminar flow;

on the other hand, small Reynolds indicate that viscosity has a larger effect, thus resulting in a turbulent flow. The convective forces will be considered to be greater than the viscous, as such the Reynolds number will be considered $Re \gg 1$ in most of the domain, with the exception of the boundary layer (where viscous effects are known to be greater than convective and requires a separate in depth study).

- **Mach number:** It expresses the speed of the fluid relative to the speed of sound of the fluid. If the Mach number is one, then the flow velocity is equal to the velocity of sound in the fluid. If it is less than one, then the flow is called subsonic flow, and if it is greater than one the flow is called supersonic flow. This number allows the classification of the fluid motion attending to its compressibility: for $M < 0.3$ the fluid is considered to be incompressible; for $0.3 \leq M \leq 0.8$ the fluid is considered subsonic; for $0.8 \leq M \leq 1.2$ the fluid is considered to be transonic and phenomena such as shock-waves can be expected. Aircrafts generally have their cruise speed set around 0.75-0.8 Mach. Nonetheless, it is important to note that the effect of compressibility does not provide results significantly closer to reality, but only increases the degree of mathematical difficulty of the problem. Consequently, the air will be considered to be incompressible.
- Air can be considered to be an ideal barotropic fluid where pressure depends on density, $p = p(\rho)$. This leads to the conclusion that air will be treated as an irrotational flow $\nabla \times \vec{V} = 0$. Thus, velocity in all the domain derives from a potential, $\vec{V} = \nabla\Phi$.

Whereas steady aerodynamics considers the flow around a flying body to be unchanging over time (consequently, aerodynamic forces and moments acting on the body remain the same over time), unsteady aerodynamics considers a variation of the motion of the flow over time. This results in a phase lag in the aerodynamic response of the body with respect to its motion [17].

In the case of atmospheric turbulence explained in Chapter 2, the aircraft is randomly excited over a wide frequency spectrum. The optimum means of accurately depicting the fluid as far as atmospheric turbulence is concerned is a model of unsteady aerodynamics.

The aeroelastic response of aircrafts to unsteady aerodynamics is of utmost importance, as most severe aerolastic problems are closely linked to this flow condition (all kinds of flutter, buffeting, etc.). This chapter will consider unsteady aerodynamics with regard to atmospheric turbulence.

The application of the different hypotheses to the Navier-Stokes equations (4.1) and (4.2) and associated aerodynamic procedures, such as the eddies solution [14], results in the obtention of the following equations and boundary conditions, which shall be solved in order to obtain the unsteady aerodynamic forces and moments:

1. Differential equation: $\nabla^2\phi_1 = 0$;
2. Boundary condition on the airfoil: $(-c/2 \leq x \leq c/2, z = 0)$: $\phi_{1z}(x, 0; t)$. This boundary condition will vary and will be defined in the following Section;
3. Boundary condition at infinity: $(x^2 + z^2 \rightarrow \infty)$: $\phi_1(x, z; t) \rightarrow 0$;
4. Boundary condition in the wake $(x \geq c/2, z = 0)$: $\Delta\phi_{1t}(x, 0; t) + U_\infty\Delta\phi_{1x}(x, 0; t) = 0$,

where ϕ_1 represents the contribution of the unsteady effect to the potential of the fluid and ϕ_{1z} , ϕ_{1x} and ϕ_{1t} are its partial derivatives with regard to z , x and t , respectively. The frequency of the motion is ω , which will be referred to in its dimensionless expression,

$$k = \frac{\omega c}{U_\infty 2}.$$

As the present study will only consider turbulence acting on the vertical axis, the most important contribution to the potential is ϕ_{1z} . ϕ_{1z} is defined as the vertical speed induced by all the eddies contributing to the motion of the airfoil (both acting on the airfoil and the wake).

The calculation of the unsteady potential flow allows the obtention of the pressure coefficient, associated to unsteady aerodynamics, which will then result in the obtention of the incremental pressure coefficient between the extrados and the intrados of the airfoil.

For what this study encompasses, the aerodynamic model is strictly obtained through the integration of the incremental pressure coefficient, which will be explained in the following Sections. Although, other methods could be considered, such as different analytical procedures [18] and numerical solutions, it was deemed that the obtention of the incremental pressure coefficient would be an enlightful result, with several applications which will be laid in the following Chapters.

4.2 Unsteady aerodynamics modelling

The unsteady nature of the flow coexists with the randomness of the atmospheric turbulence. Nonetheless, it is considered that, prior to the appearance of the turbulence, the airfoil is immersed in a leveled stationary flight condition. As a consequence the aerodynamic model shall take both, the contribution of the turbulence and the natural reaction of the airfoil to a variation of its generalized forces.

When an aircraft is encountered with turbulence, it suffers two types of aerodynamical reactions. At first sight, the aircraft is only subjected to the variation of lift and aerodynamical moment produced by the atmospheric turbulence. Nonetheless, at the same time is suffering a variation of these generalized forces due to the motion of the aircraft itself. The forces associated to the motion of the aircraft mainly occur due to variations in velocity and/or acceleration in the bending and torsional modes in the model subjected to study in the present work.

From now on, lift and aerodynamic momentums will be referred to as:

- those produced by the turbulence or continuous gust: Q_h^G and Q_α^G ;
- those produced by the motion of the aircraft itself: Q_h^M and Q_α^M .

Moreover, all the variables associated to the motion of the aircraft itself will be referred to with the superscript M and those associated to the turbulence will be referred to with the superscript G .

Recalling the hypothesis of harmonic motion explained in Chapter 2, from now on, variables will be expressed in the frequency domain, taking for example the generalized variable shown in Eq (4.3):

$$q_j(t) = \Re [\tilde{q}_j e^{+i\omega t}] = \Re [\tilde{q}_j e^{+iks}], \quad (4.3)$$

where i is the imaginary unit and ω the frequency of the oscillation. The value of the variable is the real part of the complex number laid out. Introducing the reduced frequency k and the dimensionless time $s = \frac{U_\infty t}{c/2}$, the expression of the variable in the frequency domain is as shown.

Henceforth, all variables will be expressed in their frequency domain form. Temporal derivatives in the dimensionless time will be as shown in Eqs. (4.4) and (4.5):

$$\dot{q}_j(t) = ik\tilde{q}_j e^{iks}, \quad (4.4)$$

$$\ddot{q}_j(t) = -k^2\tilde{q}_j e^{iks}. \quad (4.5)$$

$$(4.6)$$

Aerodynamic forces and momentums for both contributions will be calculated through the expressions shown in Eqs. (4.7) and (4.8).

$$\tilde{Q}_h(k) = q_\infty \frac{c}{2} \int_{-1}^{+1} \Delta\tilde{C}_{p1}(\hat{x}, k)(-1)d\hat{x}, \quad (4.7)$$

$$\tilde{Q}_\alpha(k) = q_\infty \left(\frac{c}{2}\right)^2 \int_{-1}^{+1} \Delta\tilde{C}_{p1}(\hat{x}, k)(\Lambda_1 - \hat{x})d\hat{x}, \quad (4.8)$$

where q_∞ is the dynamic pressure, $\frac{c}{2}$ is the semichord of the airfoil and $\Delta\tilde{C}_{p1}(\hat{x}, k)$ is the incremental pressure coefficient.

In order to do it, the dimensionless preasure increase must be calculated for each case. Equation (4.9) shows the expression of the adimensional pressure increase where i is the complex unit, $\hat{x}$ is the dimensionless position along the airfoil, k is the reduced frequency of the motion, $\hat{\xi}$ is the dimensionless position of the wake, $C(k)$ is the Theodorsen function (which will be explained in the following sections) and $\tilde{\phi}_{1z}$ is the vertical velocity induced by all the vortices located in the profile and wake,

$$\begin{aligned} \Delta\tilde{C}_{p1}(\hat{x}, k) = & \frac{4}{\pi} \int_{-1}^1 \frac{\tilde{\phi}_{1z}(\hat{x}, 0; k)}{U_\infty} \left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{1}{\hat{x}-\hat{\xi}} - ik \ln \frac{1-\hat{x}\hat{\xi} + \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}}{1-\hat{x}\hat{\xi} - \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}} \right] d\hat{\xi} \\ & + \frac{4}{\pi} [1 - C(k)] \left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} \int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{\tilde{\phi}_{1z}(\tilde{x}, 0; k)}{U_\infty} d\hat{\xi} \right]. \end{aligned} \quad (4.9)$$

From a mathematical point of view, the main difference between the action of aerodynamic forces induced by turbulence and the motion of the airfoil itself relies on the vertical speed induced by these, $\tilde{\phi}_{1z}$. For those forces associated to the motion of the airfoil itself the vertical speed is:

$$\tilde{\phi}_{1z} = ik\tilde{z}_1(\hat{x}; k) + \frac{d\tilde{z}_1}{dx}(\hat{x}; k), \quad (4.10)$$

where $\tilde{z}_1$ is the the vertical speed of the airfoil expressed in the frequency domain.

On the other hand, in the forces associated to atmospheric turbulence the vertical speed is as shown in Eq. (4.11):

$$\tilde{\phi}_{1z} = \tilde{w}_G e^{-ik\hat{\xi}}, \quad (4.11)$$

where $\tilde{w}_G$ is the intensity of the turbulence. This variable will be substituted by one of the existing models for turbulence presented in Chapter 2.

The following sections explain the calculation of all the contributions to the aerodynamic forces.

4.2.1 Unsteady aerodynamics associated to the motion of the airfoil

The purpose of this Section is the obtention of the aerodynamic forces associated to the motion of the airfoil. This translates as the additional aerodynamic forces produced change of the initial position of the airfoil. When an aircraft encounters turbulence, the bending and the torsional generalized coordinates are prompted to vary. This variation produces a variation of the aerodynamical forces, which is associated and proportional to the variation of the generalized coordinates.

In order to obtain the forces produced by this phenomena, the vertical speed associated to it Eq. (4.10), must be introduced in the expression for the incremental pressure coefficient Eq. (4.9). The vertical speed associated to the motion of the airfoil itself requires the $\tilde{z}_1$ parameter to be defined [14]:

$$\tilde{z}_1(\hat{x}, k) = -\tilde{h}(k) - \tilde{\alpha}(k)(\hat{x} - \Lambda_1)\frac{c}{2}.$$

Introducing this expression in Eq. (4.10), Eq. (4.12) is obtained:

$$\tilde{\phi}_{1z}(\hat{x}, k) = -ik\tilde{h} - (\hat{x} - \Lambda_1 + 1)\frac{c}{2}\tilde{\alpha}. \quad (4.12)$$

The introduction of Eq. (4.12) in the incremental pressure coefficient and its subsequent integration, results in the obtention of the incremental pressure coefficient associated to the motion itself. In order to obtain this result, the integrals provided in Appendix A have been used. The pressure coefficient is shown in Eq. (4.13),

$$\begin{aligned} \Delta\tilde{C}_{p1}^M(\hat{x}, k) &= \frac{4}{\pi} \int_{-1}^1 \frac{\left(-ik\tilde{h} - (\hat{x} - \Lambda_1 + 1)\frac{c}{2}\tilde{\alpha}\right)}{U_\infty} \\ &\left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{1}{\hat{x}-\hat{\xi}} - ik \ln \frac{1-\hat{x}\hat{\xi} + \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}}{1-\hat{x}\hat{\xi} - \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}} \right] d\hat{\xi} + \\ &\frac{4}{\pi} [1 - C(k)] \left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} \int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{\left(-ik\tilde{h} - (\hat{x} - \Lambda_1 + 1)\frac{c}{2}\tilde{\alpha}\right)}{U_\infty} d\hat{\xi} \right]. \quad (4.13) \end{aligned}$$

In Eq. (4.14) the incremental pressure coefficient associated to the motion itself is shown, where mathematical calculations have been performed. This result allows the calculation of the aerodynamic forces produced due to the motion of the aircraft itself,

$$\begin{aligned} \Delta\tilde{C}_{p1}^M(\hat{x}, k) &= \frac{4}{\pi} \frac{\left(-ik\tilde{h} - (\hat{x} - \Lambda_1 + 1)\frac{c}{2}\tilde{\alpha}\right)}{U_\infty} \\ &\left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} (-\pi) - ik \frac{\pi\sqrt{1-\hat{x}^2}(\hat{x}\sqrt{\hat{x}^2-1} - \hat{x}^2 + 1)}{\hat{x}^2 - 1} \right] + \\ &\frac{4}{\pi} [1 - C(k)] \left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} \pi \frac{\left(-ik\tilde{h} - (\hat{x} - \Lambda_1 + 1)\frac{c}{2}\tilde{\alpha}\right)}{U_\infty} \right]. \quad (4.14) \end{aligned}$$

The introduction of the dimensionless pressure variation associated to the motion of the airfoil, $\Delta\tilde{C}_{p1}^M(\hat{x}, k)$, in Eqs. (4.7) and (4.8) results in the obtention of the aerodynamic forces associated to the motion of the airfoil, shown in Eqs. (4.15) and (4.16):

$$\tilde{Q}_h^M(k) = q_\infty \frac{c}{2} \left(-2\pi \left[-k^2 \frac{\tilde{h}}{c/2} + ik\tilde{\alpha} - \Lambda_1(-k^2\tilde{\alpha}) \right] - 4\pi C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik\tilde{\alpha} \right\} \right), \quad (4.15)$$

$$\begin{aligned} \tilde{Q}_\alpha^M(k) = q_\infty \left(\frac{c}{2} \right)^2 & \left(-2\pi \left[\Lambda_1 k^2 \frac{\tilde{h}}{c/2} + ik \left(\frac{1}{2} - \Lambda_1 \right) \tilde{\alpha} \right. \right. \\ & \left. \left. + \left(\frac{1}{8} - \Lambda_1^2 \right) (-k^2\tilde{\alpha}) \right] - 4\pi \left(\frac{1}{2} + \Lambda_1 \right) C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik\tilde{\alpha} \right\} \right). \end{aligned} \quad (4.16)$$

The terms dependant on the Theodorsen function are called circulatory terms, while the remaining terms are called non-circulatory terms. Circulatory terms derive from the eddy and vortex formulation for lift and aerodynamic momentum. On the other hand, non-circulatory terms are those associated to the motion of the air surrounding the airfoil, which act as apparent masses. Thus, they induce additional lift and momentums.

4.2.2 Unsteady aerodynamics associated to the gust

The main purpose of this section is the calculation of the unsteady aerodynamic forces derived from a harmonic motion. These are based on the initial assumption of turbulence as a "quasi-harmonic" motion, explained in Chapter 2. Thus, the adimensional pressure will have to be calculated with this assumption. The procedure used to calculate this pressure coefficient will be the one used in [14], by the vortex distribution method. Hence, a distribution of vortex-type singularities is introduced in the fluid zones where boundary conditions are imposed, which are the airfoil itself (slip condition) and the wake (zero pressure jump).

In the case of atmospheric turbulence, the vertical speed resulting of this motion will be as shown in Eq.(4.11). As a result, the dimensionless pressure variation which shall be integrated is presented in Eq. (4.17):

$$\begin{aligned} \Delta\tilde{C}_{p1}^G(\hat{x}, k) = & \frac{4}{\pi} \int_{-1}^1 \frac{\tilde{w}_G}{U_\infty} e^{-ik\hat{\xi}} \left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{1}{\hat{x}-\hat{\xi}} - ik \ln \frac{1-\hat{x}\hat{\xi} + \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}}{1-\hat{x}\hat{\xi} - \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}} \right] d\hat{\xi} \\ & + \frac{4}{\pi} [1 - C(k)] \left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} \int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{\tilde{w}_G}{U_\infty} e^{-ik\hat{\xi}} d\hat{\xi} \right]. \end{aligned} \quad (4.17)$$

The complexity of these integrals has led to the development of several alternate methods to obtain the generalized forces for the problem. Sears and Von Karman [19] modelled both the unsteady forces and momentums induced by a harmonic motion as series of cosines. The alignment of that hypothesis with those of the gust, allowed them

to obtain the generalized forces without obtaining the intermediate result, the incremental pressure coefficient due to a harmonic motion. This result has several applications and will be calculated in this section.

The integrals subjected to study will be those indicated in Eqs. (4.18) - (4.20):

$$I_1(\hat{x}, k) = \int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{e^{-ik\hat{\xi}}}{\hat{x}-\hat{\xi}} d\hat{\xi}, \quad (4.18)$$

$$I_2(\hat{x}, k) = \int_{-1}^1 \ln \frac{1-\hat{x}\hat{\xi} + \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}}{1-\hat{x}\hat{\xi} - \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}} e^{-ik\hat{\xi}} d\hat{\xi}, \quad (4.19)$$

$$I_3(\hat{x}, k) = \int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} e^{-ik\hat{\xi}} d\hat{\xi}. \quad (4.20)$$

Therefore, as shown in Eq. (4.21), the dimensionless pressure increase can be written as a function of these integrals:

$$\Delta \tilde{C}_{p1}^G(\hat{x}, k) = \frac{4}{\pi} \frac{\tilde{w}_G}{U_\infty} \left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} I_1(\hat{x}, k) - ik I_2(\hat{x}, k) \right] + \frac{4}{\pi} [1 - C(k)] \left[\sqrt{\frac{1+\hat{x}}{1-\hat{x}}} \frac{\tilde{w}_G}{U_\infty} I_3(\hat{x}, k) \right]. \quad (4.21)$$

The main complexity of these integrals lays in the exponential factor associated to the harmonic motion. The most efficient approach to the resolution of these integrals is the use of Bessel series.

The Bessel series is a particular kind of generalized Fourier series [20] (an infinite series expansion on a finite interval) based on Bessel functions. These arise as an easier approach to solve partial differential equations, particularly in cylindrical coordinate systems.

Bessel established that an exponential function can be expressed with the Bessel series, where the Bessel functions (the coefficients determining the contribution of cosines of different orders) depend on the frequency of the motion [20]:

$$e^{ik \cos(\theta)} = J_0(k) + 2 \sum_{n=1}^{\infty} (-i)^n J_n(k) \cos(n\theta), \quad (4.22)$$

where $J_n(k)$ is Bessel's function of n^{th} order, for a positive value of n .

Bessel functions of different orders are linked through the properties laid in Eqs. from (4.23) to (4.28):

$$J_n(-k) = (-1)^n J_n(k), \quad (4.23)$$

$$2nJ_n(k)/k = J_{n-1}(k) + J_{n+1}(k), \quad (4.24)$$

$$\frac{d}{dk} [k^n J_n(k)] = k^n J_{n-1}(k), \quad (4.25)$$

$$\frac{d}{dk} [k^{-n} J_n(k)] = k^{-n} J_{n+1}(k), \quad (4.26)$$

$$\frac{d}{dk} [J_0(k)] = -J_1(k), \quad (4.27)$$

$$2 \frac{d}{dk} [J_n(k)] = J_{n-1}(k) - J_{n+1}(k). \quad (4.28)$$

Then, with the aim of solving the integrals, it is necessary to make a change of variable in Eq. (4.21) where $\xi = \cos(\theta)$. In order to obtain an easier mathematical expression, $e^{-ik\cos(\theta)}$ will be replaced with the relation expressed in Eq. (4.22), except for $I_2(\hat{x}, k)$ (4.19). At this stage, the integrals that have to be solved are:

$$I_1(\hat{x}, k) = \int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{e^{-ik\hat{\xi}}}{\hat{x}-\hat{\xi}} d\hat{\xi} = \int_0^\pi \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \frac{e^{-ik\cos\theta}}{\hat{x}-\cos\theta} (-\sin\theta) d\theta, \quad (4.29)$$

$$I_2(\hat{x}, k) = \int_{-1}^1 \ln \frac{1-\hat{x}\hat{\xi} + \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}}{1-\hat{x}\hat{\xi} - \sqrt{1-\hat{\xi}^2}\sqrt{1-\hat{x}^2}} e^{-ik\hat{\xi}} d\hat{\xi}, \quad (4.30)$$

$$I_3(\hat{x}, k) = \int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} e^{-ik\hat{\xi}} d\hat{\xi} = \int_0^\pi \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} e^{-ik\cos\theta} (-\sin\theta) d\theta. \quad (4.31)$$

In addition, the definition of Bessel functions (4.32) as an integral of an exponential function has to be taken into account, as it will be used in the calculation of the integrals:

$$J_n(-k) = \frac{i^{-n}}{\pi} \int_0^\pi e^{-ik\cos\theta} \cos(n\theta) d\theta. \quad (4.32)$$

The results obtained for the integrals will be shown in the following paragraphs. Firstly, $I_1(\hat{x}, k)$ will be solved as shown in Eqs.(4.33) - (4.35) :

$$\begin{aligned} I_1(\hat{x}, k) &= \int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{e^{-ik\hat{\xi}}}{\hat{x}-\hat{\xi}} d\hat{\xi} = \int_{-1}^1 \frac{e^{-ik\hat{\xi}}}{\sqrt{1-\hat{\xi}^2}} \frac{d\hat{\xi}}{\hat{x}-\hat{\xi}} + \int_{-1}^1 \hat{\xi} \frac{e^{-ik\hat{\xi}}}{\sqrt{1-\hat{\xi}^2}} \frac{d\hat{\xi}}{\hat{x}-\hat{\xi}} = \\ &= \int_{-1}^1 \frac{e^{-ik\hat{\xi}}}{\sqrt{1-\hat{\xi}^2}} \left(\frac{1+\hat{x}}{\hat{x}-\hat{\xi}} - 1 \right) d\hat{\xi} = (1+\hat{x}) \int_0^\pi \frac{e^{-ik\cos\theta}}{\hat{x}-\cos\theta} d\theta - \int_0^\pi e^{-ik\cos\theta} d\theta = \\ &= \int_0^\pi \frac{e^{-ik\cos\theta}}{\hat{x}-\cos\theta} d\theta - \pi J_0(k). \end{aligned} \quad (4.33)$$

As can be seen in Eq. (4.33), in order to solve $I_1(\hat{x}, k)$ another integral needs to be calculated. Using Glauert's integral [21] the result of this integral is shown in Eq. (4.34):

$$I_4(k, \phi) = \int_{-1}^1 \frac{e^{-ik\hat{\xi}}}{\sqrt{1-\hat{\xi}^2}} \frac{d\hat{\xi}}{\hat{x}-\hat{\xi}} = 2\pi \sum (-i)^n J_n(k) \frac{\sin(n\phi)}{\sin\phi}, \quad (4.34)$$

where $\hat{x} = \cos\phi$.

Introducing this result in Eq. (4.33), the resulting integral is as shown in Eq. (4.35):

$$I_1(\hat{x}, k) = -J_0(k)\pi + (1+\hat{x})2\pi \sum (-i)^n J_n(k) \frac{\sin(n\phi)}{\sin\phi}. \quad (4.35)$$

With respect to $I_2(\hat{x}, k)$ an integration by parts where $du = e^{-ik\hat{\xi}}$ and $v = \Lambda(\hat{x}, k)$ will

be used:

$$I_2(\hat{x}, k) = \int_{-1}^1 \ln \frac{1 - \hat{x}\hat{\xi} + \sqrt{1 - \hat{\xi}^2}\sqrt{1 - \hat{x}^2}}{1 - \hat{x}\hat{\xi} - \sqrt{1 - \hat{\xi}^2}\sqrt{1 - \hat{x}^2}} e^{-ik\hat{\xi}} d\hat{\xi} = \left[-\frac{\Lambda(\hat{x}, \hat{\xi})}{ik} e^{-ik\hat{\xi}} \right]_{-1}^1 + \int_{-1}^1 \frac{e^{-ik\hat{\xi}}}{ik} \frac{d\Lambda}{d\hat{\xi}} d\hat{\xi} = \frac{\sqrt{1 - \hat{x}^2}}{ik} \int_{-1}^1 \frac{e^{-ik\hat{\xi}}}{\sqrt{1 - \hat{\xi}^2}} \frac{d\hat{\xi}}{\hat{x} - \hat{\xi}}. \quad (4.36)$$

It can be seen that the result of integral $I_2(\hat{x}, k)$ is also dependant on the result presented in Eq. (4.34). Introducing this expression in Eq. (4.36):

$$I_2(\hat{x}, k) = \frac{\sqrt{1 - \hat{x}^2}}{ik} \left(2\pi \sum (-i)^n J_n(k) \frac{\sin(n\phi)}{\sin\phi} \right). \quad (4.37)$$

Finally, the last integral is solved as showed in Eq. (4.38):

$$I_3(\hat{x}, k) = \int_{-1}^1 \sqrt{\frac{1 + \hat{\xi}}{1 - \hat{\xi}}} e^{-ik\hat{\xi}} d\hat{\xi} = \int_0^\pi e^{-ik \cos \theta} d\theta + \int_0^\pi \cos \theta e^{-ik \cos \theta} d\theta = \pi J_0(k) - \pi i J_1(k). \quad (4.38)$$

Introducing the values obtained in Eqs. (4.35) - (4.38) in Equation (4.21):

$$\Delta \tilde{C}_{p1}^G(\hat{x}, k) = 4 \frac{\tilde{\omega}_G}{U_\infty} \sqrt{\frac{1 - \hat{x}}{1 + \hat{x}}} [iJ_1(k) + C(k) [J_0(k) - iJ_1(k)]] = 4 \frac{\tilde{\omega}_G}{U_\infty} \sqrt{\frac{1 - \hat{x}}{1 + \hat{x}}} S(k), \quad (4.39)$$

where $S(k) = [iJ_1(k) + C(k) [J_0(k) - iJ_1(k)]]$ is the Sears Function, which will be explained in the following Section.

Finally, the result for the $\Delta \tilde{C}_{p1}^G$ will be used to calculate the aerodynamic forces $\tilde{Q}_h^G$ and $\tilde{Q}_\alpha^G$, applying the definition of the aerodynamic forces shown in Eqs. (4.40) and (4.40):

$$\tilde{Q}_h^G = \int_{-1}^1 \Delta \tilde{C}_{p1}(-1) d\hat{x} = -4\pi q_\infty \frac{c}{2} \frac{\tilde{\omega}_G}{U_\infty} S(k), \quad (4.40)$$

$$\tilde{Q}_\alpha^G = \int_{-1}^1 \Delta \tilde{C}_{p1}(\Lambda_1 - \hat{x}) d\hat{x} = 4\pi q_\infty \left(\frac{c}{2} \right)^2 \frac{\tilde{\omega}_G}{U_\infty} S(k) \left(\Lambda_1 + \frac{1}{2} \right). \quad (4.41)$$

Nonetheless, the aerodynamic loads, to which the airfoil be subjected to, will be the addition of those associated with the turbulence, and those associated to the natural motion of the airfoil.

4.2.3 Theodorsen and Sears functions

As presented in the previous results, both Theodorsen and Sears functions are of great importance in the study of unsteady aerodynamics of airfoils.

The Theodorsen function [22] determines the magnitude attenuations and phase lags in circulatory aerodynamic forces and moments due to variations in torsion and bending

for a typical section in unsteady incompressible potential flow. This function can be expressed in terms of Hankel functions as follows:

$$C(k) = \frac{H_1^2(k)}{H_1^2(k) + iH_0^2(k)}, \quad (4.42)$$

where $H_n^2(k)$ are the second species Hankel functions of n th order. These can be expressed in terms of the Bessel functions $H_n^2(k) = J_n(k) - iY_n(k)$.

An interesting approach is to express the Theodorsen function by distinguishing between the real and the imaginary part of it:

$$C(k) = F(k) + iG(k). \quad (4.43)$$

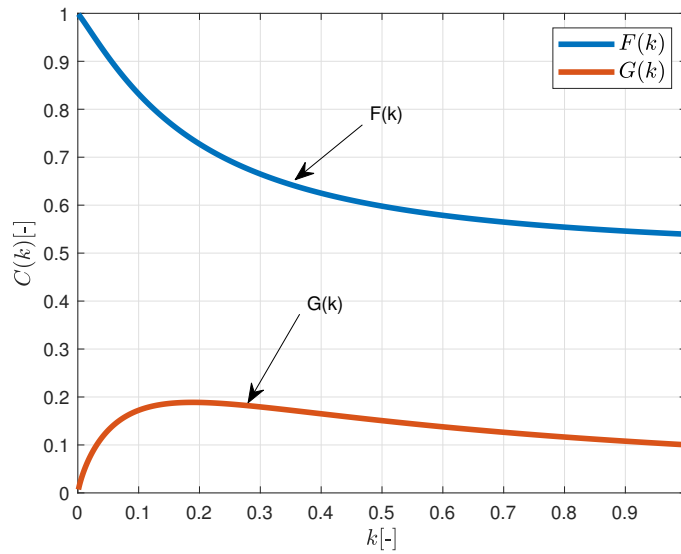


Figure 4.1: Representation of Theodorsen function's real and imaginary parts vs reduced frequency.

Figure 4.1 shows the representation of the real and imaginary part of the Theodorsen function. These tend asymptotically to 0.5 when the reduced frequency is increased, at the same time that the argument and the imaginary part approach zero. This behavior causes the circulatory lift component to be modified depending on the value of k , affecting both the amplitude and the phase lag of the aerodynamic forces and moments.

It is also important to understand what the Sears' function (4.44) implies and what physical meaning lays behind it. It is considered a transfer function which takes into account the unsteady lift on an airfoil in a periodic gust. As a result, it can be constructed by combining the apparentmass and wake-effect lift components with the quasi-steady lift derived from the Kutta-Joukowski theorem [23]:

$$S(k) = [iJ_1(k) + C(k) [J_0(k) - iJ_1(k)]] . \quad (4.44)$$

The scope of this function is defined by the previously established hypotheses. Despite this, its inherent simplicity renders it highly versatile. Similar to the previously mentioned Theodorsen function, it solely relies on the reduced frequency of the incoming gust, retaining its mathematical straightforwardness. Beyond its extensive utilization for

forecasting unsteady loads in gust-response scenarios, the Sears function finds extensive relevance across a spectrum of research domains, spanning from active flow control to aeroacoustics.

Furthermore, it also is interesting to separate the Sears function in its real and imaginary parts:

$$S(k) = D(k) + iL(k). \quad (4.45)$$

Shown in 4.2 is the representation both the real and the imaginary parts of the Sears function.

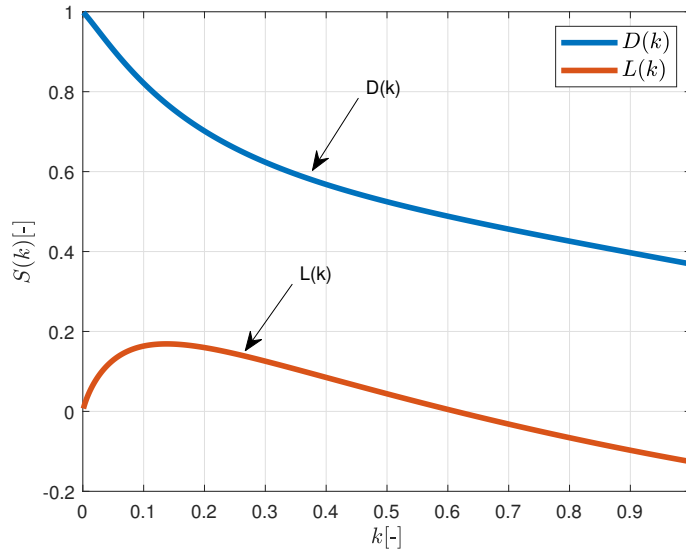


Figure 4.2: Representation of Sears function's real and imaginary parts vs reduced frequency.

The primary distinction between these two functions lies in their respective objectives. The Theodorsen function concentrates on incorporating alterations in magnitude and phase caused by harmonic airfoil movements, whereas the Sears function aims to introduce similar phase shifts resulting from harmonic gusts. The Theodorsen function is tailored to analyze the airfoil's harmonic motion, whereas the Sears function is designed to address the harmonic motion of the gust itself.

4.3 Aerodynamic model assembly

The final expressions for the aerodynamic forces the airfoil will be subjected to will be explained in this section. Therefore, the final aerodynamic loads result in Eq. (4.46) for the bending motion and Eq. (4.47) for the torsion motion:

$$\tilde{Q}_h = \tilde{Q}_h^M + \tilde{Q}_h^G, \quad (4.46)$$

$$\tilde{Q}_\alpha = \tilde{Q}_\alpha^M + \tilde{Q}_\alpha^G. \quad (4.47)$$

Shown in Eqs. (4.48) and (4.49) are the final expressions for the generalized forces and momentums the airfoil is subjected to, after adding both contributions to the motion of the airfoil when subjected to turbulence:

$$\begin{aligned} \tilde{Q}_h = \frac{1}{2}\rho_\infty U_\infty^2 \frac{c}{2} & \left(-2\pi \left[-k^2 \frac{\tilde{h}}{c/2} + ik\tilde{\alpha} - \Lambda_1(-k^2\tilde{\alpha}) \right] \right. \\ & \left. - 4\pi C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik\tilde{\alpha} \right\} - 4\pi \frac{\tilde{\omega}_G}{U_\infty} S(k) \right), \end{aligned} \quad (4.48)$$

$$\begin{aligned} \tilde{Q}_\alpha = \frac{1}{2}\rho_\infty U_\infty^2 \left(\frac{c}{2} \right)^2 & \left(-2\pi \left[\Lambda_1 k^2 \frac{\tilde{h}}{c/2} + ik \left(\frac{1}{2} - \Lambda_1 \right) \tilde{\alpha} + \left(\frac{1}{8} - \Lambda_1^2 \right) (-k^2\tilde{\alpha}) \right] \right. \\ & \left. - 4\pi \left(\frac{1}{2} + \Lambda_1 \right) C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik\tilde{\alpha} \right\} + 4\pi \frac{\tilde{\omega}_G}{U_\infty} S(k) \left[\Lambda_1 + \frac{1}{2} \right] \right), \end{aligned} \quad (4.49)$$

where the dynamic pressure has been replaced by its flight speed dependant expression U_∞ , $q_\infty = \frac{1}{2}\rho_\infty U_\infty^2$.

