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A General Form for the Extra Element Theorem

Ricardo Riaza

Abstract—**In this brief we prove that both forms of Middlebrook's formula for the transfer impedance of a two-port can be seen as particular instances of a more general expression involving a generic reference immittance, not necessarily an open-circuit or a short-circuit. This generalized formula holds under minimal regularity assumptions.**

Keywords: Middlebrook's formulae, extra element theorem, two-port, transfer function, homogeneous coordinates.

I. Introduction

Middlebrook's extra element theorem [10], [11] is an analytical result addressing the dependence of a transfer function or a driving point immittance on a number of "extra" circuit elements. Besides, of course, the characterization of transfer functions [1], [7], [10], [11], [21], this technique can be used e.g. for filter design [7], [21], the analysis of converters [7], [12], [13], the determination of frequency responses of microwave circuits [21], or the analysis of feedback circuits [9], [21].

In this brief we focus on the case in which there is a single extra element. Consider, specifically, the (open-circuit) transimpedance of a two-port, defined as the ratio $T = v_2/i_{s_1}$ with $i_2 = 0$. Find an example in Figure 1, Section III. Here i_{s_1} is the current injected by an ideal current source at the input port (port 1 throughout the whole document) and v_2 is the voltage at the output (port 2) when open-circuited. Assume that the goal is to examine the dependence of T on a given device, namely the one located at branch 3. With this in mind, we look at the circuit as a three-port, the third port corresponding to the branch accommodating this device. Middlebrook's first formula can be written as

$$T = T^{\infty}\frac{z_3 + z_n}{z_3 + z_d} \quad \text{or} \quad T = T^{\infty}\frac{1 + z_n y_3}{1 + z_d y_3}, \tag{1}$$

where T^{∞} stands for the transimpedance in the absence of any element at port 3 (hence the "extra element" term for this device), z_3 or y_3 is the value of the impedance or admittance eventually connected at port 3, z_n is the impedance seen at port 3 under *null double injection conditions*, and z_d is the Thévenin impedance at port 3 when $i_{s_1} = 0$ and port 2 is open-circuited (find detailed discussions in [1], [7], [10], [11], [21]). We will refer to these two impedances as the *double null impedance* and the *single injection impedance*, inspired in Middlebrook's original terminology.

The setup above assumes that the *reference immittance* is an open-circuit; that is, the factor T^{∞} is measured with a null admittance at port 3. The dual version of the formula uses a short-circuit as the reference immittance, and reads as

$$T = T^{0}\frac{1 + y_n z_3}{1 + y_d z_3} \quad \text{or} \quad T = T^{0}\frac{y_3 + y_n}{y_3 + y_d}. \tag{2}$$

Here, T^0 is the transimpedance measured when the extra element is short-circuited. Leaving details for later, just think of y_n and y_d as the inverses of the impedances z_n and z_d.

In the forms given above, we are assuming that the three parameters arising in either formula (viz., T^{∞}, z_n, z_d in (1), or T^0, y_n, y_d in (2)) are well-defined, with T^{∞} or T^0 not vanishing. But such a general assumption need not hold in practice: for a two-port amounting to an impedance in parallel with both ports, neither the short-circuit nor the open-circuit are valid reference immittances, as it will be shown later. This shows that both (1) and (2) may fail to hold even for a rather simple two-port.

This brief stems from the need to provide a careful analysis of the working hypotheses that support the formulae above. This analysis will show that both (1) and (2) can be seen as particular cases of a more general formula, namely

$$T = T^{*}\frac{p_n q_3 + q_n p_3}{p_d q_3 + q_d p_3}, \tag{3}$$

which allows the reference immittance to take any value, exception made of at most two values. The p- and q-parameters arising here owe to the use of a homogeneous description of the extra element and of the double null and the single injection immittances. In Section II we present some background on this homogeneous approach [15], [16]. The main results can be found in Section III, which also includes a running example. Concluding remarks are compiled in Section IV.

II. Background

Homogeneous circuit modeling. The basic idea behind the modeling approach proposed in [15], [16] (find some precedents in [4], [14], [18]) is to describe symbolically a generic immittance by writing Ohm's law in the form $pv - qi = 0$, where the complex parameters p and q do not vanish simultaneously. If $p \neq 0$ (resp. if $q \neq 0$) then one may resort to the impedance form $v = zi$, with $z = q/p$ (resp. to the admittance form $i = yz$, with $y = p/q$). By keeping both parameters we avoid a loss of generality when setting up the circuit equations. In particular, both a short-circuit ($p \neq 0$, $q = 0$) and an open-circuit ($p = 0$, $q \neq 0$) can be handled within the same framework, something which is not feasible when using either an impedance or an admittance description.

