UNIVERSIDAD POLITÉCNICA DE MADRID
Escuela Técnica Superior Ingenieros Industriales

Virtual Nonholonomic Constraints: A geometric approach

DOCTORAL THESIS

Submitted for the degree of Doctor by:

Efstratios Stratoglou
Master's Degree in Theoretical Mathematics

Madrid, 2025

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Doctoral Degree in Automatic Control and Robotics

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Under the supervision of:
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Title: Virtual Nonholonomic Constraints: A geometric approach

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Doctoral Programme: Automatic Control and Robotics

Thesis Supervision:

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External Reviewers:

Thesis Defense Committee:

Thesis Defense Date:

to my family

τα πάντα ρεί
ΗΡΑΚΛΕΙΤΟΣ

Panta rei (todo fluye)
HERÁCLITO

Acknowledgement

I would like to dedicate this page of my PhD thesis to those that help me explicitly as well as implicitly during this academic journey.

First and foremost, I would like to thank my supervisor, Leonardo Colombo, to whom I am deeply grateful for his support, guidance and trust. I feel extremely fortunate that we cross one another's bath and I met a generous, sharp-witted and kind person. Without his expertise, advice and encouragement, the works covered in this thesis would have definitely never come to fruition. Our conversations have been instrumental in shaping my knowledge and attitude towards research. I wouldn't have made it this far if it weren't for him, he can make the impossible, possible.

A great thank you to Alexandre Anahory Simoes for hosting me in his office, for listening very carefully to any question that I might had and answering with even more care and for guiding and consulting. His help during this time was highly valuable to the development of this work. Frankly, I credit a lot of my knowledge to him.

I would like to express my sincere gratitude to ICMAT and the Geometric, Mechanics, and Control (GMC) network for giving me the opportunity to get to know a big academic family that truly revealed the research world to me. Also, I wish to thank one of my main collaborators Antony Bloch for his inspirations and insights on nonholonomic mechanics, related control topics and real life applications he shared in Miraflores. Lastly, I would like to thank Omayra Yago Nieto and Damiano Rigo for their help and support.

I owe special gratitude to Maria Barbero, Dimitris Boskos, Dimos Dimarogonas, Juan Ignacio Giribet, Jacob Goodman, Manuel de León, Juan Carlos Marrero, David Martin de Diego, Max Mendel, Tom Mestdag, Sotirios Natsiavas, Nicola Sansonetto, Rodrigo Sato Martin de Almagro, Miguel Vaquero, Mariana Costa Villegas and particularly to Emmanouil Maou for making true the Workshop in Geometric Control and Mechanics held at the American College of Thessaloniki. I would never forget Emmanouil's generosity and understanding, he once again showed what a great team leader he is.

Ultimately, I am profoundly grateful to my family, for their unconditional love, patience, and support especially during the most challenging moments of this journey.

Abstract

This thesis centers around the concepts of Geometric Control, Nonholonomic Mechanics, and Geometric Mechanics. Geometric Mechanics establishes the mathematical foundation for understanding the intrinsic structure and behavior of mechanical systems. In a nutshell, it offers geometric tools for a deeper understanding of the dynamics and control of mechanical systems. Nonholonomic Mechanics extend the analysis of these systems to systems subject to constraints on the velocities that are not integrable. Lastly, Geometric Control incorporates insights from both geometric and nonholonomic mechanics into the design of controllers that align with the underlying geometry of the system.

In this dissertation, we develop a rigorous mathematical theory for virtual nonholonomic constraints. Virtual constraints are invariant relations imposed on a control system via feedback as opposed to real physical constraints acting on the system. Virtual nonholonomic constraints are defined as a controlled invariant distribution associated with an affine connection mechanical control system. We develop geometric tools sufficient to guarantee the existence and uniqueness of a control law that turns the constraint submanifold into virtual nonholonomic constraints and characterize the dynamics for nonholonomic systems in terms of virtual nonholonomic constraints, i.e., we characterize when can we obtain nonholonomic dynamics from virtual nonholonomic constraints. The theory is developed for systems evolving in general manifolds for constraints that are linear, affine and nonlinear in the velocities. We extend the theory in the framework of Lie groups and Riemannian homogeneous spaces.

Finally, we discuss the problem of stabilizing a system around desired virtual nonholonomic constraints. In brief, we focus on the stabilization of systems around desired manifolds of the phase space, determined by virtual nonholonomic constraints. We prove the existence of a control law ensuring that the system adheres to the constraints. Additionally, we show that if the system already satisfies the constraints at a given point, the control law aligns with the unique control law derived along the dissertation, which guarantees the existence of a virtual nonlinear nonholonomic constraint.

Resumen

Esta tesis se centra en los conceptos de Control Geométrico, Mecánica Noholónoma y Mecánica Geométrica. La Mecánica Geométrica establece la base matemática para comprender la estructura intrínseca y el comportamiento de los sistemas mecánicos. En pocas palabras, ofrece herramientas geométricas para una comprensión más profunda de la dinámica y el control de los sistemas mecánicos. La Mecánica Noholonómica extiende el análisis de estos sistemas a aquellos sujetos a restricciones en las velocidades que no son integrables. Por último, el Control Geométrico incorpora ideas tanto de la mecánica geométrica como de la mecánica noholonómica en el diseño de controladores que se alinean con la geometría subyacente del sistema.

En esta disertación, desarrollamos una teoría matemática rigurosa para las restricciones noholonómicas virtuales. Las restricciones virtuales son relaciones invariantes impuestas a un sistema de control mediante retroalimentación, en contraposición a las restricciones físicas reales que actúan sobre el sistema. Las restricciones noholonómicas virtuales se definen como una distribución invariante controlada asociada a un sistema de control mecánico basado en una conexión afín. Desarrollamos herramientas geométricas para garantizar la existencia y unicidad de una ley de control que transforme la subvariedad de restricción en restricciones noholonómicas virtuales y caracterizamos la dinámica de los sistemas noholonómicos en términos de restricciones noholonómicas virtuales, es decir, caracterizamos cuándo es posible obtener una dinámica noholonómica a partir de restricciones noholonómicas virtuales. La teoría se desarrolla para sistemas que evolucionan en variedades diferenciables abstractas para restricciones que son lineales, afines y no lineales en las velocidades. Además, extendemos la teoría al marco de grupos de Lie y espacios homogéneos Riemannianos.

Finalmente, discutimos el problema de estabilizar un sistema en torno a restricciones noholonómicas virtuales deseadas. En resumen, nos enfocamos en la estabilización de sistemas alrededor de las variedades deseadas del espacio de fases, determinadas por restricciones noholonómicas virtuales. Demostramos la existencia de una ley de control que garantiza que el sistema se adhiera a las restricciones. Adicionalmente, mostramos que si el sistema ya satisface las restricciones en un punto dado, la ley de control se alinea con la única ley de control derivada a lo largo de la disertación, la cual garantiza la existencia de una restricción noholonómica virtual.

Table of Contents

List of Figures

Chapter 1

Introduction

This dissertation studies virtual nonholonomic constraints through the prism of geometric control theory and develops useful techniques for the analysis of mechanical control systems and nonholonomic systems by employing methodologies from geometric mechanics.

One of the great advances in classical mechanics occurred when the geometric aspects of mechanical systems began to be studied. Geometric mechanics is exactly the field of applied mathematics that uses the tools of differential geometry to study the dynamics and control of mechanical systems. It provides a clear and unified framework for understanding the geometry underlying control systems, in particular, control systems governed by the principles of Hamiltonian and Lagrangian mechanics. By focusing on the geometric structures of the system, this mathematical framework offers deep insights into the behavior of complex dynamics, such as those involving constraints, symmetries, and conservation laws. The configuration space of a mechanical control system is a smooth manifold which provides a general framework for describing control systems that do not evolve in usual Euclidean spaces with zero curvature. For instance, systems with rotational motion, symmetries or constraints, like unmanned aerial vehicles, robotic manipulators, unmanned surface vehicles, and underwater vehicles, among many others.

Many mechanical control systems have some kind of symmetries that are often associated with Lie groups, and represent transformations like rotations or translations. Geometric control uses tools like Lie group actions and Lie algebras to study systems with these symmetries. The dynamics of systems with symmetries can be simplified using Noether's theorem, which relates symmetries to conserved quantities (e.g., angular momentum). In these situations, we can take advantage of the symmetries and translate the dynamics from a Lie group to its Lie algebra, a vector space, where classical numerical methods can be applied. For related works on mechanical systems with symmetries one can see Marsden and Ratiu, 1999, Holm et al., 2009, Cendra et al., 2001, A. M. Bloch et al., 1996.

Mechanical systems often are subject to constraints, two essential categories are holonomic and nonholonomic. In this work, we primarily focus on nonholonomic constraints which is the main objective of nonholonomic mechanics. Nonholonomic constraints are constraints on the velocities of a system rather than on the positions and they are typically non-integrable, meaning they cannot be expressed purely in terms of position coordinates which is the case for the holonomic ones. While the study of mechanical systems with holonomic constraints is well known and admits a unified analysis, the theory behind mechanical systems with nonholonomic constraints is lacking a unique theoretical framework due to loss of integrability and not satisfaction of symplectic properties. These constraints are naturally described using distributions on manifolds, and their analysis involves geometric tools like affine connections on Riemannian manifolds. In Lagrangian mechanics, nonholonomic constraints are incorporated using Lagrange multipliers, the Lagrange-d'Alembert principle (see A. Bloch, 2015), which modifies standard variational calculus or the form of Chetaev's equation (see Cendra, Ibort, de León, and de Diego, 2004). In Hamiltonian mechanics with nonholonomic constraints, the dynamics may not preserve the standard Hamiltonian structure and the energy is not necessarily a conserved quantity (even when the constraints are ideal) thus special formulations (like the use of Poisson or nonholonomic brackets, see for instance Van Der Schaft and Maschke, 1994, A. M. Bloch, 2000, Cantrijn et al., 1999a) are required, making analysis more subtle. Typical examples of nonholonomic constraints are those imposed by rolling and sliding of the mechanical systems e.g. rolling without slipping, knife-edge constraints, robotic arms or mobile robots. There is an extensive related literature, here we mention some related work on nonholonomic mechanics: A. Bloch, 2015, Neimark and Fufaev, 2004, Cortés, Martínez, et al., 2002, Cortés et al., 2000, Cortés, de León, et al., 2002.

State-of-the-art and main objective of the thesis

Virtual constraints are relations imposed on a control system that become invariant via feedback control, as opposed to physical constraints acting on the system. The concept of virtual holonomic constraints goes back to the work of Appell, 1911 and later it was examined more extensively in the thesis of Beghin, 1922 as a constraint that can be applied via the exertion of external forces.

Virtual holonomic constraints (VHC) have emerged as a useful tool for motion control in hybrid systems for bipedal locomotion (see for instance Westervelt et al., 2003, Canudas-de-Wit, 2004 Chevallereau et al., 2003, Chevallereau et al., 2009, Westervelt et al., 2018 Hera et al., 2013, Chevallereau et al., 2018, Razavi et al., 2016). In the works of A. S. Shiriaev et al., 2010, Nielsen and Maggiore, 2008 and Westerberg et al., 2009 the authors study virtual holonomic constraints trhough transverse linearization for motion planning and control problems. The path following problem for a planar vertical take-off and landing aircraft (PVTOL) applicable to a class of smooth Jordan curves is examined at Consolini et al., 2010 and a controlled bicycle that traces a strictly convex Jordan curve with bounded roll angle and bounded speed is presented at Consolini and Maggiore, 2010. More recently Mohammadi et al., 2013 studies the constraint dynamics (i.e., the system's dynamics resulting on the enforcement of the VHC) and presents necessary and sufficient conditions for them to be Lagrangian. A stability analysis using virtual constraints is explored at Mohammadi et al., 2017 and A. Shiriaev et al., 2005, while A. S. Shiriaev et al., 2006 and Freidovich et al., 2008 review periodic motion planning. In Hamed et al., 2020 feedback control algorithms are developed that allow cooperative locomotion of quadrupedal robots coupled to each other by holonomic constraints. These constraints are imposed via optimal distributed feedback controllers, based on nonlinear control and quadratic programming.

In subsequent years this approach was more formally studied from a geometric control formalism. In Maggiore and Consolini, 2013, the authors studied conditions under which holonomic constraints can be made invariant and stabilizable by feedback, and sufficient conditions for the constrained dynamics to coincide with an Euler-Lagrange system are provided. In Consolini and Costalunga, 2015 the authors presented virtual holonomic constraints in an affine connection geometric framework, where the trajectories of the closed-loop virtual constrained system are described as geodesics of a connection obtained from the mechanical system's Levi-Civita connection. The former work is an extension of Maggiore and Consolini, 2013 to the case of systems with underactuation degree greater than one and it examines the sufficient conditions for a constraint to be virtual (they call it feasible). Additionally, the authors define an induced connection whose geodesics describe the trajectories of the constraint dynamics and study conditions under which the metrizability of the affine connection guarantees that the constrainted dynamics is Lagrangian.

Even though the literature on virtual holonomic constraints is much more extensive than that of the nonholonomic ones, the study of the latter has received significant attention during the last decade. Thinking of virtual nonholonomic constraints one should consider a class of virtual constraints that depend on velocities rather than only on the configurations of the system. Those constraints were introduced in Griffin and Grizzle, 2015 to design a velocity-based swing foot placement in bipedal robots where the nonholonomic constraint is given by the momenta equations defined through a Legendre transformation of the original Lagrangian system describing the continuous-time of the hybrid dynamics for the bipedal robot. This last development was based purely on the application, and no theoretical results were studied for a general theory of virtual nonholonomic constraints. In Moran-MacDonald, 2021 the authors give a first mathematical definition of virtual nonholonomic constraint for Hamiltonian systems, but similarly as in Griffin and Grizzle, 2015 the authors only focus on the application of the designed controller to a practical example.

The notion of virtual nonholonomic constraints is closely related to the notion of zero dynamics, the connection can be seen at Isidori, 2000 as well as at Akbari Hamed and Ames, 2020 where it discusses zero dynamics associated with relative degree one and two nonholonomic outputs for exponential stabilization of given periodic orbits for hybrid models of bipedal locomotion and it synthesizes nonholonomic hybrid zero dynamics controllers to stabilize periodic orbits based on reduced-order stability analysis. In a series of works, namely at Horn and Gregg, 2022; Horn et al., 2019, 2020, virtual nonholonomic constraints were used to encode velocity-dependent stable walking gaits via momenta conjugate to the unactuated degrees of freedom of legged robots and prosthetic legs in a similar fashion as Griffin and Grizzle, 2015. In details, Horn et al., 2019 presents hybrid zero dynamics based on virtual nonholonomic constraints for planar bipedal robots with one degree of underactuation. Also they discuss assumptions that make the virtual nonholonomic constraints invariant according to rigid impacts with the terrain. At Horn et al., 2020, the authors investigate a method of designing relative-degree-two virtual nonholonomic constraints that allow for stable bipedal locomotion across fields with variable incline disturbances. They induce a methodology for designing virtual nonholonomic constraints, via an optimization problem, in order to achieve stable walking across variable inclined ground floor. In extend to that, Horn and Gregg, 2022 provides a method to design a virtual nonholonomic constraint controller that produces multiple distinct stance-phase trajectories for corresponding walking speeds. In the paper, an optimization problem is formed to design a single stance-phase nonholonomic virtual constraint for three distinct walking speed trajectories (slow, normal, and fast) by the use of the segmental conjugate momentum for the prosthesis.

The main objectives of this dissertation are to provide a rigorous mathematical framework for designing virtual nonholonomic constraints on differentiable manifolds of interest to robotics applications and to study the stabilization of systems around desired manifolds of the phase space, as determined by virtual nonholonomic constraints.

The thesis's specific objectives align with the theoretical and applied results obtained in each chapter. Below is a breakdown of the particular objectives of the thesis.

Particular objectives of the thesis

- Define and analyze virtual nonholonomic constraints: In particular, formalize the concept of virtual nonholonomic constraints concerning the state-of-the-art as a controlled invariant distribution in an affine connection mechanical control systems and prove the existence and uniqueness of control laws that guarantee the invariance of these constraints in closed-loop systems.
- Introduce new affine connections, which describe the constrained dynamics of the system and establish conditions under which nonholonomic dynamics can emerge from virtual constraints.
- Extend the theory of virtual nonholonomic constraints to systems evolving on Lie groups. In particular, establish conditions that simplify the requirement for transforming nonholonomic constraints into virtual ones and characterize the closed-loop system dynamics in terms of reduced nonholonomic dynamics induced by virtual constraints.
- Develop a geometric framework to describe nonholonomic trajectories on Riemannian homogeneous spaces and establish the connection between geodesics, mechanical trajectories, and nonholonomic dynamics in Riemannian homogeneous spaces.
- Analyze virtual nonholonomic constraints in Riemannian homogeneous spaces, generalizing the results beyond Lie groups. In particular, extend the theory of virtual nonholonomic constraints to Riemannian homogeneous spaces and analyze the existence and uniqueness of control laws in this setting.
- Develop a stabilization theory for virtual nonholonomic constraints. Introduce a generalized control law that ensures exponential stabilization of virtual constraints and prove that if the system initially satisfies the constraints, the derived control law is the unique control law that preserves them.

Methodology

The thesis employs a geometric approach based on Riemannian geometry and Lie group theory to study the control of mechanical systems inspired by robotics applications. The methodology includes:

- Theoretical formulation: Development of a formal definition of virtual nonholonomic constraints and the necessary and sufficient conditions for their existence.

- Control law design: Derivation of feedback control laws ensuring that the system adheres to the constraints.
- Mathematical characterization: Analysis of the constrained dynamics and their geometric interpretation, particularly in Lie groups and Riemannian homogeneous spaces.
- Stabilization analysis: Design control strategies for geometric stabilization of systems around desired constraint manifolds.
- Applications and simulations: Validation of the theoretical results with simulation results and applications to specific mechanical systems inspired by robotics applications.

Outline of the thesis

Next, we present the structure of the present work and summarize the main goals addressed in every chapter.

- In Chapter 2, we provide the necessary framework to develop the theory and present key features from the existing literature that our work builds upon. Among others, the induced topics are from Geometric Mechanics, Riemannian Geometry, Nonholonomic Mechanics, Geometric Control, Lie Groups and Homogeneous spaces. Apart from the basic preliminaries, we give some new results on the last two fields. Regarding mechanical systems on Lie groups, we express the geodesics on the configuration space by defining a bilinear map that yields from the right trivialization of the Levi-Civita connection for a right-invariant metric on Lie groups. These results are presented in Theorems 2.6.9 and 2.6.15. Furthermore, concerning Riemannian homogeneous spaces, we demonstrate the relationship between the trajectories of a mechanical system in a homogeneous space and the horizontal trajectories of a mechanical system on the Lie group in Theorem 2.7.11.
- In Chapter 3, we introduce the concept of virtual nonholonomic constraints, covering both linear and affine cases. We specifically analyze control systems subject to these constraints and introduce the existence and uniqueness of control laws that ensure the closed-loop system's trajectories satisfying them. Section 3.1 sets the theoretical background for the development of virtual nonholonomic constraints in the linear case, where the constraints depend linearly on the system's velocities — a feature commonly observed in many nonholonomic systems. We formally define virtual nonholonomic constraints as a controlled invariant distribution associated with an affine connection mechanical control system. Additionally, we prove the existence and uniqueness of a control law that enforces such constraints in the sense that once the initial condition satisfies the constraints the whole trajectory of the closed-loop system never escapes the constraint distribution. Furthermore, we introduce the concept of an induced constraint connection, which characterizes the closed-loop system's trajectories as solutions of the mechanical system defined by this connection. We also present conditions under which nonholonomic dynamics can be derived from virtual nonholonomic constraints. In Section 3.2, we extend the results obtained for the linear case to the framework of virtual affine nonholonomic constraints. Although the overall structure of the control

mechanical system remains consistent with the linear case, the key difference lies in the nature of the constraint equations.

- Chapter 4 takes the theory of virtual nonholonomic constraints, that developed in the previous chapter, a step further by extending it to nonlinear nonholonomic constraints. The underlying geometry in this case is more challenging, as the constraints are represented by a submanifold of the tangent bundle rather than a distribution on the configuration space. Consequently, the transversality condition posed in Theorems 3.1.8 and 3.2.8 is replaced by a translated version of it to the tangent space of the constraint submanifold with the use of the new concept of velocity-dependent distribution to guarantee the existence and uniqueness of a control law that transforms nonlinear nonholonomic constraints into virtual ones. In addition, we form the equation of the nonholonomic trajectories of the closed-loop system in terms of an affine connection and establish conditions under which these trajectories coincide with solutions of Chetaev's equations. In Section 4.4, we highlight the theory through applications, including the precessional motion of a rigid body, the double pendulum and flocking motion, at Subsections 4.4.1, 4.4.2 and 4.4.3 respectively, with simulation results provided for the latter two cases. Ultimately, we present a geometric characterization of virtual nonholonomic constraints using almost tangent structures.

- In Chapter 5, we elaborate the theory of virtual nonholonomic constraints on Lie groups. By exploiting the system's symmetry, that we naturally encounter in many physical phenomena, the theory established on Riemannian manifolds is significantly simplified. Specifically, we define virtual nonholonomic constraints on Lie groups as a controlled invariant subspace associated with an affine connection mechanical control system evolving within the Lie algebra. Rather than a constraint distribution and an input distribution, we consider vector subspaces of the Lie algebra and establish the existence and uniqueness of a control law that enforces a virtual nonholonomic constraint under certain conditions on these subspaces. The transversality condition formed in Chapters 3 and 4 boils down to Whitney sum to simplify the requirement for turning nonholonomic constraints to virtual ones. Additionally, we characterize the closed-loop system's trajectories as solutions of a mechanical system governed by an induced constrained connection and examine the conditions under which reduced nonholonomic dynamics can be derived from virtual nonholonomic constraints. Furthermore, we extend the theoretical framework to include virtual affine nonholonomic constraints. In this case, the constraints form an affine vector space of Lie algebra and the Whitney sum is transferred to the associated vector space. In Subsection 5.2, we apply this theory to two examples: a homogeneous rigid body and a rigid body with a rotor, incorporating linear and affine nonholonomic constraints, respectively.

- One of the objectives of Chapter 6 is to fill the gap in the literature concerning nonholonomic systems on Riemannian homogeneous spaces — particularly to describe their nonholonomic trajectories. We link geodesics and nonholonomic trajectories on Riemannian homogeneous spaces with those on the Lie group that decodes the structure of the homogeneous space. The second part of the chapter focuses on developing the theory of virtual nonholonomic constraints in this setting. The main goal is to establish the existence and uniqueness of a control law that preserves the constraint distribution and connect it with the associated on the relevant Lie group. Additionally, we characterize the resulting closed-loop nonholonomic dynamics, governed by this unique control law, in terms of an affine connection. To illustrate the theory, we elaborate two examples: a sphere rolling over another sphere in Section 6.4 and a blade moving on a sphere in Section 6.5.

- In Chapter 7, we investigate the stabilizability of virtual nonholonomic constraints. Specifically, we introduced a more general control law designed to ensure the system's compliance with the constraint. This control law provides exponential stability to the system, i.e. the dynamics converge exponentially fast to the constraint manifold or constraint distribution for all cases considered in the thesis, that is, linear, affine, and nonlinear nonholonomic constraints. Additionally, we prove that if the system evolves to satisfy the constraint at a given moment, this control law is the unique law that keeps the trajectories on the constraint manifold. We exemplify the theory with two applications the control of a flocking motion and a boat moving on a stream.

Relevance of the results

This research is highly relevant to the automatic control, geometric control, and robotics communities, as it:

- Expands the understanding of nonholonomic constraints beyond conventional physical constraints, introducing a virtual control-based perspective.
- Provides a geometric formulation that can be applied to autonomous vehicles, robotic manipulators, and legged robots, among other nonholonomic systems.
- Enhances the stability and control of constrained mechanical systems, enabling improved motion planning and trajectory tracking for autonomous robots and multi-agent systems.
- Bridges the gap between geometric mechanics and modern control theory, contributing to the development of more efficient and robust controllers for constrained dynamical systems.
- Gives new geometric insights into control theory by establishing a formal definition of virtual constraints and delivering the necessary mathematical developments for their application.
- Facilitates interdisciplinary applications by providing a mathematical framework that can be extended to fields such as biomechanics, aerospace engineering, and robotic locomotion, enabling novel implementations of virtual constraints in diverse mechanical systems.

Originality of the thesis

This work introduces a novel geometric framework to define, analyze, and control systems through virtual nonholonomic constraints in differentiable manifolds including Lie groups and Riemannian homogeneous spaces. Unlike previous approaches, which primarily focused on specific robotic applications, this dissertation provides a unified theoretical foundation that extends to a broader class of nonholonomic mechanical systems. The research contributes to both theoretical advancements and practical applications by establishing new control laws that guarantee the existence and stability of the constraint manifold determined by these constraints.

This dissertation has contributed to five journal papers in top journals of the research field of Automatic Control and Geometric Control. In addition, the work of this work has been published in the proceedings of the 2025 European Control Conference and two book chapters of Springer.

Papers derived from this thesis[1]

Journal Papers

- Anahory Simoes*, A., Stratoglou*, E., Bloch, A., & Colombo, L. J. (2023). Virtual nonholonomic constraints: A geometric approach. *Automatica*, 155, 111166.
- Stratoglou, E., Anahory Simoes, A., Bloch, A., & Colombo, L. J. (2024). On the geometry of virtual nonlinear nonholonomic constraints. *IEEE Transactions on Automatic Control,* 1–8. https://doi.org/10.1109/TAC.2024.3508543
- Stratoglou, E., Anahory Simoes, A., Bloch, A., & Colombo, L. (2023). Virtual constraints on Lie groups. Submitted to *SIAM Journal on Control and Optimization* on December 2023. Preprint available at arXiv preprint arXiv:2312.17531.
- Stratoglou, E., Anahory Simoes, A., Bloch, A., & Colombo, L. J. (2024). Nonholonomic mechanics and virtual constraints on Riemannian homogeneous spaces. Submitted to *Automatica* in October 2024. Preprint available at https:// arxiv.org/ abs/ 2411.05485.
- Stratoglou, E., Anahory Simoes, A., Bloch, A., & Colombo, L. J. (2025). Geometric stabilization of virtual nonlinear nonholonomic constraints. Submitted to *IEEE Control Systems Letters*, on January 2025. Preprint available at https:// arxiv.org/ abs/2502.03902.

Conference Papers

- Anahory Simoes*, A., Bloch, A., Colombo, L., & Stratoglou*, E. (2024). Geometric stabilization of virtual linear nonholonomic constraints. Submitted to *European Control Conference 2025.* Preprint available at https://arxiv.org/abs/2411.01692.

Book Chapters

- Stratoglou, E., Anahory Simoes, A., Bloch, A., & Colombo, L. (2023). Virtual affine nonholonomic constraints. *6th International Conference on Geometric Science of Information*, 89–96. Springer.
- Stratoglou, E., Anahory Simoes, A., Bloch, A., & Colombo, L. J. (2025). Virtual nonlinear nonholonomic constraints from a symplectic point of view. Submitted to the *7th International Conference on Geometric Science of Information,* on March 2025.

[1]* Authors contributed equaly

Some of the results of the thesis were communicated in the following conferences:

Conferences

- Nonholonomic mechanics and virtual constraints on Riemannian homogeneous spaces.
  RSME's 7th Congress of Young Researchers, Bilbao Spain, January 2025.
- On the stabilizability of virtual nonholonomic constraints.
  XIX Young Researchers Workshop in Geometry, Dynamics and Field Theory, Verona, Italy, January 2025.
- Virtual Nonholonomic Constraints.
  XVI International ICMAT Summer School on Geometry, Dynamics and Field Theory, La Residencia La Cristalera in Miraflores de la Sierra, Madrid, Spain, June 2024
- Virtual constraints on Lie groups,
  XVIII International Young Researchers Workshop in Geometry, Dynamics and Field Theory, University of Warsaw, Poland, February 2024.
- Virtual Nonholonomic Constraints: A Geometric Approach.
  XV International ICMAT Summer School on Geometry, Dynamics and Control, Miraflores de la Sierra, Madrid, Spain, July 2023.

Chapter 2

Geometric Description of Mechanical Systems

2.1 Calculus on differentiable manifolds

In this section we present the fundamental object in differential geometry, a differentiable manifolds and give all the essential differential geometric tools to work on them. Differentiable (smooth) manifolds, roughly speaking, are sets or better given, abstract surfaces that locally look like open subsets of Euclidean space.

Differentiable manifolds generalize the notion of curves and surfaces from Euclidean spaces to arbitrary spaces of finite as well as infinite dimension. One great motivation to work with manifolds, besides the fact that they appear naturally in many physical problems, is the fact that one can deal with concepts that are coordinate invariant and describe intrinsic phenomena in an coordinate free way which terms out to be a great advantage. Readers can find extended analysis on differential geometry in Abraham et al., 1993, Abraham and Marsden, 1978, Boothby, 2003, Conlon, 2008, Marsden and Ratiu, 1999 Bullo and Lewis, 2005 and Helgason, 1979.

Before the definition of a smooth manifold let us first give the notion of **coordinate charts**. Consider a set M, a coordinate chart for M is a pair (U, ϕ), where U is a subset of M and $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a homomorphism, that is, a continuous, bijective (one-to-one and onto) function between U and $\phi(U)$ with continuous inverse. Two charts (U, ϕ) and (V, ψ) such that $U \cap V \neq \emptyset$ are called **compatible** if $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is a diffeomorphism and a collection of compatible charts that covers M forms an **atlas** $\mathcal{A}$ (see Figure 2.1).

Definition 2.1.1. A set M is a **differentiable (smooth) manifold** of dimension n if there is an atlas with the above properties.

We usually write $\phi(p) = (x^1(p), x^2(p), \ldots, x^n(p))$ and call x^i the **coordinates** of the point $p \in U \subset M$. To make use of the local identification with an open subset of $\mathbb{R}^n$ often we omit ϕ and keep the coordinates of a point when we work locally on a smooth manifold using the chart (U, ϕ) that covers the region of interest.

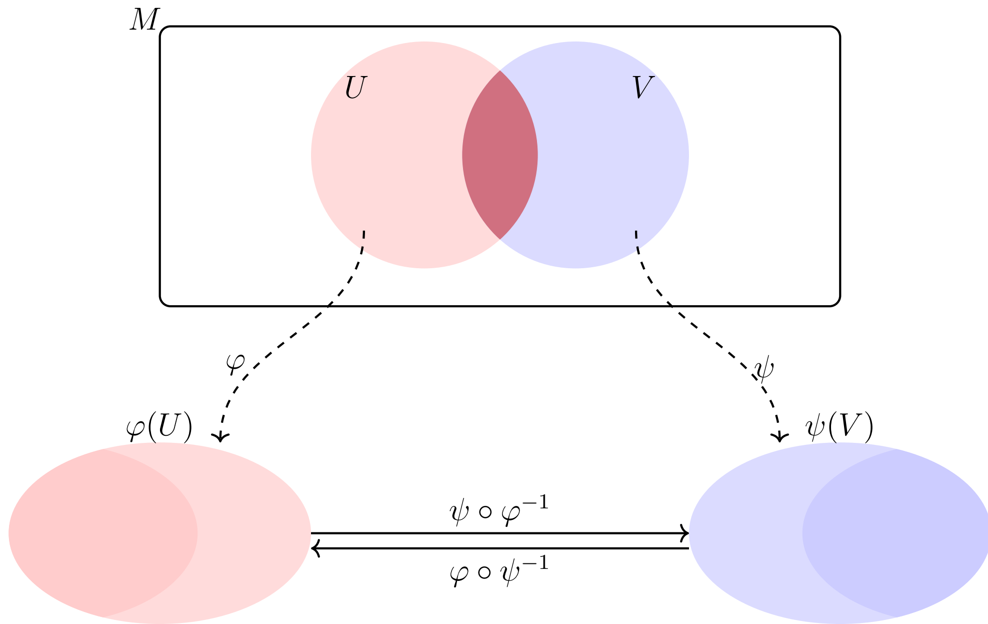


Figure 2.1: Compatible charts on a manifold M.

Just as for sets and vector spaces there is the notion of subset and subspace, for manifolds there are submanifolds that inherent the differentiable structure from the manifold they belong.

Definition 2.1.2. A subset N of an n- dimensional smooth manifold M is a smooth **submanifold** of dimension $k < n$ if, for every point $x \in N$, there exist a chart (U, ϕ) for M containing x such that $\phi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $\phi(U \cap N) = \phi(U) \cap (\mathbb{R}^k \times \{0\})$.

Two curves $t \mapsto \gamma_1(t)$ and $t \mapsto \gamma_2(t)$ in a manifold M are called **equivalent** at the point p if

$$\gamma_1(0) = \gamma_2(0) = p \quad \text{and} \quad \frac{\partial}{\partial t}(\phi \circ \gamma_1)\Big|_{t=0} = \frac{\partial}{\partial t}(\phi \circ \gamma_2)\Big|_{t=0}$$

for some chart ϕ. This definition is chart independent and defines a en equivalence relation.

A **tangent vector** v at a point p of a manifold M is an equivalence class of curves at p. The collection of all these tangent vector forms the **tangent space** to M at p, denoted by T_pM and it has a vector space structure. For a tangent vector $v \in T_pM$ let γ be a curve of the equivalence class of v, the **components** of v, $v^1, \ldots, v^n$, are given by taking the derivatives of the composition $\phi \circ \gamma$, which are the components in the Euclidean space, namely,

$$v^i = \frac{\partial}{\partial t}(\phi \circ \gamma)\Big|_{t=0}$$

where $i = 1, \ldots, n$. The **tangent bundle** is the defined to be the disjoint union of all tangent spaces

$$TM = \bigcup_{p \in M} T_p M$$

and the **tangent bundle projection** (or **natural projection**) is the map $\pi_{TM} : TM \to M$ defined by $\pi_{TM}(v) = p$ for $v \in T_p M$. We denote by $C^\infty(M)$ the set of all smooth real valued function on M. For a smooth function $f \in C^\infty(M)$ i.e. $f : M \to \mathbb{R}$, we define its derivative at any point $p \in M$ to be the map $d_p f : T_p M \to T_{f(x)}\mathbb{R}$. We call df the **differential** of f and for $v \in T_p M$ we call $df(p) \cdot v$ the **directional derivative** of f in the direction of v. In local coordinates the directional derivative is given by

$$df(p) \cdot v = \sum_{i=1}^{n} \frac{\partial (f \circ \phi^{-1})}{\partial x^i} v^i$$

where ϕ is a chart at p and $d_p f = df(p)$. Thus, we can describe completely the directional derivative by using the operators $\frac{\partial}{\partial x^i}$. So, the basis of $T_p M$ is

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$$

and we write $v = v^i \frac{\partial}{\partial x^i}$. Where in the last equality we have used the **summation convention** according to which when an index appears twice in a single term it implies summation of that term over all the values of the index.

Considering the dual as vector space of $T_p M$ we get the **cotangent space** at a point p, $T_p^* M$. The elements of the cotangent space are called **covectors** (or **one-forms**) and are linear maps such that if $\alpha \in T_p^* M$, then $\alpha : T_p M \to \mathbb{R}$. To represent elements of the cotangent space we use the dual basis associated with the basis $\{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$ of the tangent space, which will be denoted as $\{dx^1, \ldots, dx^n\}$ such that $\langle \partial/\partial x^i, dx^j \rangle = \delta_{ij}$ where $\langle \cdot, \cdot \rangle : T_p M \times T_p^* M \to \mathbb{R}$ is the natural pairing between vectors and covectors and δ_{ij} stands for the Kronecker delta. Note here that the differential of a function $f : M \to \mathbb{R}$ as defined above is a covector.

2.1.1 Maps between manifolds

Let $f : M \to N$ be a map from a manifold M to a manifold N and $I \subseteq \mathbb{R}$ containing the origin. We call f differentiable (or smooth) if there are coordinate charts (U, ϕ) and (V, ψ) from M and N respectively such that $p \in M, f(p) \in N, f(U) \subset V$ and the map $F = \psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is differentiable. The derivative of f at a point $p \in M$ is defined to be the linear map

$$d_p f : T_p M \to T_{f(p)} N,$$

defined as follows. For a vector $v \in T_p M$ choose a curve $\gamma : I \to M$ from the respective equivalence class where $\gamma(0) = p$. The vector is called the velocity vector for this curve at t=0,

$\partial\gamma/\partial t|_{t=0} = v$. The image of this vector through the differential of f is the velocity vector of the curve $f \circ \gamma : I \to N$, that is

$$d_p f \cdot v = \frac{\partial(f \circ \gamma)}{\partial t}\bigg|_{t=0},$$

which it is independent of the choice of the curve. The derivative of f at a point p will be denoted also as $T_p f$.

For a smooth map $f : M \to N$ we say that f is of **rank** r if the linear map $d_p f$ has rank r i.e. the rank of the Jacobian matrix is r at $p \in M$, and we write $\text{rank}df = r$. If the rank remains constant for all points of M we say that f is of **constant rank**. If the differential of a map is surjective ($\text{rank}df = \dim N$) then the map is called **submersion**. A particular case of submersions are those which are also surjectives are the called **projections**. Recall the natural projection $\pi_{TM} : TM \to M$ of the tangent bundle to a manifold M defined above. In case the differential of a map, df, is injective ($\text{rank}df = \dim M$) then f is called **immersion**.

When the map $f : M \to N$ is a smooth injective immersion and $f(M)$ is a submanifold then f is called smooth **embedding**. A subset $S \subset N$ is said to be a smooth **immersed submanifold** if there exist a manifold M and a smooth injective immersion $f : M \to N$ such that $f(M) = S$. In this last case f is called **inclusion** which are usually denoted by $i : M \hookrightarrow N$. When f is a diffeomorphism then $\dim M = \dim N = \text{rank}df$, we say that M and N are diffeomorphic and we write $M \simeq N$.

2.1.2 Fields and Bundles

In this subsection be present the notion of bundles in particular we define the concept of vector bundles, sections and fiber bundles. In brief, a vector bundle consists of the total space, the base space and a projection map that connect the two spaces. Vector bundles generalize the idea of tangent bundles given before and sections the idea of vector fields (defined in the next subsection). Fiber bundles are an even more general than vector bundles and can be used in more complex settings. More detailed analysis on these concepts one can see the references cited at the beginning of the section.

A **vector bundle** consists of a **base space** B, a **total space** V and a projection $\pi : V \to B$ such that:

- π is a smooth surjective submersion and
- for every $b \in B$ the fiber $\pi^{-1}(b)$ is vector space.

Locally the total space can be expressed as $V \simeq B \times \mathbb{R}^n$, where n is the dimension of the fiber (we consider the dimension of the fibers constant). A smooth **section** of a vector bundle $\pi : V \to B$ is a map $\xi : B \to V$ such that $\pi \circ \xi = id_B$, where id_B is the identity map on B. We denote the set of sections of V by $\Gamma(V)$. The base space B can be seen as a submanifold of V, we shall call it the **zero section** and it can be given by all points $v \in V$ such that $\phi(v) = (b, 0)$ for some coordinate chart (U, ϕ) of the total space. Tangent bundles are vector bundles where the fiber of each element of the base space is the tangent space at that point, namely for all $p \in M,\ \pi^{-1}(p) = T_pM$ for the bundle $\pi : TM \to M$.

If we have two subbundles of the tangent bundle of M we define their **Whitney sum** or **direct sum bundle** which is a straightforward generalization of the direct sum of two vector spaces. Namely, consider the subbundles V and W of M then their Whitney sum is the subbundle $V \oplus W$ whose fiber over each $p \in M$ is the direct sum $V_p \oplus W_p$ of the vector spaces V_p and W_p.

A generalization of a vector bundle is the fiber bundle where the fiber are manifolds. A locally trivial **fiber bundle** consists of

- M, B and F smooth manifolds
- $\pi : M \to B$ a smooth surjective submersion
- a coordinate chart for B, (U, ϕ), and a diffeomorphism ψ such that the next diagram commutes

$$\begin{CD} \pi^{-1}(U) @>\psi>> U \times F \\ @V{\pi}VV @VV{pr_1}V \\ U @>>{id}> U \end{CD}$$

where pr_1 is the projection to the first component and id is the identity map.

2.1.3 Vector fields and Lie bracket

Vector fields are perhaps the most utilized geometric objects in control theory. In a concise way, a vector field assigns a tangent vector to every point on a manifold in a smooth way. One of the reasons that vector fields are so widely used is the rich interpretation they have, namely they can be studied as differential operators, one can use them to differentiate functions, but also other vector fields. In addition, they are in 1-1 correspondence to ordinary differential equations and they appear naturally in physical phenomena. The Lie bracket of two vector fields is given as particular composition of differential operators and measures how much a vector field change along another.

A smooth **vector field** X on a manifold M is a map $X : M \to TM$ that assigns a vector X_p (or $X(p)$) at every element $p \in M$, i.e. $\pi_{TM} \circ X = id_M$ where $\pi_{TM} : TM \to M$ is the natural projection. We denote the set of all vector fields on M by $\mathfrak{X}(M)$. In local coordinates provided by a chart (U, ϕ) of the manifold M we can write a vector field as

$$X_p = X^i(p) \frac{\partial}{\partial x^i},$$

where $X^i \in C^\infty(M)$ are called **coordinate functions** (or **components**) of X with respect to the chart. An **integral curve** for a vector field $X \in \mathfrak{X}(M)$ is a map $\gamma : I \to M, I \subset \mathbb{R}$ such that

$$\dot{\gamma}(t) = X(\gamma(t)).$$

The existence and uniqueness of an integral curve is guaranteed locally by the existence and uniqueness theorem for ordinary differential equations (see Marsden and Ratiu, 1999, A. Bloch, 2015, Arnold, 1992). The **flow** of X is a set of diffeomorphisms $\psi_t : M \to M$ such that $\psi_t(p)$ is the integral curve of X with initial position p.

Consider a set of vector fields $\{X_1, \dots, X_n\}$ on an n-dimensional manifold M. This set will be called a **frame** for the tangent bundle TM if for every point $p \in M$ the set of vectors $\{X_1(p), \dots, X_n(p)\}$ consists of a basis for the vector space T_pM.

Let $X \in \mathfrak{X}(M)$ be a vector field on M and $f \in C^\infty(M)$ a smooth function, the **Lie derivative** of f with respect to (or along) X is the function given by

$$\mathcal{L}_X f = X_p f = df(p) \cdot X_p.$$

In local coordinates, where $X = X^i \frac{\partial}{\partial x^i}$, the Lie derivative is given by

$$\mathcal{L}_X f(p) = df(p) \cdot X = \frac{\partial f}{\partial x^i} X^i(p).$$

Given two vector fields $X, Y \in \mathfrak{X}(M)$ we define the **Lie bracket** of X and Y, which we denote by $[X, Y]$, as the vector field given by

$$[X, Y]f = X(Yf) - Y(Xf). \tag{2.1}$$

Considering the coordinate chart (U, ϕ) where $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$, the vector field $[X, Y]$ is locally given by

$$[X, Y] = \left(\frac{\partial Y^i}{\partial x^j} X^j - \frac{\partial X^i}{\partial x^j} Y^j \right) \frac{\partial}{\partial x^i}.$$

This vector field gives the Lie derivative of Y along X and sometimes is denoted, also, by $\mathcal{L}_X Y$. Next let us state some useful properties of the Lie bracket.

Proposition 2.1.3. *For real valued functions $f, g \in C^\infty(M)$ and vector fields $X, Y, Z \in \mathfrak{X}(M)$ we have*

1. $[X, Y] = -[Y, X]$ *(**skew-symmetry**),*
2. $[X + Y, Z] = [X, Z] + [Y, Z]$ *(**linearity**),*
3. $[fX, gY] = fg[X, Y] + f(\mathcal{L}_X g)Y - g(\mathcal{L}_Y)X$,
4. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(**Jacobi identity**).*

Consider a smooth map $f : M \to N$ between manifolds M and N and two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. We call X and Y **f-related** if for all $p \in M$ they satisfy

$$Y_{f(p)} = d_p f \cdot X_p.$$

If $f : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$ we define a new vector field on N which is called the **push forward** of the vector field X by f given by

$$(f_* X)_q = d_p f \cdot X_p,$$

where $f(p) = q$. Accordingly, for f a diffeomorphism and a vector field $Y \in \mathfrak{X}(N)$, we define the **pull back** of Y by f which is given by

$$(f^* Y)_p = (d_p f)^{-1} Y_q,$$

where $f(p) = q$.

We give the definition of a **covector field** or **one-form**, similarly to that of vector field, as follows. A one-form is a map that assigns a covector α_p (or $\alpha(p)$) at ever point $p \in M$ i.e. $\alpha : M \to T^*M$. The collection of all covector fields is denoted by $\Omega^1(M)$. Given a coordinate chart (U, ϕ) for M a covector field can be expressed as

$$\alpha_p = \alpha_i dx^i$$

where $\alpha_i \in C^\infty(M)$ are called **coordinate functions** (or **components**) of α in the given chart.

Since the tangent bundle is a manifold on its own, often it is very useful to "lift" (as we call it) vector fields on a manifold to its tangent bundle. The **vertical lift** of a vector field $X \in \mathfrak{X}(M)$ to TM is defined by

$$X^V_{v_p} = \left.\frac{d}{dt}\right|_{t=0} (v_p + tX(p)).$$

Accordingly, the vertical lift of a one-form $\alpha \in \Omega^1(Q)$ is defined as the pullback of α to TQ, i.e.

$$\alpha^V = (\pi_{TM})^* \alpha.$$

In local coordinates the above expressions, for $X = X^i \frac{\partial}{\partial x^i}$ and $\alpha = \alpha_i dx^i$, take the forms

$$X = X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad \alpha^V = \alpha_i dx^i$$

respectively. In a similar fashion, we define the **complete lift** of vector and covector fields. The complete lift of a vector field, X, is

$$X^c = X^i \frac{\partial}{\partial x^i} + \dot{x}^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial \dot{x}^i},$$

and the one of a one-form $\alpha \in \Omega^1(M)$ is

$$\alpha^c = \alpha^i dx^i + \dot{x}^j \frac{\partial \alpha^i}{\partial x^j} dx^i.$$

In the context of mechanical systems, we find a special type of vector fields that are always defined on the tangent bundle TM, considered as a manifold itself. A **second-order vector field** (SODE) Γ on the tangent bundle TM is a vector field on the tangent bundle satisfying the property that $T\pi_{TM}(\Gamma(v_p)) = v_p$, $p \in M$. The expression of any SODE in coordinates is the following:

$$\Gamma(x^i, \dot{x}^i) = \dot{x}^i \frac{\partial}{\partial x^i} + f^i(x^i, \dot{x}^i) \frac{\partial}{\partial \dot{x}^i}, \tag{2.2}$$

where $f^i : TM \to \mathbb{R},\ i = 1, \ldots, n$ are n smooth functions.

2.1.4 Distributions and codistributions

In what follows we give the notion of distribution and codistribution which are subbundles of TM and T^*M for a manifold M. They appear naturally in mechanics and in the theory of controllability. In the concept of mechanical systems when they are subject to nonintegrable constraints on the velocities, distributions describe the admitted velocities for the system. Their role in the theory of controllability is beyond the scope of this thesis. Both distributions and codistribution play an important role in this dissertation and below we bring them into focus.

Let M be a smooth manifold. A **distribution** $\mathcal{D}$ is an assignment to each point $p \in M$, of a subspace $\mathcal{D}_p$ of T_pM. For every p there exist a set of smooth vector fields $\{X_a\}_{a\in A}$ such that in a neighborhood $\mathcal{U}$ of p, we have

$$\mathcal{D}_p = \operatorname{span}\{X_a(p) : a \in A\},\ x \in \mathcal{U}.$$

These vector fields are called **local generators** of $\mathcal{D}$ about p. If the dimension of the vector space $\mathcal{D}_p$ for all $p \in M$ is constant the distribution is called **regular**.

Next we give two properties of distributions that are related but not obviously so, these are the notion of involutivity and integrability. A smooth distribution $\mathcal{D}$ is **involutive** if it is closed under the Lie bracket operator i.e. for every $X, Y \in \Gamma(\mathcal{D})$, $[X, Y] \in \Gamma(\mathcal{D})$. Recall that $\Gamma(\mathcal{D})$ is the set of sections of $\mathcal{D}$. A **local integral submanifold** of a smooth distribution $\mathcal{D}$ is a submanifold S of M such that $T_pS = \mathcal{D}_p$ for every $p \in S$. If for every point of M there exist a local integral submanifold of $\mathcal{D}$ then $\mathcal{D}$ is called **integrable**. The next Theorem due to Frobenius, 1877 links these two notions.

Theorem 2.1.4 (**Frobenius's Theorem**). *A smooth distribution $\mathcal{D}$ is integrable if and only if it is involutive.*

If a distribution $\mathcal{D}$ is integrable then we can extend the local integral submanifolds to **maximal integral submanifolds** which are immersed submanifolds on M. A **foliation** is the collection of all maximal integral manifolds through all points of M.

Considering a smooth manifold M, the definition of **codistribution** is analogous to that of distribution. Here we assign a subspace $\tilde{\mathcal{D}}_p$ of the cotangent bundle T_p^*M to a each point $p \in M$. With the help of codistributions we define the annihilator of a distribution. Let $\mathcal{D}$ be a distribution, the **annihilator** of $\mathcal{D}$ is a codistribution denoted by $\mathcal{D}^\circ$ and it is given by

$$\mathcal{D}_p^\circ = (\mathcal{D}_p)^\circ = \{\alpha \in T_p^*M : \alpha(v) = \langle \alpha, v\rangle = 0, \forall v \in \mathcal{D}_p\}, \tag{2.3}$$

for every $p \in M$.

Example 2.1.5. The simplest example of a distribution is that of an integrable distribution such as all tangent vectors in $\mathbb{R}^3$ with vanishing third coordinate, i.e., $\dot{z} = 0$. However, observe that these vectors are all tangent to the planes defined by $z =$ constant. In the other hand, the distribution defined by the constraint $\dot{z} - y\dot{x} = 0$ can not be written as the tangent plane to some plane or surface of $\mathbb{R}^3$. So, this is an example of a non-integrable distribution.

2.1.5 Tensors, wedge products and exterior derivatives

A very important concept for calculus and differential geometry is the notion of tensors. Tensors generalize operators to manifolds of arbitrary dimension.

Let M be a smooth manifold. A **tensor of type** (r, k) at some point $p \in M$ is a multilinear map

$$t : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{r} \times \underbrace{T_pM \times \cdots \times T_pM}_{k} \to \mathbb{R}$$

the set of all (r, k) - tensors at $p \in M$ is denoted by $T_k^r(T_pM)$. A map is multilinear when it is linear in each of its factors. A basic operation we can use to tensors is the **tensor product** which is defined as follows. For two tensors $t_1 \in T_k^r(T_pM)$ and $t_2 \in T_\kappa^\rho(T_pM)$ the tensor product is a new tensor $t_1 \otimes t_2$ of type $(r + \rho, k + \kappa)$, $t_1 \otimes t_2 \in T_{k+\kappa}^{r+\rho}(T_pM)$, defined by

$$\begin{aligned}(t_1 \otimes t_2) = (\alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^\rho, v_1, \ldots, v_k, u_1, \ldots, u_\kappa) = \\ t_1(\alpha^1, \ldots, \alpha^r, v_1, \ldots, v_k)t_2(\beta^1, \ldots, \beta^\rho, u_1, \ldots, u_\kappa).\end{aligned}$$

Using the tensor product we give here an operator that will be useful later on for this dissertation. Let M be a smooth manifold, the **canonical almost tangent structure** or **vertical endomorphism** $J : TTM \to TTM$ is a type $(1, 1)$- tensor field on TM whose expression in local coordinates is $J = dx^i \otimes \frac{\partial}{\partial \dot{x}^i}$. For instance, if Γ is a SODE vector field then $J(\Gamma) = \dot{x}^i \frac{\partial}{\partial \dot{x}^i}$. Accordingly, the dual of this map, is given by $J^* : T^*TM \to T^*TM$ such that for a $(0, 1)$-form on TM, $\Delta = t_i dx^i + s_i d\dot{x}^i$, we get $J^*(\Delta) = s_i dx^i$.

An important subset of tensors are the **skew-symmetric** (or **alternating**) $(0, k)$ - tensors which are called **differential** k**-forms** or just k**-forms** and are denoted by $\Omega^k(M)$. A k-form is skew-symmetric when it changes sign when two of its entries interchange, i.e.

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for $\alpha \in \Omega^k(M)$, on the contrary if it does not change sign it is called **symmetric**. Note here that the covector fields defined previously are $(0,1)$-tensors and the vector fields are $(1,0)$-tensors.

The **alternating map** $A : T^0_k(T_pM) \to \Omega^k(M)$ is defined by

$$A(t)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \mathrm{sign}(\sigma) t(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

where Σ_k is the set of k-permutations. A is linear, $A\Big|_{\Omega^k(M)} = id$ and $A \circ A = A$.

The **wedge** or **exterior product** between two forms, $t_1 \in \Omega^k(M)$ and $t_2 \in \Omega^l(M)$, is the form $t_1 \wedge t_2 \in \Omega^{k+l}(M)$ given by

$$t_1 \wedge t_2 = \frac{(k+l)!}{k!l!} A(t_1 \otimes t_2).$$

The wedge product of two differential forms

1. is bilinear and associative and
2. for $t_1 \in \Omega^k(M), t_2 \in \Omega^l(M)$, we have that $t_1 \wedge t_2 = (-1)^{kl} t_2 \wedge t_1$.

The direct sum of $\Omega^k(M)$ for $k = 0, 1, \ldots$ equipped with the wedge product form the **algebra of exterior differential forms** denoted by $\Omega(M)$ and it is an infinite-dimensional real vector space.

The **exterior derivative**, denoted by $\mathbf{d}$, is the unique family of mappings from k-forms on M to $(k+1)$-forms on M, i.e. $\mathbf{d}^k(M) : \Omega^k(M) \to \Omega^{k+1}(M)$, such that the next properties are satisfied

1. a 0-form on M is considered to be a real valued function on M, i.e. $t \in C^r(M)$ (for any differentiability class $r > 1$.), then dt is the one-form that is the differential of t,
2. the exterior derivative is linear
$$\mathbf{d}(at_1 + bt_2) = a\mathbf{d}t_1 + b\mathbf{d}t_2,$$
for real numbers a, b,
3. it satisfies the antiderivation product rule
$$\mathbf{d}(t_1 \wedge t_2) = \mathbf{d}t_1 \wedge t_2 + (-1)^k t_1 \wedge \mathbf{d}t_2,$$
where $t_1 \in \Omega^k(M), t_2 \in \Omega^l(M)$,
4. $\mathbf{d}^2 = 0$, which means $\mathbf{d}(\mathbf{d}t) = 0$ for any k-form t,
5. the exterior derivative is a local operator, i.e. $\mathbf{d}t(p)$ depends on the differential form around the the element p, formally if U is an open in M then
$$\mathbf{d}\left(t\Big|_U\right) = (\mathbf{d}t)\Big|_U.$$

In local coordinates, a k-form t is given by

$$t = t_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

and the exterior derivative in local coordinates is given by

$$\mathbf{d}t = \frac{\partial t_{i_1 \dots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where $t_{i_1 \dots i_k}$ are real valued functions. A k-form t is called **closed** if $\mathbf{d}t = 0$ and if there is an other $(k-1)$-form s such that $t = \mathbf{d}s$ it is called **exact**. From property 4 above we see that every exact form is closed.