Chapter 5

Aeroelastic model

The main purpose of this chapter is to obtain the final equations of motion for the 2-degrees-of-freedom airfoil subjected to atmospheric turbulence. As explained in the previous Chapters, a mechanical system involves both external and internal forces. Although both the structural (Chapter 3) and aerodynamic models (Chapter 4) have been developed, it should be noted that in this case study the internal forces are those provided by the structural model and the external forces are the aerodynamic forces (which contain the effects of atmospheric turbulence).

First of all, the harmonic motion hypothesis has to be implemented in the structural model, since it is already included in the aerodynamic model. In Eq. (5.1) the equations of motion derived from the application of the harmonic motion hypothesis to Eq. (3.7) are shown, without further development of the aerodynamic model:

$$-\omega^2 \begin{bmatrix} M & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \tilde{h} \\ \tilde{\alpha} \end{Bmatrix} + i\omega \begin{bmatrix} M \frac{\omega_h^2}{\omega} g_h & 0 \\ 0 & I_\alpha \frac{\omega_\alpha^2}{\omega} g_\alpha \end{bmatrix} \begin{Bmatrix} \tilde{h} \\ \tilde{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix} \begin{Bmatrix} \tilde{h} \\ \tilde{\alpha} \end{Bmatrix} = \begin{Bmatrix} \tilde{Q}_h \\ \tilde{Q}_\alpha \end{Bmatrix}. \quad (5.1)$$

The inertia, damping and stiffness matrices have as a common factor the system's response in bending and torsion in the frequency domain. Therefore, these equations can be expressed in a more compact way, as shown in Eq. (5.2). In addition, the following parameters are substituted into this equation $\omega = \frac{kU_\infty}{c/2}$, $\omega_h = \frac{k_h U_\infty}{c/2}$ and $\omega_\alpha = \frac{k_\alpha U_\infty}{c/2}$. The correlations between the stiffnesses of the system and its natural frequencies are also introduced, $K_h = M \left(\frac{k_h U_\infty}{c/2} \right)^2$ y $K_\alpha = I_\alpha \left(\frac{k_\alpha U_\infty}{c/2} \right)^2$:

$$\begin{bmatrix} -\left(\frac{kU_\infty}{c/2} \right)^2 M + ikM \frac{\left(\frac{k_h U_\infty}{c/2} \right)^2}{\frac{kU_\infty}{c/2}} g_h + M \left(\frac{k_h U_\infty}{c/2} \right)^2 & -\left(\frac{kU_\infty}{c/2} \right)^2 S_\alpha \\ -\left(\frac{kU_\infty}{c/2} \right)^2 S_\alpha & -\left(\frac{kU_\infty}{c/2} \right)^2 I_\alpha + ikI_\alpha \frac{\left(\frac{k_\alpha U_\infty}{c/2} \right)^2}{\frac{kU_\infty}{c/2}} g_\alpha + I_\alpha \left(\frac{k_\alpha U_\infty}{c/2} \right)^2 \end{bmatrix} \begin{Bmatrix} \tilde{h} \\ \tilde{\alpha} \end{Bmatrix} = \begin{Bmatrix} \tilde{Q}_h \\ \tilde{Q}_\alpha \end{Bmatrix}, \quad (5.2)$$

where $\tilde{Q}_h$ and $\tilde{Q}_\alpha$ were defined in Eqs. (4.48) and (4.49).

5.1 Dimensionless equations of motion

There are certain dimensionless parameters of widespread use, which allow the expression of the main characteristics of the model. The Eqs. (5.4), (5.5) and (5.3) present the dimensionless parameters that will be used in the forthcoming Eqs.

$$\lambda = \frac{M}{\rho_{\infty} c^2 \pi} \quad (5.3)$$

the λ parameter represents the ratio of the mass per unit of wingspan divided by the mass of air per unit of wingspan contained in a circle with a radius equal to c . The variation of this parameter, for flights with slight fluctuations of the flight level as are those caused by turbulence, represents the variation of the mass of the aircraft. The consumption of fuel during flight varies the weight of the aircraft, as a result, it is interesting to study the fluctuation of the response to turbulence, when varying the λ parameter.

$$x_{\alpha} = \frac{S_{\alpha}}{M \frac{c}{2}}, \quad (5.4)$$

the parameter x_{α} is the ratio of the crossed inertia product of the airfoil divided by the mass per unit of wingspan. This parameter aims to provide an estimate of the dimensionless distance between the Elastic Axis(EA) and the Center of Gravity(CG) of the airfoil. Positive values of x_{α} imply that the CG is behind the EA, relatively to the chord of the airfoil. On the other hand, negative values represent that the CG is in advance with respect to the EA, chordwise. The x_{α} parameter is of great importance for the present work as S_{α} is the term that structurally couples the model subjected to study. As a result, the fluctuation of the response for different values of x_{α} is an interesting result, which will be presented in the following Chapter.

$$r_{\alpha}^2 = \frac{4I_{\alpha}}{Mc^2}, \quad (5.5)$$

the parameter r_{α}^2 represents the dimensionless radius of rotation of the airfoil. Meanwhile this is an important parameter in the airfoil, the present work will not cover the variation of the rotational inertia of the airfoil.

Using these parameters, the equations will be adimensionalized separately: firstly, the bending equation will be adimensionalized and, thereafter, the torsional equation will be adimensionalized.

$$\begin{aligned} & \tilde{h} \left(- \left(\frac{kU_{\infty}}{c/2} \right)^2 M + ikM \frac{\left(\frac{k_h U_{\infty}}{c/2} \right)^2}{\frac{kU_{\infty}}{c/2}} g_h \right. \\ & \left. + M \left(\frac{k_h U_{\infty}}{c/2} \right)^2 \right) + \tilde{\alpha} \left(- \left(\frac{kU_{\infty}}{c/2} \right)^2 S_{\alpha} \right) = \frac{1}{2} \rho_{\infty} U_{\infty}^2 \pi \frac{c}{2} \left(-2 \left[-k^2 \frac{\tilde{h}}{c/2} + ik\tilde{\alpha} - \Lambda_1(-k^2\tilde{\alpha}) \right] \right. \\ & \left. - 4C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik\tilde{\alpha} \right\} - 4 \frac{\tilde{\omega}_G}{U_{\infty}} S(k) \right), \end{aligned} \quad (5.6)$$

In Eq. (5.6) the complete expression associated with the bending motion is shown. To adimensionalize it, both members of the equation are divided by $M \left(\frac{U}{c/2} \right)^2$. The result is shown in the Eq. (5.7):

$$\begin{aligned}
& \tilde{h} \left(-k^2 + ik_h^2 g_h + k_h^2 \right) + \tilde{\alpha} \left(-k^2 \frac{S_\alpha}{M} \right) \\
&= \frac{\frac{1}{2} \rho_\infty \pi \frac{c}{2}}{M \left(\frac{c}{2} \right)^2} \left(-2 \left[-k^2 \frac{\tilde{h}}{c/2} + ik \tilde{\alpha} - \Lambda_1 (-k^2 \tilde{\alpha}) \right] \right. \\
&\quad \left. - 4C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik \tilde{\alpha} \right\} - 4 \frac{\tilde{\omega}_G}{U_\infty} S(k) \right). \tag{5.7}
\end{aligned}$$

By dividing by $k^2 c/2$ and identifying the previously mentioned dimensionless parameters, the dimensionless equation associated with the bending motion is obtained:

$$\begin{aligned}
& \frac{\tilde{h}}{c/2} \left(-1 + \left(\frac{k_h}{k} \right)^2 (1 + ig_h) \right) \\
&+ \tilde{\alpha} (-x_\alpha) = \frac{\frac{1}{2} \rho_\infty \pi \frac{c}{2}}{M \left(\frac{c}{2} \right)^2 k^2 c/2} \left(-2 \left[-k^2 \frac{\tilde{h}}{c/2} + ik \tilde{\alpha} - \Lambda_1 (-k^2 \tilde{\alpha}) \right] \right. \\
&\quad \left. - 4C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik \tilde{\alpha} \right\} - 4 \frac{\tilde{\omega}_G}{U_\infty} S(k) \right). \tag{5.8}
\end{aligned}$$

After identifying the terms and regrouping in such a way that $\frac{\tilde{\omega}_G}{U_\infty}$ is left as a linear combination of $\frac{\tilde{h}}{c/2}$ and $\tilde{\alpha}$, the expression shown in Eq. (5.9), is obtained:

$$\begin{aligned}
& \frac{\tilde{h}}{c/2} \left(-1 + \left(\frac{k_h}{k} \right)^2 (1 + ig_h) - \frac{1}{4\lambda} + \frac{iC(k)}{2\lambda k} \right) \\
&+ \tilde{\alpha} \left(-x_\alpha + \frac{i}{4\lambda k} + \frac{\Lambda_1}{4\lambda} + \frac{C(k)}{2\lambda k^2} + \frac{iC(k)(1/2 - \Lambda_1)}{2\lambda k} \right) = -\frac{\tilde{\omega}_G}{2\lambda k^2} S(k). \tag{5.9}
\end{aligned}$$

Similarly, the equation corresponding to the airfoil's torsion can be adimensionalized. In Eq. (5.10) the complete expression associated to the airfoil torsion is shown:

$$\begin{aligned}
& \tilde{h} \left(- \left(\frac{kU_\infty}{c/2} \right)^2 S_\alpha \right) + \tilde{\alpha} \left(- \left(\frac{kU_\infty}{c/2} \right)^2 I_\alpha + ik I_\alpha \frac{\left(\frac{k_\alpha U_\infty}{c/2} \right)^2}{\frac{kU_\infty}{c/2}} g_\alpha + I_\alpha \left(\frac{k_\alpha U_\infty}{c/2} \right)^2 \right) = \\
& \frac{1}{2} \rho_\infty U_\infty^2 \pi \left(\frac{c}{2} \right)^2 \left(-2 \left[\Lambda_1 k^2 \frac{\tilde{h}}{c/2} + ik \left(\frac{1}{2} - \Lambda_1 \right) \tilde{\alpha} + \left(\frac{1}{8} - \Lambda_1^2 \right) (-k^2 \tilde{\alpha}) \right] \right. \\
&\quad \left. - 4 \left(\frac{1}{2} + \Lambda_1 \right) C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik \tilde{\alpha} \right\} + 4 \frac{\tilde{\omega}_G}{U_\infty} S(k) \left[\Lambda_1 + \frac{1}{2} \right] \right). \tag{5.10}
\end{aligned}$$

The approach taken for the adimensionalization of this equation is very similar to the one used with the bending equation. To begin with, the whole expression has to be divided by $M \left(\frac{U_\infty}{c/2} \right)^2$:

$$\begin{aligned}
\tilde{h} \left(-k^2 \frac{S_\alpha}{M} \right) + \tilde{\alpha} \left(\frac{I_\alpha}{M} (-k^2 + ik_\alpha^2 g_\alpha + k_\alpha) \right) = \\
\frac{\frac{1}{2} \rho_\infty U_\infty^2 \pi}{M \left(\frac{U_\infty}{c/2} \right)^2} \left(\frac{c}{2} \right)^2 \left(-2 \left[\Lambda_1 k^2 \frac{\tilde{h}}{c/2} + ik \left(\frac{1}{2} - \Lambda_1 \right) \tilde{\alpha} + \left(\frac{1}{8} - \Lambda_1^2 \right) (-k^2 \tilde{\alpha}) \right] \right. \\
\left. - 4 \left(\frac{1}{2} + \Lambda_1 \right) C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik \tilde{\alpha} \right\} + 4 \frac{\tilde{\omega}_G}{U_\infty} S(k) \left[\Lambda_1 + \frac{1}{2} \right] \right). \quad (5.11)
\end{aligned}$$

The subsequent step is to divide the resulting expression by $k^2 \left(\frac{c}{2} \right)^2$.

$$\begin{aligned}
\frac{\tilde{h}}{c/2} (-x_\alpha) + \tilde{\alpha} \left(r_\alpha^2 \left(-1 + \left(\frac{k_\alpha}{k} \right)^2 (1 + ig_\alpha) \right) \right) = \\
\frac{\frac{1}{2} \rho_\infty U_\infty^2 \pi}{M \left(\frac{U_\infty}{c/2} \right)^2 k^2 \left(\frac{c}{2} \right)^2} \left(-2 \left[\Lambda_1 k^2 \frac{\tilde{h}}{c/2} + ik \left(\frac{1}{2} - \Lambda_1 \right) \tilde{\alpha} + \left(\frac{1}{8} - \Lambda_1^2 \right) (-k^2 \tilde{\alpha}) \right] \right. \\
\left. - 4 \left(\frac{1}{2} + \Lambda_1 \right) C(k) \left\{ ik \frac{\tilde{h}}{c/2} + \tilde{\alpha} + \left(\frac{1}{2} - \Lambda_1 \right) ik \tilde{\alpha} \right\} + 4 \frac{\tilde{\omega}_G}{U_\infty} S(k) \left[\Lambda_1 + \frac{1}{2} \right] \right). \quad (5.12)
\end{aligned}$$

Lastly, the terms are rearranged and all dimensionless numbers are identified. The dimensionless equation corresponding to the torsional motion of the system is as shown in Eq. (5.13):

$$\begin{aligned}
\frac{\tilde{h}}{c/2} \left(-x_\alpha - \frac{\Lambda_1}{4\lambda} - \frac{i(1/2 + \Lambda_1)C(k)}{2\lambda k} \right) + \tilde{\alpha} \left(r_\alpha^2 \left(-1 + \left(\frac{k_\alpha}{k} \right)^2 (1 + ig_\alpha) \right) \right) \\
+ \frac{i(1/2 - \Lambda_1)}{4\lambda k} - \frac{(1/8 + \Lambda_1^2)}{4\lambda} - \frac{(1/2 + \Lambda_1)C(k)}{2\lambda k^2} - \frac{C(k)i(1/4 - \Lambda_1^2)}{2\lambda k} \Bigg) = \frac{\tilde{\omega}_G}{U_\infty} \frac{S(k)}{2\lambda k^2} \left[\Lambda_1 + \frac{1}{2} \right]. \quad (5.13)
\end{aligned}$$

Assembling the whole model as a matrix system, the correlation matrix between the degrees-of-freedom and the atmospheric turbulence model, A Eq. (5.14) is obtained and the result is as shown in Eq. (5.14):

$$\begin{bmatrix} A_{hh} & A_{h\alpha} \\ A_{h\alpha} & A_{\alpha\alpha} \end{bmatrix} \begin{Bmatrix} \tilde{h}/\frac{c}{2} \\ \tilde{\alpha} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tilde{w}_G, \quad (5.14)$$

where A_{hh} , $A_{h\alpha}$, $A_{\alpha h}$ and $A_{\alpha\alpha}$ are the algebraic, frequency (ω) dependant, terms that correlate the degrees-of-freedom of the system with the external loads, the turbulence model.

$$A_{hh} = \frac{\left(-1 + \left(\frac{k_h}{k} \right)^2 (1 + ig_h) - \frac{1}{4\lambda} + \frac{iC(k)}{2\lambda k} \right) U_\infty}{-\frac{S(k)}{2\lambda k^2}}, \quad (5.15)$$

$$A_{h\alpha} = \frac{\left(-x_\alpha + \frac{i}{4\lambda k} + \frac{\Lambda_1}{4\lambda} + \frac{C(k)}{2\lambda k^2} + \frac{iC(k)(1/2 - \Lambda_1)}{2\lambda k}\right) U_\infty}{-\frac{S(k)}{2\lambda k^2}}, \quad (5.16)$$

$$A_{\alpha h} = \frac{\left(-x_\alpha - \frac{\Lambda_1}{4\lambda} - \frac{i(1/2 + \Lambda_1)C(k)}{2\lambda k}\right) U_\infty}{\frac{S(k)}{2\lambda k^2} \left[\Lambda_1 + \frac{1}{2}\right]}, \quad (5.17)$$

$$A_{\alpha\alpha} = \frac{\left(r_\alpha^2 \left(-1 + \left(\frac{k_\alpha}{k}\right)^2 (1 + ig_\alpha)\right) + \frac{i(1/2 - \Lambda_1)}{4\lambda k} - \frac{(1/8 + \Lambda_1^2)}{4\lambda} - \frac{(1/2 + \Lambda_1)C(k)}{2\lambda k^2} - \frac{C(k)i(1/4 - \Lambda_1^2)}{2\lambda k}\right) U_\infty}{\frac{S(k)}{2\lambda k^2} \left[\Lambda_1 + \frac{1}{2}\right]}. \quad (5.18)$$

5.2 Transfer matrix

A special mention shall be given to the transfer function or, more concretely for the case at hand, the transfer matrix. The transfer matrix method is widely spread over engineering, as it can be used for the analysis of wave propagation of all kinds. For coupled mechanical systems, this method allows the calculation of the response of the aforementioned system without the need to work on its modal space (the modal space where the system is decoupled, the base of this space are the eigenvectors of the system). Transfer functions or matrices relate the system's response to the input signal in a such a way that the system response is obtained as the product of the input signal and the transfer matrix. Hence, mathematically the response of the system will have the same form as the input.

5.2.1 Matrix assembly

Up to this point, both the structural, aeroelastic model and the aerodynamic loads have been obtained. Thus the obtention of the transfer matrix is the next step forward. The previously explained adimensionalization process is highly valuable as it allows analysis of a wide range of cases, without having to significantly modify the equations. Shown in Figure 5.1 is a scheme representing the considerations of the system subjected to study. Transfer matrices include all relevant information about the system.

Considerations of the model are taken into account in the transfer matrix. In particular, the transfer matrix of the two degree-of-freedom airfoil subjected to unsteady aerodynamics includes all the abovementioned constraints. Therefore, this matrix is obtained through the equations of motion of the airfoil, Eqs. (5.9) and (5.13). The transfer function or matrix can be obtained both for systems where harmonic motion can be assumed and non-dynamic systems, static equilibrated systems. Given that the system subjected to study foresees that the response is harmonic, the transfer matrix is then obtained:

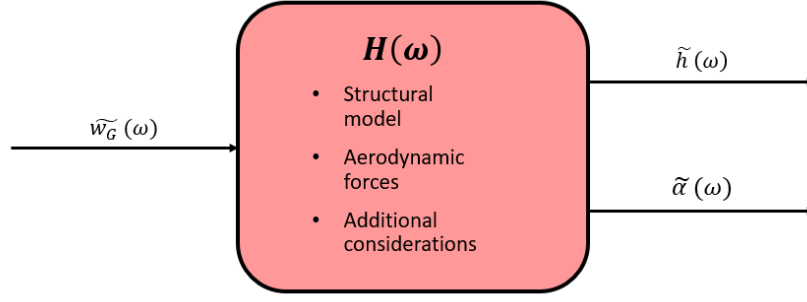


Figure 5.1: Transfer matrix

$$\begin{Bmatrix} \tilde{h}/\frac{c}{2} \\ \tilde{\alpha} \end{Bmatrix} = \begin{bmatrix} H_{hh} & H_{h\alpha} \\ H_{h\alpha} & H_{\alpha\alpha} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tilde{w}_G. \quad (5.19)$$

As can be easily identified, the matrix resulting from inverting the previously explained matrix A , will be the dimensionless transfer matrix and it is designated as H (5.20):

$$H(i\omega) = A(i\omega)^{-1}. \quad (5.20)$$

In Eq. (5.19), the terms H_{hh} , $H_{h\alpha}$, $H_{h\alpha}$ and $H_{\alpha\alpha}$ are the terms that correlate the turbulence model with the system's response (degrees of freedom). These terms are as shown in Eqs. (5.21)-(5.24), where $\Lambda_A = A_{hh}A_{\alpha\alpha} - A_{h\alpha}A_{\alpha h}$ is considered to be the determinant of A , the correlation matrix:

$$H_{hh} = \frac{\left(r_\alpha^2 \left(-1 + \left(\frac{k_\alpha}{k} \right)^2 (1 + ig_\alpha) \right) + \frac{i(1/2 - \Lambda_1)}{4\lambda k} - \frac{(1/8 + \Lambda_1^2)}{4\lambda} - \frac{(1/2 + \Lambda_1)C(k)}{2\lambda k^2} - \frac{C(k)i(1/4 - \Lambda_1^2)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2} \left[\Lambda_1 + \frac{1}{2} \right] \Lambda_A}, \quad (5.21)$$

$$H_{h\alpha} = - \frac{\left(-x_\alpha + \frac{i}{4\lambda k} + \frac{\Lambda_1}{4\lambda} + \frac{C(k)}{2\lambda k^2} + \frac{iC(k)(1/2 - \Lambda_1)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2} \Lambda_A}, \quad (5.22)$$

$$H_{\alpha h} = - \frac{\left(-x_\alpha - \frac{\Lambda_1}{4\lambda} - \frac{i(1/2 + \Lambda_1)C(k)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2} \left[\Lambda_1 + \frac{1}{2} \right] \Lambda_A}, \quad (5.23)$$

$$H_{\alpha\alpha} = \frac{\left(-1 + \left(\frac{k_h}{k} \right)^2 (1 + ig_h) - \frac{1}{4\lambda} + \frac{iC(k)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2} \Lambda_A}. \quad (5.24)$$

The relation between the degrees-of-freedom and the external harmonic loads is easily obtained from Eq. (5.19):

$$\frac{\tilde{h}}{c/2} = H_{hh} \frac{\tilde{w}_G}{U_\infty} + H_{h\alpha} \frac{\tilde{w}_G}{U_\infty}, \quad (5.25)$$

$$\tilde{\alpha} = H_{h\alpha} \frac{\tilde{w}_G}{U_\infty} + H_{\alpha\alpha} \frac{\tilde{w}_G}{U_\infty}. \quad (5.26)$$

It is important to note that the transfer matrix that has been obtained correlates the displacement generalized coordinates and the turbulence model as shown in Eq. (5.19). From now on, this matrix will be referred to as $H^{(0)}(i\omega)$. Nonetheless, it is usual to find that the design requirements establish a limit on the structural loads that the aircraft is subjected to. The value of these loads can be obtained by calculating the transfer matrix which allows the determination of the values of acceleration undergone by the system:

$$\frac{\ddot{q}(t)}{w_G} = \frac{U_\infty^2}{(c/2)^2} \frac{\ddot{q}(s)}{w_G} = -\frac{U_\infty^2}{(c/2)^2} k^2 \frac{\tilde{q}(k)}{\tilde{w}_G}. \quad (5.27)$$

Taking into account the relation between the frequency domain response and the second derivative of a generalized coordinate shown in Eq. (5.27), the transfer matrix for the acceleration of the system is as established in Eq. (5.28):

$$H^{(2)}(i\omega) = -\frac{U_\infty^2}{(c/2)^2} k^2 H^{(0)}(i\omega). \quad (5.28)$$

From now on, all $H(i\omega)$ and associated variables, refer to the acceleration transfer function.

5.3 Bending and torsion response

The study of the aeroelastic response to atmospheric turbulence is very relevant in both the frequency and time domain. This is because in the frequency domain an exhaustive and efficient study of this response is available. On the other hand, the aircraft response to turbulence is really significant in the time plane. The aeroelastic response must comply with certain design parameters imposed by the standards in force. These are mainly reflected in the maximum loads that the structure can withstand, i.e., a maximum acceleration. Moreover, the main advantage that the study of this phenomena in the frequency domain poses is that it allows the analysis of the response in a wide spectrum of frequencies. Whereas in the time domain this task would be very tedious, the coverage of a broad spectrum of frequencies can prevent undesired or unexpected behaviours of the structure.

This section describes the calculation of the aeroelastic response to turbulence in the frequency domain, the transformation of this response to the time domain and the description of the decoupled model and the rigid body motion of the airfoil.

5.3.1 Frequency domain

As explained in Chapter 2, the response of an aircraft to atmospheric turbulence is most efficiently obtained as the response of a system to a random excitation, which is determined

by obtaining the Power Spectral Density (PSD) of each of the generalized coordinates [24].

The PSD of both bending and torsional modes can be calculated as shown in Eqs. (5.29) and (5.30):

$$PSD_{\ddot{h}}(\omega) = |H_{hG}(i\omega)|^2 PSD_G(\omega), \quad (5.29)$$

$$PSD_{\ddot{\alpha}}(\omega) = |H_{\alpha G}(i\omega)|^2 PSD_G(\omega), \quad (5.30)$$

where $|H_{hG}(i\omega)|$ and $|H_{\alpha G}(i\omega)|$ stand for the sum in absolute value of the terms of the transfer matrix as shown in Eqs. (5.31) and (5.32):

$$|H_{hG}(i\omega)| = |H_{hh}(i\omega) + H_{h\alpha}(i\omega)|, \quad (5.31)$$

$$|H_{\alpha G}(i\omega)| = |H_{\alpha h}(i\omega) + H_{\alpha\alpha}(i\omega)|. \quad (5.32)$$

As previously exposed, the mean value of the bending and torsional response's amplitude is near zero. Thus, the values are squared to produce a positive quantity. To obtain a linear value, a square root is applied to the mean square. This calculation results in the root-mean-square (RMS) value.

Although the results will be explained in the following Chapter, shown in Figure 5.2 are the PSDs for a specific example of the present study.

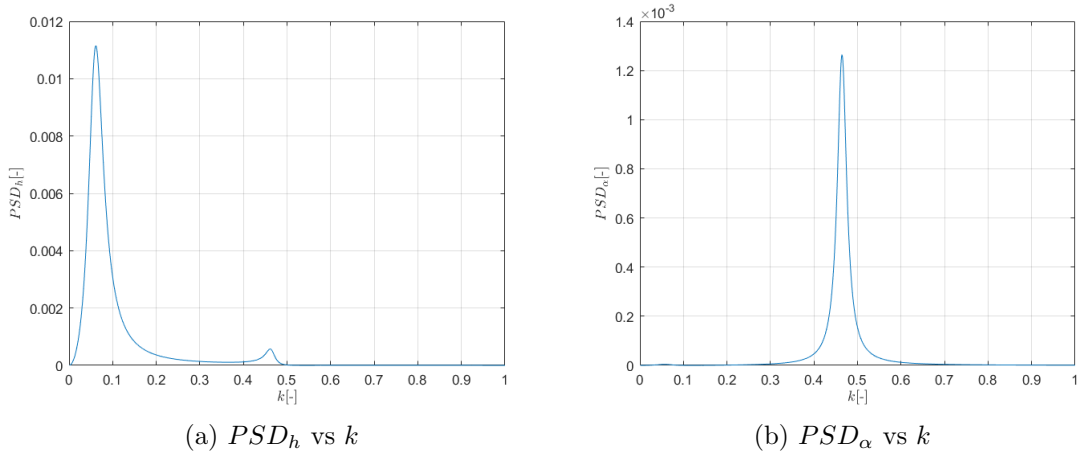


Figure 5.2: PSD of bending and torsional response

5.3.2 Time domain

As can be expected, the response of an airfoil to atmospheric turbulence is extremely important when evaluating the safety and reliability of an aircraft. Turbulence has an impact on the aircraft's structure which can be displayed through direct visual damage to the structure (failure of the structure, fatigue implications, etc.). As a result, it becomes clear that turbulence must be prevented or contained. Physically the aircraft encountered with turbulence is subjected to a series of rapidly-changing loads. Through the model that has been developed in this project, what can be evaluated is the extent of the impact of atmospheric turbulence on the aircraft. This can be achieved through the determination of the standard deviation of the acceleration turbulence causes on the aircraft's structure.

As exposed in Chapter 2, atmospheric turbulence is modeled according to a series of properties which allow its analysis through a Gaussian distribution, which is the normal

distribution of probability. The normal distribution is characterized by its symmetry about the mean, showing that data near the mean are more frequent in occurrence than data far from the mean. The Gaussian distribution can be particularized by the value of the mean (μ , which refers to the arithmetic average value of the variable, as stated in Chapter 2, for the case at hands it is null) and its standard deviation (σ , which refers to the measure of the amount of variation or dispersion of a set of values).

Thus, the need to obtain the standard deviation of the output signals becomes clear. The methodology implemented is the conversion of the PSD from the frequency domain to the time domain. According to the definition of the PSD laid out in Eq. (2.3), it represents the variance of a signal in the frequency domain. At the same time, it is known that the square root of the variance results in the standard deviation.

Even though the PSD represents the variance of the signal in the frequency domain, the method used to obtain said value in the time domain is not the Fourier transform. This is due to the different nature of both the PSD and the variance. Meanwhile, for two different frequencies there are two different values of the PSD, it is not the same with the variance or the standard deviation. The reason for this is the stationarity of the statistical values of the response. For a whole spectrum of frequencies a sole value of the variance or standard deviation is obtained. Recalling the relation between the variance and the PSD:

$$\langle x \rangle^2 = \sigma_x^2 = \int_{-\infty}^{\infty} PSD(\omega) d\omega, \quad (5.33)$$

it can be observed that the value of the variance corresponds to the area below the function delimited by the PSD variation over frequency. Shown in Figure 5.3 is the representation of said areas that constitute the variances for each degree-of-freedom.

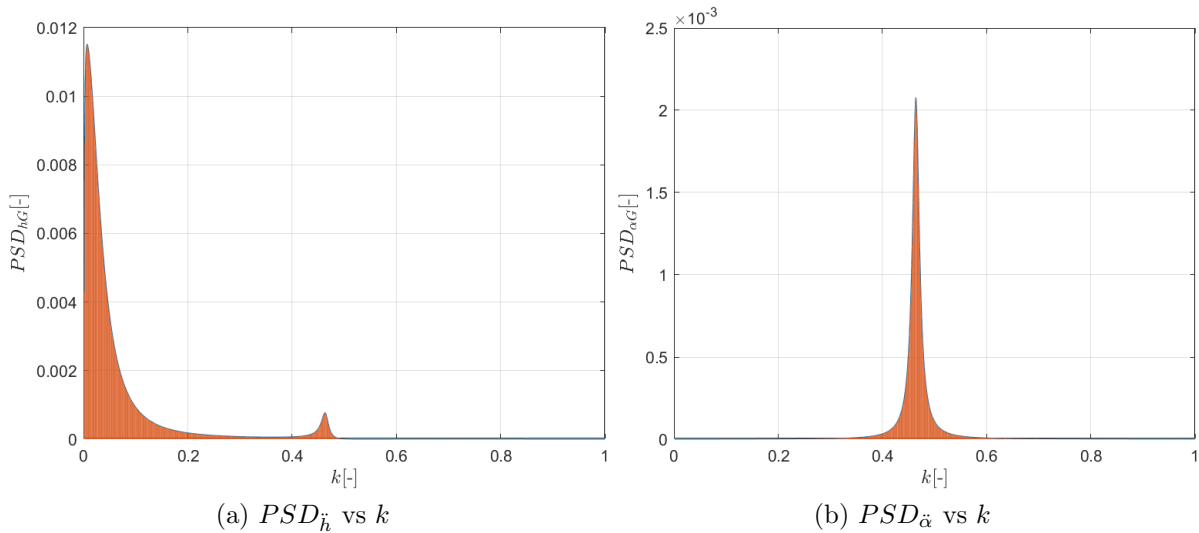


Figure 5.3: Graphical representation of the variances of the model

The obtention of these variances and standard deviations results in the obtention of the normal distribution for each degree-of-freedom. In the following chapter, an analysis of the results obtained will be carried out. Nonetheless, the resulting normal distributions for a specific example obtained through this study is shown in Figure 5.4.

Design requirements regarding turbulence

This section aims to explain the design criteria imposed by the current regulatory body, EASA (European Aviation Safety Agency). EASA regulates many aspects related to

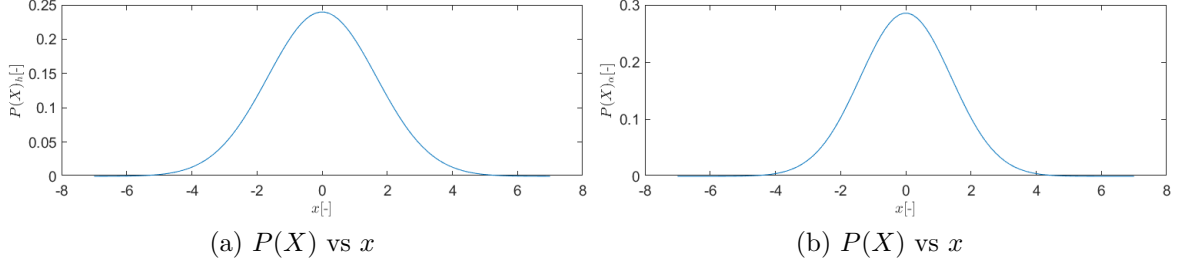


Figure 5.4: Probability distribution of the bending and torsional response to atmospheric turbulence

aviation and aeronautics. Specifically in the regulatory part CS-25, the certification specifications required for large aircrafts are exposed. Specific criteria regarding gusts and turbulence loads are laid out in *CS 25.341* [12].