We borrow from projective geometry (cf. [8], [17], [19]) the notation $(p : q)$ to describe the immittance using homogeneous coordinates. This means that $(p : q) = (\tilde{p} : \tilde{q})$ if and only if there exists a nonzero constant μ such that $p = \mu\tilde{p}$ and $q = \mu\tilde{q}$. In this setting, the immittance takes values within a complex projective line. Note that only the ratio between the parameters p and q

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matters: both are defined only up to a nonvanishing multiplicative constant. A way to fix the constant is to set, when $p \neq 0$ (that is, when the device admits an impedance description), the values $p = 1$ and $q = z$, whereas if $q \neq 0$ one can choose $p = y$ and $q = 1$. We will do this at several points in the paper. When proceeding this way, a short-circuit and an open-circuit are defined by the values $p = 1$, $q = 0$ and $p = 0$, $q = 1$, respectively.

The ideas above can be easily extended to accommodate independent sources by writing $pv - qi = s$ for some nonzero excitation term s. This accommodates, in particular, ideal voltage sources and ideal current sources by setting $p = 1$, $q = 0$, $s = v_s$ and $p = 0$, $q = 1$, $s = i_s$, respectively. Here v_s and i_s are the voltage and the current injected by the respective sources.

These notions can be naturally combined with Kirchhoff laws in the circuit modeling process. Following the ideas sketched in Section I, let us focus on the description of three-ports, where ports 1 and 2 are the input and the output, respectively, and port number 3 is the one accommodating the extra element. We further assume that the input port is the only branch eventually accommodating an independent source. The extra element is assumed to be uncoupled from the rest of the circuit and, similarly, the characteristics of all internal devices are assumed independent of the ones describing the devices connected at the ports. With this setup, the circuit equations can be written as a $2m$-equation system with $2m$ unknowns, namely

$$\begin{pmatrix} A_1 & A_2 & A_3 & A_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_1 & B_2 & B_3 & B_r \\ -q_{11} & -q_{12} & 0 & 0 & p_{11} & p_{12} & 0 & 0 \\ -q_{21} & -q_{22} & 0 & 0 & p_{21} & p_{22} & 0 & 0 \\ 0 & 0 & -q_3 & 0 & 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & -Q_r & 0 & 0 & 0 & P_r \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_r \\ v_1 \\ v_2 \\ v_3 \\ v_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ s_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

At this point we refer the reader to the running example in Section III for illustration. In (4), i_1, i_2, i_3 are the currents at the input port, the output port and the extra branch, respectively, whereas the vector i_r comprises the remaining circuit currents. The same subscript convention applies to the voltages v_1, v_2, v_3 and v_r. The first m rows in the coefficient matrix accommodate the submatrices resulting from the splitting of a reduced incidence matrix A (the circuit is assumed to be connected and therefore the dimension of A is $(n-1) \times m$) and a reduced cycle matrix B (with dimension $(m-n+1) \times m$) as $A = \begin{pmatrix} A_1 & A_2 & A_3 & A_r \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & B_2 & B_3 & B_r \end{pmatrix}$, with the same criterion as above; that is, A_1, A_2 and A_3 just amount to the first, second and third columns of A, whereas A_r comprises the remaining $m-3$ columns. The same applies to B. The reader may find background on the use of such matrices in circuit theory in [2], [3], [5], [6]. In light of this notation and by letting $i = (i_1, i_2, i_3, i_r)$ and $v = (v_1, v_2, v_3, v_r)$ stand for the full vectors of currents and voltages, it is clear that the first m equations of (4) simply express Kirchhoff laws in the form $Ai = 0$, $Bv = 0$.