As we have defined the push forward and the pull back for vector fields next we give the analogous definitions for differential k-forms. Let $f : M \to N$ be a smooth map between manifolds. The **push forward** $f_*\omega$ of a k-form $\omega \in \Omega^k(M)$ is defined by

$$\left(f_*\omega(f(p))\right)(u_1, \dots, u_k) = \omega(p)\left(T_{f(p)}f^{-1}(u_1), \dots, T_{f(p)}f^{-1}(u_k)\right),$$

where $u_i \in T_{f(p)}N$ for $i = 1, \dots, k$ and $p \in M$. Respectively, the **pull back** $f^*\omega$ of a k-form $\omega \in \Omega^k(N)$ is defined by

$$\left(f^*\omega(p)\right)(v_1, \dots, v_k) = \omega(f(p))(T_pf(v_1), \dots, T_pf(v_k)),$$

where $v_i \in T_pM$ for $i = 1, \dots, k$ and $p \in M$. Next, we define a natural operator given a vector filed and a differential form. Let $\omega \in \Omega^k(M)$ and a vector field $X \in \mathfrak{X}(M)$, the **inner product** or **contraction** of X and ω is a $(k-1)$-form denoted by $i_X\omega \in \Omega^{k-1}(M)$ and given by

$$i_X\omega(p)(v_1, \dots, v_{k-1}) = \omega(p)\left(X(p), v_1, \dots, v_{k-1}\right),$$

where $v_i \in T_pM$ for $i = 1, \dots, k-1$ and $p \in M$. The operator of contraction, i_X, satisfies the antiderivation product rule i.e. $i_X(\omega_1 \wedge \omega_2) = (i_X\omega_1) \wedge \omega_2 + (-1)^k\omega_1 \wedge (i_X\omega_2)$, where $\omega_1 \in \Omega^k(M)$ and $\omega_2 \in \Omega^l(M)$ and also for $f \in C^\infty(M)$, we have that $i_{(fX)}\omega = f(i_X\omega)$.

We are closing this section by giving the definition of a distinguished part of the set of differential 2-forms, the symplectic forms. A **symplectic form** on a manifold M is the 2-form, ω, which has the properties

1. ω is closed i.e. $d\omega = 0$ and
2. ω is nondegenerate, i.e. $\omega_z(v, w) = 0$ for all $v \in T_zM$ then $w = 0$.

The pair (M, ω) is called **symplectic manifold**.

2.2 Riemannian Geometry

The Riemannian geometry is the study of smooth manifolds endowed with a Riemannian metric which, in essence, is a inner product on the tangent space at each point of the manifold that varies smoothly. This metric helps to define locally distances and angles, lengths of curves and surface areas as well as volumes on manifolds. In brief, Riemannian metrics give the ability to use many of the notions we know from Euclidean spaces to abstract multi-dimensional manifolds. Reference for Riemannian geometry used along this thesis are do Carmo, 1992, Boothby, 2003, Abraham and Marsden, 1978, Helgason, 1979.

2.2.1 Riemannian metrics

The concept of Riemannian metric was first introduced by Riemann, 1892 where at that time the notion of manifold was a very abstract and the whole theory was at its early stage. The modern perception of a Riemannian manifold took considerable amount of time to evolve to its present form see for instance Tu, 2017, Gallot et al., 2004, J. Lee, 1997.

Definition 2.2.1. A *Riemannian metric* on a smooth manifold Q is a (0,2)-tensor field $\mathcal{G}$ that is

- symmetric i.e. $\mathcal{G}(X,Y)=\mathcal{G}(Y,X)$ for all $X,Y\in T_qQ, q\in Q$ and
- positive-definite i.e. $\mathcal{G}(X,X)>0$ for all $X\in T_qQ, q\in Q$.

A manifold equipped with a given Riemannian metric is called a *Riemannian manifold* and it is denoted by $(Q,\mathcal{G})$.

Observe that a Riemannian metric is a smooth familly of inner products in the sence that, for any element of the manifold, $q\in Q$, the Riemannian metric restricted to T_qQ gives an inner product on T_qQ. For this reason, we often denote the metric by $\langle\cdot,\cdot\rangle$ instead of $\mathcal{G}$ when there is no chance of confusion. The Riemannian metric can be defined as a map that assigns to every pair of smooth sections of Q, a real-valued function on Q, i.e.: $\mathcal{G}:\Gamma(TQ)\times\Gamma(TQ)\to C^\infty(Q)$ given by $\mathcal{G}(X,Y)$ for all $X,Y\in\Gamma(TQ)$.

Let $(Q,\mathcal{G})$ be a Riemannian manifold, $q\in Q$ a point of Q and $\dim Q=n$, we define the *length* or *norm* of any tangent vector $X\in T_qQ$ to be $||X||=\mathcal{G}(X,X)^{1/2}=\langle X,X\rangle^{1/2}$. We say that two vectors $X,Y\in T_qQ$ are *orthogonal* if $\langle X,Y\rangle=0$ and that the vectors $X_1,\dots,X_n$ are *orthonormal* if they are of length 1 and pairwise orthogonal, namely, $\langle X_i,X_j\rangle=\delta_{ij}$ where $1\leq i,j\leq n$ and δ_{ij} is the Kronecker delta.

If $\{\xi_1,\dots,\xi_n\}$ is a local frame for the tangent bundle TQ, and $\{\eta^1,\dots,\eta^n\}$ is its dual coframe, a Riemannian metric can be written locally as $\mathcal{G}=\mathcal{G}_{ij}\eta^i\otimes\eta^j$, where $\mathcal{G}_{ij}=\langle\xi_i,\xi_j\rangle$. $\mathcal{G}_{ij}$ is symmetric in i and j and depends smoothly on $q\in Q$. In particular, in a coordinate frame, $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ and due to the symmetry of $\mathcal{G}_{ij}$, $\mathcal{G}$ has the form $\mathcal{G}=\mathcal{G}_{ij}dx^idx^j$.

Very often we encounter Riemannian manifolds that are a product of other Riemannian manifolds or have that structure. Consider two Riemannian manifolds $(Q_1,\mathcal{G}_1)$ and $(Q_2,\mathcal{G}_2)$ and the product manifold $Q=Q_1\times Q_2$. There is a natural choice for the metric of Q. For $X_1,Y_1\in TQ_1$ and $X_2,Y_2\in TQ_2$ the **product metric** is given by $\mathcal{G}\left((X_1,Y_1),(X_2,Y_2)\right)=$

$\mathcal{G}_1(X_1, X_2) + \mathcal{G}_2(X_2, Y_2)$, where we have identified $T(Q_1 \times Q_2)$ and $TQ_1 \times TQ_2$.

An **isometry** between two Riemannian manifolds $(Q_1, \mathcal{G}_1)$ and $(Q_2, \mathcal{G}_2)$ is a diffeomorphism $\phi : (Q_1, \mathcal{G}_1) \to (Q_2, \mathcal{G}_2)$ such that $\phi^*\mathcal{G}_2 = \mathcal{G}_1$. In other words an isometry preserves the metric in the way that

$$\mathcal{G}_1(v, u) = \mathcal{G}_2\left(d\phi(v), d\phi(u)\right).$$

Riemannian metrics induce isomorphisms between the tangent bundle, TQ, and the cotangent bundle, T^*Q, of the manifold, this is a way to convert tangent vectors into cotangent vectors (and vice versa). These isomorphisms are called **musical isomorphisms** (or canonical isomorphisms). We define first the **flat map** $\flat : TQ \to T^*Q$ by $X^\flat(Y) = \langle X, Y\rangle$ for all $X, Y \in TQ$ and the **sharp map** $\sharp : T^*Q \to TQ$ as the inverse of the flat map.

In the following, we will make use of a special technique to lift a Riemannian metric on a manifold to a metric on the tangent bundle TQ. The complete lift of a Riemannian metric $\mathcal{G}$ on Q is denoted by $\mathcal{G}^c$ and it is almost a Riemannian metric, since it does not satisfy the positive-definiteness property. Given natural bundle coordinates on TQ, its local expression is

$$\mathcal{G}^c = \dot{q}^k \frac{\partial \mathcal{G}_{ij}}{\partial q^k} dq^i \otimes dq^j + \mathcal{G}_{ij} dq^i \otimes d\dot{q}^j + \mathcal{G}_{ij} d\dot{q}^i \otimes dq^j$$

.

Proposition 2.2.2. *For a Riemannian metric $\mathcal{G}$ on Q, vector fields $X, Y \in \mathfrak{X}(Q)$ and a one-form $\alpha \in \Omega^1(Q)$ we have*

1. $(\alpha(X))^V = \alpha^c(X^V)$,
2. $\mathcal{G}^c(X^V, Y^c) = \mathcal{G}^c(X^c, Y^V) = [\mathcal{G}(X, Y)]^V$,
3. $\mathcal{G}^c(X^V, Y^V) = 0$,
4. $[\sharp_{\mathcal{G}}(\alpha)]^V = \sharp_{\mathcal{G}^c}(\alpha^V)$.

Proof. For the first three, see Proposition 2.5.5 in de León and Rodrigues, 1989. For 4., given any $Y \in \mathfrak{X}(Q)$, it is enough to prove the equality using the inner product with the lifts Y^c and Y^V, because if $\{Y^a\}$ was a local basis of vector fields, then $\{(Y^a)^c, (Y^a)^V\}$ would also be a local basis of vector fields on TQ. On the one hand, $\mathcal{G}^c\left([\sharp_{\mathcal{G}}(\alpha)]^V, Y^V\right) = 0 = \alpha^V(Y^V) = \mathcal{G}^c\left(\sharp_{\mathcal{G}^c}(\alpha^V), Y^V\right)$. On the other hand, $\mathcal{G}^c\left([\sharp_{\mathcal{G}}(\alpha)]^V, Y^c\right) = [\mathcal{G}(\sharp_{\mathcal{G}}(\alpha), Y)]^V = [\alpha(Y)]^V = \alpha^V(Y^c) = \mathcal{G}^c\left(\sharp_{\mathcal{G}^c}(\alpha^V), Y^c\right)$. Hence, the results follow by non-degeneracy of $\mathcal{G}^c$. □

2.2.2 Affine differential geometry

The concept of affine connection plays a significant role in this thesis. Covariant derivatives and affine connections are two classical objects in Riemannian geometry and affine differential geometry. One important feature that one can observe, is the 1-1 correspondence between affine connection (with specific properties) and Riemannian metrics.

Consider a vector bundle over a manifold Q, $\pi : E \to Q$. A **connection** on Q is a map $\nabla : \mathfrak{X}(Q) \times \Gamma(E) \to \Gamma(E)$, $(X, Y) \mapsto \nabla_X Y$ which satisfies the next properties

1. $\nabla_{fX_1+gX_2}Y = f\nabla_{X_1}Y + g\nabla_{X_2}Y$, i.e. $C^\infty(Q)$-linear in X,
2. $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$, i.e. $\mathbb{R}$-linear in Y and
3. $\nabla_X fY = f\nabla_X Y + (\mathcal{L}f)Y$

for each $X_1, X_2 \in \mathfrak{X}(Q)$, $f, g \in C^\infty(Q)$, and $a, b \in \mathbb{R}$. The new vector field $\nabla_X Y$ is called **covariant derivative** of Y with respect to X.

Now if we choose for vector bundle the tangent bundle of the manifold Q, we get the **affine connection** on Q and we define the map $\nabla : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q)$ which satisfies the properties $(1) - (3)$ in the above definition.

We equip Q with local coordinates $(q^1, \ldots, q^n)$ and we have that

$$\nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \Gamma^k_{ij} \frac{\partial}{\partial q^k}$$

is a smooth vector field on Q where Γ^k_{ij} are real-valued smooth functions and are called **Christoffel symbols** for ∇. For two vector fields X and Y where

$$X = X^i \frac{\partial}{\partial q^i} \quad \text{and} \quad Y = Y^i \frac{\partial}{\partial q^i}$$

the covariant derivative of Y with respect to X is locally given by

$$\nabla_X Y = \left(\frac{\partial Y^k}{\partial x^i} X^i + \Gamma^k_{ij} X^i Y^j \right) \frac{\partial}{\partial x^k}.$$

In view of the above we give a different approach to the covariant derivative which will be proved useful in subsequent computations. Consider a vector $v_q \in T_qQ$ and a vector field $Y \in \mathfrak{X}(Q)$, we define the vector $\nabla_{v_q} Y = \nabla_X Y \in T_qQ$ where X is a smooth vector field such that $X_q = v_q$. This is an important observation because the vector v_q can represent a velocity vector of a curve on Q. Recall here that a **vector field along a curve** $\gamma : I \to Q$ is an assignment to every point $\gamma(t)$ a vector of $T_{\gamma(t)}Q$.

Lemma 2.2.3 (Lemma 3.100 in Bullo and Lewis, 2005)**.** *Let ∇ be a smooth affine connection on a manifold Q and let $\gamma : I \to Q$ be a smooth curve. There exist a smooth map that assigns to each vector field X along γ a new vector field along γ denoted by $\nabla_{\gamma'(t)} X(t)$ with the next properties:*

(i) $\nabla_{\gamma'(t)} (X(t)) + Y(t))$,

(ii) $\nabla_{\gamma'(t)}(f(t)X(t)) = \dot{f}(t)X(t) + f(t)\nabla_{\gamma'(t)}X(t)$ *and*

(iii) $\nabla_{\gamma'(t)}X(t) = \nabla_{\gamma'(t)}Y(\gamma(t))$, *for every vector field Y such that $Y \circ \gamma = X$,*

where X, Y smooth vector fields along γ and $f : I \to \mathbb{R}$. The vector field $\nabla_{\gamma'(t)}X(t)$ along γ is the ***covariant derivative*** *of X along γ.*

In local coordinates we write

$$\nabla_{\gamma'(t)}X(t) = \left(\dot{X}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{x}^i(t)X^j(t) \right) \frac{\partial}{\partial x^k},$$

where $\gamma(t) = \gamma(x^1(t), \dots, x^n(t))$ a local representation of the curve γ.

Definition 2.2.4. Let ∇ be a smooth affine connection on a manifold Q, a smooth curve $\gamma : I \to Q$ is said to be a **geodesic** of the affine connection if it satisfies $\nabla_{\gamma'(t)}\gamma'(t) = 0$.

With the local representation of the curve γ given above, the second-order differential equations determining a geodesic are

$$\ddot{x}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{x}^i(t)\dot{x}^j(t) = 0, \qquad \text{for} \quad k \in \{1, 2, \dots, n\}.$$

Next we define an affine connection which is of particular interest in geometric control, namely the Levi-Civita connection.

Theorem 2.2.5 (Theorem 3.104 in Bullo and Lewis, 2005)**.** *For a Riemannian manifold* $(Q, \mathcal{G})$ *there exist a unique smooth affine connection* $\nabla^{\mathcal{G}}$ *on* Q *such that*

1. *it is torsion-free (or symmetric), i.e.* $\nabla^{\mathcal{G}}_X Y - \nabla^{\mathcal{G}}_Y X = [X, Y]$ *for all* $X, Y \in \mathfrak{X}(Q)$*, and*
2. $\nabla^{\mathcal{G}}\mathcal{G} = 0 \Leftrightarrow \mathcal{L}_X(\mathcal{G}(Y, Z)) = \mathcal{G}(\nabla^{\mathcal{G}}_X Y, Z) + \mathcal{G}(Y, \nabla^{\mathcal{G}}_X Z)$.

This affine connection it is called ***Levi-Civita connection*** *associated with the Riemannian manifold* $(Q, \mathcal{G})$*.*

The second property is often called "compatibility with the metric" and it says that the parallel transport is an isometry i.e. the inner products at different tangent spaces are preserved. The Christoffel symbols for $\nabla^{\mathcal{G}}$ are given by

$$\Gamma^k_{ij} = \frac{1}{2}\mathcal{G}^{kl}\left(\frac{\partial \mathcal{G}_{il}}{\partial q^j} + \frac{\partial \mathcal{G}_{jl}}{\partial q^i} - \frac{\partial \mathcal{G}_{ij}}{\partial q^l}\right).$$

2.3 Mechanical systems on differentiable manifolds

In this chapter, we present the theory of mechanical systems starting from the Lagrangian and Hamiltonian formalism, which are considered to be the cornerstone of mechanics. The Lagrangian part is using variational principles for its development, while the Hamiltonian part takes advantage of the geometric structures namely the symplectic or Poisson structures. In addition, we develop the concepts of mechanical systems with interactions with the environment (i.e. subjected to external forces) which will be very useful later on when we talk about control theory. The theory of mechanical systems is deployed in local coordinates and in a coordinate free setting when we deal with general manifolds. The reader is referred to one of the standard works such as A. Bloch, 2015, Bullo and Lewis, 2005, Abraham and Marsden, 1978, Holm et al., 2009, Marsden and Ratiu, 1999, Holm, 2011, Crampin, 1983.

2.3.1 Lagrangian mechanics

We start this section by developing the theory of Lagrangian and Hamiltonian mechanics necessary for this thesis, and we add external forces to show how interactions with the surroundings interfere with the system.

To define a mechanical system we need a configuration space (or state space) Q which in this work will be a Euclidean space or a smooth manifold and a function called Lagrangian $L : TQ \to \mathbb{R}$, where TQ is a tangent bundle of Q. Thus, a **mechanical system** is a pair (Q, L). If there are neither constraints nor external forces that act on the system, the **trajectories** of the mechanical system are solutions of the **Euler-Lagrange** equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \tag{2.4}$$

where they are written in local coordinates (q^i) for Q and $(q^i, \dot{q}^i)$ for the tangent bundle TQ. Sometimes it seems like these equations appear in thin air and that they have no physical interpretation. However, this is not the case because equations (2.4) give a formalism for a generalization of Newton's second law which states that

$$\ddot{q} = -\frac{dV}{dt}.$$

If one identifies the kinetic energy of the mechanical system T and the potential energy V, then one defines the mechanical type Lagrangian function to be $L = T - V$, from which one can deduce Newton's second law. However, Hamilton's principle provides the necessary background in an appropriate context.

Hamilton's principle says that the trajectories of a mechanical system are given by minimizing the action functional defined over the set of variations of curves with fixed end-points. Let us denote this set of differentiable curves with fixed and-points $q_a, q_b \in Q$ by

$$C^\infty(q_a, q_b) = \{q : [a, b] \to Q : q(a) = q_a, q(b) = q_b\}.$$

Let $q : [a, b] \to Q$ be a smooth curve in Q, a **variation** of q is a smooth map $\vartheta : J \times [a, b] \to Q$ such that

1. $J \subset \mathbb{R}$, an interval with the property that $0 \in J$,
2. $\vartheta(0, t) = q(t)$ for all $t \in [a, b]$,
3. $\vartheta(s, a) = q(a)$ and $\vartheta(s, b) = q(b)$ for all $s \in J$.

Given this variation ϑ, the associated **infinitesimal variation** is the vector field along q given by

$$\delta\vartheta(t) = \left.\frac{d}{ds}\right|_{s=0} \vartheta(s, t) \in T_{\gamma(t)}Q.$$

For more about variations we are referring to A. Bloch, 2015, Holm et al., 2009 or any reference at the beginning of the section.

Definition 2.3.1 (Hamilton's principle)**.** Consider a manifold Q and a Lagrangian function $L : TQ \to \mathbb{R}$. A curve $q \in C^\infty(q_a, q_b)$ is a trajectory of the mechanical system with Lagrangian L if it is a critical point of the **action functional** $\mathcal{S} : C^\infty(q_a, q_b) \to \mathbb{R}$ given by

$$\mathcal{S}(q) = \int_a^b L(q(t), \dot{q}(t))dt,$$

for all variations of curves in $C^\infty(q_a, q_b)$.

In other words, the Hamilton's principle states:

$$\frac{d}{ds}\bigg|_{s=0} \int_a^b L(\vartheta(s,t), \dot{\vartheta}(s,t))dt = 0$$

for all variations. The Euler-Lagrange equations derive by a straightforward calculation from minimizing $\mathcal{S}(q)$, for details one can see Bullo and Lewis, 2005, Holm et al., 2009, A. Bloch, 2015.

It turns out that the Euler-Lagrange equations are coordinate invariant, in the sense that in different local coordinates on the configuration space Q the trajectories of the mechanical system are merely a reparametrization of the same trajectories. This can be seen also by the Hamilton's principle where the problem of minimizing the action functional is obviously independent of local coordinates.

A Lagrangian is called of **mechanical type** when it is of the form $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$ where $K : TQ \to \mathbb{R}$ is the kinetic energy of the system and $V : Q \to \mathbb{R}$ is the potential energy.

2.3.2 Forced Lagrangian systems

When a mechanical system is subject to external forces, Hamilton's principle is replaced by the Lagrange-d'Alembert principle that tells us how the forces should act on the Euler-Lagrange equations. The concept of external forces is very important when we come to discussing of control problems because we see forces as something by which we can interfere to the system in order to achieve certain tasks.

An **external force** is a map $F : TQ \to T^*Q$ where Q is the configuration space of the mechanical system. If (q^i) are local coordinates of Q then the external force is locally given by $F = F_i dq^i$ where F_i are real valued functions on TQ which are called the **components** of the force F. A mechanical system with Lagrangian function $L : TQ \to \mathbb{R}$ and subjected to external force F is called **forced Lagrangian system** and it is denoted by (Q, L, F) or simply by (L, F) if there is no confusion. The trajectories of this mechanical system are curves on the configuration space Q that satisfies Lagrange-d'Alembert principle below.

Definition 2.3.2 (Lagrange-d'Alembert principle)**.** Let Q be the configuration space of a mechanical system with Lagrangian function $F : TQ \to \mathbb{R}$ and F be an external force acting to the system. A smooth curve $\gamma : [a, b] \to Q$ is a trajectory of the force mechanical system (Q, L, F) if it satisfies

$$\frac{d}{ds}\bigg|_{s=0} \int_a^b L(\vartheta(s,t), (\dot{\vartheta}(s,t))dt + \int_a^b F(\dot{\gamma}(t)) \cdot \delta\vartheta dt = 0, \tag{2.5}$$

for all smooth variations $\vartheta : J \times [a, b] \to Q$ of γ, where $F \cdot \delta\vartheta = F_i \delta\vartheta^i$. If we consider coordinates (q^i) on Q then Lagrange-d'Alembert principle states that the curve γ should satisfy the equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = F_i, \tag{2.6}$$

Equations (2.6) are called **forced Euler-Lagrange** equations.

2.3.3 Geometric formalism for forced Lagrangian systems

Consider a mechanical system (L, Q) with local coordinates (q^i). Recall the almost tangent structure $J : TTQ \to TTQ$ which is a type $(1,1)$- tensor field on TQ whose expression in local coordinates is $J = dq^i \otimes \frac{\partial}{\partial \dot{q}^i}$ defined at the Subsection 2.1.5. We define the **Liouville vector field** on TQ as $\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$, and the **Poincaré-Cartan 2-form** (or **canonical symplectic Lagrangian form**) $\omega_L = -d\theta_L$, where $\theta_L = J^*(dL)$ and J^* is the adjoint operator of J. The **Lagrangian energy** of the mechanical system is defined to be the quantity $E_L = \Delta(L) - L$. In local coordinates we have

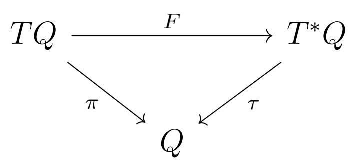
$$\theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i, \quad \omega_L = dq^i \wedge d\left(\frac{\partial L}{\partial \dot{q}^i}\right), \quad E_L = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L(q^i, \dot{q}^i).$$

The Euler-Lagrange equations are intrinsically described as the equations for the flow of the vector field X_{E_L}:

$$i_{X_{E_L}} \omega_L = dE_L. \tag{2.7}$$

The Lagrangian L will be called **regular** or **non-degenerate** if the Hessian matrix with entries $W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \dot{q}^j}$ is non-singular. In this case, the Poincaré-Cartan 2-form, ω_L, is a symplectic form on TQ. In what follows we assume that the Lagrangian is regular.

Consider an external force, as given before, $F : TQ \to T^*Q$ and the natural projection maps $\pi : TQ \to Q$ and $\tau : T^*Q \to Q$ so we have the next commutative diagram.



Using this external force we construct a **semibasic 1-form** $\mu_F \in \Omega^1(TQ)$ by

$$\langle \mu_F(v_q), X_{v_q} \rangle = \langle F(v_q), T\pi(X_{v_q}) \rangle, \quad \text{for all} \quad X_{v_q} \in T_{v_q}TQ$$

and in coordinates we have

$$\mu_F = F_i(q, \dot{q}) dq^i$$

(for more about the semibasic 1-forms one can see de León and Rodrigues, 1989, Section 4.2.2). Consider the vertical vector field Z_F^V given by the equation $i_{Z_F^V} \omega_L = -\mu_F$ or equivalently

$$Z_F^V(v_q) = W_{ij} \frac{d}{dt}\bigg|_{t=0} (v_q + tF(v_q))$$

which in coordinates reads

$$Z_F^V = W_{ij} F_j \frac{\partial}{\partial \dot{q}^j},$$

where W_{ij} are the components of the inverse of the Hessian matrix of the Lagrangian. Hence, the solutions of the forced Euler-Lagrange equations (2.6) are the integral curves of the vector field $X_{E_L} + Z_F^V$ determined by

$$i_{(X_{E_L} + Z_F^V)} \omega_L = dE_L - \mu_L.$$

2.3.4 Hamiltonian mechanics

A useful way with rich background to think about mechanical systems is through the Hamiltonian approach. Extensive work has been done in that field e.g. Abraham and Marsden, 1978, A. Bloch, 2015, Bullo and Lewis, 2005, Marsden and Ratiu, 1999, Roubtsov and Dutykh, 2021. On special occasions, Lagrangian and Hamiltonian approaches appear to be equivalent. For the latter way, we work on the cotangent space of the configuration Q of the system.

Given a **Hamiltonian function** $H : T^*Q \to \mathbb{R}$ where we equip T^*Q with local coordinates (q^i, p_i) **Hamilton's equations** are

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.8}$$

In this setting, the Hamiltonian function is the total energy of the system. The equivalency of the Lagrangian and the Hamiltonian formalisms is made through the hyperregularity of the Lagrangian function.

Consider a mechanical system (Q, L) with local coordinates (q^i), if the Lagrangian L is hyperregular then, by the implicit function theorem, we induce the change of variables $\mathbb{F}L : TQ \to T^*Q$, $(q^i, \dot{q}^i) \mapsto (q^i, p_i)$ where p_i is given by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

and it is called **momentum.** This change of variables from velocities of the system to momentum is called the **Legendre transformation.** The **energy** of this mechanical system determined by L is defined to be the quantity

$$E(q^i, \dot{q}^i) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L(q^i, \dot{q}^i)$$

which is a real-valued function. If we apply the Legendre transformation to the energy function we get the Hamiltonian function $H : T^*Q \to \mathbb{R}$ given by

$$E\left(\mathbb{F}L^{-1}(q^i, p_i)\right) = H(q^i, p_i) = \dot{q}^i(q^i, p_i)p_i - L(q^i, \dot{q}^i(q^i, p_i)).$$

Using this Hamiltonian function we have the same Hamilton's equations as in (2.8) which, by straightforward calculations, we see that they are equivalent to the Euler-Lagrange equations. Note here that is $\dim Q = n$ the Euler-Lagrange equations are n 2^{nd} order differential equations while the Hamiltonian equations are $2n$ 1^{st} order differential equations. Also, note that when the Lagrangian of a system is of mechanical type, i.e. kinetic minus potential, the Hamiltonian is equal to the total energy of the system i.e. kinetic plus potential.

For a coordinate free expression of the Hamiltonian counterpart we need some results from symplectic geometry. Consider a symplectic manifold (P, ω) and let $f \in C^\infty(P)$, $z \in P$. The **Hamiltonian vector field** of f is the unique vector field, X_f, on P that satisfies

$$\omega_z(X_f, v) = df(z) \cdot v,$$

for all $v \in T_zP$, using the contraction of X_f and ω (defined in Subsection 2.1.5), we get

$$i_{X_f}\omega = df \tag{2.9}$$

and **Hamilton's equations** for f is the set of differential equations on P defined by

$$\dot{z} = X_f(z).$$

The coordinate-free version of the Legendre transformation is a map $\mathbb{F}L : TQ \to T^*Q$ defined by

$$\langle \mathbb{F}L(u_q), v_q \rangle = \frac{d}{dt}\bigg|_{t=0} L(u_q + tv_q).$$

If the Lagrangian L is hyperregular, the Legendre transformation is a diffeomorphism. The energy of the system $E : TQ \to \mathbb{R}$ is given by

$$E(v_p) = \langle \mathbb{F}L(v_p), v_q \rangle - L(v_q),$$

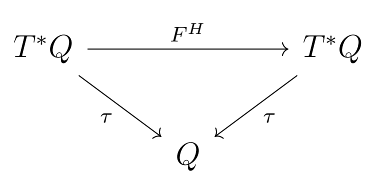
while the Hamiltonian function is $H : T^*Q \to \mathbb{R}$ given by

$$H = E \circ \mathbb{F}L^{-1}.$$

Connecting the symplectic geometry formulation given previously with the notions of mechanical systems above we have that the Hamiltonian vector field for the Hamiltonian $H = E \circ \mathbb{F}L^{-1}$ is given by the equation (2.9).

2.3.5 Forced Hamiltonian systems

Consider the external force given by $F^H : T^*Q \to T^*Q$ which is locally described by $F(q^i, p_i) = F_i(q^i, p_i)dp_i$ such that the following diagram is commutative



where $\tau : T^*Q \to Q$. Then, the Hamilton equations read

$$\dot{q}^i = \frac{\partial H}{\partial p_i}(q^i, p_i), \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}(q^i, p_i) + F_i^H(q^i, p_i). \tag{2.10}$$

For an intrinsic description of a Hamiltonian system consider the canonical exact symplectic structure $\omega_Q = -d\theta_Q$ where θ_Q is the canonical 1-form (or Liouville 1-form) on T^*Q defined by

$$(\theta_Q)_{\alpha_q}(X_{\alpha_q}) = \langle \alpha_q, T_{\alpha_q}\tau(X_{\alpha_q})\rangle,$$

for $X_{\alpha_q} \in T_{\alpha_q}T^*Q, \alpha_q \in T_q^*Q$, $\tau : T^*Q \to Q$. In local coordinates (q^i, p_i) on T^*Q we have

$$\theta_Q = p_i dq^i, \quad \omega_Q = dq^i \wedge dp_i.$$

The Hamiltonian vector field X_H is given by solutions of

$$i_{X_H}\omega_Q = dH,$$

which locally can be written as

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

An external force is associated to a semibasic 1-form $\beta_{F^H} \in \Omega^1(T^*Q)$ given by

$$\langle \beta_{F^H}(\alpha_q), W\rangle = \langle F^H(\alpha_q), T_{\alpha_q}\tau W\rangle, \text{ for all } W \in T_{\alpha_q}T^*Q,$$

which in coordinates is written as $\beta_{F^H} = F_i dq^i$. Consider the vertical vector field Y_F^V which is given by the equation $i_{Y_F^V}\omega_Q = -\beta_{F^H}$ or equivalently

$$Y_H^V(\alpha_q) = \frac{d}{dt}\bigg|_{t=0}(\alpha_q + tF^H(\alpha_q))$$

and in coordinates it is given by

$$Y_F^V = F_i^H \frac{\partial}{\partial p_i}.$$

The solutions of the forced Hamiltonian equations are given by the vector field, $X_H + Y_F^V$, determined by the equation

$$i_{(H+Y_F^V)}\omega_Q = dH - \beta_{F^H}.$$

2.3.6 Mechanics on Riemannian manifolds

In the cases where the configuration space is not an Euclidean space but rather a Riemannian manifold, the situation described in the previous subsection generalizes taking into account the Riemannian metric provided. These cases appear in many control problems (see for instance A. Bloch, 2015, Bullo and Lewis, 2005). One advantage of working in this framework is the coordinate-free character of the theory. Since the configuration space is a Riemannian manifold $(Q, \mathcal{G})$ the Lagrangian is given by $L(v_q) = \frac{1}{2}\mathcal{G}(v_q, v_q)$ and by convention we will call it kinetic energy.

Even though the concept of geodesics is purely geometric, it appears very naturally in the theory of mechanical systems. In this thesis, for different mechanical systems (e.g. evolving in Riemannian manifolds, Lie groups, or homogeneous spaces) we express their trajectories in the form of geodesics. The next proposition gives the solutions of the Euler-Lagrange equations for L.

Proposition 2.3.3 (Proposition 4.43 in Bullo and Lewis, 2005)**.** *Let Q be a Riemannian manifold with Riemannian metric $\mathcal{G}$ and let L be the associated Lagrangian. The solutions of the Euler-Lagrange equations are exactly the geodesics of the Levi-Civita connection associated with the Riemannian metric $\mathcal{G}$, $\nabla^{\mathcal{G}}$, namely*

$$\nabla^{\mathcal{G}}_{\dot{\gamma}}\dot{\gamma} = 0.$$

In the case where there are external forces acting to the system, we have the generalization of the Lagrange-d'Alembert principle to Riemannian manifolds.

Proposition 2.3.4 (Proposition 4.59 in Bullo and Lewis, 2005)**.** *Under the same hypothesis of Proposition 2.3.3 and with an external force $F : TQ \to T^*Q$, a smooth curve γ satisfies the Lagrange-d'Alembert principle if*

$$\nabla^{\mathcal{G}}_{\dot{\gamma}}\dot{\gamma} = \sharp_{\mathcal{G}}(F(\dot{\gamma}(t))),$$

where $\sharp_{\mathcal{G}} : \Omega^1(Q) \to \mathfrak{X}(Q)$ is the sharp map associated to the Riemannian metric $\mathcal{G}$.

Whenever we consider a potential energy for a system, we can add potential forces which are resulting from this potential energy. If the potential function is $V : Q \to \mathbb{R}$ the respective potential force is given by $F(q, \dot{q}) = -dV(q)$ and the **gradient vector field** of V is given by

$$\text{grad}_{\mathcal{G}} V = \sharp_{\mathcal{G}}(dV).$$

Here the Lagrange-d'Alembert principle holds and the forced Euler-Lagrange equations are

$$\nabla^{\mathcal{G}}_{\dot{\gamma}}\dot{\gamma} = -\text{grad}_{\mathcal{G}} V(\gamma(t)). \tag{2.11}$$

For a mechanical system which is subject to potential forces and external forces, we have the next corollary.

Corollary 2.3.5. Consider a Riemannian manifold $(Q, \mathcal{G})$ and the associated Lagrangian L with potential function $V : Q \to \mathbb{R}$ and an external force $F : TQ \to T^*Q$. The trajectories that satisfy the Lagrange-d'Alembert principle are given by solutions of

$$\nabla^{\mathcal{G}}_{\dot{\gamma}}\dot{\gamma} = -\text{grad}_{\mathcal{G}} V(\gamma(t)) + \sharp_{\mathcal{G}}(F(\dot{\gamma}(t))), \tag{2.12}$$

where $\sharp_{\mathcal{G}} : \Omega^1(Q) \to \mathfrak{X}(Q)$ is the sharp map associated to the Riemannian metric $\mathcal{G}$.

2.4 Nonholonomic mechanics

Suppose we have a Lagrangian system on a configuration space Q which is subject to constraints on the velocities. For simplicity, assume the system constraints are linear on the velocities (also called Pfaffian constraints) and they are presented by m equations of the form

$$\mathcal{S}(q)\dot{q} = 0 \quad \Leftrightarrow \quad \mu^a_k(q)\dot{q}^k = 0, \tag{2.13}$$

where $a = 1, \dots, m$, $\mathcal{S}$ is an $m \times n$ matrix with entries μ_k^a and $\dot{q}$ is considered to be a column vector.

If there is a function on the position only, $h(q) = 0$, such that its time derivative,

$$\frac{\partial h}{\partial q^k}\dot{q}^k = 0, \tag{2.14}$$

gives the same constraint distribution as the constraints (2.13) then we call these constraints **holonomic**, if not we call them **nonholonomic**.

There is also another distinction for the constraints of a system. If the constraints are dependent on time then they are called **rheonomic** whereas if they are independent of time they are called **scleronomic**. In this thesis, all the constraints will be scleronomic.

The set of all **virtual displacements** for a holonomic or nonholonomic constraint is defined to be the distribution $\mathcal{D}$ of vectors δq, that satisfy the constraints, for every $q \in Q$ and it is given by

$$\mathcal{D}_q = \{\delta q \in T_q Q : \mathcal{S}(q) \cdot \delta q = 0\}. \tag{2.15}$$

In the nonholonomic case and under the assumption of the **nonholonomic principle** which says that the constraint force, F, lies in the annihilator of the space of the virtual displacements (see A. Bloch, 2015 for instance) we have that F is a linear combination of the rows of $\mathcal{S}(q)$ i.e. $F = \lambda \mathcal{S}(q)$, where F is regarded as a cotangent row vector at q, λ is a row vector with real entries called "Lagrange multiplier". The **nonholonomic equations** or the **Lagrange-d'Alembert equations** for the mechanical system subject to constraints linear on the velocities are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \lambda \mathcal{S}(q), \quad \mathcal{S}(q)\dot{q} = 0. \tag{2.16}$$

Note here that we have $2n + m$ unknowns and n second order differential equations (which can be seen as $2n$ first order equations) and m constraint equations or $2n + m$ first order differential equations.

2.4.1 Nonholonomic mechanics on Riemannian manifolds

We begin this subsection by defining the constraint distribution which represents the most frequent type of constraints namely the constraints that are linear on the velocities.

Definition 2.4.1 (Definition 4.69, Bullo and Lewis, 2005)**.** Consider a smooth manifold Q, a smooth **constraint distribution** is a distribution $\mathcal{D}$ on Q of which the annihilator $\mathcal{D}^\circ$ is a codistribution. A curve $q : I \to Q$ satisfies the constraint $\mathcal{D}$ if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all $t \in I$.

Definition 2.4.2. A **nonholonomic mechanical system** on a smooth manifold Q is given by the triple $(\mathcal{G}, V, \mathcal{D})$, where $\mathcal{G}$ is a Riemannian metric on Q, representing the kinetic energy of the system, $V : Q \to \mathbb{R}$ is a smooth function representing the potential energy, and $\mathcal{D}$ a regular distribution on Q describing the nonholonomic constraints.

Using the Riemannian metric $\mathcal{G}$ we can define two complementary orthogonal projectors $\mathcal{P}\colon TQ \to \mathcal{D}$ and $\mathcal{Q}\colon TQ \to \mathcal{D}^{\perp}$, with respect to the tangent bundle orthogonal decomposition $\mathcal{D} \oplus \mathcal{D}^{\perp} = TQ$.

In the presence of a constraint distribution $\mathcal{D}$, the Euler-Lagrange equations (2.11) must be slightly modified as follows. Consider the **nonholonomic connection** $\nabla^{nh} : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q)$ defined by (see Definition 4.85, Bullo and Lewis, 2005 for more details)

$$\nabla^{nh}_X Y = \nabla^{\mathcal{G}}_X Y + (\nabla^{\mathcal{G}}_X \mathcal{Q})(Y), \tag{2.17}$$

with $X, Y \in \mathfrak{X}(Q)$. Then, the trajectories for the nonholonomic mechanical system associated with the Lagrangian given by $L(v_q) = \frac{1}{2}\mathcal{G}(v_q, v_q) - V(q)$, $v_q \in T_qQ$ and the distribution $\mathcal{D}$ must satisfy the following equation

$$\nabla^{nh}_{\dot{q}} \dot{q} + \mathcal{P}(\text{grad}_{\mathcal{G}} V(q(t))) = 0. \tag{2.18}$$

The next result shows some properties of the nonholonomic connection that will be used along the thesis.

Proposition 2.4.3 (Proposition 4.85, Bullo and Lewis, 2005)**.** *Consider the Riemannian manifold $(Q, \mathcal{G})$ and the regular constraint distribution $\mathcal{D}$ on Q. For the nonholonomic connection ∇^{nh} we have the following:*

1. $\nabla^{nh}_X Y = \mathcal{P}\left(\nabla^{nh}_X Y\right)$ *and thus* $\nabla^{nh}_X Y \in \Gamma(\mathcal{D})$,
2. $\left(\nabla^{nh}_X \mathcal{Q}\right)(Y) \in \Gamma(\mathcal{D}^{\perp})$,

where $X \in \Gamma(TQ)$ and $Y \in \Gamma(\mathcal{D})$.

2.5 Virtual holonomic constraints

In this section, we give a background on virtual holonomic constraints (from now on referred to as VHC) necessary for the present thesis. Virtual constraints are related to a control system. We begin by recalling the basic definitions of nonlinear control theory from a geometric perspective. For more details, we recommend A. Bloch, 2015 and Bullo and Lewis, 2005.

2.5.1 Basic definitions on nonlinear geometric control

Generally speaking, a control system can be seen as a dynamical system whose dynamical laws are not prescribed and fixed in advance like problems we come across in classical physics, but rather they depend on parameters that are called controls and may vary. The advantage of working with controls is the fact that one can interfere to the behavior of the system. One of the main objections in control theory is to understand the effects of the controls on the dynamics of the system.

Many can be said about control systems since there are many concepts in which one is concerned in analyzing control systems e.g. accessibility, controllability, stabilizability, feedback linearizability, path planning, optimal control, nonlinear stability, output regulation, robustness, adaptive control control, etc. Here we restrict ourselves to the basic definition of a nonlinear control system for feedback control and stabilizability.

Consider the state space of the system to be an n-dimensional smooth manifold Q. A **nonlinear control system** (also called **affine nonlinear control system**) on Q is a differential equation of the following type

$$\dot{q} = F(q) + u_a F^a(q), \tag{2.19}$$
$$y = h(q), \tag{2.20}$$

where $q \in Q$, $a = 1, \ldots, m$. The function $u = u(q, \dot{q}, t) : TQ \times \mathbb{R} \to U$ with U an open subset of $\mathbb{R}^m$ containing the origin is called **control input** and the functions F and F^a are smooth $\mathbb{R}^n$ valued functions i.e. smooth vector fields on Q. The vector field F is usually called **drift vector filed** while F^a are called **control vector fields.** Equation (2.19) is called **input function** while equation (2.20) is commonly refereed as **output function** with h to be an $\mathbb{R}^k$ valued function and $y \in \mathbb{R}^k$. For a given control law, an integral curve of equation (2.19) is called **closed-loop trajectory.**

Consider a mechanical system whose configuration space is a Riemannian manifold $(Q, \mathcal{G})$, an external force $F^0 : TQ \to T^*Q$, and a **control force** $F : TQ \times U \to T^*Q$ of the form

$$F(q, \dot{q}, u) = \sum_{a=1}^{m} u_a F^a(q, \dot{q}) \tag{2.21}$$

where $F^a(q, \dot{q}) \in T^*Q$ with $m < n$, $U \subset \mathbb{R}^m$ the set of controls and $u_a \in \mathbb{R}$ with $1 \leq a \leq m$ the control inputs, consider the associated mechanical control system of the form

$$\nabla_{\dot{q}} \dot{q} = Y^0(q, \dot{q}) + u_a Y^a(q, \dot{q}), \tag{2.22}$$

where $Y^0(q, \dot{q}) = \sharp_{\mathcal{G}}(F^0(q, \dot{q}))$ and $Y^a = \sharp_{\mathcal{G}}(F^a(q, \dot{q}))$. Equation (2.22) forms a system of second-order differential equations whose solutions are the trajectories of a SODE vector field (such as in (2.2)) of the form

$$\Gamma(q, \dot{q}, u) = G(q, \dot{q}) + u_a (Y^a)^V_{(q,\dot{q})}. \tag{2.23}$$

We call each $Y^a = \sharp_{\mathcal{G}}(F^a)$ a control force vector field, and G is the vector field determined by the unactuated forced mechanical system, that is,

$$\nabla_{\dot{q}} \dot{q} = Y^0(q, \dot{q}).$$

Definition 2.5.1. The distribution $\mathcal{F} \subseteq TQ$ generated by the vector fields $Y^a = \sharp_{\mathcal{G}}(F^a)$ is called the *input distribution* associated with the mechanical control system (2.22).

2.5.2 Virtual holonomic constraints

Virtual constraints are relations among the links of the mechanism that are dynamically imposed through feedback control. Their function is to coordinate the evolution of the various links throughout a step—which is another way of saying that they reduce the degrees of freedom—with the goal of achieving a closed-loop mechanism that naturally gives rise to a desired motion. In Canudas-de-Wit, 2004 virtual holonomic constraints are used for constructing orbitally stable feedback laws for designing periodic walking gaits in bipedal robots.

In Maggiore and Consolini, 2013 it is studied conditions under which holonomic constraints are made invariant and stabilizable by feedback, and also sufficient conditions for the constraint dynamics to correspond to an Euler-Lagrange system. More precisely, they examine mechanical systems evolving on an n-dimensional configuration space Q, with $n-1$ controls $u \in \mathbb{R}^{n-1}$ (degree of underactuation 1). It is considered the controlled mechanical system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Fu, \tag{2.24}$$

where $F : Q \to \mathbb{R}^{n \times n-1}$ is a smooth function and it has rank $n-1$, L is a smooth Lagrangian of mechanical type $L(q,\dot{q}) = K(q,\dot{q}) - V(q)$ with $K(q,\dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q}$ and $D(q)$ the inertia matrix which is positive definite for all $q \in Q$ and $u \in \mathbb{R}^{n-1}$. The dynamics associated to this system can be rewritten in the usual form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \text{grad}V(q) = F(q)u. \tag{2.25}$$

They make a distinction between virtual holonomic constraints (VHC) and regular VHC. The definition of the former is as follows.

Definition 2.5.2 (Definition 2.1, Maggiore and Consolini, 2013)**.** A **virtual holonomic constraint of order k** for the system (2.25) is a relation $h(q) = 0$, where $h : Q \to \mathbb{R}^k$ is smooth, $\text{rank}(dh_q) = k$ for all $q \in h^{-1}(0)$ and the set

$$\mathcal{M} = \{(q,\dot{q}) : h(q) = 0, dh_q\dot{q} = 0\}$$

is controlled invariant. That is to say, there exists a smooth feedback $u(q,\dot{q})$ such that $\mathcal{M}$ is positively invariant for the closed-loop system.

A special demand is made for a VHC to be regular and this is done to meet controller design techniques as we comment below.

Definition 2.5.3 (Definition 3.1, Maggiore and Consolini, 2013)**.** A relation $h(q) = 0$ is a **regular VHC** if the output function $e = h(q)$ yields vector relative degree $\{2,\dots,2\}$ everywhere on the set $\mathcal{M}$. $\mathcal{M}$ is called **constraint manifold** associated with the VHC $h(q) = 0$.

Remark 2.5.4. The requirement of the output function $e = h(q)$ to yield a vector relative degree $\{2,\dots,2\}$ on $\mathcal{M}$ is used to make the output a input-output feedback linearizable and the associated zero dynamics manifold is precisely $\mathcal{M}$ as discussed in Section 6.1 of Isidori, 2000.

Proposition 2.5.5. *Suppose $h : Q \to \mathbb{R}^k$ is smooth and rank $dh_q = k$, for all $q \in h^{-1}(0)$. Then $h(q) = 0$ is regular VHC of order k for each element in the level set $h^{-1}(0)$, if $n-1-k$*

of the acceleration directions forced by the control input are transversal to the tangent space of $h^{-1}(0)$.

An immediate extension to Maggiore and Consolini, 2013 is the paper of Consolini and Costalunga, 2015 where, in brief, it is studied the case of systems with underactuation degree greater than one and the notion of virtual constraints for a mechanical system it is approached geometrically to provide a coordinate-free framework. In particular, they present virtual holonomic constraints in an affine geometric background where the constraint dynamics are described as geodesics of a connection obtained by the Levi-Civita connection of the mechanical system. Considering the configuration space, Q, equipped with a metric, they use the same controlled mechanical system (2.24) and express it in terms of the Levi-Civita connection, namely,

$$\nabla_{\dot{q}}\dot{q} = -\mathrm{grad}V + Fu, \tag{2.26}$$

$(q, \dot{q}) \in TQ$ (see Bullo and Lewis, 2005, page 224, equation (4.24) for more details).

To start with, they define the concept of controlled invariant manifold and transversality below. Two subbundles on a manifold Q are transversal if their Whitney sum produces the whole tangent space of Q.

Definition 2.5.6 (Definition 1)**.** A manifold $\mathcal{M}$ is said to be controlled invariant for the system (2.24) if there exist a control function $\hat{u} : T\mathcal{M} \to \mathbb{R}^m$ such that the solution of (2.24) with this control is well defined and stays in $\mathcal{M}$ for all time, i.e. $q(t) \in \mathcal{M}, \forall t \geq 0$, for any initial value $(q(0), \dot{q}(0)) \in T\mathcal{M}$.

Definition 2.5.7. Let $\mathcal{F}$ be a subbundle of the tangent bundle TQ and let $\mathcal{M}$ be an embedded submanifold of Q. We call $\mathcal{F}$ and $\mathcal{M}$ **transversal** if

$$\mathcal{F}_q + T_q\mathcal{M} = T_qQ, \text{ for all } q \in \mathcal{M}.$$

Proposition 2.5.8 (Proposition 1, Consolini and Costalunga, 2015)**.** *Let* $\mathcal{M}$ *be an embedded submanifold of* Q *of dimension* $n - m$*. If the input subbundle* $\mathcal{F}$ *and* $\mathcal{M}$ *are transversal then* $\mathcal{M}$ *is control invariant for system (2.24). Moreover, the control law* $\hat{u}$ *that renders* $\mathcal{M}$ *invariant is unique.*

If the premise of Proposition 2.5.8 above is satisfied then the solutions of equation (2.24) with the unique input $u = \hat{u}(q(t), \dot{q}(t))$ and initial conditions on the tangent bundle of $\mathcal{M}$, $T\mathcal{M}$, belong to $T\mathcal{M}$ for all time and they are called **constraint dynamics** and satisfy the equation below

$$\nabla_{\dot{q}}\dot{q} = -\mathrm{grad}V + F\hat{u}, \tag{2.27}$$

Note here that, by this time, Consolini and Costalunga, 2015 develop a general theory about controlled invariant submanifolds and there is no reference to virtual constraints. Subsequently, the authors define an *induced connection* whose geodesics describe the trajectories of the constraint dynamics. Whenever a system is subject to ideal constraints we know that the trajectories of the constrained Lagrangian system can be expressed as geodesic curves of an affine connection (see Remark 4.91, Bullo and Lewis, 2005).

Considering the input subbundle $\mathcal{F}$ and the embedded submanifold $\mathcal{M}$, which represents the constraint submanifold formed by holonomic constraints, to be transversal, then there are unique projections $P_{\mathcal{F}} : TQ \to \mathcal{F}$ and $P_{\mathcal{M}} : TQ \to T\mathcal{M}$ such that for every $q \in Q$ and every vector $v_q \in T_qQ$ we have $v_q = P_{\mathcal{M}}(v_q) + P_{\mathcal{F}}(v_q)$. In consequence of this unique decomposition, one can define the **induced connection** with which the constraint dynamics will be described as geodesics.

Proposition 2.5.9 (Proposition 2, Consolini and Costalunga, 2015)**.** *Let Q be the configuration space of the mechanical controlled system equipped with a Riemannian metric, also let ∇ be the Levi-Civita connection related to that metric and $\mathcal{M}$ be a virtual holonomic constraint for that system. For any two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$ define*

$$\overset{h}{\nabla}_X Y = P_{\mathcal{M}}(\nabla_X Y),$$

then

- *$\overset{h}{\nabla}$ is a connection on $\mathcal{M}$ and*
- *if ∇ is torsionless then $\overset{h}{\nabla}$ is torsionless.*

With the help of this induced connection the constraint dynamics (2.27) can be written in a compact form as the next proposition states.

Proposition 2.5.10 (Proposition 3 & Remark 5, Consolini and Costalunga, 2015)**.** *The trajectories of the constraint dynamics (2.27) satisfy*

$$\overset{h}{\nabla}_{\dot{q}}\dot{q} = -P_{\mathcal{M}}(grad V).$$

If the potential $V = 0$ the above constraint dynamics reduce to

$$\overset{h}{\nabla}_{\dot{q}}\dot{q} = 0,$$

which means that the constraint dynamics correspond to the geodesic curves of the induced connection $\overset{h}{\nabla}$.

2.6 Lie Groups

In this subsection, we present an important class of smooth manifolds, Lie groups. In short, Lie groups are smooth manifolds that have also the structure of a group. Lie groups appear naturally in mechanical systems when the latter present symmetries and they allow certain simplifications of these systems. In this dissertation we encounter mainly Lie groups as configuration spaces of mechanical systems. Some references from literature that approaches this concept as we do here are Bullo and Lewis, 2005, Holm et al., 2009, Marsden and Ratiu, 1999, A. Bloch, 2015, Holm, 2008, Abraham and Marsden, 1978. In this thesis, we only consider finite-dimensional Lie groups.

A **Lie group** G is a smooth manifold that is also a group such that the group multiplication and the inverse map given by

$$m(g,h) = gh, \qquad i(g) = g^{-1}$$

are smooth. For every element $g \in G$ of the Lie group we define the **left and right translation maps**

$$\begin{aligned} L_g : G &\to G \quad \text{and} \quad & R_g : G &\to G \\ h &\mapsto gh & h &\mapsto hg \end{aligned}$$

respectively. Note that since $L_g \circ L_h = L_{gh}$ and $R_g \circ R_h = R_{hg}$ we have that $(L_g)_{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$. Also, notice that left and right translations commute since $L_g \circ R_h = R_h \circ L_g$. From the chain rule we get

$$T_{gh}L_{g^{-1}} \circ T_h L_g = T_h(L_{g^{-1}} \circ L_g) = Id,$$

thus $T_h L_g$ is invertible (likewise $T_h R_g$) and so it is an isomorphism. If the group is abelian (which means that $gh = hg$, $\forall g, h \in G$) then $L_g = R_g$. A **Lie group homomorphism**, between two Lie groups, is a smooth map that it respects the group structure, formally is given as follows. Given two Lie groups $(G, \cdot)$ and $(H, \star)$ a map $\rho : G \to H$ is a Lie group homomorphism if is satisfies $\rho(a \cdot b) = \rho(a) \star \rho(b)$ for all $a, b \in G$.

A vector field X on a Lie group G will be called **left-invariant** if for every element $g \in G$ we have $L_g^* X = X$ i.e.

$$(T_h L_g)X(h) = X(gh)$$

for every $h \in G$. The set of all left-invariant vector fields on G will be denoted by $\mathfrak{X}_L(G)$. Likewise the set of **right-invariant** vector fields is denoted by $\mathfrak{X}_R(G)$ and for all $X \in \mathfrak{X}_R(G)$ we have that $R_g^* X = X$ i.e.

$$(T_h R_g)X(h) = X(hg)$$

for every $h \in G$.

Next, we consider the notion of the Lie algebra in a general framework and later we define the Lie algebra of a Lie group.

A **Lie algebra** V is a $\mathbb{R}-$vector space together with a bilinear operator $[\cdot,\cdot] : V \times V \to \mathbb{R}$ called the **bracket** and satisfies the properties:

1. $[\xi, \eta] = -[\eta, \xi]$ (**skew-symmetry**) and
2. $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0$ (**Jacobi identity**)

for all $\xi, \eta, \zeta \in V$. A **Lie subalgebra** of a Lie algebra V is a subspace of V which is closed under the bracket operator.

Note that the set of all vector fields on G, $\mathfrak{X}(G)$, equipped with the Lie bracket defined at the equation (2.1) is a Lie algebra and the subset of all left-invariant vector fields, $\mathfrak{X}_L(G)$, is a subalgebra with the same Lie bracket. Also, $\mathbb{R}^3$ equipped with the vector cross-product operator is a Lie algebra and the set of linear maps, $L(V,V)$, on a vector space V is a Lie algebra with operation the commutator $[A,B] = A \circ B - B \circ A$.

For any Lie group G there exists an isomorphism between the tangent space at the identity, T_eG, and the left-invariant vector fields $\mathfrak{X}_L(G)$, namely, the map $(\cdot_L) : T_eG \to \mathfrak{X}_L(G)$ which is given by $\xi \mapsto \xi_L$ such that $\xi_L(g) = (L_g)_*\xi$. With the help of this map we define a bracket on T_eG by

$$[\xi, \eta]_{T_eG} = [\xi_L, \eta_L]_{\mathfrak{X}_L(G)}.$$

The tangent space at the identity to a Lie group G will be denoted by $\mathfrak{g}$ and equipped with the above bracket becomes a Lie algebra which is called the **Lie algebra of the Lie group** G**.** When there is no chance of confusion the subscript of the brackets will be omitted. In an equivalent way we can identify the right-invariant vector fields of a Lie group with the T_eG and define the respective bracket.

2.6.1 Lie groups actions

Here we present the theory of Lie group actions on manifolds. The action of a group on a set can be seen as an assignment to every element of the group an invertible transformation of this set hence the group operator corresponds to composition of transformations. For example a group of matrices can be viewed as the group of linear transformations of a Euclidean space.