Regarding turbulence, as it is not possible to obtain the maximum and minimum loads induced, the method used to ensure that the aircraft does not suffer structural damage is to determine a limit load.

The aforementioned limit load shall be used to calculate the probability of its exceedance, if an aircraft is encountered with turbulence. This ensures that the loads the structure has to withstand will not cause structural damage that would compromise the proper performance of the aircraft and that the certified maneuvering envelope is not exceeded.

5.4 Decoupled models

This Section presents the two decoupled models for the airfoil subjected to atmospheric turbulence, for the bending and torsional motions. As the present work aims to study the effect of the coupling on the response to turbulence, it is considered interesting to compare the results of the coupled model with those of the decoupled.

5.4.1 Bending

This subsection presents the Eqs. describing the bending motion of the decoupled system. It is important to understand that the coupling present in the model is derived from the structural model presented in Chapter 3 and the aerodynamic model explained in Chapter 4. As a result, the decoupling of the model is carried out by nullifying the $H_{h\alpha}$:

$$H_{hh} = \frac{\left(r_\alpha^2 \left(-1 + \left(\frac{k_\alpha}{k} \right)^2 (1 + i g_\alpha) \right) + \frac{i(1/2 - \Lambda_1)}{4\lambda k} - \frac{(1/8 + \Lambda_1^2)}{4\lambda} - \frac{(1/2 + \Lambda_1)C(k)}{2\lambda k^2} - \frac{C(k)i(1/4 - \Lambda_1^2)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2} \left[\Lambda_1 + \frac{1}{2} \right] \Lambda_A}, \quad (5.34)$$

$$H_{h\alpha} = 0. \quad (5.35)$$

The Eqs. (5.34) and (5.35) describe the bending motion for the decoupled model.

5.4.2 Torsion

This subsection presents the Eqs. describing the torsion motion of the decoupled system. The procedure followed for the obtention of the torsion motion, for the decoupled model, is the same used for the bending motion, nullifying $H_{\alpha h}$.

$$H_{\alpha h} = 0, \quad (5.36)$$

$$H_{\alpha\alpha} = \frac{\left(-1 + \left(\frac{k_h}{k} \right)^2 (1 + ig_h) - \frac{1}{4\lambda} + \frac{iC(k)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2}} \cdot \frac{1}{\Lambda_A}. \quad (5.37)$$

The Eqs. (5.36) and (5.37) describe the torsional motion for the decoupled model.

5.5 Rigid body motion

This Section presents the Eqs. of the two degrees-of-freedom model, where the bending natural frequency is null. It is important to note that this model is obtained by introducing the value for the natural frequency $\omega_h = 0$.

$$H_{hh} = \frac{\left(r_\alpha^2 \left(-1 + \left(\frac{k_\alpha}{k} \right)^2 (1 + ig_\alpha) \right) + \frac{i(1/2 - \Lambda_1)}{4\lambda k} - \frac{(1/8 + \Lambda_1^2)}{4\lambda} - \frac{(1/2 + \Lambda_1)C(k)}{2\lambda k^2} - \frac{C(k)i(1/4 - \Lambda_1^2)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2} \left[\Lambda_1 + \frac{1}{2} \right]} \cdot \frac{1}{\Lambda_A}, \quad (5.38)$$

$$H_{h\alpha} = - \frac{\left(-x_\alpha + \frac{i}{4\lambda k} + \frac{\Lambda_1}{4\lambda} + \frac{C(k)}{2\lambda k^2} + \frac{iC(k)(1/2 - \Lambda_1)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2}} \cdot \frac{1}{\Lambda_A}, \quad (5.39)$$

$$H_{\alpha h} = - \frac{\left(-x_\alpha - \frac{\Lambda_1}{4\lambda} - \frac{i(1/2 + \Lambda_1)C(k)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2} \left[\Lambda_1 + \frac{1}{2} \right]} \cdot \frac{1}{\Lambda_A}, \quad (5.40)$$

$$H_{\alpha\alpha} = \frac{\left(-1 - \frac{1}{4\lambda} + \frac{iC(k)}{2\lambda k} \right) U_\infty}{\frac{S(k)}{2\lambda k^2}} \cdot \frac{1}{\Lambda_A}. \quad (5.41)$$

Eqs. (5.38), (5.39), (5.40) and (5.41) show the Eqs. of motion of the rigid body model, considering that the hypothesis, $\omega_h = 0$, must also be introduced in the determinant of the matrix A , Λ_A .

Chapter 6

Results

The main purpose of this Chapter is the analysis and discussion of the results obtained through the work. The following results will be analyzed and deeply explained:

- the incremental pressure coefficient produced by a harmonic motion, $\Delta\tilde{C}_{p1}^G(\hat{x}, k)$;
- the dimensionless aerodynamic forces induced by a harmonic motion, $\tilde{Q}_h(k)$ and $\tilde{Q}_\alpha(k)$;
- the transfer acceleration functions for the bending and torsional motions, $|H_{hG}|^2$ and $|H_{\alpha G}|^2$;
- the Power Spectral Densities of the response, PSD_h and PSD_α ;
- and the standard deviations of both of the bending and torsional degrees-of-freedom.

All the results presented in this Chapter have been obtained from the model developed in the previous chapters, using the methods which will be described in the Section 6.1, which are based on the computational implementation of the model.

Furthermore, these results will be studied for different test cases, since it is very interesting to compare them by varying different parameters that define each case. It is intended to study the influence of geometrical parameters and those associated with them (such as wingspan, airfoil chord, airspeed, airplane ceiling, etc) and mass parameters (the mass of the airplane and the static inertia product, responsible in the structural model for the coupling between bending and torsion). In addition, during the flight, an aircraft suffers mass variations continuously due to fuel consumption. Therefore, it is interesting to study how the variation of the mass will impact the response of the aircraft to turbulence. In addition, this mass variation will be reflected on those parameters which are dependant on the aircraft's mass, such as r_α^2 and x_α . Whereas both parameters will be subjected to changes, it has been considered more relevant the impact of the variation of the x_α parameter. The reason for this is that the project aims to study the effect of the coupling of both the bending and torsional generalized coordinates and reflect to which extent the coupling can vary the response of the aircraft. As a consequence the variation of the x_α parameter can provide a wider understanding of the impact of this coupling. For this reason, the values that have been selected for x_α consider not only variations in the mass of the aircraft, but also variations of the position of the elastic axis. It is known that this variation corresponds to varying the airfoil, but it is considered that this information is relevant for the present study. In the Test Cases section, the chosen cases are specified: the Boeing B737-300 aircraft and the Cessna 172.

Additionally, the results will be analyzed and compared with two additional models: the model developed in this work decoupled and studied only for bending; and an analytical approach of one degree of freedom of rigid solid, presented in [14].

6.1 Test Cases

In this section the main Test Cases of study the results of the work are presented. Two aircrafts with very different geometrical and mass characteristics will be used, whose response to turbulence due to these differences may result in an interesting comparison. First, the main parameters of the B737-300 [25] aircraft will be explained and presented. Second, the same parameters particularized for the Cessna 172 [25] will be presented. Finally, the analytical one degree-of-freedom model for rigid solid will be explained.



Figure 6.1: Boeing 737-300 and Cessna 172

The parameters that have been selected for both aircrafts, for the purpose of comparison. For the rigid body motion, the damping does not intervene in the model are shown in Table 6.1.

Table 6.1: General parameters

Bending natural frequency, ω_h	5 rad/s
Torsional natural frequency, ω_α	30 rad/s
Bending damping coefficient, g_h	0.01 -
Torsional damping coefficient, g_α	0.01 -
Dimensionless torsion inertia, r_α^2	0.75 -

The values established for the natural frequencies present a lower value for the bending one, rather than the torsional, as found in [26]. In addition, it is considered that these values are distanced enough to appreciate the contributions of each mode to the response of the airfoil. The value selected for the r_α^2 was chosen with the same criteria. Furthermore, these values have been selected for both aircrafts for the purpose of comparison between these. Moreover, the values selected for the damping coefficients are representative as these consider a damping of 1%.

6.1.1 Boeing 737-300

The Boeing 737-300 is a narrow-body airliner produced by Boeing Commercial Airplanes. Single-aisled and twin-engined this airliner is a medium range aircraft (around 4000 km of range) and is considered to be a large aeroplane. The main characteristics that must be taken into account regarding the impact of turbulence on this aircraft are shown in Table 6.2.

Table 6.2: Boeing 737-300 Specifications

Chord of the 3/4 airfoil of the wing, c	4.71 m
Wingspan, b	28.88 m
Maximum Take-Off Weight, $MTOW$	56470 kg
Maximum Landing Weight, MLW	51720 kg
Maximum Zero Fuel Weight, $MZFW$	47625 kg
Cruise Speed, U_∞	253.3 m/s

For both aircrafts, the speed selected is the cruise speed, as it is deemed interesting to frame the study of the aeroelastic response to turbulence in a realistic flight condition, in terms of velocities. As can be seen in previous Chapters, the aircraft flight speed is involved in multiple parameters and the aircraft response has a significative variation with speed. Therefore, the cruising speed has been estimated to represent a sufficiently representative case. These speeds have been extracted from the book *Janes* [25]. As previously explained, the implementation of dimensionless parameters provides a more efficient means of studying the model and allows for a clearer understanding of the physical phenomenon of turbulence. Comparison between different cases is thus rendered simpler and more explanatory.

Table 6.3 shows the dimensionless parameters that will be used for the study of the aeroelastic response of the Boeing 737-300 to atmospheric turbulence.

Table 6.3: Boeing 737-300 Dimensionless parameters

$\lambda[-]$	$\lambda_1 = 22.9030$	$\lambda_2 = 22.4285$	$\lambda_3 = 21.9398$	$\lambda_4 = 20.9765$
$x_\alpha[-]$	$x_{\alpha1} = -0.5$	$x_{\alpha2} = -0.25$	$x_{\alpha3} = 0.25$	$x_{\alpha4} = 0.5$

The values selected for λ correspond to a series of masses, that have been considered representative, starting from the $MTOW$ which is associated with λ_1 and the last one, λ_4 , corresponds to the MLW . The values selected for the x_α parameter, which represent the dimensionless distance between the EA and the CG, convey different coupling scenarios relevant for the present work.

6.1.2 Cessna 172

The Cessna 172 is a four-seat, high-wing, single-engine aircraft with fixed tricycle landing gear, designed primarily for general aviation purposes. Comparing the Cessna 172 with Boeing 737-300, it is observed that both mass-related and geometrical parameters differ

greatly. The comparison between these two aircrafts is interesting with the purpose of understanding what the variation of structural and mass-related parameters entails.

Table 6.4: Cessna 172 Specifications

Chord of the 3/4 airfoil of the wing, c	1.472 m
Wingspan, b	10.90 m
Maximum Take-Off Weight, $MTOW$	1156.661 kg
Maximum Landing Weight, MLW	1156.661 kg
Maximum Zero Fuel Weight, $MZFW$	762.0352 kg
Cruise Speed, U_∞	57.1033 m/s

The dimensionless parameters utilized for investigating the aeroelastic response of the Cessna 172 to atmospheric turbulence are presented in Table 6.5.

Table 6.5: Cessna 172 Dimensionless parameters

$\lambda[-]$	$\lambda_1 = 16.1885$	$\lambda_2 = 15.3954$	$\lambda_3 = 13.9959$	$\lambda_4 = 12.5963$
$x_\alpha[-]$	$x_{\alpha1} = -0.5$	$x_{\alpha2} = -0.25$	$x_{\alpha3} = 0.25$	$x_{\alpha4} = 0.5$

As explained for the B737, the values selected for λ descend starting from the $MTOW$ associated to λ_1 and the last one, λ_4 , corresponds to the 900 kg. The values selected for the x_α parameter, convey different coupling scenarios relevant for the present work.

6.1.3 Analytical one degree-of-freedom model

An approach of the turbulence response for the rigid body motion is developed in this section [14]. Comparisons to this model will be conducted with the null value of the natural bending frequency, in the formulation developed by the present author.

All equations or formulæ pertaining to the analytical model will be denoted with the letter a , i.e $H_a^{(2)}(\omega)$. The transfer acceleration function for the described model is shown in Eq. (6.1):

$$H_a^{(2)}(\omega) = \frac{U_\infty}{\left(\frac{c}{2}\right)^2} \frac{S(k)}{2 \left[\lambda + \frac{1}{4} - i \frac{C(k)}{2\lambda k} \right]} = \frac{\ddot{h}/c/2}{\tilde{w}_G}. \quad (6.1)$$

Moreover, the variance of the acceleration of this model is as shown in Eqs. (6.2) - (6.4):

$$\langle \ddot{h} \rangle = \frac{4U_\infty^2 \langle w_G^2 \rangle}{\pi \left(\frac{c}{2}\right)^2} \frac{1}{(4\lambda + 1)^2} \left[\frac{a^2(3a^2 - 1)}{(a^2 - 1)^2} P_1(a, r_L) - \frac{5a^2 - 3}{(a^2 - 1)^2} P_1(a, r_L) + \frac{2}{a^2 - 1} P_2(a, r_L) \right], \quad (6.2)$$

where

$$P_1(a, r_L) = \frac{2\pi r_L}{1 + 4\pi^2 r_L^2 a^2} \left(\ln 2\pi r_L + \ln a + \frac{1}{4ar_L} \right), \quad (6.3)$$

$$P_2(a, r_L) = \frac{8\pi^3 r_L^3}{(1 + 4\pi^2 r_L^2 a^2)^2} \left(\ln 2\pi r_L + \ln a + \frac{1}{4ar_L} \right) + \frac{\pi r_L}{a^2 (1 + 4\pi^2 r_L^2 a^2)} \left(\frac{1}{4ar_L} - 1 \right). \quad (6.4)$$

Auxiliar parameters were defined to simplify the mathematical instrument developed for the obtention of the result, these are:

$$r_L = c/(2L),$$

and

$$a = 2/[r_L(2\lambda + 1)].$$

For the obtention of the solution, the approximations shown in Eqs. (6.5) and (6.6) were adopted. It is important to mention that, as a numerical inaccuracy could arise from these approximations.

$$|S(k)|^2 \approx (1 + 2\pi k)^{-1}, \quad (6.5)$$

$$|k(4\lambda + 1) - 2iC(k)|^2 \approx 4 + k^2(4\lambda + 1)^2. \quad (6.6)$$

6.2 Methodology

The purpose of this section is to briefly explain the computation method and to obtain the results of the turbulence response model presented in this project.

6.2.1 Computational implementation

The implementation of this study has been carried out computationally, using MATLAB.

The methodology adopted for this project encompasses key stages aimed at modeling aircraft responses to turbulence. It commences by discretizing continuous functions through the definition of discrete reduced frequency, (k) , values, which lays the foundation for subsequent matrix manipulations. These manipulations yield transfer coefficients, capturing intricate relationships governing diverse aeroelastic properties.

It is important to note that the discretization of continuous functions is a central aspect of the code's computations. Through the definition of k values as from a numerical zero to $k = 1$, by the step of 0.001, the continuous relationships transform into discrete increments. This process allows the approximation of the continuous functions, though it's crucial to recognize that the selected step size (0.001 in this case) affects precision.

Moreover, the numerical integration method that has been used for the PSD is the trapezoidal scheme, integrated in MATLAB. As a result, it is assumed that this could interfere with the obtention of the correct, exact number. This will be taken into account when recalling the conclusions of this project.

6.2.2 Outline of the results

The results will be presented in the following way:

- Incremental pressure coefficient induced by a harmonic motion, $\Delta\tilde{C}_{p1}$, Section 6.3.

- Dimensionless unsteady aerodynamic forces, Section 6.4:
 - $\tilde{Q}_h(k)$ and $\tilde{Q}_\alpha(k)$ vs k for both B737 and C172.
 - $\tilde{Q}_h(k)$ and $\tilde{Q}_\alpha(k)$ vs k for B737 changing the mass parameter λ .
 - $\tilde{Q}_h(k)$ and $\tilde{Q}_\alpha(k)$ vs k for C172 changing the mass parameter λ .
- Acceleration transfer matrix, Section 6.5:
 - Bending motion, Section 6.5.1:
 - * $|H|_{hG}^2$ Boeing 737-300: variation of the x_α , λ and the comparison of the coupled and decoupled systems.
 - * $|H|_{hG}^2$ Cessna 172: variation of the x_α , λ and the comparison of the coupled and decoupled systems.
 - Torsional motion, Section 6.5.2:
 - * $|H|_{\alpha G}^2$ Boeing 737-300: variation of the x_α , λ and the comparison of the coupled and decoupled systems.
 - * $|H|_{\alpha G}^2$ Cessna 172: variation of the x_α , λ and the comparison of the coupled and decoupled systems.
 - Rigid body motion, Section 6.5.3:
 - * $|H|_{hG}^{a2}$ Boeing 737-300: variation of the x_α , λ and the comparison of the coupled, decoupled and analytical systems.
 - * $|H|_{hG}^{a2}$ Cessna 172: variation of the x_α , λ and the comparison of the coupled, decoupled and analytical systems.
- Power Spectral Densities, Section 6.6:
 - Bending motion, Section 6.6.1:
 - * PSD_h Boeing 737-300: variation of the x_α , λ and the comparison of the coupled and decoupled systems.
 - * PSD_h Cessna 172: variation of the x_α , λ and the comparison of the coupled and decoupled systems.
 - Torsional motion, Section 6.6.2:
 - * PSD_α Boeing 737-300: variation of the x_α , λ and the comparison of the coupled and decoupled systems.
 - * PSD_α Cessna 172: variation of the x_α , λ and the comparison of the coupled and decoupled systems.
 - Rigid body motion, Section 6.6.3:
 - * PSD_h Boeing 737-300: variation of the x_α , λ and the comparison of the coupled, decoupled and analytical systems.
 - * PSD_h Cessna 172: variation of the x_α , λ and the comparison of the coupled, decoupled and analytical systems.
- Standard deviations, Section 6.7:
 - Boeing 737-300, Section 6.7.1:

- * σ_h Bending motion: variation of the x_α and λ .
 - * σ_h Rigid body motion: variation of the x_α and λ .
 - * σ_α Torsional motion: variation of the x_α and λ .
- Cessna 172, Section 6.7.2:
- * σ_h Bending motion: variation of the x_α and λ .
 - * σ_h Rigid body motion: variation of the x_α and λ .
 - * σ_α Torsional motion: variation of the x_α and λ .

6.3 Pressure coefficient

The unsteady pressure coefficient increment between the intrados and the extrados of the airfoil when it is subjected to a harmonic motion is presented in Figure 6.2.

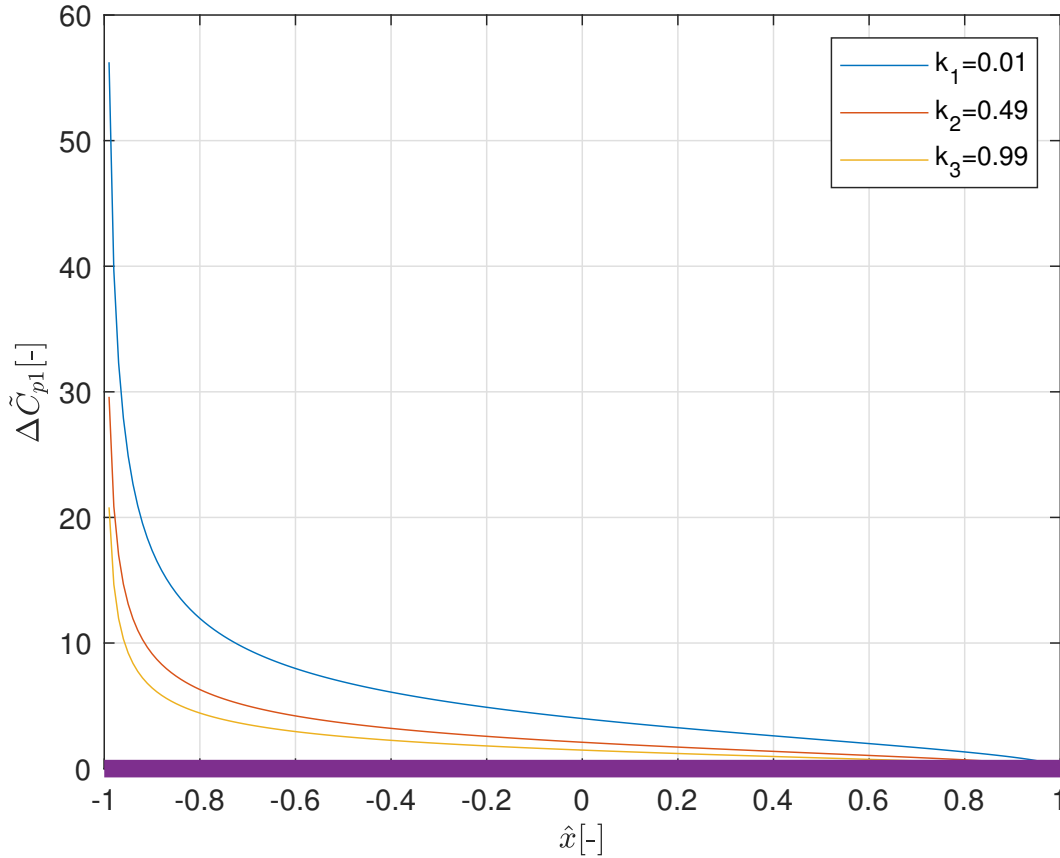


Figure 6.2: $\Delta\tilde{C}_{p1}(\hat{x}, k)$ vs k

The figure shows the dimensionless increase for three different reduced frequencies $k_1 = 0.01$, $k_2 = 0.49$ and $k_3 = 0.99$. These frequencies have been selected because it is considered that these are representative for the selected domain, which is from $k = 0$ to $k = 1$. The bottom line represents the airfoil subjected to study. The aerolastic model does not take into account the thickness, nor the curvature of the airfoil. Thus, it is modelled as a flat plate with the equivalent mass, stiffness and damping of the real airfoil. A discontinuity in the leading edge, where the pressure coefficient trends to infinity, is detected. The discontinuity that can be observed for $\hat{x} = 0$ is due to the modeling of the airfoil as a flat plate.

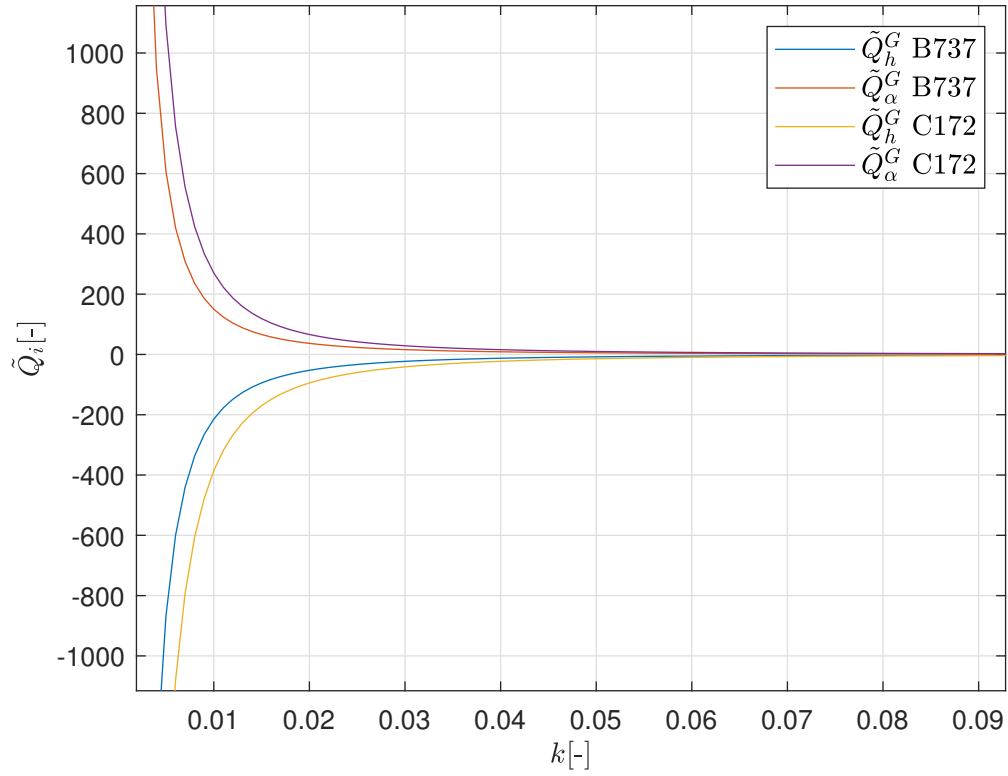
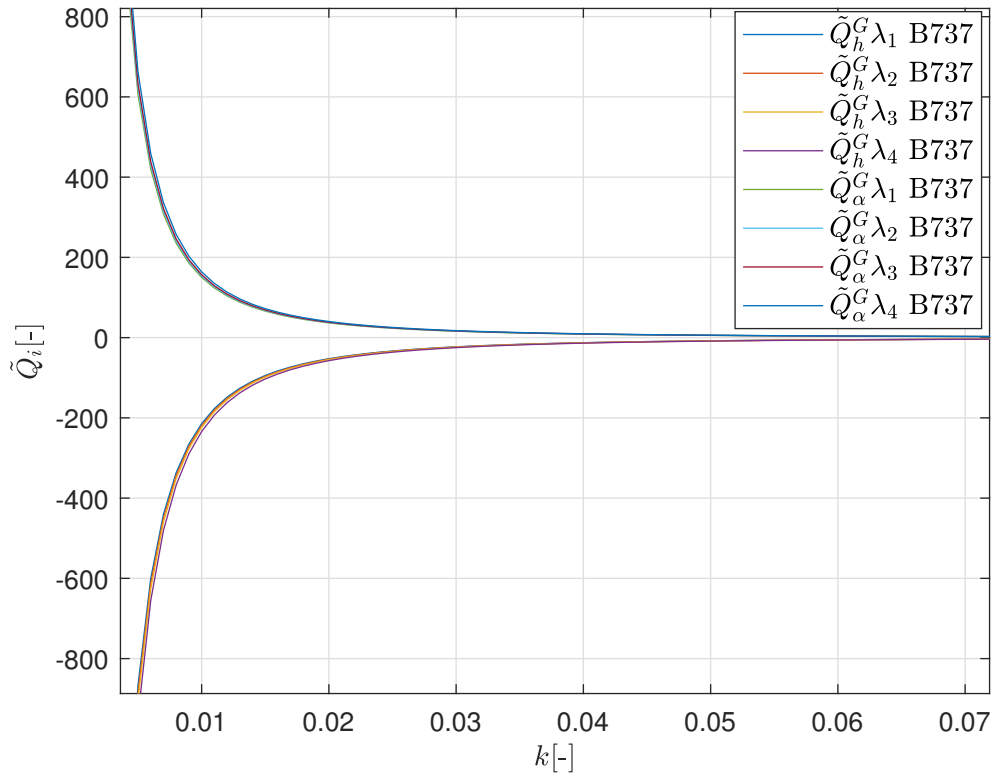
Beyond, it can be observed that the increase of pressures that the profile undergoes when it is subjected to a harmonic motion decreases when the reduced frequency of the motion increases. Additionally, it is shown that it decreases as the $\hat{x}$ coordinate, that is, the dimensionless position relative to the chord of the profile, increases. In addition, the value of the pressure increase nullifies for $\hat{x} = 1$. The cause for this is the introduction of the Kutta hypothesis in the aerodynamic equations leading to the dimensionless pressure coefficient. The Kutta hypothesis contemplates that the pressure jump at the trailing edge of the airfoil is zero [17]. This hypothesis is linked to the theory of vortices and circulations associated with Lift, which can be found further developed in books on aerodynamics [17], [23].

The achievement of this result is of great interest for the present work as well as a relevant contribution. This is because the pressure increase shown in Figure 6.2 is the one suffered by a slender body immersed in a fluid whose motion can be considered harmonic. Therefore, can have different applications beyond the one presented in this work. For example, the motion of fishes immersed in the water [27], flags subjected to unsteady aerodynamics [28], or the flapping motion of the wings of birds [28].

6.4 Unsteady aerodynamic forces

The aerodynamic forces associated with the motion caused by the gust versus the reduced frequency, for both the Boeing 737-300 and the Cessna 172 are presented in Figure 6.3. As can be seen, both bending and torsional generalized forces have a very similar shape, but with the opposite sign, as the dimensionless torsional generalized force is obtained by multiplying the bending expression by $-\left[\Lambda_1 + \frac{1}{2}\right]$. It is worth noting that the dimensionless aerodynamic forces caused by a harmonic motion have far greater impact at low reduced frequencies.

With respect to the difference between the aerodynamic forces that each aircraft is subjected to, it can be observed in Figure 6.3 that the difference is not highly significant. According to the data presented at the beginning of this Chapter, it is known that the mass parameters faced by the C172 airfoil are lower than those of the B737. Therefore, it is observed that for lower mass parameters, the slope of the curve is lower, it will be subjected to higher aerodynamic forces. Moreover, the range of frequencies in which these forces are acting is higher.

Figure 6.3: $\tilde{Q}_h(k)$ and $\tilde{Q}_\alpha(k)$ vs k for both B737 and C172Figure 6.4: $\tilde{Q}_h(k)$ and $\tilde{Q}_\alpha(k)$ vs k for B737 changing the mass parameter λ

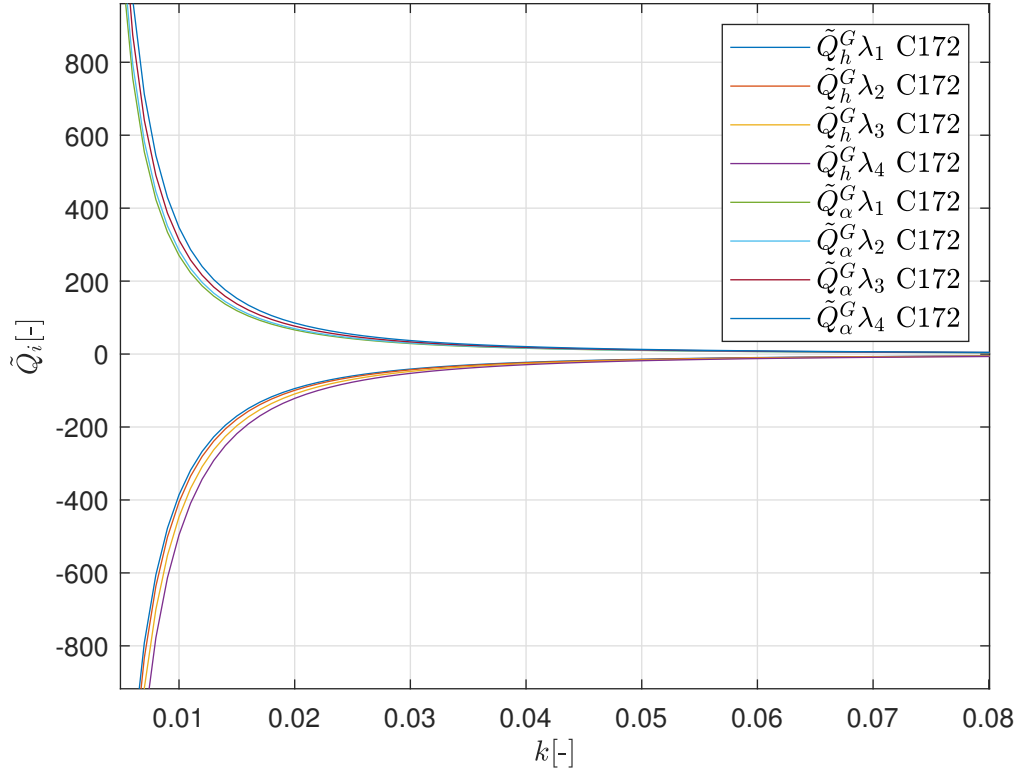


Figure 6.5: $\tilde{Q}_h(k)$ and $\tilde{Q}_\alpha(k)$ vs k for C172 changing the mass parameter λ

Figures 6.4 and 6.5 show the aerodynamic forces caused by a harmonic motion to which each aircraft is subjected. In these figures, the variation of these with the mass parameter λ is shown. As previously discussed, for both cases λ_1 is the mass parameter corresponding to the MTOW of each aircraft. The weights of each aircraft have been selected in such a way as to obtain a descending order of mass parameters, with λ_4 being the lowest in both cases. It can be observed that the slope of the curve decreases with decreasing mass parameter.