The last m equations describe the characteristics of the different circuit elements. The parameters arising in the $(m+1)$-th, $(m+2)$-th and $(m+3)$-th rows of the coefficient matrix are aimed at capturing a number of working setups which will arise in later analyses. For the moment, the (let us say) main setting is obtained with the following values of the parameters. First, by setting $p_{11} = 0$ and $q_{11} = 1$, together with $p_{12} = q_{12} = 0$, the $(m+1)$-th equation just describes an independent current source injecting a current s_1 at port 1. In turn, the values of p_{22} and q_{22} (with $p_{21} = q_{21} = 0$) describe in homogeneous terms a load immittance at port 2. In particular, to model an open-circuit one sets $p_{22} = 0$ and $q_{22} = 1$. Finally, the last $m-2$ rows correspond to the equations $p_3 v_3 - q_3 i_3 = 0$ and $P_r v_r - Q_r i_r = 0$, which stand for the characteristics of the extra element, located at port 3, and of the internal circuit branches. Again, the case in which the extra element is absent (or, equivalently, it amounts to an open-circuit) is obtained with $p_3 = 0$, $q_3 = 1$, whereas a short-circuit at port 3 is defined by the values $p_3 = 1$, $q_3 = 0$.

Determinantal formulae for transfer functions. Determinantal expressions for the different transfer functions are discussed in [16] using the homogeneous terms here advocated. In particular, the open-circuit transfer impedance T referred to in the Introduction is shown in the aforementioned paper to read as η/δ, where both η and δ arise as the determinant of the coefficient matrix of (4) for certain parameter values. Specifically, the η-determinant is obtained with $p_{12} = q_{22} = 1$ and $p_{11} = p_{21} = p_{22} = q_{11} = q_{12} = q_{21} = 0$. Together with $s_1 = 0$, the $(m+1)$-th and $(m+2)$-th equations in (4) amount to $v_2 = 0$ and $i_2 = 0$: this describes the connection of a norator at port 1 and a nullator at port 2 (cf. [13], [20]). In turn, the δ-determinant corresponds to the case in which ports 1 and 2 are open-circuited, what is captured by the values $q_{11} = q_{22} = 1$, $p_{11} = p_{12} = p_{21} = p_{22} = q_{12} = q_{21} = 0$, together with $s_1 = 0$.

III. The extra element theorem with generic reference immittances

We show in this section that both (1) and (2) can be seen as particular instances of a more general formula (specifically, (10) below: see Remark 1). The different notions and results to be discussed will be illustrated by means of the running example defined by the circuit displayed in Figure 1. The branches labeled as 3, 4 and 5 accommodate immittances and we avoid specifying *a priori* an impedance or an admittance description for any of them. Needless to say, the transfer impedance of such a circuit is easy to compute and our aim is to provide a simple example aimed at helping the reader follow the discussion.

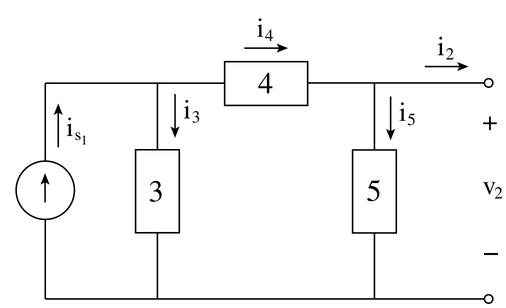

Fig. 1. A two-port example, with a current source connected at the input port.

By way of motivation, let us consider the particular case in which the immittance in branch 5 is an open-circuit; that is, set temporarily $y_5 = 0$. In this case, the transimpedance T of the two-port is easily seen to amount to the impedance z_3 (and note that T

is not well-defined in the specific case that branch 3 is itself open-circuited). Now, assume that we want to provide a Middlebrook-type formula for the transimpedance in the particular case that $y_5 = 0$, the immittance in branch 3 playing the role of the extra element. In order to fix the reference immittance, to be denoted as z_3^*, one needs to guarantee that the corresponding value of the transimpedance, namely T^*, is well-defined and nonzero; however, since T is not well-defined when $y_3 = 0$ and equals z_3 otherwise, neither an open-circuit nor a short-circuit are valid reference immittances in this case. This means that neither (1) nor (2) apply when $y_5 = 0$. By contrast, *all* but the open-circuit and the short-circuit would be valid reference immittances in this scenario. This idea is the key for the broader formulation of the extra element theorem stated in Theorem 1 below.