Consider a manifold M and a Lie group G, a **(left) action** of G on M (or a G **action** on M) is a smooth mapping $\Psi : G \times M \to M$ such that

(i) $\Psi(e, x) = x$ for all $x \in M$ and e is the identity element of the group G,

(ii) $\Psi(g, \Psi(h, x)) = \Psi(gh, x)$ for all $g, h \in G$ and $x \in M$ and

(iii) for every $g \in G$ the map $\Psi_g : M \to M$, defined by $\Psi_g(x) = \Psi(g, x)$, is a diffeomorphism.

Note that condition (ii) may also be written as $\Psi_g \circ \Psi_h = \Psi_{gh}$ and that the notation $gx := \Psi_g(x) = \Psi(g, x)$ is often used.

A **right action** of G on M is a smooth mapping satisfying the same conditions as above except condition (ii), which is replaced by:

$$(ii)' \ \ \Psi(g, \Psi(h, x)) = \Psi(hg, x) \text{ for all } g, h \in G \text{ and } x \in M.$$

A convenient notation for a right action is $xg := \Psi_g(x) = \Psi(g, x)$ and so the condition $(ii)'$ then reads $(xh)g = x(hg)$. A left or right action by a group G is simply called G action and note that if the group G is abelian then the two actions (left and right) coincide (see left and right translation maps at the beginning of Section 2.6).

Remark 2.6.1. Any left action $(g, x) \mapsto gx$ gives rise to a right action given by $(g, x) \mapsto g^{-1}x$.

For the subsequent will use the left action notation, but everything can be defined for a right action as well.

Let G be a Lie group that acts on a manifold M, for any element $x \in M$ we define the **orbit** of x by

$$\text{Orb}(x) = \{gx : g \in G\}.$$

The orbit of an element is a subset of M and it is also called **group orbit** through x. For any element of M there is one more subset of particular interest, the stabilizer. Let $x \in M$ the **stabilizer** (**isotropy** or **symmetry**) **group** of x is given by

$$\mathrm{Stab}(x) = \{g \in G : gx = x\}.$$

An action $\Psi : G \times M \to M$ is said to be:

(i) **transitive** if for any $x, y \in M$ there exist $g \in G$ such that $gx = y$,

(ii) **free** if it has no isotropic points, i.e. for all x if $gx = x$ then $g = e$,

(iii) **faithful** (or **effective**) if for all $g \in G$ such that $g \neq e$, there exists $x \in M$ such that $gx \neq x$,

(iv) **proper** if, whenever the sequences $\{x_n\}$ and $\{g_n x_n\}$ converge in M, the sequence $\{g_n\}$ has a convergent subsequence in G.

Remark 2.6.2. Note that a transitive action has only one group orbit.

Remark 2.6.3. A faithful action can also be defined by: if Ψ_g is the identity transformation on M, then $g = e$.

A Lie group action of G on a vector space V is called linear if the map Ψ_g is a linear map for all $g \in G$. A linear action of a group G on the Euclidean space $\mathbb{R}^n$ is called a **representation** of G. If this is the case for an action Ψ then Ψ_g corresponds to an element of $GL(n)$, the set of $n \times n$ invertible matrices with real entries.

Example 2.6.4. 1. Consider the left action of the special orthogonal group SO(3) which is the set $\{R \in GL(n, \mathbb{R}) : RR^T = R^T R = I \text{ with } \det A = 1\}$ on $\mathbb{R}^3$ given by left multiplication $(R, x) \mapsto Rx$, $x \in \mathbb{R}^3$. This action is faithful but neither transitive nor free. It is not transitive because there is not matrix $R \in \mathrm{SO}(3)$ which that $\mathbf{0}^T R = e_1^T$ where $\mathbf{0}$ is the zero vector and $e_1 = (1, 0, 0)$. For no being free one can see that there are more that one matrices of the form $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ such that $Re_1 = e_1$, where $*$ can be any number such that $R \in \mathrm{SO}(3)$.

2. Consider the special Euclidean group SE(2) which is the set of 3×3 matrices of the form

$$S = \begin{pmatrix} R & r \\ 0 & 1 \end{pmatrix},$$

where $R \in \mathrm{SO}(2)$ and $r \in \mathbb{R}^2$. The action of SE(2) on the real plane $\mathbb{R}^2$ via the assignment $((R, r), x) \mapsto Rx + r$, is transitive and faithful but not free. This action is not free because the isotropy group of any element $x \in \mathbb{R}^2$ has at least two elements $(I, 0), (\mathbf{0}, x) \in \mathrm{Stab}(x)$, where $\mathbf{0}$ is the zero 2×2 matrix and I is the identity 2×2 matrix.

2.6.2 Group action on itself

An interesting class of Lie group actions appear when they act on themselves. We have already defined the left and right translation maps at the beginning of this section, which for convenience they are repeated here:

$$\begin{aligned} L_g &: G \to G \\ h &\mapsto L_g(h) = gh \end{aligned} \qquad \text{and} \qquad \begin{aligned} R_g &: G \to G \\ h &\mapsto R_g(h) = hg. \end{aligned}$$

These maps can be seen as actions of the Lie group G on itself. The right and left translation on G can be lifted to an action on the tangent bundle TQ and the cotangent bundle T^*Q. For the right translation we have the **tangent lifted right translation** which is given by

$$\begin{aligned} &G \times TG \to TG \\ &(g,(h,v)) \mapsto (hg, vg) := (R_g(h), T_h R_g(v) = \left(R_g(h), \frac{d}{dt}(c(t)g)\bigg|_{t=0} \right), \end{aligned}$$

while the **cotangent lifted right translation** is given by

$$\begin{aligned} &G \times T^*G \to T^*G \\ &(g,(h,\alpha)) \mapsto (hg, \alpha g) := (R_g(h), T^*_{hg} R_{g^{-1}}(\alpha)), \end{aligned}$$

where

$$\left\langle T^*_{hg} R_{g^{-1}}(\alpha), w \right\rangle = \langle \alpha, T_{hg} R_{g^{-1}}(w) \rangle \quad \text{for all} \quad w \in T_{hg}G.$$

For the associate **tangent and cotangent left translations** we have

$$\begin{aligned} &G \times TG \to TG \\ &(g,(h,v)) \mapsto (gh, gv) := (L_g(h), T_h L_g(v) = \left(L_g(h), \frac{d}{dt}(gc(t))\bigg|_{t=0} \right), \end{aligned}$$

and

$$\begin{aligned} &G \times T^*G \to T^*G \\ &(g,(h,\alpha)) \mapsto (gh, g\alpha) := (L_g(h), T^*_{gh} L_{g^{-1}}(\alpha)), \end{aligned}$$

where

$$\left\langle T^*_{gh} L_{g^{-1}}(\alpha), w \right\rangle = \langle \alpha, T_{gh} L_{g^{-1}}(w) \rangle \quad \text{for all} \quad w \in T_{gh}G,$$

respectively. These lifted actions will be of particular use at Subsection 2.6.4 where we analyze the properties of a right/left invariant Lagrangian of a mechanical system. Now let us define another action, the conjugation, of G on itself by using both left and right translations.

The action of G on itself by **conjugation** (or **inner automorphism**) is

$$\begin{aligned} &G \times G \to G \\ &(g,h) \mapsto I_g(h) := \left(L_g \circ R_{g^{-1}}\right)(h) = ghg^{-1}, \end{aligned}$$

where the orbits of this action are called **conjugacy classes**. This action gives rise to an action of G on its Lie algebra $\mathfrak{g}$, namely, the **Adjoint action** of G on $\mathfrak{g}$ which is given by

$$\begin{aligned} &G \times \mathfrak{g} \to \mathfrak{g} \\ &(g,\xi) \mapsto \mathrm{Ad}_g \xi := T_e I_g(\xi) = T_e \left(L_g \circ R_{g^{-1}}\right)(\xi). \end{aligned}$$

Also we can define the **Co-Adjoint action** of G on $\mathfrak{g}^*$ which is the inverse dual of the Adjoint action and it is given by

$$\begin{aligned} G \times \mathfrak{g}^* &\to \mathfrak{g}^* \\ (g,\mu) &\mapsto \mathrm{Ad}^*_{g^{-1}}\mu, \end{aligned}$$

where $\langle \mathrm{Ad}^*_{g^{-1}}\mu, \xi\rangle = \langle \mu, \mathrm{Ad}_{g^{-1}}\xi\rangle$, for all $\mu \in \mathfrak{g}^*,\ \xi \in \mathfrak{g}$ and where $\langle\cdot,\cdot\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the pairing between elements of $\mathfrak{g}$ and $\mathfrak{g}^*$.

Next we define the infinitesimal generator which encaptures the infinitesimal description of an action. The **infinitesimal generator** map is called the **adjoint action** (or **adjoint operator**) of $\mathfrak{g}$ on itself (even though it is not an action) and it is given by

$$\begin{aligned} &\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \\ &(\xi,\eta) \mapsto \mathrm{ad}_\xi(\eta) = \xi_{\mathfrak{g}}(\eta) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp t\xi}\eta. \end{aligned}$$

Also we define the **coadjoint operator** which is the dual of the adjoint operator above and is given by

$$\begin{aligned} \mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* &\to \mathfrak{g}^* \\ (\xi,\mu) &\mapsto \mathrm{ad}^*_\xi(\mu), \end{aligned}$$

such that for all $\xi \in \mathfrak{g}$ the map $\mathrm{ad}_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ satisfies $\langle \mathrm{ad}^*_\xi\mu, \eta\rangle = \langle \mu, \mathrm{ad}_\xi\eta\rangle$, where $\eta \in \mathfrak{g}$.

Example 2.6.5. Let us first define the hat map which identifies the Lie algebra of the special orthogonal group, $\mathfrak{so}(3)$, given by skew-symmetric 3×3 matrices, with the vector space $\mathbb{R}^3$. The hat map is denoted by $(\ \hat{}\ ) : \mathbb{R}^3 \to \mathfrak{so}(3)$ and given by

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

Consider the left action of the Lie group SO(3) to itself given by left multiplication. The Adjoint action of SO(3) on its Lie algebra $\mathfrak{so}(3)$ is given by

$$\begin{aligned} &\mathrm{SO}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(3) \\ &\qquad (R,\hat{\Omega}) \mapsto \mathrm{Ad}_R\hat{\Omega} = R\hat{\Omega}R^{-1} = \widehat{(R\Omega)}, \end{aligned}$$

and the Co-Adjoint action is given by

$$\begin{aligned} &\mathrm{SO}(3) \times \mathfrak{so}(3)^* \to \mathfrak{so}(3)^* \\ &\qquad (R,\breve{\Pi}) \mapsto \mathrm{Ad}_{R^{-1}}\breve{\Pi} = (R\breve{\Pi}), \end{aligned}$$

where we have used the breve map $(\ \breve{}\ ) : \mathbb{R}^3 \to \mathfrak{so}(3)^*$ which is an isomorphism given by $\Pi = (\Pi_1,\Pi_2,\Pi_3) \mapsto \breve{\Pi}$ such that

$$\langle \breve{\Pi}, \hat{\Omega}\rangle = \Pi_1\Omega_1 + \Pi_2\Omega_2 + \Pi_3\Omega_3.$$

Moreover, given $\hat{\Omega}, \hat{w} \in \mathfrak{so}(3)$ and $\breve{\Pi} \in \mathfrak{so}(3)^*$, the adjoint and the coadjoint operators for $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^*$ are given by

$$\mathrm{ad}_{\hat{\Omega}}\hat{w} = [\hat{\Omega},\hat{w}] = \widehat{\Omega \times w} \quad \text{and} \quad \mathrm{ad}^*_{\hat{\Omega}}\breve{\Pi} = (\Pi \breve{\times} \Omega),$$

respectively.

2.6.3 Riemannian geometry on Lie groups

When Lie groups are endowed with a Riemannian metric they present the structure of a Riemannian manifold with additional properties necessary for this dissertation.

Let G be a Lie group with Lie algebra $\mathfrak{g} := T_eG$, where e is the identity element of G. Consider the left-translation by g, $L_g : G \to G$. Given any inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, left-translation provides us with a Riemannian metric $\langle \cdot, \cdot \rangle$ on G via the relation:

$$\langle X_g, Y_g \rangle := \langle T_g L_{g^{-1}} X_g, T_g L_{g^{-1}} Y_g \rangle_{\mathfrak{g}}, \tag{2.28}$$

for all $g \in G, X_g, Y_g \in T_gG$, where $T_gL_{g^{-1}}$ denotes the tangent map of $L_{g^{-1}}$ at the point g. Such a Riemannian metric is called **left-invariant**, and it follows naturally that there is a one-to-one correspondence between left-invariant Riemannian metrics on G and inner products on the Lie algebra $\mathfrak{g}$, and that the left translation map $L_g : G \to G$ is an isometry for all $g \in G$ by construction. Any Lie group equipped with a left-invariant metric is complete as a Riemannian manifold. The same occurs with a right-translation of an inner product given by the relation

$$\langle X_g, Y_g \rangle := \langle T_g R_{g^{-1}} X_g, T_g R_{g^{-1}} Y_g \rangle_{\mathfrak{g}}, \tag{2.29}$$

which provide us with a **right-invariant** Riemannian metric on G.

It is well-known that the map $(\cdot_R) : \mathfrak{g} \to \mathfrak{X}_R(G)$ defined by $\xi_R(g) = (R_g)_*\xi$ for all $\xi \in \mathfrak{g}$, $g \in G$ is a vector spaces isomorphism between the Lie algebra of G and the set of its right-invariant vector fields (see Holm et al., 2009, Marsden and Ratiu, 1999, Helgason, 1979). A similar discussion is valid for the left-translation map and left-invariant vector fields, being the map $(\cdot_L) : \mathfrak{g} \to \mathfrak{X}_L(G)$ an isomorphism of Lie algebras since the Lie bracket on $\mathfrak{g}$ is defined according to $[\xi, \eta] = [\xi_L, \eta_L](e)$.

Lemma 2.6.6. *Given $\xi, \eta \in \mathfrak{g}$, the Lie bracket of right invariant vector fields $[\xi_R, \eta_R]$ is a right-invariant vector field and*

$$[\xi_R, \eta_R] = -\left([\xi, \eta]\right)_R. \tag{2.30}$$

Proof. Consider the Lie group G equipped with the smooth operation

$$g \odot h = hg, \quad \forall g, h \in G.$$

G equipped with $\odot$ is also a Lie Group. Thus, denoting by $\bar{L}_g : G \to G$ the left multiplication with respect to the operation $\odot$ and by $R_g : G \to G$ the right multiplication with respect to the original operation on G we have that $R_g = \bar{L}_g$. Hence, it is not difficult to deduce that given $\xi \in \mathfrak{g}$, we have that $\xi_R = \xi_{\bar{L}}$, where $\xi_{\bar{L}}$ is defined by $\xi_{\bar{L}}(g) = T_e\bar{L}_g(\xi)$. Now, consider the inverse map $i : G \to G$ whose tangent map at the idenity is $T_ei = -id_{\mathfrak{g}}$. The inverse map is an isomorphism of Lie groups implying that T_ei is an isomorphism of Lie algebras. In particular, if we equip $\mathfrak{g}$ also with the Lie bracket $[\cdot, \cdot]_{\bar{\mathfrak{g}}}$ associated with the operation $\odot$, T_ei is an isomorphism of Lie algebras between $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{g}, [\cdot, \cdot]_{\bar{\mathfrak{g}}})$. Hence,

$$[\xi, \eta]_{\mathfrak{g}} = -T_ei([\xi, \eta]_{\mathfrak{g}}) = -[T_ei(\xi), T_ei(\eta)]_{\bar{\mathfrak{g}}} = -[\xi, \eta]_{\bar{\mathfrak{g}}}.$$

Therefore,

$$[\xi_R, \eta_R] = [\xi_{\bar{L}}, \eta_{\bar{L}}] = ([\xi, \eta]_{\bar{\mathfrak{g}}})_{\bar{L}} = ([\xi, \eta]_{\bar{\mathfrak{g}}})_R = -([\xi, \eta]_{\mathfrak{g}})_R.$$

□

Remark 2.6.7. The last Lemma shows that $(\cdot_R) : \mathfrak{g} \to \mathfrak{X}_R(G)$ is an anti-isomorphism of Lie algebras.

Remark 2.6.8. For left-invariance we have that, given $\xi, \eta \in \mathfrak{g}$, the Lie bracket of left invariant vector fields $[\xi_L, \eta_L]$ is a left-invariant vector field and

$$[\xi_L, \eta_L] = ([\xi, \eta])_L . \tag{2.31}$$

The isomorphism between $\mathfrak{g}$ and $\mathfrak{X}_R(G)$ allows us to construct a bilinear map $\nabla^{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by:

$$\nabla^{\mathfrak{g}}_\xi \eta := (\nabla_{\xi_R} \eta_R)(e), \tag{2.32}$$

for all $\xi, \eta \in \mathfrak{g}$, where ∇ is the Levi-Civita connection on G corresponding to the right-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ defined in (2.28).

The following result guarantees the consistency of this definition:

Theorem 2.6.9. *Let G be a Lie group equipped with a right-invariant metric associated with the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Then the Levi-Civita connection is a right-invariant connection and there exists a bilinear map denoted by $\nabla^{\mathfrak{g}}_\xi \eta$ such that*

$$(\nabla^{\mathfrak{g}}_\xi \eta)_R = \nabla_{\xi_R} \eta_R$$

and given by the equation

$$\nabla^{\mathfrak{g}}_\xi \eta = -\frac{1}{2}[\xi, \eta] + \frac{1}{2}\sharp\left[ad^*_\xi \flat(\eta)\right] + \frac{1}{2}\sharp\left[ad^*_\eta \flat(\xi)\right]. \tag{2.33}$$

Proof. First, we prove that the Levi-Civita connection is right-invariant, i.e., given two right-invariant vector fields such as ξ_R and η_R, where $\xi, \eta \in \mathfrak{g}$, the vector field $\nabla_{\xi_R} \eta_R$ is also right-invariant. This is proven by the formula

$$\begin{aligned} 2\langle \nabla_{\xi_R} \eta_R, \zeta_R \rangle = \mathcal{L}_{\xi_R} \langle \eta_R, \zeta_R \rangle + \mathcal{L}_{\eta_R} \langle \zeta_R, \xi_R \rangle - \mathcal{L}_{\zeta_R} \langle \xi_R, \eta_R \rangle + \langle [\xi_R, \eta_R], \zeta_R \rangle \\ - \langle [\xi_R, \zeta_R], \eta_R \rangle - \langle [\eta_R, \zeta_R], \xi_R \rangle, \end{aligned}$$

to compute $\langle \nabla_{\xi_R} \eta_R, \zeta_R \rangle$ where $\zeta \in \mathfrak{g}$. The first three terms on the right-hand side vanish because they are nothing but the Lie derivative of a constant function (the right-invariant inner product). The last three terms are also right-invariant. Thus the expression on the left-hand side is right invariant. Consequently, $\nabla_{\xi_R} \eta_R$ must also be right invariant. Denote by $\nabla^{\mathfrak{g}}_\xi \eta$ the restriction to e of $\nabla_{\xi_R} \eta_R$. It is clear that it is bilinear and that $(\nabla^{\mathfrak{g}}_\xi \eta)_R$ gives $\nabla_{\xi_R} \eta_R$.

In addition, using right invariance of the metric and equation (2.30) we deduce

$$2\langle \nabla^{\mathfrak{g}}_\xi \eta, \zeta \rangle = -\langle [\xi, \eta], \zeta \rangle + \langle [\xi, \zeta], \eta \rangle + \langle [\eta, \zeta], \xi \rangle \tag{2.34}$$

from which it is a straightforward computation to check that

$$\nabla^{\mathfrak{g}}_{\xi}\eta = -\frac{1}{2}[\xi,\eta] + \frac{1}{2}\sharp\left[\mathrm{ad}^*_{\xi}\flat(\eta)\right] + \frac{1}{2}\sharp\left[\mathrm{ad}^*_{\eta}\flat(\xi)\right].$$

□

Although $\nabla^{\mathfrak{g}}$ is not a connection, we shall refer to it as the **Riemannian $\mathfrak{g}$-connection**, as in J. Goodman and Colombo, 2024 for left-invariant metrics, corresponding to ∇ because of the following properties inherited from the Levi-Civita connection properties:

Lemma 2.6.10. *$\nabla^{\mathfrak{g}}:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is $\mathbb{R}$-bilinear, and for all $\xi,\eta,\sigma\in\mathfrak{g}$, the following relations hold:*

1. $\nabla^{\mathfrak{g}}_{\xi}\eta - \nabla^{\mathfrak{g}}_{\eta}\xi = -\left[\xi,\eta\right]_{\mathfrak{g}}$,
2. $\langle\nabla^{\mathfrak{g}}_{\sigma}\xi,\eta\rangle_{\mathfrak{g}} + \langle\xi,\nabla^{\mathfrak{g}}_{\sigma}\eta\rangle_{\mathfrak{g}} = 0$.

Proof. The $\mathbb{R}$-bilinearity of $\nabla^{\mathfrak{g}}$ follows immediately from (2.32). For 1., observe that

$$\nabla^{\mathfrak{g}}_{\xi}\eta - \nabla^{\mathfrak{g}}_{\eta}\xi = \left(\nabla_{\xi_R}\eta_R - \nabla_{\eta_R}\xi_R\right)(e) = \left[\xi_R,\eta_R\right](e) = -\left[\xi,\eta\right]_{\mathfrak{g}}$$

since ∇ is the torsion-free. For 2., note that since ∇ is compatible with $\langle\cdot,\cdot\rangle$, we have that

$$\begin{aligned}\mathcal{L}_{\sigma_R}\langle\xi_R,\eta_R\rangle(e) &= \langle\nabla_{\sigma_R}\xi_R(e),\eta_R(e)\rangle + \langle\xi_R(e),\nabla_{\sigma_R}\eta_R(e)\rangle\\ &= \langle\nabla^{\mathfrak{g}}_{\sigma}\xi,\eta\rangle + \langle\xi,\nabla^{\mathfrak{g}}_{\sigma}\eta\rangle,\end{aligned}$$

where $\mathcal{L}$ stands for the Lie derivative of vector fields. But since ξ_R,η_R are right-invariant, $\langle\xi_R,\eta_R\rangle$ is a constant function, so $\mathcal{L}_{\sigma_R}\langle\xi_R,\eta_R\rangle\equiv 0$. □

Remark 2.6.11. We may consider the Riemannian $\mathfrak{g}$-connection as an operator $\nabla^{\mathfrak{g}}: C^\infty([a,b],\mathfrak{g})\times C^\infty([a,b],\mathfrak{g})\to C^\infty([a,b],\mathfrak{g})$ in a natural way, namely, if $\xi,\eta\in C^\infty([a,b],\mathfrak{g})$, we can write $(\nabla^{\mathfrak{g}}_{\xi}\eta)(t) := \nabla^{\mathfrak{g}}_{\xi(t)}\eta(t)$ for all $t\in[a,b]$. With this notation, Lemma 2.6.10 works identically if we replace $\xi,\eta,\sigma\in\mathfrak{g}$ with $\xi,\eta,\sigma\in C^\infty([a,b],\mathfrak{g})$. ◇

Remark 2.6.12. Analogously, under the same hypothesis of Theorem 2.6.9 but for left-invariant metric we have that the Levi-Civita connection is left-invariant and there exist a bilinear map

$$(\nabla^{\mathfrak{g}}_{\xi}\eta)_L = \nabla_{\xi_L}\eta_L$$

which is given by the equation

$$\nabla^{\mathfrak{g}}_{\xi}\eta = \frac{1}{2}[\xi,\eta] - \frac{1}{2}\sharp\left[\mathrm{ad}^*_{\xi}\flat(\eta)\right] - \frac{1}{2}\sharp\left[\mathrm{ad}^*_{\eta}\flat(\xi)\right]. \tag{2.35}$$

Note the abuse of notation for the right and left invariant bilinear maps. In what follows it will be always clear from the context whether we deal with right or left invariance.

2.6.4 Mechanical systems on Lie groups

We devote this section to the study of mechanical systems whose configuration space is a Lie group, $Q = G$. A Lie group, as presented in Section 2.6, is a smooth manifold with a structure of a group where it possesses certain symmetries and has a rich geometry. Regarding their use in mechanical systems and in this thesis, we will take advantage of these symmetries to reduce the equations of motion.

Consider a Lie group G to be the configuration space of a mechanical system together with a Lagrangian function $L : TG \to \mathbb{R}$ and let this Lagrangian to be right invariant i.e. $L(gh, \dot{g}h) = L(g, \dot{g}), h \in G$ where we have considered the right translation map and the tangent lifted right action given at Subsection 2.6.2. Recall that from the Hamilton's principle we want to minimize the action functional $\mathcal{S}(q) = \int_a^b L(q(t), \dot{q}(t))dt$, over all variations of curves. The critical points of $\mathcal{S}$ are curves that satisfy the Euler-Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{g}}\right) + \frac{\partial L}{\partial g} = 0$ (see Subsection 2.3.1). Since the Lagrangian is right invariant we can define a new function which is called **reduced Lagrangian** $l : \mathfrak{g} \to \mathbb{R}$ and it is given by

$$l(\xi) = L(e, \dot{g}g^{-1}) = L(g, \dot{g}),$$

where $\xi = \dot{g}g^{-1} \in \mathfrak{g}$. Similarly to Hamilton's principle we define the **reduced action functional** $\mathcal{S}_{\text{red}}(\xi) = \int_a^b l(\xi)dt$ over variations of the form $\delta\xi = \dot{\eta} - [\xi, \eta]_{\mathfrak{g}}$, where η is an arbitrary variation. The curves that minimize $\mathcal{S}_{\text{red}}$ satisfy the so-called **Euler-Poincaré equations** $\frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) = -\text{ad}^*_\xi \frac{\partial l}{\partial \xi}$ (Holm et al., 1998). Moreover, it can be seen that there is a one-to-one correspondence between a curve g that satisfies the Euler-Lagrange equations associated to L and the curve $\xi := \dot{g}g^{-1}$ that satisfies the Euler-Poincaré equations associated to l (for a proof see for instance Holm et al., 2009, Holm et al., 1998). The equation $\dot{g} = \xi g$ is known as the **reconstruction equation**.

Next, consider a mechanical system which evolves in a Riemannian manifold $(G, \mathcal{G})$ with $\mathcal{G}$ a right invariant Riemannian metric on G. Since the Riemannian metric is right invariant the Lagrangian given by $L(v_q) = \frac{1}{2}\mathcal{G}(v_q, v_q)$ is right invariant as well. Thus, we define the reduced Lagrangian $l : \mathfrak{g} \to \mathbb{R}$ as previously.

Given a basis $\{e_i\}$ of $\mathfrak{g}$, we may write any vector field Z on G as $X = X^i(e_i)_R$, where $X^i : G \to \mathbb{R}$ are the components of the vector field and we have used the summation convention. If $g : [a, b] \to G$ is a smooth curve, then we can write a vector field X along the smooth curve g as $X = X^i e_i g$, where $e_i g = T_e R_g(e_i)$. In particular, we may write its velocity as $\dot{g} = \xi^i(e_i)_R$ where $\xi^i : [a, b] \to \mathbb{R}$ are the components of the velocity vector. Let $\xi : [a, b] \to \mathfrak{g}$ be the curve given by $\xi(t) = T_g R_{g^{-1}}(\dot{g})$. Then, in our chosen basis for $\mathfrak{g}$, the curve ξ can be equivalently written as $\xi = \xi^i e_i$. We denote $\dot{\xi} = \dot{\xi}^i e_i$, which may be written in a coordinate-free fashion via $\dot{\xi}(t) = \frac{d}{dt}\left(R_{g(t)^{-1}*}\dot{g}(t)\right)$.

We now wish to understand how the Levi-Civita connection ∇, associated with the right invariant Riemannian metric, along a curve is related to the Riemannian $\mathfrak{g}$-connection $\nabla^{\mathfrak{g}}$. This relation is summarized in the following result:

Lemma 2.6.13. *Consider a Lie group G with Lie algebra $\mathfrak{g}$ and right-invariant Levi-Civita* *connection ∇. Let $g : [a, b] \to G$ be a smooth curve and X a smooth vector field along g. Then the following relation holds for all $t \in [a, b]$:*

$$\nabla_{\dot{g}} X(t) = \left(\dot{\eta}(t) + \nabla^{\mathfrak{g}}_{\xi}\eta(t)\right) g(t), \tag{2.36}$$

where $\xi(t) = \dot{g}(t)g(t)^{-1}$ and $\eta(t) = X(t)g(t)^{-1}$.

Proof. Let $\{e_i\}$ be a basis for $\mathfrak{g}$. Then, we can write $\xi(t) = \xi^i(t)e_i$ and $\eta(t) = \eta^i(t)e_i$. Expanding out the left-hand side of (2.36), and using that $R_{g^*}(\eta^j e_j) = \eta^j R_{g^*}(e_j)$ we get

$$\nabla_{\dot{g}} X(t) = \nabla_{\dot{g}} \left(\eta^j (e_j)_R\right)(t) = \dot{\eta}^j(t)(e_j)_R + \eta^j(t)\nabla_{\dot{g}}(e_j)_R.$$

Using that $\dot{g}(t) = \xi^i(t)(e_i)_R$ for all $t \in [a, b], 1 \leq i \leq n$, we deduce

$$\nabla_{\dot{g}} (e_j)_R = \xi^i(t)\nabla_{(e_i)_R} (e_j)_R = \left(\xi^i(t)\nabla^{\mathfrak{g}}_{e_i} e_j\right)_R,$$

where the last equality follows from the definition of the Riemannian $\mathfrak{g}$-connection. Equation (2.36) then follows by the bi-linearity of $\nabla^{\mathfrak{g}}$ together with the expression of η and ξ with respect to the basis $\{e_i\}$. □

Remark 2.6.14. For a left-invariant Levi-Civita conection the above Lemma 2.6.13 gives the equation

$$\nabla_{\dot{g}} X(t) = g(t)\left(\dot{\eta}(t) + \nabla^{\mathfrak{g}}_{\xi}\eta(t)\right), \tag{2.37}$$

where $\xi(t) = g(t)^{-1}\dot{g}(t)$ and $\eta(t) = g(t)^{-1}X(t)$.

From the previous Lemma, if $g(t)$ is a geodesic with respect to the Levi-Civita connection, then $R_{g^*}\left(\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi\right) = 0$, where $\xi := \dot{g}g^{-1}$, and since right-translation is a diffeomorphism, we obtain the Euler-Poincaré equations for geodesics:

Theorem 2.6.15. *Suppose that $g : [a, b] \to G$ is a geodesic, and let $\xi := \dot{g}g^{-1}$. Then, ξ satisfies*

$$\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi = 0 \tag{2.38}$$

on $[a, b]$ or equivalently

$$\dot{\xi}(t) + \sharp\left[ad^*_{\xi(t)}\flat(\xi(t))\right] = 0. \tag{2.39}$$

Proof. Equation (2.38) follows directly from Lemma 2.6.13. Using the expression (2.33) of the $\mathfrak{g}$-connection we immediately deduce

$$\nabla^{\mathfrak{g}}_{\xi}\xi = \sharp\left[\mathrm{ad}^*_{\xi(t)}\flat(\xi(t))\right].$$

□

Remark 2.6.16. Since the Riemannian metric is right-invariant, the Lagrangian is right-invariant, and so the reduced Lagrangian takes the form $l(\xi) = \frac{1}{2}\|\xi\|^2$, it is straight-forward to show that $\dfrac{\partial l}{\partial \xi} = \flat(\xi)$, so that the Euler-Poincaré equations associated to l are given by $\flat(\dot{\xi}) = -\mathrm{ad}^*_{\xi}\flat(\xi)$, which is equivalent to (2.38). Notice that the Euler-Poincaré equations corresponding to l naturally live on the dual of the Lie algebra $\mathfrak{g}^*$. It is only through the metric that we are able to convert them into equations on $\mathfrak{g}$.

2.7 Homogeneous spaces

In this section, we consider homogeneous spaces which appear when a group acts on a manifold transitively. Homogeneous spaces are not Lie groups themselves; nonetheless, they have some kind of symmetries in the sense that locally they look the same in terms of the Lie group action. In other words, any two points of the manifold are similar with respect to the properties that are preserved by the Lie group.

Let G be a connected Lie group. A **homogeneous space** H of G is a smooth manifold on which G acts transitively. Thus, we see that any Lie group is itself a homogeneous space, where the transitive action is provided by left or right translation.

Consider a manifold H and a transitive left-action on it, namely, $\Psi : G \times H \to H$ which we denote by $gx := \Psi_g(x)$. Moreover, suppose that K is a closed Lie subgroup of G and consider the action $\Psi' : G \times G/K \to G/K$ satisfying $\Phi'_g([h]) = [gh]$ for all $g, h \in G$, where $[g]$ is the coset of this action. By its construction this action is transitive and thus the quotient G/K is a homogeneous space. Connecting these two actions, we choose K to be the stabilizer group of an arbitrary element $x \in H$ then it can be shown that $G/\text{Stab}(x) \cong H$ as differentiable manifolds, where $\text{Stab}(x) := \{g \in G \mid gx = x\}$. Hence, we may assume without loss of generality that $H = G/K$ is a homogeneous space of G for some closed Lie subgroup K. Let $\pi : G \to H$ be the canonical projection map.

Example 2.7.1. Consider the special Euclidean group $SE(3)$ composed of rotations and translations in the space $\mathbb{R}^3$. An element of $SE(3)$ is a pair (R, r) where R is a rotation matrix in $SO(3)$ and $r \in \mathbb{R}^3$. Consider the action of SE(3) on $\mathbb{R}^3$ given by the map

$$\Psi : \text{SE}(3) \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \Psi_{(R,r)}(x) = Rx + r.$$

This action performs a rotation R on the vector $x \in \mathbb{R}^3$ followed by a translation by r. This action is transitive, thus $\mathbb{R}^3$ has the structure of a homogeneous space. Thus, $\mathbb{R}^3$ can be seen as the quotient $\mathbb{R}^3 \simeq \text{SE}(3)/K$ where K is a subgroup of SE(3) such that $K = \{(k, 0) : k \in \text{SO}(3)\}$, thus $K \simeq \text{SO}(3)$. Moreover, $K = \text{Stab}(0)$ for $0 \in \mathbb{R}^3$ which is the stabilizer subgroup of the action Ψ. The projection map $\pi : \text{SE}(3) \to \mathbb{R}^3$ is given by $\pi(R, r) = R0 + r = r$.

Furthermore, we have that $\text{SE}(3) \simeq \text{SO}(3) \times \mathbb{R}^3$, using the hat map $\hat{(\cdot)} : \mathbb{R}^3 \to \mathfrak{so}(3)$ and the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$ (see Holm et al., 2009 for instance), we have that the basis for the Lie algebra $\mathfrak{se}(3)$ consists of the elements $\bar{e}_1 = (0, e_1), \bar{e}_2 = (0, e_2), \bar{e}_3 = (0, e_3), \bar{e}_4 = (\hat{e}_1, 0), \bar{e}_5 = (\hat{e}_2, 0), \bar{e}_6 = (\hat{e}_3, 0)$. Hence, from the projection map π, we deduce $T_I\pi(\bar{e}_1) = e_1, T_I\pi(\bar{e}_2) = e_2, T_I\pi(\bar{e}_3) = e_3, T_I\pi(\bar{e}_4) = T_I\pi(\bar{e}_5) = T_I\pi(\bar{e}_6) = 0$, where I denotes the 4×4 identity matrix. □

Using the canonical projection map we define two subbundles of the tangent bundle TG. The **vertical subspace** at $g \in G$ is given by $\text{Ver}_g := \ker(\pi_*|_g)$, which helps us to define the **vertical bundle** as $VG := \bigsqcup_{g \in G}\{g\} \times \text{Ver}_g$. Suppose G is equipped with a Riemannian metric $\langle \cdot, \cdot \rangle_G$, then we define the **horizontal subspace** at any point $g \in G$ (with respect to this metric) as the orthogonal complement of Ver_g. That is, $\text{Hor}_g := \text{Ver}_g^{\perp}$, where Hor_g contains all vectors that are orthogonal to every vector in Ver_g. Accordingly, we define the

horizontal bundle as $HG := \bigsqcup_{g\in G}\{g\}\times \mathrm{Hor}_g$. Since both the vertical and horizontal bundles are subbundles of the tangent bundle TG, it is clear that $T_gG = \mathrm{Ver}_g \oplus \mathrm{Hor}_g$ for all $g \in G$, so that the Lie algebra $\mathfrak{g}$ of G admits the decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$, where $\mathfrak{s}$ is the Lie algebra of K and $\mathfrak{h} \cong T_{\pi(e)}H$. For later use we denote the orthogonal projections onto the vertical and horizontal subspaces by $\mathcal{V}$ and $\mathcal{H}$ respectively.

A section $Z \in \Gamma(TG)$ is called a **horizontal vector field** if $Z(g) \in \mathrm{Hor}_g$ for all $g \in G$ and we write $Z \in \Gamma(HG)$. Consider a vector field $Y \in \Gamma(TG)$, Y will be π**-related** to some $X \in \Gamma(TH)$ if $\pi_* Y_g = X_{\pi(g)}$ for all $g \in G$. Moreover, if this Y belongs to the horizontal bundle, i.e. $Y \in \Gamma(HG)$, we say that Y is a **horizontal lift** of X. Also we define a horizontal lift of a smooth curve $q : [a,b] \to H$ as a smooth curve $\tilde{q} : [a,b] \to G$ such that $\pi \circ \tilde{q} = q$ and $\dot{\tilde{q}}(t)$ is horizontal for all $t \in [a,b]$. We have the following results from J. Goodman and Colombo, 2024.

Lemma 2.7.2. *Let H be a homogeneous space of G and $X \in \Gamma(TH)$. Then:*

1. *For all $X \in \Gamma(TH)$, there exists a unique horizontal lift $\tilde{X}$ of X. That is, the map $\tilde{\cdot} : \Gamma(TH) \to \Gamma(HG)$ sending $X \mapsto \tilde{X}$ is $\mathbb{R}$-linear and injective.*
2. *For all smooth curves $q : [a,b] \to H$ and $q_0 \in \pi^{-1}(\{q(a)\})$, there exists a unique horizontal lift $\tilde{q} : [a,b] \to H$ of q satisfying $\tilde{q}(a) = q_0$, called the horizontal lift of X which is π-related to X.*

Observe here that the left action $\Psi : G \times H \to H$ and the left translation matrix $L_g : G \to G$ satisfy the equality $\Psi_g \circ \pi = \pi \circ L_g$ for all $g \in G$ and taking differential of both sides we obtain the following commutative diagrams

$$\begin{array}{ccc} G & \xrightarrow{\pi} & H \\ {\scriptstyle L_g}\downarrow & & \downarrow{\scriptstyle \Psi_g} \\ G & \xrightarrow[\pi]{} & H \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Hor}_g & \xrightarrow{d\pi_g} & T_{\pi(g)}H \\ {\scriptstyle L_{g^{-1}*}}\downarrow & & \downarrow{\scriptstyle \Psi_{g^{-1}*}} \\ \mathfrak{h} & \xrightarrow[d\pi_e]{} & T_{\pi(e)}H. \end{array} \tag{2.40}$$

Consider a curve $\tilde{\eta} : [a,b] \to \mathfrak{h}$ and its projection $\eta : [a,b] \to T_{\pi(e)}H$ under $(\pi_*)_e$. Since the action Ψ_g is transitive we can translate η to a vector field X along any curve $q : [a,b] \to H$ through $X(t) = \Psi_{\tilde{q}(t)*}\eta(t)$ for $t \in [a,b]$ where $\tilde{q}$ is the horizontal lift of q. The vector field X can be lifted horizontally to a unique vector field $\tilde{X}$ along $\tilde{q}$ and from (2.40) we get that $L_{g^{-1}*}\tilde{X} = \tilde{\eta}$. This is presented in the next Lemma 2.7.3.

Lemma 2.7.3. *Suppose that $q : [a,b] \to H$ and $\tilde{q} : [a,b] \to G$ is a horizontal lift of q. Then, for any $\tilde{\eta} : [a,b] \to \mathfrak{h}$, there exists a unique $X \in \Gamma(q)$ such that its horizontal lift $\tilde{X}$ along $\tilde{q}$ satisfies $L_{\tilde{q}(t)^{-1}*}\tilde{X}(t) = \tilde{\eta}(t)$ for all $t \in [a,b]$.*

2.7.1 Riemannian Homogeneous Spaces

Let G be a connected Lie group and $H = G/K$ a homogeneous space of G with left action $\Psi : G \times H \to H$. Since H is a smooth manifold, we can equip it with a Riemannian metric. We are interested in those metrics $\langle \cdot, \cdot \rangle_H$ which preserve the structure of the homogeneous space. In other words, we would like to choose $\langle \cdot, \cdot \rangle_H$ so that the projection map $\pi : G \to H$ is a **Riemannian submersion**,which means that $\pi_*|_g$ is a linear isometry between Hor_g and $T_{\pi(g)}H$ for all $g \in G$ (that is, a linear map between Hor_g and and $T_{\pi(g)}H$ that preserves lengths). Hence, we call H a **Riemannian homogeneous space**.

It is clear that if H is a Riemannian homogeneous space, then $\langle \mathcal{H}(X), \mathcal{H}(Y) \rangle_G = \langle \pi_* X, \pi_* Y \rangle_H$ for all $X, Y \in T_g G, g \in G$ and as a result $\left\langle \tilde{X}, \tilde{Y} \right\rangle_G = \left\langle X, Y \right\rangle_H$ for all $X, Y \in T_g G, g \in G$. The metric $\langle \cdot, \cdot \rangle_H$ is said to be G-**invariant** if it is invariant under the left-action Ψ_g for all $g \in G$. It can be shown that every homogeneous space $H = G/K$ that admits a G-invariant metric is **reductive**. That is, the Lie algebra admits a decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$, where $\mathfrak{s}$ is the Lie algebra of K, and $\mathfrak{h}$ satisfies $[\mathfrak{s}, \mathfrak{h}] \subset \mathfrak{h}$. In particular, this implies that $\mathfrak{h} \cong T_{\pi(e)}(G/K)$ are vector spaces.

Example 2.7.4. Continuing with Example 2.7.1, equip the Lie group SE(3) with the usual left-invariant metric determined by the inner product

$$\langle (\hat{\Omega}_1, r_1), (\hat{\Omega}_2, r_2) \rangle_{\mathfrak{se}(3)} = \Omega_1^T \Omega_2 + r_1^T r_2,$$

where $\Omega_1, \Omega_2 \in \mathbb{R}^3$ and $r_1, r_2 \in \mathbb{R}^3$. Using this metric we define an inner product on $T_{\mathbf{0}}\mathbb{R}^3, \mathbf{0} \in \mathbb{R}^3$ through the relation $\langle (T_I\pi)(\xi_1), (T_I\pi)(\xi_2) \rangle_{T_{\mathbf{0}}\mathbb{R}^3} = \langle \xi_1, \xi_2 \rangle_{\mathfrak{se}(3)}$ for all $\xi_1, \xi_2 \in \mathfrak{se}(3)$ and we extend this inner product to an SE(3)-invariant Riemannian metric on $\mathbb{R}^3$ by $\langle X, Y \rangle_{\mathbb{R}^3} = \langle (T_r\Psi_{(0,r)})(X), (T_r\Psi_{(0,r)})(Y) \rangle_{T_{\mathbf{0}}\mathbb{R}^3}$ where $X, Y \in T_r\mathbb{R}^3$, $r \in \mathbb{R}^3$. Thus, since $T_r\Psi_{(0,r)} = Id$, we have that $\langle X, Y \rangle_{\mathbb{R}^3} = X^T Y$, which is the standard Euclidean product on $\mathbb{R}^3$.

With this Riemannian structure, we have that $\mathfrak{s} = \ker(T_I\pi) = \mathrm{span}\{\bar{e}_4, \bar{e}_5, \bar{e}_6\} \simeq \mathfrak{so}(3)$ and we define $\mathfrak{h} = \mathfrak{s}^\perp$ such that $\mathfrak{h} = \mathrm{span}\{\bar{e}_1, \bar{e}_2, \bar{e}_3\} \simeq \mathbb{R}^3$. The adjoint operator for the Lie algebra $\mathfrak{se}(3)$ is given by $\mathrm{ad} : \mathfrak{se}(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3)$,

$$\mathrm{ad}_{(\hat{\Pi}, t)}(\hat{\Omega}, s) = (\mathrm{ad}_{\hat{\Pi}}\hat{\Omega}, \hat{\Pi}s - \hat{\Omega}t),$$

where $\mathrm{ad}_{\hat{\Pi}}\hat{\Omega}$ is the adjoint operator on $\mathfrak{so}(3)$ and $t, s \in \mathbb{R}^3$ (see Holm et al., 2009 for instance). Since $\mathfrak{s} = \mathrm{span}\{\bar{e}_4, \bar{e}_5, \bar{e}_6\} \simeq \mathfrak{so}(3)$, the vertical and the horizontal spaces of SE(3) are given by

$$\mathrm{Ver}_g = \mathrm{span}\{g\bar{e}_4, g\bar{e}_5, g\bar{e}_6\} \simeq \mathrm{SO}(3) \quad \text{and} \quad \mathrm{Hor}_g = \mathrm{span}\{g\bar{e}_1, g\bar{e}_2, g\bar{e}_3\} \simeq \mathbb{R}^3,$$

respectively, where $g \in \mathrm{SE}(3)$. The horizontal projection is given by $\mathcal{H}(\hat{\Omega}, r) = (0, r)$. □

Remark 2.7.5. From J. Goodman and Colombo, 2024 we know that the existence of a G-invariant metric on H and the existence of a left-invariant metric on G for which H is a Riemannian homogeneous space are strongly linked, i.e. there is an equivalency between the two.

Example 2.7.6. Let us equip the Lie group $G = \mathrm{SO}(3)$ with the left-invariant metric $\langle \hat{\Omega}_1, \hat{\Omega}_2 \rangle_{\mathrm{SO}(3)} = \Omega_1^T \Omega_2$ for $\Omega_1, \Omega_2 \in \mathbb{R}$, where we use the hat map defined at the Example 2.6.5 for the identification $\mathfrak{so}(3) \cong \mathbb{R}^3$. Consider the action of SO(3) on $\mathbb{R}^3$ by the usual matrix multiplication given by the map $\Psi : \mathrm{SO}(3) \times \mathbb{R} \to \mathbb{R}$ where $(R, x) \mapsto Rx$ and consider also the Lie subgroup $K \subset \mathrm{SO}(3)$ given by

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\},$$

which is isomorphic with the Lie group $\mathrm{SO}(2)$. As it can be easily checked K is the isotropy group (stabilizer) of the action Ψ of the vector $e_3 = (0,0,1)^T$, i.e. $K = \mathrm{Stab}(e_3)$ and if we restrict it to the unit sphere $S^2 \subset \mathbb{R}^3$, $\Psi\big|_{S^2}$ is transitive. Hence, the quotient space $H = \mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2$ is a homogeneous space and the projection map $\pi : \mathrm{SO}(3) \to S^2$ is given by $\pi(R) = Re_3$ for $R \in \mathrm{SO}(3)$. Let us equip the Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ with the usual orthonormal basis $\{\hat{\Omega}_1, \hat{\Omega}_2, \hat{\Omega}_3\}$ given by

$$\hat{\Omega}_1 = \hat{e}_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{\Omega}_2 = \hat{e}_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \hat{\Omega}_3 = \hat{e}_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $e_i, i = 1,2,3$ is the canonical basis of $\mathbb{R}^3$. From the projection map we have that $\pi_*\hat{\Omega}_1 = -e_2$, $\pi_*\hat{\Omega}_2 = e_1$ and $\pi_*\hat{\Omega}_3 = 0$ hence, $\mathfrak{s} = \ker(\pi_*\big|_I) = \mathrm{span}\{\hat{\Omega}_3\}$ and $\mathfrak{h} = \mathfrak{s}^{\perp}$. Since we have a left-invariant metric on the Lie group G, we define a G-invariant metric on the homogeneous space H by the following process. First, consider an inner product on $T_{e_3}S^2$ defined through the relation $\langle X, Y\rangle_{T_{e_3}S^2} := \langle \pi_*^{-1}X, \pi_*^{-1}Y\rangle_{\mathrm{SO}(3)}$ for all $X, Y \in T_{e_3}S^2$. Since $T_{e_3}S^2 \cong \mathbb{R}^2$ the set $\{e_1, e_2\}$ forms a basis for $T_{e_3}S^2$ so the vectors X, Y can be written as $X = X^1e_1 + X^2e_2$ and $Y = Y^1e_1 + Y^2e_2$, and the inner product on $T_{e_3}S^2$ is given by

$$\begin{aligned}\langle X, Y\rangle_{T_{e_3}S^2} &= X^1Y^1\langle e_1, e_1\rangle_{T_{e_3}S^2} + X^1Y^2\langle e_1, e_2\rangle_{T_{e_3}S^2} + X^2Y^1\langle e_2, e_1\rangle_{T_{e_3}S^2} + X^2Y^2\langle e_2, e_2\rangle_{T_{e_3}S^2} \\ &= X^1Y^1\langle \hat{\Omega}_1, \hat{\Omega}_1\rangle_{\mathrm{SO}(3)} + X^1Y^2\langle \hat{\Omega}_1, \hat{\Omega}_2\rangle_{\mathrm{SO}(3)} + X^2Y^1\langle \hat{\Omega}_2, \hat{\Omega}_1\rangle_{\mathrm{SO}(3)} + X^2Y^2\langle \hat{\Omega}_2, \hat{\Omega}_2\rangle_{\mathrm{SO}(3)} \\ &= X^1Y^1 + X^2Y^2,\end{aligned}$$

which tells us that $\langle\cdot,\cdot\rangle_{T_{e_3}S^2}$ is the standard Euclidean metric with respect to the basis $\{e_1, e_2\}$. Next, we define an $\mathrm{SO}(3)$-invariant Riemannian metric on S^2 by extending this inner product to all S^2, i.e.

$$\langle X, Y, \rangle_{S^2} = \langle R^TX, R^TY\rangle_{T_{e_3}S^2} = R^TX \cdot R^TY = X \cdot Y,$$

for every $X, Y \in T_pS^2$ and $R \in \mathrm{SO}(3)$ such that $\pi(R) = p$.

Denote the Levi-Civita connections on H and G with respect to these metrics by ∇ and $\tilde{\nabla}$, respectively. Also, we denote by $\tilde{\nabla}^{\mathfrak{g}}$ the bilinear map defined by the left invariant connection $\tilde{\nabla}$ as in Remark 2.6.12. In a Riemannian homogeneous space, due to the fact that π is a Riemannian submersion we have the remarkable property that if a geodesic on G is horizontal at some point, then it is horizontal at all points. In particular, the horizontal lift of geodesics on H are geodesics on G (see O'Neill, 1967). These geodesics are usually called **horizontal geodesics**.

Theorem 2.7.7. *Let H be a Riemannian homogeneous space with respect to a Lie group action by G. If $g : [a,b] \to G$ is a curve on G, $q(t) = \pi(g(t))$ the projection of g on H and $\xi(t) := (T_{g(t)}L_{g^{-1}(t)})(\dot{g}(t))$, then the following statements are equivalent:*

1. *The curve* $g : [a, b] \to G$ *is a horizontal geodesic,*
2. $\xi(t) \in \mathfrak{h}$ *for all* $t \in [a, b]$ *and*

$$\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi = 0. \tag{2.41}$$

3. $g(t) = \tilde{q}(t)$ *and the curve* q *is a geodesic for* ∇.

Proof. The proof that statements (1) and (2) are equivalent derives from equation (2.38) (note the different symbol for the connection because, here, we have two distinct connections $\tilde{\nabla}$ and ∇) and the fact that geodesics horizontal at one point remain horizontal for all times. If $g(t)$ is a horizontal geodesic with respect to the Levi-Civita connection, then

$$(T_e L_g)\left(\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi\right) = 0, \quad \dot{g} \in HG.$$

Since left-translation is a diffeomorphism, we have the desired result.

The equivalence between (1) and (3) is a general fact for Riemannian submersions that can be found on O'Neill, 1967. □

Example 2.7.8. Continuing with Example 2.7.1 and Example 2.7.4, consider a horizontal geodesic $g : [a, b] \to \mathrm{SE}(3)$ given by $g(t) = (R, r(t))$ where R is a constant element of $\mathrm{SO}(3)$ and $r(t) \in \mathbb{R}^3$. The left translation to the identity gives $\xi = (T_g L_{g^{-1}})(\dot{g}) = (R^T, -R^T r(t))(0, \dot{r}(t)) = \left(0, R^T \dot{r}\right)$ thus ξ is an element of the horizontal subspace at the identity, i.e. $\xi \in \mathfrak{h} = Hor_e = \mathrm{span}\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$. From (2.35), $\tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi = 0$, since $\mathrm{ad}^*_{\xi}\xi = 0$. Thus, (2.41) reads $\ddot{r} = 0$. Let $q(t) = \pi(g(t))$. So, $q(t) = r(t) \in \mathbb{R}^3$. Clearly, since $\ddot{r} = 0$, we have that r is a geodesic on $\mathbb{R}^3$. In addition, the horizontal lift of $q(t)$ to the point $(R, r(0)) \in \mathrm{SE}(3)$ is $\tilde{q}(t) = (R, r(t))$. □

The next Proposition relates the covariant derivative of vector fields in G and H and will be useful later.

Proposition 2.7.9 (Lemma 45 from Chapter 7, O'Neill, 1983)**.** *Let* ∇ *and* $\tilde{\nabla}$ *be the Levi-Civita connections on* H *and* G*, respectively. Then,*

1. $T\pi(\tilde{\nabla}_{\tilde{X}}\tilde{Y}) = \nabla_X Y$, $X, Y \in \mathfrak{X}(H)$.
2. $\mathcal{H}\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla_X Y}$ *, where the right-hand side denotes the horizontal lift of the* $\nabla_X Y$.

2.7.2 Mechanical systems on homogeneous spaces

For the rest of the chapter, assume that H is a homogeneous manifold and there is a Lie group G that acts on H and that both manifolds are equipped with Riemannian metrics making $\pi : G \to H$ a Riemannian submersion.

Let $V : H \to \mathbb{R}$ be a potential function on the homogeneous space H. Via the projection map π, this potential function induces a potential function on the Lie group G denoted by $\tilde{V} = V \circ \pi$. Let $q : [a, b] \to H$ be a trajectory of a mechanical system with mechanical type Lagrangian function $L : TH \to \mathbb{R}$ of the form $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$. Then, the curve q

satisfies the equation

$$\nabla_{\dot{q}}\dot{q}(t) = -\text{grad } V(q(t)), \tag{2.42}$$

where grad is the gradient with respect to the metric on H.

The next result establishes that the gradient vector field $\widetilde{\text{grad}}\,\tilde{V}$, where $\widetilde{\text{grad}}$ is the gradient with respect to the metric on G, is a horizontal vector field.

Lemma 2.7.10. *If $\tilde{V} = V \circ \pi$ is the potential function induced by $V : H \to \mathbb{R}$, then the vector field $\widetilde{grad}\,\tilde{V} \in \Gamma(HG)$, where $\widetilde{grad}$ is the gradient with respect to the metric on G.*

Proof. By construction, since given a vertical vector field $Y \in VG$, we have that

$$\langle \widetilde{\text{grad}}\,\tilde{V}, Y\rangle = d\tilde{V}(Y) = dV(T\pi(Y)) = 0.$$

□

Due to the fact that the gradient vector field of the potential $\tilde{V}$ is horizontal, one might deduce that the horizontal lift $g = \tilde{q}$ of a curve q satisfying (2.42) satisfies the mechanical equation

$$\tilde{\nabla}_{\dot{g}}\dot{g} = -\widetilde{\text{grad}}\,\tilde{V}(g(t)). \tag{2.43}$$

Writing this equation entirely in the Lie algebra of G it is often impossible in real applications since the potential function $\tilde{V}$ typically does not possess any invariance property with respect to the group multiplication. Interesting applications in geometric control occur when the potential possesses partial symmetries A. M. Bloch et al., 2017, Colombo and Stratoglou, 2023, J. R. Goodman and Colombo, 2024.