6.5 Acceleration transfer matrix

In this Section, the results for the acceleration transfer matrix are presented. The Section is subdivided so as to present first the results of the bending motion (both for B737 and C172) and then the results of the torsional motion (both for B737 and C172).

6.5.1 Bending motion

In this Section, the acceleration transfer matrix, referred to the bending motion for both test cases will be presenting analyzing:

1. $|H|_{hG}^2$ vs k changing x_α .
2. $|H|_{hG}^2$ vs k changing λ .
3. $|H|_{hG}^2$ vs k comparing the coupled and decoupled responses.

Boeing 737-300

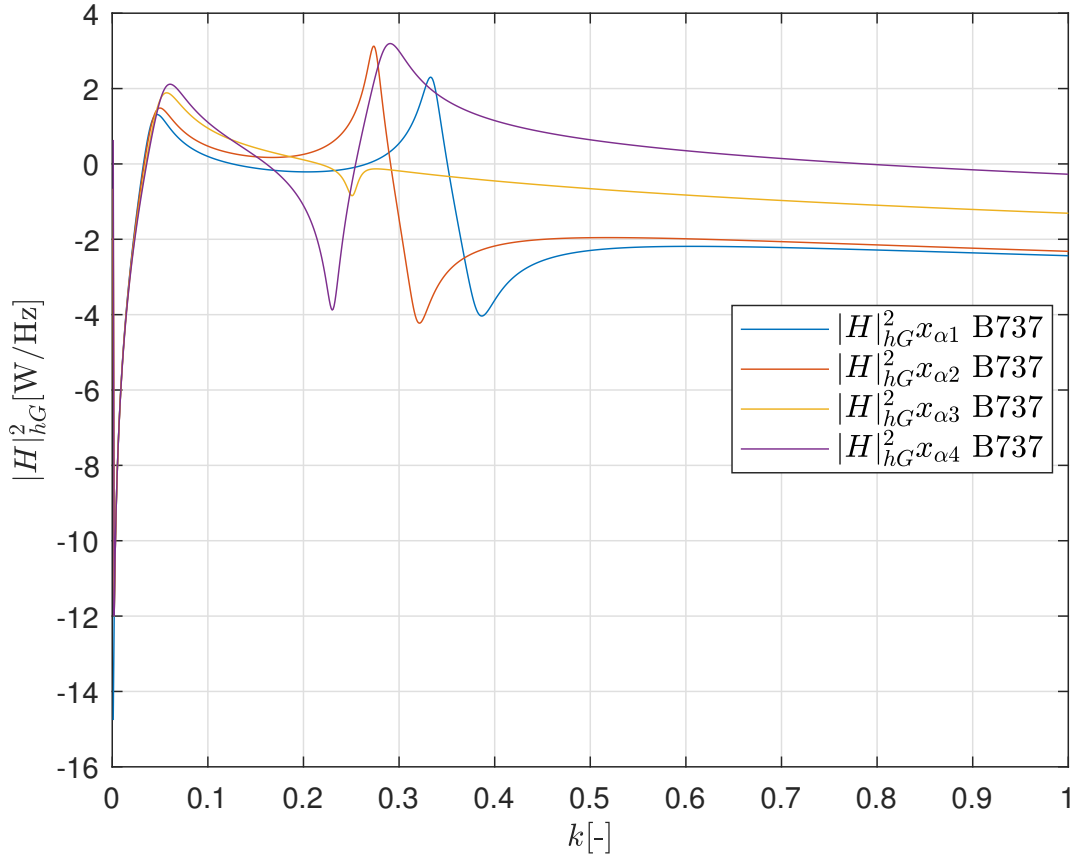
Figure 6.6: $|H|_{hG}^2$ vs k for the B737 changing x_α

Figure 6.6 shows the bending transfer acceleration function in logarithmic scale, changing the x_α parameter, having fixed the mass parameter at λ_1 . First and foremost, it can be seen that, as expected, the response of the airfoil is amplified when the system reaches its natural reduced frequencies $k_h = 0.047$ and $k_\alpha = 0.28$. Depending on the value of the x_α parameter and the geometrical parameters the amplification of the response can happen in lower or higher frequencies. Moreover, the value of this parameter can also affect the amplitude of the response. Nonetheless, it shall be noted that generally the amplitude of the response is greater around the k_α reduced frequency. It is also interesting to point out that when the motion is not in the range that includes the natural reduced frequencies of the system, the response is stable and does not induce great changes in the bending coordinate.

As previously explained, x_α is the dimensionless crossed inertia product, which is responsible for the structural coupling of the system. It can be seen in Figure 6.6 that when $x_\alpha = 0.5$ it seems that the response suffers the greatest variations in amplitude. Not only is the torsional reduced frequency closer to the bending one (making it more probable that the turbulence acts on both), but also the amplitudes for both frequencies are the greatest of all. Moreover, when the dimensionless crossed inertia product is $x_\alpha = 0.25$ the airfoil suffers less changes in the amplitude. In this case the bending response is much greater than the torsional. As seen in Figure 6.6 the aircraft will suffer less loads due to the turbulence for $x_{\alpha 3}$.

For the purpose of comparison of these results with those of the Cessna 172, it must be

noted that the maximum and minimum amplitude peaks are achieved at around $|H|_{hG}^2 = 3$ and $|H|_{hG}^2 = -4$.

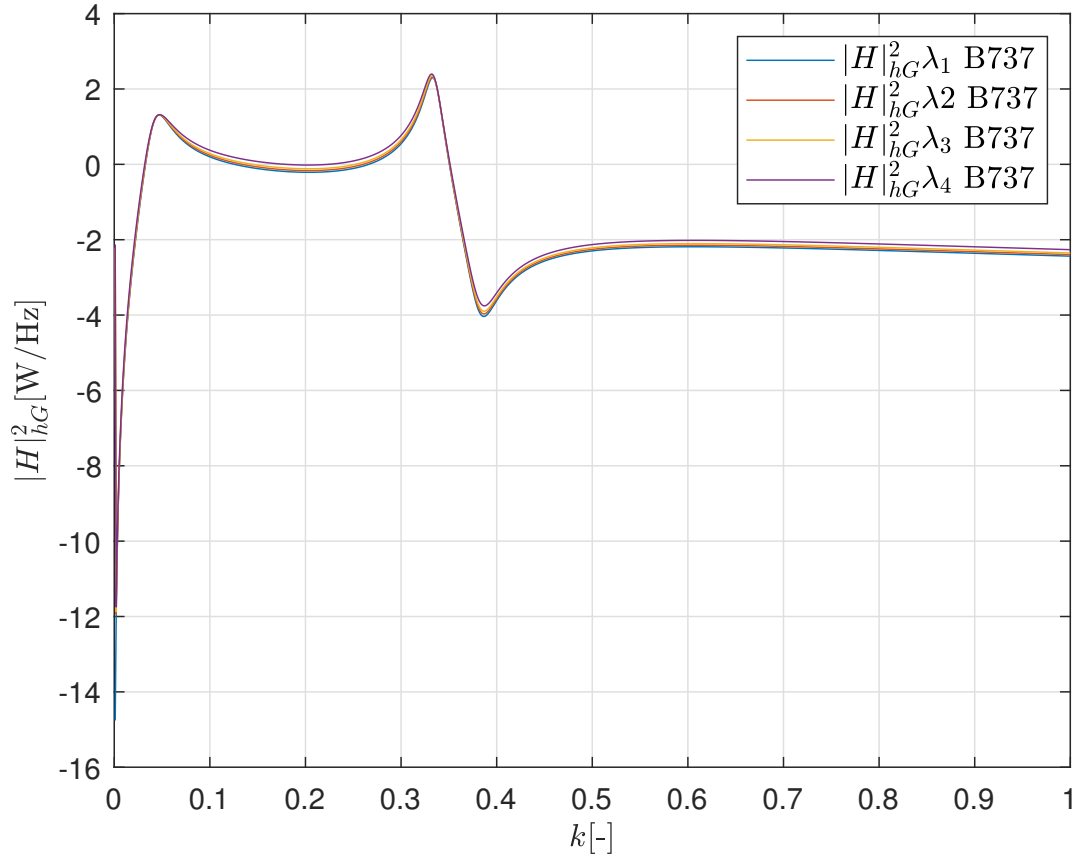


Figure 6.7: $|H|_{hG}^2$ vs k for the B737 changing λ

Figure 6.7 presents the logarithmic-scale bending transfer acceleration function, with the parameter λ being varied while keeping the crossed inertia product parameter fixed at $x_{\alpha 1}$. As can be seen, the variation of λ does not produce a great variation in the bending response of the system. Nonetheless, it can be seen that the least structurally aggressive response corresponds to the greatest mass, MTOW (λ_1).

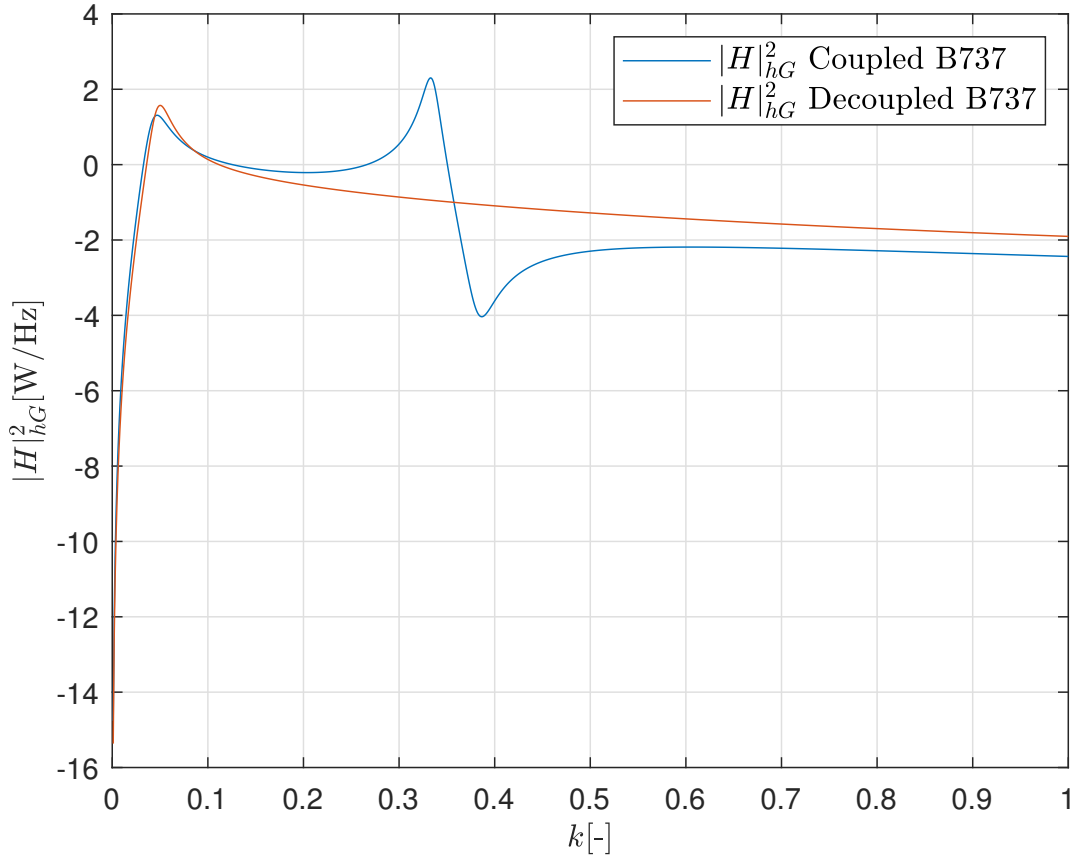
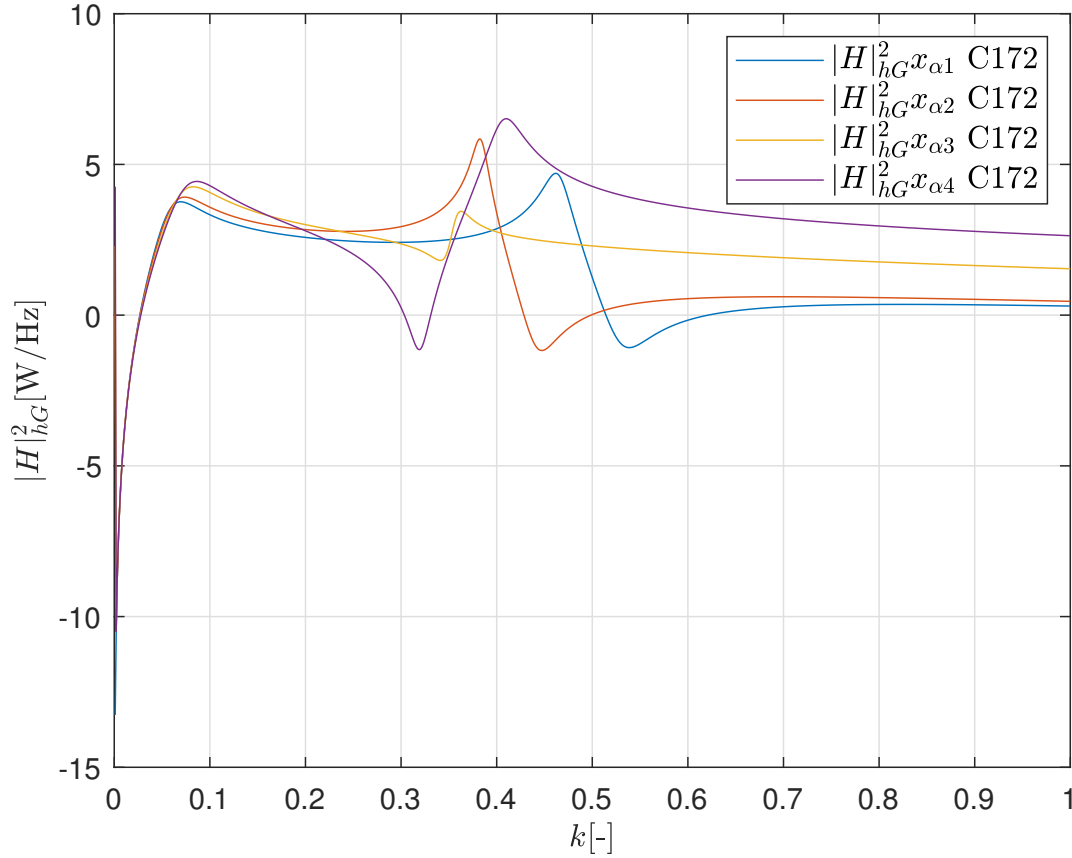


Figure 6.8: $|H|_{hG}^2$ vs k for the B737 for the coupled and decoupled systems

Figure 6.8 depicts the bending acceleration function for the B737, presenting both the coupled and decoupled system responses. It is evident and anticipated that the coupled function undergoes two amplitude changes in its response, whereas the decoupled function experiences only one. It's worth emphasizing that the coupled function attains a higher value when excited at the reduced frequency k_α compared to k_h . This discrepancy suggests the possibility of the coupled system enduring greater loads than the decoupled system. However, it is essential to note that the bending transfer function serves as an interim outcome, needing validation through the final Power Spectral Density (PSD) results for significant consideration.

Cessna 172

Figure 6.9: $|H|_{hG}^2$ vs k for the C172 changing x_α

Similarly to the results obtained for the B737, for the Cessna 172, it can be seen in Figure 6.9 the bending transfer acceleration function in logarythim scale, varying the x_α parameter and having fixed the mass parameter at λ_1 .

Foremost, it is observed that, in line with the predictions, the response of the system is amplified in proximity to its intrinsic reduced frequencies, $k_h = 0.065$ and $k_\alpha = 0.39$. The extent of response amplification hinges on the value of the x_α parameter and geometrical factors, leading to amplification occurrences at either lower or higher frequencies.

However, it is important to note that the response magnitude is notably heightened in the k_α reduced frequency. Moreover, it can be observed that when motion is analysed beyond the system's natural reduced frequencies range, the response remains stable, inducing minimal perturbations in the bending coordinate.

The same behaviour as the B737 is obtained, being the most adverse case with $x_{\alpha 4} = 0.5$, where maximal amplitudes for both frequencies are achieved. On the other hand, the most favorable value for the dimensionless crossed inertia product is $x_{\alpha 3} = 0.25$. For that value of x_α , the bending response markedly surpasses the torsional counterpart. Analogous to the observations in Figure 6.6, the aircraft will endure diminished turbulence-induced loads for $x_{\alpha 3}$.

In comparison with the results obtained for the B737, it is obvious that the magnitude of the response is greater for the Cessna 172. The maximum and minimum amplitudes are approximately $|H|_{hG}^2 = 7$ and $|H|_{hG}^2 = -2$. Even though this response has a greater lower value, the range is bigger and thus a more aggressive response to turbulence is expected.

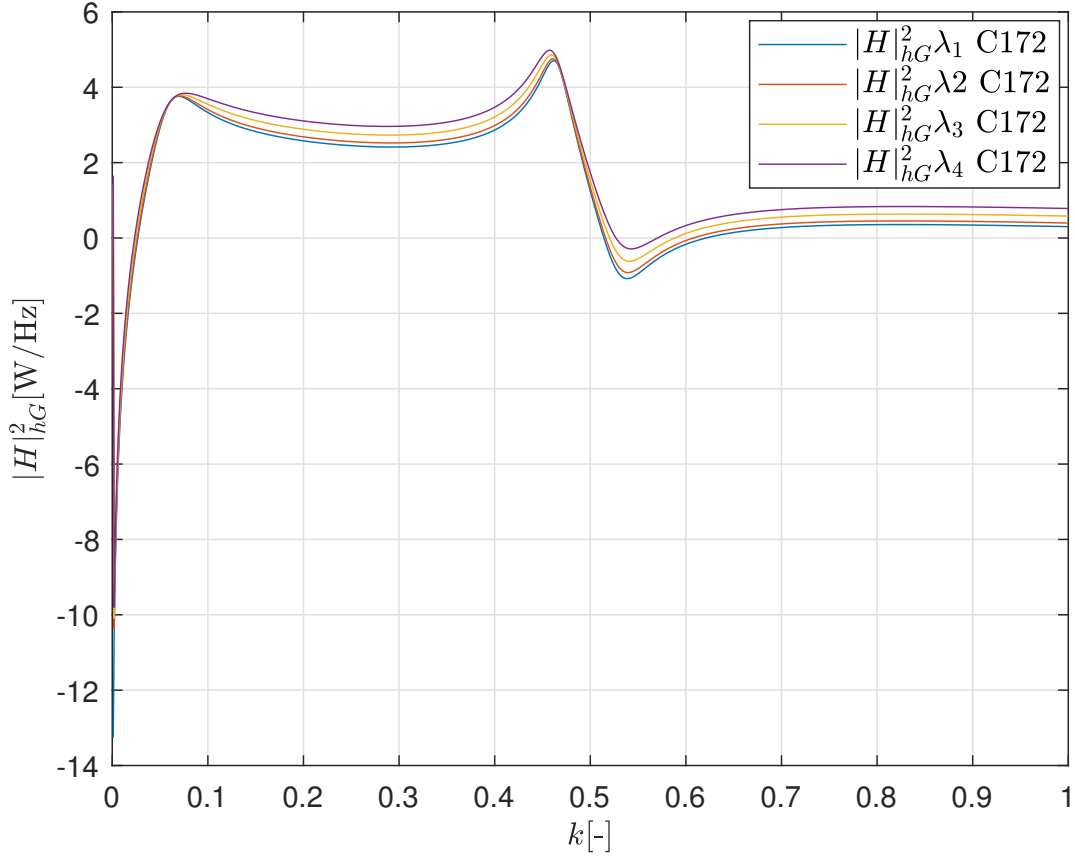


Figure 6.10: $|H|_{hG}^2$ vs k for the C172 changing λ

Figure 6.10 depicts the logarithmic-scale bending transfer acceleration function, showcasing the impact of varying the parameter λ while maintaining a fixed crossed inertia product parameter at $x_{\alpha 1}$. It is shown that the variation of λ does not lead to substantial variations in the bending response of the system. However, a discernible trend is observed: the least structurally demanding response corresponds to the highest mass, which is the Maximum Takeoff Weight (MTOW) denoted as λ_1 .

While the obtained response closely resembles that of the B737, it becomes apparent that the behavior of the system shows more substantial fluctuations when changing the mass parameter within the valley situated between its natural frequencies. This discrepancy is particularly pronounced in the Cessna 172 compared to the B737.

Figure 6.11 illustrates the bending acceleration function for the Cessna 172, presenting responses from both the coupled and decoupled systems. Similar to the earlier case, the coupled response displays two amplitude variations, while the decoupled response showcases a single change. Notably, the coupled system's amplitude is higher when excited at the reduced frequency k_{α} compared to k_h , suggesting the potential for increased loads compared to the decoupled system. As previously stated, it must be taken into account that this is an intermediate result and no conclusions shall be drawn until the PSD results are presented.

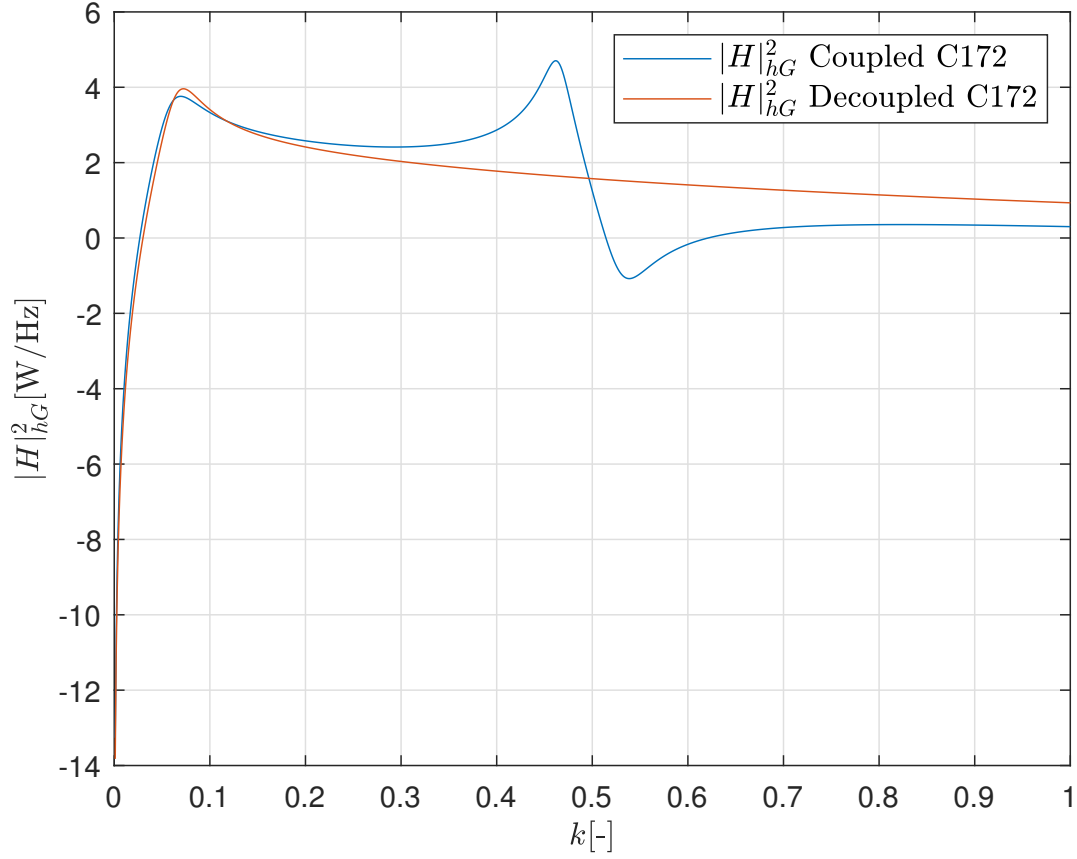


Figure 6.11: $|H|_{hG}^2$ vs k for the C172 comparing the coupled and decoupled functions

6.5.2 Torsional motion

In this section the results derived from the application of the model for both the Boeing 737-300 and the Cessna 172 are presented. For each aircraft, the acceleration transfer matrix, referred to the torsional motion will be presenting analysing:

1. $|H|_{\alpha G}^2$ vs k changing x_α .
2. $|H|_{\alpha G}^2$ vs k changing λ .
3. $|H|_{\alpha G}^2$ vs k for the coupled and decoupled systems.

Boeing 737-300

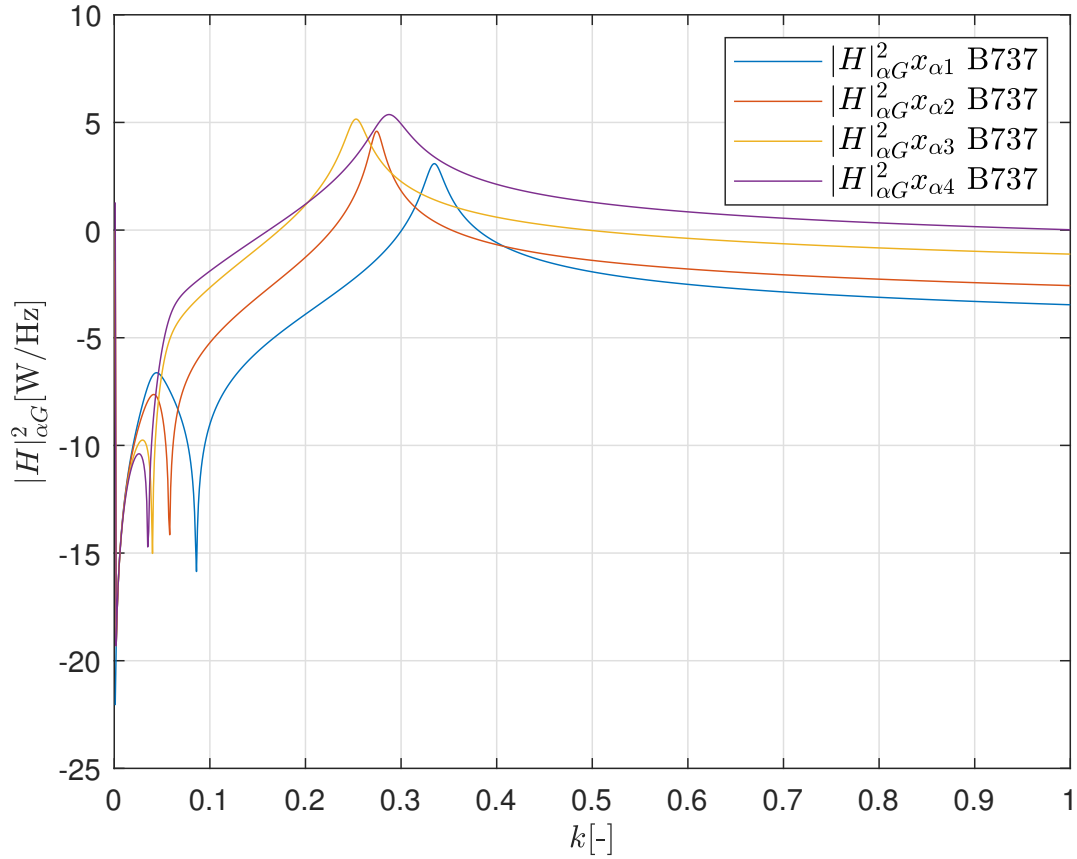
Figure 6.12: $|H|_{\alpha G}^2$ vs k for the B737 changing x_{α}

Figure 6.12 shows the torsional transfer acceleration function in logarythim scale, changing the x_{α} parameter and having fixed the mass parameter at λ_1 . In comparison with the bending transfer function shown in Figure 6.6, it can be seen that the variation of the response when acting on the bending and consequent torsional frequency is much greater. Values in the logarythim scale of the transfer function when acting on the k_h frequency are around $|H|_{\alpha G}^2 = -15$. When acting on the k_{α} these reach the value of $|H|_{\alpha G}^2 = 5$. This range is significantly wider than the one showed for the bending motion. As previously stated, these shall be backed by those of the PSD.

It can be seen in Figure 6.6 that most undesirable response of the system is when $x_{\alpha 4} = 0.5$. Nonetheless, in this case it cannot be as easily distinguished as in the bending motion results. The outcomes concerning to the torsional response of the B737 demonstrate a significant impact when altering the x_{α} parameter.

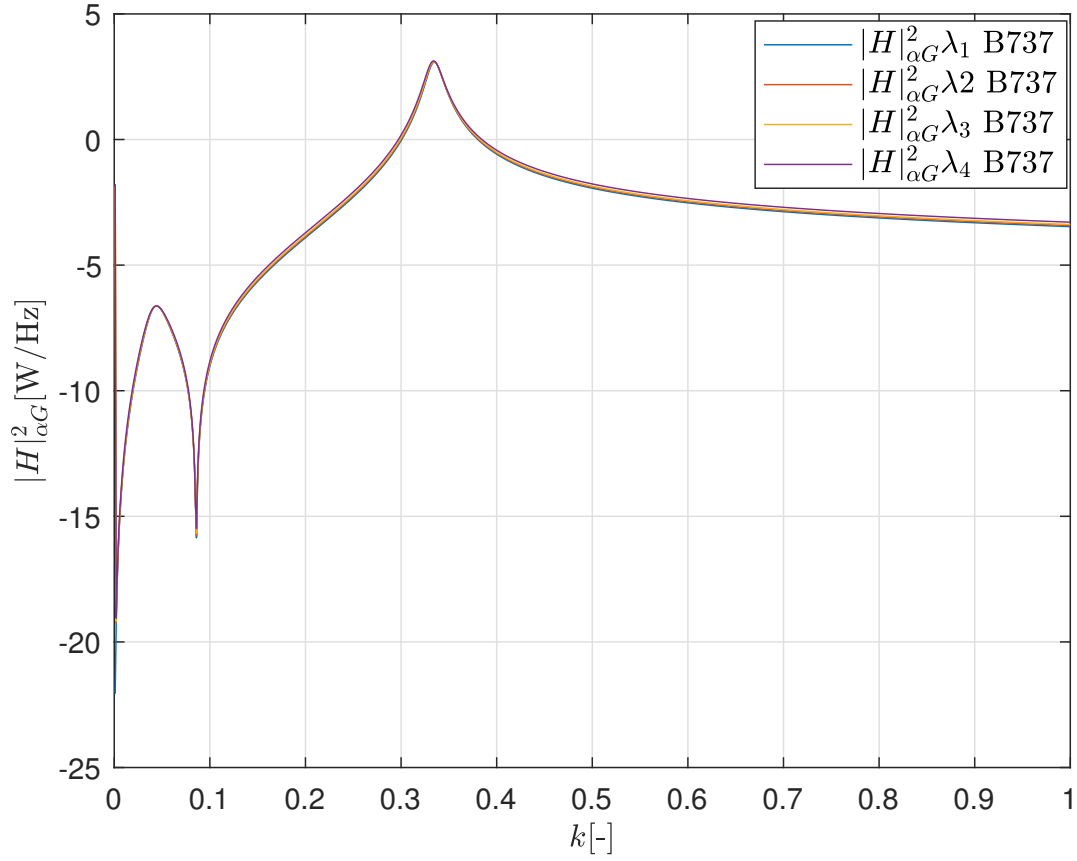


Figure 6.13: $|H|_{\alpha G}^2$ vs k for the B737 changing λ

Figure 6.13 illustrates the logarithmic-scale torsional transfer acceleration function, with the parameter λ being varied while keeping the crossed inertia product parameter fixed at $x_{\alpha 1}$. As can be seen, varying the mass parameter yields negligible variations in the torsional response of the system. In contrast to the bending transfer function, where the optimal condition could be distinctly identified, the perturbation in the torsional response is so small that it lacks significant relevance for consideration.