Regular two-ports, regular immittances. Remember that the transfer impedance T can be written as the ratio η/δ (cf. Section II). Let us denote

$$\eta^0 = \eta\big|_{\substack{p_3=1\\ q_3=0}},\ \eta^\infty = \eta\big|_{\substack{p_3=0\\ q_3=1}},\ \delta^0 = \delta\big|_{\substack{p_3=1\\ q_3=0}},\ \delta^\infty = \delta\big|_{\substack{p_3=0\\ q_3=1}}. \tag{5}$$

With this notation, the transfer impedance when the extra element is an open-circuit or a short-circuit can be written as $T^\infty = \eta^\infty/\delta^\infty$ and $T^0 = \eta^0/\delta^0$, respectively, provided that the denominators do not vanish. For an arbitrary extra immittance $(p_3 : q_3)$, and in light of the form of the coefficient matrix of (4), we will write η and δ as

$$\eta = p_3\eta^0 + q_3\eta^\infty, \quad \delta = p_3\delta^0 + q_3\delta^\infty, \tag{6}$$

using elementary properties of determinants.

As a minimal working assumption, it is reasonable to restrict the attention to two-ports for which at least one value of the extra immittance $(p_3 : q_3)$ yields a well-defined and nonvanishing transimpedance T. When this is the case, the two-port will be said to be *regular* (with respect to the branch accommodating the extra element). On account of the expression $T = \eta/\delta$, this means that both η and δ do not vanish for at least one homogeneous pair $(p_3 : q_3)$. Using (6), the regularity requirement on the two-port is seen to be equivalent to the assumption that

$$(\eta^0, \eta^\infty) \neq (0, 0) \neq (\delta^0, \delta^\infty), \tag{7}$$

which supports the concept of a *regular immittance*.

Definition 1. *Given a regular two-port, an immittance* $(p_3 : q_3)$ *is said to be* η-regular *(resp.* δ-regular*) if* $p_3\eta^0 + q_3\eta^\infty \neq 0$ *(resp. if* $p_3\delta^0 + q_3\delta^\infty \neq 0$*). An admittance is said to be* regular *if it is simultaneously* η*-regular and* δ*-regular.*

Under the regularity assumption (7) on the two-port, there exists exactly one immittance which is not η-regular, and also one immittance (not necessarily different from the previous one) which is not δ-regular. Altogether, for a regular two-port there may exist at most two nonregular immittances.

Regularity conditions for the running example. Consider the example in Figure 1. For the sake of generality, let us give all three immittances a homogeneous form: that is, write them as $(p_3 : q_3)$, $(p_4 : q_4)$ and $(p_5 : q_5)$. In the analysis we will use the reduced incidence and cycle matrices

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

Note that the rows of A yield the KCL equations $Ai = 0$ in the form $i_1 + i_3 + i_4 = 0$ and $i_2 - i_4 + i_5 = 0$, whereas B is responsible for the KVL equations $Bv = 0$, viz. $v_1 - v_3 = 0$, $-v_3 + v_4 + v_5 = 0$ and $v_2 - v_5 = 0$. Worth noting also is that i_r and v_r in Section II amount here to (i_4, i_5) and (v_4, v_5), respectively. The η- and δ-determinants can be easily checked to read as

$$\eta = q_3p_4q_5,\ \delta = p_3p_4q_5 + p_3q_4p_5 + q_3p_4p_5,$$

with $\eta^0 = 0$, $\eta^\infty = p_4q_5$, $\delta^0 = p_4q_5 + q_4p_5$ and $\delta^\infty = p_4p_5$. Now, looking at the first condition in (7) it becomes clear that, for the two-port to be regular, the conditions $p_4 \neq 0 \neq q_5$ must hold; in other words, by restricting the attention to regular two-ports, branch number 4 can be described as an impedance, with $p_4 = 1$ and $q_4 = z_4$, whereas branch number 5 admits a description as an admittance, with $p_5 = y_5$ and $q_5 = 1$. Note that in this setting one has $\delta^0 = 1 + z_4y_5$ and $\delta^\infty = y_5$, and therefore the second requirement in (7) is automatically met (indeed, if $\delta^\infty = y_5 = 0$ then $\delta^0 \neq 0$). Additionally, assuming the two-port to be regular, for an immittance to be η-regular the condition $\eta = q_3\eta^\infty = q_3 \neq 0$ must hold (notice that this rules out Middlebrook's second formula, (2), for this example, regardless of the other parameter values). Hence, η-regular immittances can be given an admittance description with $p_3 = y_3$ and $q_3 = 1$. Under this hypothesis, the δ-regularity condition finally shows that the extra immittance, located at branch number 3, will be regular if and only if

$$y_3(1 + z_4y_5) + y_5 \neq 0. \tag{8}$$

Main result. The regularity conditions (7) for a two-port always make it possible to take as the reference immittance $(p_3^* : q_3^*)$ a regular one, that is, an immittance for which the relations

$$\eta^* = p_3^*\eta^0 + q_3^*\eta^\infty \neq 0, \quad \delta^* = p_3^*\delta^0 + q_3^*\delta^\infty \neq 0 \tag{9}$$

hold. Note that these assumptions guarantee that the reference transimpedance T^* in (10) is well-defined and does not vanish. Also well-defined are the parameters p_n, q_n, p_d and q_d below, which provide homogeneous expressions for the double null immittance and the single injection immittance.