Theorem 2.7.11. *Let H be a Riemannian homogeneous space w.r.t a Lie group action by G and $V : H \to \mathbb{R}$ a potential function. If $g : [a, b] \to G$ is a curve on G, $q(t) = \pi(g(t))$ the projection of g on H, and $\xi(t) := (T_{g(t)}L_{g^{-1}(t)})(\dot{g}(t))$, then the following statements are equivalent:*

1. *The curve $g : [a, b] \to G$ is a horizontal trajectory of the mechanical system* (2.43)*, that is $\dot{g}(t) \in HG$ for all $t \in [a, b]$.*
2. *$\xi \in \mathfrak{h}$ on $[a, b]$ and satisfies*

$$\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi = -(T_g L_{g^{-1}}) \left(\widetilde{grad}\,\tilde{V}(g(t))\right). \tag{2.44}$$

3. *If $q(t) = \pi(g(t))$ then $g(t) = \tilde{q}(t)$ and the curve q satisfies* (2.42)*.*

Proof. The equivalence of the statements (1) and (2) is a direct consequence of (2.37) and of equation (2.43). Indeed,

$$\tilde{\nabla}_{\dot{g}}\dot{g} = T_e L_g \left(\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi\right)$$

from where it is clear that $\tilde{\nabla}_{\dot{g}}\dot{g} = -\widetilde{\text{grad}}\,\tilde{V}(g(t))$ if and only if $\dot{\xi}+\tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi = -(T_g L_{g^{-1}}) \left(\widetilde{\text{grad}}\,\tilde{V}(g(t))\right)$ since

$$(T_e L_g)(T_g L_{g^{-1}}) \left(\widetilde{\text{grad}}\,\tilde{V}(g(t))\right) = \widetilde{\text{grad}}\,\tilde{V}(g(t)).$$

The equivalence of the statements (1) and (3) can be seen firstly from the fact that if $q(t) = \pi(g(t))$ then $g(t) = \tilde{q}(t)$ if and only if g is horizontal.

Secondly, using Lemma 2.7.10 stating that $\widetilde{\text{grad}}\, \tilde{V}$ is a horizontal vector field and since by construction $\tilde{V} = V \circ \pi$, we conclude that $(T\pi)(\widetilde{\text{grad}}\, \tilde{V}) = \text{grad}\, V$. Thus, $\widetilde{\text{grad}}\, \tilde{V}$ is the horizontal lift of grad V. Hence, if g satisfies equation (2.43), by Proposition 2.7.9(2), the horizontal lift of $\nabla_{\dot{q}}\dot{q}(t)$ must coincide with the horizontal lift of $-\widetilde{\text{grad}}\, \tilde{V}$. Therefore, equation (2.42) must hold.

Conversely, suppose that equation (2.42) holds. By Theorem 2.7.7, we know that the geodesic vector field of $\tilde{\nabla}$ is tangent to HG, since geodesics with initial velocity in HG, remain in HG for all time. Furthermore, from Stratoglou, Anahory Simoes, et al., 2023, the vector field whose trajectories are the solution of equation (2.43) has the form

$$\tilde{\Gamma} = G - (\widetilde{\text{grad}}\, \tilde{V}(g(t)))^V,$$

where G is the geodesic vector field of $\tilde{\nabla}$ and $(\widetilde{\text{grad}}\, \tilde{V}(g(t)))^V$ is the vertical lift of the vector field $\widetilde{\text{grad}}\, \tilde{V}(g(t))$. Thus, Γ is the sum of two vector fields that are tangent to HG, implying that Γ is itself tangent to HG. Consequently, if the trajectories of Γ, that is, of equation (2.43), are horizontal at one point, they are horizontal at all points. In particular, since the trajectories of equation (2.43) project onto trajectories of equation (2.42), we must have that the horizontal lift $g = \tilde{q}$ must be a solution of (2.43). □

Chapter 3

Virtual nonholonomic constraints: Linear and Affine cases

In this chapter, we introduce the concept of virtual nonholonomic constraints, detailing both the linear and affine cases. We consider control systems that are subject to linear and affine constraints and we prove the existence and uniqueness of a control law such that trajectories of the closed-loop system satisfy the respective constraints. The approach is geometric and is based on the concept of transversality which has been already used in the literature for holonomic constraints as it was presented at the Section 2.5. Most of the examples of nonholonomic constraints in the literature of nonholonomic systems fall under the cases examined in this chapter (see the books A. Bloch, 2015 and Neimark and Fufaev, 2004 for instance, and also Jarzębowska and McClamroch, 2000; Jarzębowska, 2005, 2006; Jarzębowska, 2008; Jarzębowska et al., 2019 for modeling and control applications).

3.1 Linear nonholonomic constraints

In this section, we develop the theory of virtual nonholonomic constraints in the linear case where the constraints depend linearly on the velocities of the system and they appear in many nonholonomic systems (A. Bloch, 2015, Jurdjevic, 1997). After giving a rigorous definition for virtual nonholonomic constraints and presenting it as a controlled invariant distribution associated with an affine connection mechanical control system, we prove the existence and uniqueness of a control law defining virtual nonholonomic constraints. In the following, we introduce an induced constraint connection and characterize the trajectories of the closed-loop system as solutions of the mechanical system associated with this connection. Lastly, we show when we can obtain nonholonomic dynamics from virtual nonholonomic constraints.

Recall from Subsection 2.4 that when a systems evolves in an n dimensional manifold Q and it is subject to some linear nonholonomic constraint determined by the zero level set of a function $\Phi : TQ \to \mathbb{R}^m$, given by $\Phi = (\phi^1, \dots, \phi^m)$ i.e. the equations

$$\phi^a(q, \dot{q}) = \mu^a_i(q)\dot{q}^i = 0, \tag{3.1}$$

$a = 1 \dots, m$, the dynamics are constrained in the sense that the solutions must satisfy specific properties. From a geometric point of view, these constraints are defined by a nonintegrable regular distribution $\mathcal{D}$ on Q of constant rank $(n-m)$. Namely, a curve $q : I \to Q$ satisfies the constraints if $\dot{q}(t) \in \mathcal{D}_{q(t)}$, for $t \in I$ and $q = (q^1, \ldots, q^n)$ are local coordinates.

Recall also here that the annihilator of $\mathcal{D}$ is denoted by $\mathcal{D}^o$ and is locally given at each point of Q by $\mathcal{D}^o_q = \text{span}\,\{\mu^a(q) = \mu^a_i dq^i\,; 1 \leq a \leq m\}$, where μ^a are linearly independent differential one-forms at each point of Q. For more details on distributions, see Section 2.1.4.

Now, consider a mechanical system that is subject to nonholonomic constraints given by Definition 2.4.2 and which for simplicity is restated here.

Definition 3.1.1. A **nonholonomic mechanical system** on a smooth manifold Q is given by the triple $(\mathcal{G}, V, \mathcal{D})$, where $\mathcal{G}$ is a Riemannian metric on Q, representing the kinetic energy of the system, $V : Q \to \mathbb{R}$ is a smooth function representing the potential energy, and $\mathcal{D}$ a regular distribution on Q describing the nonholonomic constraints.

3.1.1 Virtual nonholonomic constraints

Next, we present the rigorous construction of virtual nonholonomic constraints. On contrary to the case of standard nonholonomic constraints of the form (3.1), the concept of virtual constraint is always associated with a controlled system, rather than with the distribution defined by the constraints.

Given an external force $F^0 : TQ \to T^*Q$ and a control force $F : TQ \times U \to T^*Q$ of the form

$$F(q, \dot{q}, u) = \sum_{a=1}^{m} u_a F^a(q) \tag{3.2}$$

where $F^a \in \Omega^1(Q)$ with $m < n$, $U \subset \mathbb{R}^m$ the set of controls and $u_a \in \mathbb{R}$ with $1 \leq a \leq m$ the control inputs, consider the associated mechanical control system of the form

$$\nabla^{\mathcal{G}}_{\dot{q}(t)} \dot{q}(t) = Y^0(q(t), \dot{q}(t)) + u_a(t) Y^a(q(t)), \tag{3.3}$$

with $Y^0 = \sharp(F^0)$ and $Y^a = \sharp(F^a)$ the corresponding force vector fields. For details see Proposition 2.3.4.

Now we will define the concept of virtual nonholonomic constraint which is represented as a controlled invariant distribution.

Definition 3.1.2. A **virtual nonholonomic constraint** associated with the mechanical control system (3.3) is a controlled invariant distribution $\mathcal{D} \subseteq TQ$ for that system, that is, there exists a control function $u^* : \mathcal{D} \to \mathbb{R}^m$ such that the solution of the closed-loop system satisfies $\psi_t(\mathcal{D}) \subseteq \mathcal{D}$, where $\psi_t : TQ \to TQ$ denotes its flow.

Remark 3.1.3. A particular example of mechanical control system appearing in applications is determined by a mechanical Lagrangian function $L : TQ \to \mathbb{R}$. In this case, the control system is given by the controlled Euler-Lagrange equations, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F(q, \dot{q}, u). \tag{3.4}$$

If the curve $q : I \to Q$ is a solution of the controlled Euler Lagrange equations (3.4), it may be shown that it satisfies the mechanical equation (see Bullo and Lewis, 2005 for instance)

$$\nabla^{\mathcal{G}}_{\dot{q}(t)}\dot{q}(t) + \mathrm{grad}_{\mathcal{G}}V(q(t)) = u_a(t)Y^a(q(t)). \tag{3.5}$$

These are the equations of a mechanical control system as in (3.3), where the force field Y^0 is simply given by $-\mathrm{grad}_{\mathcal{G}}V(q(t))$. In this case, we call (3.5) a controlled Lagrangian system.$\diamond$

At the following subsection, we present some illustrative examples to exemplify the notion of virtual nonholonomic constraints.

3.1.2 Examples

In the next examples, we consider control mechanical systems with nonholonomic constraints. The reader may recall the definitions of *input distribution* given at Definition 2.5.1, which is the span of vector fields defined by the control force.

Example 3.1.4. Consider in $SE(2) \cong \mathbb{R}^2 \times \mathbb{S}^1$ the mechanical Lagrangian function

$$L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I\dot{\theta}^2}{2}$$

together with the control force

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = u(\sin\theta dx - \cos\theta dy + d\theta).$$

The corresponding controlled Lagrangian system is

$$m\ddot{x} = u\sin\theta, \quad m\ddot{y} = -u\cos\theta, \quad I\ddot{\theta} = u$$

and, as we will show, it has the following virtual nonholonomic constraint

$$\sin\theta\dot{x} - \cos\theta\dot{y} = 0.$$

The input distribution $\mathcal{F}$ is generated just by one vector field

$$Y = \frac{\sin\theta}{m}\frac{\partial}{\partial x} - \frac{\cos\theta}{m}\frac{\partial}{\partial y} + \frac{1}{I}\frac{\partial}{\partial\theta},$$

while the virtual nonholonomic constraint is the distribution $\mathcal{D}$ defined as the set of tangent vectors $v_q \in T_qQ$ where $\mu(q)(v_q) = 0$, with $\mu = \sin\theta dx - cos\theta dy$. Thus, we may write it as

$$\mathcal{D} = \mathrm{span}\Big\{X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial\theta}\Big\}.$$

We may check that $\mathcal{D}$ is controlled invariant for the controlled Lagrangian system above. In fact, the control law

$$u^*(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y})$$

makes the distribution invariant under the closed-loop system, since in this case, the dynamical SODE vector field (see (2.23) for the SODE vector field) arising from the controlled Euler-Lagrange equations given by

$$\Gamma = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{\theta}\frac{\partial}{\partial\theta} + \frac{u^*\sin\theta}{m}\frac{\partial}{\partial\dot{x}} - \frac{u^*\cos\theta}{m}\frac{\partial}{\partial\dot{y}} + \frac{u^*}{I}\frac{\partial}{\partial\dot{\theta}}$$

is tangent to $\mathcal{D}$. This is deduced from the fact that $\Gamma(\sin\theta\dot{x} - \cos\theta\dot{y}) = 0$. $\diamond$

Example 3.1.5. Consider in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ the mechanical Lagrangian function

$$L(x, y, \theta, \varphi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I\dot{\theta}^2}{2} + \frac{J\dot{\varphi}^2}{2}$$

together with the control force

$$F(x, y, \theta, \varphi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}, u) = u_1(dx - \cos\varphi d\theta + d\varphi) + u_2(dy - \sin\varphi d\theta + d\varphi).$$

The controlled Lagrangian system is then

$$m\ddot{x} = u_1,\ m\ddot{y} = u_2,\ I\ddot{\theta} = -u_1\cos\varphi - u_2\sin\varphi,\ J\ddot{\varphi} = u_1 + u_2.$$

The virtual nonholonomic constraints associated to this system are defined by the following equations

$$\dot{x} = \dot{\theta}\cos\varphi, \quad \dot{y} = \dot{\theta}\sin\varphi.$$

Therefore, the input distribution $\mathcal{F}$ is the set

$$\mathcal{F} = \operatorname{span}\Big\{Y^1 = \frac{1}{m}\frac{\partial}{\partial x} - \frac{\cos\varphi}{I}\frac{\partial}{\partial\theta} + \frac{1}{J}\frac{\partial}{\partial\varphi},\ Y^2 = \frac{1}{m}\frac{\partial}{\partial y} - \frac{\sin\varphi}{I}\frac{\partial}{\partial\theta} + \frac{1}{J}\frac{\partial}{\partial\varphi}\Big\},$$

and the constraint distribution $\mathcal{D}$ is defined by the 1-forms $\mu^1 = dx - \cos\varphi d\theta$ and $\mu^2 = dy - \sin\varphi d\theta$, thus

$$\mathcal{D} = \Big\{X_1 = \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta},\ X_2 = \frac{\partial}{\partial\varphi}\Big\}.$$

We may verify, using a similar argument as Example 3.1.4, that $\mathcal{D}$ is in fact controlled invariant under the control law

$$u_1^* = -m\dot{\theta}\dot{\varphi}\sin\varphi, \quad u_2^* = m\dot{\theta}\dot{\varphi}\cos\varphi.$$

$\diamond$

Example 3.1.6. Let us see an example of a mechanical control system which is not a Lagrangian system. Consider again the mechanical control system proposed in Example 3.1.4 but now with an additional damping term determined by the vector field

$$Y^0 = -\frac{\gamma}{m}(\dot{x}dx + \dot{y}dy),$$

where $\gamma > 0$ is a damping constant. The mechanical control system has the following equations of motion

$$m\ddot{x} = u\sin\theta - \gamma\dot{x}, \quad m\ddot{y} = -u\cos\theta - \gamma\dot{y}, \quad I\ddot{\theta} = u.$$

It is not difficult to check that the control law

$$u^*(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y})$$

still makes the distribution invariant under the flow of the closed-loop system. $\diamond$

3.1.3 Existence and uniqueness of a feedback control making the constraints invariant

It is often very useful if we have conditions under which we are guaranteed that a distribution $\mathcal{D}$ is controlled invariant for the controlled Lagrangian system (3.5). The next result not only states the existence of a control function making $\mathcal{D}$ invariant, but it also states that it is unique. After the definition of transversality given at Definition 2.5.7, we given the respective notion for distributions.

Definition 3.1.7. Two distributions $\mathcal{A}_1$ and $\mathcal{A}_2$ on the manifold Q are said to be transversal if they are complementary, in the sense that $TQ = \mathcal{A}_1 \oplus \mathcal{A}_2$.

Theorem 3.1.8. *If the distribution $\mathcal{D}$ and the control input distribution $\mathcal{F}$ are transversal, then there exists a unique control function making the distribution a virtual nonholonomic constraint associated with the mechanical control system* (3.3)*.*

Proof. Suppose that $TQ = \mathcal{D} \oplus \mathcal{F}$ and that trajectories of the control system (3.3) may be written as the integral curves of the vector field Γ defined by (2.23). For each $v_q \in \mathcal{D}_q$, we have that

$$\Gamma(v_q) \in T_{v_q}(TQ) = T_{v_q}\mathcal{D} \oplus \operatorname{span}\Big\{(Y^a)^V_{v_q}\Big\},$$

with $Y^a = \sharp(F^a)$ and $(Y^a)^V$ is the vertical lift of the vector field. Using the uniqueness decomposition property arising from transversality, we conclude there exists a unique vector $u^*(v_q) = (u_1^*(v_q), \cdots, u_m^*(v_q)) \in \mathbb{R}^m$ such that

$$\Gamma(v_q) = G(v_q) + u_a^*(v_q)(Y^a)^V_{v_q} \in T_{v_q}\mathcal{D},$$

where Γ and G are as in equation (2.23). If $\mathcal{D}$ is defined by m constraints of the form $\phi^b(v_q) = 0$, $1 \leq b \leq m$, then the condition above may be rewritten as

$$d\phi^b(G(v_q) + u_a^*(v_q)(Y^a)^V_{v_q}) = 0,$$

which is equivalent to

$$u_a^*(v_q)d\phi^b((Y^a)^V_{v_q}) = -d\phi^b(G(v_q)).$$

Note that, the equation above is a linear equation of the form $A(v_q)u = b(v_q)$, where $b(v_q)$ is the vector $(-d\phi^1(G(v_q)), \ldots, -d\phi^m(G(v_q))) \in \mathbb{R}^m$ and $A(v_q)$ is the $m \times m$ matrix with entries $A_a^b(v_q) = d\phi^b((Y^a)^V_{v_q}) = \mu^b(q)(Y^a)$, where the last equality may be deduced by computing the expressions in local coordinates. That is, if $(q^i, \dot{q}^i)$ are natural bundle coordinates for the tangent bundle, then

$$\begin{aligned} d\phi^b((Y^a)^V_{v_q}) &= \left(\frac{\partial \mu_i^b}{\partial q^j}\dot{q}^i dq^j + \mu_i^b d\dot{q}^i\right)\left(Y^{a,k}\frac{\partial}{\partial \dot{q}^k}\right) \\ &= \mu_i^b Y^{a,i} = \mu^b(q)(Y^a). \end{aligned}$$

In addition, $A(v_q)$ has full rank, since its columns are linearly independent. In fact suppose that

$$c_1 \begin{bmatrix} \mu^1(Y^1) \\ \vdots \\ \mu^m(Y^1) \end{bmatrix} + \cdots + c_m \begin{bmatrix} \mu^1(Y^m) \\ \vdots \\ \mu^m(Y^m) \end{bmatrix} = 0,$$

which is equivalent to

$$\begin{bmatrix} \mu^1(c_1Y^1 + \cdots + c_mY^m) \\ \vdots \\ \mu^m(c_1Y^1 + \cdots + c_mY^m) \end{bmatrix} = 0.$$

However, by transversality we have $\mathcal{D} \cap \mathcal{F} = \{0\}$ which implies that $c_1Y^1 + \cdots + c_mY^m = 0$. Since $\{Y_i\}$ are linearly independent we conclude that $c_1 = \cdots = c_m = 0$ and A has full rank. But, since A is an $m \times m$ matrix, and $\mathcal{D}$ is a regular distribution, it must be invertible. Therefore, there is a unique vector $u^*(v_q)$ satisfying the matrix equation and $u^* : \mathcal{D} \to \mathbb{R}^m$ is smooth since it is the solution of a matrix equation depending smoothly on v_q. □

Remark 3.1.9. Note that in Examples 3.1.4 and 3.1.5, the constraint distribution $\mathcal{D}$ and the control input distribution $\mathcal{F}$ are transversal. Thus the control laws obtained in there are unique by Theorem 3.1.8. ◇

The transversality condition is essential in order to have existence and uniqueness of the control law making the constraint distribution control invariant. If they are not transversal then a control law making $\mathcal{D}$ control invariant may not exist or may not be unique as the next examples show.

Example 3.1.10 (Non-existence)**.** Consider the Lagrangian function L and the distribution $\mathcal{D}$ given in Example 3.1.4, but now let the control force be

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = u(\cos\theta dx + \sin\theta dy),$$

so that the controlled Lagrangian system is now

$$m\ddot{x} = u\cos\theta, \quad m\ddot{y} = u\sin\theta, \quad I\ddot{\theta} = 0.$$

Note that, in this case, the control input distribution $\mathcal{F}$ is generated by the vector field

$$Y = \frac{\cos\theta}{m}\frac{\partial}{\partial x} + \frac{\sin\theta}{m}\frac{\partial}{\partial y}.$$

Hence, $\mathcal{F} \subseteq \mathcal{D}$. Suppose that a control law u^* making the distribution control invariant exists. Differentiating the constraints, we get

$$\cos\theta\dot{x} + \sin\theta\ddot{x} + \sin\theta\dot{y} - \cos\theta\ddot{y} = 0,$$

and substituting by the closed-loop system we get

$$0 = \cos\theta\dot{x} + \frac{u^*\sin\theta\cos\theta}{m} + \sin\theta\dot{y} - \frac{u^*\sin\theta\cos\theta}{m},$$

which is satisfied only when $\cos\theta\dot{x} + \sin\theta\dot{y} = 0$. Therefore, there is no control law u^* making the distribution control invariant. ◇

Example 3.1.11 (Non-uniqueness)**.** Consider again the situation given in Example 3.1.4 but now with the control force

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = u_1(\sin\theta dx - \cos\theta dy + d\theta) + u_2(\sin\theta dx - \cos\theta dy).$$

In this case, we have that $TQ = \mathcal{D} + \mathcal{F}$ but $\mathcal{D} \cap \mathcal{F} \neq \{0\}$. Two examples of control laws making $\mathcal{D}$ control invariant are

$$u_1^* = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y}), \quad u_2^* = 0$$

and

$$u_1^* = 0, \quad u_2^* = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y}).$$

◇

3.1.4 The induced constrained connection

In this subsection, we introduce a new connection that will help us describe the constraint dynamics of a mechanical system. Firstly, we give the definition of this connection and some basic properties, and next, we characterize the trajectories of the closed-loop system. In the remainder of this section, suppose that the distribution $\mathcal{D}$ describing the virtual nonholonomic constraints and the input distribution $\mathcal{F}$ given by the control force are transversal. Therefore, the projections $P_{\mathcal{F}} : TQ \to \mathcal{F}$ and $P_{\mathcal{D}} : TQ \to \mathcal{D}$ associated with the direct sum, $TQ = \mathcal{D} + \mathcal{F}$, are well defined.

In the same fashion as with the nonholonomic connection we define the **induced constrained connection** associated to the distribution $\mathcal{D}$ and the input distribution $\mathcal{F}$ which is given by

$$\overset{c}{\nabla}_X Y = \nabla^{\mathcal{G}}_X Y + (\nabla^{\mathcal{G}}_X P_{\mathcal{F}})(Y), \tag{3.6}$$

where $\nabla^{\mathcal{G}}$ is the Levi-Civita connection associated to the Riemannian metric $\mathcal{G}$. The induced constrained connection is a linear connection on Q with the special property that $\mathcal{D}$ is geodesically invariant for $\overset{c}{\nabla}$, i.e., if a geodesic of $\overset{c}{\nabla}$ starts on $\mathcal{D}$ then it stays in $\mathcal{D}$ for all time (see Lewis, 1998).

We have the following useful lemma that we will use later on.

Lemma 3.1.12. *If $X, Y \in \Gamma(\mathcal{D})$ then*

$$\overset{c}{\nabla}_X Y = P_{\mathcal{D}}(\nabla^{\mathcal{G}}_X Y).$$

Proof. If $X, Y \in \Gamma(\mathcal{D})$ we have that

$$\begin{aligned}\overset{c}{\nabla}_X Y =& \nabla^{\mathcal{G}}_X Y + (\nabla^{\mathcal{G}}_X P_{\mathcal{F}})(Y) \\ =& \nabla^{\mathcal{G}}_X Y + \nabla^{\mathcal{G}}_X (P_{\mathcal{F}}(Y)) - P_{\mathcal{F}}(\nabla^{\mathcal{G}}_X Y),\end{aligned}$$

where we have used the definition of covariant derivative of a map of the form $T : TQ \to TQ$ in the last equality. Noting that $P_{\mathcal{F}}(Y) = 0$ since Y is a section of $\pi_{\mathcal{D}}$, we conclude that $\overset{c}{\nabla}_X Y = P_{\mathcal{D}}(\nabla^{\mathcal{G}}_X Y)$. □

The last lemma implies in particular that $\overset{c}{\nabla}$ is well-defined as a connection on sections of $\pi_{\mathcal{D}}$ in the sense that the restriction $\overset{c}{\nabla}|_{\Gamma(\mathcal{D}) \times \Gamma(\mathcal{D})}$ also takes values in $\Gamma(\mathcal{D})$. Next, we give an alternative proof of a lemma that has also been shown at Lewis, 1998 to emphasize the fact that the constrained connection is not symmetric, in general.

Lemma 3.1.13. *If the constrained connection* $\overset{c}{\nabla}$ *is symmetric, then the constraint distribution* $\mathcal{D}$ *is integrable.*

Proof. The torsion of the constrained connection is given by

$$T^c(X,Y) = \overset{c}{\nabla}_X Y - \overset{c}{\nabla}_Y X - [X,Y].$$

Suppose that $X, Y \in \Gamma(\mathcal{D})$. In this case

$$\begin{aligned} T^c(X,Y) &= P_{\mathcal{D}}(\nabla^{\mathcal{G}}_X Y - \nabla^{\mathcal{G}}_Y X) - [X,Y] \\ &= P_{\mathcal{D}}([X,Y]) - [X,Y] \\ &= -P_{\mathcal{F}}([X,Y]), \end{aligned}$$

where we used the fact that $\nabla^{\mathcal{G}}$ is symmetric in the first equality. It is clear now that if $\overset{c}{\nabla}$ is symmetric then $[X,Y]$ must be a section of $\mathcal{D}$, which implies that $\mathcal{D}$ is integrable. □

In the following, we characterize the closed-loop dynamics as solutions of the mechanical system associated with the induced constrained connection.

Theorem 3.1.14. *A curve* $q : I \to Q$ *is a trajectory of the closed-loop system for the Lagrangian control system* (3.5) *making* $\mathcal{D}$ *invariant if and only if it satifies*

$$\overset{c}{\nabla}_{\dot{q}(t)}\dot{q}(t) + P_{\mathcal{D}}(grad_{\mathcal{G}} V(q(t))) = 0. \tag{3.7}$$

Proof. If $q : I \to Q$ is a trajectory of the closed-loop system for (3.5) with $\dot{q}(t) \in \mathcal{D}_{q(t)}$ then it satisfies

$$\nabla^{\mathcal{G}}_{\dot{q}(t)}\dot{q}(t) + \mathrm{grad}_{\mathcal{G}} V(q(t)) = u^*_a(t) Y^a(q(t)),$$

where $u^* : \mathcal{D} \to \mathbb{R}^m$ is the unique control law making $\mathcal{D}$ invariant. Attending to the fact that $\dot{q}(t) \in \mathcal{D}_{q(t)}$ we have that

$$\begin{aligned} \overset{c}{\nabla}_{\dot{q}(t)}\dot{q}(t) =& P_{\mathcal{D}}(\nabla^{\mathcal{G}}_{\dot{q}(t)}\dot{q}(t)) \\ =& - P_{\mathcal{D}}(\mathrm{grad}_{\mathcal{G}} V(q(t))) + P_{\mathcal{D}}(u^*_a(t) Y^a(q(t))) \\ =& - P_{\mathcal{D}}(\mathrm{grad}_{\mathcal{G}} V(q(t))), \end{aligned}$$

where we have used Lemma 3.1.12 in the first equality and $P_{\mathcal{D}}(Y^a) = 0$ in the last one.

Conversely, if the curve q satisfies (3.7), we have

$$P_{\mathcal{D}}(\nabla^{\mathcal{G}}_{\dot{q}(t)}\dot{q}(t) + \mathrm{grad}_{\mathcal{G}} V(q(t))) = 0,$$

where we used Lemma 3.1.12. Since $\ker P_{\mathcal{D}} = \mathcal{F}$, there exist $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$ such that

$$\nabla^{\mathcal{G}}_{\dot{q}(t)}\dot{q}(t) + \mathrm{grad}_{\mathcal{G}} V(q(t)) = u_a Y^a.$$

By Theorem 3.1.8, we conclude that $u = u^*$, since the control law making $\mathcal{D}$ invariant is unique. □

Remark 3.1.15. Suppose $\mathcal{D}$ is an integrable distribution and assume $\mathcal{M}$ is a maximal integrable manifold of $\mathcal{D}$. If $\overset{h}{\nabla}$ denotes the holonomic connection on $\mathcal{M}$ defined in Proposition 2.5.9 (see also Consolini and Costalunga, 2015, Consolini et al., 2018), as

$$\overset{h}{\nabla}_X Y = P_{\mathcal{M}}(\nabla^{\mathcal{G}}_X Y), \quad X, Y \in \mathfrak{X}(\mathcal{M}),$$

then Lemma 3.1.12 implies that the two connections are the same when $\overset{c}{\nabla}$ is restricted to vector fields on $\mathcal{M}$. ⋄

3.1.5 The constrained connection in coordinates

In this section we will compute the Christoffel symbols of the induced connection defined at the previous subsection. Given any coordinate chart (q^i) on Q the Christoffel symbols are determined by the values of the connection taken over the standard basis of the tangent space $\{\frac{\partial}{\partial q^1}, \cdots, \frac{\partial}{\partial q^n}\}$. It is not difficult to prove the following useful expression

$$\overset{c}{\nabla}_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = P_{\mathcal{D}}\left(\nabla^{\mathcal{G}}_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j}\right) + \nabla^{\mathcal{G}}_{\frac{\partial}{\partial q^i}}\left(P_{\mathcal{F}}\left(\frac{\partial}{\partial q^j}\right)\right).$$

Example 3.1.16. Consider once again the control system given in Example 3.1.4. The Levi-Civita connection $\nabla^{\mathcal{G}}$ associated with this system has vanishing Christoffel symbols. Considering the coordinates $q = (x, y, \theta)$ on $SE(2)$, we have that

$$\overset{c}{\nabla}_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \nabla^{\mathcal{G}}_{\frac{\partial}{\partial q^i}}\left(P_{\mathcal{F}}\left(\frac{\partial}{\partial q^j}\right)\right).$$

Note that the natural coordinate vector fields for $SE(2)$ may be decomposed in a unique way, under the direct sum $\mathcal{D} \oplus \mathcal{F}$, and this decomposition is given by

$$\begin{aligned}
\frac{\partial}{\partial x} &= \cos\theta X_1 - \frac{m\sin\theta}{I} X_2 + m\sin\theta Y, \\
\frac{\partial}{\partial y} &= \sin\theta X_1 + \frac{m\cos\theta}{I} X_2 - m\cos\theta Y, \\
\frac{\partial}{\partial \theta} &= X_2.
\end{aligned}$$

Hence, we obtain the following non-vanishing Christoffel symbols for the constrained connection $\overset{c}{\nabla}$

$$\begin{aligned}
\Gamma^x_{\theta x} &= 2\sin\theta\cos\theta, & \Gamma^x_{\theta y} &= \sin^2\theta - \cos^2\theta, \\
\Gamma^y_{\theta x} &= \sin^2\theta - \cos^2\theta, & \Gamma^y_{\theta y} &= -2\sin\theta\cos\theta, \\
\Gamma^\theta_{\theta x} &= \frac{m\cos\theta}{I}, & \Gamma^\theta_{\theta y} &= \frac{m\sin\theta}{I}.
\end{aligned}$$

If we introduce the coordinates $q = (x, y, \theta, \varphi)$ in Example 3.1.5 and following the same reasoning we get

$$\begin{aligned}
P_{\mathcal{F}}\left(\frac{\partial}{\partial x}\right) &= \frac{IJm + Jm^2\sin^2(\varphi)}{L(\varphi)}Y^1 - \frac{Jm^2\sin(\varphi)\cos(\varphi)}{L(\varphi)}Y^2\\
P_{\mathcal{F}}\left(\frac{\partial}{\partial y}\right) &= \frac{Im - Jm^2\sin(\varphi)\cos(\varphi)}{L(\varphi)}Y^1 + \frac{-Im + Jm^2\cos^2(\varphi)}{L(\varphi)}Y^2\\
P_{\mathcal{F}}\left(\frac{\partial}{\partial \theta}\right) &= \frac{-IJm\cos(\varphi) - Im\sin(\varphi)}{L(\varphi)}Y^1 + \frac{Im\sin(\varphi)}{L(\varphi)}Y^2\\
P_{\mathcal{F}}\left(\frac{\partial}{\partial \varphi}\right) &= \frac{-IJ - Jm\sin^2(\varphi)}{L(\varphi)}Y^1 + \frac{Jm\sin(\varphi)\cos(\varphi)}{L(\varphi)}Y^2,
\end{aligned}$$

with $L(\varphi) = -I + Jm\cos^2(\varphi) - m\sin^2(\varphi) + m\sin(\varphi)\cos(\varphi)$. Where, the non-vanishing Christoffel symbols are given by

$$\begin{aligned}
\Gamma^x_{\varphi x} =& \frac{2Jm\sin\varphi\cos\varphi}{L} - \frac{(IJ + Jm\sin^2\varphi)L'}{L^2},\\
\Gamma^y_{\varphi x} =& \frac{Jm(\sin^2\varphi - \cos^2\varphi)}{L} + \frac{Jm\sin\varphi\cos\varphi L'}{L^2},\\
\Gamma^\theta_{\varphi x} =& \frac{Jm\sin\varphi}{L} + \frac{Jm\cos\varphi L'}{L^2},\\
\Gamma^\varphi_{\varphi x} =& \frac{m^2(2\sin\varphi\cos\varphi + \sin^2\varphi - \cos^2\varphi)}{L} - \frac{m(I + m\sin^2\varphi - m\sin\varphi\cos\varphi)L'}{L^2},\\
\Gamma^x_{\varphi y} =& \frac{Jm(\sin^2(\phi) - \cos^2(\phi))}{L} + \frac{(I - Jm\sin(\phi)\cos(\phi))\,L'}{L^2},\\
\Gamma^y_{\varphi y} =& -\frac{2Jm\sin(\phi)\cos(\phi)}{L} + \frac{(-I + Jm\cos^2(\phi))\,L'}{L^2},\\
\Gamma^\theta_{\varphi y} =& \frac{2Jm^2\sin^2(\phi)\cos(\phi)}{IL} - \frac{(-Im + Jm^2\cos^2(\phi))\cos(\phi)}{IL} - \frac{(-Im + Jm^2\cos^2(\phi))\,L'\sin(\phi)}{IL^2}\\
&+ \frac{(Im - Jm^2\sin(\phi)\cos(\phi))\sin(\phi)}{IL} - \frac{(Im - Jm^2\sin(\phi)\cos(\phi))\,L'\cos(\phi)}{IL^2}\\
&- \frac{(Jm^2\sin^2(\phi) - Jm^2\cos^2(\phi))\cos(\phi)}{IL},\\
\Gamma^\varphi_{\varphi y} =& \frac{(Jm^2\cos^2(\phi) - Jm^2\sin(\phi)\cos(\phi))\,L'}{JL^2} + \frac{m^2(\sin^2(\phi) - \cos^2(\phi) - 2\sin(\phi)\cos(\phi))}{L},\\
\Gamma^x_{\varphi\theta} =& \frac{IJ\sin(\phi) - I\cos(\phi)}{L} + \frac{(-IJ\cos(\phi) - I\sin(\phi))\,L'}{L^2},\\
\Gamma^y_{\varphi\theta} =& \frac{I\cos(\phi)}{L} + \frac{IL'\sin(\phi)}{L^2},\\
\Gamma^\theta_{\varphi\theta} =& -\frac{(2 + 2J)m\sin(\phi)\cos(\phi)}{L} - \frac{mL'\sin^2(\phi)}{L^2} + \frac{m(\cos^2(\phi) - \sin^2(\phi))}{L}\\
&+ \frac{(Jm\cos(\phi) + m\sin(\phi))\,L'\cos(\phi)}{L^2},\\
\Gamma^\varphi_{\varphi\theta} =& \frac{Im\cos(\phi)}{JL} + \frac{ImL'\sin(\phi)}{JL^2} + \frac{IJm\sin(\phi) - Im\cos(\phi)}{JL} + \frac{(-IJm\cos(\phi) - Im\sin(\phi))\,L'}{JL^2},
\end{aligned}$$

$$\Gamma^x_{\varphi\varphi} = -\frac{2J\sin(\phi)\cos(\phi)}{L} + \frac{(-IJ - Jm\sin^2(\phi))\,L'}{mL^2},$$
$$\Gamma^y_{\varphi\varphi} = \frac{J(\cos^2(\phi) - \sin^2(\phi))}{L} + \frac{JL'\sin(\phi)\cos(\phi)}{L^2},$$
$$\Gamma^\theta_{\varphi\varphi} = \frac{Jm\sin^3(\phi)}{IL} - \frac{JmL'\sin^2(\phi)\cos(\phi)}{IL^2},$$
$$+ \frac{(-IJ - Jm\sin^2(\phi))\sin(\phi)}{IL} - \frac{(-IJ - Jm\sin^2(\phi))\,L'\cos(\phi)}{IL^2},$$
$$\Gamma^\varphi_{\varphi\varphi} = -\frac{m\sin^2(\phi)}{L} - \frac{2m\sin(\phi)\cos(\phi)}{L} + \frac{m\cos^2(\phi)}{L} + \frac{mL'\sin(\phi)\cos(\phi)}{L^2}$$
$$+ \frac{(-IJ - Jm\sin^2(\phi))\,L'}{JL^2}.$$

3.1.6 Existence of a nonholonomic Lagrangian structure for the dynamics on the constraint distribution

Here, we examine when it is feasible to obtain nonholonomic dynamics from virtual nonholonomic constraints; in other words, we want to characterize the dynamics of nonholonomic systems in terms of virtual nonholonomic constraints. A key factor here will be the relative position of the input distribution and the constraint distribution, as the next proposition shows. If the input distribution is orthogonal to the virtual nonholonomic constraint distribution then the constrained dynamics is precisely the nonholonomic dynamics with respect to the original Lagrangian function.

Proposition 3.1.17. *If the input distribution $\mathcal{F}$ is orthogonal to the virtual constraint distribution $\mathcal{D}$ with respect to the metric $\mathcal{G}$, then the trajectories of the constrained mechanical system* (3.7) *are the nonholonomic equations of motion.*

Proof. If $\mathcal{F} = \mathcal{D}^\perp$, then the projectors $P_{\mathcal{D}}$ and $\mathcal{P}$ coincide (as well as the projectors $P_{\mathcal{F}}$ and $\mathcal{Q}$). Thus, the constrained connection $\overset{c}{\nabla}$ is precisely the nonholonomic connection ∇^{nh}. This implies that the trajectories of the constrained connection are nonholonomic trajectories. □

Remark 3.1.18. The fact that $\mathcal{F} = \mathcal{D}^\perp$ is independent of the chosen metric. Once you fix the control force F and let the control input distribution be obtained using the musical isomorphism $\sharp$ as in Section 3.1.1, then $\mathcal{F}$ is orthogonal to $\mathcal{D}$ if and only if $F^a \in \mathcal{D}^o$, for $a = 1, \cdots, m$. ◇

Although the orthogonal condition $\mathcal{F} = \mathcal{D}^\perp$ is sufficient in order for the constrained dynamics to be the nonholonomic dynamics, it is not necessary as the following result shows.

Proposition 3.1.19. *Suppose there exists a modified potential function $\tilde{V}$ satisfying*

$$\mathcal{P}(grad_{\mathcal{G}}\tilde{V}) = P_{\mathcal{D}}(grad_{\mathcal{G}}V). \tag{3.8}$$

Then the nonholonomic trajectories with respect to $(\mathcal{G}, \tilde{V}, \mathcal{D})$ coincide with the constrained dynamics (3.7) *if and only if $\nabla^{\mathcal{G}}_X\mathcal{Q}(X) = \nabla^{\mathcal{G}}_X P_{\mathcal{F}}(X)$ for all $X \in \Gamma(\mathcal{D})$.*

Proof. It is not difficult to see that $\nabla^{\mathcal{G}}_X \mathcal{Q}(X) = \nabla^{\mathcal{G}}_X P_{\mathcal{F}}(X)$ if and only if the two connections satisfy $\overset{c}{\nabla}_X X = \nabla^{nh}_X X$. Therefore, the equation

$$\overset{c}{\nabla}_{\dot{q}(t)} \dot{q}(t) + P_{\mathcal{D}}(\text{grad}_{\mathcal{G}} V(q(t))) = 0$$

holds if and only if

$$\nabla^{nh}_{\dot{q}(t)} \dot{q}(t) + \mathcal{P}(\text{grad}_{\mathcal{G}} \tilde{V}(q(t))) = 0$$

also holds.

Conversely, if the trajectory $q(t)$ satisfies both equation, then

$$\nabla^{nh}_{\dot{q}(t)} \dot{q}(t) = \overset{c}{\nabla}_{\dot{q}(t)} \dot{q}(t)$$

is also satisfied. Using tensoriality of the difference tensor

$$D(X, Y) = \overset{c}{\nabla}_X Y - \nabla^{nh}_X Y,$$

we may evaluate D point-wise so that

$$D(X_q, X_q) = (\overset{c}{\nabla}_X X - \nabla^{nh}_X X)(q).$$

Choosing the trajectory $q(t)$ with initial point q and initial veclocity $X_q \in \mathcal{D}_q$, which is always possible thanks to the existence and uniqueness theorem for ODE, we deduce that $D(X_q, X_q) = 0$ for any $X_q \in \mathcal{D}_q$. Hence, $D(X, X) = 0$ which is equivalent to $\nabla^{\mathcal{G}}_X \mathcal{Q}(X) = \nabla^{\mathcal{G}}_X P_{\mathcal{F}}(X)$. □

In the absence of a potential function, i.e., $V = 0$, the nonholonomic trajectories coincide with the constrained dynamics if and only if $\nabla^{\mathcal{G}}_X \mathcal{Q}(X) = \nabla^{\mathcal{G}}_X P_{\mathcal{F}}(X)$ for any $X \in \Gamma(\mathcal{D})$.

Note that the previous characterization of when both dynamics have the same trajectories may be equivalently written as

$$\mathcal{P}(\nabla^{\mathcal{G}}_X X) = P_{\mathcal{D}}(\nabla^{\mathcal{G}}_X X) \text{ or } \mathcal{Q}(\nabla^{\mathcal{G}}_X X) = P_{\mathcal{F}}(\nabla^{\mathcal{G}}_X X)$$

for any $X \in \Gamma(\mathcal{D})$.

Corollary 3.1.20. If the geodesic vector field associated with $\nabla^{\mathcal{G}}$ is tangent to $\mathcal{D}$, then the nonholonomic trajectories coincide with the constrained geodesics and they are both the geodesics of $\nabla^{\mathcal{G}}$ with initial velocity in $\mathcal{D}$.

Proof. We just have to establish that the geodesic vector field associated with $\nabla^{\mathcal{G}}$ is tangent to $\mathcal{D}$ if and only if $\nabla^{\mathcal{G}}_X X \in \Gamma(\mathcal{D})$ for every $X \in \Gamma(\mathcal{D})$. Then this is equivalent to $\mathcal{Q}(\nabla^{\mathcal{G}}_X X) = 0$ and also to $P_{\mathcal{F}}(\nabla^{\mathcal{G}}_X X) = 0$. Hence, by the previous result, the geodesics with initial velocity in $\mathcal{D}$ of ∇^{nh} coincide with the geodesics with initial velocity in $\mathcal{D}$ of $\overset{c}{\nabla}$.

Now, $\nabla^{\mathcal{G}}_X X \in \Gamma(\mathcal{D})$ for every $X \in \Gamma(\mathcal{D})$ if and only if $\mathcal{D}$ is geodesically invariant with respect to $\nabla^{\mathcal{G}}$ (see Lewis, 1998, Theorem 5.4). Using standard results on differential geometry, $\mathcal{D}$ is geodesically invariant with respect to $\nabla^{\mathcal{G}}$ if and only if the geodesic vector field associated with $\nabla^{\mathcal{G}}$ is tangent to $\mathcal{D}$. □

Remark 3.1.21. One important feature of the theory of virtual holonomic constraints presented in Consolini et al., 2018 is that if the induced connection has the same trajectories as the Levi-Civita connection with respect to the induced metric on the constraint submanifold $\mathcal{M} \subseteq Q$, then the two connections are the same. However, its argument relies on the fact that the induced connection is symmetric. Therefore, the result does not follow in the nonholonomic case whenever the distribution is not integrable.

The next example illustrates Proposition 3.1.17.

Example 3.1.22. Consider the Chaplygin sleigh, a celebrated example of a nonholonomic mechanical system evolving on the configuration manifold $SE(2)$ with Lagrangian function as in Example 3.1.4 but now we consider the control force

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = u(\sin\theta dx - \cos\theta dy).$$

The corresponding controlled Lagrangian system is

$$m\ddot{x} = u\sin\theta, \quad m\ddot{y} = -u\cos\theta, \quad I\ddot{\theta} = 0.$$

The input distribution $\mathcal{F}$ is generated just by one vector field

$$Y = \frac{\sin\theta}{m}\frac{\partial}{\partial x} - \frac{\cos\theta}{m}\frac{\partial}{\partial y},$$

while the virtual nonholonomic constraint is the same distribution $\mathcal{D}$ as in Example 3.1.4. We may check that the control law

$$u^*(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y})$$

makes the distribution invariant under the closed-loop system. In addition, by Proposition 3.1.17 the resulting system is precisely the nonholonomic equation (2.18) for the Chaplygin system, since the input distribution spanned by Y is orthogonal to the virtual nonholonomic constraints. ◇

Remark 3.1.23. There are plenty of ways to impose a virtual nonholonomic constraint on a mechanical control system in order to obtain a nonholonomic system. In the last example, one could choose the control force to be

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u_1, u_2) = u_1\sin\theta dx + u_2\cos\theta dy$$

and the corresponding controlled Lagrangian system would be

$$m\ddot{x} = u_1\sin\theta, \quad m\ddot{y} = u_2\cos\theta, \quad I\ddot{\theta} = 0.$$

Then, the control law

$$u_1^*(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y}), \quad u_2^* = -u_1^*$$

makes the the closed-loop system coincide again with the nonholonomic equations for the Chaplygin system. Note that the input distribution is now generated by the vector fields

$$Y^1 = \frac{\sin\theta}{m}\frac{\partial}{\partial x} \quad \text{and} \quad Y^2 = \frac{\cos\theta}{m}\frac{\partial}{\partial y}.$$

Since they do not generate a transversal distribution to $\mathcal{D}$, we should not expect the control law to be unique. ◇

Remark 3.1.24. Under the conditions of Proposition 3.1.17, certain mechanical control systems may be driven to desired stable trajectories by imposing virtual nonholonomic constraints and using the proper control force.

For instance, for the mechanical control system that appears in Example 3.1.22, we may drive the system to an asymptotically stable trajectory characterized by $\dot{\theta} = 0$. In fact, defining the variables $v = \dot{x}\cos\theta + \dot{y}\sin\theta$ and $\omega = \dot{\theta}$, the Chaplygin sleigh equations of motion might be written as

$$\dot{\omega} = -\frac{ma}{I + ma^2} v\omega, \quad \dot{v} = a\omega^2,$$

for which the points with $\omega = 0$ are equilibria. Moreover, from a stability analysis, we deduce that the system exhibits asymptotic stability.

However, not every nonholonomic system exhibits asymptotically stable behavior. As discussed in e.g. Zenkov et al., 1998 one may have a stable (but not asymptotically stable) dynamics or a mix of stable and asymptotically stable dynamics. Therefore, the applicability of our method is closely related to the kind of trajectories that you wish to obtain. Thus, when a mechanical control system is given, satisfying the conditions of Proposition 3.1.17, we should first examine the qualitative properties of the associated nonholonomic system. Typical behavior includes asymptotic stability, periodic or quasi-periodic orbits, and conservation of first integrals such as the energy or the nonholonomic momentum. In a wide class of examples, virtual nonholonomic constraints enable us to use energy-momentum methods from Zenkov et al., 1998 to decide when it is possible to obtain stable or asymptotically stable trajectories. ◇

3.2 Virtual Affine Nonholonomic Constraints

In the present section, we extend the results obtained in the last section to the case of virtual affine nonholonomic constraints. The setup in control mechanical system is as before while the difference lays on the nature of the constraint equation.

3.2.1 Affine nonholonomic constraints

Consider systems that are subjected to **affine nonholonomic constraints**, i.e for each $q \in Q$ the velocities belong to an affine subspace $\mathcal{A}_q$ of the tangent space T_qQ. Thus, $\mathcal{A}_q$ can be written as a sum of a vector field $X \in \mathfrak{X}(Q)$ and a nonintegrable distribution $\mathcal{D}$ on Q, i.e. $\mathcal{A}_q = X(q) + \mathcal{D}_q$, where $\mathcal{D}$ is of constant rank r, with $1 < r < n$. In this case, we say that the affine space $\mathcal{A}_q$ is modelled on the vector subspace $\mathcal{D}_q$. In local coordinates $\mathcal{D}$ can be expressed as the null space of a q-dependent matrix $\mathcal{S}(q)$ of dimension $m \times n$ and rank$\mathcal{S}(q) = m$, with $m = n - r$ as $\mathcal{D}_q = \{\dot{q} \in T_qQ : \mathcal{S}(q)\dot{q} = 0\}$. The rows of $\mathcal{S}(q)$ can be represented by the coordinate functions of m independent 1-forms given at (7.6). The affine distribution is

$$\mathcal{A}_q = \{\dot{q} \in T_qQ : \mathcal{S}(q)(\dot{q} - X(q)) = 0\},$$

hence $\mathcal{A} = \{(q, \dot{q}) \in TQ : \Phi(q, \dot{q}) = 0\}$, with

$$\Phi(q, \dot{q}) = \mathcal{S}(q)\dot{q} + Z(q)$$

and $Z(q) = -\mathcal{S}(q)X(q) \in \mathbb{R}^m$. More information on affine constraints can be found, for instance, at Fassò and Sansonetto, 2015, Fassò et al., 2018.

Definition 3.2.1. A mechanical system with **affine nonholonomic constraints** on a smooth manifold Q is given by the triple $(\mathcal{G}, V, \mathcal{A})$, where $\mathcal{G}$ is a Riemannian metric on Q, representing the kinetic energy of the system, $V : Q \to \mathbb{R}$ is a smooth function representing the potential energy, and $\mathcal{A}$ is an affine distribution on Q describing the affine nonholonomic constraints.

3.2.2 Virtual affine nonholonomic constraints

Next, we present a detailed construction of virtual affine nonholonomic constraints. As will be clear from the definition of virtual affine nonholonomic constraints, given below, their existence is essentially linked to a controlled system, rather than to the affine distribution $\mathcal{A}$ defined by the constraints.

Definition 3.2.2. A **virtual affine nonholonomic constraint** associated with the mechanical control system (3.3) is a controlled invariant affine distribution $\mathcal{A} \subseteq TQ$ for that system, that is, there exists a control function $u^* : \mathcal{A} \to \mathbb{R}^m$ such that the solution of the closed-loop system satisfies $\psi_t(\mathcal{A}) \subseteq \mathcal{A}$, where $\psi_t : TQ \to TQ$ denotes its flow.

Before we proceed to the theorem which gives the necessary conditions for the existence and uniqueness of a control law that turns an affine distribution into a controlled invariant affine distribution (i.e. a virtual affine nonholonomic constraint), we present some necessary preliminaries.

Definition 3.2.3. If W is an affine subspace of the vector space V modeled on the vector subspace W_0, then the **dimension** of the affine subspace W is defined to be the dimension of the model vector subspace W_0.

Two affine subspaces W_1 and W_2 of a vector space V are **transversal** if

1. $V = W_1 + W_2$.
2. $\dim V = \dim W_1 + \dim W_2$, i.e., the dimensions of W_1 and W_2 are complementary with respect to the ambient space dimension.

Remark 3.2.4. If W_1 and W_2 are subspaces of V then the previous definition implies that $V = W_1 \oplus W_2$.

Remark 3.2.5. If W_1 and W_2 are affine spaces modeled on vector subspaces W_{10} and W_{20}, respectively, then W_1 and W_2 are transversal if and only if $V = W_{10} \oplus W_{20}$.

Proposition 3.2.6. *For two distributions $\mathcal{A}$ and $\mathcal{F}$ on a manifold Q where $\mathcal{A}$ is an affine distribution with $\mathcal{D}$ the associated model distribution and X a vector field on Q satisfying $\mathcal{A} = X + \mathcal{D}$, the transversality condition for $\mathcal{A}$ and $\mathcal{F}$ is an inherited property from the transversality of the model distribution $\mathcal{D}$ and vice versa, namely, $\mathcal{A}$ and $\mathcal{F}$ are transversal if and only if $\mathcal{D}$ and $\mathcal{F}$ are transversal.*

Proof. First, consider that $\mathcal{A}$ and $\mathcal{F}$ are transversal, which means that for every $q \in Q$ we have

$$T_qQ = \mathcal{A}_q + \mathcal{F}_q = X(q) + \mathcal{D}_q + \mathcal{F}_q;$$

hence, for every $v_q \in T_qQ$ there exist vectors $d_q \in \mathcal{D}_q$ and $f_q \in \mathcal{F}_q$ such that

$$v_q = X(q) + d_q + f_q \;\Leftrightarrow\; v_q - X(q) = d_q + f_q.$$

Since $v_q - X(q) \in T_qQ$ and v_q is arbitrary we have $T_qQ = \mathcal{D}_q + \mathcal{F}_q$ for every $q \in Q$. Together with the fact that $\dim \mathcal{D}_q + \dim \mathcal{F}_q = \dim T_qQ$, we have that $\mathcal{D}$ and $\mathcal{F}$ are transversal. Conversely, suppose that $\mathcal{D}$ and $\mathcal{F}$ are transversal. Note that this is the same as $TQ = \mathcal{D} \oplus \mathcal{F}$. Hence, as before, for $v_q \in T_qQ$ there are $d_q \in \mathcal{D}_q$ and $f_q \in \mathcal{F}_q$ such that

$$v_q = d_q + f_q \;\Leftrightarrow\; v_q + X(q) = X(q) + d_q + f_q.$$

By the same argument as above, $v_q + X(q) \in T_qQ$ and v_q is arbitrary. Thus, together with the dimension condition, we conclude that $\mathcal{A}$ and $\mathcal{F}$ are transversal. □

By the next proposition, we lift the transversality condition of two distributions to the tangent space of TQ.

Proposition 3.2.7. *Consider two distributions $\mathcal{A}$ and $\mathcal{F}$ where the first is an affine distribution as defined previously at Proposition 3.2.6 i.e. $\mathcal{A} = X + \mathcal{D}$, with $X \in \mathfrak{X}(Q)$ and $\mathcal{D}$ its associated model distribution. For $v_q \in \mathcal{A}$ we have that if $\mathcal{A}$ and $\mathcal{F}$ are transversal, then*

$$T_{v_q}(TQ) = T_{v_q}\mathcal{A} \oplus \mathcal{F}^V_{v_q},$$

where $\mathcal{F}^V_{v_q}$ is the vertical lift of $\mathcal{F}_{v_q}$.

Proof. From the structure of $\mathcal{A}$, i.e., from the fact that each $v_q \in \mathcal{A}_q$ can be written as $v_q = Z(q) + d_q$ where $d_q \in \mathcal{D}_q$, we may conclude that $\mathcal{A}_q$ is a r-dimensional manifold, where r is the rank of the distribution $\mathcal{D}$. Thus $\mathcal{A}$ is a fiber bundle whose base space is the n dimensional manifold Q and whose fibers are r dimensional affine subspaces. Hence,

$$\dim(T_{v_q}\mathcal{A}) = \dim(T_{d_q}\mathcal{D}) = n + r$$

and since $\dim \mathcal{F}^V_{v_q} = n - r = m$, we have that

$$\dim T_{v_q}(TQ) = \dim T_{v_q}\mathcal{A} + \dim \mathcal{F}^V_{v_q} = n + r + m = 2n.$$

So, in order to prove that both subspaces are transversal it suffices to prove that their intersection contains only the zero tangent vector. In fact, suppose that $v_q \in \mathcal{A}$ and $X_{v_q} \in T_{v_q}\mathcal{A}$. Since $\mathcal{A}$ is defined to be the set of vectors satisfying the equation $\Phi = 0$, the tangent vector satisfies $T_{v_q}\Phi(X_{v_q}) = 0$. If, in addition, $X_{v_q} \in \mathcal{F}^V_{v_q}$, then it can be written as

$$X_{v_q} = c^i \sharp(f_i)^V_{v_q}.$$

However,

$$T_{v_q}\Phi(\sharp(f_i)^V_{v_q}) = (S(q)\sharp(f_i))^V_{v_q}$$

from whence it follows that if $c^i\sharp(f_i)^V_{v_q}$ was in the null space of the linear map $T_{v_q}\Phi$, then $c^i\sharp(f_i)$ would be in the null space of $S(q)$ which is false, since these are vectors in $\mathcal{D}_q$ and $\mathcal{F}$ and $\mathcal{D}$ are transversal using the previous proposition. Thus, $X_{v_q} = 0$. □

The next theorem gives a necessary and sufficient condition for the existence and uniqueness of a feedback control that turns the affine constraint distribution into a virtual affine nonholonomic constraint.