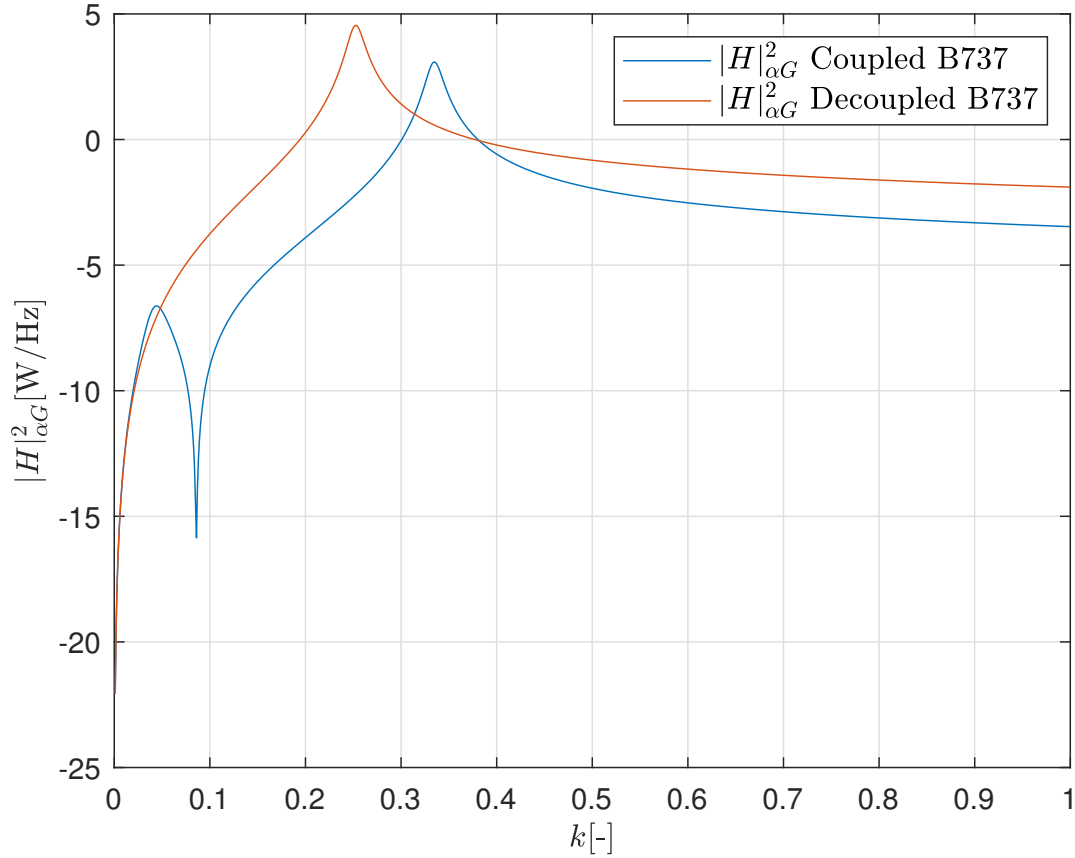


Figure 6.14: $|H|_{\alpha G}^2$ vs k for the B737 for the coupled and decoupled systems

Figure 6.14 shows the torsional acceleration function for the Boeing 737-300, presenting responses from both the coupled and decoupled systems. As expected, the coupled response presents two amplitude variations, while the decoupled response showcases a single change around the k_α . It is also interesting to note that the coupling shifts the natural frequency, with respect to that of the decoupled system. In contrast to the bending scenario, the uncoupled model attains a notably higher value than the maximum achieved by the coupled model. Nevertheless, it is essential to acknowledge that this value is constrained to an upper bound of $|H|_{\alpha G}^2 = 5$, whereas in the bending case, this limit was $|H|_{\alpha G}^2 = 2$. Consistent with prior findings, it is necessary to treat this as an interim finding, refraining from deriving conclusive inferences until the conclusive results are obtained.

Cessna 172

Figure 6.15 shows the torsional transfer acceleration function in logarithmic scale, changing the x_α parameter, having fixed the mass parameter at λ_1 . As expected, the torsional transfer function evaluated in the C172, presents greater values of the response. Figure 6.15 shows that in the logarithmic scale of the transfer function, values around $|H|_{\alpha G}^2 = -15$ are evident at the k_h frequency, while they reach approximately $|H|_{\alpha G}^2 = 10$ at the k_α frequency. This amplitude range is notably broader than what was observed for the bending motion and is also wider than that recorded for the B737. As emphasized before, these findings need support from the Power Spectral Density (PSD) outcomes to further validate their significance.

Figure 6.15 indicates that the system's most undesirable response may be when $x_{\alpha 4} =$

0.5. However, unlike the bending motion results, this scenario is not as distinctly discernible.

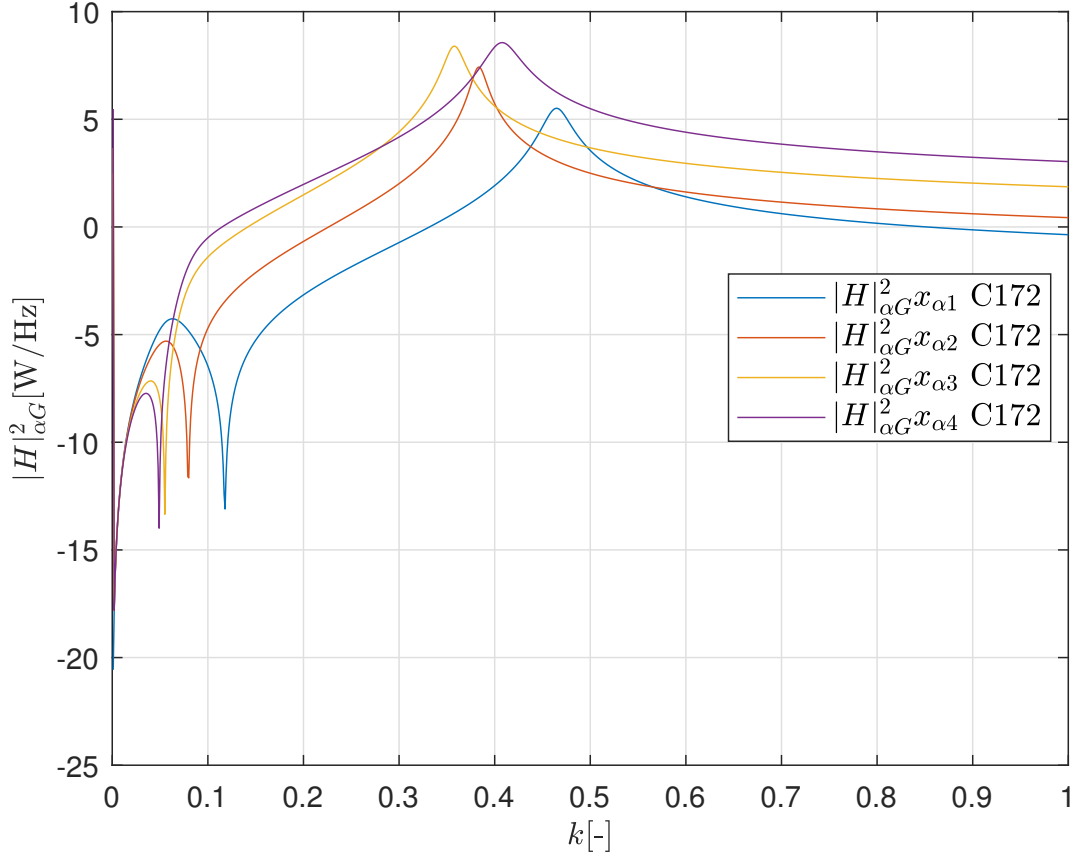


Figure 6.15: $|H|_{\alpha G}^2$ vs k for the C172 changing x_{α}

Figure 6.16 illustrates the logarithmic-scale bending transfer acceleration function, with the parameter λ being varied while keeping the crossed inertia product parameter fixed at $x_{\alpha 1}$. Whereas the task of distinguishing the least and most favorable scenarios was not possible within the B737 analysis, the presented Figure offers a contrasting perspective. In the frequencies between k_h and k_{α} , a noticeable differentiation becomes discernible. Notwithstanding this, it is important to note that the peak amplitudes remain uniform across the spectrum of λ values. Consequently, it is not deemed relevant to establish a preferential or adverse case.

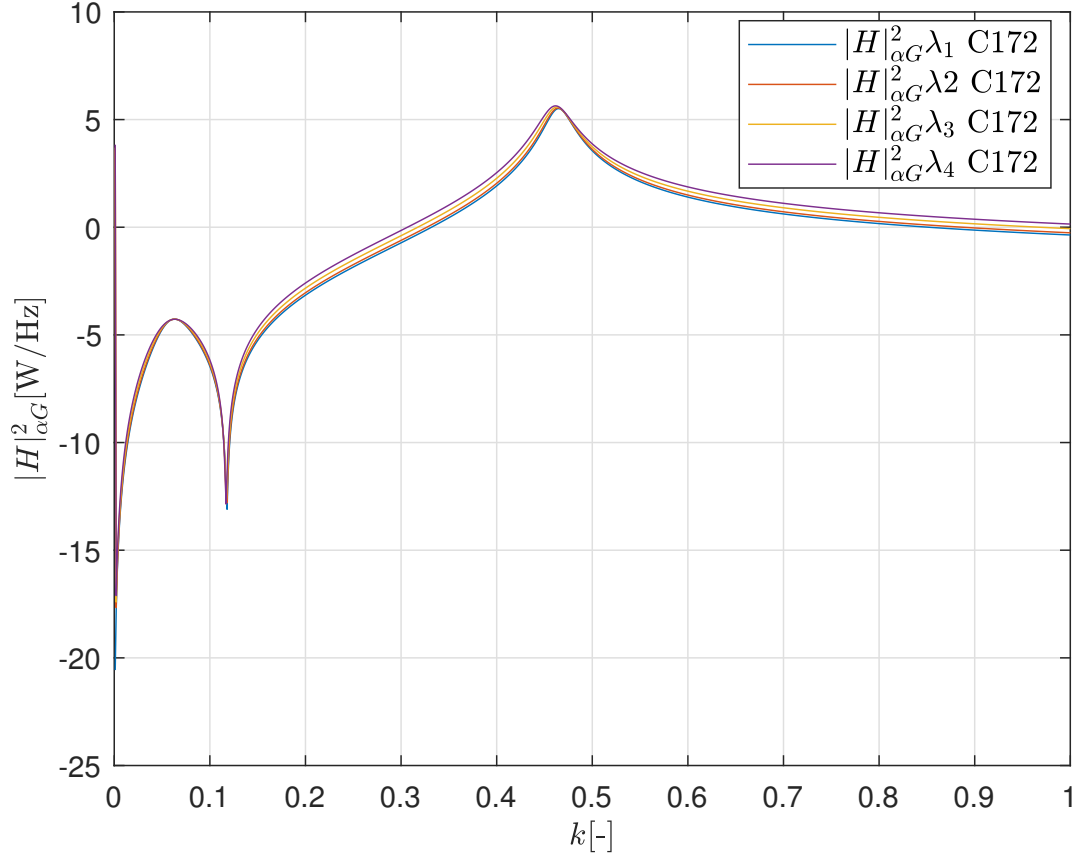


Figure 6.16: $|H|_{\alpha G}^2$ vs k for the C172 changing λ

In Figure 6.17 the torsional acceleration profile of the Cessna 172 is presented, encompassing responses from both the coupled and decoupled system configurations. As anticipated, the coupled response manifests a distinctive oscillatory behavior characterized by two amplitude peaks and troughs, whereas the decoupled response showcases a singular amplitude shift around the k_α frequency.

In direct contrast to the findings in the bending analysis, the torsional uncoupled model exhibits significantly elevated response amplitudes in comparison to the coupled counterpart. However, it's imperative to acknowledge that this heightened response is nearing $|H|_{\alpha G}^2 = 7$, whereas the bending scenario had the upper bound set at $|H|_{\alpha G}^2 = 5$.

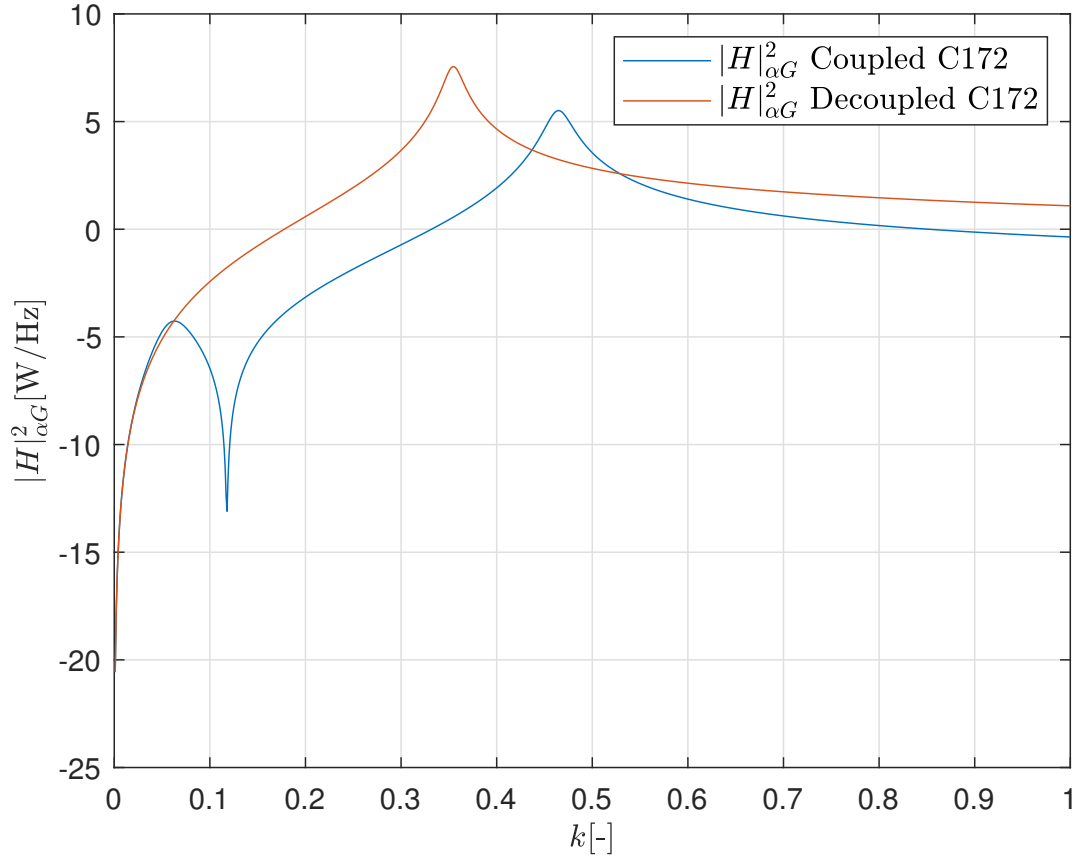


Figure 6.17: $|H|_{\alpha G}^2$ vs k for the C172 for the coupled and decoupled systems

6.5.3 Acceleration rigid body transfer function

This section refers to the results derived from the application of both the model developed in this study and the analytical model, for both the Boeing 737-300 and the Cessna 172. For each aircraft, the acceleration transfer matrix, referred to the rigid body motion will be presenting analysing $|H|_{hG}^2$ against k for the coupled, decoupled and analytical responses.

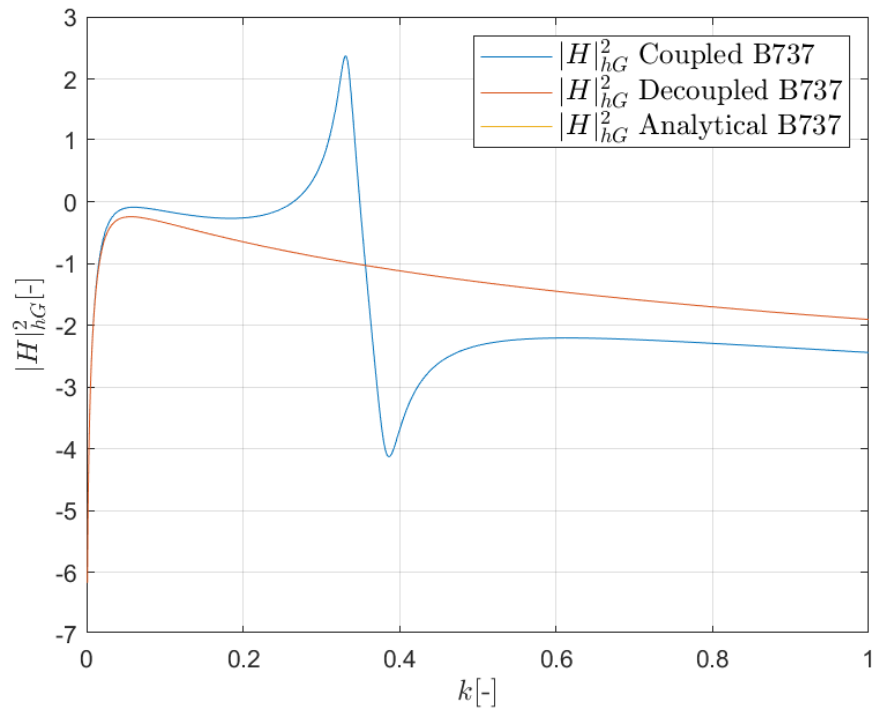
Boeing 737-300

Figure 6.18: $|H|_{hG}^2$ vs k for the B737 comparing the coupled, decoupled and analytical response for rigid body motion

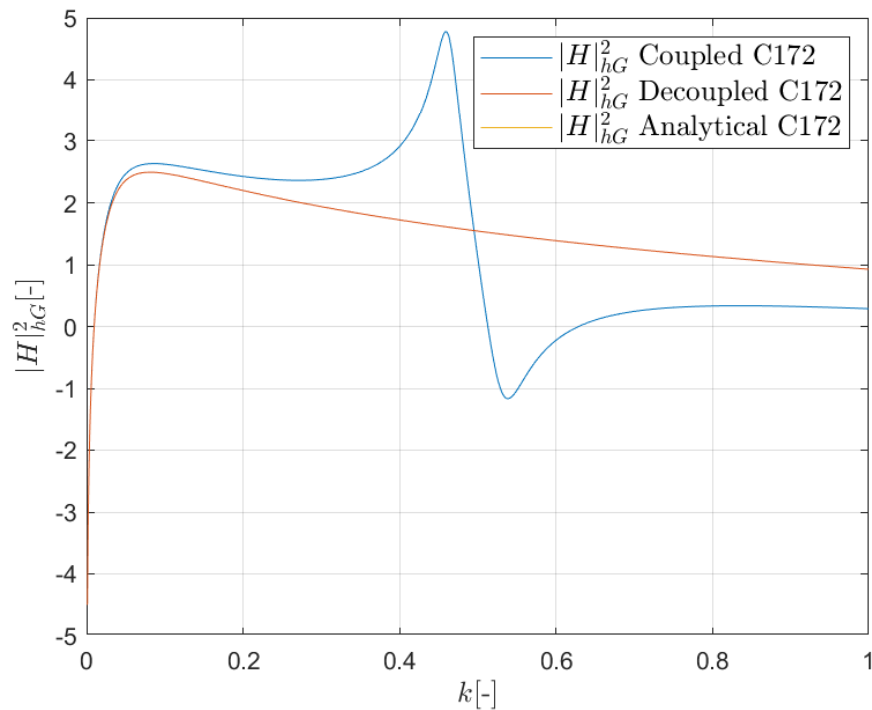
Cessna 172

Figure 6.19: $|H|_{hG}^2$ vs k for the C172 comparing the coupled, decoupled and analytical response for rigid body motion

Shown in Figures 6.18 6.19 are the responses of said aircrafts when the $\tilde{h}$ generalized coordinate is studied with rigid body frequency, $k_h = 0$. These figures show each three functions: the coupled response, the decoupled response (obtained by nulling the terms associated to the $\tilde{\alpha}$ generalized coordinate of the model) and the analytical response [14]. The motion associated to the rigid body mode is depicted by the nearly vertical surge in the transfer function's value at $k = 0$.

It can be seen that as expected, the coupled response has an amplitud variation in the surroundings of the k_α reduced frequency. Moreover, it is remarkable that both the decoupled and the analytical response match exactly, thus confirming the validity of the model. As a result, the analytical function is not possible to see as it is immediately below the decoupled function.

6.6 Power Spectral Density

6.6.1 Bending PSD

This section presents to the results derived from the application of the model for both the Boeing 737-300 and the Cessna 172. For each aircraft, the PSD, referred to the bending motion will be presented analysing:

1. PSD_h vs k changing x_α
2. PSD_h vs k changing λ
3. PSD_h vs k for the coupled and decoupled systems.

Boeing 737-300

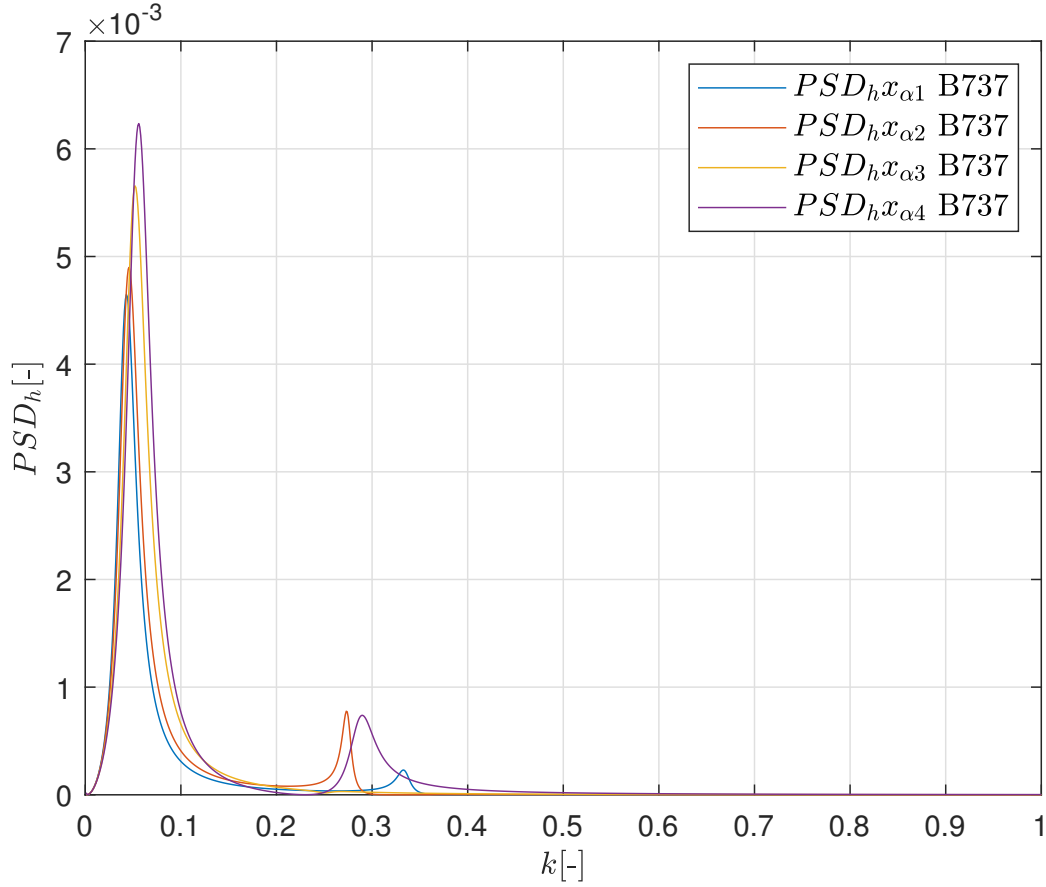
Figure 6.20: PSD_h vs k for the B737 changing x_α

Figure 6.20 shows the bending Power Spectral Density (PSD), changing the x_α parameter, having fixed the mass parameter at λ_1 . As explained in the previous results, it becomes evident that the most energetic frequencies correspond to the natural frequencies, inherent to the system. As previously noted, it is now confirmed in light of this result that most undesirable response of the system is when $x_{\alpha 4} = 0.5$. In Chapter 5, it was explained that the area below the PSD function corresponds to the variance of the motion. This relationship is manifestly demonstrated by the largest area corresponding to $x_{\alpha 4}$. Aligned as well with previous results, the most favorable value of the dimensionless crossed inertia product is $x_{\alpha 3} = 0.25$. In this case the bending response is much greater than the torsional, which tends towards null. Referencing Figure 6.20, it is evident that the aircraft's motion variance reaches its minimum point for $x_{\alpha 3}$.

For the purpose of comparison with the Cessna 172 outcomes, it is crucial to note that the maximum peak is achieved at a level exceeding $PSD_h = 6 \times 10^{-3}$.

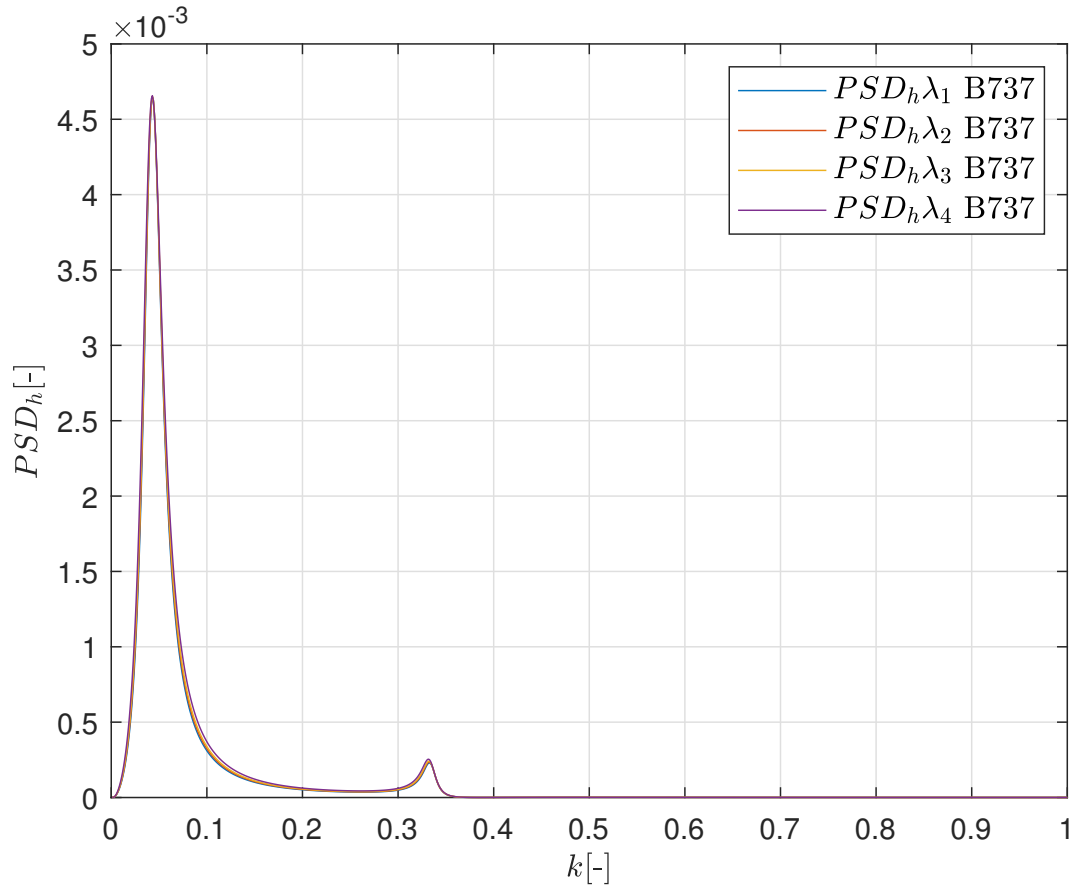


Figure 6.21: PSD_h vs k for the B737 changing λ

Figure 6.21 shows the bending Power Spectral Density (PSD), changing the λ parameter, having fixed the mass parameter at $x_{\alpha 1}$. As shown in Figure 6.21 small variations of the mass parameters do not involve significant variations of the response.

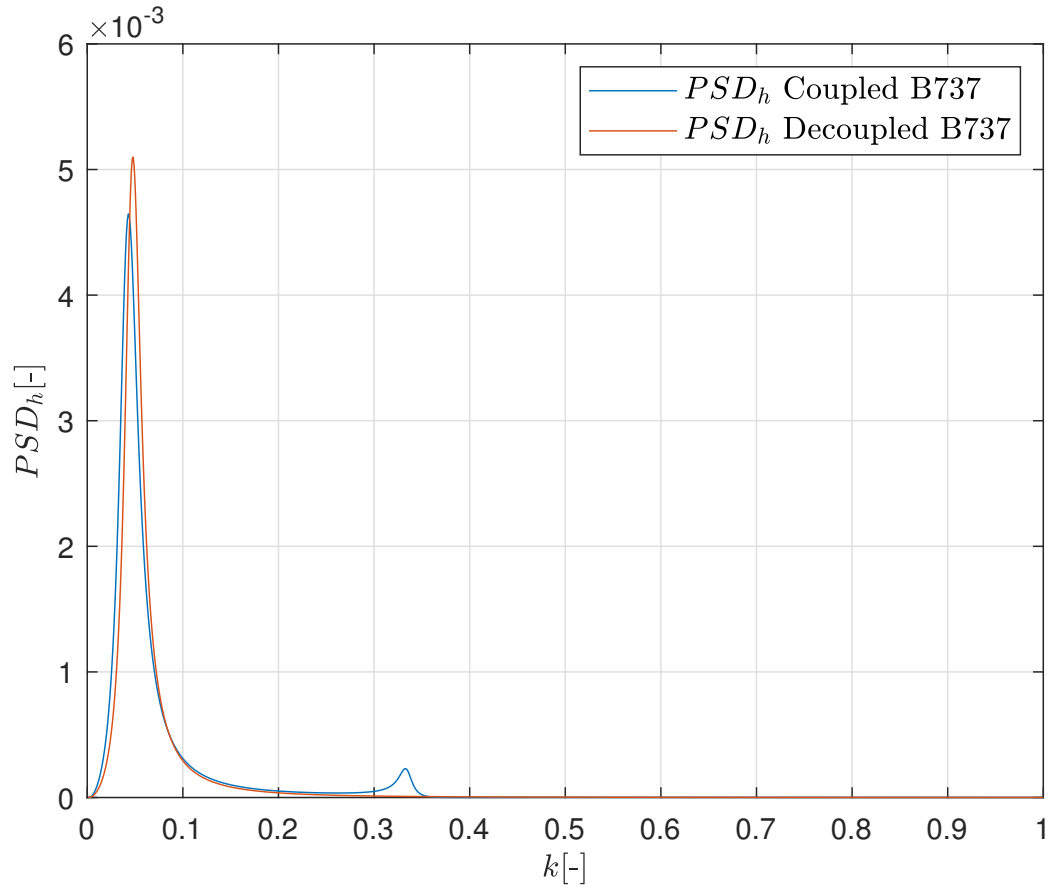


Figure 6.22: PSD_h vs k for the B737 for the coupled and decoupled systems

Figure 6.22 illustrates the bending PSD for the Boeing 737-300, presenting responses from both the coupled and decoupled systems. Similar to the earlier case, the coupled response displays two energetic focal points, while the decoupled response showcases only the one corresponding to the bending natural reduced frequency. Whereas in previous results the amplitude of the coupled response was greater, Figure 6.22 shows that the biggest area under the function corresponds to the decoupled model, as the area associated to the torsional frequency is considerably small.

Cessna 172

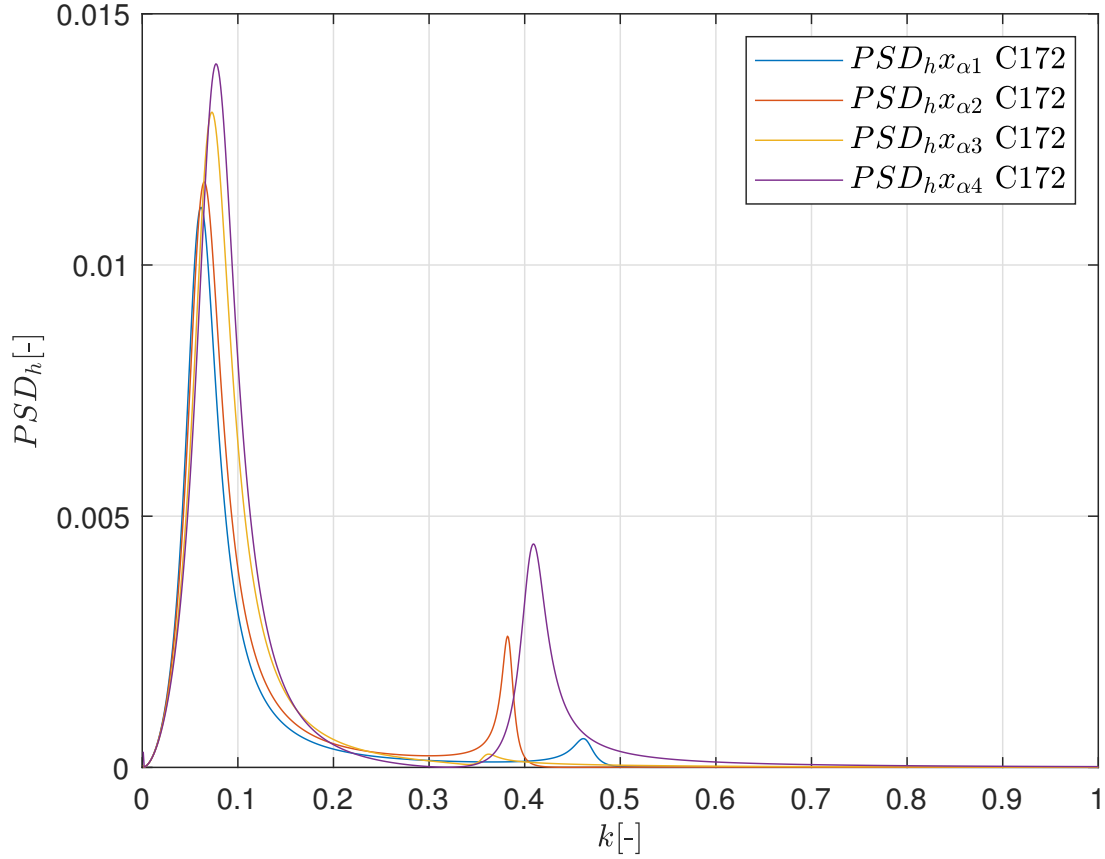
Figure 6.23: PSD_h vs k for the C172 changing x_α

Figure 6.23 shows the bending Power Spectral Density (PSD), changing the x_α parameter, having fixed the mass parameter at λ_1 .