Theorem 1. *Assume that* $(p_3^* : q_3^*)$ *is a regular immittance for a given regular two-port. Let* η^* *and* δ^* *be given in (9), and set* $T^* = \eta^*/\delta^*$*. Then:*

(a) *the identity*

$$T = T^* \frac{p_nq_3 + q_np_3}{p_dq_3 + q_dp_3}, \tag{10}$$

with

$$p_n = \frac{\eta^\infty}{\eta^*},\quad q_n = \frac{\eta^0}{\eta^*},\quad p_d = \frac{\delta^\infty}{\delta^*},\quad q_d = \frac{\delta^0}{\delta^*}, \tag{11}$$

holds for any δ*-regular immittance* $(p_3 : q_3)$*;*

(b) $(p_n : q_n)$ *are homogeneous coordinates of the double null immittance;*

(c) $(p_d : q_d)$ *are homogeneous coordinates of the single injection immittance.*

Remark 1. Before proceeding with the proof of Theorem 1, it is worth detailing the sense in which (10) provides a generalized form of the extra element theorem. Notice that we are writing the extra immittance as $(p_3 : q_3)$, that is, in homogeneous terms. By restricting the attention to cases in which it admits an impedance description, we may set $p_3 = 1$, $q_3 = z_3$ to recast (10) as

$$T = T^* \frac{p_n z_3 + q_n}{p_d z_3 + q_d}. \tag{12}$$

Now, provided that the conditions $\eta^\infty \neq 0 \neq \delta^\infty$ are met, the open-circuit $(p_3^* : q_3^*) = (0 : 1)$ is a regular immittance, since in this case (9) holds because $\eta^* = \eta^\infty \neq 0$ and also $\delta^* = \delta^\infty \neq 0$. Under these assumptions, the open-circuit may be properly taken as the reference immittance. With this choice, one has $T^* = T^\infty$. Additionally, the first and third relations in (11) amount in this case to $p_n = p_d = 1$ and, for reasons to be detailed later (cf. Remark 2), one also gets $q_n = z_n$ and $q_d = z_d$. Altogether, this shows that in this case (12) is nothing but the first Middlebrook formula (1), written in the first of the two forms displayed there. The case in which the extra element is written in terms of an admittance y_3 is treated in exactly the same manner. In turn, the second classical formula, namely (2), follows analogously from the assumptions $\eta^0 \neq 0 \neq \delta^0$ and the subsequent choice of the short-circuit as the reference immittance.

Proof of Theorem 1. The claim in (a) follows from the expression $T = \eta/\delta$ given for the transimpedance in Section III together with (6), which make it possible to write

$$T = \frac{\eta}{\delta} = \frac{p_3\eta^0 + q_3\eta^\infty}{p_3\delta^0 + q_3\delta^\infty} = \frac{\eta^*}{\delta^*} \frac{p_3 \dfrac{\eta^0}{\eta^*} + q_3 \dfrac{\eta^\infty}{\eta^*}}{p_3 \dfrac{\delta^0}{\delta^*} + q_3 \dfrac{\delta^\infty}{\delta^*}}.$$

This is allowed by the nonzero conditions on η^* and δ^* stated in (9). Using $T^* = \eta^*/\delta^*$ together with (11), one readily gets (10).