Theorem 3.2.8. *If the affine distribution $\mathcal{A}$ and the control input distribution $\mathcal{F}$ are transversal, then there exists a unique control function making the distribution a virtual affine nonholonomic constraint associated with the mechanical control system* (3.3)*.*

Proof. Suppose that $\mathcal{A}$ and $\mathcal{F}$ are transversal and that trajectories of the control system (3.3) may be written as the integral curves of the SODE vector field Γ defined by (2.23). From Proposition 3.2.7 we have

$$T_{v_q}(TQ) = T_{v_q}\mathcal{A} \oplus \mathcal{F}^V_{v_q},$$

where $v_q \in \mathcal{A}$ and $\mathcal{F}^V_{v_q} = \text{span}\{(Y^a)^V_{v_q}\}$. Using the uniqueness decomposition property arising from transversality, we conclude there exists a unique vector $u^*(v_q) = (u^*_1(v_q), \cdots, u^*_m(v_q)) \in \mathbb{R}^m$ such that $\Gamma(v_q) = G(v_q) + u^*_a(v_q)(Y^a)^V_{v_q} \in T_{v_q}\mathcal{A}$. Next, we show that Γ depends smoothly on v_q. If $\mathcal{A}$ is defined by m constraints of the form $\phi^b(v_q) = 0, \ 1 \leq b \leq m$, then the condition above may be rewritten as $d\phi^b(G(v_q) + u^*_a(v_q)(Y^a)^V_{v_q}) = 0$, which is equivalent to

$$u^*_a(v_q)d\phi^b((Y^a)^V_{v_q}) = -d\phi^b(G(v_q)).$$

Note that, the equation above is a linear equation of the form $P(v_q)u = b(v_q)$, where $b(v_q)$ is the vector $(-d\phi^1(\Gamma(v_q)), \dots, -d\phi^m(\Gamma(v_q))) \in \mathbb{R}^m$ and $P(v_q)$ is the $m \times m$ matrix with entries $P^b_a(v_q) = d\phi^b((Y^a)^V_{v_q}) = \mu^b(q)(Y^a)$, where the last equality may be deduced by computing the expressions in local coordinates. That is, if $(q^i, \dot{q}^i)$ are natural bundle coordinates for the tangent bundle, then

$$\begin{aligned} d\phi^b((Y^a)^V_{v_q}) &= \left(\frac{\partial \mu^b_i}{\partial q^j}\dot{q}^i dq^j + \frac{\partial Z_i}{\partial q^j}dq^j + \mu^b_i d\dot{q}^i\right)\left(Y^{a,k}\frac{\partial}{\partial \dot{q}^k}\right) \\ &= \mu^b_i Y^{a,i} = \mu^b(q)(Y^a). \end{aligned}$$

In addition, $P(v_q)$ has full rank, since its columns are linearly independent. In fact suppose that

$$c_1 \begin{bmatrix} \mu^1(Y^1) \\ \vdots \\ \mu^m(Y^1) \end{bmatrix} + \cdots + c_m \begin{bmatrix} \mu^1(Y^m) \\ \vdots \\ \mu^m(Y^m) \end{bmatrix} = 0,$$

which is equivalent to

$$\begin{bmatrix} \mu^1(c_1Y^1 + \cdots + c_mY^m) \\ \vdots \\ \mu^m(c_1Y^1 + \cdots + c_mY^m) \end{bmatrix}.$$

However, from Proposition 3.2.6 we have $\mathcal{D} \cap \mathcal{F} = \{0\}$ which implies $c_1Y^1 + \cdots + c_mY^m = 0$. Since $\{Y_i\}$ are linearly independent we conclude that $c_1 = \cdots = c_m = 0$ and P has full rank. However, since P is an $m \times m$ matrix, and $\mathcal{D}$ is a regular distribution, it must be invertible.

Therefore, there is a unique vector $u^*(v_q)$ that satisfies the matrix equation and $u^* : \mathcal{D} \to \mathbb{R}^m$ is smooth since it is the solution of a matrix equation that depends smoothly on v_q. Hence, Γ is a smooth vector field tangent to $\mathcal{A}$ and its flow remains in $\mathcal{A}$. □

3.2.3 An example

Here we illustrate with an example how an affine distribution prescribed by the constraints of a mechanical system becomes a virtual affine nonholonomic constraint by a feedback control. The uniqueness and existence are guaranteed by Theorem 3.2.8.

Consider a boat with a payload on the sea with a position-dependent stream. The position of the boat's center of mass is modeled by the configuration manifold $\mathbb{R}^2$ to which we add an orientation to obtain a complete description of its location in space, so that the system total configuration manifold is $\mathbb{R}^2 \times \mathbb{S}$ with local coordinates $q = (x, y, \theta)$. The sea's current is modeled by the vector field $C : \mathbb{R}^2 \to \mathbb{R}^2$, $C = (C^1(x, y), C^2(x, y))$.

The boat is well modeled by a forced mechanical system with Lagrangian function $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\dot{\theta}^2$, where m is the boat's mass, I is the moment of inertia, and the external force is denoted by $F^{ext} = W^1 dx + W^2 dy$ accounting for the action of the current on the center of mass of the boat and to which we add a control force $F = u(\sin\theta dx - \cos\theta dy + d\theta)$.

The functions W^1 and W^2 are defined according to

$$\begin{cases} W^1 &= m\ d\,(\sin^2\theta C^1 - \sin\theta\cos\theta C^2)\,(\dot{q}) \\ W^2 &= m\ d\,(-\sin\theta\cos\theta C^1 + \cos^2\theta C^2)\,(\dot{q}). \end{cases},$$

where d represents the differential of the functions inside the parenthesis. The external force assures that in the absence of controls, the dynamics of the boat satisfies the following kinematic equations

$$\begin{cases} \dot{x} = & \sin^2\theta C^1 - \sin\theta\cos\theta C^2 \\ \dot{y} = & -\sin\theta\cos\theta C^1 + \cos^2\theta C^2, \end{cases}$$

whenever the initial velocities in the x and y direction vanish. The corresponding controlled forced Lagrangian system is

$$m\ddot{x} = u\sin\theta + W^1, \quad m\ddot{y} = -u\cos\theta + W^2, \quad I\ddot{\theta} = u,$$

and, as we will show, it has the following virtual affine nonholonomic constraint

$$\sin\theta\dot{x} - \cos\theta\dot{y} = C^2\cos\theta - C^1\sin\theta.$$

The input distribution $\mathcal{F}$ is generated just by one vector field

$$Y = \frac{\sin\theta}{m}\frac{\partial}{\partial x} - \frac{\cos\theta}{m}\frac{\partial}{\partial y} + \frac{1}{I}\frac{\partial}{\partial\theta},$$

while the virtual nonholonomic constraint is the affine space $\mathcal{A}$ modelled on the distribution $\mathcal{D}$ defined as the set of tangent vectors $v_q \in T_qQ$ where $\mu(q)(v) = 0$, with $\mu = \sin\theta dx - cos\theta dy$. Thus, we may write it as

$$\mathcal{D} = \operatorname{span}\left\{X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial\theta}\right\}.$$

The affine space is given as the zero set of the function $\Phi(q,v) = \mu(q)(v) + Z(q)$ with $Z(q) = \cos\theta C^2(x,y) - \sin\theta C^1(x,y)$ or, equivalently, as the set of vectors v_q satisfying $v_q - C(q) \in \mathcal{D}_q$.

We may check that $\mathcal{A}$ is controlled invariant for the controlled Lagrangian system above. In fact, the control law $u^*(x,y,\theta,\dot{x},\dot{y},\dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y})$ makes the affine space invariant under the closed-loop system, since in this case, the dynamical vector field arising from the controlled Euler-Lagrange equations given by

$$\Gamma = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{\theta}\frac{\partial}{\partial \theta} + \left(\frac{u^*\sin\theta + W^1}{m}\right)\frac{\partial}{\partial \dot{x}} + \left(-\frac{u^*\cos\theta - W^2}{m}\right)\frac{\partial}{\partial \dot{y}} + \frac{u^*}{I}\frac{\partial}{\partial \dot{\theta}}$$

is tangent to $\mathcal{A}$. This is deduced from the fact that

$$\Gamma(\sin\theta\dot{x} - \cos\theta\dot{y} + \cos\theta C^2(x,y) - \sin\theta C^1(x,y)) = 0.$$

Chapter 4

Virtual nonholonomic constraints: Nonlinear case

We saw in the last chapter the definitions of virtual linear and affine nonholonomic constraints in mechanical systems and we presented sufficient and necessary condition for the existence and uniqueness of a feedback control function that makes them control invariant. The goal of this chapter is to address the nonlinear case and complete the coverage of all possible cases of nonholonomic constraints in mechanical systems on Riemannian manifolds. Namely, here we examine the nonholonomic mechanical systems subject to nonlinear constraints, define the virtual nonlinear nonholonomic constraints and finally extend Theorems 3.1.8 and 3.2.8 so that a respective control law is guaranteed.

4.1 Nonlinear nonholonomic mechanics

Consider a mechanical system described by a mechanical type Lagrangian $L(q,\dot{q}) = K(q,\dot{q}) - V(q)$, $q \in Q$. A nonlinear nonholonomic constraint on this mechanical system is a submanifold $\mathcal{M}$ of the tangent bundle TQ from which the velocity of the system is forced to stay in. Mathematically, the constraint may be written as the set of points where a function of the type $\Phi : TQ \to \mathbb{R}^m$ vanishes, where $m < n = \dim Q$. That is, $\mathcal{M} = \Phi^{-1}(\{0\})$. If every point in $\mathcal{M}$ is regular, i.e., the tangent map $T_p\Phi$ is surjective for every $p \in \mathcal{M}$, then $\mathcal{M}$ is a submanifold of TQ with dimension $2n - m$ by the regular level set theorem.

Let $\Phi = (\phi^1, \dots, \phi^m)$ denote the coordinate functions of the constraint Φ. The coordinate expression of the equations of motion of a system with nonholonomic constraints are called Chetaev's equations and they are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \lambda_a \frac{\partial \phi^a}{\partial \dot{q}}, \tag{4.1a}$$

$$\phi^a(q,\dot{q}) = 0, \tag{4.1b}$$

(see de León, 2011,Cendra, Ibort, de León, and Martín de Diego, 2004, Chetaev, 1932). The right hand side acts like a constraint force that forces the system to remain inside the constraint submanifold. From the physical perspective, these forces are characterized by doing no mechanical work on the system.

In the following, we will consider a mapping that to each point v_q on the submanifold $\mathcal{M}$ assigns a vector subspace of T_qQ. In differential geometry, such a map resembles a distribution on Q restricted to $\mathcal{M}$, but unlike a distribution, it also depends on the velocity. Thus we will call it a velocity-dependent distribution. From now on, let $S(v_q)$ be a subspace of T_qQ, with $v_q \in \mathcal{M}$, defined by

$$S(v_q) = \{X \in T_qQ \mid \left\langle \frac{\partial \phi^a}{\partial \dot{q}^i}(v_q)dq^i, X \right\rangle = 0,\ a = 1, \dots, m\}.$$

The subspaces $S(v_q)$ act as a linearization of the constraint submanifold $\mathcal{M}$ at each point v_q as the next remark describes.

Remark 4.1.1. If the constraint submanifold $\mathcal{M}$ was actually linear, i.e., the function ϕ depends linearly on the velocities, then we would have that $S(v_q) = \mathcal{M}$. This supports the interpretation of the vector subspaces $S(v_q)$ as a linearization of the submanifold $\mathcal{M}$.

Chetaev's equations may be written in Riemannian form using a geodesic-like equation as follows:

Theorem 4.1.2. *A curve $q : I \to Q$ is a solution of Chetaev's equations for a mechanical type Lagrangian with kinetic energy determined by a Riemannian metric $\mathcal{G}$ on Q and a potential function V if and only if $\Phi(q, \dot{q}) = 0$ and it satisfies the equation*

$$\nabla_{\dot{q}}\dot{q} + \textit{grad } V \in S(\dot{q})^{\perp}, \tag{4.2}$$

where $S(\dot{q})^{\perp}$ is the orthogonal velocity-dependent distribution to $S(\dot{q})$ with respect to the Riemannian metric $\mathcal{G}$, and ∇ is the corresponding Levi-Civita connection.

Proof. Suppose the Lagrangian L is determined by a Riemannian metric $\mathcal{G}$ on Q and a potential function V, so that its local expression is

$$L(q, \dot{q}) = \frac{1}{2}\mathcal{G}_{ij}\dot{q}^i\dot{q}^j - V(q).$$

Chetaev's equations consist of Euler-Lagrange equations plus a reaction force term responsible for enforcing the constraints. Expanding (4.1), we eventually get

$$\ddot{q}^i - \mathcal{G}^{ij}\left[\frac{1}{2}\frac{\partial \mathcal{G}_{lk}}{\partial q^j}\dot{q}^l\dot{q}^k - \frac{\partial \mathcal{G}_{lj}}{\partial q^k}\dot{q}^l\dot{q}^k - \frac{\partial V}{\partial q^j}\right] = \lambda_a \mathcal{G}^{ij}\frac{\partial \phi^a}{\partial \dot{q}^j},$$

where $\mathcal{G}^{ij}$ is the inverse matrix of $\mathcal{G}_{ij}$. The left-hand side can be recognized to be the coordinate expression of the vector field $\nabla_{\dot{q}}\dot{q} + \text{grad}V$ (see Bullo and Lewis, 2005 for details).

Considering the coordinate expression of the sharp map associated to the Riemannian metric, $\sharp_{\mathcal{G}}$, it is easy to deduce that $\sharp_{\mathcal{G}}(\frac{\partial \phi^a}{\partial \dot{q}^i}dq^i) = \mathcal{G}^{ij}\frac{\partial \phi^a}{\partial \dot{q}^j}\frac{\partial}{\partial q^i}$. So, the right-hand side is the coordinate expression of the vector fields $\sharp_{\mathcal{G}}(\frac{\partial \phi^a}{\partial \dot{q}^i}dq^i)$. Moreover, using the fact that the vector field $\sharp_{\mathcal{G}}(\alpha)$

is characterized by $\mathcal{G}(\sharp_{\mathcal{G}}(\alpha), X) = \langle \alpha, X \rangle$, for any $X \in \mathfrak{X}(Q)$, we conclude that $\sharp_{\mathcal{G}}(\frac{\partial \phi^a}{\partial \dot{q}^i} dq^i)$ are the vectors spanning the orthogonal space $S(\dot{q})^{\perp}$ to $S(\dot{q})$.

Thus, we deduce that

$$\nabla_{\dot{q}}\dot{q} + \text{grad}V = \lambda_a \sharp_{\mathcal{G}} \left(\frac{\partial \phi^a}{\partial \dot{q}^i} dq^i \right),$$

which implies the statement of the theorem. □

Remark 4.1.3. Equation (4.2) has the same interpretation as its Lagrangian counterpart. On the left hand-side we have the covariant acceleration and a conservative force term coming from a potential function V. On the right-hand side, a constraint force appears responsible for making the system stay on the constraint submanifold $\mathcal{M}$. Moreover, in equation (4.2) we can observe that this force should be a linear combination of vector fields that are orthogonal to the subspace $S(v_q)$ previously defined.

4.2 Virtual nonlinear nonholonomic constraints

Consider the control and external forces as they were presented in Chapter 3. In brief, we have an external force $F^0 : TQ \to T^*Q$ and a control force $F : TQ \times U \to T^*Q$ of the usual form

$$F(q, \dot{q}, u) = \sum_{a=1}^{m} u_a F^a(q, \dot{q})$$

with the set of controls $F^a(q, \dot{q}) \in T^*Q$ ($m < n$, $U \subset \mathbb{R}^m$) and the control inputs $u_a \in \mathbb{R}$ ($1 \leq a \leq m$). Also, consider the associated mechanical control system (3.3), repeated here for convenience

$$\nabla_{\dot{q}}\dot{q} = Y^0(q, \dot{q}) + u_a Y^a(q, \dot{q}), \tag{4.3}$$

with $Y^0(q, \dot{q}) = \sharp_{\mathcal{G}}(F^0(q, \dot{q}))$ and $Y^a = \sharp_{\mathcal{G}}(F^a(q, \dot{q}))$.

Definition 4.2.1. A **virtual nonholonomic constraint** associated with the mechanical control system (4.3) is a controlled invariant submanifold $\mathcal{M} \subseteq TQ$ for that system, that is, there exists a control function $u^* : \mathcal{M} \to \mathbb{R}^m$ such that the solution of the closed-loop system satisfies $\psi_t(\mathcal{M}) \subseteq \mathcal{M}$, where $\psi_t : TQ \to TQ$ denotes its flow.

The next theorem guarantees the existence and uniqueness of a control law making the constraint submanifold $\mathcal{M}$ control invariant.

Theorem 4.2.2. *If the velocity-dependent distribution, $S(v_q)$, is transversal to the control input distribution $\mathcal{F}$ and $T_{v_q}\mathcal{M} \cap \mathcal{F}^V = \{0\}$, then there exists a unique smooth control function making $\mathcal{M}$ a virtual nonholonomic constraint associated with the mechanical control system* (4.3).

Proof. Under the hypothesis of the theorem, we have that $TTQ = T\mathcal{M} \oplus \mathcal{F}^V$ and that trajectories of the control system (4.3) may be written as the integral curves of the SODE vector field Γ defined by (2.23). For each $v_q \in \mathcal{M}_q$, we have that

$$\Gamma(v_q) \in T_{v_q}(TQ) = T_{v_q}\mathcal{M} \oplus \text{span}\left\{ (Y^a)^V_{v_q} \right\},$$

with $Y^a = \sharp(F^a)$. Using the uniqueness decomposition property arising from transversality, we conclude there exists a unique vector $u^*(v_q) = (u_1^*(v_q), \cdots, u_m^*(v_q)) \in \mathbb{R}^m$ such that

$$\Gamma(v_q) = G(v_q) + u_a^*(v_q)(Y^a)^V_{v_q} \in T_{v_q}\mathcal{M}.$$

If $\mathcal{M}$ is defined by m constraints of the form $\phi^b(v_q) = 0$, $1 \leq b \leq m$, then the condition above may be rewritten as

$$d\phi^b(G(v_q) + u_a^*(v_q)(Y^a)^V_{v_q}) = 0,$$

which is equivalent to $u_a^*(v_q)d\phi^b((Y^a)^V_{v_q}) = -d\phi^b(G(v_q))$. Note that, the equation above is a linear equation of the form $A(v_q)u = b(v_q)$, where $b(v_q)$ is the vector $(-d\phi^1(G(v_q)), \ldots, -d\phi^m(G(v_q))) \in \mathbb{R}^m$ and $A(v_q)$ is the $m \times m$ matrix with entries $A_a^b(v_q) = d\phi^b((Y^a)^V_{v_q}) = \frac{\partial \phi^b}{\partial \dot{q}}(q, \dot{q})(Y^a)$, where the last equality may be deduced by computing the expressions in local coordinates. That is, if $(q^i, \dot{q}^i)$ are natural bundle coordinates for the tangent bundle, then

$$\begin{aligned} d\phi^b((Y^a)^V_{v_q}) &= \left(\frac{\partial \phi^b}{\partial q^j}dq^j + \frac{\partial \phi^b}{\partial \dot{q}^i}d\dot{q}^i\right)\left(Y^{a,k}\frac{\partial}{\partial \dot{q}^k}\right) \\ &= \frac{\partial \phi^b}{\partial \dot{q}^i}Y^{a,i} = \frac{\partial \phi^b}{\partial \dot{q}}(q, \dot{q})(Y^a). \end{aligned}$$

In addition, $A(v_q)$ has full rank, since its columns are linearly independent. In fact suppose that

$$c_1 \begin{bmatrix} \frac{\partial \phi^1}{\partial \dot{q}}(Y^1) \\ \vdots \\ \frac{\partial \phi^m}{\partial \dot{q}}(Y^1) \end{bmatrix} + \cdots + c_m \begin{bmatrix} \frac{\partial \phi^1}{\partial \dot{q}}(Y^m) \\ \vdots \\ \frac{\partial \phi^m}{\partial \dot{q}}(Y^m) \end{bmatrix} = 0,$$

which is equivalent to

$$\begin{bmatrix} \frac{\partial \phi^1}{\partial \dot{q}}(c_1Y^1 + \cdots + c_mY^m) \\ \vdots \\ \frac{\partial \phi^m}{\partial \dot{q}}(c_1Y^1 + \cdots + c_mY^m) \end{bmatrix} = 0.$$

Moreover, by transversality we have $T_{v_q}\mathcal{M} \cap \mathcal{F}^V = \{0\}$ which implies that $c_1Y^1 + \cdots + c_mY^m = 0$. Since $\{Y_i\}$ are linearly independent we conclude that $c_1 = \cdots = c_m = 0$ and A has full rank. But, since A is an $m \times m$ matrix, and $\mathcal{M}$ is a constrained submanifold, it must be invertible. Therefore, there is a unique vector $u^*(v_q)$ satisfying the matrix equation and $u^* : \mathcal{M} \to \mathbb{R}^m$ is smooth since it is the solution of a matrix equation depending smoothly on v_q. □

Remark 4.2.3. The transversality assumption appearing in the previous theorem has not appeared in the literature in the context of virtual holonomic constraints but it is equivalent to the assumption of (vector) relative degree $\{1, \ldots, 1\}$ appearing in the literature of zero dynamics manifolds (see Isidori, 2000) concerning control systems evolving in Euclidean spaces. It is simple to show that if $Y^a \in \mathcal{F}$ are the vectors spanning the input distribution, then the relative degree of Φ is $\{1, \ldots, 1\}$ if $\langle d\Phi(v_q), (Y^a)^V_{v_q} \rangle \neq 0$ for all a. This is equivalent to our transversality assumption. In this sense, Theorem 4.2.2 is a geometric generalization of Proposition 6.1.2. in Isidori, 2000 applied to simple mechanical control systems on a Riemannian manifold.

Now we introduce a lemma that will simplify the hypothesis used in the statement of Theorem 4.2.2. This simplification will be very helpful in particular examples.

Lemma 4.2.4. $T_{v_q}\mathcal{M} \cap \mathcal{F}^V = \{0\}$ *if and only if* $\mathcal{F} \cap S(v_q) = \{0\}$.

Proof. Suppose first that $T_{v_q}\mathcal{M} \cap \mathcal{F}^V = \{0\}$. Then if $X \in \mathcal{F}$, $\langle d\phi(v_q), X^V_{v_q} \rangle = 0$ only if $X = 0$ by the assumption. In coordinates this means that $\frac{\partial \phi}{\partial \dot{q}^i} X^i = 0$. By definition of $S(v_q)$ we deduce that X is in $S(v_q)$ only if $X = 0$. Reversing the argument, we also conclude that if $\mathcal{F} \cap S(v_q) = \{0\}$ then $T_{v_q}\mathcal{M} \cap \mathcal{F}^V = \{0\}$. □

Example 4.2.5. Consider a particle moving in three dimensional space and subject to the gravitational potential. Its configuration space is $Q = \mathbb{R}^3$ with $q = (x, y, z) \in Q$. The Lagrangian $L : TQ \to \mathbb{R}$, is given by

$$L(q, \dot{q}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz,$$

and we consider the constraint that is imposed by $\Phi(q, \dot{q}) = 0$ with

$$\Phi(q, \dot{q}) = a^2\left(\dot{x}^2 + \dot{y}^2\right) - \dot{z}^2.$$

Consider also the control force $F : TQ \times U \to T^*Q$

$$F(q, \dot{q}, u) = u\left(xdx + ydy + dz\right).$$

The controlled Euler-Lagrange equations are

$$m\ddot{x} = ux, \quad m\ddot{y} = uy, \quad m\ddot{z} = -gm + u.$$

The velocity-dependent distribution is given by $S(v_q) = \operatorname{span}\{X_1, X_2\}$, where

$$X_1 = \dot{z}\frac{\partial}{\partial y} + a^2\dot{y}\frac{\partial}{\partial z} \quad X_2 = \dot{z}\frac{\partial}{\partial x} + a^2\dot{x}\frac{\partial}{\partial z},$$

and the input distribution $\mathcal{F}$ is generated by the vector field $Y = \dfrac{x}{m}\dfrac{\partial}{\partial x} + \dfrac{y}{m}\dfrac{\partial}{\partial y} + \dfrac{1}{m}\dfrac{\partial}{\partial z}$. Since $\mathcal{F} \cap S(v_q) = \{0\}$ the unique control law that makes the constraint manifold invariant with the choice of the control force F is given by

$$u^* = -\frac{mg\dot{z}}{a^2x\dot{x} + a^2y\dot{y} - \dot{z}}.$$

Remark 4.2.6. In previous work, virtual nonholonomic constraints appeared under different definitions. At Moran-MacDonald, 2021 virtual nonholonomic constraints are presented in Hamiltonian formalism, namely, a virtual nonholonomic constraint is a set of the form

$$\tilde{\mathcal{M}} = \{(q, p) \in Q \times \mathbb{R}^n \mid \tilde{\Phi}(q, p) = 0\},$$

for which there exists a control law making it invariant under the flow of the closed-loop controlled Hamiltonian equations. This constraint might be rewritten using the cotangent bundle T^*Q and $\tilde{\Phi}$ might be seen as a function $\tilde{\Phi} : T^*Q \to \mathbb{R}^m$. In addition, $\tilde{\Phi}$ should satisfy rank $d\tilde{\Phi}(q,p) = m$ for all $(q,p) \in \tilde{\mathcal{M}}$.

When we deal with linear nonholonomic constraints our definition coincides with this definition for the particular case where the function $\tilde{\Phi}$ is linear on the fibers, i.e., a linear function on the momenta p_i. In order to see it, one should rewrite the virtual nonholonomic constraints and the control system on the cotangent bundle.

Indeed, consider the Hamiltonian function $H : T^*Q \to \mathbb{R}$ obtained from a Lagrangian function in the following way

$$H(q,p) = p\dot{q}(q,p) - L(q, \dot{q}(q,p)),$$

where $\dot{q}(q,p)$ is a function of (q,p) given by the inverse of the Legendre transformation

$$p = \frac{\partial L}{\partial \dot{q}}.$$

The controlled Hamiltonian equations are given by

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} + F^0(q, \dot{q}(q,p)) + u_a F^a(q),$$

where F^0 is an external force map. Now, any distribution $\mathcal{D} \subseteq TQ$ might be defined as the set

$$\mathcal{D} = \{(q, \dot{q}) \in TQ \mid \mu^a(q)(\dot{q}) = 0\},$$

where μ^a with $1 \leqslant a \leqslant m$ are m linearly independent one-forms. The cotangent version of the distribution is the set

$$\tilde{\mathcal{D}} = \{(q,p) \in T^*Q \mid \mu^a(q)(\dot{q}(q,p)) = 0\}.$$

Therefore, we set

$$\tilde{\Phi}(q,p) = (\mu^1(q)(\dot{q}(q,p)), \cdots, \mu^m(q)(\dot{q}(q,p))).$$

We just have to check if rank $d\tilde{\Phi} = m$. Note that each component of $\tilde{\Phi}$ is linear on fibers if the Lagrangian function (and thus, the corresponding Hamiltonian function) is of mechanical type, i.e., $L = \dot{q}^T M \dot{q} - V(q)$, where M is the mass matrix and it represents the Riemannian metric on coordinates, then the Legendre transform is just $p = M\dot{q}$ and its inverse is $\dot{q} = M^{-1}p$. Therefore,

$$\tilde{\Phi}(q,p) = (\mu^1 M^{-1} p, \cdots, \mu^m M^{-1} p).$$

Hence, the submatrix of the Jacobian formed by the partial derivatives with respect to the momenta p are formed by the rows

$$M^{-1}\mu^1, \cdots M^{-1}\mu^m,$$

which are linearly independent. Thus this submatrix has rank m and this implies that the Jacobian matrix $d\tilde{\Phi}$ has rank greater than m. However, since it is formed by m rows, the rank of $d\tilde{\Phi}$ must be exactly m and $\tilde{\mathcal{M}}$ is a virtual nonholonomic constraint according to Moran-MacDonald, 2021 if there is a control law making it invariant.

When we encounter nonlinear nonholonomic constraints instead of a distribution $\mathcal{D}$ we have a constraint submanifold $\mathcal{M} \subseteq TQ$. The submanifold $\mathcal{M}$ might be defined as the set $\mathcal{M} = \{(q, \dot{q}) \in TQ \mid \Phi(q, \dot{q}) = 0\}$, with $\Phi = (\phi^1, \ldots, \phi^m)$ and where $d\phi^a$ with $1 \leqslant a \leqslant m$ are m linearly independent constraints. The cotangent version of the constraint manifold is the set $\tilde{\mathcal{M}} = \{(q, p) \mid \Phi(q, \dot{q}(q, p)) = 0\}$. Therefore, we set $\tilde{\Phi}(q, p) = \Phi(q, \dot{q}(q, p)) = (\phi^1(q, \dot{q}(q, p)), \cdots, \phi^m(q, \dot{q}(q, p)))$.

4.3 Constraint dynamics in terms of the affine connection and Chetaev's equation

In the following, we characterize the closed-loop dynamics obtained using the unique control law derived from Theorem 4.2.2 in terms of the affine connection.

From now on, let us assume that the distributions $S(v_q)$ and $\mathcal{F}$ are transversal. Consider the velocity-dependent projection maps

$$\mathcal{P}(v_q) : T_qQ \to S(v_q) \text{ and } \mathcal{Q}(v_q) : T_qQ \to \mathcal{F},$$

for each $v_q \in \mathcal{M}$, related with the decompositions $TQ = S(v_q) \oplus \mathcal{F}$. Associated with these projections, we may define the generalized projection map

$$P : \mathcal{M} \times TQ \to TQ \text{ given by } P(v_q, w_q) = \mathcal{P}(v_q)(w_q).$$

Notice that if $\mathcal{M}$ were a linear constraint manifold then $P(v_q, \cdot)|_{\mathcal{M}} = Id|_{\mathcal{M}}$ since $\mathcal{M} = S(v_q)$ for any $v_q \in \mathcal{M}$. In this sense, $P(v_q, \cdot)|_{\mathcal{M}}$ measures the non-linearity of the constraint set $\mathcal{M}$ and $P(v_q, w_q)$ is the perturbation of w_q by the nonlinear constraints $\mathcal{M}$ at v_q.

Using the previous constructions, for each vector field $X, Y, Z \in \mathfrak{X}(Q)$ with $Z \in \mathcal{M}$ consider the generalized projection of the covariant derivative $\nabla_X Y$ at Z, i.e, the vector field $P(Z, \nabla_X Y)$, where ∇ denotes the Levi-Civita connection with respect to the Riemannian metric $\mathcal{G}$.

Definition 4.3.1. A **nonlinear constrained geodesic** on $\mathcal{M}$ associated with the generalized projection P and the Riemannian metric $\mathcal{G}$ is a trajectory $q(t)$ satisfying

$$P(\dot{q}, \nabla_{\dot{q}}\dot{q}) = 0, \text{ and } \dot{q}(t) \in \mathcal{M}$$

which is equivalent to the equations

$$\nabla_{\dot{q}}\dot{q} \in \mathcal{F}, \text{ and } \dot{q}(t) \in \mathcal{M}.$$

From Theorem 4.2.2, the constraint set $\mathcal{M}$ is invariant under the flow of the above equations.

Theorem 4.3.2. *Assume that the distributions $S(v_q)$ and $\mathcal{F}$ are transversal. A curve $q : I \to Q$ is a trajectory of the closed-loop system for the Lagrangian control system* (4.3) *making $\mathcal{M}$ invariant if and only if it satisfies*

$$P(\dot{q}, \nabla_{\dot{q}}\dot{q} + grad\ V) = 0, \quad \dot{q}(0) \in \mathcal{M}. \tag{4.4}$$

Proof. If $q : I \to Q$ is a trajectory of the closed-loop system for (4.3) with $\dot{q}(t) \in \mathcal{M}$ then it satisfies

$$\nabla_{\dot{q}(t)}\dot{q}(t) + \text{grad } V(q(t)) = u_a^*(t)Y^a(q(t)),$$

where $u^* : \mathcal{D} \to \mathbb{R}^m$ is the unique control law making $\mathcal{M}$ invariant. Attending to the fact that $\dot{q}(t) \in \mathcal{M}$ we have that

$$\begin{aligned} P(\dot{q}, \nabla_{\dot{q}}\dot{q}) =& \mathcal{P}(\dot{q})(\nabla_{\dot{q}(t)}\dot{q}(t)) \\ =& -\mathcal{P}(\dot{q})(\text{grad } V(q(t))) + \mathcal{P}(\dot{q})(u_a^*(t)Y^a(q(t))) \\ =& -\mathcal{P}(\dot{q})(\text{grad } V(q(t))) = -P(\dot{q}, \text{grad } V(q(t))), \end{aligned}$$

where we have used the fact that $\mathcal{P}(\dot{q})(Y^a) = 0$ in the last equality since $\ker \mathcal{P}(\dot{q}) = \mathcal{F}$.

Conversely, if the curve q satisfies (4.4), we equivalently have that

$$\mathcal{P}(\dot{q})(\nabla_{\dot{q}(t)}\dot{q}(t) + \text{grad } V(q(t))) = 0.$$

Since $\ker \mathcal{P}(\dot{q}) = \mathcal{F}$, there exist $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$ such that

$$\nabla_{\dot{q}(t)}\dot{q}(t) + \text{grad } V(q(t)) = u_a Y^a.$$

By Theorem 4.2.2, we conclude that $u = u^*$, since the control law making $\mathcal{M}$ invariant is unique. □

The next proposition shows that if the input distribution $\mathcal{F}$ coincides with the distribution $S(v_q)^\perp$ for all $v_q \in \mathcal{M}$ then the constrained dynamics is precisely the nonholonomic dynamics with respect to the original Lagrangian function.

Proposition 4.3.3. *If $\mathcal{F} = S^\perp$ then the trajectories of the feedback controlled mechanical system* (4.4) *are the nonholonomic equations of motion* (4.2).

Proof. From the geometric formulation of Chetaev's equations in (4.2) we have that if $S^\perp$ equals $\mathcal{F}$ then equation (4.4) becomes equation (4.2). □

Giving a general intrinsic description of the closed-loop system derived from Theorem 4.2.2 might be very hard due to the arbitrary non-linearity of the constraint set $\mathcal{M}$. In the following, we will give an intrinsic description for a particular case of single-input mechanical control systems.

Theorem 4.3.4. *Assume that the distributions $S(v_q)$ and $\mathcal{F}$ are transversal. Suppose $\Phi : TQ \to \mathbb{R}$ is the function $\Phi(q, \dot{q}) = B(\dot{q}, \dot{q})$ where B is a symmetric $(0, 2)$-tensor, and that the constraint is given by $\mathcal{M} = \Phi^{-1}(c)$. Then a curve q is a trajectory of the closed-loop system if and only if $\dot{q} \in \mathcal{M}$ and*

$$\nabla_{\dot{q}}\dot{q} = \left(\frac{B(\dot{q}, grad\ V)}{B(\dot{q}, Y)} - \frac{1}{2}\frac{(\nabla_{\dot{q}}B)(\dot{q}, \dot{q})}{B(\dot{q}, Y)} \right) Y - grad\ V, \tag{4.5}$$

where Y is the vector field generating $\mathcal{F}$.

Proof. Differentiating the equation $B(\dot{q},\dot{q}) = c$ we get

$$0 = \frac{d}{dt}B(\dot{q},\dot{q}) = (\nabla_{\dot{q}}B)(\dot{q},\dot{q}) + 2B(\dot{q},\nabla_{\dot{q}}\dot{q}).$$

Denoting by u^* the unique control law obtained in Theorem 4.2.2, the closed-loop system satisfies $\nabla_{\dot{q}}\dot{q} = u^*Y - \text{grad } V$. Therefore

$$(\nabla_{\dot{q}}B)(\dot{q},\dot{q}) + 2B(\dot{q}, u^*Y - \text{grad } V) = 0.$$

Using the linearity of the tensor B, we get that

$$u^* = \frac{B(\dot{q}, \text{grad } V)}{B(\dot{q}, Y)} - \frac{1}{2}\frac{(\nabla_{\dot{q}}B)(\dot{q},\dot{q})}{B(\dot{q},Y)}.$$

□

This particular case shows that for some choices of the tensor B appearing in the statement of Theorem 4.3.4, the term containing the control law cannot be incorporated in a (non-Levi Civita) affine connection on $\mathcal{M}$ since its form can have a non-quadratic dependence on the velocities. Hence, the non-linear case behaves differently from its linear counterpart, where this is possible (see Consolini et al., 2018 and Consolini and Costalunga, 2015) and the intrinsic description must be much more complex.

Example 4.3.5. Consider, as in Example 4.2.5, a particle moving in three dimensional space and subject to the gravitational potential, with the same Lagrangian $L : TQ \to \mathbb{R}$,

$$L(q,\dot{q}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

but consider now a constraint that makes the magnitude of the velocity constant zero, namely, $\Phi(q,\dot{q}) = 0$ with

$$\Phi = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - c = 0, \quad c > 0.$$

Thus B is the symmetric tensor represented by the identity matrix I, since $\Phi = \dot{q}^T\dot{q} = \dot{q}^T I\dot{q}$. The constraint manifold is given by $\mathcal{M} = \{(q,\dot{q}) \in TQ \ : \ \Phi(q,\dot{q}) = 0\}$ and consider the control force $F : TQ \times U \to T^*Q$

$$F(q,\dot{q},u) = u(\dot{x}dx + \dot{y}dy + \dot{z}dz).$$

The controlled Euler-Lagrange equations are

$$m\ddot{x} = u\dot{x}, \quad m\ddot{y} = u\dot{y}, \quad m\ddot{z} = -gm + u\dot{z}.$$

The input distribution, $\mathcal{F}$, is generated by the vector field $Y = \frac{\dot{x}}{m}\frac{\partial}{\partial x} + \frac{\dot{y}}{m}\frac{\partial}{\partial y} + \frac{\dot{z}}{m}\frac{\partial}{\partial z}$ and note that $\mathcal{F} \cap S(v_q) = \{0\}$. Applying equation (4.5), the unique control law that makes the constraint manifold $\mathcal{M}$ invariant is $u^* = \frac{mg\dot{z}}{c}$. Thus the closed loop system is

$$\ddot{q} = \frac{g\dot{z}}{c}\dot{q} - g(0,0,1)^T.$$

Remark 4.3.6. For a constraint set as in Theorem 4.3.4, the velocity dependent distribution $S(\dot{q})$ has the form $S(\dot{q}) = \{v \in T_qQ \mid \dot{q}Bv = 0\}$. Therefore $S(\dot{q})^\perp$ is a one-dimensional space generated by $B\dot{q}$. In the previous example, the input distribution, $\mathcal{F}$ coincides with $S(\dot{q})^\perp$ and from Proposition 4.3.3, the local expression of the equations (4.2) and (4.5) will be the same. Note also that, when there is no potential $V = 0$, the symmetric form B does not depend on the position q of the configuration manifold and the configuration manifold is $\mathbb{R}^n$. Then the geodesics of the Levi-Civita connection ∇ that start in $\mathcal{M}$, remain in $\mathcal{M}$ for all time.

4.4 Applications

The previous results on virtual nonlinear nonholonomic constraints can be used to enforce a desired relation between state variables through a control force whenever the interplay between forces and constraints satisfies our assumptions. In the following, we give various applications of how a desired constraint can be enforced.

4.4.1 Precessional motion of a rigid body about a fixed point

Consider a precessional motion of a rigid body about a fixed point as in Jarzębowska, 2002. Fix an inertial system XYZ and a fixed body system xyz. The motion is completely described by the Euler angles (ϕ, θ, ψ) where θ is the tilt angle between the rotational axis and the vertical axis, ϕ is the azimuth angle from the line of nodes (the intersection of the planes xy and XY) and ψ the angle of rotation about the z-axis. The configuration space is $Q = SO(3)$ parametrized by Euler angles and the angular velocity of the body is given by $\omega = (\omega_1, \omega_2, \omega_3)$ with

$$\begin{cases} \omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\ \omega_3 &= \dot{\phi}\cos\theta + \dot{\psi}, \end{cases}$$

and by assuming that the z-axis is the symmetry axis of the body, i.e. $I_1 = I_2 = I_{12}$ the Lagrangian $L : TQ \to \mathbb{R}$ is

$$\begin{aligned} L &= \frac{1}{2}I_{12}(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 - G(q) \\ &= \frac{1}{2}I_{12}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - G(q) \end{aligned}$$

where $q = (\phi, \theta, \psi)$, $G(q) = mg\rho\cos\theta$ is the potential energy due to gravity, m is the mass of the body and ρ is the distance from the pivot point to the center of mass.

Consider the control force $F : TQ \times \mathbb{R} \to T^*Q$ given by

$$F(q, \dot{q}, u) = u(f_1 d\phi + f_2 d\theta + f_3 d\psi),$$

which will be determined such that it makes the nonholonomic constraint below invariant. The input distribution $\mathcal{F}$ is defined as the span of the vector field $Y = (Kf_1 - 2\cos\theta K f_3)\frac{\partial}{\partial\phi} + \frac{f_2}{I_{12}}\frac{\partial}{\partial\theta} + \frac{f_3}{I_3}\frac{\partial}{\partial\psi}$, where $K = (I_{12}\sin^2\theta + I_3\cos^2\theta)^{-1}$.

The Euler-Lagrange equations are

$$\begin{aligned} I_{12}\ddot{\phi}\sin^2\theta + \dot{\phi}\dot{\theta}\sin\theta\cos\theta(2I_{12}-I_3) - \dot{\psi}\dot{\theta}\cos\theta &= u(f_1 - f_3\cos\theta), \\ I_{12}\ddot{\theta} + \dot{\phi}^2\sin\theta\cos\theta(I_3 - I_{12}) + I_3\dot{\phi}\dot{\psi}\sin\theta + mg\rho\sin\theta &= uf_2, \\ I_{12}\ddot{\psi}\sin^2\theta - I_{12}\dot{\phi}\dot{\theta}\sin^3\theta - \dot{\phi}\dot{\theta}\sin\theta\cos^2\theta(2I_{12}-I_3) + I_3\dot{\psi}\dot{\theta}\cos^2\theta &= uf_3I_{12}I_3^{-1}\sin^2\theta \\ &\quad - u\cos\theta(f_1 - f_3\cos\theta). \end{aligned}$$

For a spinning body to perform precession the configuration variables should satisfy $\Phi(q,\dot{q}) = 0$ where

$$\Phi(q,\dot{q}) \;=\; \ddot{\phi}\dot{\theta}\sin\theta \;-\; \ddot{\theta}\dot{\phi}\sin\theta \;+\; 2\dot{\phi}\dot{\theta}^2\cos\theta \;+\; \dot{\phi}^3\sin^2\theta\cos\theta \;-\; \cot\theta\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right)^{3/2}.$$

When $\dot{\phi} = \text{const}$, $\dot{\psi} = \text{const}$ and $\dot{\theta} = 0$ we have a regular precession, while for a pseudo-regular precession (i.e. nutation) we need $\dot{\theta} \neq 0$ and $\dot{\phi}, \dot{\psi}$ to be arbitrary (for more details see Jarzębowska, 2002, Grioli, 2011 equation (118)). We consider the first case here, for ease of calculating. Hence $\Phi(q,\dot{q})$ reduces to

$$\Phi(q,\dot{q}) = a\dot{\phi}^3 - a\dot{\psi}^3,$$

where $a = \sin^2\theta\cos\theta$ and we consider for simplicity $\dot{\psi}\sin\theta > 0$. The constraint manifold is $\mathcal{M} = \{(q,\dot{q}) \in TQ \;:\; \Phi(q,\dot{q}) = 0\}$ and the velocity-dependent distribution is given by $S(v_q) = \text{span}\{X_1, X_2\}$, with $X_1 = \dfrac{\partial}{\partial\dot{\theta}}$ and $X_2 = \dot{\psi}^2\dfrac{\partial}{\partial\dot{\phi}} + \dot{\phi}\dfrac{\partial}{\partial\dot{\psi}}$.

The control law u^* that makes the constraint manifold invariant is obtained by the transversality condition and is given by

$$u^* = \frac{I\dot{\phi}\dot{\theta}s\theta c\theta\left(\dot{\psi}c\theta - \dot{\phi}\right) + \dot{\phi}\dot{\theta}\dot{\psi}\left(\dot{\phi}c\theta + I_{12}\dot{\psi}s^3\theta\right) - I_3\dot{\psi}^3\dot{\theta}c^2\theta}{(f_3c\theta - f_1)(\dot{\phi}^2 + \dot{\psi}c\theta) - \dot{\psi}f_3I_{12}I_3^{-1}s^2\theta},$$

where $s\theta$ and $c\theta$ stand for $\sin\theta$ and $\cos\theta$ respectively and $I = 2I_{12} - I_3$. In the particular case where $f_1 = 1$, $f_3 = 0$ and f_2 is an arbitrary function (the choice of the control force is such that every time imposes a specific behavior for the system), such that $\mathcal{F} \cap S(v_q) = \{0\}$ hence the hypothesis of Theorem 4.2.2 is satisfied, we obtain the unique control law u^* that makes the constraint submanifold invariant under the flow of the closed-loop system.

4.4.2 The double pendulum

Consider the double pendulum that resembles the gymnastic acrobot but the actuator is at the shoulder rather that at the elbow. Following Moran-MacDonald, 2021 and Moran-MacDonald et al., 2024 we consider the configuration manifold of the system $Q = \mathbb{S}^1 \times \mathbb{S}^1$ with $q = (q_1, q_2) \in Q$, where q_1 represents the angle for the shoulder and q_2 the angle for the elbow, see Figure 4.1.

The Lagrangian $L : TQ \to \mathbb{R}$, is given by

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q),$$

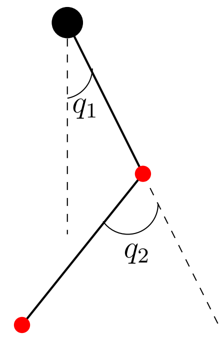


Figure 4.1: Double Pendulum actuated at the shoulder.

where

$$D(q) = \begin{bmatrix} ml^2(3+2\cos q_2) & ml^2(1+\cos q_2) \\ ml^2(1+\cos q_2) & ml^2 \end{bmatrix}$$
$$V(q) = -mgl\left(2\cos q_1 + \cos\left(q_1+q_2\right)\right)$$

are the inertia matrix and the potential energy, respectively. For simplicity we set $m = l = 1$. The constraint $\Phi : TQ \to \mathbb{R}$ is given by the equation

$$\Phi(q, \dot{q}) = q_2 - \arctan\left[(3+2\cos q_2)\dot{q}_1 + (1+\cos q_2)\dot{q}_2\right]$$

and the control force $F : TQ \times \mathbb{R} \to T^*Q$ is given by

$$F(q, \dot{q}, u) = u dq_1.$$

The controlled Euler-Lagrange equations are

$$D(q)\ddot{q} + P(q, \dot{q}) = B,$$

with

$$P(q, \dot{q}) = \begin{bmatrix} -2s_2\dot{q}_1\dot{q}_2 - s_2(\dot{q}_2)^2 + g(2s_1 + s_{12}) \\ -s_2\dot{q}_1\dot{q}_2 + gs_{12} \end{bmatrix}, B = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

where, as shorthand, we write $s_1 = \sin q_1, s_2 = \sin q_2$ and $s_{12} = \sin(q_1 + q_2)$. The constraint manifold is $\mathcal{M} = \{(q, \dot{q}) \in TQ \ : \ \Phi(q, \dot{q}) = 0\}$ and its tangent space, at every point $(q, \dot{q}) \in \mathcal{M}$, is given by $T_{(q,\dot{q})}\mathcal{M} = \{v \in TTQ \ : \ d\Phi(v) = 0\} = \text{span}\{X_1, X_2, X_3\}$, with

$$X_1 = \frac{\partial}{\partial q_1}, \quad X_2 = (1+\cos q_2)\frac{\partial}{\partial \dot{q}_1} - (3+2\cos q_2)\frac{\partial}{\partial \dot{q}_2},$$

$$X_3 = (3+2\cos q_2)\frac{\partial}{\partial q_2} + \left[A + \sin q_2(2\dot{q}_1 + \dot{q}_2)\right]\frac{\partial}{\partial \dot{q}_1},$$

where $A = 1 + \left[(3+2\cos q_2)\dot{q}_1 + (1+\cos q_2)\dot{q}_2\right]^2$. The input distribution $\mathcal{F}$ is generated by the vector field

$$Y = \frac{\partial}{\partial q_1} - (1+\cos q_2)\frac{\partial}{\partial q_2}.$$

Note here that the vertical lift of the input distribution, $\mathcal{F}^V$, which is generated by

$$Y^V = \frac{\partial}{\partial \dot{q}_1} - (1 + \cos q_2)\frac{\partial}{\partial \dot{q}_2},$$

is transversal to the tangent space of the constraint manifold, $T\mathcal{M}$, thus, by Theorem 4.2.2 there is a unique control law making the constraint manifold a virtual nonholonomic constraint.

The unique control law that makes the constraint manifold invariant is

$$\hat{u} = -\frac{\dot{q}_2(c_2^2 - 2)(t_2^2 - 1) + qC}{d}$$

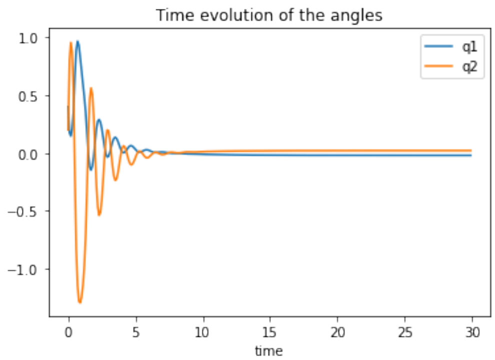
where $C = s_{12}c_2(2 - c_2^2) + s_1c_2(2c_2 - 1) - 5s_1 - 2s_2c_1$, d is the determinant of the inertia matrix $D(q)$ and we write $c_1 = \cos q_1, c_2 = \cos q_2$ and $t_2 = \tan q_2$, for simplicity.

We have run a simulation of the double pendulum using a fourth-order Runge-Kutta method for $N = 100$ steps using a time step of $h = 0.1$ and physical constants $m = l = 1$ and $g = 10$. We used as initial conditions $q_1 = 0.4, q_2 = 0$ and $\dot{q}_2 = 10$. The velocity $\dot{q}_2$ is computed by solving the equation $\Phi(q, \dot{q}) = 0$, so that initial conditions are in the constraint submanifold. Figure (4.2a) shows the time evolution of the angles q_1 and q_2, while the Figures (4.2b) and (4.2c) show the phase space $(q_1, \dot{q}_1)$ and $(q_2, \dot{q}_2)$, respectively. The last two figures show the energy and constraint evolution in time. The simulation shows that the controlled motion has an equilibrium point, occurring near $q_1 = q_2 = 0$. The system dissipates energy as a result of the control forces acting on it. The constraint is approximately preserved, though we cannot observe exact preservation since the method is not specifically designed to preserve it. It is interesting to note that as we increase the value of the initial angles, the system eventually reaches an equilibrium point but it occurs at points distant from $q_1 = q_2 = 0$, where a constant control force must always be active. The reader can see the code and a video of the simulation on the page https://github.com/StratosSim/VNHC-nonlinear.

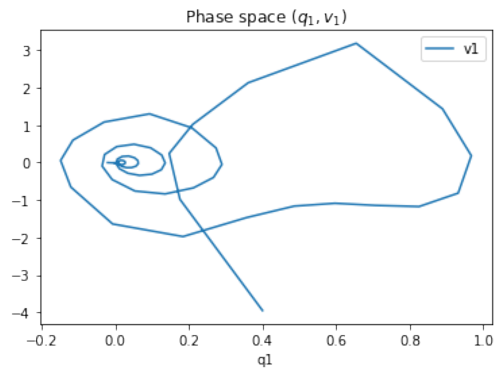
4.4.3 Enforcing flocking motion with virtual nonholonomic constraints

Flocking, swarming, and schooling are common emergent collective motion behaviors exhibited in nature. These natural collective behaviors can be leveraged in multirobot systems to safely transport large cohesive groups of robots within a workspace. The study of flocking motion in multi-agent systems has been studied in various disciplines, including robotics (Zhu et al., 2020), control theory (Beckers et al., 2022), and computational biology (Rahmani et al., 2020). Flocking refers to the coordinated movement of agents within a system, inspired by natural phenomena such as bird flocks, fish schools, and insect swarms. This behavior emerges from local interactions among agents. To capture these effects, Reynolds, 1987 introduced three heuristic rules: cohesion; alignment; and separation, to reproduce flocking motions in computer graphics. Here we introduce an application that can be useful in imposing virtual constraints for flocking motion in multi-agent systems like in Tanner et al., 2007.

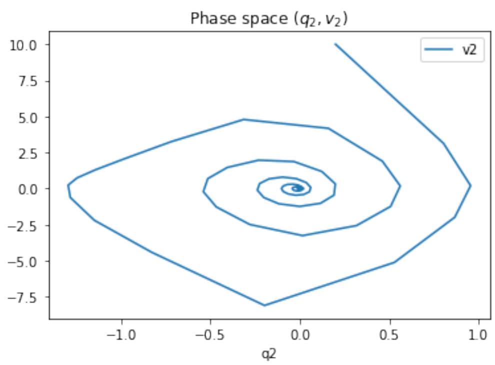
Consider two particles moving under the influence of gravity and which we desire to constrain to move with parallel velocity. Suppose that the motion of the two particles evolves in a



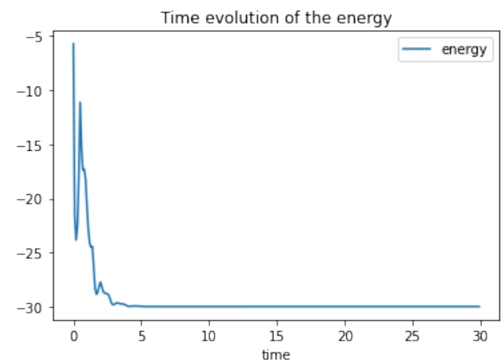
(a) angles time evolution



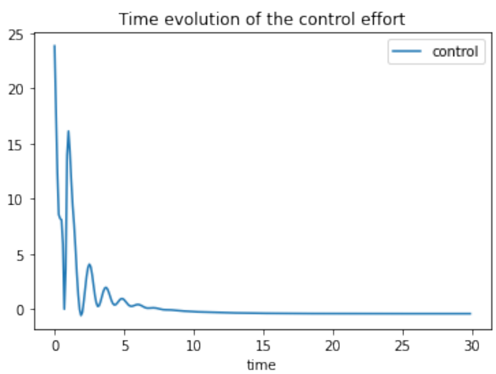
(b) phase space $(q_1, \dot{q}_1)$



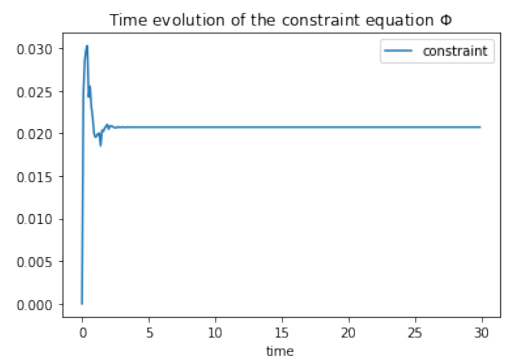
(c) phase space $(q_2, \dot{q}_2)$



(d) energy evolution in time



(e) control in time



(f) constraint in time

Figure 4.2: Double pendulum simulation figures

plane parametrized by (x, z). The position of the particles is given by $q_1 = (x_1, 0, z_1)$ and $q_2 = (x_2, 0, z_2)$, respectively, so the configuration space can be considered as $Q = \mathbb{R}^4$ with $q = (q_1, q_2) \in Q$.