The results obtained for the PSD of the bending motion for C172 are in alignment with those obtained for B737. Both worst case scenarios are achieved when the x_α parameter is equal to $x_{\alpha 4} = 0.5$. Furthermore, the most favorable value of the dimensionless crossed inertia product is $x_{\alpha 3} = 0.25$. This value is expected to derive in the lowest value for the variance and thus, have a less aggressive response, in structural terms.

Previous results pointed towards a more threatening and energetic response of the aircraft when the mass decreases. This can be backed up by the result now presented. It can be seen that the maximum value reached for the C172 is almost ten times greater than the one obtained for the B737.

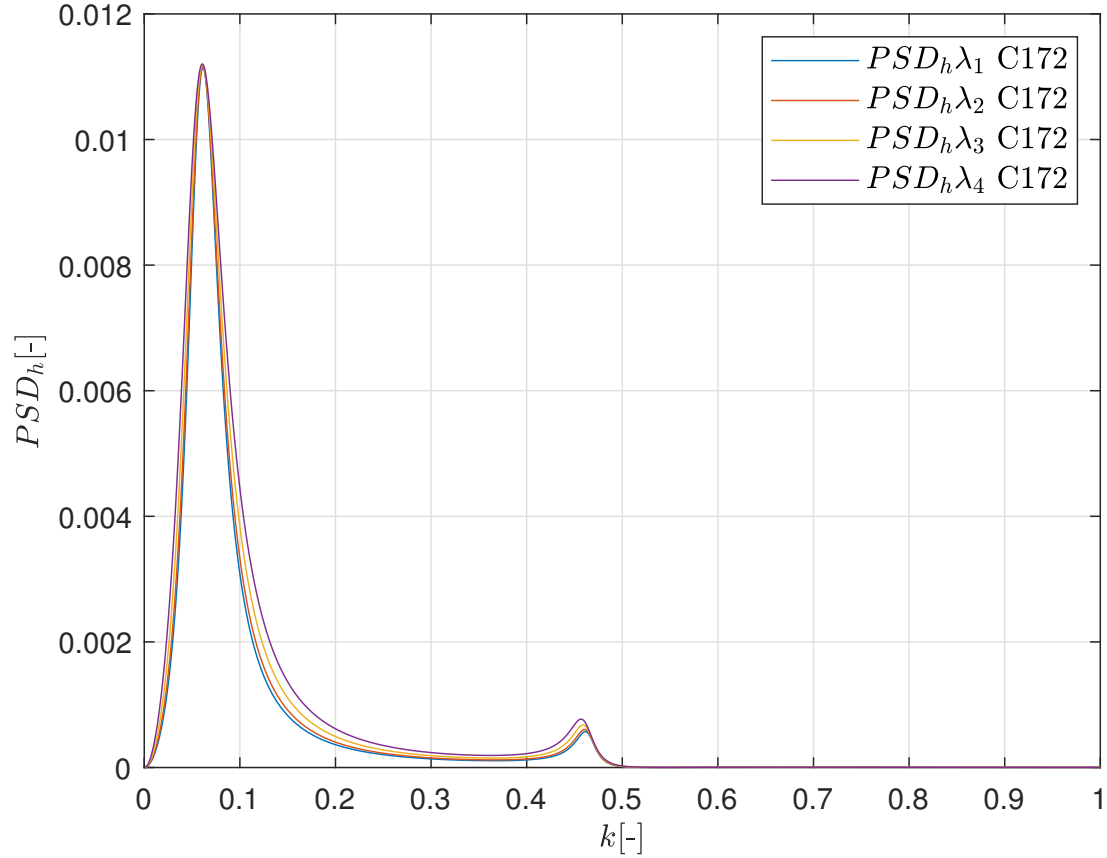


Figure 6.24: PSD_h vs k for the C172 changing λ

Figure 6.24 shows the bending Power Spectral Density (PSD), changing the λ parameter, having fixed the mass parameter at $x_{\alpha 1}$. As evident from Figure 6.24, marginal adjustments to the mass parameters do not result in substantial fluctuations in the response. For the model at hands, it is observed that the mass variations that can occur in an aircraft do not show a relevant impact in the response to atmospheric turbulence. The only impact, related to the variation of mass, that has been noted to the moment is the difference of mass and geometrical parameters between the B737 and C172.

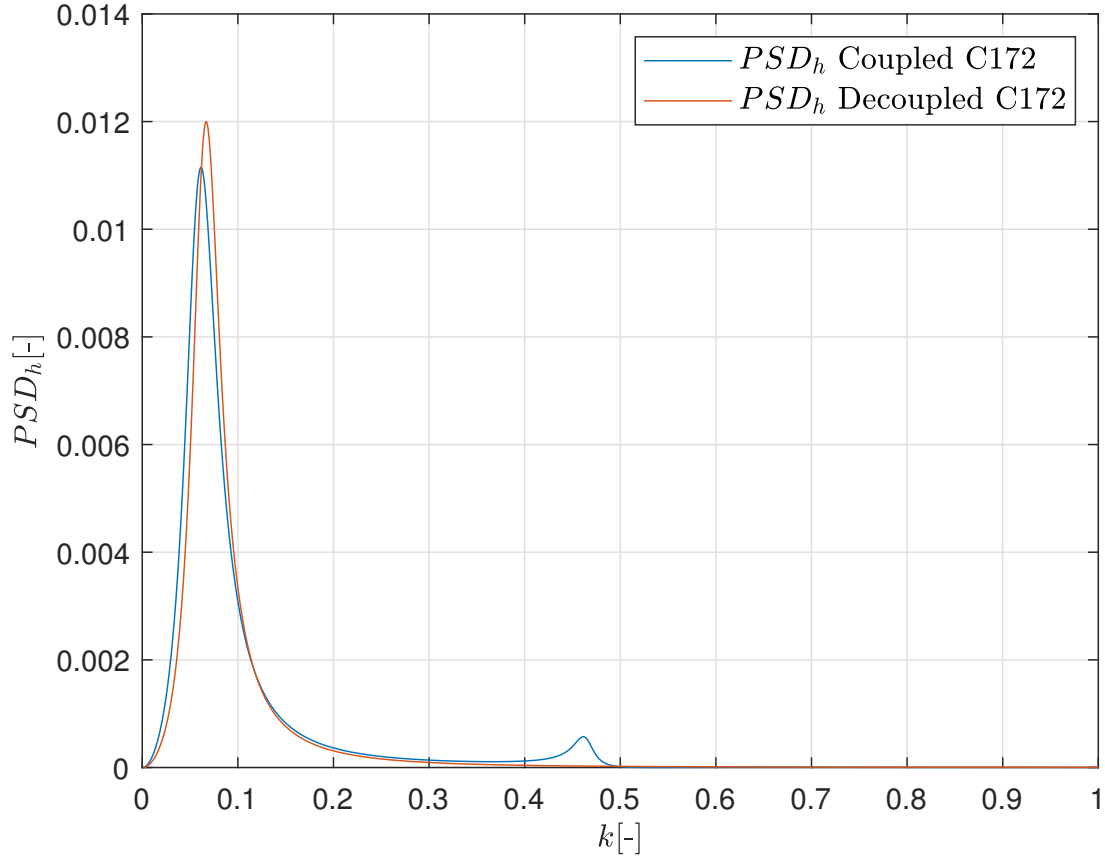


Figure 6.25: PSD_h vs k for the C172 for the coupled and decoupled systems

Figure 6.25 illustrates the bending PSD for the Cessna 172, presenting responses from both the coupled and decoupled systems. As in the previous case, the coupled response shows two prominent peaks of energy, while the decoupled response highlights a singular peak aligned with the bending natural reduced frequency.

In contrast with earlier findings, where the coupled response boasted a larger amplitude, Figure 6.25 reveals that the most expansive area beneath the function corresponds to the decoupled model. This is attributed to the significantly smaller area associated with the torsional frequency.

6.6.2 Torsional PSD

This section refers to the results derived from the application of the model for both the Boeing 737-300 and the Cessna 172. For each aircraft, the PSD, referred to the torsional motion will be presented analysing:

1. PSD_α vs k changing x_α .
2. PSD_α vs k changing λ .
3. PSD_α vs k for the coupled and decoupled systems.

Boeing 737-300

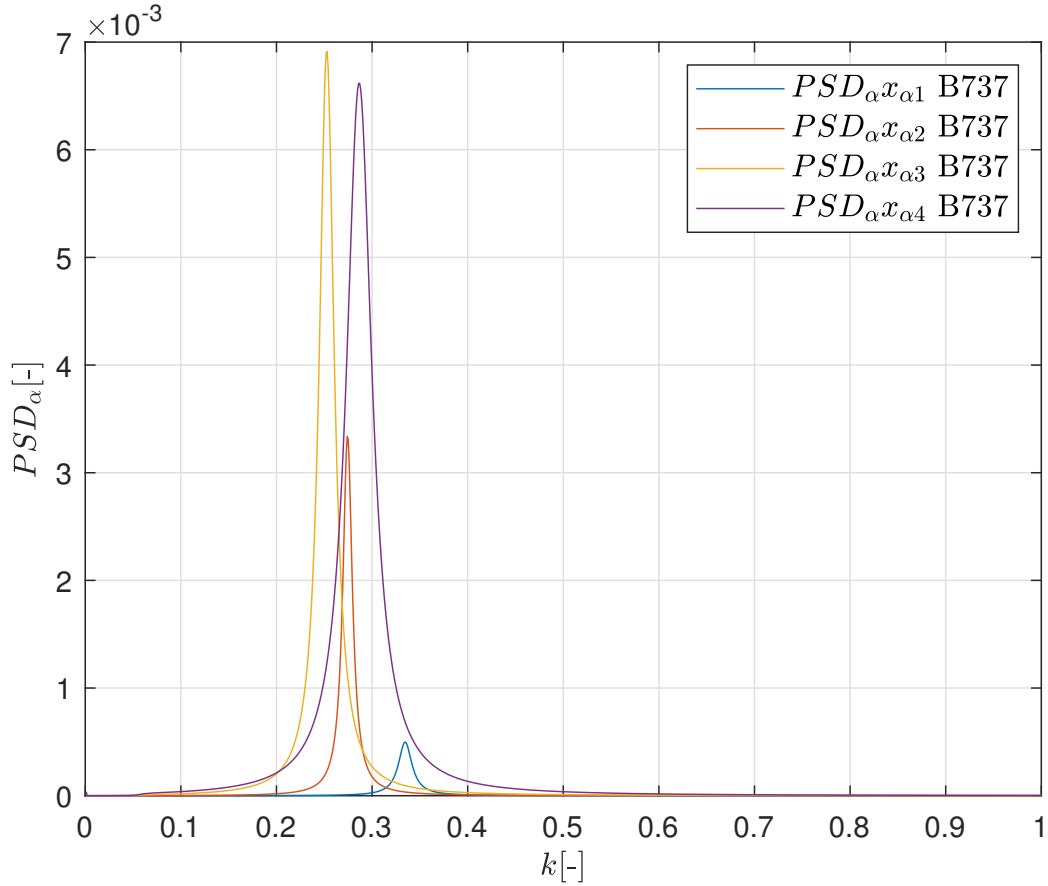
Figure 6.26: PSD_{α} vs k for the B737 changing x_{α}

Figure 6.26 shows the torsional PSD, changing the x_{α} parameter, having fixed the mass parameter at λ_1 . In comparison with the bending PSD, it can be seen that energy associated to the bending frequency is almost negligible and that the most of the energy associated to this motion is center around the torsional reduced frequency. Values of the PSD are around $PSD_{\alpha} = 7 \times 10^{-3}$, whereas for the bending motion these are around $PSD_{\alpha} = 5 \times 10^{-3}$.

It can be seen in Figure 6.6 that most undesirable response of the system is when $x_{\alpha 4} = 0.5$. Meanwhile, the most desirable case scenario is $x_{\alpha 1} = -0.5$.

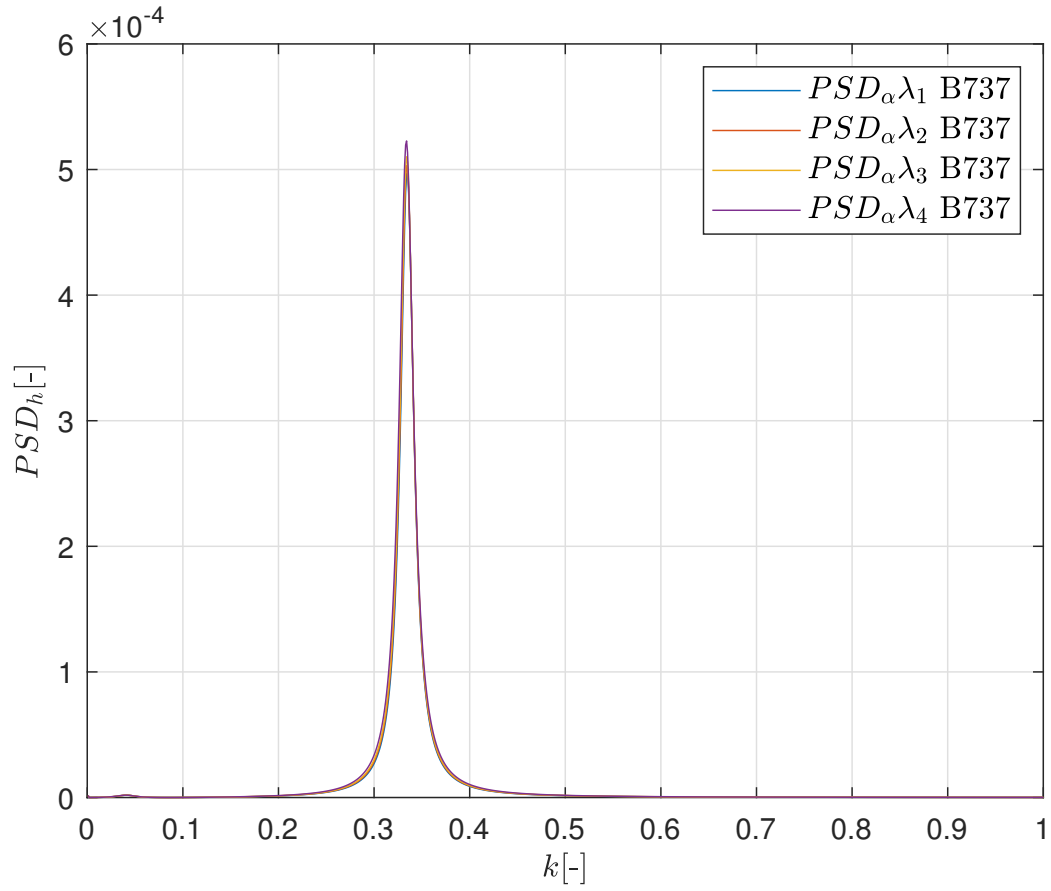


Figure 6.27: PSD_{α} vs k for the B737 changing λ

Figure 6.27 shows the torsional PSD, changing the λ parameter, having fixed the mass parameter at $x_{\alpha 1}$. As shown in Figure 6.27 small variations of the mass parameters do not involve significant variations of the response.

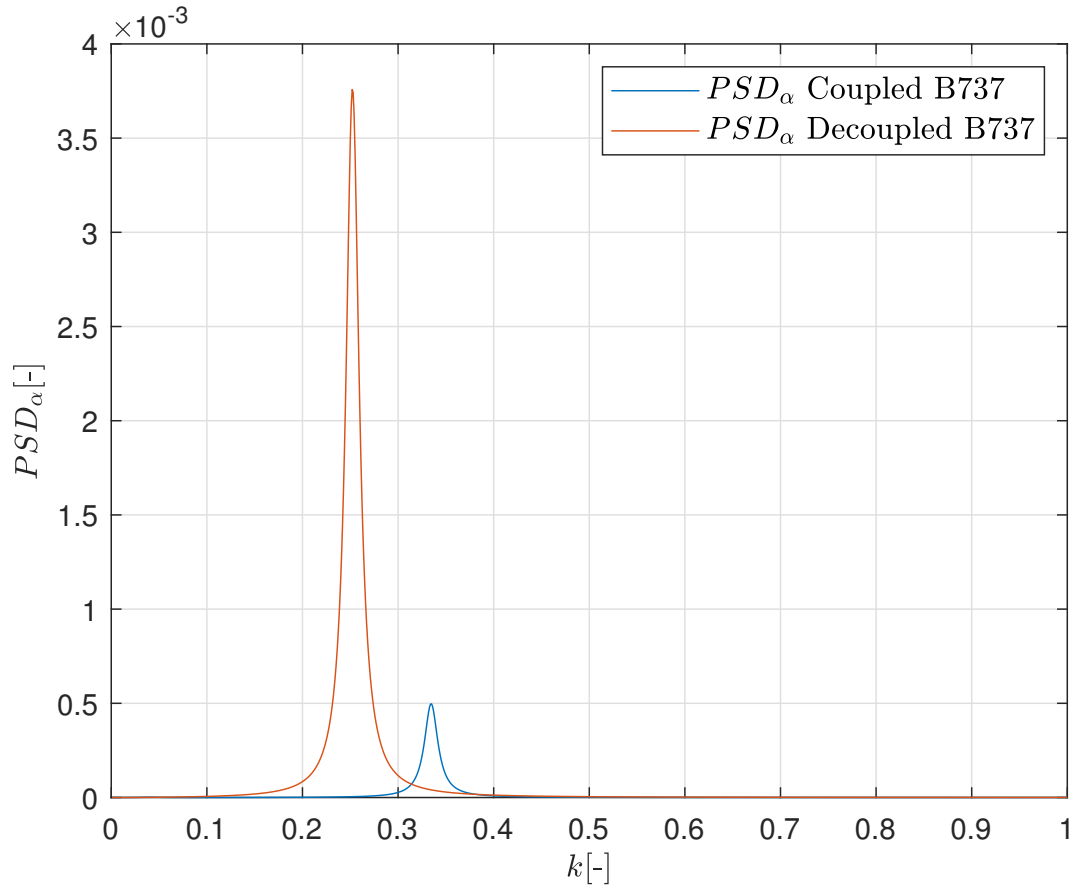


Figure 6.28: PSD_α vs k for the B737 for the coupled and decoupled systems

Figure 6.22 illustrates the torsional PSD for the B737, presenting responses from both the coupled and decoupled systems. In contrast with the bending PSD, the coupled response displays only one energetic focal point, as does the decoupled response, both in the surroundings of the torsional natural reduced frequency. Nonetheless, the coupled PSD seems to be displaced from the torsional natural reduced frequency. Figure 6.22 shows that the biggest area under the function corresponds to the decoupled model, as the area associated to the coupled model is considerably small. The maximum value obtained for the torsional PSD of the B737 is rounding $PSD_\alpha = 4 \times 10^{-3}$.

Cessna 172

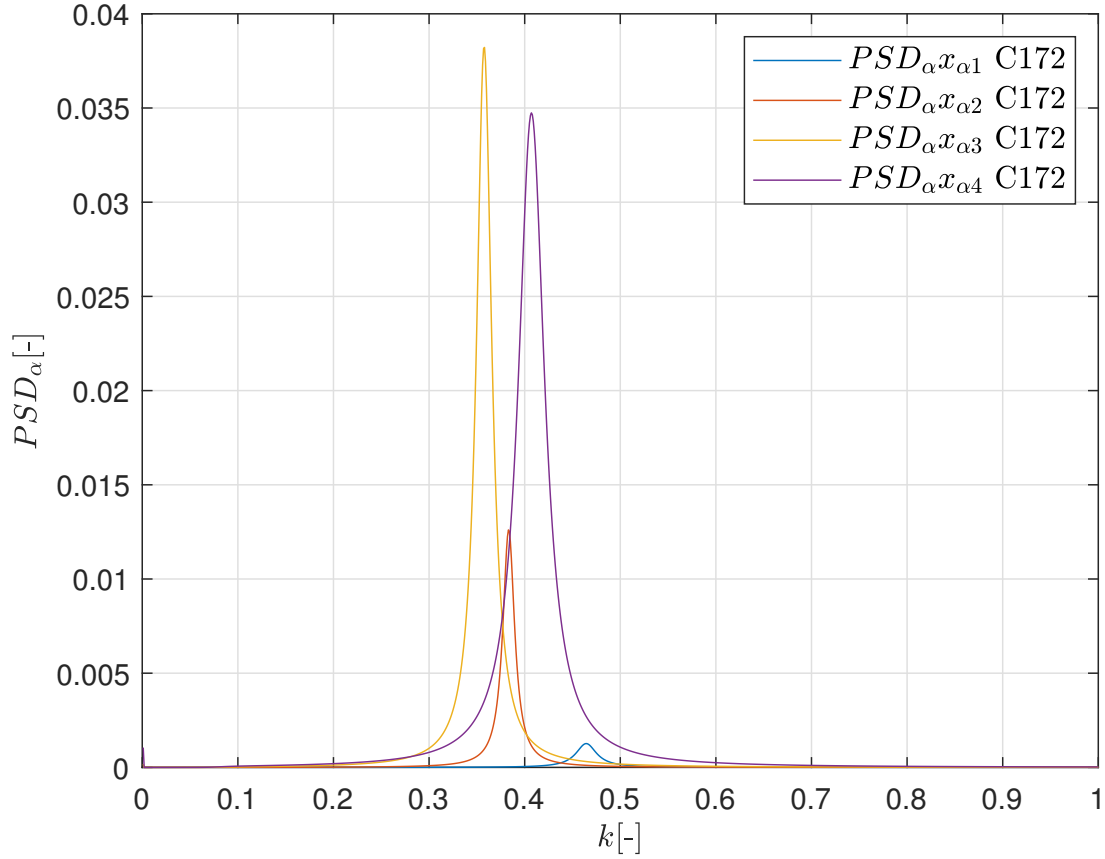
Figure 6.29: PSD_α vs k for the C172 changing x_α

Figure 6.29 shows the torsional PSD, changing the x_α parameter, having fixed the mass parameter at λ_1 . In comparison with the bending PSD, it can be seen that energy associated to the bending frequency is almost negligible and that the most of the energy associated to this motion is center around the torsional reduced frequency. Values of the PSD are around 0.04, whereas for the bending motion these are around 0.015. In this case, it is obvious that the bending motion is associated to a higher energy and thus will be more aggressive for the structure.

It can be seen in Figure 6.9 that most undesirable response of the system is when $x_{\alpha 4} = 0.5$. Meanwhile, the most desirable case scenario is $x_{\alpha 1} = -0.5$.

In alignment with previous transfer function results, it is observed that the values of the PSD reached by the C172 are considerably higher, compared to those of the B737.

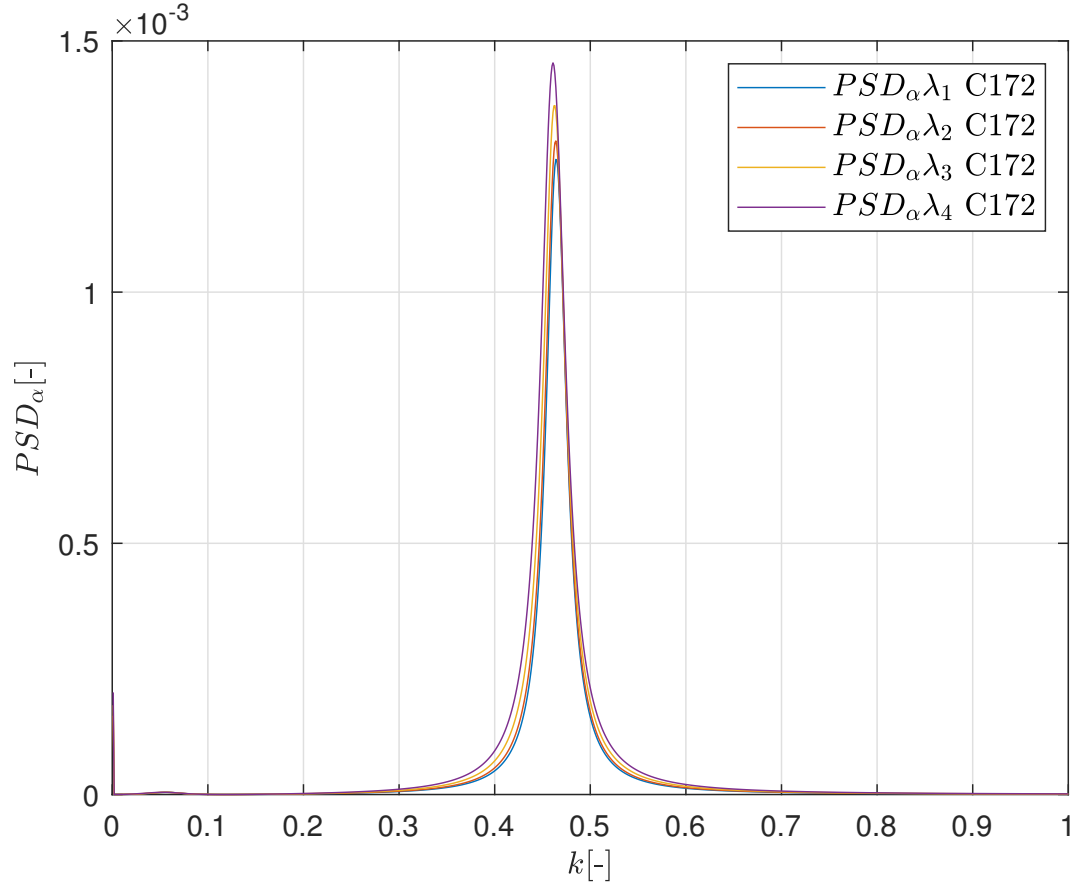


Figure 6.30: PSD_{α} vs k for the C172 changing λ

Figure 6.30 shows the torsional PSD, changing the λ parameter, having fixed the mass parameter at $x_{\alpha 1}$. As can be seen in Figure 6.30, marginal adjustments to the mass parameters do not result in substantial fluctuations in the response.

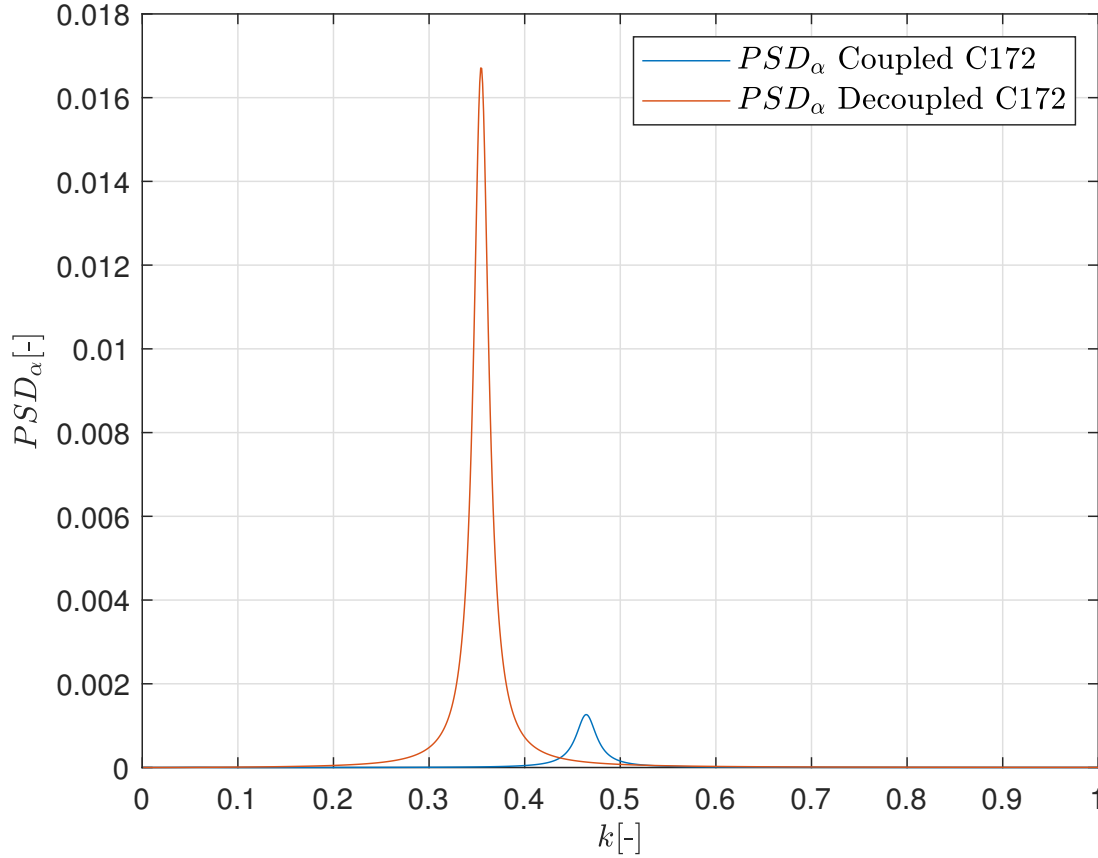


Figure 6.31: PSD_{α} vs k for the C172 comparing the coupled and decoupled functions

Depicted in Figure 6.17, the torsional PSD of the C172 is presented, encompassing responses from both the coupled and decoupled system configurations. Contrary to expected behaviour, the coupled response only shows a sole energetic peak, which is displaced from the expected torsional natural reduced frequency. Moreover, the decoupled response showcases a wider area under the function centered around the k_{α} frequency.

The torsional uncoupled model exhibits significantly elevated response amplitudes in comparison to the coupled counterpart. However, it's imperative to acknowledge that this heightened response is nearing $PSD_{\alpha} = 0.016$, whereas the bending scenario had the upper bound set at $PSD_{\alpha} = 0.012$.

6.6.3 Rigid body PSD

This section refers to the results derived from the application of the model for both the Boeing 737-300 and the Cessna 172. For each aircraft, the PSD, referred to the rigid body motion will be presented analysing: PSD_h vs k for the coupled, decoupled and analytical systems.