To prove (b), note that the immittance seen at port 3 under null double injection conditions is given by the linear relation between v_3 and i_3 defined by the one-port equations

$$\begin{pmatrix} A_1 & A_2 & A_3 & A_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_1 & B_2 & B_3 & B_r \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Q_r & 0 & 0 & 0 & P_r \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_r \\ v_1 \\ v_2 \\ v_3 \\ v_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{13}$$

where the equations corresponding to the rows with a single 1 or -1 describe the double null conditions $v_2 = 0$, $i_2 = 0$. We claim that such a linear relation between v_3 and i_3 can be written as $\eta^\infty v_3 + \eta^0 i_3 = 0$. Indeed, fix any solution of (13) with $(v_3, i_3) \neq (0, 0)$ (note that if $v_3 = i_3 = 0$ then the relation $\eta^\infty v_3 + \eta^0 i_3 = 0$ holds trivially). By adding to the equations in (13) the relation $-v_3 i_3 + i_3 v_3 = 0$, the coefficient matrix of the resulting linear system turns out to be square and singular, since the product vanishes but the second factor is a nonzero vector (note that at least i_3 or v_3 is not zero). The determinant of this coefficient matrix can be seen to be $\eta^\infty v_3 + \eta^0 i_3$, what means that the relation $\eta^\infty v_3 + \eta^0 i_3 = 0$ holds, as we aimed to show. Thus, $(\eta^\infty : \eta^0)$ are homogeneous coordinates for the double null immittance, and so are $(p_n : q_n)$ (cf. (11)) in light of $(\eta^\infty, \eta^0) = \eta^*(p_n, q_n)$.

The proof of (c) is entirely analogous to the one above: just note that in this case the immittance seen at port 3 under single injection conditions is defined from the one-port equations obtained with the parameter values $p_{11} = p_{12} = q_{12} = 0$, $q_{11} = 1$, instead of the values $p_{11} = q_{11} = q_{12} = 0$, $p_{12} = 1$ used in (13). The corresponding augmented system naturally leads to the relation $\delta^\infty v_3 + \delta^0 i_3 = 0$, what means that both $(\delta^\infty : \delta^0)$ and $(p_d : q_d)$ are homogeneous coordinates for the single injection immittance. We leave the details to the reader. □

Remark 2. As shown above, both $(\eta^\infty : \eta^0)$ and $(p_n : q_n)$ are homogeneous coordinates for the double null immittance. Now, if $\eta^\infty \neq 0$, we may divide the left-hand side of $\eta^\infty v_3 + \eta^0 i_3 = 0$ by η^∞ to get $v_3 + (\eta^0/\eta^\infty) i_3 = 0$, what means that the double null immittance admits an impedance description, with $z_n = \eta^0/\eta^\infty$. Analogously, if $\eta^0 \neq 0$ then the double null immittance can be written as $y_n = \eta^\infty/\eta^0$. By the same token, $\delta^\infty \neq 0$ and $\delta^0 \neq 0$ support respectively the expressions $z_d = \delta^0/\delta^\infty$ and $y_d = \delta^\infty/\delta^0$ for the single injection immittance. Note also that under the assumptions $\eta^\infty \neq 0 \neq \delta^\infty$, one may choose the open-circuit as the reference immittance, meaning that $\eta^* = \eta^\infty$ and $\delta^* = \delta^\infty$. Using (11), one then gets $p_n = 1$, $q_n = \eta^0/\eta^\infty = z_n$, $p_d = 1$, $q_d = \delta^0/\delta^\infty = z_d$, as we claimed in Remark 1.

Additional remarks on the running example. As shown above, the regularity assumptions on the example of Figure 1 support the use of an admittance description for the devices in branches 3 and 5, and an impedance description for the one in branch 4. In this scenario, the condition stated in (8) is the one precisely characterizing regular immittances. Specifically, a reference admittance y_3^* is admissible (that is, it qualifies for a generalized Middlebrook formula) if and only if the condition

$$y_3^*(1 + z_4 y_5) + y_5 \neq 0 \tag{14}$$

holds. This makes it clear that in the particular case that branch 5 is open-circuited, as it was discussed at the beginning of Section III, the value $y_3^* = 0$ is not valid as a reference admittance because (14) fails to hold. For this reason, Middlebrook's first formula, (1), does not apply when $y_5 = 0$. In general, for an arbitrary value of y_5, an admittance y_3^* will be admissible if and only if (14) is met. Moreover, the general formula (10) will apply regardless of the actual value of y_5, always under the condition stated in (14).