The Lagrangian $L : TQ \to \mathbb{R}$, is given by

$$L(q, \dot{q}) = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 - G(q)$$

where $G(q) = m_1 g z_1 + m_2 g z_2$ is the potential energy due to gravity and $m_i, i = 1, 2$ are the masses of the particles, respectively. The constraint $\Phi : TQ \to \mathbb{R}$ is given by the equation

$$\Phi(q, \dot{q}) = \dot{x}_1 \dot{z}_2 - \dot{x}_2 \dot{z}_1$$

to enforce an alignment in the velocities and the control force is just $F : TQ \times \mathbb{R} \to T^*Q$ given by

$$F(q, \dot{q}, u) = u(f_1 dx_1 + f_2 dz_1 + f_3 dx_2 + f_4 dz_2).$$

The controlled Euler-Lagrange equations are

$$\begin{aligned} m_1 \ddot{x}_1 &= u f_1, \quad m_1 \ddot{z}_1 + m_1 g = u f_2, \\ m_2 \ddot{x}_2 &= u f_3, \quad m_2 \ddot{z}_2 + m_2 g = u f_4. \end{aligned} \tag{4.6}$$

The constraint manifold is $\mathcal{M} = \{(q, \dot{q}) \in TQ \ : \ \Phi(q, \dot{q}) = 0\}$ and its tangent space, at every point $(q, \dot{q}) \in \mathcal{M}$, is given by $T_{(q,\dot{q})}\mathcal{M} = \{v \in TTQ \ : \ d\Phi(v) = 0\} = \text{span}\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$, with

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial z_1}, \quad X_3 = \frac{\partial}{\partial x_2}, \quad X_4 = \frac{\partial}{\partial z_2}$$

$$X_5 = \dot{x}_2 \frac{\partial}{\partial \dot{x}_1} + \dot{z}_2 \frac{\partial}{\partial \dot{z}_1} + \dot{x}_1 \frac{\partial}{\partial \dot{x}_2} + \dot{z}_1 \frac{\partial}{\partial \dot{z}_2},$$

$$X_6 = \dot{z}_1 \frac{\partial}{\partial \dot{x}_1} + \dot{x}_1 \frac{\partial}{\partial \dot{z}_1} + \dot{z}_2 \frac{\partial}{\partial \dot{x}_2} + \dot{x}_2 \frac{\partial}{\partial \dot{z}_2},$$

$$X_7 = \dot{x}_1 \frac{\partial}{\partial \dot{x}_1} + \dot{z}_1 \frac{\partial}{\partial \dot{z}_1} - \dot{x}_2 \frac{\partial}{\partial \dot{x}_2} - \dot{z}_2 \frac{\partial}{\partial \dot{z}_2}.$$

The input distribution $\mathcal{F}$ is generated by the vector field

$$Y = \frac{f_1}{m_1} \frac{\partial}{\partial x_1} + \frac{f_2}{m_1} \frac{\partial}{\partial z_1} + \frac{f_3}{m_2} \frac{\partial}{\partial z_2} + \frac{f_4}{m_2} \frac{\partial}{\partial z_2}.$$

Note here that the vertical lift of the input distribution, $\mathcal{F}^V$, which is generated by

$$Y^V = \frac{f_1}{m_1} \frac{\partial}{\partial \dot{x}_1} + \frac{f_2}{m_1} \frac{\partial}{\partial \dot{z}_1} + \frac{f_3}{m_2} \frac{\partial}{\partial \dot{x}_2} + \frac{f_4}{m_2} \frac{\partial}{\partial \dot{z}_2},$$

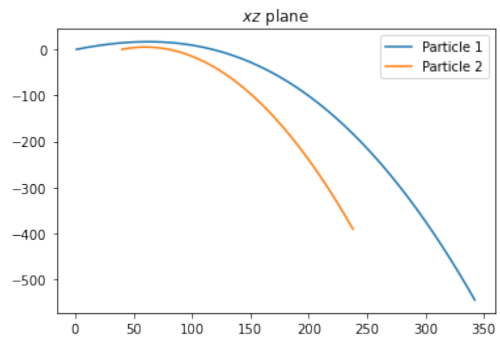
is transversal to the tangent space of the constraint manifold, $T\mathcal{M}$. By Theorem 4.2.2 there is a unique control law making the constraint manifold a virtual nonholonomic constraint. The control law that makes the constraint manifold invariant is

$$\hat{u} = (\dot{z}_2 f_1 - \dot{z}_1 f_3 + \dot{x}_1 f_4 - \dot{x}_2 f_2)^{-1} (\dot{x}_1 - \dot{x}_2) \, gm.$$

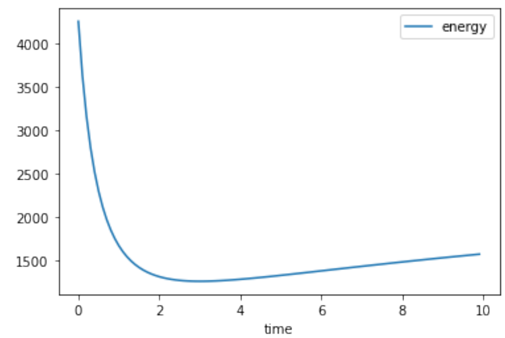
For $f_1 = f_2 = 1$ and $f_3 = f_4 = 0$ we get $F(q, \dot{q}, u) = u(dx_1 + dz_1)$ and

$$\hat{u} = (\dot{z}_2 - \dot{x}_2)^{-1} (\dot{x}_1 - \dot{x}_2) \, gm.$$

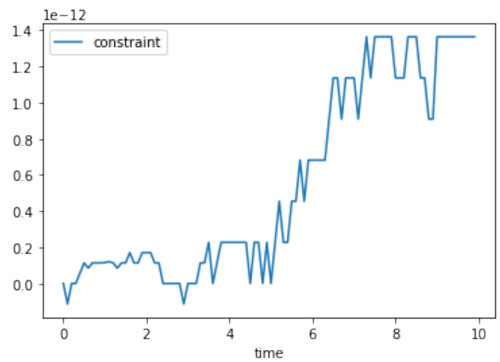
We have simulated the closed-loop dynamics with the preferred feedback control law using a standard fourth-order Runge-Kutta method with initial positions $(x_1, x_2, z_1, z_2) = (1, 40, 0, 0)$ and with initial velocities $(\dot{x}_1, \dot{x}_2, \dot{z}_1, \dot{z}_2) = (80, 20, 40, 10)$. In Fig. 4.3a we show the controlled trajectories for both particles where can be seen the velocities' compliance with the constraint. The total energy of the system is depicted in Fig. 4.3b while the preservation of the constraint during the simulation time is shown in Fig. 4.3c. Fluctuations of the values of the constraint function appear due to simulation computational process and are restricted to a minor interval as expected. The control function is depicted in Fig. 4.3d where it tends to zero since the motion tends to become vertical and gravity takes over.



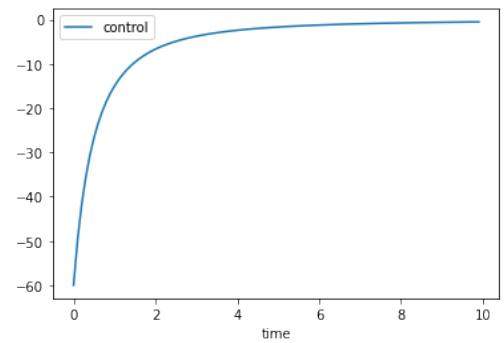
(a) Controlled trajectory of the two particles



(b) Total energy of the system



(c) The value of the constraint function Φ during the simulation time



(d) Control function during simulation time.

Figure 4.3: Flocking motion simulation figures

4.5 Symplectic characterization of virtual nonholonomic constraints

Next, we provide a geometric characterization of virtual nonlinear nonholonomic constraints from a symplectic perspective. Under a transversality assumption, we will show that there is a unique control law making the trajectories of the associated closed-loop system satisfy the virtual nonlinear nonholonomic constraints. We will characterize them in terms of the symplectic structure on TQ induced by a Lagrangian function and the almost-tangent structure. In particular, we will show that the closed-loop vector field satisfies a geometric equation of Chetaev type. Moreover, the closed-loop dynamics is obtained as the projection of the uncontrolled dynamics to the tangent bundle of the constraint submanifold defined by the virtual constraints.

Recall Chetaev's equations which are the equations of motion of a system with nonholonomic constraints defined at (4.1)

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \lambda_a \frac{\partial \phi^a}{\partial \dot{q}}, \\ &\phi^a(q, \dot{q}) = 0, \end{aligned} \tag{4.7}$$

where $(\phi^1, \dots, \phi^m)$ denote the coordinate functions of the constraint Φ. In what follows, we give an intrinsic/geometric formalism of the Chetaev's equations above. Recall from Subsection 2.3.3 that the Euler-Lagrange equations are geometrically described as the equations for the flow of the vector field X_{E_L}:

$$i_{X_{E_L}}\omega_L = dE_L,$$

where E_L is the energy of the system and ω_L the canonical symplectic form. Consider the dual of the canonical almost tangent structure J given at Subsection 2.1.5, with $J : TTQ \to TTQ$ and local expression in coordinates $J = dq^i \otimes \frac{\partial}{\partial \dot{q}^i}$. Its dual map, is given by $J^* : T^*TQ \to T^*TQ$. So, for every coordinate function of the constraint Φ, we have that $J^*(d\phi^a) = \frac{\partial \phi^a}{\partial \dot{q}^i} dq^i$. Notice also that $J^*(d\phi^a)(X^V) = 0$, for the vertical lift of every vector field $X \in \mathfrak{X}(Q)$. The equations of motion of a nonholonomic mechanical system with nonlinear constraints are integral curves of a vector field Γ_{nh} defined by the equations

$$\begin{aligned} &i_{\Gamma_{nh}}\omega_L - dE_L = \lambda_a J^*(d\phi^a) \\ &\Gamma_{nh} \in T\mathcal{M}, \end{aligned} \tag{4.8}$$

where λ_a are Lagrange multiplier's to be determined and E_L, ω_L are as before. These equations have a well-defined solution if $\sharp_{\omega_L}(J^*(d\phi^a)) \cap T\mathcal{M} = \{0\}$, where $\sharp_{\omega_L}$ is the sharp map defined by the 2-form ω_L.

Next we reproduce Example 4.2.5 with the formalism presented above and we exemplify Proposition 4.3.3 by comparing the equations (4.2) and (4.4).

Example 4.5.1. Consider, the Example 4.2.5, where a particle moving in three dimensional space and subject to the gravitational potential, with the same Lagrangian $L : TQ \to \mathbb{R}$,

$$L(q, \dot{q}) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

the constraint to be $\Phi = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - c = 0, \quad c > 0$ with constraint manifold given by

$$\mathcal{M} = \{(q, \dot{q}) \in TQ \ : \ \Phi(q, \dot{q}) = 0\}$$

and recall also the control force $F : TQ \times U \to T^*Q$ $F(q, \dot{q}, u) = uf = u(\dot{x}dx + \dot{y}dy + \dot{z}dz)$. The controlled Euler-Lagrange equations are

$$m\ddot{x} = u\dot{x}, \quad m\ddot{y} = u\dot{y}, \quad m\ddot{z} = -gm + u\dot{z}.$$

The tangent space of the constraint manifold $\mathcal{M}$ is given by

$$\begin{aligned} T_{(q,\dot{q})}\mathcal{M} &= \{v \in TTQ \ : \ d\Phi(v) = 0\} \\ &= \operatorname{span}\{X_1, X_2, X_3, X_4, X_5\}, \end{aligned}$$

where $(q, \dot{q}) \in \mathcal{M}$ and

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z},$$

$$X_4 = \dot{y}\frac{\partial}{\partial \dot{x}} - \dot{x}\frac{\partial}{\partial \dot{y}} \quad X_5 = \dot{z}\frac{\partial}{\partial \dot{y}} - \dot{y}\frac{\partial}{\partial \dot{z}}$$

The input distribution, $\mathcal{F}$, and the vertical lift of the input distribution, $\mathcal{F}^V$, are generated respectively by the vector fields

$$Y = \frac{\dot{x}}{m}\frac{\partial}{\partial x} + \frac{\dot{y}}{m}\frac{\partial}{\partial y} + \frac{\dot{z}}{m}\frac{\partial}{\partial z} \qquad \text{and} \qquad Y^V = \frac{\dot{x}}{m}\frac{\partial}{\partial \dot{x}} + \frac{\dot{y}}{m}\frac{\partial}{\partial \dot{y}} + \frac{\dot{z}}{m}\frac{\partial}{\partial \dot{z}}.$$

The control law that makes the constraint manifold invariant is

$$\hat{u} = \frac{mg\dot{z}}{c}.$$

For $S^\perp$, the orthogonal to S, we write the differential of Φ, namely, $d\Phi = 2\dot{x}d\dot{x} + 2\dot{y}d\dot{y} + 2\dot{z}d\dot{z}$ and its image through the dual of the canonical almost tangent structure $J = dq \otimes \frac{\partial}{\partial \dot{q}}$,

$$J^*(d\Phi) = 2\dot{x}dx + 2\dot{y}dy + 2\dot{z}dz.$$

Hence, $S^\perp$ is generated by

$$\sharp_{\mathcal{G}^c}(J^*(d\Phi)) = \frac{\dot{x}}{m}\frac{\partial}{\partial \dot{x}} + \frac{\dot{y}}{m}\frac{\partial}{\partial \dot{y}} + \frac{\dot{z}}{m}\frac{\partial}{\partial \dot{y}}.$$

Notice that the vertical lift of the input distribution, $\mathcal{F}^V$, is equal to $S^\perp$ and from Proposition 4.3.3, the local expression of the equations (4.2) and (4.4) will be the same. Indeed, the equations of the corresponding nonholonomic systems are

$$\begin{cases} m\ddot{x} = \lambda\dot{x} \\ m\ddot{y} = \lambda\dot{y} \\ m\ddot{z} + mg = \lambda\dot{z} \\ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - c = 0 \end{cases},$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier to be determined using the constraints, while the equations of the controlled system with control force determined by F are given by

$$\begin{cases} m\ddot{x} = -u^*\dot{x} \\ m\ddot{x} = -u^*\dot{y} \\ m\ddot{x} + mg = -u^*\dot{z} \end{cases},$$

where u^* is the unique feedback control making the constraints invariant under the flow. The two systems are equivalent on the submanifold $\mathcal{M}$ i.e. the trajectories of the constrained mechanical system (4.4) and the nonholonomic equations of motion (4.2) coincide on the constraint manifold. ◇

From now on, we assume that S and $\mathcal{F}^V$ are transversal. In the next result, we characterize the closed-loop dynamics arising from Theorem 4.2.2 as trajectories of a vector field satisfying an equation similar to that of the nonholonomic equations (4.8).

Lemma 4.5.2. *Consider the Lagrangian function $L : TQ \to \mathbb{R}$ of mechanical type $L = K(q,\dot{q}) - V(q)$, where the kinetic energy is given by a Riemannian metric $\mathcal{G}$ and consider the symplectic form ω_L of TQ. For any one-form $f \in \Omega^1(Q)$ we have that $\sharp_{\omega_L}(f^V) = \sharp_{\mathcal{G}^c}(f^V)$.*

Proof. In local coordinates the flat map of ω_L is represented by the matrix $\flat_{\omega_L} = \begin{pmatrix} A & \mathcal{G}_{ij} \\ -\mathcal{G}_{ij} & 0 \end{pmatrix}$, where A is a skew symmetric matrix and $\mathcal{G}_{ij}$ is the matrix representation of the Riemannian metric. Hence, the sharp map is given by $\sharp_{\omega_L} = \begin{pmatrix} 0 & -\mathcal{G}^{ij} \\ \mathcal{G}^{ij} & \mathcal{G}^{ij}A^{-1}\mathcal{G}^{ij} \end{pmatrix}$ where $\mathcal{G}^{ij}$ is the inverse of $\mathcal{G}_{ij}$ and for any one-form $f \in \Omega^1(Q)$ with $f = f^i dq_i$ we have that $\sharp_{\omega_L} f^V = \begin{pmatrix} 0 \\ \mathcal{G}^{ij} f^j \end{pmatrix}$. On the other hand, the sharp map of the Riemannian metric $\mathcal{G}$ is given by $\sharp_{\mathcal{G}} = \mathcal{G}^{ij}$ and so $[\sharp_{\mathcal{G}}(f)]^V = \sharp_{\omega_L} f^V$. Finally, from 4. of Proposition 2.2.2 we have $\sharp_{\omega_L}(f^V) = \sharp_{\mathcal{G}^c}(f^V)$. □

Theorem 4.5.3. *A SODE vector field Γ of the form* (2.23) *corresponding to the closed-loop system of the Lagrangian control system* (4.3) *makes $\mathcal{M}$ invariant if and only if it satisfies*

$$i_\Gamma \omega_L - dE_L = -u_a^*(f^a)^V, \ \Gamma \in TM, \tag{4.9}$$

or, equivalently, $i_\Gamma \omega_L - dE_L \in \flat_{\mathcal{G}^c}(\mathcal{F}^V)$, where $\flat_{\mathcal{G}^c}(\mathcal{F}^V) = span\{\flat_{\mathcal{G}^c}(Y^V)\} = span\{(f^a)^V\}$, $\mathcal{F}^V$ the distribution on TQ spanned by the vector fields $\{\sharp_{\mathcal{G}^c}(f^a)^V\}$, and u_a^* being the unique control law from Theorem 4.2.2.

Proof. Let G be the vector field defined by the free system $i_\Gamma \omega_L = dE_L$, i.e. $\sharp_{\omega_L}(dE_L) = G$ and Γ the one defined by the equation $i_\Gamma \omega_L - dE_L = -u_a(f^a)^V$. Thus, Γ is of the form $\Gamma(v_q) = G(v_q) + u_a(Y^a)^V_{v_q}$, for $v_q \in TQ$, where $(Y^a)^V = (\sharp_{\mathcal{G}}(f^a))^V = \sharp_{\mathcal{G}^c}((f^a)^V) = \sharp_{\omega_L}((f^a)^V)$ and the last equality holds by Lemma 4.5.2. From Theorem 4.2.2 there exists a unique control function u_a^* that makes $\mathcal{M}$ a virtual nonholonomic constraint, i.e. the vector field $\Gamma \in \mathfrak{X}(\mathcal{M})$ satisfies $i_\Gamma \omega_L - dE_L = -u_a^*(f^a)^V$, and and it is of the form

$$\Gamma(v_q) = G(v_q) + u_a^*(Y^a)^V_{v_q} \in T_{v_q}\mathcal{M},$$

for $v_q \in \mathcal{M}$. Equivalently, $i_\Gamma \omega_L - dE_L \in \flat_{\mathcal{G}^c}(\mathcal{F}^V)$ since for all $a = 1, \dots, m$ $\flat_{\mathcal{G}^c}[(Y^a)^V] = \flat_{\mathcal{G}^c}[(\sharp_{\mathcal{G}} f^a)^V] = \flat_{\mathcal{G}^c}[\sharp_{\mathcal{G}^c}(f^a)^V] = (f^a)^V$ where we have used property 4. of Proposition 2.2.2. □

Note that equation (4.9) can be equivalently written in the form

$$i_\Gamma \omega_L - dE_L \in J^*\hat{\mathcal{F}}^o, \ \Gamma \in T\mathcal{M}, \tag{4.10}$$

where $\hat{\mathcal{F}}^o = \text{span}\{d\hat{f}^a\}$ and $\hat{f}^a$ are the fiberwise linear functions on TQ defined by $\hat{f}^a(v_q) = \langle f^a(q), v \rangle$. Equations (4.10) look like the symplectic equations that appear in de León et al., 1997 in the context of constrained mechanical systems. Although they are slightly different, many of the constructions obtained by these authors follow in our case. In particular, we can characterize the closed-loop dynamics as the projection of the uncontrolled dynamics to the tangent bundle $T\mathcal{M}$.

Consider the distribution $\mathcal{S} = \sharp_{\omega_L}(J^*\hat{\mathcal{F}}^o)$. It is not difficult to prove that $\mathcal{S} = \mathcal{F}^V$. In particular, the transversality assumption appearing in Theorem 4.2.2 is equivalent to $\mathcal{S} \cap T\mathcal{M} = \{0\}$, which implies the decomposition

$$TTQ|_{\mathcal{M}} = \mathcal{S}|_{\mathcal{M}} \oplus T\mathcal{M}.$$

Now, choosing the vector field $\{(Y^a)^V\}$ as a local basis for the distribution $\mathcal{S}$, we can define the associated projections $\mathcal{Q} : TTQ \to \mathcal{S}$ and $\mathcal{P} : TTQ \to T\mathcal{M}$ given by

$$\mathcal{Q} = C_{ab}(Y^a)^V \otimes d\phi^b$$

and $\mathcal{P} = Id - \mathcal{Q}$, where the matrix C_{ab} is the inverse matrix of

$$C^{ab} = (Y^b)^V(\phi^a) = \mathcal{G}^{ij} f_i^b \frac{\partial \phi^a}{\partial \dot{q}^j} = -\sharp_{\omega_L}(J^* d\hat{f}^b)(\phi^a).$$

Note that this matrix is invertible to the fact that the Riemannian metric is invertible. Finally, we can prove the following result which shows that the closed-loop dynamics results from the projection to $T\mathcal{M}$ of the uncontrolled dynamics and gives a formula for the unique control law guaranteed from Theorem 4.2.2.

Proposition 4.5.4. *The vector field* Γ *defined by* (4.10) *satisfies*

$$\Gamma = \mathcal{P}(G) = G + C_{ab}G(\phi^b)(Y^a)^V.$$

Proof. The vector field Γ satisfies $\Gamma = G + u_a^*(Y^a)^V$. Applying the projection $\mathcal{P}$ to both sides of this equality and using $\Gamma \in T\mathcal{M}$ to impose $\mathcal{P}(\Gamma) = \Gamma$, we deduce that $u_a^* = C_{ab}G(\phi^b)$ which proves the result. □

Chapter 5

Virtual Constraints on Lie groups

Control systems defined on Lie groups offer a comprehensive framework for various systems, including control design for spacecraft and unmanned autonomous vehicles, such as aerial and underwater vehicles J. R. Goodman et al., 2023, T. Lee et al., 2006, Reyhanoglu, 1997, Egeland et al., 1996. Typically, the configuration space for these systems is represented globally by a matrix Lie group, which facilitates coordinate-free formulations of the dynamics that govern system behavior. We refer the reader to these books for a detailed treatment of this concept Jurdjevic, 1997, Bullo and Lewis, 2005, A. Bloch, 2015, T. Lee et al., 2018, Murray et al., 1994.

When a system's configuration lies on a Lie group, the application of left or right translations enables a globalization of solutions. This means that even when utilizing local charts for minor maneuvers, the Lie group framework permits navigation across the entire configuration space without the need to reformulate control strategies. This is possible because the system's position can consistently be expressed relative to the identity element of the Lie group. Furthermore, if the configuration space of these mechanical systems exhibits symmetry, it becomes feasible to exploit this symmetry to reduce the system's degrees of freedom. This reduction allows for the analysis of a lower-dimensional system, thereby decreasing computational costs and mitigating the risk of singularities by operating within a coordinate-free framework in the corresponding Lie algebra of the Lie group.

At this chapter we develop the theory of virtual nonholonomic constraints on Lie groups. The results obtained are analogous to the previous ones on Riemannian manifolds but here the geometric structure of the configuration space allows simplifications in many aspects of the theory. More precisely, we define virtual nonholonomic constraints on Lie groups as a controlled invariant subspace associated with an affine connection mechanical control system on the Lie algebra associated with the Lie group. We demonstrate the existence and uniqueness of a control law defining a virtual nonholonomic constraint and we characterize the trajectories of the closed-loop system as solutions of a mechanical system associated with an induced constrained connection. Moreover, we characterize when we can obtain reduced nonholonomic dynamics from virtual nonholonomic constraints. Finally, the theory is developed on right-invariance accordingly all results can be obtained for left-invariant as well.

5.1 Nonholonomic constraints on Lie groups

Consider a mechanical control system where the configuration space is a Lie group G and a right invariant Lagrangian function $L : TG \to \mathbb{R}$. Equip the configuration space with a right invariant Riemannian metric $\mathcal{G}$ and let ∇ denote the Levi-Civita connection. Following the discussion we did at Subsection 2.6.4, we consider a right invariant distribution $\mathcal{D}$ on the Lie group G, that is, for each $g \in G$, the fiber at g, denoted by $\mathcal{D}_g$, is defined by

$$\mathcal{D}_g := T_eR_g(\mathfrak{d}),$$

where $\mathfrak{d}$ is a subspace of the Lie algebra $\mathfrak{g}$. Using the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on the Lie algebra we define the orthogonal subset to $\mathfrak{d}$ with respect to the inner product and denote it by $\mathfrak{d}^{\perp}$ as $\mathfrak{d}^{\perp} = \{\xi \in \mathfrak{g} \ : \ \langle \xi, \eta \rangle_{\mathfrak{g}} = 0, \ \forall \eta \in \mathfrak{d}\}$, then $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{d}^{\perp}$. Finally, consider the orthogonal projections:

$$\mathfrak{P} : \mathfrak{g} \to \mathfrak{d} \text{ and } \mathfrak{Q} : \mathfrak{g} \to \mathfrak{d}^{\perp}. \tag{5.1}$$

Next we give a Lemma which will be useful in the sequel:

Lemma 5.1.1. *Let $\mathcal{D}$ be a right invariant distribution. Given a right invariant metric on G consider the associated orthogonal distribution $\mathcal{D}^{\perp}$ and the orthogonal projections $\mathcal{P} : TG \to \mathcal{D}$ and $\mathcal{Q} : TG \to \mathcal{D}^{\perp}$. Then the following statements hold:*

1. *The orthogonal distribution $\mathcal{D}^{\perp}$ is right-invariant and $\mathcal{D}^{\perp}_g = T_eR_g(\mathfrak{d}^{\perp})$.*
2. *The Lie algebra projections satisfy*

$$\mathfrak{P} = T_gR_{g^{-1}} \circ \mathcal{P} \circ T_eR_g \text{ and } \mathfrak{Q} = T_gR_{g^{-1}} \circ \mathcal{Q} \circ T_eR_g. \tag{5.2}$$

Proof. 1. Consider two vector fields X, Y on TG such that $X \in \mathcal{D}$ and $Y \in \mathcal{D}^{\perp}$. If $G_{\mathbb{I}}$ is the right-invariant metric we have $G_{\mathbb{I}}(X_g, Y_g) = 0$ for some $g \in G$. From the right invariance of the metric we get $\langle T_gR_{g_{-1}}(X_g), T_gR_{g_{-1}}(Y_g) \rangle_{\mathfrak{g}} = 0$ and since $\mathcal{D}$ is right invariant $\langle \xi, T_gR_{g_{-1}}(Y_g) \rangle_{\mathfrak{g}} = 0$ for some $\xi \in \mathfrak{g}$. The vector fields X, Y are arbitrary so $T_gR_{g_{-1}}(Y_g) \in \mathfrak{d}^{\perp}$, hence $Y_g = T_eR_g(\eta)$ for some $\eta \in \mathfrak{d}^{\perp}$ thus, $\mathcal{D}^{\perp}$ is right-invariant and $\mathcal{D}^{\perp}_g = T_eR_g(\mathfrak{d}^{\perp})$.

2. Let $\xi \in \mathfrak{g}$, since $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{d}^{\perp}$ we have the decomposition $\xi = \xi^{\top} + \xi^{\perp}$ where $\xi \in \mathfrak{d}$ and $\xi^{\perp} \in \mathfrak{d}^{\perp}$, and so $T_eR_g(\xi) = \xi_R = \xi_R^{\top} + \xi_R^{\perp}$. The distribution $\mathcal{D}$ and its orthogonal, $\mathcal{D}^{\perp}$, are right invariant i.e. $\mathcal{D}_g = T_eR_g(\mathfrak{d})$ and $\mathcal{D}^{\perp}_g = T_eR_g(\mathfrak{d}^{\perp})$ thus we have that $T_eR_g(\xi) = T_eR_g(\xi^{\top} + \xi^{\perp}) = \xi_R^{\top} + \xi_R^{\perp}$. So $\mathcal{P}(T_eR_g(\xi)) = \xi_R^{\top} \in \mathcal{D}_g$ and $T_gR_{g^{-1}}(\xi_R^{\top}) = \xi^{\top} \in \mathfrak{d}$, and since $\mathfrak{P}(\xi) = \xi^{\top}$ the first equation (5.2) holds.

For the second equation of (5.2) consider the projection of $T_eR_g(\xi)$ on $\mathcal{D}^{\perp}$, namely $\mathcal{Q}(T_eR_g(\xi)) = \xi_R^{\perp} \in \mathcal{D}^{\perp}$ and $T_gR_{g^{-1}}(\xi_R^{\perp}) = \xi^{\perp} \in \mathfrak{d}^{\perp}$, since $\mathfrak{Q}(\xi) = \xi^{\perp}$ the second equation holds. □

Using the above splitting of the Lie algebra on the subspace $\mathfrak{d}$ and its orthogonal complement $\mathfrak{d}^{\perp}$ and the associated projections, we define a map that allows the description of the nonholonomic trajectories of a constrained mechanical system what is defined later on, and in the next proposition, we link this map with the Riemannian $\mathfrak{g}$-connection given in Subsection 2.6.3.

Definition 5.1.2. We define the **nonholonomic $\mathfrak{d}$-connection** $\nabla^{\mathfrak{d}}:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ to be a bilinear map satisfying

$$\nabla^{\mathfrak{d}}_{\xi}\eta=\left(\nabla^{nh}_{\xi_R}\eta_R\right)(e), \tag{5.3}$$

where ∇^{nh} is the nonholonomic connection corresponding to the Levi-Civita connection ∇ on G, given from (2.17) at Section 3.1.

Note that we call $\nabla^{\mathfrak{d}}$ connection even though it is not one, see comment after Theorem 2.6.9.

Proposition 5.1.3. *Define the bilinear map* $(\nabla^{\mathfrak{g}}_{\xi}\mathfrak{Q})(\eta)=\nabla^{g}_{\xi}(\mathfrak{Q}(\eta))-\mathfrak{Q}(\nabla_{\xi}\eta)$*, where* $\nabla^{\mathfrak{g}}$ *is the Riemannian* $\mathfrak{g}$*-connection corresponding to the Levi-Civita connection* ∇ *on* G*. Then*

$$\nabla^{\mathfrak{d}}_{\xi}\eta=\nabla^{\mathfrak{g}}_{\xi}\eta+(\nabla^{\mathfrak{g}}_{\xi}\mathfrak{Q})(\eta).$$

Proof. If $\xi,\eta\in\mathfrak{d}$ we have that

$$\nabla^{\mathfrak{d}}_{\xi}\eta=\nabla^{\mathfrak{g}}_{\xi}\eta+(\nabla^{\mathfrak{g}}_{\xi}\mathfrak{Q})(\eta)=\nabla^{\mathfrak{g}}_{\xi}\eta+\nabla^{\mathfrak{g}}_{\xi}(\mathfrak{Q}(\eta))-\mathfrak{Q}(\nabla^{\mathfrak{g}}_{\xi}\eta),$$

where we have used the definition of covariant derivative of a map in the last equality. Noting that $\mathfrak{Q}(\eta)=0$ since $\eta\in\mathfrak{d}$, we prove the equality. □

It is clear that the nonholonomic $\mathfrak{d}$-connection is the projection of the Riemannian $\mathfrak{g}$-connection, namely, $\nabla^{\mathfrak{d}}_{\xi}\eta=\mathfrak{P}(\nabla^{\mathfrak{g}}_{\xi}\eta)$, for all $\xi,\eta\in\mathfrak{d}$. Therefore, we obtain the explicit expression

$$\nabla^{\mathfrak{d}}_{\xi}\eta=\frac{1}{2}\mathfrak{P}(-[\xi,\eta]_{\mathfrak{g}}+\mathrm{ad}^{\dagger}_{\xi}\eta+\mathrm{ad}^{\dagger}_{\eta}\xi),\quad \xi,\eta\in\mathfrak{d}, \tag{5.4}$$

where $\mathrm{ad}^{\dagger}_{\xi}\eta=\sharp[\mathrm{ad}^{*}_{\xi}\flat(\eta)]$. We now express Lemma 2.6.13 in terms of the Riemannian $\mathfrak{d}$-connection and the nonholonomic connection:

Lemma 5.1.4. *Let* $g:[a,b]\to G$ *be a curve and* X *a smooth vector field along* g *satisfying* $X(t)\in\mathcal{D}_{g(t)}$*. Suppose that* $\xi(t)=\dot{g}(t)g(t)^{-1}$ *and* $\eta(t)=X(t)g(t)^{-1}$*. Then the following relation holds for all* $t\in[a,b]$*:*

$$\nabla^{nh}_{\dot{g}}X(t)=\left(\dot{\eta}(t)+\nabla^{\mathfrak{d}}_{\xi}\eta(t)\right)g(t). \tag{5.5}$$

As an important case of the previous Lemma, we deduce the following theorem:

Theorem 5.1.5. *Suppose that* $g:[a,b]\to G$ *is a nonholonomic trajectory with respect to a right-invariant metric and distribution* $\mathcal{D}$ *and let* $\xi(t)=\dot{g}(t)g(t)^{-1}$*. Then,* ξ *satisfies*

$$\dot{\xi}+\nabla^{\mathfrak{d}}_{\xi}\xi=0, \tag{5.6}$$

or, equivalently,

$$\dot{\xi}+(\mathfrak{P}\circ\sharp)\left[ad^{*}_{\xi(t)}\flat(\xi(t))\right]=0,\quad \xi\in\mathfrak{d}. \tag{5.7}$$

Proof. Suppose that $\{e_i\}$ is a basis for $\mathfrak{g}$, so $\xi(t)=\xi^i(t)e_i$ and $\dot{g}(t)=T_eR_{g(t)}\xi(t)=T_eR_g(\xi^ie_i)=\xi^iT_eR_g(e_i)=\xi^i(e_i)_R$. Since $g(t)$ is a nonholonomic trajectory we have $\nabla^{nh}_{\dot{g}}\dot{g}=0$ and

$$\begin{aligned}\nabla^{nh}_{\dot{g}}\dot{g}&=\mathcal{P}(\nabla_{\dot{g}}\dot{g})=\mathcal{P}\left(\dot{\xi}^i(e_i)_R+\xi^i\xi^j\nabla_{(e_j)_R}(e_i)_R\right)=\mathcal{P}\left(T_eR_g(\dot{\xi}^ie_i+\xi^i\xi^j\nabla_{e_j}e_i)\right)\\&=(\mathcal{P}\circ T_eR_g)\,(\dot{\xi}+\nabla^{\mathfrak{g}}_{\xi}\xi),\end{aligned}$$

so

$$0 = (T_g R_{g^{-1}} \circ \mathcal{P} \circ T_e R_g)\,(\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi) = \mathfrak{P}(\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi) = \mathfrak{P}(\dot{\xi}) + \mathfrak{P}(\nabla^{\mathfrak{g}}_{\xi}\xi).$$

Since $\nabla^{\mathfrak{d}}_{\xi}\xi = \mathfrak{P}(\nabla^{\mathfrak{g}}_{\xi}\xi)$ and $\xi \in \mathfrak{d}$ so $\mathfrak{P}(\xi) = \xi$ we get equation (5.6):

$$\dot{\xi} + \nabla^{\mathfrak{d}}_{\xi}\xi = 0.$$

By the definition of $\nabla^{\mathfrak{d}}$, for $\xi, \eta \in \mathfrak{g}$, we have

$$\nabla^{\mathfrak{d}}_{\xi}\eta = \nabla^{\mathfrak{g}}_{\xi}\eta + (\nabla^{\mathfrak{g}}_{\xi}\mathfrak{Q})(\eta) = \mathfrak{P}(\nabla^{\mathfrak{g}}_{\xi}\eta) + \mathfrak{Q}(\nabla^{\mathfrak{g}}_{\xi}\eta) + \nabla^{\mathfrak{g}}_{\xi}\mathfrak{Q}(\eta) - \mathfrak{Q}(\nabla^{\mathfrak{g}}_{\xi}\eta).$$

If $\eta \in \mathfrak{d}$ we get $\nabla^{\mathfrak{d}}_{\xi}\eta = \mathfrak{P}(\nabla^{\mathfrak{g}}_{\xi}\eta)$ and from the expression (5.4) we obtain

$$\nabla^{\mathfrak{d}}_{\xi}\xi = (\mathfrak{P} \circ \sharp)\left[\mathrm{ad}^*_{\xi}\flat(\xi)\right].$$

Hence, from the last expression and equation (5.6) we get the equation (5.7) . □

5.1.1 Virtual linear nonholonomic constraints on Lie groups

Consider the mechanical system on the Lie group G given by (2.38) of Theorem 2.6.15, together with a control force $F : \mathfrak{g} \times U \to \mathfrak{g}^*$ of the form

$$F(\xi, u) = \sum_{a=1}^{m} u_a F^a(\xi) \tag{5.8}$$

where $F^a \in \mathfrak{g}^*$ with $m < n$, $U \subset \mathbb{R}^m$ the set of controls and $u_a \in \mathbb{R}$ with $1 \leq a \leq m$ the control inputs. Thus, we have the controlled mechanical system

$$\dot{g} = T_e R_g(\xi), \quad \dot{\xi} + \sharp\left[\mathrm{ad}^*_{\xi(t)}\flat(\xi(t))\right] = u_a f^a, \tag{5.9}$$

where $f^a = \sharp(F^a|_{\mathfrak{g}}) \in \mathfrak{g}^*$ are a set of $m < n$ vectors, there vector are spanning the subspace

$$\mathfrak{f} = \mathrm{span}\{f^1, \ldots, f^m\}.$$

Definition 5.1.6. The subspace $\mathfrak{f}$, of the Lie algebra $\mathfrak{g}$, given by $\mathfrak{f} = \mathrm{span}\{f_1, \ldots, f_m\}$, is called **control input subspace** associated with the mechanical control system (5.9).

Definition 5.1.7. A virtual nonholonomic constraint associated with the mechanical control system of type (5.9) is a controlled invariant subspace $\mathfrak{d}$ of $\mathfrak{g}$, that is, there exist a control law making the subspace $\mathfrak{d}$ invariant under the flow of the closed-loop system, i.e. $\xi(0) \in \mathfrak{d}$ and $\xi(t) \in \mathfrak{d}$, $\forall t \geq 0$.

Theorem 5.1.8. *Suppose $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{d}$. Then there exists a unique control law u^* making $\mathfrak{d}$ a virtual contraint for the controlled mechanical system* (5.9).

Proof. Let $\dim \mathfrak{d} = k$ and $\dim \mathfrak{f} = m = n - k$. Consider the covectors $\mu^1 \ldots, \mu^m \in \mathfrak{g}^*$ spanning the annihilator subspace of $\mathfrak{d}$. $\xi(t)$ is a curve on $\mathfrak{g}$ satisfying $\xi(t) \in \mathfrak{d}$ for all time if and only if it satisfies $\mu^a(\xi(t)) = 0$ for all $a = 1, \ldots, m$. Differentiating this equation and assuming that $\xi(t)$ is a solution of the closed loop system (5.9) for an appropriate choice of control law u, we have

$$-\mu^a \left(\sharp \left[\mathrm{ad}^*_{\xi(t)} \flat(\xi(t)) \right] \right) + u_b \mu^a(f^b) = 0.$$

Since $\sharp \left[\mathrm{ad}^*_{\xi(t)} \flat(\xi(t)) \right] \in \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{d}$ there is a unique way to decompose this vector as the sum

$$\sharp \left[\mathrm{ad}^*_{\xi(t)} \flat(\xi(t)) \right] = \eta(t) + \tau_b(t) f^b,$$

with $\eta \in \mathfrak{d}$. In addition, note that the coefficients τ_b may be regarded as a function of $\mathfrak{g}$. In fact, its definition is associated with the projection to $\mathfrak{f}$ together with the choice of $\{f^b\}$ as a basis for $\mathfrak{f}$. Therefore, $\mu^a(\dot{\xi}(t)) = 0$ if and only if

$$(\tau_b - u_b)\mu^a(f^b) = 0.$$

Since $\mu^a(f^b)$ is an invertible matrix, we conclude that $\tau_b = u_b$ proving the existence and uniqueness of a control law making $\mathfrak{d}$ a virtual constraint. □

From now on suppose that the subspace $\mathfrak{d}$ describing the virtual nonholonomic constraints and the input subspace $\mathfrak{f}$ are transversal. Therefore, the projections $\mathfrak{p} : \mathfrak{g} \to \mathfrak{d}$ and $\mathfrak{q} : \mathfrak{g} \to \mathfrak{f}$ associated to the direct sum are well-defined. Using the Riemannian $\mathfrak{g}$-connection we define the bilinear map $\nabla^{\mathfrak{c}}$

$$\nabla^{\mathfrak{c}}_{\xi} \eta = \nabla^{\mathfrak{g}}_{\xi} \eta + (\nabla^{\mathfrak{g}}_{\xi} \mathfrak{q})(\eta), \tag{5.10}$$

where $\xi, \eta \in \mathfrak{g}$. We refer to that bilinear map as **induced constrained connection** associated to the subspace $\mathfrak{d}$ and the input subspace $\mathfrak{f}$. The induced constrained connection is a linear connection on $\mathfrak{g}$ with the special property that $\mathfrak{d}$ is geodesically invariant for $\nabla^{\mathfrak{c}}$, i.e., if a geodesic of $\nabla^{\mathfrak{c}}$ starts on $\mathfrak{d}$ then it stays in $\mathfrak{d}$ for all time (see Lewis, 1998).

Lemma 5.1.9. *If $\xi, \eta \in \mathfrak{d}$ then*

$$\nabla^{\mathfrak{c}}_{\xi} \eta = \mathfrak{p}(\nabla^{\mathfrak{g}}_{\xi} \eta).$$

Proof. If $\xi, \eta \in \mathfrak{d}$ we have that

$$\nabla^{\mathfrak{c}}_{\xi} \eta = \nabla^{\mathfrak{g}}_{\xi} \eta + (\nabla^{\mathfrak{g}}_{\xi} \mathfrak{q})(\eta) = \nabla^{\mathfrak{g}}_{\xi} \eta + \nabla^{\mathfrak{g}}_{\xi}(\mathfrak{q}(\eta)) - \mathfrak{q}(\nabla^{\mathfrak{g}}_{\xi} \eta),$$

where we have used the definition of covariant derivative of a map in the last equality. Noting that $\mathfrak{q}(\eta) = 0$ since $\eta \in \mathfrak{d}$, we prove the equality. □

Remark 5.1.10 (**Virtual holonomic constraints on Lie groups.**). We may use a right-invariant distribution to enforce a virtual holonomic constraint, that is, to guarantee that a trajectory of the closed-loop system (5.9), with the control law determined by Theorem 5.1.8 remains inside a desired submanifold of the configuration space, as long as its tangent space is right-invariant.

If $\mathfrak{d}$ is a Lie subalgebra of $\mathfrak{g}$, i.e., for any ξ, $\eta \in \mathfrak{d}$ their Lie bracket $[\xi, \eta]$ is also in $\mathfrak{d}$, we deduce that the right-invariant distribution $\mathcal{D}$ is integrable, where the distribution is defined according to $\mathcal{D}_g = T_eR_g(\mathfrak{d})$ for each $g \in G$. Hence, there exists a foliation of G such that the submanifold passing through a point $g \in G$ is denoted by $\mathcal{F}(g)$ and satisfies $T_h\mathcal{F}(g) = \mathcal{D}_h$ for any $h \in \mathcal{F}(g)$.

In particular, the trajectory of the closed-loop system (5.9), with the control law determined by Theorem 5.1.8 and starting at g_0 with initial velocity $\xi(0) \in \mathfrak{d}$ satisfies $\dot{g} = T_eR_g(\xi) \in \mathcal{D}_g$, since $\xi(t) \in \mathfrak{d}$. Therefore, the trajectory $g(t)$ is a curve starting in the submanifold $\mathcal{F}(g_0)$ that is always tangent to the integrable distribution $\mathcal{D}_g$. Hence, $g(t)$ must be entirely contained in $\mathcal{F}(g_0)$.

Remark 5.1.11. Note that if the control input subspace $\mathfrak{f}$ is orthogonal to the controlled invariant subspace $\mathfrak{d}$ then the constrained dynamics is precisely the nonholonomic dynamics. Indeed, since $\mathfrak{f}$ and $\mathfrak{d}$ are orthogonal with respect to the metric $\langle\cdot,\cdot\rangle$ then $\mathfrak{f} = \mathfrak{d}^{\perp}$ and the projections $\mathfrak{P}$ and $\mathfrak{p}$ coincide (as well as the projections $\mathfrak{Q}$ and $\mathfrak{q}$). Thus, the induced constrained connection $\nabla^{\mathfrak{c}}$ is precisely the nonholonomic $\mathfrak{d}$-connection $\nabla^{\mathfrak{d}}$. Hence, the dynamics derived from the two connections coincide.

5.1.2 Virtual affine nonholonomic constraints on Lie groups

In this section, we briefly introduce virtual affine nonholonomic constraints on Lie groups. All the results follow by a slight modification of the findings at the subsection above. In what follows, the control force F is still of the form (5.8).

Definition 5.1.12. A virtual affine nonholonomic constraint associated with the control system (5.9) is a controlled invariant affine subspace $\mathfrak{a} \subseteq \mathfrak{g}$ for that system, that is, there exists a control law making $\mathfrak{a}$ invariant for the closed-loop system, i.e., $\xi(t) \in \mathfrak{a}$, whenever $\xi(0) \in \mathfrak{a}$.

Theorem 5.1.13 guarantees the existence and uniqueness of a control law that turns an affine subspace into a controlled invariant affine subspace (virtual affine nonholonomic constraint).

Theorem 5.1.13. *If the affine subspace $\mathfrak{a}$ and the control input subspace $\mathfrak{f}$ are transversal, then there exists a unique control law making the affine subspace a virtual affine nonholonomic constraint associated with the control system* (5.9).

Proof. Suppose that $\mathfrak{a}$ is modelled on a vector subspace $\mathfrak{d} \subseteq \mathfrak{g}$. Let $\dim \mathfrak{d} = k$ and $\dim \mathfrak{f} = m = n - k$, by transversality of $\mathfrak{a}$ and $\mathfrak{f}$. Consider the covectors $\mu^1 \ldots, \mu^m \in \mathfrak{g}^*$ spanning the annihiliator subspace of $\mathfrak{d}$ and let $a_0 \in \mathfrak{a}$. A vector $v \in \mathfrak{g}$ is in the affine subspace $\mathfrak{a}$ if and only if $v - a_0 \in \mathfrak{d}$ or, equivalently, if $\mu^a(v - a_0) = 0$ for all $a = 1, \ldots, m$.

The curve $\xi(t)$ on $\mathfrak{g}$ satisfyies $\xi(t) \in \mathfrak{a}$ for all time if and only if it satifies $\mu^a(\xi(t) - a_0) = 0$ for all $a = 1, \ldots, m$. Differentiating this equation and supposing that $\xi(t)$ is a solution of the closed loop system (5.9) for an appropriate choice of control law u, we have that

$$-\mu^a\left(\sharp\left[\mathrm{ad}^*_{\xi(t)}\flat(\xi(t))\right]\right) + u_b\mu^a(f^b) = 0.$$

Since $\sharp\left[\mathrm{ad}^*_{\xi(t)}\flat(\xi(t))\right] \in \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{d}$ there is a unique way to decompose this vector as the sum

$$\sharp\left[\mathrm{ad}^*_{\xi(t)}\flat(\xi(t))\right] = \eta(t) + \tau_b(t)f^b,$$

with $\eta \in \mathfrak{d}$. In addition, note that the coefficients τ_b may be regarded as a function of $\mathfrak{g}$. In fact, its definition is associated with the projection of $\mathfrak{f}$ together with the choice of $\{f^b\}$ as a basis for $\mathfrak{f}$. Therefore, $\mu^a(\dot{\xi}(t)) = 0$ if and only if

$$(\tau_b - u_b)\mu^a(f^b) = 0.$$

Since $\mu^a(f^b)$ is an invertible matrix, we conclude that $\tau_b = u_b$ proving existence and uniqueness of a control law making $\mathfrak{a}$ a virtual constraint. □

Remark 5.1.14. Note that, from the proof of Theorem 5.1.13 we deduce that the same control law making a vector subspace $\mathfrak{d}$ control invariant is also responsible for making any affine subspace of the form $\mathfrak{a} = a_0 + \mathfrak{d}$ control invariant, whenever $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{f}$.

5.2 Examples

At this section the theory is presented through some examples of mechanical systems evolving on Lie groups. The first example is a homogeneous rigid body on SE(3) with linear nonholonomic constraints while the second is a rigid body with a rotor on SE(3) $\times\, \mathbb{S}^1$ subject to affine nonholonomic constraints.

5.2.1 Mechanical systems evolving on SE(3)

Consider a homogeneous rigid body moving on the configuration space $G = \mathrm{SE}(3) \simeq SO(3) \times \mathbb{R}^3$. The motion of this body is completely described by its position in $\mathbb{R}^3$ together with its relative orientation. Suppose that a point q in G is parametrized by $q = (R, x, y, z)$. Let $\omega = (\omega_1, \omega_2, \omega_3)$ be the angular velocity of the body, m be its mass and mk^2 be its inertia about any axis. All variables are measured with respect to the inertial frame. The configuration space G can be identified with the special Euclidean group SE(3) by writing $q \in Q$ as

$$q = \begin{bmatrix} R & r \\ 0 & 1 \end{bmatrix},$$

with $r = (x, y, z)$. The Lie algebra $\mathfrak{g} = \mathfrak{se}(3)$ is then the set of matrices of the form

$$\xi = \begin{bmatrix} \hat{\omega} & \dot{r} \\ 0 & 1 \end{bmatrix},$$

where $\hat{\omega}$ is the skew-symmetric matrix obtained from the identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$ through the hat map, see Example 2.6.5, and $\dot{r} = (\dot{x}, \dot{y}, \dot{z})$. In addition, consider the following inner product on $\mathfrak{g}$ given by:

$$\langle \xi, \xi \rangle = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mk^2(\omega_1^2 + \omega_2^2 + \omega_3^2).$$

Thus, the mechanical control system takes the form

$$\begin{cases} m\ddot{x} = m(\omega_2\dot{z} - \omega_3\dot{y}) + u_x \\ m\ddot{y} = m(\omega_3\dot{x} - \omega_1\dot{z}) + u_y \\ m\ddot{z} = m(\omega_1\dot{y} - \omega_2\dot{x}) + u_z \\ mk^2\omega_1 = u_1 \\ mk^2\omega_2 = u_2 \\ mk^2\omega_3 = u_3 \end{cases}$$

Consider the basis of $\mathfrak{g}$ denoted by $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ given by

$$e_1 = (\widehat{(1,0,0)}, 0),\ e_2 = (\widehat{(0,1,0)}, 0),\ e_3 = (\widehat{(0,0,1)}, 0), e_4 = (\hat{0}, 1, 0, 0),\ e_5 = (\hat{0}, 0, 1, 0),\ e_6 = (\hat{0}, 0, 0, 1).$$

Now suppose we would like to impose the constraints $\begin{cases} \omega_1 + \dot{y} = 0 \\ \omega_2 - \dot{x} = 0 \\ \omega_3 = 0 \\ \dot{z} = 0 \end{cases}$ which define the constraint vector subspace $\mathfrak{d} \subseteq \mathfrak{g}$ spanned by the vectors: $v_1 = e_1 - e_5$ and $v_2 = e_2 + e_4$. A complementary vector space to $\mathfrak{d}$ is $\mathfrak{f} = \langle\{e_1 + e_5, e_2 - e_4, e_3, e_6\}\rangle$. Thus the control system with respect to this choice of the controlled directions is

$$\begin{cases} mk^2\omega_1 = u_1 \\ mk^2\omega_2 = u_2 \\ mk^2\omega_3 = u_3 \\ m\ddot{x} = m(\omega_2\dot{z} - \omega_3\dot{y}) - u_2 \\ m\ddot{y} = m(\omega_3\dot{x} - \omega_1\dot{z}) - u_1 \\ m\ddot{z} = m(\omega_1\dot{y} - \omega_2\dot{x}) + u_z \end{cases}$$

We find that this control system makes the vector space $\mathfrak{d}$ control invariant under the control law

$$u_1 = u_2 = u_3 = 0, \quad u_z = m(\omega_1^2 + \omega_2^2)$$

5.2.2 Rigid body with a rotor

This example is inspired in A. Bloch, 2015 example of a rigid body with a rotor (see also A. M. Bloch et al., 1992), In fact, the feedback control studied there can be reinterpreted in the context of virtual nonholonomic constraints as a control law making a particular affine subspace control invariant.

The configuration space of a rigid body with an internal rotor is $G = SO(3) \times \mathbb{S}^1$ which is the cartesian product of two Lie groups equipped with the product Lie group operation. Consider on the Lie algebra $\mathfrak{g}$ the inner product

$$\langle \xi, \xi \rangle = \lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2 + 2J\omega_3\dot{\alpha} + J\dot{\alpha}^2,$$

where $\xi = (\omega, \dot{\alpha}) = (\omega_1, \omega_2, \omega_3, \dot{\alpha}) \in \mathfrak{g} \equiv \mathbb{R}^4$ is a vector in the Lie algebra. For this inner product to be positive definite we shall assume that $\lambda_i > 0$ for $i = 1, 2, 3$ and $J\lambda_3 - J^2 > 0$. The adjoint action $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ on the Lie algebra is given by

$$ad_\xi \eta = (\gamma \times \omega, 0),$$

where $\eta = (\gamma, \dot{\beta}) \in \mathfrak{g}$.The uncontrolled mechanical system reads

$$\begin{aligned}
\dot{\omega}_1 &= -\frac{1}{\lambda_1}\left((\lambda_3 - \lambda_2)\omega_2\omega_3 + J\omega_2\dot{\alpha}\right) \\
\dot{\omega}_2 &= -\frac{1}{\lambda_2}\left((\lambda_1 - \lambda_3)\omega_1\omega_3 - J\omega_1\dot{\alpha}\right) \\
\dot{\omega}_3 &= -\frac{J}{D}\left(\lambda_2 - \lambda_1\right)\omega_1\omega_2 \\
\ddot{\alpha} &= \frac{J}{D}\left(\lambda_2 - \lambda_1\right)\omega_1\omega_2,
\end{aligned}$$

where $D = J\lambda_3 - J^2$.

Next, we would like to control the rotor such that it satisfies the constraint

$$(J - k\lambda_3)\omega_3 + J(1 - k)\dot{\alpha} = p, \tag{5.11}$$

for some $k \in \mathbb{R}$ and $p \in \mathbb{R}$. Consider the basis of $\mathfrak{g}$ given by

$$e_1 = (\widehat{(1, 0, 0)}, 0),\ e_2 = (\widehat{(0, 1, 0)}, 0),\ e_3 = (\widehat{(0, 0, 1)}, 0),\ e_4 = (\hat{0}, 1),$$

where the hat map is the same as in the Example 5.2.1.