Boeing 737-300

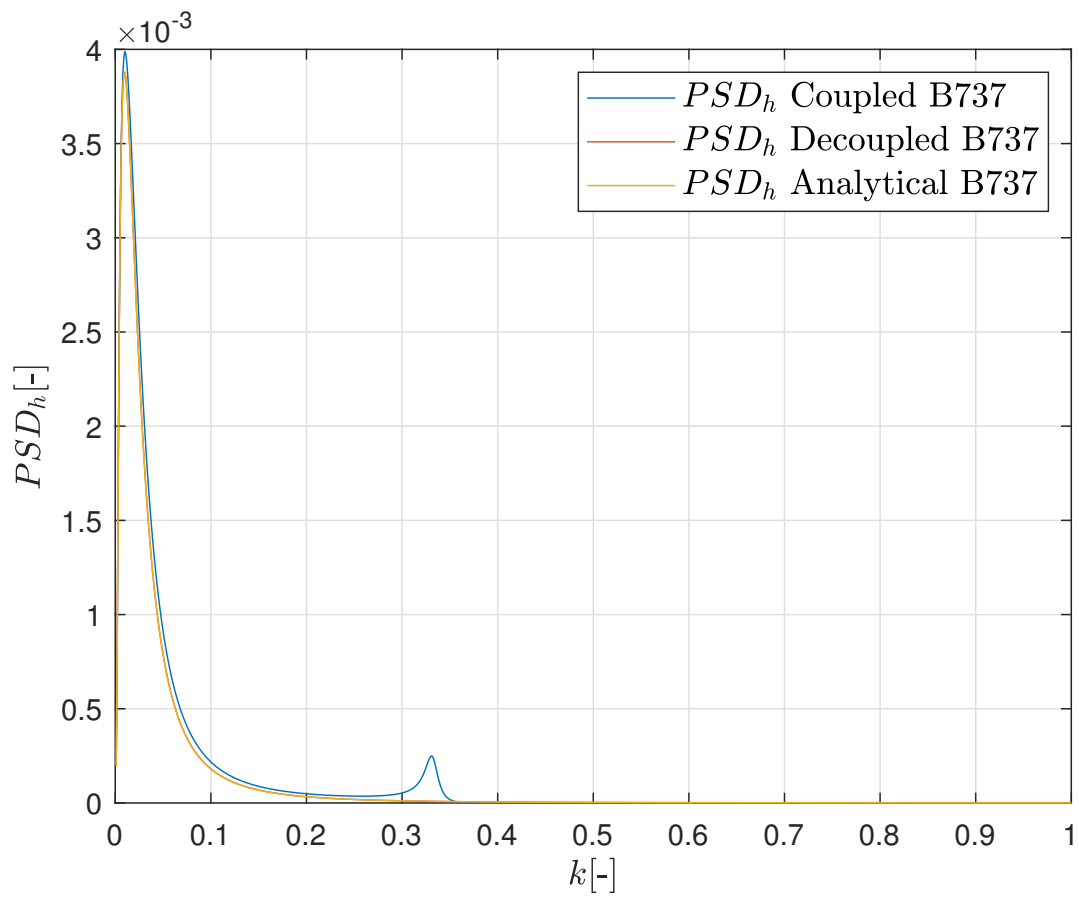


Figure 6.32: PSD_h vs k for the B737 comparing the coupled, decoupled and analytical response for rigid body motion

Cessna 172

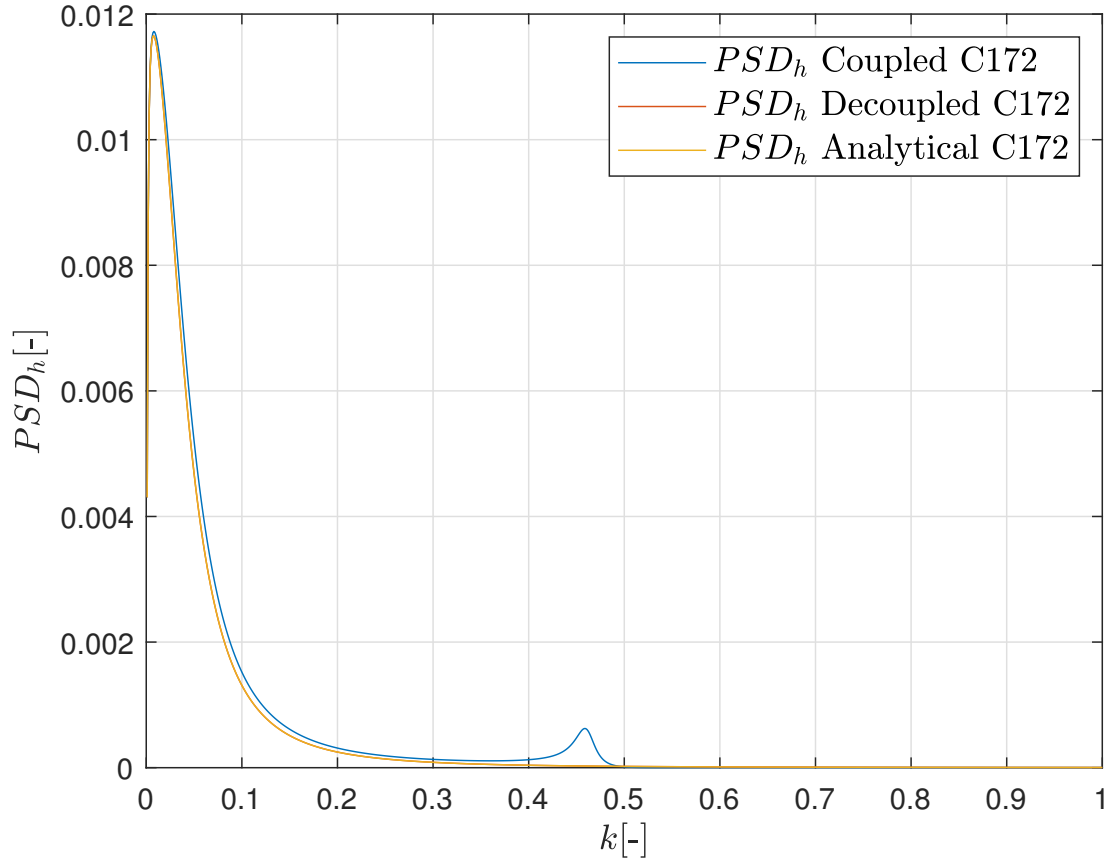


Figure 6.33: PSD_h vs k for the C172 comparing the coupled, decoupled and analytical response for rigid body motion

Shown in Figures 6.32 and 6.33 are the Power Spectral Densities of the two aircrafts when the $\tilde{h}$ generalized coordinate is studied with rigid body frequency, $k_h = 0$. These figures illustrate three distinct functions: the coupled response, the decoupled response, and the analytical response.

The behavior associated with the rigid body mode becomes evident through the sharp upward surge in the transfer function's value at $k = 0$.

As anticipated, the coupled response exhibits variations in amplitude around the vicinity of the k_α reduced frequency. Notably, the decoupled and analytical responses exhibit precise alignment, thereby affirming the model's accuracy. Consequently, the analytical function isn't easily distinguishable as it closely aligns with the decoupled function. Whereas in the bending motion it is most probable that the decoupled model is more energetic, it cannot be assured in the rigid body motion. It can be seen in both Figures 6.32 and 6.33 that the energetic peak associated to the rigid body frequency is almost exactly the same for the three functions. The additional energetic peak associated to the torsional motion of the coupled system may produce a more aggressive response of the aircraft when considered coupled.

6.7 Standard deviation

This section refers to the results derived from the application of the model for both the Boeing 737-300 and the Cessna 172. For each aircraft, the standard deviation, referred to the bending, rigid body and torsional motion will be presented analysing:

1. σ_h for the bending motion changing x_α and λ .
2. σ_h for the rigid body motion changing x_α and λ .
3. σ_α for the torsional motion changing x_α and λ .

6.7.1 Boeing 737-300

Bending motion

Table 6.6: Boeing 737-300 σ_h for the bending motion changing x_α , for λ_1

	$x_{\alpha 1}$	$x_{\alpha 2}$	$x_{\alpha 3}$	$x_{\alpha 4}$
Coupled	0.1336	0.1452	0.1586	0.1769
Decoupled	0.1291	0.1291	0.1291	0.1291

Table 6.6 shows the bending standard deviations of the Boeing 737-300, when changing the x_α parameter, having fixed the mass parameter at λ_1 .

It can be seen that the increasement of the value of the crossed inertia parameter, starting on the value of -0.5 up to 0.5 , entails an augmentation of the value of the standard deviation. It is also worth noting that the values obtained for the standard deviation for the bending case are higher for the coupled system, rather than the decoupled one.

As expected the value of the standard deviation of the decoupled system does not vary when the crossed inertia parameter varies. Even though, the difference between the standard deviations of the coupled and decoupled system for $x_{\alpha 1}$ is only of approximately 3%, when observing the results for $x_{\alpha 4}$ the difference is 37%. All percentages haven been calculated with respect to the standard deviation of the decoupled model.

The standard deviation is upper bounded by 0.17686 and lower bounded by 0.12314.

Table 6.7: Boeing 737-300 σ_h for the bending motion changing λ , for $x_{\alpha 1}$

	λ_1	λ_2	λ_3	λ_4
Coupled	0.1336	0.1351	0.1367	0.1401
Decoupled	0.1291	0.1305	0.1319	0.1348

Table 6.7 shows the bending standard deviations of the Boeing 737-300, when changing the λ parameter, having fixed the crossed inertia parameter at $x_{\alpha 1}$.

As previously discussed, altering the aircraft's mass does not result in a significant change in the bending turbulence response, as shown earlier. The greatest difference among the standard deviations for different λ values is only 0.1.

Rigid body motion

Table 6.8: Boeing 737-300 σ_h for the rigid body motion changing x_α , for λ_1

	$x_{\alpha 1}$	$x_{\alpha 2}$	$x_{\alpha 3}$	$x_{\alpha 4}$
Coupled	0.1305	0.1408	0.1525	0.1707
Decoupled	0.1219	0.1219	0.1219	0.1219
Analytical	0.1871	0.1871	0.1871	0.1871

Table 6.8 shows the rigid body standard deviations of the Boeing 737-300, when changing the x_α parameter, having fixed the mass parameter at λ_1 .

It is evident that as the crossed inertia parameter's value increases, ranging from $x_\alpha = -0.5$ to $x_\alpha = 0.5$, there is a corresponding rise in the standard deviation value.

It is also worth noting that the values obtained for the standard deviation in the rigid motion case are higher for the analytical system (0.1871), only neared, although not surpassed, by the value of the standard deviation derived from the coupled model with the highest x_α value, 0.5.

It is seen that the highest value of the standard deviation for both the bending and rigid body motions is the one obtained for the analytical model.

Table 6.9: Boeing 737-300 σ_h for the rigid body motion changing λ , for $x_{\alpha 1}$

	λ_1	λ_2	λ_3	λ_4
Coupled	0.1305	0.1321	0.1339	0.1376
Decoupled	0.1219	0.1233	0.1248	0.1279
Analytical	0.1871	0.1887	0.1903	0.1937

Table 6.9 shows the rigid body standard deviations of the Boeing 737-300, when changing the λ parameter, having fixed the crossed inertia parameter at $x_{\alpha 1}$.

As previously stated, the variation of the mass parameter does not pose a significant variation of the response and, thus, will not be subjected to further analysis.

Torsional motion

Table 6.10: Boeing 737-300 σ_h for the torsional motion changing x_α , for λ_1

	$x_{\alpha 1}$	$x_{\alpha 2}$	$x_{\alpha 3}$	$x_{\alpha 4}$
Coupled	0.0375	0.0814	0.1467	0.1870
Decoupled	0.1016	0.1016	0.1016	0.1016

Table 6.10 shows the torsional standard deviations of the Boeing 737-300, when changing the x_α parameter, having fixed the mass parameter at λ_1 .

Whereas in the bending response the coupled system provided higher values of the standard deviation, in the torsional motion the standard deviation is higher for the decoupled system. The variation in the response introduced by the x_α parameter observed previously, has the same behavior for both bending, rigid solid and torsional motions.

The standard deviation is upper bounded by 0.1016 for the decoupled system and 0.0375 for the coupled system. Nonetheless, it is worth noting that the values are notably lower than those provided by the bending response, and more significantly so by those of the rigid body motion.

Table 6.11: Boeing 737-300 σ_h for the torsional motion changing λ , for $x_{\alpha 1}$

	λ_1	λ_2	λ_3	λ_4
Coupled	0.0375	0.0380	0.0386	0.0397
Decoupled	0.1016	0.1028	0.1041	0.1067

Table 6.11 shows the torsional standard deviations of the Boeing 737-300, when changing the λ parameter, having fixed the crossed inertia parameter at $x_{\alpha 1}$.

As mentioned earlier, changes in the mass parameter do not introduce a significant variation in the response. Therefore, no further analysis will be conducted on this aspect.

6.7.2 Cessna 172

Bending motion

Table 6.12: Cessna 172 σ_h for the bending motion changing x_α , for λ_1

	$x_{\alpha 1}$	$x_{\alpha 2}$	$x_{\alpha 3}$	$x_{\alpha 4}$
Coupled	0.2262	0.2443	0.2590	0.2953
Decoupled	0.2180	0.2180	0.2180	0.2180

Table 6.12 shows the bending standard deviations of the Cessna 172, when changing the x_α parameter, having fixed the mass parameter at λ_1 .

The trend is evident: increasing the value of the crossed inertia parameter from $x_\alpha = -0.5$ to $x_\alpha = 0.5$ leads to an increase in the standard deviation value. Notably, for the bending case, the standard deviation values are higher in the coupled system compared to the decoupled system.

As anticipated, the standard deviation of the decoupled system remains constant while varying the crossed inertia parameter. However, when comparing the results for $x_{\alpha 1}$, the difference in standard deviations between the coupled and decoupled systems is only about 3.67%. In contrast, for $x_{\alpha 4}$, this difference grows to 35.32%.

The standard deviation is upper bounded by 0.29529 and lower bounded by 0.21799. Whereas the general trends are similar to those observed for the Boeing 737-300, the values obtained are notably greater. This is thought to be due to the lack of robustness of the Cessna 172, in comparison to the Boeing 737. It has been observed that higher values of the λ parameter conclude in less aggressive and more stable responses to turbulence.

Table 6.13: Cessna 172 σ_h for the bending motion changing λ , for $x_{\alpha 1}$

	λ_1	λ_2	λ_3	λ_4
Coupled	0.2262	0.2321	0.2440	0.2578
Decoupled	0.2180	0.2232	0.2333	0.2450

Table 6.13 displays the standard deviations for bending effects while varying the λ parameter, with the crossed inertia parameter set at $x_{\alpha 1}$.

As mentioned earlier, modifying the aircraft's mass does not lead to a substantial alteration in the bending turbulence response, as indicated in the previous discussion. The most notable difference in standard deviations among various λ values is merely 0.3. Nonetheless, as the Cessna 172 is a small aircraft when compared to the Boeing 737-300, the same mass variation in the latter has less effects, than on the Cessna 172. It can also be seen that values for the standard deviation are higher for the Cessna 172. As a result, the response is more exacerbated in the Cessna 172.

Rigid body motion

Table 6.14: Cessna 172 σ_h for the rigid body motion changing x_α , for λ_1

	$x_{\alpha 1}$	$x_{\alpha 2}$	$x_{\alpha 3}$	$x_{\alpha 4}$
Coupled	0.2354	0.2510	0.2645	0.3001
Decoupled	0.2225	0.2225	0.2225	0.2225
Analytical	0.0817	0.0817	0.0817	0.0817

Table 6.14 shows the rigid body standard deviations of the Cessna 172, when changing the x_α parameter, having fixed the mass parameter at λ_1 . As expected, the value obtained for the coupled system is greater than the one obtained for the decoupled. Moreover, the previously identified trend is appreciated: the value of the standard deviation increases when the x_α parameter increases.

It is worth noting that whereas in the Boeing 737-300 the approximations considered by the analytical model did not induce a significative deviation from the results. Nonetheless, it is seen that these approximations are not appropriate for smaller aircrafts, which are associated with lower masses. It can be deduced that this result is not reliable because in Figure 6.33, it can be seen that the PSD graph for the decoupled model and the analytical solution match exactly. This would oblige for the standard deviations of both the decoupled and the analytical models to match exactly. Moreover, when the standard deviation for the analytical model was obtained by calculating the area below the PSD, the value obtained was 0.225.

Table 6.15: Cessna 172 σ_h for the rigid body motion changing λ , for $x_{\alpha 1}$

	λ_1	λ_2	λ_3	λ_4
Coupled	0.2354	0.2416	0.2535	0.2678
Decoupled	0.2225	0.2274	0.2377	0.2495
Analytical	0.0817	0.0832	0.0861	0.0895

Table 6.15 shows the rigid body standard deviations of the Cessna 172, when changing the λ parameter, having fixed the crossed inertia parameter at $x_{\alpha 1}$.

As previously noted, alterations in the mass parameter do not bring about a substantial change in the response. Hence, no additional examination will be carried out concerning this aspect.

Torsional motion

Table 6.16: Cessna 172 σ_h for the torsional motion changing x_{α} , for λ_1

	$x_{\alpha 1}$	$x_{\alpha 2}$	$x_{\alpha 3}$	$x_{\alpha 4}$
Coupled	0.0627	0.1502	0.2966	0.3697
Decoupled	0.1955	0.1955	0.1955	0.1955

Table 6.16 shows the torsional standard deviations of the Cessna 172, when changing the x_{α} parameter, having fixed the mass parameter at λ_1 .

Whereas in the bending response the coupled system provided higher values of the standard deviation, in the torsional motion the standard deviation is higher for the decoupled system. The variation in the response introduced by the x_{α} parameter observed previously, has the same behavior for both bending, rigid solid and torsional motions.

The standard deviation is upper bounded by 0.1955 for the decoupled system and 0.3697 for the coupled system. It is remarkable to note that the standard deviation values of the coupled system experience a substantial increase when the parameter x_{α} is varied, reaching a value of 0.3697 for the case of $x_{\alpha 4}$.

Table 6.17: Cessna 172 σ_h for the torsional motion changing λ , for $x_{\alpha 1}$

	λ_1	λ_2	λ_3	λ_4
Coupled	0.0627	0.0649	0.0690	0.0690
Decoupled	0.1955	0.2012	0.2123	0.2249

Table 6.17 shows the torsional standard deviations of the Cessna 172, when changing the λ parameter, having fixed the crossed inertia parameter at $x_{\alpha 1}$.

It is noteworthy to highlight that in the case of $x_{\alpha} = -0.5$, the standard deviation is significantly lower for the coupled case, when compared to the decoupled system.

As previously indicated, adjustments to the mass parameter do not result in a substantial alteration of the response. Hence, there is no need for further analysis in this regard.

Chapter 7

Conclusions and future lines of work

This chapter aims to summarize the results that have been obtained and to present the findings and conclusions of the project, along with future lines of work.

7.1 Conclusions

The present work has allowed the deciphering of the coupling effects associated to atmospheric turbulence. The conclusions drawn as a result from the study carried out, represent an important contribution to the study of atmospheric turbulence, as the effect of multiple degrees-of-freedom has not been deeply explored. The conclusions extracted from the work are as follows:

- The incremental pressure coefficient derived from a harmonic motion of the flow, $\Delta\tilde{C}_{p1}^G$, was obtained through an exhaustive use of the Bessel functions and the integral of Glauert, in Chapter 4. This result shows that $\Delta\tilde{C}_{p1}^G$ decreases when the reduced frequency increases k . This result is of great importance as it involved complex mathematical instruments and it describes the motion of a slender body immersed in a flow subjected to harmonic motions. These motions include the motion of fishes immersed in the water [27], flags subjected to unsteady aerodynamics [28], the flapping motion of the wings of birds [28], etc.
- The dimensionless aerodynamic forces associated to the gust, influenced by the harmonic motion induced by turbulence, have been observed to be greater for the C172, when compared to those of the Boeing 737-300. The obtention of this result has shed light over a key point when studying the effect of turbulence on aerostructures. As can be seen in the Equations presented in Chapter 5, the dimensionless aerodynamic forces are inversely proportional to the mass parameter λ . As a result, it is clear that a lower value of λ is translated in a higher value of the dimensionless aerodynamic forces.
- The results presented in Chapter 6 illustrate a trend regarding the influence of the torsional mode on the bending response, and viceversa, for both aircrafts, B737 and C172. Focusing on the Acceleration transfer matrixs, it is observed that the influence of the torsional mode in the transfer function is significant, as well as is the bending mode on the torsional response. Nonetheless, the results obtained for the PSDs of both the bending and torsional motions, differ from these results. The results for the PSDs, for the bending motion, illustrate that the influence of the torsional mode in this motion, whereas existing, contains a significantly reduced energetic contribution than would be estimated on the basis of the transfer function.

Along these lines, the contribution of the bending mode, in the results obtained for the torsional motion PSD, is almost negligible. This result is supported by the results presented for the transfer function for the torsional mode. It is observed that the energetic contribution of the bending mode in the torsional Acceleration transfer matrix is upper and lower bounded by -5 and -15 in logarithm scale, which equals to 10^{-5} and 10^{-15} .

- The coupled system provides a more attenuated response of the model. The transfer acceleration function associated to the bending motion of the airfoil, when considering the coupled system, achieves a higher peak values than the decoupled system. Nonetheless, the effect of the torsional mode on the bending motion triggers a second amplitude variation in the system, which leads to an attenuation of the response, for reduced frequencies higher than k_α . In accordance with this result, for the torsional motion, the transfer acceleration function illustrates higher values for the decoupled system, due to the action of the bending mode which induces an amplitude variation. Following the same pattern as in the bending function, the response of the coupled system is attenuated for values of the reduced frequencies higher than k_α . The reason for these attenuations is that the coupling of the modes introduces aerodynamic terms which are proportional to first derivatives of both the bending and torsional modes, $\dot{h}$ and $\dot{\alpha}$, which constitute the mechanism of aerodynamic damping.
- Regarding the rigid body motion, the results presented in Chapter 6, illustrate that the Acceleration transfer matrix for the decoupled system matches exactly with the analytical expression. Similarly, the PSDs obtained for both aircrafts, for the decoupled system match exactly with the PSD obtained by the analytical model, when considering rigid body motion.
- The results presented for the Power Spectral Densities show that both the bending and torsional responses are mainly derived from the energy associated to the bending and torsional frequencies, respectively. Whereas the bending PSD shows an influence of the energy associated to the torsional mode, the torsional response shows almost no influence of the bending mode. This conclusion is coherent with the conclusion derived from the observation of the transfer functions.
- The variation of the x_α parameter has a very diverse effect. In most of the cases presented in Chapter 6, the value of $x_{\alpha 4} = 0.5$ causes a more pronounced response than the other values for both the B737 and the C172. However, it should be noted that the value of $x_{\alpha 3} = 0.25$ is the most favorable of all those evaluated for both airplanes. This is because the aircraft response for $x_{\alpha 3}$, presents the lowest values for the transfer function and for the PSD. Furthermore, for the value of 0.25, the influence of torsional mode on the bending response is the lowest presented, in the coupled airfoil model.
- All the results presented in Chapter 6 demonstrate that the variation of the weight on both B737 and C172, does not induce a significant variation of the response, for the selected ranges of weights. These ranges are deemed representative for the presented aircrafts, as these cover the weights starting on the Maximum Take-Off Weight decreasing until Maximum Landing Weight is reached. In accordance with the previously exposed conclusion, the variation of the λ parameter is more significant for the C172, as its range encompasses lighter weights. Nonetheless, the variation of the λ parameter, does not provide significant fluctuations the response to atmospheric turbulence.

- The comparison between the responses of both aircrafts B737 and C172 illustrates the differences between the encounter of atmospheric turbulence for a heavier and lighter aircrafts, respectively. As expressed in the previously mentioned conclusion, the fact that the λ parameter is inversely proportional to the aerodynamic forces is reflected in all the results exposed in Chapter 6. Thus, it is clearly observed that transfer acceleration functions and PSDs, of the bending, torsional and rigid body motions, cover a wider range of values for the C172, than the B737. It is concluded that the response to turbulence is less energetic and more damped for higher values of the λ parameter.
- The standard deviations presented in Chapter 6 allow the identification of a pattern for both aircrafts, when the analyzing the coupled and decoupled systems. It is observed for the bending motion that the standard deviations obtained, for the coupled system, are greater than those of the decoupled system. In contrast, when referring to the torsional motion the standard deviations obtained for the decoupled system are greater than those of the coupled system, for $x_{\alpha 1}$ and $x_{\alpha 2}$. This is a result of the substantial increase that the standard deviation suffers when increasing the x_{α} parameter.
- Regarding the standard deviations for the rigid body motions, for both aircrafts, there is a clear discrepancy between the expected values and those that have been obtained for the analytical model. Whereas transfer functions and PSDs coincide for the decoupled and the analytical models, the values of the standard deviation of the decoupled system and the analytical model differ, especially for the C172 showing a clear incoherency. After deeper research, it has been concluded that the approximations used for the analytical model, laid out in Chapter 6.1.3 induce a significant numerical discrepancy. Concretely, for the C172, the numerical error introduced by these approximations has a greater impact than on the B737.

7.2 Future lines of work

This section aims to cover some of the future lines of work deemed interesting for the enhancing of the study and the extension its utilities.

- The project carried out for the study of the coupling of modes in the response to atmospheric turbulence, consider sonly the bending and torsional modes. Whereas these are deemed representative for a first approximation, the study of models that consider more degrees-of-freedom, such as the deflection of an aileron, is of great interest.
- For the obtention of more precise and reliable results, it is deemed interesting to obtain and work with higher precision schemes for numerical integration. In last instance, the resolution of the present model analytically, without using approximations, would result in the most precise and robust model, as there would not be an associated numerical error.
- The analysis and variation of all the structural parameters, both mass-related and geometrical, would derive the complete comprehension of the phenomena. It would also allow the design of an airfoil which is best-suited for its application, as a compromise solution of the previously mentiones parameters.

- The calculation of the limit loads established for each aircraft in [12]. In addition, the calculation of the probability of these loads being surpassed should be calculated, verifying whether the *CS-25* criteria are satisfied.
- The extension of the two-dimensional model to full wings would allow the calculation of moments and forces at the wing attachment. This would allow accurate calculation of the structural damage to the wing, if encountered with atmospheric turbulence. For example, it could be accomplished by the application of the strip theory [29].

In conclusion, the effect of the coupling of the degrees-of-freedom in an airfoil subjected to turbulence is of utmost importance, as it provides an attenuated response of the airfoil.

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Appendix A

Finite integrals

Shown in Appendix A are the finite integrals intervening in the calculation of the aerodynamic forces of the model subjected to study in this project.

$$\int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{d\hat{\xi}}{\hat{x}-\hat{\xi}} = -\pi \quad (\text{A.1})$$

$$\int_{-1}^1 \sqrt{\frac{1+\hat{\xi}}{1-\hat{\xi}}} \frac{\hat{\xi} d\hat{\xi}}{\hat{x}-\hat{\xi}} = -\pi(1+\hat{x}) \quad (\text{A.2})$$

$$\int_{-1}^1 \sqrt{\frac{1-\hat{\xi}}{1+\hat{\xi}}} \frac{\hat{\xi} d\hat{\xi}}{\hat{x}-\hat{\xi}} = -\pi(1-\hat{x}) \quad (\text{A.3})$$

$$\int_{-1}^1 \sqrt{\frac{1-\hat{\xi}}{1+\hat{\xi}}} \hat{\xi} d\hat{\xi} = -\frac{\pi}{2} \quad (\text{A.4})$$

Appendix B

MATLAB code

```
1      %% Aeroelastic response to atmospheric turbulence
2
3      clear
4      close all
5
6      %% auxiliar
7      j          = sqrt(-1);
8
9      %%imaginary unit
10
11     %% b737 characteristics
12     U_inf_B737   = 0.745*340;
13
14     %%cruise speed m/s
15     rho_inf      = 1.225;
16
17     %%z = 0 kg/m3
18     b_B737       = 28.88;
19
20     %%wingspan B737-300 m
21     c_B737       = 4.71;
22
23     %%chord m
24     MLW_B737     = 51720;
25
26     %%Maximum Landing Weight B737-300
27     MZFW_B737    = 47625;
28
29     %%Maximum Zero Fuel Weight B737-300
30     MTOW_B737    = 56470;
31
32     %%Maximum Take-Off Weight B737-300
33     M_B737       = [MTOW_B737 55300 54095 MLW_B737]/
34                     b_B737; %kg /m
35
36     w_h_B737     = 5;
37
38     %%rad/s
```

```

19      w_alpha_B737    = 30;

      %rad/s
20      k_h_B737        = w_h_B737*c_B737/(2*U_inf_B737);
                                     %bending reduced
      frequency
21      k_alpha_B737    = w_alpha_B737*c_B737/(2*U_inf_B737);
                                     %torsional reduced
      frequency
22      g_h_B737        = 0.01;

      %bending damping coefficient
23      g_alpha_B737    = 0.01;

      %torsional damping coefficient
24      lamd_B737       = [M_B737(1) M_B737(2) M_B737(3) M_B737
                          (4)]/(rho_inf*pi*c_B737^2); %mass dimensionless
      parameter
25      r_alpha2_B737   = 0.75;

      %inertial dimensionless parameter
26      x_alpha         = [-0.5 -0.25 0.25 0.5];
                                     %crossed
      inertial dimensionless parameter
27      Lamd1           = 0.2;

      %dimensionless position of the EA
28      z_mo_B737       = 11300;

      %B737-300 service ceiling m
29      g0               = 9.81;

      %gravity value at sea level m/s2
30
31      %% c172 characteristics
32      U_inf_C172       = 57.103333;

      %cruise speed m/s
33      b_C172           = 10.90;

      %wingspan C172 m
34      c_C172           = 1.472;

      %chord m
35      MLW_C172         = 1156.661;

      %Maximum Landing Weight C172
36      MZFW_C172        = 762.0352;

      %Maximum Zero Fuel Weight C172

```

```

37 MTOW_C172      = 1156.661;

      %Maximum Take-Off Weight C172
38 M_C172         = [MTOW_C172 1100 1000 900]/b_C172;
                                     %kg /m
39 w_h_C172       = 5;

      %rad/s
40 w_alpha_C172   = 30;

      %rad/s
41 k_h_C172       = w_h_C172*c_C172/(2*U_inf_C172);
                                     %bending reduced
      frequency
42 k_alpha_C172   = w_alpha_C172*c_C172/(2*U_inf_C172);
                                     %torsional reduced
      frequency
43 g_h_C172       = 0.01;

      %bending damping coefficient
44 g_alpha_C172   = 0.01;

      %torsional damping coefficient
45 lamd_C172      = [M_C172(1) M_C172(2) M_C172(3) M_C172
      (4)]/(rho_inf*pi*c_C172^2); %mass dimensionless
      parameter
46 r_alpha2_C172  = 0.75;

      %inertial dimensionless parameter
47 z_mo_C172      = 3962.4;

      %C172 service ceiling m
48
49 %% TURBULENCE parameters
50 L = 762;                                     %reference
      turbulence lenght (2500 ft) in m
51 sigm_w = 1*0.3048;                           %standard
      deviation of the turbulent vertical speed
52
53 %% DIMENSIONLESS INCREASE OF PRESSURES CAUSED BY A
      HARMONIC MOTION
54 k = 0.001:0.001:1;                           %reduced
      frequency
55 n = length(k);                               %length of
      the reduced frequency vector
56 C = zeros(n,1);                             %Theodorsen
      function initialization
57 S = zeros(n,1);                             %Sears
      function initialization
58

```

```

59     for ii=1:n
60         C(ii,1) = besselh(1,2,k(ii))/(besselh(1,2,k(ii))+1j*
            besselh(0,2,k(ii)));           %Theodorsen
            function
61         S(ii,1) = 1j*besselj(1,k(ii)) + C(ii,1)*(besselj(0,k(
            ii))-1j*besselj(1,k(ii)));      %Sears function
62     end
63
64     x = -1:0.01:1;
                                           %
        discretizacion de la cuerda adimensional del
        perfil
65
66     deltCp = zeros (n,length(x));
67
68     for ii=1:n
69         for h=1:length(x)
70             deltCp(ii,h)=4*sqrt((1-x(h))/(1+x(h)))*S(ii); %
                pressure increase depending on frequency
71         end
72     end
73
74     %% Dimensionless aerodynamic forces associated to the
        gust
75     Q_h_ad_B737      = zeros (n,4);
76     Q_alpha_ad_B737  = zeros (n,4);
77     for u=1:4
78         for ii=1:n
79             %bending
80             Q_h_ad_B737 (ii, u)      = -S(ii,1)/(2*lamd_B737(u)*k(
                ii)^2);
81
82             %torsion
83             Q_alpha_ad_B737 (ii,u)   = (Lamd1+1/2)*S(ii,1)/(2*
                lamd_B737(u)*k(ii)^2);
84         end
85     end
86
87     %% Dimensionless aerodynamic forces associated to the
        gust
88     Q_h_ad_C172      = zeros (n,4);
89     Q_alpha_ad_C172  = zeros (n,4);
90     for u=1:4
91         for ii=1:n
92             %bending
93             Q_h_ad_C172 (ii, u)      = -S(ii,1)/(2*lamd_C172(u)*k(
                ii)^2);
94
95             %torsion

```

```

96 Q_alpha_ad_C172 (ii,u) = (Lamd1+1/2)*S(ii,1)/(2*
    lamd_C172(u)*k(ii)^2);
97 end
98 end
99 %% Transfer function acceleration B737
100 A_x_2_B737 = zeros(2*n, 2*n,4, 4);
    %correlation matrix
    coupled
101 A_x_2_des_B737 = zeros(2*n, 2*n,4, 4);
    %correlation matrix decoupled
102 H_x_2_B737 = zeros(2*n, 2*n,4, 4);
    %transfer matrix coupled
103 H_x_2_des_B737 = zeros(2*n, 2*n, 4, 4);
    %transfer matrix decoupled
104 H_hh_2_B737 = zeros (n, 1, 4, 4);
105 H_aa_2_B737 = zeros (n, 1, 4, 4);
106 H_ha_2_B737 = zeros (n, 1, 4, 4);
107 H_ah_2_B737 = zeros (n, 1, 4, 4);
108
109 H_h_2_1gd1_B737 = zeros(n, 1, 4, 4);
    %Acceleration transfer
    matrix for rigid body assumption (fogeda_2015)
110
111 DM = 1;
112 DM_des = 0;
113 for u=1:4
114 for z = 1:4
115 p = 0;
116 for ii=1:n
117
118
119 A_x_2_B737(ii+p:ii+1+p,ii+p:ii+1+p, z, u) =
    [(-1+(k_h_B737/k(ii))^2*(1+1j*g_h_B737)+1j*C(ii,1)
    /(2*lamd_B737(u)*k(ii))-1/(4*lamd_B737(u)))/
    Q_h_ad_B737(ii, u)...
120 DM*(1j/(4*lamd_B737(u)*k(ii))+Lamd1/(4*lamd_B737(u))+
    C(ii,1)/(2*lamd_B737(u)*k(ii)^2)+C(ii,1)*(1/2-
    Lamd1)*1j/(2*lamd_B737(u)*k(ii))-x_alpha(z))/
    Q_h_ad_B737(ii, u);
121 DM*(Lamd1/(4*lamd_B737(u))-(1/2+Lamd1)*C(ii,1)*1j/(2*
    lamd_B737(u)*k(ii))-x_alpha(z))/Q_alpha_ad_B737(ii
    , u)...
122 (r_alpha2_B737*((k_alpha_B737/k(ii))^2*(1+1j*
    g_alpha_B737)-1)+(1/2-Lamd1)*1j/(4*lamd_B737(u)*k(
    ii))-(1/8+Lamd1^2)/(4*lamd_B737(u))-(1/2+Lamd1)*C(
    ii,1)/(2*lamd_B737(u)*k(ii)^2)-(1/4-Lamd1^2)*1j
    /(2*lamd_B737(u)*k(ii)))/Q_alpha_ad_B737(ii, u)];
123
124 A_x_2_des_B737(ii+p:ii+1+p,ii+p:ii+1+p, z, u) =
    [(-1+(k_h_B737/k(ii))^2*(1+1j*g_h_B737)+1j*C(ii,1)