The example allows for a discussion of the different forms that the generalized formula may take when resorting to classical descriptions for the extra element and for the double null and single injection immittances. Up to $2^4 = 16$ different forms arise; two of them are

$$T = T^* \frac{\dfrac{1 + z_n y_3}{1 + z_n y_3^*}}{\dfrac{y_3 + y_d}{y_3^* + y_d}}, \quad T = T^* \frac{\dfrac{1 + z_n y_3}{1 + z_n y_3^*}}{\dfrac{1 + z_d y_3}{1 + z_d y_3^*}}, \tag{15}$$

depending on whether the single injection immittance is described in admittance or in impedance terms: by contrast, we are assuming that both y_3 and y_3^* are given an admittance description and that the double null immittance z_n is written as an impedance. In the example, the single injection immittance admits an admittance description if $1 + z_4 y_5 \neq 0$, and an impedance description if $y_5 \neq 0$. Note also that $p_n = 1$, $q_n = 0$ yield $z_n = 0$.

To be specific, let us consider two concrete setups. Assume first that branches 3 through 5 in Figure 1 are resistors: we replace therefore y_3, z_4 and y_5 by G_3, R_4 and G_5. We do not allow negative resistance or conductance but want to include in the analysis the case in which G_5 eventually vanishes (in order to model an open-circuit): remember that this case is not covered by the classical Middlebrook formulae (1)-(2) and note also that z_d is not well-defined if $G_5 = 0$. We want to characterize the effect that a deviation in G_3 (which may be seen as the parallel conductance in a possibly nonideal current source) with respect to a nominal value G_3^* causes in T. Using the first form in (15) with $z_n = 0$, $y_d = G_d = G_5/(1 + R_4 G_5)$, one gets

$$\frac{T}{T^*} = \frac{G_3^* + G_d}{G_3 + G_d}, \tag{16}$$

a formula which holds irrespective of whether G_3^* vanishes or not (again, Middlebrook's original setup just accommodates the case $G_3^* = 0$, and only if $G_5 \neq 0$). Setting e.g. $R_4 = 10^3\,\Omega$, $G_5 = 10^{-3}\,S$ we get $G_d = 0.5 \cdot 10^{-3}\,S$; assuming two scenarios defined by $G_3^* = 0$ and $G_3^* = 10^{-3}\,S$, respectively, with $G_3 = G_3^* + 10^{-5}\,S$ in both of them, one gets $T/T^* = 0.9804$ in the former case and 0.9934 in the latter.

Let now branches 4 and 5 accommodate an inductor and a capacitor, respectively, and use phasor descriptions to write $z_4 = j\omega L$, $y_5 = j\omega C$, with $L > 0$ and $C > 0$. Again, we let branch 3 accommodate a resistor with conductance G_3. We are now interested in describing the ratio T/T^* for any nonzero frequency; this includes the resonant value $\omega = 1/\sqrt{LC}$, which annihilates the expression $1 + z_4 y_5$, ruling out the use of y_d and of the first form of (15), contrary to the setup above. We may use instead the second form depicted in (15) with the values $z_n = 0$ and $z_d = (1 + z_4 y_5)/y_5 = j(\omega^2 LC - 1)/(\omega C)$. This yields

$$\frac{T}{T^*} = \frac{\omega C + j(\omega^2 LC - 1)G_3^*}{\omega C + j(\omega^2 LC - 1)G_3}. \tag{17}$$

In particular, the resonant frequency is handled without problems in this context: when $\omega^2 = 1/LC$ one easily gets $T/T^* = 1$. This is due to the fact that the series connection of L and C behaves as a short-circuit and no current flows through branch 3, making the actual value of G_3 immaterial in this particular case.

Let us emphasize that, by contrast, the homogeneous form for $(p_d : q_d)$ used in the general formula (10) avoids the need to restrict the attention to either one of the two particular forms defined by (15) above, thereby preventing the resulting loss of generality. Indeed, by using $p_n = 1$, $q_n = 0$ and

$$p_d = \frac{y_5}{y_3^*(1 + z_4 y_5) + y_5},\ q_d = \frac{1 + z_4 y_5}{y_3^*(1 + z_4 y_5) + y_5}, \tag{18}$$

allowed by (14), one easily gets $T/T^* = 1/(p_d + q_d y_3)$, an expression which applies in broad generality and may be checked to cover all the setups described above.

IV. Concluding remarks

We have presented in this brief a general form for the extra element theorem which comprises Middlebrook's formulae (1) and (2) as particular cases. Our approach unveils the exact working assumptions supporting the different formulae, and should have a natural extension to problems with several extra elements [1], [7], [11], [21]. The ideas here discussed might also be of interest in other related circuit-theoretic problems, involving e.g. sensitivity analyses, and should be helpful in the fields where the extra element theorem finds applications, as mentioned in Section I.

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