The constraint defined by equation 5.11 defines an affine subspace of $\mathfrak{g}$ whose model vector subspace is $\mathfrak{d}$ with

$$\mathfrak{d} = \mathrm{span}\left\{ e_1, e_2, e_4 + \frac{J(1-k)}{J - k\lambda_3} e_3 \right\}.$$

Let us choose a complementary input force vector space $\mathfrak{f}$ defined by

$$\mathfrak{f} = \mathrm{span}\left\{ -\frac{J}{D} e_3 + \frac{\lambda_3}{D} e_4 \right\}.$$

Under this control force, the controlled mechanical system becomes

$$\begin{aligned}
\dot{\omega}_1 &= -\frac{1}{\lambda_1}\left((\lambda_3 - \lambda_2)\omega_2\omega_3 + J\omega_2\dot{\alpha}\right) \\
\dot{\omega}_2 &= -\frac{1}{\lambda_2}\left((\lambda_1 - \lambda_3)\omega_1\omega_3 - J\omega_1\dot{\alpha}\right) \\
\dot{\omega}_3 &= -\frac{J}{D}\left(\lambda_2 - \lambda_1\right)\omega_1\omega_2 - \frac{J}{D}u \\
\ddot{\alpha} &= \frac{J}{D}\left(\lambda_2 - \lambda_1\right)\omega_1\omega_2 + \frac{\lambda_3}{D}u.
\end{aligned}$$

By taking the time derivative of the constraint equation (5.11) and substituting the equation of motion we derive the control law

$$u = k(\lambda_1 - \lambda_2)\omega_1\omega_2$$

which is the unique control law making the affine constraint control invariant.

Remark 5.2.1. Note that the control law we have obtained coincides with the control law obtained in Example 9.2.1 at A. Bloch, 2015. The closed-loop equations satisfy:

1. the projection of the equations to $\mathfrak{so}(3)$ is a Hamiltonian system with respect to the standard Lie-Poisson structure.
2. For $k = J/\lambda_3$, $\dot{\alpha}$ is constant.
3. For $p = 0$, the equilibrium $(0, M, 0)$ is stable if $k > 1 - J/\lambda_3$.

Chapter 6

Virtual nonholonomic constraints on Riemannian homogeneous spaces

Nonholonomic systems on Riemannian homogeneous spaces have not been considered in the literature before, but nevertheless some examples have been considered in the context of geometric control such as a sphere rolling on another sphere in Jurdjevic, 1997 (see also A. Bloch, 2015 Section 7.4, Rojo and Bloch, 2010), but a detailed geometric description of the dynamics of such systems has not been analyzed.

In this chapter we develop the theory of virtual nonholonomic constraints on Riemannian homogeneous spaces. The goal is to show the existence and uniqueness of a control law preserving the invariant distribution and characterize the closed-loop nonholonomic dynamics obtained using the unique control law in terms of an affine connection.

In Stratoglou, Simoes, Bloch, and Colombo, 2024, we have studied virtual *holonomic* constraints on Riemannian homogeneous spaces but the result have not been considered in this thesis because here we only focus on virtual nonholonomic constraints, here, we extend the work on virtual nonholonomic constraints. The main challenge in considering this class of constraints is the geometry behind the problem. In particular, we first need to develop a mathematical theory for nonholonomic systems on Riemannian homogeneous spaces. We developed new examples of such systems that were never studied neither from the modeling point of view nor in the control literature.

6.1 Nonholonomic systems on Riemannian homogeneous spaces

The reader is refereed to Chapter 2, Section 2.7 for the necessary background that we use here on Riemannian homogeneous spaces and they are refereed to Subsection 2.4 and Chapter 5 for the basics on nonholonomic mechanics and Lie groups respectively. For convenience we write down some key features. Given a constraint distribution $\mathcal{D}$ the *nonholonomic connection*

$\nabla^{nh}:\mathfrak{X}(Q)\times\mathfrak{X}(Q)\to\mathfrak{X}(Q)$ is given by

$$\nabla^{nh}_X Y=\nabla^{\mathcal{G}}_X Y+(\nabla^{\mathcal{G}}_X\mathcal{Q})(Y) \tag{6.1}$$

where $\mathcal{Q}:TQ\to\mathcal{D}^{\perp}$ and we know that the trajectories for the nonholonomic mechanical system associated with the Lagrangian given by $L(v_q)=\frac{1}{2}\mathcal{G}(v_q,v_q)-V(q)$, $v_q\in T_qQ$ and the distribution $\mathcal{D}$ must satisfy the equation

$$\nabla^{nh}_{\dot{q}}\dot{q}+\mathcal{P}(\mathrm{grad}_{\mathcal{G}}V(q(t)))=0, \tag{6.2}$$

where $\mathcal{P}:TQ\to\mathcal{D}$. Also, recall that the Riemannian $\mathfrak{d}$-connection given at Definition 5.1.2 is the bilinear map satisfying

$$\nabla^{\mathfrak{d}}_{\xi}\eta=\left(\nabla^{nh}_{\xi_R}\eta_R\right)(e), \tag{6.3}$$

Definition 6.1.1. Consider a Lie group G acting on a homogeneous space H. A distribution $\mathcal{D}$ on H is said to be **G-invariant** if $\mathcal{D}_{g\cdot\pi(e)}=(\Phi_g)_*(\mathcal{D}_{\pi(e)})$.

We may consider the horizontal lift of the distribution $\mathcal{D}$ to G. Given $g\in G$ such that $\pi(g)=q$, we have that $\tilde{\mathcal{D}}_g=\{\tilde{v}_g\in\mathrm{Hor}_g\ |(T_g\pi)(\tilde{v}_g)\in\mathcal{D}_q\ \}$. Essentially, $\tilde{\mathcal{D}}_g$ is a subspace of Hor_g composed of the horizontal lift of vectors in $\mathcal{D}_q$.

Proposition 6.1.2. *The distribution $\tilde{\mathcal{D}}_g$ is left-invariant.*

Proof. Let $\mathfrak{d}=\tilde{\mathcal{D}}_e$. Note that $T_e\pi(\mathfrak{d})=\mathcal{D}_{\pi(e)}$ by definition and also that, by G-invariance of $\mathcal{D}$, we have that $\mathcal{D}_{\pi(g)}=(T_{\pi(e)}\Phi_g)(\mathcal{D}_{\pi(e)})$ for all $g\in G$. Thus $\mathcal{D}_{\pi(g)}=(T_e(\Phi_g\circ\pi))(\mathfrak{d})$. In addition, since π commutes with the action Φ_g and the left action, we also have that

$$\mathcal{D}_{\pi(g)}=T_g\pi(T_eL_g(\mathfrak{d})).$$

Notice that $T_eL_g(\mathfrak{d})\in\mathrm{Hor}_g$ since HG is left invariant, and also $T_g\pi|_{\mathrm{Hor}_g}$ maps $\tilde{\mathcal{D}}_g$ isomorphically to $\mathcal{D}_{\pi(g)}$. Both these facts imply that $\tilde{\mathcal{D}}_g=T_eL_g(\mathfrak{d})$. □

As a consequence of the previous proposition we have the next Corollary.

Corollary 6.1.3. There exists a subspace $\mathfrak{d}$ of the Lie algebra $\mathfrak{g}$ such that $\tilde{\mathcal{D}}_g=T_eL_g(\mathfrak{d})$.

Throughout this section we will consider a G-invariant distribution $\mathcal{D}$ on H and since the horizontal lift of $\mathcal{D}$ is a left-invariant distribution $\tilde{\mathcal{D}}$ on G, $\mathfrak{d}$ will be the restriction to the identity of $\tilde{\mathcal{D}}$. The orthogonal complement of $\mathfrak{d}$ with respect to the inner product on $\mathfrak{g}$ will be denoted by $\mathfrak{d}^{\perp}$. Note that $\mathfrak{d}^{\perp}$ has non-zero intersections with both the horizontal space $\mathfrak{h}=H_eG$ and the vertical space $\mathfrak{s}=\mathrm{Ver}_e$, while $\mathfrak{d}\subseteq\mathfrak{h}$.

Consider on G, the orthogonal projections $\tilde{P}:TG\to\tilde{\mathcal{D}}$ and $\tilde{Q}:TG\to\tilde{\mathcal{D}}^{\perp}$. With this projections, we are able to define the nonholonomic connection $\tilde{\nabla}^{nh}$ given by the analogous expression to that from (6.1):

$$\tilde{\nabla}^{nh}_X Y=\tilde{\nabla}_X Y+(\tilde{\nabla}_X\tilde{Q})(Y). \tag{6.4}$$

Similarly, on the manifold H, we can define the nonholonomic connection ∇^{nh} with respect to the Riemannian connection ∇ and the orthogonal projections $\mathcal{P}:TH\to\mathcal{D}$ and $\mathcal{Q}:TH\to\mathcal{D}^{\perp}$. We have the following results relating geodesics with respect to both nonholonomic connections.

Lemma 6.1.4 (Bullo and Lewis, 2019, Section 4.5)**.** *Given a Riemannian manifold Q, letting ∇ be the Levi-Civita connection and $\mathcal{D}$ a nonintegrable distribution then a curve $q:[a,b]\to Q$ is a geodesic of the nonholonomic connection ∇^{nh} if and only if*

$$\nabla_{\dot{q}}\dot{q}\in\mathcal{D}^{\perp} \text{ and } \dot{q}\in\mathcal{D}.$$

Lemma 6.1.5. *A curve $q:[a,b]\to H$ is a geodesic associated with ∇^{nh} and the constraint distribution $\mathcal{D}$ if and only if its tangent lift $\tilde{q}$ is a geodesic with respect to $\tilde{\nabla}^{nh}$ and the constraint distribution $\tilde{\mathcal{D}}$.*

Proof. Suppose $q(t), t\in[a,b]$ is a curve in H such that its tangent lift $\tilde{q}$ is a geodesic with respect to $\tilde{\nabla}^{nh}$ and the constraint distribution $\tilde{\mathcal{D}}$. Thus $\tilde{\nabla}^{nh}_{\dot{\tilde{q}}}\dot{\tilde{q}}=0$ and $\dot{\tilde{q}}\in\tilde{\mathcal{D}}$.

By Lemma 6.1.4, we have that $\tilde{\nabla}_{\dot{\tilde{q}}}\dot{\tilde{q}}\in\tilde{\mathcal{D}}^{\perp}$ and $\dot{\tilde{q}}\in\tilde{\mathcal{D}}$. Using Proposition 2.7.9 and also $T\pi(\tilde{\mathcal{D}}^{\perp})=\mathcal{D}^{\perp}$, we conclude that $\nabla_{\dot{q}}\dot{q}\in\mathcal{D}^{\perp}$ and $\dot{q}\in\mathcal{D}$. Finally, using Lemma 6.1.4 again, we deduce that q satisfies $\nabla^{nh}_{\dot{q}}\dot{q}=0$.

Conversely, if q satisfies $\nabla^{nh}_{\dot{q}}\dot{q}=0$ then, by Lemma 6.1.4, it also satisfies $\nabla_{\dot{q}}\dot{q}\in\mathcal{D}^{\perp}$ and $\dot{q}\in\mathcal{D}$. Therefore, the horizontal lift of $\nabla_{\dot{q}}\dot{q}$, denoted by $\widetilde{\nabla_{\dot{q}}\dot{q}}$ belongs to $\tilde{\mathcal{D}}^{\perp}$. But since by the second statement of Proposition 2.7.9, we have that $\tilde{\nabla}_{\dot{\tilde{q}}}\dot{\tilde{q}}=\widetilde{\nabla_{\dot{q}}\dot{q}}+V$ with $V\in VG$, and noting that $VG\subseteq\tilde{\mathcal{D}}^{\perp}$, we must have that $\tilde{\nabla}_{\dot{\tilde{q}}}\dot{\tilde{q}}\in\tilde{\mathcal{D}}^{\perp}$. Obviously, we have also that $\dot{\tilde{q}}\in\tilde{\mathcal{D}}$. □

Next we show the following result relating the corresponding nonholonomic trajectories in each space.

Theorem 6.1.6. *Let H be a Riemannian homogeneous space with respect to a Lie group action by G, $\tilde{\mathcal{D}}\subseteq HG$ a left-invariant distribution on G and $V:H\to\mathbb{R}$ a potential function. If $g:[a,b]\to G$ is a curve on G, $q(t)=\pi(g(t))$ the projection of g on H and $\xi(t):=(T_{g(t)}L_{g^{-1}(t)})(\dot{g}(t))$, then the following statements are equivalent:*

1. *$g:[a,b]\to G$ is a nonholonomic trajectory associated to the mechanical Lagrangian $L(v_g)=\frac{1}{2}\mathcal{G}(v_g,v_g)-V(g)$, $v_g\in T_gG$ and the distribution $\tilde{\mathcal{D}}$.*
2. *$\xi\in\mathfrak{d}$ on $[a,b]$ and satisfies*

$$\dot{\xi}+\tilde{\nabla}^{\mathfrak{d}}_{\xi}\xi=-\mathfrak{P}\left((T_gL_{g^{-1}})\left(\widetilde{grad}\,\tilde{V}(g(t))\right)\right),$$

where $\mathfrak{P}:\mathfrak{g}\to\mathfrak{d}$ is the projection associated with the decomposition $\mathfrak{g}=\mathfrak{d}\oplus\mathfrak{d}^{\perp}$ and $\tilde{\nabla}^{\mathfrak{d}}$ is the $\mathfrak{d}$-connection associated with $\tilde{\nabla}^{\mathfrak{g}}$ and $\mathfrak{d}$ defined by (6.3).

3. *If $q(t)=\pi(g(t))$, then $\dot{q}\in\mathcal{D}=T\pi(\tilde{\mathcal{D}})$, $g(t)=\tilde{q}(t)$ and q satisfies equations (6.2) associated with the G-invariant metric on H and the distribution $\mathcal{D}$.*

Proof. We first prove the equivalence of the statements (1) and (2) and later the equivalence between (1) and (3). If (1) holds, then the curve g satisfies

$$\tilde{\nabla}^{nh}_{\dot{g}}\dot{g}=\tilde{P}(\tilde{\nabla}_{\dot{g}}\dot{g})$$

which implies that g satisfies

$$\tilde{P}(\tilde{\nabla}_{\dot{g}}\dot{g}) = -\tilde{P}\left(\widetilde{\text{grad}}\,\tilde{V}(g(t))\right).$$

Using the properties of $\tilde{\nabla}$, one deduces that

$$\tilde{P}(T_eL_g(\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi)) = -\tilde{P}\left(\widetilde{\text{grad}}\,\tilde{V}(g(t))\right).$$

Now, since $\tilde{P} \circ (T_eL_g) = (T_eL_g) \circ \mathfrak{P}$ we also have that

$$(T_eL_g \circ \mathfrak{P})(\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi) = -\tilde{P}\left(\widetilde{\text{grad}}\,\tilde{V}(g(t))\right).$$

Equivalently, applying $T_gL_{g^{-1}}$ to both sides and using again the expression relating the projections

$$\mathfrak{P}(\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi) = -\mathfrak{P} \circ T_gL_{g^{-1}}\left(\widetilde{\text{grad}}\,\tilde{V}(g(t))\right).$$

Finally, since $\dot{g} \in \tilde{\mathcal{D}}$ we have that $\xi \in \mathfrak{d}$ and $\dot{\xi} \in \mathfrak{d}$. Therefore, using the fact that $\nabla^{\mathfrak{d}}_{\xi}\eta = \mathfrak{P}(\nabla^{\mathfrak{g}}_{\xi}\eta)$, for all $\xi, \eta \in \mathfrak{d}$ we have that

$$\dot{\xi} + \tilde{\nabla}^{\mathfrak{d}}_{\xi}\xi = -\mathfrak{P} \circ T_gL_{g^{-1}}\left(\widetilde{\text{grad}}\,\tilde{V}(g(t))\right).$$

Reversing all the arguments, we conclude that (2) also implies (1).

Suppose (1) holds. The curve $g(t)$ is horizontal and by definition $\mathcal{D} = T\pi(\tilde{\mathcal{D}})$ so $g(t) = \tilde{q}(t)$ and $\dot{q} \in \mathcal{D}$. Also, if g is a nonholonomic trajectory, it satisfies

$$\tilde{\nabla}^{nh}_{\dot{g}}\dot{g} = -\tilde{P}\left(\widetilde{\text{grad}\tilde{V}}\right).$$

It is not difficult to prove a similar result to that from Lemma 6.1.5 in the presence of a potential function. Indeed, the previous equation is equivalent to

$$\tilde{\nabla}_{\dot{g}}\dot{g} + \widetilde{\text{grad}}\,\tilde{V} \in \tilde{\mathcal{D}}^{\perp}$$

and $\dot{g} \in \tilde{\mathcal{D}}$. The fact that the gradient vector field $\widetilde{\text{grad}}\,\tilde{V}$ projects onto grad V and from Proposition 2.7.9, we have that

$$\nabla_{\dot{q}}\dot{q} + \text{grad}\ V(q(t)) \in \mathcal{D}^{\perp} \text{ and } \dot{q} \in \mathcal{D}$$

which implies that

$$\nabla^{nh}_{\dot{q}}\dot{q} = -\mathcal{P}\left(\text{grad}V(q(t))\right).$$

Conversely, if (3) holds then $g(t)$ is not only horizontal but also $\dot{g}(t) \in \tilde{\mathcal{D}}$. Let $h(t)$ be a curve on G satisfying the equation

$$\tilde{\nabla}^{nh}_{\dot{h}}\dot{h} = -\tilde{P}\left(\widetilde{\text{grad}\tilde{V}}\right),$$

with intial condition $h(0) = g(0)$ and $\dot{h}(0) = \dot{g}(0) \in \tilde{\mathcal{D}}$. $h(t)$ is a curve whose velocity lies in $\tilde{\mathcal{D}}$ for all t and projects to the unique solution $p(t)$ of the equation

$$\nabla^{nh}_{\dot{p}}\dot{p} = -\mathcal{P}\left(\text{grad}V(p(t))\right),$$

with initial position $p(0) = q(0)$ and initial velocity $\dot{p}(0) = \dot{q}(0)$. Hence, $p(t) = q(t)$, and by uniqueness of the horizontal lift $h(t) = g(t)$. □

6.2 Virtual nonholonomic constraints on Riemannian homogeneous spaces

Suppose that H is a homogeneous manifold, acted by a Lie group G and $\pi : G \to H$ denotes the associated projection. Consider the mechanical system given by equation (2.42) in Subsection 2.7.2 on the homogeneous manifold H and a control force $F : TH \times U \to T^*H$ which is of the form

$$F(q, \dot{q}, u) = \sum_{a=1}^{m} u_a F^a(q) \tag{6.5}$$

where F^a is a one-form on H with $m < n$, $U \subset \mathbb{R}^m$ the set of controls and $u_a \in \mathbb{R}$ with $1 \leq a \leq m$ the control inputs. Suppose that the force vector fields $Y^a = \sharp(F^a)$ are G-invariant. In particular, the input distribution $\mathcal{F}$ is G-invariant and $\mathcal{F}_q = (T_{\pi(e)}\Phi_g)(\mathcal{F}_{\pi(e)})$, for $q \in H$ and $g \in G$ such that $q = \Phi_g(\pi(e))$. In addition, suppose that $\mathcal{D}$ is also a G-invariant distribution on H so that $\mathcal{D}_q = (T_{\pi(e)}\Phi_g)(\mathcal{D}_{\pi(e)})$, where $q \in H$.

Using the identification between $\mathfrak{h}$ and $T_{\pi(e)}H$ described in the theory of Riemannian homogeneous spaces, there exists a subspace $\mathfrak{d}$ of $\mathfrak{h}$ such that $T_e\pi(\mathfrak{d}) = \mathcal{D}_{\pi(e)}$. Likewise, there exist a subspace $\mathfrak{f} \subseteq \mathfrak{h}$ such that $T_e\pi(\mathfrak{f}) = \mathcal{F}_{\pi(e)}$. These identifications are particularly important for reproducing the trajectories of the Lie algebra on the homogeneous space and vice-versa.

On the Lie algebra $\mathfrak{g}$, consider the controlled mechanical system of the form

$$\dot{g} = T_e L_g \xi, \quad \dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi + T_g L_{g^{-1}}\left(\widetilde{\text{grad}}\ \tilde{V}(g(t))\right) = \tilde{u}_a f^a, \tag{6.6}$$

where the vectors $f^a \in \mathfrak{h}$ are defined by $T_e\pi(f^a) = Y^a(\pi(e))$ and span the control input subspace $\mathfrak{f} = \text{span}\{f^1, \dots, f^m\}$. This system evolves inside the horizontal bundle, provided that ξ starts in $\mathfrak{h}$. However, we are interested in something else. We would like to know if there exists a control law, that is a function $\tilde{u} : \mathfrak{d} \to U$, forcing the trajectories to remain in $\mathfrak{d}$.

Definition 6.2.1. The above subspace $\mathfrak{f}$ of the horizontal space $\mathfrak{h} \subseteq \mathfrak{g}$ is called the **control input subspace** associated with the mechanical control system (6.6).

Definition 6.2.2. A **virtual nonholonomic constraint** associated with the mechanical system of type (6.6) is a controlled invariant subspace $\mathfrak{d}$ of $\mathfrak{h}$, that is, there exists a control law making the subspace $\mathfrak{d}$ invariant under the flow of the closed-loop system, i.e. $\xi(0) \in \mathfrak{d}$ and $\xi(t) \in \mathfrak{d}$, $\forall t \geq 0$.

With this definition we can state the main result of this section: the existence of a control law making $\mathfrak{d}$ a virtual nonholonomic constraint.

Theorem 6.2.3. *Suppose $\mathfrak{h} = \mathfrak{f} \oplus \mathfrak{d}$. Then there exists a unique control law $\tilde{u}^*$ making $\mathfrak{d}$ a virtual nonholonomic constraint for the controlled mechanical system* (6.6).

Proof. Let $\dim \mathfrak{d} = d$ and $\dim \mathfrak{f} = m = k - d$. Consider the covectors $\mu^1 \dots, \mu^m \in \mathfrak{h}^*$ spanning the annihilator subspace of $\mathfrak{d}$. $\xi(t)$ is a curve on $\mathfrak{h}$ satisfying $\xi(t) \in \mathfrak{d}$ for all time if and only if it satisfies $\mu^a(\xi(t)) = 0$ for all $a = 1, \dots, m$. Differentiating this equation and supposing that $\xi(t)$ is a solution of the closed loop system (6.6) for an appropriate choice of control law $\tilde{u}$, we have that

$$-\mu^a\left(\tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi + (T_gL_{g^{-1}})\left(\widetilde{\text{grad}}\;\tilde{V}(g(t))\right)\right) + \tilde{u}_b\mu^a(f^b) = 0.$$

Since $\tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi + (T_gL_{g^{-1}})\left(\widetilde{\text{grad}}\;\tilde{V}(g(t))\right) \in \mathfrak{h}$ and $\mathfrak{h} = \mathfrak{f} \oplus \mathfrak{d}$, there is a unique way to decompose this vector as the sum

$$\tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi + (T_gL_{g^{-1}})\left(\widetilde{\text{grad}}\;\tilde{V}(g(t))\right) = \eta(t) + \tau_b(t)f^b,$$

with $\eta(t) \in \mathfrak{d}$. In addition, note that the coefficients τ_b may be regarded as functions on $\mathfrak{h}$. In fact, its definition is associated with the projection to $\mathfrak{f}$ together with the choice of $\{f^b\}$ as a basis for $\mathfrak{f}$. Therefore, $\mu^a(\dot{\xi}(t)) = 0$ if and only if

$$(\tau_b - \tilde{u}_b)\mu^a(f^b) = 0.$$

Since $\mu^a(f^b)$ is an invertible matrix, we conclude that $\tau_b = \tilde{u}_b$ proving existence and uniqueness of a control law making $\mathfrak{d}$ a virtual constraint. □

By construction, if $(g(t), \xi(t))$ is a trajectory of the controlled mechanical system (6.6), then $g(t)$ is a horizontal curve. We will prove next that g is in fact the horizontal lift of the curve $q(t) = \pi(g(t))$ and that q is the trajectory of a controlled mechanical system of the form

$$\nabla_{\dot{q}(t)}\dot{q}(t) = -\text{grad}\; V(q(t)) + u_a(t)Y^a(q(t)).$$

In addition, if $\xi(t) \in \mathfrak{d}$ for all t, then $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t. Then, by uniqueness of the control law making the distribution $\mathcal{D}$ a virtual nonholonomic constraint, the control law $\tilde{u}^* \in U$ given in Theorem 6.2.3 must also be the unique control law making $\mathcal{D}$ control invariant. We summarize these facts in the next result.

Theorem 6.2.4. *Let $\tilde{u}^* : \mathfrak{d} \to U$ be the unique control law given by Theorem 6.2.3 and $(g, \xi) : [a, b] \to G \times \mathfrak{g}$ the solution of the associated closed-loop system*

$$\begin{aligned} &\dot{g} = (T_eL_g)(\xi) \\ &\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi + (T_gL_{g^{-1}})\left(\widetilde{grad}\;\tilde{V}(g(t))\right) = \tilde{u}^*_a(\xi)f^a, \end{aligned} \tag{6.7}$$

with $\xi(0) \in \mathfrak{d}$. If $q(t) = \pi(g(t))$ then $\dot{q}(t) \in \mathcal{D}_{q(t)}$, $g(t) = \tilde{q}(t)$ and q satisfies

$$\nabla_{\dot{q}(t)}\dot{q}(t) = -grad\; V(q(t)) + u^*_a(t)Y^a(q(t)).$$

with $u^*_a(t) := u^*_a(q(t), \dot{q}(t)) = \tilde{u}^*_a(\xi(t))$.

Proof. By construction of the $\mathfrak{g}$-connection, if (g, ξ) satisfies equation (6.6), then g satisfies

$$\tilde{\nabla}_{\dot{g}}\dot{g} + \widetilde{\text{grad}}\;\tilde{V}(g(t)) = \tilde{u}^*_a((T_gL_{g^{-1}})(\dot{g}))(f^L)^a,$$

where $(f^L)^a = T_eL_g(f^a)$. Moreover, since $\xi(t) \in \mathfrak{d}$ in $[a, b]$ we have that $\dot{g}(t) \in \tilde{\mathcal{D}}$. In particular, g is horizontal and, hence, $g(t) = \tilde{q}(t)$.

By Proposition 2.7.9, $T\pi$ projects $\tilde{\nabla}_{\dot{\tilde{q}}}\dot{\tilde{q}}$ to $\nabla_{\dot{q}}\dot{q}$. In addition, from Lemma 2.7.10, $\widetilde{\text{grad}}\ \tilde{V}(\tilde{q}(t))$ is the horizontal vector field projected onto grad $V(q(t))$ by $T\pi$. Finally,

$$T_g\pi(f^L)^a(g) = T_g\pi(T_eL_g(f^a)) = T_{\pi(e)}\Phi_g(T_e\pi(f^a)),$$

where the last equality comes from the relation $\pi \circ L_g = \Phi_g \circ \pi$. Now, using that $Y^a(e) = T_e\pi(f^a)$ and the G-invariance of Y^a, implying that $Y^a(g) = T_{\pi(e)}\Phi_g(Y^a(e))$ we deduce that $T_g\pi(f^L)^a(g) = Y^a$.

If we define $u_a^*(q(t), \dot{q}(t)) := \tilde{u}_a^*(T_{\tilde{q}(t)}L_{\tilde{q}(t)^{-1}}\dot{\tilde{q}}(t))$, we obtain the desired result. □

6.3 The induced constrained connection

Let us define the projections $\mathfrak{p}_{\mathfrak{d}} : \mathfrak{g} \to \mathfrak{d}$ and $\mathfrak{p}_{\mathfrak{f}} : \mathfrak{g} \to \mathfrak{f} \oplus \mathfrak{s}$, associated with the direct sum $\mathfrak{h} = \mathfrak{d} \oplus \mathfrak{f}$. Note that, from the first decomposition of the Lie algebra in terms of horizontal and vertical spaces, i.e., $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$, we obtain the three-part decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{d} \oplus (\mathfrak{f} \oplus \mathfrak{s})$ to which the above projections are associated.

Now let us define the Lie algebra connection called $\tilde{\nabla}^{\mathfrak{d},\mathfrak{f}}$-connection given by

$$\tilde{\nabla}^{\mathfrak{d},\mathfrak{f}}_{\xi}\eta = \tilde{\nabla}^{\mathfrak{g}}_{\xi}\eta + (\tilde{\nabla}^{\mathfrak{g}}_{\xi}\mathfrak{p}_{\mathfrak{f}})(\eta),$$

Theorem 6.3.1. *The trajectory of the closed-loop system* (6.7) *is a trajectory of the equations*

$$\begin{aligned} &\dot{g} = (T_eL_g)(\xi), \\ &\dot{\xi} + \tilde{\nabla}^{\mathfrak{d},\mathfrak{f}}_{\xi}\xi + \mathfrak{p}_{\mathfrak{d}}\left((T_gL_{g^{-1}})\left(\widetilde{grad}\ \tilde{V}(g(t))\right)\right) = 0. \end{aligned} \tag{6.8}$$

Proof. Similarly to Lemma 3.1.12 we have that $\xi \in \mathfrak{d}$, then $\tilde{\nabla}^{\mathfrak{d},\mathfrak{f}}_{\xi}\xi = \mathfrak{p}_{\mathfrak{d}}\left(\tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi\right)$. Attending to the fact that $\xi(t) \in \mathfrak{d}$ along the solutions of the closed loop system, then we have that

$$\dot{\xi} + \tilde{\nabla}^{\mathfrak{d},\mathfrak{f}}_{\xi}\xi = \dot{\xi} + \mathfrak{p}_{\mathfrak{d}}\left(\tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi\right) = \mathfrak{p}_{\mathfrak{d}}\left(\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi\right)$$

And using that $\mathfrak{p}_{\mathfrak{d}}(f_a) = 0$, we deduce

$$\dot{\xi} + \tilde{\nabla}^{\mathfrak{d},\mathfrak{f}}_{\xi}\xi = -\mathfrak{p}_{\mathfrak{d}}\left((T_gL_{g^{-1}})\left(\widetilde{\text{grad}}\ \tilde{V}(g(t))\right)\right)$$

□

Remark 6.3.2. Consider the affine connection $\nabla^{\mathcal{D},\mathcal{F}}$ on H associated with the projections $P_{\mathcal{D}} : TH \to \mathcal{D}$ and $P_{\mathcal{F}} : TH \to \mathcal{F}$. Then the trajectories (g, ξ) of (6.8) generate a trajectory $q(t) = \pi(g(t))$ which satisfies

$$\nabla^{\mathcal{D},\mathcal{F}}_{\dot{q}(t)}\dot{q}(t) = -P_{\mathcal{D}}(\text{grad } V(q(t))).$$

This is a consequence of Theorems 6.2.4 and 6.3.1 together with Theorem 2 in Simoes et al., 2023.

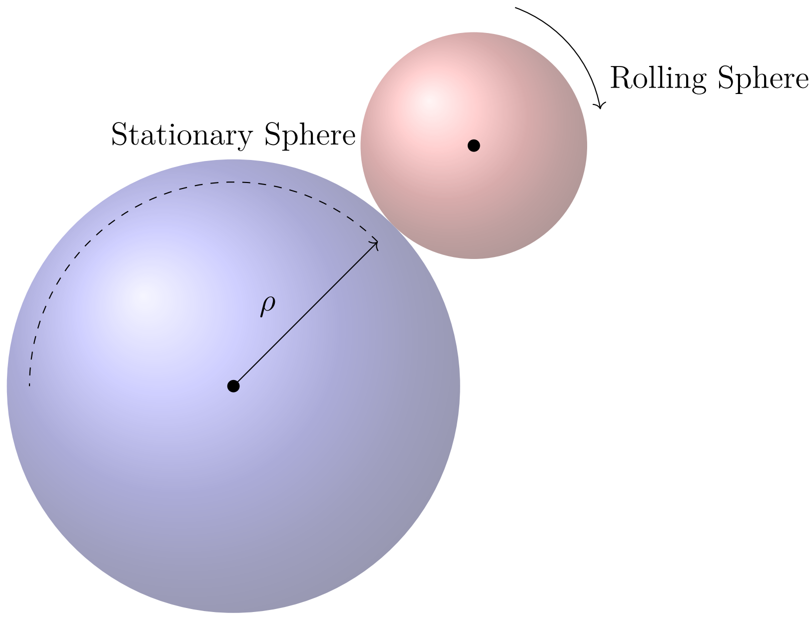


Figure 6.1: Rolling sphere on a stationary sphere. The red sphere rolls on the blue sphere, with the dashed arc showing the trajectory.

6.4 A sphere rolling over another sphere

Consider a sphere rolling on another sphere. For simplicity we consider the radius of the rolling sphere to be 1 and that of the stationary sphere equal to $\rho > 1$. The rolling sphere is equipped with an orthonormal frame fixed at its center. The motion of the sphere is completely described by its position on the standing sphere given by a normalized vector $r \in \mathbb{S}^2$ and by a rotation matrix $R \in \mathrm{SO}(3)$ which gives the orientation of the frame of the rolling sphere related to a fixed frame at the center of the stationary one. Thus, the configuration space is $H = \mathbb{S}^2 \times \mathrm{SO}(3)$.

The Lie group SO(3) acts transitively on the 2-sphere $\mathbb{S}^2$ by left multiplication, hence the latter is a homogeneous space i.e. it can be seen as the quotient of the SO(3) and a closed subgroup of SO(3), namely $\mathbb{S}^2 \simeq \mathrm{SO}(3)/\tilde{K}$, where the subgroup $\tilde{K}$ is isomorphic to SO(2) expressing its elements $k \in \tilde{K}$ by $k = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\theta \in \mathbb{S}^1$. For the second component of the configuration manifold H, that is SO(3), which is a Lie group, the same occurs trivially with the left-multiplication on itself and the homogeneous structure is $\mathrm{SO}(3) \simeq \mathrm{SO}(3)/I$, where I is the identity matrix. Ultimately, the Lie group $G = \mathrm{SO}(3) \times \mathrm{SO}(3)$ acts transitively on H by the Lie group action

$$\Psi : G \times H \to H, \quad \text{with} \quad \Psi_{(S,R)}(r, T) = (Sr, RT).$$

For $e_3 = [0\ 0\ 1]^T$, we define $K = \mathrm{Stab}((e_3, I))$ which is the stabilizer subgroup of the action Ψ. The projection map $\pi : G \to H$ is given by $\pi(S, R) = (Se_3, RI)$. We will denote the projection $SO(3) \to \mathbb{S}^2$ also by $\tilde{\pi}$, i.e., $\tilde{\pi}(S) = Se_3, S \in \mathrm{SO}(3)$. Using the hat map, we identify $\mathfrak{so}(3) \cong \mathbb{R}^3$. Consider the orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ of $\mathfrak{so}(3)$, where e_1, e_2, e_3 is the standard basis on $\mathbb{R}^3$ i.e.

$$\hat{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \hat{e}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so, we have $\pi_*(\hat{e}_1, \cdot) = (-e_2, \cdot), \pi_*(\hat{e}_2, \cdot) = (e_1, \cdot)$ and $\pi_*(\hat{e}_3, \cdot) = (0, \cdot)$. Suppose the Lie group $G = \mathrm{SO}(3) \times \mathrm{SO}(3)$ is equipped with a left-invariant metric given by the inner product on $\mathfrak{g} = \mathfrak{so}(3) \times \mathfrak{so}(3)$, i.e.

$$\langle (\hat{\Pi}_1, \hat{\Omega}_1), (\hat{\Pi}_2, \hat{\Omega}_2) \rangle_{\mathfrak{g}} = \langle \hat{\Pi}_1, \hat{\Pi}_2 \rangle_{\mathfrak{so}(3)} + \langle \hat{\Omega}_1, \hat{\Omega}_2 \rangle_{\mathfrak{so}(3)} = \Pi_1^T \Pi_2 + \Omega_1^T \mathbb{J} \Omega_2$$

for all $\Pi_1, \Pi_2, \Omega_1, \Omega_2 \in \mathbb{R}^3$ and $\mathbb{J}$ the moment of inertia tensor. Using this left-invariant metric of the Lie group G we define an inner product on $T_{e_3}\mathbb{S}^2 \times \mathfrak{so}(3)$ via the relation

$$\langle X, Y \rangle_{T_{e_3}\mathbb{S}^2 \times \mathfrak{so}(3)} := \langle \pi_*^{-1} X, \pi_*^{-1} Y \rangle_{\mathfrak{g}} = x \cdot y + \bar{X}^T \mathbb{J} \bar{Y}$$

for all $X, Y \in T_{e_3}\mathbb{S}^2 \times \mathfrak{so}(3)$ where $X = (x, \bar{X})$ and $Y = (y, \bar{Y})$. Following J. Goodman and Colombo, 2024 we have that the first part of $\langle \cdot, \cdot \rangle_{T_{e_3}\mathbb{S}^2 \times \mathfrak{so}(3)}$ is the standard Euclidean metric with respect to the basis $\{e_1, e_2\}$. Thus, we can extend this inner product to an SO(3)-invariant Riemannian metric on $H = \mathbb{S}^2 \times \mathrm{SO}(3)$ by left-action given by

$$\begin{aligned} \langle X, Y \rangle_H &= \langle \bar{R}^{-1} X, \bar{R}^{-1} Y \rangle_{T_{e_3}\mathbb{S}^2 \times \mathfrak{so}(3)} = \langle \pi_*^{-1}(\bar{R}^{-1} X), \pi_*^{-1}(\bar{R}^{-1} Y) \rangle_{\mathfrak{g}} \\ &= S^T x \cdot S^T y + (R^T \bar{X})^T \mathbb{J} R^T \bar{Y} = x \cdot y + \bar{X}^T (R \mathbb{J} R^T) \bar{Y}, \end{aligned}$$

for all $X, Y \in T_q\mathbb{S}^2 \times T_R\mathrm{SO}(3)$, $\bar{R} = (S, R) \in G$ such that $\pi(S, R) = (q, R)$, and for $X = (x, \bar{X})$ and $Y = (y, \bar{Y})$, $\bar{R}^{-1} X = (S^{-1}, R^{-1}) \cdot (x, \bar{X}) = (S^{-1} x, R^{-1} \bar{X})$.

With the Riemannian homogeneous structure above, we have $\mathfrak{s} = \ker(\pi_*|_{\mathfrak{g}}) = \mathrm{span}\{(\hat{e}_3, 0)\}$ and we define $\mathfrak{h} = \mathfrak{s}^\perp$ such that $\mathfrak{h} = \mathrm{span}\{(\hat{e}_1, 0), (\hat{e}_2, 0), (0, \hat{e}_1), (0, \hat{e}_2), (0, \hat{e}_3)\}$. For the inner product on $\mathfrak{g}$, the flat map $\flat_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}^*$ is given by $\flat_{\mathfrak{g}}(\Pi, \Omega) = (\Pi, \mathbb{J}\Omega)$ and its inverse, the sharp map $\sharp_{\mathfrak{g}} : \mathfrak{g}^* \to \mathfrak{g}$, is given by $\sharp_{\mathfrak{g}}(\mu, \nu) = (\mu, \mathbb{J}^{-1}\nu)$, where μ and ν are vectors in $\mathbb{R}^3$ identified with the matrices $\bar{\mu}$ and $\bar{\nu}$ in $\mathfrak{g}^*$ through the dual pairing $\left\langle \bar{\mu}, \hat{\Pi} \right\rangle = \mu^T \Pi$ and $\left\langle \bar{\nu}, \hat{\Omega} \right\rangle = \nu^T \Omega$. The adjoint operator $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ for $\mathfrak{g} = \mathfrak{so}(3) \times \mathfrak{so}(3)$ is given by

$$\mathrm{ad}_\xi \eta = (\mathrm{ad}_{\hat{\Pi}_1} \hat{\Pi}_2, \mathrm{ad}_{\hat{\Omega}_1} \hat{\Omega}_2) = ([\hat{\Pi}_1, \hat{\Pi}_2], [\hat{\Omega}_1, \hat{\Omega}_2]) = (\widehat{\Pi_1 \times \Pi_2}, \widehat{\Omega_1 \times \Omega_2})$$

where $\mathrm{ad}_{\hat{\Pi}_1} \hat{\Pi}_2$ and $\mathrm{ad}_{\hat{\Omega}_1} \hat{\Omega}_2$ are the adjoint operators on $\mathfrak{so}(3)$ given by the cross product of vectors on $\mathbb{R}^3$ using the hat map, $\xi = (\hat{\Pi}_1, \hat{\Omega}_1)$ and $\eta = (\hat{\Pi}_2, \hat{\Omega}_2)$. The co-adjoint operator is given by $\mathrm{ad}^*_{(\hat{\Pi}, \hat{\Omega})}(\mu, \nu) = (\mu \times \Pi, \nu \times \Omega)$.

Since $\mathfrak{s} = \mathrm{span}\{(\hat{e}_3, 0)\}$, the vertical space of G at $g = (S, R)$ is defined by $\mathrm{Ver}_S \times \{0\} = \mathrm{span}\{(T_I L_S(\hat{e}_3), 0)\}$ and the horizontal space is defined as $\mathrm{Hor}_S \times T_R\mathrm{SO}(3)$ where $\mathrm{Hor}_S = \mathrm{span}\{(T_I L_S(\hat{e}_1), T_I L_S(\hat{e}_2))\}$. The horizontal projection can be calculated by $\mathcal{H}(\hat{\Pi}, \hat{\Omega}) = (\widehat{\Pi \times e_3}, \hat{\Omega})$.

The $\mathfrak{g}$-connection is given by

$$\tilde{\nabla}^{\mathfrak{g}}_{\xi}\eta = \frac{1}{2}\left(\widehat{(\Pi_1 \times \Pi_2)}, \widehat{\Omega_1 \times \Omega_2} - \mathbb{J}^{-1}(\mathbb{J}\Omega_1 \times \widehat{\Omega_2 - \mathbb{J}}\Omega_2 \times \Omega_1)\right),$$

where $\xi = (\hat{\Pi}_1, \hat{\Omega}_1)$ and $\eta = (\hat{\Pi}_2, \hat{\Omega}_2)$. For a horizontal curve $\bar{R} : [a, b] \to G$ we have from Remark 2.6.14 that

$$\begin{aligned}\tilde{\nabla}_{\dot{\bar{R}}}\dot{\bar{R}}(t) &= \bar{R}(t)\left(\dot{\xi} + \tilde{\nabla}^{\mathfrak{g}}_{\xi}\xi(t)\right)\\ &= (S, R)\left(\dot{\hat{\Pi}}, \dot{\hat{\Omega}} - \left(\mathbb{J}^{-1}\widehat{(\mathbb{J}\Omega \times \Omega)}\right)\right),\end{aligned}$$

where $\xi = \bar{R}^{-1}\dot{\bar{R}}$ and $\bar{R} = (S, R)$. In particular, if $\bar{R}$ is a horizontal geodesic then from the respective equation $\xi = (\hat{\Pi}, \hat{\Omega})$ satisfies

$$\dot{\Pi} = 0, \quad \mathbb{J}\dot{\Omega} = \mathbb{J}\Omega \times \Omega.$$

Note here that the second equation is the usual Euler equation for a rigid body. Suppose we want to impose the non-slipping condition expressed by the nonholonomic constraints equations

$$\dot{q} \cdot Re_1 = -e_2^T\omega, \text{ and } \dot{q} \cdot Re_2 = e_1^T\omega$$

which, in the north pole, can be written as

$$\dot{x} = -e_2^T\omega, \quad \dot{y} = e_1^T\omega.$$

The constraint in the tangent space of the north pole, T_qH, can be written as

$$\operatorname{span}\{(e_2, \hat{e}_1), (-e_1, \hat{e}_2), (0, \hat{e}_3)\} \subseteq T_qH$$

and the same constraints expressed in the Lie algebra $\mathfrak{so}(3)$ define the subspace $\mathfrak{d} = \operatorname{span}\{(-\hat{e}_1, \hat{e}_1), (-\hat{e}_2, \hat{e}_2), (0, \hat{e}_3)\} \subseteq \mathfrak{h}$. Thus, we define $\mathfrak{f} = \operatorname{span}\{(\hat{e}_1, \hat{e}_1), (\hat{e}_2, \hat{e}_2)\}$ and we look for a control law that makes the system

$$\dot{\Pi} = \mathbf{u}, \quad \mathbb{J}\dot{\Omega} = \mathbb{J}\Omega \times \Omega + \mathbf{u} \tag{6.9}$$

control invariant, where $\mathbf{u} = (u_1, u_2, 0) \in \mathbb{R}^3$. Let us consider $\mathbb{J}$ to be a diagonal matrix with entries $J_i, i = 1, 2, 3$. Differentiating the constraint equations and expressing them in terms of the Lie group, we get

$$\dot{\Pi}_2 = -e_2^T\dot{\Omega}, \quad \dot{\Pi}_1 = -e_1^T\dot{\Omega}.$$

Thus, using the equations (6.9), we have that the unique control law that makes $\mathfrak{d}$ a virtual nonholonomic constraint is

$$u_1 = \frac{J_3 - J_2}{J_1 + 1}\Omega_2\Omega_3, \quad u_2 = \frac{J_1 - J_3}{J_2 + 1}\Omega_1\Omega_3.$$

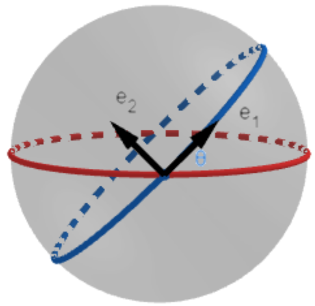


Figure 6.2: The sphere and the point of contact with the blade. The red circle is the equator, e_1 is the velocity vector of the blade, the coordinate system is given by $\{e_1, e_2\}$ and the orientation is given by the angle ϑ.

6.5 A Blade moving on a sphere

Consider a blade moving on a sphere. To analyze the system fix a great circle on the sphere, called the equator and a coordinate system fixed on the body $\{e_1, e_2\}$, attached to the point of contact of the blade. The configuration of the body is described by its position on the sphere $r \in \mathbb{S}^2$, and the angle ϑ defined as the angle between the tangent vector to the equator and the velocity vector of the geodesic passing through r with direction e_1 at the point at which the two great circles intersect. Hence, the configuration is $H = \mathbb{S}^2 \times \mathbb{S}^1$.

Regarding the first component of H, the analysis in the previous example applies here as well and $\mathbb{S}^1$ is a Lie group so H is a homogeneous space. Thus, we have that the Lie group $G = \mathrm{SO}(3) \times \mathbb{S}^1$ acts transitively on H by the action

$$\Psi : G \times H \to H, \quad \text{with} \quad \Psi_{(S,\varphi)}(r, \vartheta) = (Sr, \varphi + \vartheta).$$

For $e_3 = [0\ 0\ 1]^T$ we define $K = \mathrm{Stab}((e_3, 0))$ which is the stabilizer subgroup of the action Ψ. The projection map is $\pi : G \to H$ is given by $\pi(S, \varphi) = (Se_3, \varphi)$. By abuse of notation, we will denote the projection $SO(3) \to \mathbb{S}^2$ also by π, i.e., $\pi(S) = Se_3, S \in \mathrm{SO}(3)$. The Lie algebra is $\mathfrak{so}(3) \times \mathbb{R}$, where the first component is as in Example 6.4. Suppose the Lie group $G = \mathrm{SO}(3) \times \mathbb{S}^1$ is equipped with a left-invariant metric given by the inner product on $\mathfrak{g} = \mathfrak{so}(3) \times \mathbb{R}$, i.e.

$$\langle (\hat{\Pi}_1, \omega_1), (\hat{\Pi}_2, \omega_2) \rangle_{\mathfrak{g}} = \langle \hat{\Pi}_1, \hat{\Pi}_2 \rangle_{\mathfrak{so}(3)} + \langle \omega_1, \omega_2 \rangle_{\mathbb{R}} = \Pi_1^T \Pi_2 + \omega_1 \omega_2$$

for all $\Pi_1, \Pi_2 \in \mathbb{R}^3$ and $\omega_1, \omega_2 \in \mathbb{R}$. Using this left-invariant metric we define an inner product on $T_{e_3}\mathbb{S}^2 \times \mathbb{R}$ via the relation

$$\langle X, Y \rangle_{T_{e_3}\mathbb{S}^2 \times \mathbb{R}} := \langle \pi_*^{-1} X, \pi_*^{-1} Y \rangle_{\mathfrak{g}} = x \cdot y + \omega_1 \omega_2$$

for all $X, Y \in T_{e_3}\mathbb{S}^2 \times \mathbb{R}$, where $X = (x, \omega_1), Y = (y, \omega_2)$. As previously, we can extend this inner product to a G-invariant Riemannian metric on $H = \mathbb{S}^2 \times \mathbb{S}^1$ by left-action given by

$$\begin{aligned} \langle X, Y \rangle_H &= \langle \bar{R}^{-1} X, \bar{R}^{-1} Y \rangle_{T_{e_3}\mathbb{S}^2 \times \mathbb{R}} = \langle \pi_*^{-1}(\bar{R}^{-1} X), \pi_*^{-1}(\bar{R}^{-1} Y) \rangle_{\mathfrak{g}} \\ &= S^T x \cdot S^T y + \omega_1 \omega_2 \\ &= x \cdot y + \omega_1 \omega_2 \end{aligned}$$

for all $X, Y \in T_q\mathbb{S}^2 \times T_\theta\mathbb{S}^1$, $\bar{R} = (S, \varphi) \in G$ such that $\pi(S, \varphi) = (q, \theta)$ where $\bar{R}^{-1}X = (S^{-1}, -\varphi)(x, \omega_1) = (S^{-1}x, \omega_1)$ and $X = (x, \omega_1), Y = (y, \omega_2)$.

With the Riemannian homogeneous structure above we have that $\mathfrak{s} = \ker(\pi_*|_\mathfrak{g}) = \text{span}\{(\hat{e}_3, 0)\}$ and $\mathfrak{h} = \mathfrak{s}^\perp = \text{span}\{(\hat{e}_1, 0), (\hat{e}_2, 0), (0, 1)\}$. Associated with the inner product, the flat map $\flat_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}^*$ is given by $\flat_\mathfrak{g}(\Pi, \omega) = (\Pi, \omega)$ and its inverse, the sharp map $\sharp_\mathfrak{g} : \mathfrak{g}^* \to \mathfrak{g}$, is given by $\sharp_\mathfrak{g}(\mu, \lambda) = (\mu, \lambda)$ where μ is a vector in $\mathbb{R}^3$ identified with the matrix $\bar{\mu}$ in $\mathfrak{g}^*$ through the dual pairing $\left\langle \bar{\mu}, \hat{\Pi} \right\rangle = \mu^T \Pi$ and $\lambda \in \mathbb{R}$. The adjoint operator of $\mathfrak{g} = \mathfrak{so}(3) \times \mathbb{R}$ to itself is given by $\text{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$,

$$\text{ad}_\xi \eta = (\text{ad}_{\hat{\Pi}_1} \hat{\Pi}_2, \text{ad}_{\omega_1} \omega_2) = ([\hat{\Pi}_1, \hat{\Pi}_2], 0) = (\widehat{\Pi_1 \times \Pi_2}, 0)$$

where $\text{ad}_{\hat{\Pi}_1} \hat{\Pi}_2$ is the adjoint operator on $\mathfrak{so}(3)$, $\xi = (\hat{\Pi}_1, \omega_1)$ and $\eta = (\hat{\Pi}_2, \omega_2)$. The co-adjoint operator is given by $\text{ad}^*_{(\hat{\Pi}, \omega)}(\mu, \nu) = (\mu \times \Pi, 0)$.

Since $\mathfrak{s} = \text{span}\{(\hat{e}_3, 0)\}$ the vertical space of G is defined by $\text{Ver}_S \times \{0\} = \text{span}\{(T_I L_S(\hat{e}_3), 0)\}$ and the horizontal space is defined as $\text{Hor}_S \times \mathbb{S}^1$ where $\text{Hor}_S = \text{span}\{T_I L_S(\hat{e}_1), T_I L_S(\hat{e}_2)\}$ for $S \in \text{SO}(3)$. The horizontal projection in given by $\mathcal{H}(\hat{\Pi}, \omega) = (\widehat{\Pi \times e_3}, \omega)$ and the $\mathfrak{g}$-connection by

$$\tilde{\nabla}^\mathfrak{g}_\xi \eta = \frac{1}{2}(\widehat{\Pi_1 \times \Pi_2}, 0),$$

where $\xi = (\hat{\Pi}_1, \omega_1)$ and $\eta = (\hat{\Pi}_2, \omega_2)$. For a horizontal curve $\bar{R} : [a, b] \to G$ we have from Remark 2.6.14 that

$$\tilde{\nabla}_{\dot{\bar{R}}} \dot{\bar{R}}(t) = \bar{R}(t) \left(\dot{\xi} + \tilde{\nabla}^\mathfrak{g}_\xi \xi(t) \right) = (S, R) \left(\dot{\hat{\Pi}}, \dot{\hat{\Omega}} \right),$$

where $\xi = \bar{R}^{-1} \dot{\bar{R}}$ and $\bar{R} = (S, \vartheta)$. In particular, if $\bar{R}$ is a horizontal geodesic then from the respective equation $\xi = (\hat{\Pi}, \omega)$ satisfies

$$\dot{\Pi} = 0, \quad \dot{\omega} = 0.$$

The above equations in the homogeneous space take the form

$$R(\dot{\Pi} \times e_3) = 0, \quad \dot{\omega} = 0.$$

For simplicity consider that the equator passes from the north pole then the constraint on the tangent plane at the north pole is given by the knife edge constraint (see A. Bloch, 2015, Section 1.6) $\dot{x} \sin\vartheta = \dot{y} \cos\vartheta$. This equation defines a vector space $\mathcal{D}_{e_3}$ and the same constraints expressed in the Lie algebra $\mathfrak{g}$ define a distribution $\mathfrak{d}$ given by

$$\mathcal{D}_{e_3} = \text{span}\{X = \cos\vartheta e_1 + \sin\vartheta e_2\} \times \mathbb{R} \subset T_{e_3} H$$
$$\mathfrak{d} = \text{span}\{(-\sin\vartheta \hat{e}_1 + \cos\vartheta \hat{e}_2, 0), (0, 1)\} \subseteq \mathfrak{h}$$

respectively. Thus, we define $\mathfrak{f} = \text{span}\{(\cos\vartheta \hat{e}_1 + \sin\vartheta \hat{e}_2, 0)\}$ and we look for a control law that makes the system

$$\dot{\Pi}_1 = u \cos\vartheta, \quad \dot{\Pi}_2 = u \sin\vartheta, \quad \dot{\Pi}_3 = 0, \quad \dot{\omega} = 0 \tag{6.10}$$

control invariant. Differentiating the constraint equations and expressing them in terms of the Lie group, we get

$$\dot{\Pi}_2 \sin\vartheta + \dot{\Pi}_2 \omega \cos\vartheta + \dot{\Pi}_1 \cos\vartheta - \dot{\Pi}_1 \omega \sin\vartheta = 0$$

Thus, using the equations (6.10) we have that the unique control law making $\mathfrak{D}$ a virtual nonholonomic constraint is

$$u^* = \omega \left(\Pi_1 \sin\vartheta - \Pi_2 \cos\vartheta\right).$$

Chapter 7

Geometric stabilization for virtual nonholonomic constraints

In this chapter, we examine the stabilization of systems around desired manifolds of the phase space, determined by virtual nonlinear nonholonomic constraints. We prove the existence of a control law ensuring that the system adheres to the constraints. In addition, we demonstrate that if the system already satisfies the constraints at some point, the control law aligns with the unique control law derived in Theorem 4.2.2, which guarantees the existence of a virtual nonlinear nonholonomic constraint. The approach is focused on nonlinear constraints where linear and affine constraints appear as special cases. We derive flexible control laws containing gain matrices able to accommodate different rates of convergence. In particular, these advances allow us to cover more interesting cases in engineering and could be used to tackle more challenging problems in robotic locomotion. We will consider a general nonholonomic mechanical system as described at the beginning of Chapter 4 but for convenience some key features are presented here as well.

7.1 Stabilization of virtual nonholonomic constraints

Consider a mechanical system that evolves in a manifold Q described by a mechanical type Lagrangian $L(q,\dot{q}) = K(q,\dot{q}) - V(q)$, $q \in Q$. Also, consider nonholonomic constraints that are defined by the level set of a function of the type $\Phi : TQ \to \mathbb{R}^m$, $m < n = \dim Q$ i.e., $\mathcal{M} = \Phi^{-1}(\{0\})$ and we denote the coordinate functions of the constraint by $\Phi = (\phi^1, \ldots, \phi^m)$.