```

```

        /(2*lamd_B737(u)*k(ii))-1/(4*lamd_B737(u)))/
        Q_h_ad_B737(ii,u)...
125 DM_des*(1j/(4*lamd_B737(u)*k(ii))+Lamd1/(4*lamd_B737(
        u))+C(ii,1)/(2*lamd_B737(u)*k(ii)^2)+C(ii,1)*(1/2-
        Lamd1)*1j/(2*lamd_B737(u)*k(ii))-x_alpha(z))/
        Q_h_ad_B737(ii,u);
126 DM_des*(Lamd1/(4*lamd_B737(u))-(1/2+Lamd1)*C(ii,1)*1j
        /(2*lamd_B737(u)*k(ii))-x_alpha(z))/
        Q_alpha_ad_B737(ii,u) ...
127 (r_alpha2_B737*((k_alpha_B737/k(ii))^2*(1+1j*
        g_alpha_B737)-1)+(1/2-Lamd1)*1j/(4*lamd_B737(u)*k(
        ii))-(1/8+Lamd1^2)/(4*lamd_B737(u))-(1/2+Lamd1)*C(
        ii,1)/(2*lamd_B737(u)*k(ii)^2)-(1/4-Lamd1^2)*1j
        /(2*lamd_B737(u)*k(ii)))/Q_alpha_ad_B737(ii,u)];
128
129 H_x_2_B737(ii+p:ii+1+p,ii+p:ii+1+p, z, u) = -k(
        ii)^2*U_inf_B737/(c_B737/2)^2* inv(A_x_2_B737(ii+p
        :ii+1+p,ii+p:ii+1+p, z, u));
130 H_x_2_des_B737(ii+p:ii+1+p,ii+p:ii+1+p, z, u) = -k(
        ii)^2*U_inf_B737/(c_B737/2)^2* inv(A_x_2_des_B737(
        ii+p:ii+1+p,ii+p:ii+1+p, z, u));
131
132 H_aa_2_B737 (ii, 1, z, u) =
        H_x_2_B737 (ii+1+p, ii+1+p, z, u);
133 H_hh_2_B737 (ii, 1, z, u) =
        H_x_2_B737 (ii+p, ii+p, z, u);
134 H_ha_2_B737 (ii, 1, z, u) =
        H_x_2_B737 (ii+p, ii+1+p, z, u);
135 H_ah_2_B737 (ii, 1, z, u) =
        H_x_2_B737 (ii+1+p, ii+p, z, u);
136
137 H_h_2_1gdl_B737 (ii, 1, z, u) = -
        U_inf_B737/(c_B737/2)^2*S(ii)/(2*(lamd_B737(u)
        +1/4 - 1j*C(ii)/(2*k(ii)))); %Admitancia
        aceleracion 1gdl solido rigido
138
139 p = ii; %
        auxiliar
140 end
141 end
142 end
143
144
145
146 H_vec_2_x_B737
        = zeros(2*n, 1, 4,4);
147 H_vec_2_des_x_B737
        = zeros(2*n, 1, 4, 4);
148
149 for u=1:4

```

```

150     for z =1:4
151         H_vec_2_x_B737(:, 1,z, u) = sum(H_x_2_B737
152             (:,:, z, u),2); %auxiliar
153         H_vec_2_des_x_B737 (:, 1,z, u) = sum(
154             H_x_2_des_B737(:,:, z, u),2); %auxiliar
155     end
156 end
157 % Transfer function acceleration C172
158 A_x_2_C172 = zeros(2*n, 2*n,4, 4);
159 %correlation matrix
160 %coupled
161 A_x_2_des_C172 = zeros(2*n, 2*n,4, 4);
162 %correlation matrix decoupled
163 H_x_2_C172 = zeros(2*n, 2*n,4, 4);
164 %transfer matrix coupled
165 H_x_2_des_C172 = zeros(2*n, 2*n, 4, 4);
166 %transfer matrix decoupled
167 H_hh_2_C172 = zeros (n, 1, 4, 4);
168 H_aa_2_C172 = zeros (n, 1, 4, 4);
169 H_ha_2_C172 = zeros (n, 1, 4, 4);
170 H_ah_2_C172 = zeros (n, 1, 4, 4);
171
172 H_h_2_1gdl_C172 = zeros(n, 1, 4, 4);
173 %Acceleration transfer
174 %matrix for rigid body assumption (fogeda_2015)
175
176 DM = 1;
177 DM_des = 0;
178 for u = 1:4
179     for z = 1:4
180         p = 0;
181         for ii=1:n
182             A_x_2_C172(ii+p:ii+1+p,ii+p:ii+1+p, z, u) =
183                 [(-1+(k_h_C172/k(ii))^2*(1+1j*g_h_C172)+1j*C(ii,1)
184                     /(2*lamd_C172(u)*k(ii))-1/(4*lamd_C172(u)))/
185                     Q_h_ad_C172(ii, u)...
186                     DM*(1j/(4*lamd_C172(u)*k(ii))+Lamd1/(4*lamd_C172(u))+
187                         C(ii,1)/(2*lamd_C172(u)*k(ii)^2)+C(ii,1)*(1/2-
188                             Lamd1)*1j/(2*lamd_C172(u)*k(ii))-x_alpha(z))/
189                         Q_h_ad_C172(ii, u);
190                     DM*(Lamd1/(4*lamd_C172(u))-(1/2+Lamd1)*C(ii,1)*1j/(2*
191                         lamd_C172(u)*k(ii))-x_alpha(z))/Q_alpha_ad_C172(ii
192                             , u)...
193                     (r_alpha2_C172*((k_alpha_C172/k(ii))^2*(1+1j*
194                         g_alpha_C172)-1)+(1/2-Lamd1)*1j/(4*lamd_C172(u)*k(

```

```

ii))-(1/8+Lamd1^2)/(4*lamd_C172(u))-(1/2+Lamd1)*C(
ii,1)/(2*lamd_C172(u)*k(ii)^2)-(1/4-Lamd1^2)*1j
/(2*lamd_C172(u)*k(ii)))/Q_alpha_ad_C172(ii, u)];
182
183 A_x_2_des_C172(ii+p:ii+1+p,ii+p:ii+1+p, z, u) =
[(-1+(k_h_C172/k(ii))^2*(1+1j*g_h_C172)+1j*C(ii,1)
/(2*lamd_C172(u)*k(ii))-1/(4*lamd_C172(u)))/
Q_h_ad_C172(ii,u)...
184 DM_des*(1j/(4*lamd_C172(u)*k(ii))+Lamd1/(4*lamd_C172(
u))+C(ii,1)/(2*lamd_C172(u)*k(ii)^2)+C(ii,1)*(1/2-
Lamd1)*1j/(2*lamd_C172(u)*k(ii))-x_alpha(z))/
Q_h_ad_C172(ii,u);
185 DM_des*(Lamd1/(4*lamd_C172(u))-(1/2+Lamd1)*C(ii,1)*1j
/(2*lamd_C172(u)*k(ii))-x_alpha(z))/
Q_alpha_ad_C172(ii,u) ...
186 (r_alpha2_C172*((k_alpha_C172/k(ii))^2*(1+1j*
g_alpha_C172)-1)+(1/2-Lamd1)*1j/(4*lamd_C172(u)*k(
ii))-(1/8+Lamd1^2)/(4*lamd_C172(u))-(1/2+Lamd1)*C(
ii,1)/(2*lamd_C172(u)*k(ii)^2)-(1/4-Lamd1^2)*1j
/(2*lamd_C172(u)*k(ii)))/Q_alpha_ad_C172(ii,u)];
187
188 H_x_2_C172(ii+p:ii+1+p,ii+p:ii+1+p, z, u) = -k(
ii)^2*U_inf_C172/(c_C172/2)^2* inv(A_x_2_C172(ii+p
:ii+1+p,ii+p:ii+1+p, z, u));
189 H_x_2_des_C172(ii+p:ii+1+p,ii+p:ii+1+p, z, u) = -k(
ii)^2*U_inf_C172/(c_C172/2)^2* inv(A_x_2_des_C172(
ii+p:ii+1+p,ii+p:ii+1+p, z, u));
190
191 H_aa_2_C172 (ii, 1, z, u) =
H_x_2_C172 (ii+1+p, ii+1+p, z, u);
192 H_hh_2_C172 (ii, 1, z, u) =
H_x_2_C172 (ii+p, ii+p, z, u);
193 H_ha_2_C172 (ii, 1, z, u) =
H_x_2_C172 (ii+p, ii+1+p, z, u);
194 H_ah_2_C172 (ii, 1, z, u) =
H_x_2_C172 (ii+1+p, ii+p, z, u);
195
196 H_h_2_1gdl_C172 (ii, 1, z, u) = -
U_inf_C172/(c_C172/2)^2*S(ii)/(2*(lamd_C172(u)
+1/4 - 1j*C(ii)/(2*k(ii)))); %Admitancia
aceleracion 1gdl solido rigido
197
198 p = ii; %
auxiliar
199 end
200 end
201 end
202
203
204

```

```

205 H_vec_2_x_C172 = zeros(2*n, 1, 4,4);
206 H_vec_2_des_x_C172= zeros(2*n, 1, 4, 4);
207
208 for u=1:4
209 for z =1:4
210 H_vec_2_x_C172(:, 1,z, u) = sum(H_x_2_C172
    (:,:, z, u),2); %auxiliar
211 H_vec_2_des_x_C172 (:, 1,z, u) = sum(
    H_x_2_des_C172(:,:, z, u),2); %auxiliar
212
213 end
214 end
215
216 %% Turbulence model and calculation of PSDs B737
217 Dr_aux_B737= zeros(2*n, 1);
218 Dr_B737 = zeros(n, 1); %
    Dryden model
219 VK_B737 = zeros(n, 1); %Von
    Karman model
220 p = 0;
221 for ii = 1:n
222 Dr_B737(ii) = sigm_w^2*L/(pi*U_inf_B737
    )*(1+3*(k(ii)*L/(c_B737/2))^2)/(1+(k(ii)*L/(c_B737
    /2))^2)^2;
223 VK_B737(ii) = sigm_w^2*L/(pi*U_inf_B737
    )*(1+8/3*(1.339*k(ii)*L/(c_B737/2))^2)/(1+(1.339*k
    (ii)*L/(c_B737/2))^2)^(11/6);
224
225
226 Dr_aux_B737(ii+p:ii+1+p,1) = Dr_B737(ii,1); %
    auxiliar
227 p = ii;
228 end
229
230 Dr__B737 = log(Dr_B737);
    %auxiliar
231
232 VK__B737 = log(VK_B737);
233
234 PSD_q_B737 = zeros(2*n, 1,4, 4); %PSD
    coupled
235
236 PSD_q_des_B737 = zeros(2*n, 1,4, 4); %PSD
    decoupled
237
238 for u = 1:4
239 for z = 1:4
240 for ii= 1:2*n
241 PSD_q_B737(ii, 1,z, u) = abs(H_vec_2_x_B737(ii,
    1, z, u))^2 * Dr_aux_B737 (ii,1);

```

```

241 PSD_q_des_B737(ii, 1,z, u) = abs(H_vec_2_des_x_B737(
    ii, 1,z, u))^2 * Dr_aux_B737 (ii,1);
242 end
243 end
244 end
245
246 PSD_qi_B737          = zeros (2,n, 4, 4);          %
    auxiliar
247 H_rep_2_B737         = zeros (2,n, 4, 4);          %
    auxiliar
248 PSD_qi_des_B737      = zeros (2,n, 4, 4);          %
    auxiliar
249 H_rep_2_des_B737     = zeros (2,n, 4, 4);          %
    auxiliar
250 p=0;
251 for u=1:4
252     for z =1:4
253         for h=1:n
254             PSD_qi_B737(:,h, z, u)          = PSD_q_B737 (h+p:h+p+1,
                1,z, u);
255             H_rep_2_B737 (:,h, z, u)          = H_vec_2_x_B737 (h+p:h+
                p+1, 1,z, u);
256             PSD_qi_des_B737(:,h, z, u)      = PSD_q_des_B737 (h+p:h+
                p+1, 1,z, u);
257             H_rep_2_des_B737 (:,h, z, u) = H_vec_2_des_x_B737 (h+
                p:h+p+1, 1, z, u);
258
259             p=h;                                %auxiliar
260         end
261     end
262 end
263
264 H_rep1_2_B737         = zeros(2,n, 4, 4);          %
    auxiliar
265 H_rep1_2_des_B737     = zeros(2,n, 4, 4);          %
    auxiliar
266
267 H_hh_rep_2_B737       = zeros(n,1, 4, 4);          %
    auxiliar
268 H_aa_rep_2_B737       = zeros(n,1, 4, 4);          %
    auxiliar
269 H_ha_rep_2_B737       = zeros(n,1, 4, 4);          %
    auxiliar
270 H_ah_rep_2_B737       = zeros(n,1, 4, 4);          %
    auxiliar
271
272 for u=1:4
273     for z =1:4
274         H_rep1_2_B737 (:,:, z, u)          = abs(H_rep_2_B737
                (:,:, z, u)).^2;

```

```

275 H_rep1_2_des_B737 (:,:, z, u) = abs(H_rep_2_des_B737
    (:,:, z, u)).^2;
276 H_hh_rep_2_B737 (:, 1, z, u) = abs(H_hh_2_B737 (:,
    1, z, u)).^2;
277 H_aa_rep_2_B737 (:, 1, z, u) = abs(H_aa_2_B737 (:,
    1, z, u)).^2;
278 H_ha_rep_2_B737 (:, 1, z, u) = abs(H_ha_2_B737 (:,
    1, z, u)).^2;
279 H_ah_rep_2_B737 (:, 1, z, u) = abs(H_ah_2_B737 (:,
    1, z, u)).^2;
280
281 end
282 end
283
284
285 %% Turbulence model and calculation of PSDs C172
286 Dr_aux_C172= zeros(2*n, 1);
287 Dr_C172 = zeros(n, 1); %
    Dryden model
288 VK_C172 = zeros(n, 1); %Von
    Karman model
289 p = 0;
290 for ii = 1:n
291 Dr_C172(ii) = sigm_w^2*L/(pi*U_inf_C172
    )*(1+3*(k(ii)*L/(c_C172/2))^2)/(1+(k(ii)*L/(c_C172
    /2))^2)^2;
292 VK_C172(ii) = sigm_w^2*L/(pi*U_inf_C172
    )*(1+8/3*(1.339*k(ii)*L/(c_C172/2))^2)/(1+(1.339*k
    (ii)*L/(c_C172/2))^2)^(11/6);
293
294
295 Dr_aux_C172(ii+p:ii+1+p,1) = Dr_C172(ii,1); %
    auxiliar
296 p = ii;
297 end
298
299 Dr__C172 = log(Dr_C172);
    %auxiliar
300
301
302 PSD_q_C172 = zeros(2*n, 1,4, 4); %PSD
    coupled
303 PSD_q_des_C172 = zeros(2*n, 1,4, 4); %PSD
    decoupled
304
305 for u = 1:4
306 for z = 1:4
307 for ii= 1:2*n
308

```

```

309     PSD_q_C172(ii, 1,z, u)      = abs(H_vec_2_x_C172(ii,
310         1, z, u))^2 * Dr_aux_C172 (ii,1);
311     PSD_q_des_C172(ii, 1,z, u) = abs(H_vec_2_des_x_C172(
312         ii, 1,z, u))^2 * Dr_aux_C172 (ii,1);
313     end
314     end
315     end
316     PSD_qi_C172      = zeros (2,n, 4, 4);          %
317     auxiliar
318     H_rep_2_C172      = zeros (2,n, 4, 4);          %
319     auxiliar
320     PSD_qi_des_C172   = zeros (2,n, 4, 4);          %
321     auxiliar
322     H_rep_2_des_C172  = zeros (2,n, 4, 4);          %
323     auxiliar
324     p=0;
325     for u=1:4
326     for z =1:4
327     for h=1:n
328     PSD_qi_C172(:,h, z, u)      = PSD_q_C172 (h+p:h+p+1,
329         1,z, u);
330     H_rep_2_C172 (:,h, z, u)      = H_vec_2_x_C172 (h+p:h+
331         p+1, 1,z, u);
332     PSD_qi_des_C172(:,h, z, u)    = PSD_q_des_C172 (h+p:h+
333         p+1, 1,z, u);
334     H_rep_2_des_C172 (:,h, z, u) = H_vec_2_des_x_C172 (h+
335         p:h+p+1, 1, z, u);
336
337     p=h;                                          %auxiliar
338     end
339     end
340     end
341     H_rep1_2_C172      = zeros(2,n, 4, 4);          %
342     auxiliar
343     H_rep1_2_des_C172  = zeros(2,n, 4, 4);          %
344     auxiliar
345
346     H_hh_rep_2_C172    = zeros(n,1, 4, 4);          %
347     auxiliar
348     H_aa_rep_2_C172    = zeros(n,1, 4, 4);          %
349     auxiliar
350     H_ha_rep_2_C172    = zeros(n,1, 4, 4);          %
351     auxiliar
352     H_ah_rep_2_C172    = zeros(n,1, 4, 4);          %
353     auxiliar
354
355     for u=1:4
356     for z =1:4

```

```

343 H_rep1_2_C172 (:,:, z, u)      = abs(H_rep_2_C172
    (:,:, z, u)).^2;
344 H_rep1_2_des_C172 (:,:, z, u)  = abs(H_rep_2_des_C172
    (:,:, z, u)).^2;
345 H_hh_rep_2_C172 (:, 1, z, u)   = abs(H_hh_2_C172 (:,
    1, z, u)).^2;
346 H_aa_rep_2_C172 (:, 1, z, u)   = abs(H_aa_2_C172 (:,
    1, z, u)).^2;
347 H_ha_rep_2_C172 (:, 1, z, u)   = abs(H_ha_2_C172 (:,
    1, z, u)).^2;
348 H_ah_rep_2_C172 (:, 1, z, u)   = abs(H_ah_2_C172 (:,
    1, z, u)).^2;
349
350 end
351 end
352
353 %% Transformation from the frequency domain to the
    time domain
354 %coupled
355 psd_signal_h_2_B737            = zeros(n, 1, z, u);
356 psd_signal_alpha_2_B737       = zeros(n, 1, z, u);
357
358 for u=1:4
359 for z =1:4
360 psd_signal_h_2_B737 (:, 1, z, u)    = PSD_qi_B737(1,
    :, z, u);                %Bending
361 psd_signal_alpha_2_B737(:, 1, z, u) = PSD_qi_B737(2,
    :, z, u);                %Torsion
362
363 end
364 end
365
366 % decoupled
367 psd_signal_h_2_des_B737 = zeros(n, 1, z, u);
368 psd_signal_alpha_2_des_B737 = zeros(n, 1, z, u);
369
370
371 for u=1:4
372 for z =1:4
373 psd_signal_h_2_des_B737 (:, 1, z, u)    =
    PSD_qi_des_B737(1, :, z, u);    %Bending
374 psd_signal_alpha_2_des_B737 (:, 1, z, u) =
    PSD_qi_des_B737(2, :, z, u);    %Torsion
375 end
376 end
377
378 %% Transformation from the frequency domain to the
    time domain
379 %coupled
380 psd_signal_h_2_C172            = zeros(n, 1, z, u);

```

```

381     psd_signal_alpha_2_C172      = zeros(n, 1, z, u);
382
383     for u=1:4
384         for z =1:4
385             psd_signal_h_2_C172 (:, 1, z, u)      = PSD_qi_C172(1,
386                 :, z, u);                          %Bending
387             psd_signal_alpha_2_C172(:, 1, z, u) = PSD_qi_C172(2,
388                 :, z, u);                          %Torsion
389
390         end
391     end
392
393     % decoupled
394     psd_signal_h_2_des_C172 = zeros(n, 1, z, u);
395     psd_signal_alpha_2_des_C172 = zeros(n, 1, z, u);
396
397     for u=1:4
398         for z =1:4
399             psd_signal_h_2_des_C172 (:, 1, z, u)      =
400                 PSD_qi_des_C172(1, :, z, u);          %Bending
401             psd_signal_alpha_2_des_C172 (:, 1, z, u) =
402                 PSD_qi_des_C172(2, :, z, u);          %Torsion
403
404         end
405     end
406
407     %% Variances and standard deviations B737
408     % Use the inverse Fourier transform to obtain the
409     % time-domain signal
410     Var_h_2_B737 = zeros(4);
411     Var_alpha_2_B737 = zeros(4);
412     Var_h_2_des_B737 = zeros(4);
413     Var_alpha_2_des_B737 = zeros(4);
414
415     for u=1:4
416         for z =1:4
417             %Coupled
418             Var_h_2_B737 (z, u)      = trapz(k(1, :),
419                 psd_signal_h_2_B737(:,1,z,u))*(U_inf_B737*2/c_B737
420                 );
421             Var_alpha_2_B737 (z, u)      = trapz(k(1, :),
422                 psd_signal_alpha_2_B737(:,1,z,u))*(U_inf_B737*2/
423                 c_B737);
424
425             %Decoupled

```

```

421 Var_h_2_des_B737(z, u)      = trapz(k(1, :),
    psd_signal_h_2_des_B737(:,1,z,u))*(U_inf_B737*2/
    c_B737);
422 Var_alpha_2_des_B737 (z, u) = trapz(k(1, :),
    psd_signal_alpha_2_des_B737(:,1,z,u))*(U_inf_B737
    *2/c_B737);
423
424 end
425 end
426
427 Var_g_B737 = trapz(k(1, :), Dr_B737(:,1))*(U_inf_B737
    *2/c_B737);          %variance turbulence
    Dryden
428
429
430 sigm_h_2_B737          = sqrt(Var_h_2_B737);
431 sigm_alpha_2_B737      = sqrt(Var_alpha_2_B737);
432
433 sigm_h_2_des_B737      = sqrt(Var_h_2_des_B737);
434 sigm_alpha_2_des_B737  = sqrt(Var_alpha_2_des_B737);
435
436 sigm_g_B737            = sqrt(Var_g_B737);
437
438
439 % Variances and standard deviations C172
440 % Use the inverse Fourier transform to obtain the
    time-domain signal
441 Var_h_2_C172 = zeros(4);
442 Var_alpha_2_C172 = zeros(4);
443 Var_h_2_des_C172 = zeros(4);
444 Var_alpha_2_des_C172 = zeros(4);
445
446 for u=1:4
447 for z =1:4
448
449 %Coupled
450 Var_h_2_C172 (z, u)      = trapz(k(1, :),
    psd_signal_h_2_C172(:,1,z,u))*(U_inf_C172*2/c_C172
    );
451 Var_alpha_2_C172 (z, u)   = trapz(k(1, :),
    psd_signal_alpha_2_C172(:,1,z,u))*(U_inf_C172*2/
    c_C172);
452
453 %Decoupled
454 Var_h_2_des_C172(z, u)    = trapz(k(1, :),
    psd_signal_h_2_des_C172(:,1,z,u))*(U_inf_C172*2/
    c_C172);
455 Var_alpha_2_des_C172 (z, u) = trapz(k(1, :),
    psd_signal_alpha_2_des_C172(:,1,z,u))*(U_inf_C172
    *2/c_C172);

```

```

456
457     end
458     end
459
460     Var_g_C172 = trapz(k(1, :), Dr_C172(:,1))*(U_inf_C172
        *2/c_C172);                                %variance turbulence
        Dryden
461
462
463     sigm_h_2_C172 = sqrt(Var_h_2_C172);
464     sigm_alpha_2_C172 = sqrt(Var_alpha_2_C172);
465
466     sigm_h_2_des_C172 = sqrt(Var_h_2_des_C172);
467     sigm_alpha_2_des_C172 = sqrt(Var_alpha_2_des_C172);
468
469     sigm_g_C172 = sqrt(Var_g_C172);
470
471
472     %% Standard deviation obtained using the eqs laid out
        in Fogeda_2015 B737
473     r_L_B737 = c_B737/(2*L);
474     a_B737 = 2/(r_L_B737*(2*lamd_B737(1)+1));
475     I_1_B737 = (2*pi*r_L_B737)/(1+4*pi^2*r_L_B737^2*
        a_B737^2)*(log(2*pi*r_L_B737) + log(a_B737) +1/(4*
        a_B737*r_L_B737));
476     I_a_B737 = (8*pi^3*r_L_B737^3)/(1+4*pi^2*r_L_B737
        ^2*a_B737^2)^2*(log(2*pi*r_L_B737) + log(a_B737)
        +1/(4*a_B737*r_L_B737)) + (pi*r_L_B737)/(a_B737
        ^2*(1+4*pi^2*r_L_B737^2*a_B737^2))*(1/(4*a_B737*
        r_L_B737)-1);
477
478     PSD_h_2_1gdl_B737 = zeros(n,1);
479
480     for ii=1:n
481
482         PSD_h_2_1gdl_B737 (ii, 1) = abs(-U_inf_B737/(c_B737
            /2)^2*S(ii)/(2*(lamd_B737(1)+1/4-1j*C(ii)/(2*k(ii)
            ))))^2*Dr_B737(ii,1);
483
484     end
485
486
487     Var_h_2_num_1gdl_B737 = trapz(k(1, :),
        PSD_h_2_1gdl_B737(:, 1))*(U_inf_B737^2/c_B737);
488
489     Var_h_2_1gdl_B737 = (4*U_inf_B737^2*Var_g_B737)
        /(pi*(c_B737/2)^2*(4*lamd_B737(1)+1)^2*((a_B737
        ^2*(3*a_B737^2-1))/(a_B737^2-1)^2*I_1_B737 - (5*
        a_B737^2-3)/(a_B737^2-1)^2*I_1_B737 +2/(a_B737
        ^2-1)*I_a_B737));

```

```

490
491     sigm_h_2_1gdl_B737      = sqrt(Var_h_2_1gdl_B737);
492
493     sigm_h_2_num_1gdl_B737  = sqrt(Var_h_2_num_1gdl_B737)
494     ;
495
496     %% Standard deviation obtained using the eqs laid out
497     in Fogeda_2015 C172
498     r_L_C172      = c_C172/(2*L);
499     a_C172        = 2/(r_L_C172*(2*lamd_C172(1)+1));
500     I_1_C172      = (2*pi*r_L_C172)/(1+4*pi^2*r_L_C172^2*
501         a_C172^2)*(log(2*pi*r_L_C172) + log(a_C172) + 1/(4*
502         a_C172*r_L_C172));
503     I_a_C172      = (8*pi^3*r_L_C172^3)/(1+4*pi^2*r_L_C172
504         ^2*a_C172^2)^2*(log(2*pi*r_L_C172) + log(a_C172)
505         +1/(4*a_C172*r_L_C172))+ (pi*r_L_C172)/(a_C172
506         ^2*(1+4*pi^2*r_L_C172^2*a_C172^2))*(1/(4*a_C172*
507         r_L_C172)-1);
508
509     PSD_h_2_1gdl_C172 = zeros(n,1);
510
511     for ii=1:n
512
513         PSD_h_2_1gdl_C172 (ii, 1) = abs(-U_inf_C172/(c_C172
514             /2)^2*S(ii)/(2*(lamd_C172(1)+1/4-1j*C(ii)/(2*k(ii)
515             ))))^2*Dr_C172(ii,1);
516
517     end
518
519     Var_h_2_num_1gdl_C172  = trapz(k(1, :),
520         PSD_h_2_1gdl_C172(:, 1))*(U_inf_C172^2/c_C172);
521
522     Var_h_2_1gdl_C172      =(4*U_inf_C172^2*Var_g_C172)
523         /(pi*(c_C172/2)^2*(4*lamd_C172(1)+1)^2)*((a_C172
524         ^2*(3*a_C172^2-1))/(a_C172^2-1)^2*I_1_C172 -(5*
525         a_C172^2-3)/(a_C172^2-1)^2*I_1_C172 +2/(a_C172
526         ^2-1)*I_a_C172);
527
528     sigm_h_2_1gdl_C172      = sqrt(Var_h_2_1gdl_C172);
529
530     sigm_h_2_num_1gdl_C172  = sqrt(Var_h_2_num_1gdl_C172)
531     ;
532
533     %% Probability calculation (according to CS 25.341
534     Gust and turbulence loads) B737
535     %parameters defined by CS-25
536     U_sig_ref      = 27.43; %m/s
537     R1_B737        = MLW_B737/MTOW_B737;

```

```

523 R2_B737 = MZFW_B737/MTOW_B737;
524 F_gm_B737 = sqrt(R2_B737*tan(pi*R1_B737/4));
525 F_gz_B737 = 1-(z_mo_B737/76200);
526 F_g_B737 = 0.5*(F_gz_B737+F_gm_B737);
527
528 PL_1g_B737 = M_B737(1) * g0;
529 PL_B737 = PL_1g_B737 + U_sig_ref*F_g_B737;
530
531                                     %load
532 A_limit_B737 = PL_B737/M_B737(1);
533                                     %acceleration
534
535 prob_h_2_B737 = normcdf([-A_limit_B737
536   A_limit_B737],0,3*sigm_h_2_B737(1, 1));
537 prob_alpha_2_B737 = normcdf([-A_limit_B737
538   A_limit_B737],0,3*sigm_alpha_2_B737(1, 1));
539 prob_h_2_des_B737 = normcdf([-A_limit_B737
540   A_limit_B737],0,3*sigm_h_2_des_B737(1, 1));
541 prob_alpha_2_des_B737 = normcdf([-A_limit_B737
542   A_limit_B737],0,3*sigm_alpha_2_des_B737(1, 1));
543
544 prob_h_2_int_B737 = prob_h_2_B737(2) -
545   prob_h_2_B737(1);
546 prob_alpha_2_int_B737 = prob_alpha_2_B737(2) -
547   prob_alpha_2_B737(1);
548 prob_h_2_des_int_B737 = prob_h_2_des_B737(2) -
549   prob_h_2_des_B737(1);
550 prob_alpha_2_des_int_B737 = prob_alpha_2_des_B737
551   (2) - prob_alpha_2_des_B737(1);
552
553 prob_h_2_1gdl_B737 = normcdf([-A_limit_B737
554   A_limit_B737],0,3*sigm_h_2_1gdl_B737);
555
556 prob_h_2_1gdl_intervalo_B737 = prob_h_2_1gdl_B737(2)
557   - prob_h_2_1gdl_B737(1);
558
559 %% Probability calculation (according to CS 25.341
560   Gust and turbulence loads)C172
561 %parameters defined by CS-25
562 U_sig_ref = 27.43; %m/s
563 R1_C172 = MLW_C172/MTOW_C172;
564 R2_C172 = MZFW_C172/MTOW_C172;
565 F_gm_C172 = sqrt(R2_C172*tan(pi*R1_C172/4));
566 F_gz_C172 = 1-(z_mo_C172/76200);
567 F_g_C172 = 0.5*(F_gz_C172+F_gm_C172);
568
569 PL_1g_C172 = M_C172(1) * g0;
570 PL_C172 = PL_1g_C172 + U_sig_ref*F_g_C172;
571
572                                     %load
573 A_limit_C172 = PL_C172/M_C172(1);
574                                     %acceleration

```

```

558
559     prob_h_2_C172          = normcdf([-A_limit_C172
560         A_limit_C172],0,3*sigm_h_2_C172(1, 1));
561     prob_alpha_2_C172      = normcdf([-A_limit_C172
562         A_limit_C172],0,3*sigm_alpha_2_C172(1, 1));
563     prob_h_2_des_C172      = normcdf([-A_limit_C172
564         A_limit_C172],0,3*sigm_h_2_des_C172(1, 1));
565     prob_alpha_2_des_C172  = normcdf([-A_limit_C172
566         A_limit_C172],0,3*sigm_alpha_2_des_C172(1, 1));
567
568     prob_h_2_int_C172      = prob_h_2_C172(2) -
569         prob_h_2_C172(1);
570     prob_alpha_2_int_C172  = prob_alpha_2_C172(2) -
571         prob_alpha_2_C172(1);
572     prob_h_2_des_int_C172  = prob_h_2_des_C172(2) -
573         prob_h_2_des_C172(1);
574     prob_alpha_2_des_int_C172 = prob_alpha_2_des_C172
575         (2) - prob_alpha_2_des_C172(1);
576
577     prob_h_2_1gdl_C172     = normcdf([-
578         A_limit_C172 A_limit_C172],0,3*sigm_h_2_1gdl_C172)
579         ;
580
581     prob_h_2_1gdl_intervalo_C172 = prob_h_2_1gdl_C172
582         (2) - prob_h_2_1gdl_C172(1);
583
584     %% Auxiliar
585     H_rep1_h_2_1gdl_B737 = abs(H_h_2_1gdl_B737).^2;
586
587     H_rep1_h_2_1gdl_C172 = abs(H_h_2_1gdl_C172).^2;
588
589     aux_cp = zeros(length(x), 1);

```