Moreover, recall the velocity-dependent distribution $S(v_q)$ which is a subspace of T_qQ defined by

$$S(v_q) = \Big\{ X \in T_qQ \mid \Big\langle \frac{\partial \phi^a}{\partial \dot{q}^i}(v_q) dq^i, X \Big\rangle = 0, \ a = 1, \ldots, m \Big\},$$

with $v_q \in \mathcal{M}$ and consider the control and external forces as they were presented in Chapter 3. Briefly, $F^0 : TQ \to T^*Q$ and $F : TQ \times U \to T^*Q$ are the external force and the control force respectively where

$$F(q,\dot{q},u) = \sum_{a=1}^{m} u_a F^a(q,\dot{q})$$

with the set of controls $F^a(q,\dot{q}) \in T^*Q$ ($m < n$, $U \subset \mathbb{R}^m$) and the control inputs $u_a \in \mathbb{R}$ ($1 \leq a \leq m$). Lastly, consider the associated mechanical control system (3.3),

$$\nabla_{\dot{q}}\dot{q} = Y^0(q,\dot{q}) + u_a Y^a(q,\dot{q}), \tag{7.1}$$

with $Y^0(q,\dot{q}) = \sharp_{\mathcal{G}}(F^0(q,\dot{q}))$ and $Y^a = \sharp_{\mathcal{G}}(F^a(q,\dot{q}))$.

The next corollary follows directly from the proof of Theorem 4.2.2.

Corollary 7.1.1. Let $\mathcal{M}$ be the constraint submanifold defined by $\Phi(q,\dot{q}) = 0$ and $\mathcal{S}(v_q)$ be the velocity-dependent distribution. Suppose that $\mathcal{S}(v_q)$ is transversal to the input distribution $\mathcal{F}$ generated by the vector fields $\{Y^a\}$ and that $T_{v_q}\mathcal{M} \cap \mathcal{F}^V = \{0\}$, then the matrix

$$C^{ab} = (Y^a)^V(\phi^b)$$

where ϕ^b are the components of $\Phi(q,\dot{q})$, is invertible with smooth inverse.

Theorem 7.1.2. *Given a mechanical control system of the form* (7.1) *and a virtual constraint submanifold $\mathcal{M}$ determined by $\phi^b = 0$, $b = 1,\dots,m$, suppose that the input distribution $\mathcal{F}$, generated by the vector field $\{Y^a\}$, is transversal to the velocity-dependent distribution, $S(v_q)$.*

Then, the control law $\hat{u} : TQ \to \mathbb{R}^m$ given by the expression

$$\hat{u}_a = C_{ab}(-K\phi^b - G(\phi^b)), \tag{7.2}$$

where G is the geodesic vector field, C_{ab} is the inverse matrix of $C^{ab} = (Y^a)^V(\phi^b)$ and K is a diagonal matrix with positive design parameters k_i, $i = 1,\dots,m$, satisfies

1. *$\phi^b \to 0$ exponentially fast along the system trajectories, for $b = 1,\dots,m$.*
2. *$\hat{u}|_{\mathcal{M}}$ is the unique control law whose existence is guaranteed by Theorem 4.2.2.*

Proof. We show at first that the control law given by (7.2) exponentially stabilizes the controlled system (7.1), in the sense that it drives the system into complying with the constraint exponentially fast. A trajectory, q, of the closed loop system (7.1) is an integral curve of the SODE vector field Γ of the form (2.23). Note that G is the geodesic vector field associated to the Riemannian metric $\mathcal{G}$. Locally, we write

$$\Gamma(q,\dot{q}) = \dot{q}^i\frac{\partial}{\partial q^i} + \left(-\Gamma^j_{ik}\dot{q}^j\dot{q}^k + Y^0_i + u^a Y^a_i\right)\frac{\partial}{\partial \dot{q}^i},$$

where Γ^j_{ik} are the Christoffel symbols. Since the curve q should exponentially vanish the constraint we examine ϕ^b along the vector filed Γ. Namely,

$$\begin{aligned}\Gamma(\phi^b) &= G(\phi^b) + \hat{u}_a (Y^a)^V(\phi^b)\\ &= G(\phi^b) + C^{ab}\hat{u}_a.\end{aligned}$$

where C^{ab} is as in Corollary 7.1.1. Using the control law provided by (7.2) we have

$$\begin{aligned}\Gamma(\phi^b) &= G(\phi^b) + C^{ab}C_{ab}(-K\phi^b - G(\phi^b))\\ &= -K\phi^b.\end{aligned}$$

Thus, since $\Gamma(\phi^b) = -K\phi^b$ for $b = 1, \dots, m$, the real valued functions ϕ^b along the curve q of the closed-loop system (3.3), $\phi^b(q(t), \dot{q}(t))$ satisfy $\phi^b(t) = \phi^b(0)e^{-Kt}$ for $t \in [a, b]$. Therefore, for every $b = 1, \dots, m$, ϕ^b converges exponentially fast to zero.

Secondly, we prove that the law given by (7.2) is the unique control law guaranteed by Theorem 4.2.2. For any value on the constraint manifold, $v_q \in \mathcal{M}$, the closed-loop system with this control law reads,

$$\Gamma(v_q)(\phi^b) = -K\phi^b(v_q).$$

But, since $v_q \in \mathcal{M}$ the right-hand side of the last equation vanishes. Thus, Γ is tangent to $\mathcal{M}$. By uniqueness of the control law given in Theorem 4.2.2, we have that $\hat{u}|_{\mathcal{M}}$ is the unique control law turning $\mathcal{M}$ into a virtual nonholonomic constraint. □

Remark 7.1.3. In the case where the virtual nonholonomic constraints are linear on the velocities thus are defined by the equations of the form

$$\phi^b = \mu^b_i(q)\dot{q}^i$$

for $b = 1, \dots, m$ with $i = 1, \dots, n$, we have the analogous results. The constraint manifold $\mathcal{M}$ is replaced by the constraint distribution $\mathcal{D}$ as defined at the beginning of Section 3.1. With the hypothesis that $\mathcal{D}$ and $\mathcal{F}$ are transversal we have the following. The matrix of the Corollary 7.1.1 takes the form

$$C^{ab} = (Y^a)^V(\phi^b) = \mu^b(Y^a) \tag{7.3}$$

and it is invertible with smooth inverse from the proof of Theorem 3.1.8 and hence, Theorem 7.2 provides the expression of the control law that stabilizes virtual nonholonomic constraints.

Remark 7.1.4. In the case that $\dim \mathcal{F} = 1$ and the virtual nonholonomic constraints are linear, the expression for the control law $\hat{u}$ simplifies to

$$\hat{u} = -\frac{\phi}{\mu(Y)} - \frac{G(\phi)}{\mu(Y)}.$$

Remark 7.1.5. Regarding the case where virtual affine nonholonomic constraints are considered the velocities belong to an affine subspace $\mathcal{A}_q$ of the tangent space T_qQ, where $\mathcal{A}_q = X(q) + \mathcal{D}_q$ with $X \in \mathfrak{X}(Q)$ and $\mathcal{D}$ is a nonintegrable distribution on Q. In this case, $\mathcal{A} = \{(q, \dot{q}) \in TQ : \Phi(q, \dot{q}) = 0\}$, with

$$\Phi(q, \dot{q}) = S(q)\dot{q} + Z(q)$$

where $Z(q) = -S(q)X(q) \in \mathbb{R}^m$ and

$$\mathcal{A}_q = \{\dot{q} \in T_qQ : S(q)(\dot{q} - X(q)) = 0\}, \quad \mathcal{D}_q = \{\dot{q} \in T_qQ : S(q)\dot{q} = 0\}.$$

Thus the coordinate functions of the constraint are given by

$$\phi^b = \mu^b_i(q)\dot{q}^i + z_b(q),$$

where $Z(q) = [z_1(q), \dots, z_m(q)]^T$ and $\mu^b_i\dot{q}^i$ are the entries of $S(q)\dot{q}$, for $b = 1, \dots, m$. In this case the matrix of Lemma 7.1.1 reads

$$C^{ab} = (Y^a)^V(\phi^b) = \mu^b(Y^a) \tag{7.4}$$

and Theorem 7.1.2 holds as well. For more details on virtual affine nonholonomic constraints see Section 3.2.

The linear case presented before is a particular case of the aforementioned with $Z(q) = 0$.

Remark 7.1.6. Maggiore and Consolini, 2013 provide a stabilization control law using feedback linearization in the case of 1-dimensional virtual holonomic constraint, i.e., in the case that the constraint manifold is a curve on the configuration space. We may deduce a similar formula in the case of virtual linear nonholonomic constraints where instead of a constraint manifold $\mathcal{M}$ we have a virtual constraint $\mathcal{D}$ which is a one-dimensional distribution and, therefore, it is integrable.

Let us suppose, for convenience, that $(q_1, \ldots, q_n)$ is a local coordinate chart on the configuration manifold Q for which the virtual holonomic constraint is the tangent space of the curve defined by

$$\gamma(q_n) = (\phi_1(q_n), \ldots, \phi_{n-1}(q_n), q_n),$$

where $\phi_i : I_i \subseteq \mathbb{R} \to \mathbb{R}$ for each $i = 1, \ldots, n-1$ and I_i is an open interval. Then, $\mathcal{D}$ is the distribution spanned by the tangent vector

$$\mathcal{D}_{\gamma(q_n)} = \left\langle (\phi_1^{'}(q_n), \ldots, \phi_{n-1}^{'}(q_n), 1) \right\rangle .$$

Therefore, $\mathcal{D}_{\gamma(q_n)}$ is the set of vectors $(v_1, \ldots, v_n) \in T_{\gamma(q_n)}Q$ defined by the $n-1$ constraints

$$\phi_1^{'}(q_n)dq^n - dq^1 = 0, \ldots, \phi_{n-1}^{'}(q_n)dq^n - dq^{n-1} = 0.$$

Let $\mu^i = \phi_i^{'}(q_n)dq^n - dq^i$ for each $i = 1, \ldots, n-1$ and suppose we have a mechanical control system of type (3.3), where the input distribution $\mathcal{F}$ and $\mathcal{D}$ satisfy the transversality assumption. Then,

$$C^{ab} = \phi_b^{'}(q_n)Y_n^a - Y_b^a$$

and $\hat{\mu}^b = \phi_b^{'}(q_n)\dot{q}^n - \dot{q}^b$. Hence,

$$u_a^* = (\phi_b^{'}(q_n)Y_n^a - Y_b^a)^{-1}(-\phi_b^{'}(q_n)\dot{q}^n + \dot{q}^b - G(\phi_b^{'}(q_n)\dot{q}^n - \dot{q}^b)).$$

Now, it is a direct computation to show that our control law coincides with the control law appearing in Maggiore and Consolini, 2013 by considering the explicit form of the geodesic vector field G that they have and setting $k_1 = 0$ and $k_2 = 1$ in their formula.

We also remark that, the fact that our control law only coincides with the one in Maggiore and Consolini, 2013 after $k_1 = 0$ is related with our original goal. We are not stabilizing the configuration space constraints but only the velocity (nonholonomic) constraints.

Remark 7.1.7. If the controlled system (7.1) is subject to holonomic constraints, $\phi_h^b(q)$, instead of nonholonomic ones, the control law given in Theorem 7.1.2 takes the form

$$\hat{u}_a = C_{ab}\left(-K_1\phi_h^b(q) - K_2\dot{\phi}_h^b - G(\phi_h^b(q))\right),$$

to meet the exponential convergence requirement to zero. K_i for $i = 1, 2$ are diagonal matrices with positive design parameters.

Example 7.1.8. Following Example 3.1.22 we consider the Chaplygin sleigh which is the nonholonomic system in $SE(2)$ whose Lagrangian function is

$$L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I\dot{\theta}^2}{2}$$

and the control force

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = u(\sin\theta dx - \cos\theta dy).$$

The corresponding controlled Lagrangian system is

$$m\ddot{x} = u\sin\theta, \quad m\ddot{y} = -u\cos\theta, \quad I\ddot{\theta} = 0. \tag{7.5}$$

The constraint is given by $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$. The input distribution $\mathcal{F}$ and the constraint distribution are given by

$$\mathcal{F} = \mathrm{span}\left\{\frac{\sin\theta}{m}\frac{\partial}{\partial x} - \frac{\cos\theta}{m}\frac{\partial}{\partial y}\right\} \quad \text{and} \quad \mathcal{D} = \mathrm{span}\left\{X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial\theta}\right\},$$

respectively.

On the one hand, the control law that makes the distribution invariant under the closed-loop system is

$$u^*(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y})$$

by Theorem 3.1.8.

On the other hand, the control law that stabilizes the constraint distribution by Theorem 7.1.2 is given by

$$\hat{u}(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y}) - m(\dot{x}\sin\theta - \dot{y}\cos\theta)$$

and it coincides with u^* on values of $\mathcal{D}$ and the constraint function $\phi = \dot{x}\sin\theta - \dot{y}\cos\theta$ converges exponentially fast to zero. Indeed, differentiating the constraint function ϕ along trajectories of the controlled system and using the dynamics (7.5), we deduce that

$$\dot{\phi} = \frac{u}{m} + \dot{x}\dot{\theta}\cos\theta + \dot{y}\dot{\theta}\sin\theta.$$

Now, substituting u by the control law $\hat{u}$, we obtain that $\dot{\phi} = -\phi$ and ϕ converges exponential fast to 0 along trajectories of the closed-loop system. In Figure 7.1 we plot the projection of a trajectory of the closed-loop system into the plane xy and, in Figure 7.2, we plot the value of the constraint function $\phi(t)$ along the same trajectory. We have considered the system's mass $m = 2$ and its moment of inertia $I = 1.5$. The total simulation time was 50 seconds, with a time step of 0.01 seconds, resulting in 5000 steps. The initial conditions were $x_0 = 1, y_0 = 1, \theta_0 = \pi$ for position, and $\dot{x}_0 = 0.5, \dot{y}_0 = 8, \dot{\theta}_0 = 0.1$ for velocity.

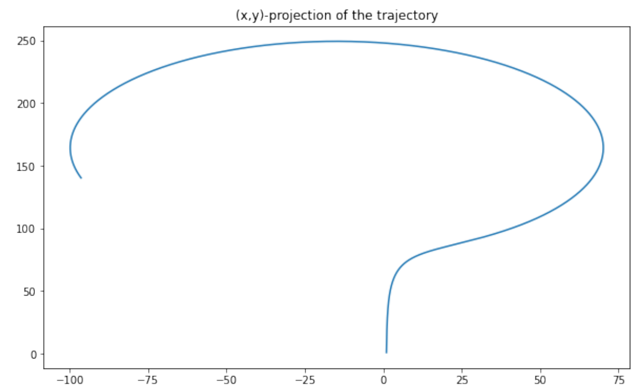


Figure 7.1: The projection of a trajectory of the closed-loop system into the plane xy in Example 7.1.8.

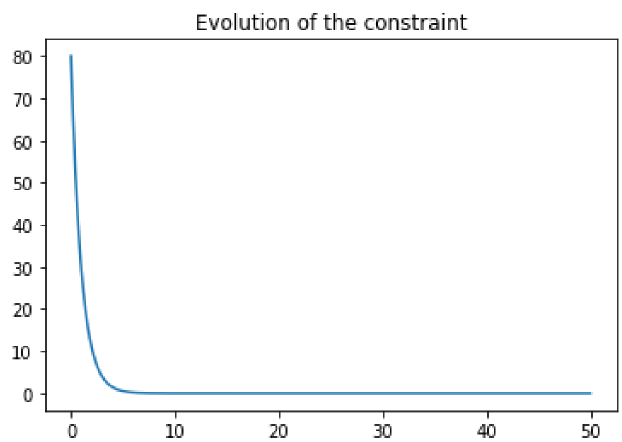


Figure 7.2: The constraint function $\phi(t)$ along the same trajectory in Example 7.1.8.

Example 7.1.9. Consider the aforementioned nonholonomic system in Example 7.1.8 with the control force

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = u(\sin\theta dx - \cos\theta dy + d\theta)$$

which is non-orthogonal i.e., the corresponding force vector field is not orthogonal to the distribution $\mathcal{D}$. The controlled Lagrangian system is

$$m\ddot{x} = u\sin\theta, \quad m\ddot{y} = -u\cos\theta, \quad I\ddot{\theta} = u$$

and the input distribution is defined as

$$\mathcal{F} = \text{span}\left\{\frac{\sin\theta}{m}\frac{\partial}{\partial x} - \frac{\cos\theta}{m}\frac{\partial}{\partial y} + \frac{1}{I}\frac{\partial}{\partial\theta}\right\}.$$

The constraint distribution and the control law that stabilize the system are the same.

Example 7.1.10. Consider the controlled vertical rolling coin as in the Example 3.1.5 whose configuration space is $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$. The Lagrangian function is given by

$$L(x, y, \theta, \varphi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I\dot{\theta}^2}{2} + \frac{J\dot{\varphi}^2}{2},$$

and consider that the the control force

$$F(x,y,\theta,\varphi,\dot{x},\dot{y},\dot{\theta},\dot{\varphi},u) = u_1(dx - \cos\varphi d\theta + d\varphi) + u_2(dy - \sin\varphi d\theta + d\varphi),$$

acts on the system. The controlled Lagrangian system is given by

$$m\ddot{x} = u_1,\ m\ddot{y} = u_2,\ I\ddot{\theta} = -u_1\cos\varphi - u_2\sin\varphi,\ J\ddot{\varphi} = u_1 + u_2,$$

and the virtual nonholonomic constraints are given by

$$\dot{x} = \dot{\theta}\cos\varphi, \quad \dot{y} = \dot{\theta}\sin\varphi.$$

The constraint distribution

$$\mathcal{D} = \left\{X_1 = \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta},\ X_2 = \frac{\partial}{\partial\varphi}\right\}$$

is controlled invariant under the control law

$$u_1^* = -m\dot{\theta}\dot{\varphi}\sin\varphi, \quad u_2^* = m\dot{\theta}\dot{\varphi}\cos\varphi$$

given by the existence and uniqueness Theorem 3.1.8. For the stability analysis we have that the matrix C^{ab} appearing in Corollary 7.1.1 is given by equation (7.3) and takes the form

$$C^{ab} = \begin{bmatrix} \frac{1}{m} + \frac{cos^2\varphi}{I} & \frac{\cos\varphi\sin\varphi}{I} \\ \frac{\cos\varphi\sin\varphi}{I} & \frac{1}{m} + \frac{sin^2\varphi}{I} \end{bmatrix}$$

and its inverse is

$$C_{ab} = \frac{m}{I+m}\begin{bmatrix} I + m\sin^2\varphi & -m\cos\varphi\sin\varphi \\ -m\cos\varphi\sin\varphi & I + m\cos^2\varphi \end{bmatrix}.$$

The stabilizing control law from Theorem 7.1.2 is then

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \frac{m}{I+m}\begin{bmatrix} I + m\sin^2\varphi & -m\cos\varphi\sin\varphi \\ -m\cos\varphi\sin\varphi & I + m\cos^2\varphi \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

where

$$c_1 = -\dot{x} + \dot{\theta}\cos\varphi - \dot{\varphi}\dot{\theta}\sin\varphi$$
$$c_2 = -\dot{y} + \dot{\theta}\sin\varphi + \dot{\varphi}\dot{\theta}\cos\varphi.$$

Therefore, simplifying the matrix calculations we have

$$\hat{u}_1 = \frac{m^2}{I+m}\left(\frac{I}{m}(-\dot{x} + \dot{\theta}\cos\varphi - \dot{\varphi}\dot{\theta}\sin\varphi) + \dot{y}\cos\varphi\sin\varphi - \dot{\varphi}\dot{\theta}\sin\varphi - \dot{x}\sin^2\varphi\right)$$
$$\hat{u}_2 = \frac{m^2}{I+m}\left(\dot{x}\cos\varphi\sin\varphi + \dot{\varphi}\dot{\theta}\cos\varphi - \dot{y}\cos^2\varphi + \frac{I}{m}(-\dot{y} + \dot{\theta}\sin\varphi + \dot{\varphi}\dot{\theta}\cos\varphi)\right).$$

In Figure 7.3 we show the projection of a trajectory of the closed-loop system into the plane xy. In Figures 7.4a and 7.4b we show the values of the constraint functions along the same trajectory. We have considered the system's mass $m = 2$ and their moments of inertia $I = 1.5$ and $J = 1.1$. The total simulation time was again 100 seconds with a time step of 0.01 seconds implying a total of 10000 steps. The initial conditions were $x_0 = 1, y_0 = 1, \theta_0 = \pi$ and $\varphi_0 = \frac{\pi}{2}$ for position, and $\dot{x}_0 = 0.5, \dot{y}_0 = 8, \dot{\theta}_0 = 0.1$ and $\dot{\varphi}_0 = -0.1$ for velocity.

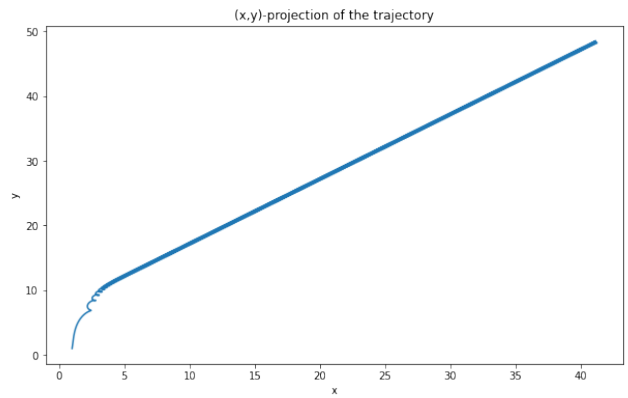
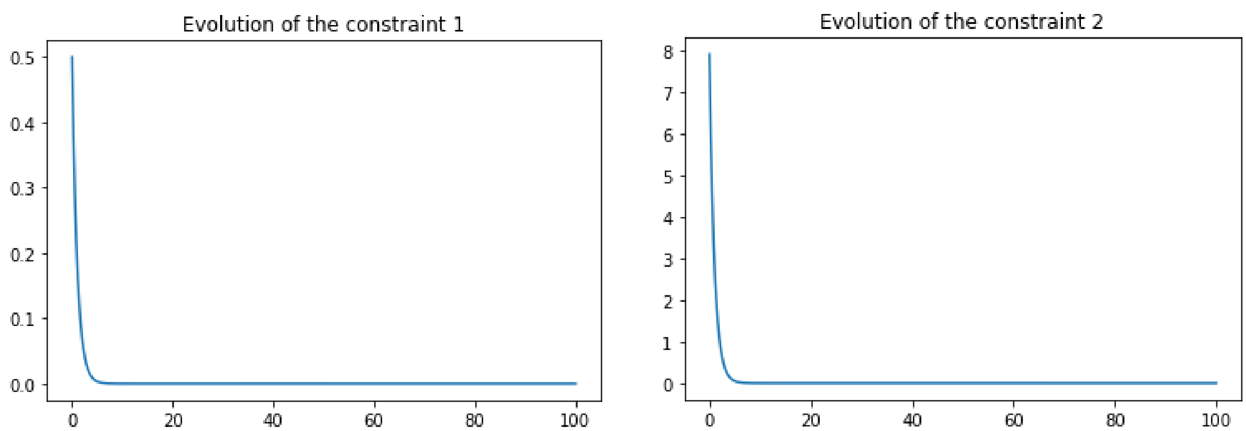


Figure 7.3: Projection of a trajectory of the closed-loop system into the plane xy. Example 7.1.10.



(a) Constraint $\hat{\mu}_1(t)$.

(b) Constraint $\hat{\mu}_2(t)$.

Figure 7.4: Constraint functions in Example 7.1.10.

7.2 Applications

In this section we examine the stability of various constraints for applications and examples studied in this thesis. We cover virtual affine and nonlinear nonholonomic constraints.

7.2.1 Geometric stabilization for flocking motion

Consider the example in the Subsection 4.4.3 of two particles moving under the influence of gravity with the imposed constraint that force them to move with parallel velocities. The Lagrangian $L : TQ \to \mathbb{R}$, is given by

$$L(q, \dot{q}) = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - m_1 g z_1 - m_2 g z_2$$

where $Q = \mathbb{R}^4$, $q_i = (x_i, 0, z_i)$ describes their position and $m_i, i = 1, 2$ are the masses of the particles. The constraint equation is $\Phi : TQ \to \mathbb{R}$,

$$\Phi(q, \dot{q}) = \dot{x}_1\dot{z}_2 - \dot{x}_2\dot{z}_1$$

and consider the general form of a control force $F : TQ \times \mathbb{R} \to T^*Q$ to be

$$F(q, \dot{q}, u) = u(f_1 dx_1 + f_2 dz_1 + f_3 dx_2 + f_4 dz_2),$$

for arbitrary coordinate functions f_j, $j = 1, 2, 3, 4$. The controlled Euler-Lagrange equations are

$$\begin{aligned} m_1\ddot{x}_1 &= uf_1, \quad m_1\ddot{z}_1 + m_1 g = uf_2, \\ m_2\ddot{x}_2 &= uf_3, \quad m_2\ddot{z}_2 + m_2 g = uf_4. \end{aligned} \tag{7.6}$$

The constraint manifold is given by $\mathcal{M} = \{(q, \dot{q}) \in TQ \ : \ \Phi(q, \dot{q}) = 0\}$ and its tangent space, at every point $(q, \dot{q}) \in \mathcal{M}$, the input distribution $\mathcal{F}$ and its vertical lift are as in Subsection 4.4.3. The unique control law making the constraint manifold a virtual nonholonomic constraint is given by

$$u^* = (\dot{z}_2 f_1 - \dot{z}_1 f_3 + \dot{x}_1 f_4 - \dot{x}_2 f_2)^{-1} (\dot{x}_1 - \dot{x}_2)\, gm.$$

and for $f_1 = f_2 = 1$ and $f_3 = f_4 = 0$ we get $F(q, \dot{q}, u) = u(dx_1 + dz_1)$ so

$$u^* = (\dot{z}_2 - \dot{x}_2)^{-1} (\dot{x}_1 - \dot{x}_2)\, gm_1.$$

For the control law that stabilizes the system we use Theorem 7.1.2, so $\hat{u}_a = C_{ab}(-\Phi - G(\Phi))$, where

$$C^{ab} = (Y)^V(\Phi) = \left(\frac{1}{m_1}\frac{\partial}{\partial \dot{x}_1} - \frac{1}{m_1}\frac{\partial}{\partial \dot{z}_1}\right) \times (\dot{x}_1\dot{z}_2 - \dot{x}_2\dot{z}_1) = \frac{\dot{z}_2 - \dot{x}_2}{m}.$$

The vector field G which is determined by the unactuated forced mechanical system is locally given by

$$G = \dot{x}_i\frac{\partial}{\partial x_i} + \dot{z}_i\frac{\partial}{\partial z_i} - g\frac{\partial}{\partial \dot{z}_i},$$

for $i = 1, 2$ hence, $G(\Phi) = g(\dot{x}_2 - \dot{x}_1)$. Thus,

$$\begin{aligned} \hat{u} &= C_{ab}(-\Phi - G(\Phi)) \\ &= m_1(\dot{z}_2 - \dot{x}_2)^{-1} (-\dot{x}_1\dot{z}_2 + \dot{x}_2\dot{z}_1 - g(\dot{x}_2 - \dot{x}_1)) \end{aligned}$$

is the unique control law that makes the constraint function $\Phi(q, \dot{q})$ converges exponentially to zero along trajectories of the controlled system.

Next, we extend the previous framework to a multi-agent system considering four particles of which one is unactuated while the others are controlled to move with parallel velocities. Suppose that the motion of the particles evolves, as before, in a plane parametrized by (x, z) so the configuration space $Q = \mathbb{R}^8$ with $q = (q_1, q_2, q_3, q_4) \in Q$. The Lagrangian $L : TQ \to \mathbb{R}$, is given by

$$L(q, \dot{q}) = \sum_{i=1}^{3} \frac{1}{2} m_i \dot{q}_i^2 - m_i g z_i$$

where $m_i, i = 1, 2, 3, 4$ are the masses of the particles. The constraint is given by the equation $\Phi : TQ \to \mathbb{R},\ \Phi(q, \dot{q}) = [\phi^1\ \phi^2\ \phi^3]^T$ where

$$\phi^b(q, \dot{q}) = \dot{x}_4\dot{z}_b - \dot{x}_b\dot{z}_4,$$

$b = 1, 2, 3$ and the control force is $F : TQ \times \mathbb{R} \to T^*Q$ given by

$$F(q, \dot{q}, u) = u_a F^a,$$

with $F^a = (q, \dot{q}) = dx_a + dz_a$, $a = 1, 2, 3$. The controlled Euler-Lagrange equations are

$$m_i \ddot{x}_i = u_i, \quad m_i \ddot{z}_i + m_i g = u_i, \quad m_4 \ddot{x}_4 = 0, \quad m_4 \ddot{z}_2 + m_4 g = 0, \tag{7.7}$$

$i = 1, 2, 3$. The constraint manifold is $\mathcal{M} = \{(q, \dot{q}) \in TQ \; : \; \Phi(q, \dot{q}) = 0\}$ and its tangent space, at every point $(q, \dot{q}) \in \mathcal{M}$, is given by $T_{(q,\dot{q})}\mathcal{M} = \{v \in TTQ \; : \; d\Phi(v) = 0\}$. For the input distribution $\mathcal{F}$ we have $\mathcal{F} = \text{span}\{Y^a\}$ where Y^a, $a = 1, 2, 3$ are the vector fields

$$Y^a = \frac{1}{m_a} \frac{\partial}{\partial x_a} + \frac{1}{m_a} \frac{\partial}{\partial z_a}.$$

Note here that the vertical lift of the input distribution, $\mathcal{F}^V$, which is generated by

$$Y^V = \frac{1}{m_a} \frac{\partial}{\partial \dot{x}_a} + \frac{1}{m_a} \frac{\partial}{\partial \dot{z}_a},$$

is transversal to the tangent space of the constraint manifold, $T\mathcal{M}$. By Theorem 4.2.2 there is a unique control law making the constraint manifold a virtual nonholonomic constraint. The control law that makes the constraint manifold invariant is

$$u^* = g \left(\dot{x}_4 - \dot{z}_4 \right)^{-1} diag(m_1, m_2, m_3) \begin{pmatrix} \dot{x}_4 - \dot{x}_1 \\ \dot{x}_4 - \dot{x}_2 \\ \dot{x}_4 - \dot{x}_3 \end{pmatrix}.$$

The control law provided by Theorem 7.1.2 stabilizes the system and it is given by $\hat{u}_a = C_{ab}(-\phi^b - G(\phi^b))$, where C_{ab} is the inverse matrix of

$$C^{ab} = (Y^a)^V(\phi^b) = (\dot{x}_4 - \dot{z}_4) diag \left(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_3} \right),$$

given at Corollary 7.3. The vector field G is given by the unactuated forced mechanical system and it is locally given by

$$G = \dot{x}_i \frac{\partial}{\partial x_i} + \dot{z}_i \frac{\partial}{\partial z_i} - g \frac{\partial}{\partial \dot{z}_i},$$

for $i = 1, 2, 3, 4$. Hence,

$$G(\phi^b) = -g \begin{pmatrix} \dot{x}_4 - \dot{x}_1 \\ \dot{x}_4 - \dot{x}_2 \\ \dot{x}_4 - \dot{x}_3 \end{pmatrix}.$$

Thus,

$$\hat{u} = \frac{diag\left(m_1, m_2, m_3\right)}{\dot{x}_4 - \dot{z}_4} \left[- \begin{pmatrix} \dot{x}_4 \dot{z}_1 - \dot{x}_1 \dot{z}_4 \\ \dot{x}_4 \dot{z}_2 - \dot{x}_2 \dot{z}_4 \\ \dot{x}_4 \dot{z}_3 - \dot{x}_3 \dot{z}_4 \end{pmatrix} + g \begin{pmatrix} \dot{x}_4 - \dot{x}_1 \\ \dot{x}_4 - \dot{x}_2 \\ \dot{x}_4 - \dot{x}_3 \end{pmatrix} \right].$$

We have simulated the closed-loop control system with the preferred feedback control law using a standard fourth-order Runge-Kutta method where particle 4 makes an unconstrained motion while the other 3 are controlled for an alignment in their velocities. The closed-loop system evolves into the xz plane. The initial positions of the particles are $(x_1, z_1) = (10, 56)$, $(x_2, z_2) = (30, 100)$, $(x_3, z_3) = (50, 100)$ and $(x_4, z_4) = (10, 90)$ and initial velocities $(\dot{x}_1, \dot{z}_1) = (0.5, 1)$, $(\dot{x}_2, \dot{z}_2) = (1, 1)$, $(\dot{x}_3, \dot{z}_3) = (-1, -1)$ and $(\dot{x}_4, \dot{z}_4) = (0.6, 0)$ in thousand units.

In Figure 7.5 we plot the trajectories of the 4 particles. In Figure 7.6, we plot a 3d trajectory that represent the values of the constraint function $\Phi = [\phi^1, \phi^2, \phi^3]^T$. We have considered that all particles' masses are $m_i = 2$ and $g = 10$. The total simulation time was 500 seconds, with a time step of 0.01 seconds, resulting in 50000 steps. We would like to remark that the trajectories cross the line $z = 0$, which under reasonable models, could correspond to the ground. We have opted to simulate motion past this point for visualization purposes to avoid starting with higher initial value for z, but we remark that the trajectories' profile would not have changed. Also, after visualizing Figure 7.5 one observes intersections in the (x, z) plane between different trajectories, and one might think this should be inconsistent with velocity alignment. However, this is not the case. We stress that these particles do not collide and pass through the intersection point at very different times.

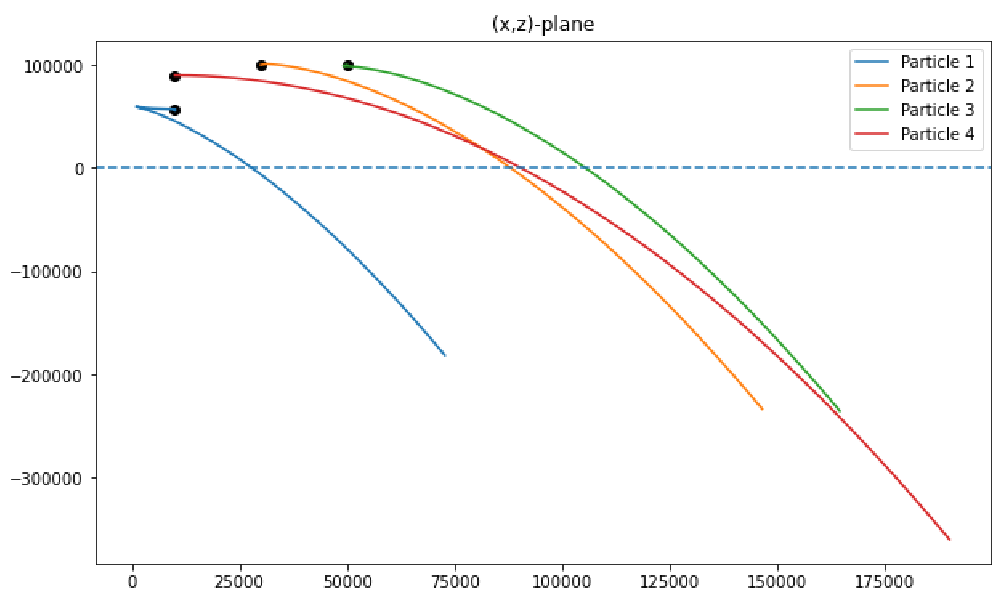


Figure 7.5: Trajectories of the 4 agents of the flocking closed-loop system. The black points indicate the initial positions.

7.2.2 Application to the control of an unmanned surface vehicle on a stream

Consider a boat with a payload on the sea with a position-dependent stream described in Example 3.2.3. The system configuration manifold is $\mathbb{R}^2 \times \mathbb{S}$ with local coordinates $q = (x, y, \theta)$.

The Lagrangian function for the forced mechanical system is $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\dot{\theta}^2$, where m is the mass and I is the moment of inertia. The current exerts an external force $F^{ext} =$

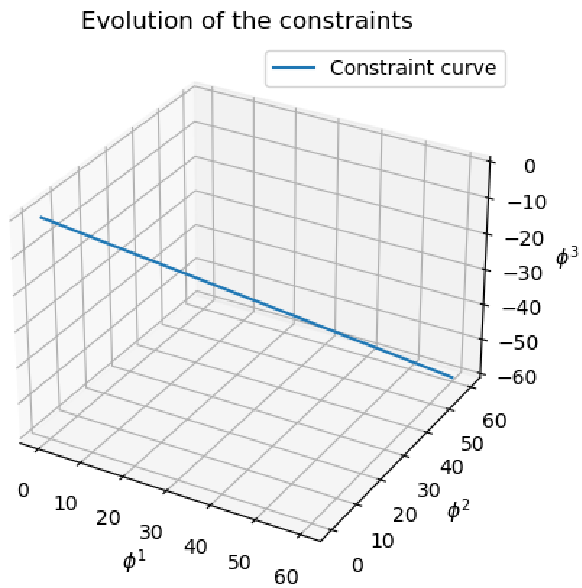


Figure 7.6: Constraint functions along the same trajectory.

$W^1dx + W^2dy$ on the center of mass of the boat and to which we add a control force $F = u(\sin\theta dx - \cos\theta dy + d\theta)$.

The functions W^1 and W^2 are given by

$$\begin{cases} W^1 &= m\ d\,(\sin^2\theta C^1 - \sin\theta\cos\theta C^2)\,(\dot{q}) \\ W^2 &= m\ d\,(-\sin\theta\cos\theta C^1 + \cos^2\theta C^2)\,(\dot{q}) \end{cases},$$

with d representing the differential of a function and C^1, C^2 characterize the sea's current as $C : \mathbb{R}^2 \to \mathbb{R}^2$, $C = (C^1(x,y), C^2(x,y))$. The controlled forced Lagrangian system is

$$m\ddot{x} = u\sin\theta + W^1, \quad m\ddot{y} = -u\cos\theta + W^2, \quad I\ddot{\theta} = u$$

and the input distribution $\mathcal{F} = \operatorname{span}\Big\{\frac{\sin\theta}{m}\frac{\partial}{\partial x} - \frac{\cos\theta}{m}\frac{\partial}{\partial y} + \frac{1}{I}\frac{\partial}{\partial\theta}\Big\}$.

The virtual affine nonholonomic constraint

$$\sin\theta\dot{x} - \cos\theta\dot{y} = C^2\cos\theta - C^1\sin\theta.$$

define an affine space $\mathcal{A}$ modeled on the distribution

$$\mathcal{D} = \operatorname{span}\Big\{X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial\theta}\Big\}.$$

By Theorem 3.2.8, the unique control law that makes the affine space $\mathcal{A}$ invariant under the closed-loop system is $u^*(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y})$.

Let us find the control law that stabilizes the system using Theorem 7.1.2, namely $\hat{u}_a = C_{ab}(-\phi^b - G(\phi^b))$. The matrix from Corollary 7.1.1 and equation (7.4) is

$$\begin{aligned}
C^{ab} &= (Y^a)^V(\hat{\mu}^b) \\
&= \left(\frac{\sin\theta}{m}\frac{\partial}{\partial \dot{x}} - \frac{\cos\theta}{m}\frac{\partial}{\partial \dot{y}} + \frac{1}{I}\frac{\partial}{\partial \dot{\theta}}\right) \times \left(\sin\theta\dot{x} - \cos\theta\dot{y} - C^2\cos\theta + C^1\sin\theta\right) \\
&= \frac{1}{m}.
\end{aligned}$$

The vector field G which is determined by the unactuated forced mechanical system locally is given by

$$G = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{\theta}\frac{\partial}{\partial \theta} + W^1\frac{\partial}{\partial \dot{x}} + W^2\frac{\partial}{\partial \dot{y}} + 0\frac{\partial}{\partial \dot{\theta}},$$

thus, for $\phi = \sin\theta\dot{x} - \cos\theta\dot{y} - C^2\cos\theta + C^1\sin\theta$ we have that

$$\begin{aligned}
G(\phi) = &- \dot{x}\partial_x C^2 c\theta + \dot{x}\partial_x C^1 s\theta - \dot{y}\partial_y C^2 c\theta + \dot{y}\partial_y C^1 s\theta + \dot{\theta}(\dot{x}c\theta + \dot{y}s\theta + C^2 s\theta + C^1 c\theta) \\
&+ W^1 s\theta - W^2 c\theta,
\end{aligned}$$

where $\partial_x C^i = \frac{\partial C^i}{\partial x}$ and $\partial_y C^i = \frac{\partial C^i}{\partial y}$ for $i = 1, 2$, $s\theta = \sin\theta$ and $c\theta = \cos\theta$.

Ultimately, $\hat{u}_a = C_{ab}(-\phi^b - G(\phi^b))$ reads

$$\begin{aligned}
\hat{u} = &- m\sin\theta\left[\dot{x} + C^1 + \dot{x}\partial_x C^1 + \dot{y}\partial_y C^1 + \dot{y}\dot{\theta} + \dot{\theta}C^2 + W^1\right] \\
&- m\cos\theta\left[-\dot{y} - C^2 - \dot{x}\partial_x C^2 - \dot{y}\partial_y C^2 + \dot{x}\dot{\theta} + \dot{\theta}C^1 - W^2\right].
\end{aligned}$$

Next, we present simulations of the closed-loop control system with the stability feedback control law using a standard fourth-order Runge-Kutta method. We have simulated two different cases.

In the fist case, the boat is on a north-east current described by $C(x, y) = (1, 1)$ measured in units per second. The initial position of the boat is $(x, y) = (1, 1)$ and its orientation $\theta = \frac{\pi}{2}$ with initial velocity $\dot{q} = (\dot{x}, \dot{y}, \dot{\theta}) = (0.8, 0.5, 0)$. The mass of the boat is $m = 10$ and the moment of inertia $I = 1.5$. The simulation time was 100 seconds, with a time step of 0.01 seconds, resulting in 10000 steps.

In Figure 7.7 we graph the trajectory of the boat under the effect of the current and the control force. In Figure 7.8 we plot the evolution of the constraint and the energy of the system over time. Note that a significant amount of energy is used to force the boat comply with the constraints.

In the second case, the current of the sea is an anticyclone (high-pressure area) described by $C(x, y) = (y, -x + y)$. The initial position and orientation of the boat was the same as before, $q = (x, y, \theta) = (1, 1, \frac{\pi}{2})$, with initial velocity $\dot{q} = (\dot{x}, \dot{y}, \dot{\theta}) = (1, 1, 0)$. The mass of the boat is $m = 20$ and the moment of inertia $I = 4$. The simulation time was set to 100 seconds, with a time step of 0.01 seconds, resulting in 10000 steps.

In Figure 7.9 we plot the trajectory of the boat on the xy plane under the effect of the current and the control force. In Figure 7.10 we plot the evolution of the constraint and the energy of the system over time.

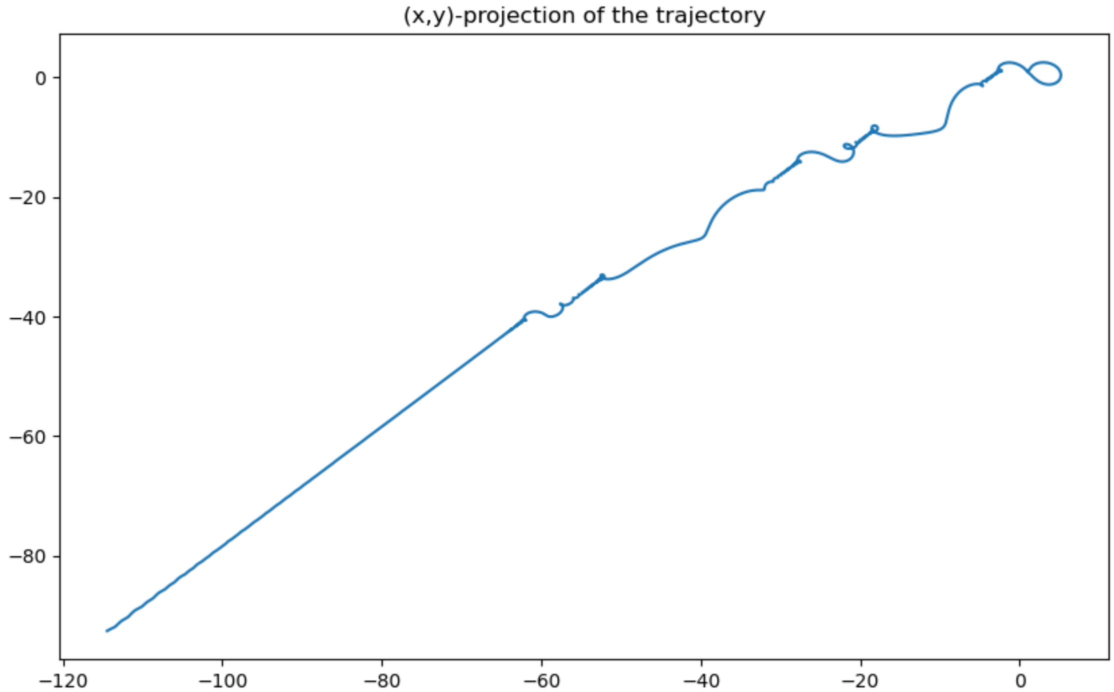
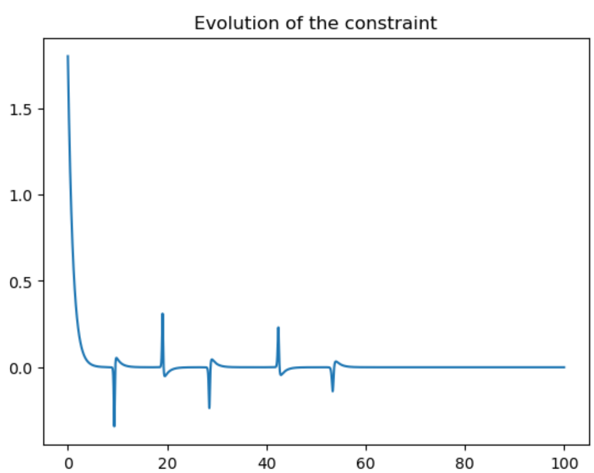
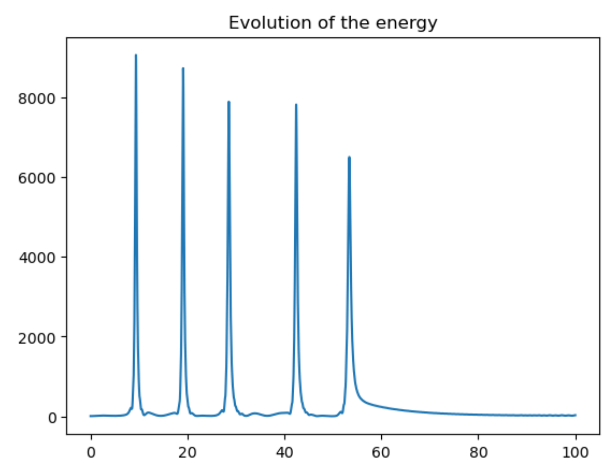


Figure 7.7: Projection of a trajectory of the closed-loop system into the plane xy of the boat on a north-east stream.



(a) Constraint $\phi(t)$.



(b) Energy of the system.

Figure 7.8: Constraints and energy for a boat on a north-east stream.

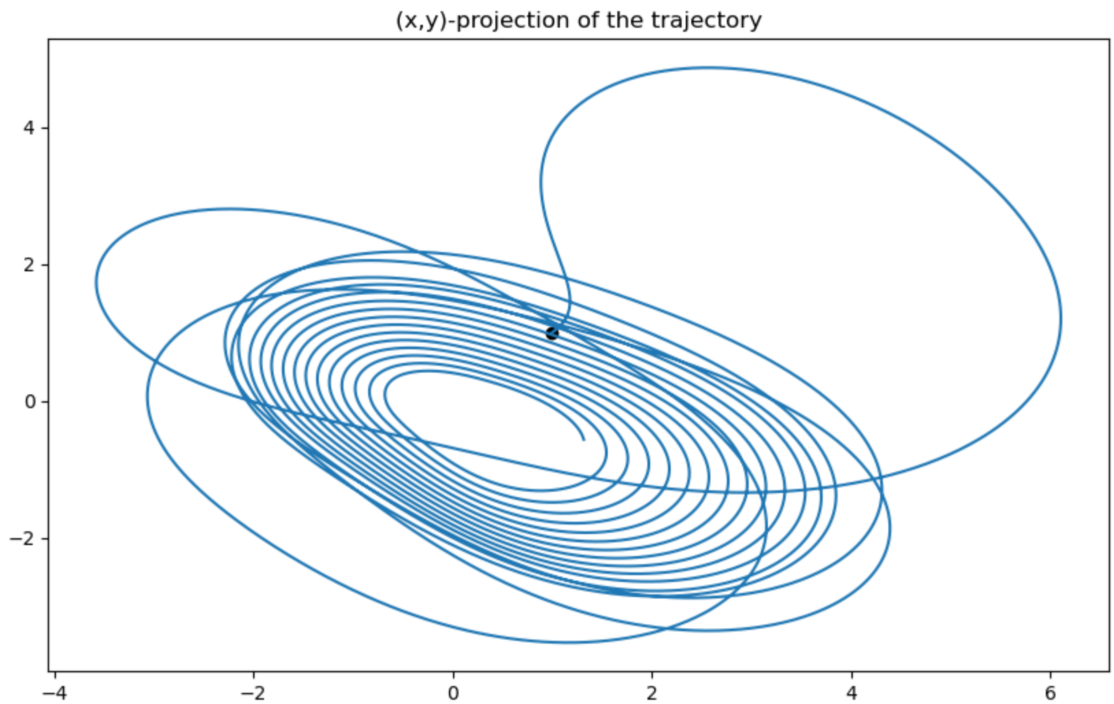
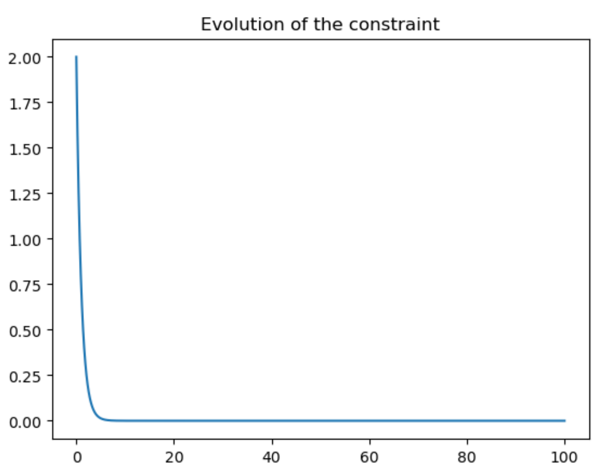
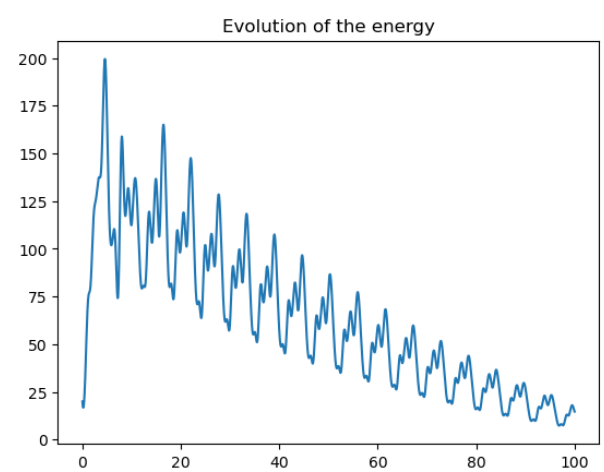


Figure 7.9: Projection of a trajectory of the closed-loop system into the plane xy of the boat on an anticyclone stream. The black dot indicates the initial position of the boat.



(a) Constraint $\phi(t)$.



(b) Energy of the system.

Figure 7.10: Constraints and energy for a boat on an anticyclone stream.

Chapter 8

Conclusions and future work

8.1 Conclusions

This last chapter is devoted to summarize the main contributions of the thesis as well as to present future research based on this work.

In a nutshell, in this thesis, we develop a rigorous mathematical theory for virtual nonholonomic constraints. Virtual constraints are invariant relations imposed on a control system via feedback as opposed to real physical constraints acting on the system. Virtual nonholonomic constraints are defined as a controlled invariant distribution associated with an affine connection mechanical control system. We develop geometric tools sufficient to guarantee the existence and uniqueness of a control law that turns the constraint submanifold into virtual nonholonomic constraints and characterize the dynamics for nonholonomic systems in terms of virtual nonholonomic constraints, i.e., we characterize when can we obtain nonholonomic dynamics from virtual nonholonomic constraints.

In Chapter 2 we gave the necessary background machinery to develop the theory and presented all features from existing literature that we have been based on. Also, we gave the right trivialization of the Levi-Civita connection for a right-invariant metric on Lie groups by defining explicitly a bilinear map and we expressed accordingly the geodesics on a Lie group, Theorem 2.6.9 and 2.6.15 respectively. Moreover, we have related trajectories of a mechanical system on the homogeneous space with horizontal trajectories of a mechanical system on the Lie group at Theorem 2.7.11.

In Chapter 3, we introduce the concept of virtual nonholonomic constraints, encompassing both linear and affine cases. Specifically, we examine control systems subjected to linear and affine constraints and establish the existence and uniqueness of control laws that ensure the trajectories of the closed-loop system satisfy the respective constraints. Section 3.1 develops the theoretical foundation for virtual nonholonomic constraints in the linear case, where the constraints are linearly dependent on the system's velocities—a feature commonly encountered in various nonholonomic systems. A rigorous definition of virtual nonholonomic constraints is provided, describing them as a controlled invariant distribution associated with an affine connection mechanical control system. Subsequently, we demonstrate the existence and uniqueness of a control law that enforces a virtual nonholonomic constraint. Furthermore, we introduced the concept of an induced constraint connection, characterizing the closed-loop system's trajectories as solutions of the mechanical system corresponding to this connection. Finally, we present conditions under which nonholonomic dynamics can be derived from virtual nonholonomic constraints. In Section 3.2, the results established for the linear case are extended to the case of virtual affine nonholonomic constraints. The overall structure of the control mechanical system remains consistent with the previous section, with the difference laying in the nature of the constraint equations. The results of Chapter 3 were published in Simoes et al., 2023 and Stratoglou, Simoes, Bloch, and Colombo, 2023.

Chapter 4 went one step further and extends the theory of linear and affine virtual nonholonomic constraints to the nonlinear nonholonomic constraints. The geometry there was more complex since the constraints cannot be described by a distribution on the configuration space but rather a submanifold of the tangent bundle. Thus, the transversality condition of Theorems 3.1.8 and 3.2.8 was lifted vertically in order to guarantee the existence and uniqueness of a control law that turns the nonlinear nonholonomic constraints into virtual ones. Moreover, we expressed the nonholonomic trajectories of the closed-loop system in terms of an affine connection and gave conditions under which they coincide with solutions of Chetaev's equations. In the Section 4.4, we illustrated the theory with applications to the precessional motion of a rigid body Subsection 4.4.1, the double pendulum Subsection 4.4.2 and flocking motion Subsection 4.4.3. where for the last two we included simulation results. Finally, we presented a geometric characterization of virtual nonholonomic constraints in terms of almost tangent structures. The results of this Chapter were published in Stratoglou, Simoes, Bloch, and Colombo, 2024b and Stratoglou et al., 2025b.

At Chapter 5 we developed the theory of virtual nonholonomic constraints on Lie groups. Here we were based on Stratoglou, Anahory Simoes, et al., 2023. Taking into account the symmetry of the system the theory obtained on Riemannian manifolds admitted many simplifications. Namely, we defined virtual nonholonomic constraints on Lie groups as a controlled invariant subspace associated with an affine connection mechanical control system evolving in the Lie algebra. Instead of a constraint distribution and an input distribution we had vector subspaces of the Lie algebra and we demonstrated the existence and uniqueness of a control law defining a virtual nonholonomic constraint under assumptions on these vector spaces. Also, we characterized the trajectories of the closed-loop system as solutions of a mechanical system associated with an induced constrained connection and we studied when we can obtain reduced nonholonomic dynamics from virtual nonholonomic constraints. Furthermore, we generalized the results of theory to virtual affine nonholonomic constraints on Lie groups. In Subsection 5.2 we applied the theory to a homogeneous rigid body and to a rigid body with a rotor with linear and affine nonholonomic constraints, respectively.

One of the goals of Chapter 6 was to complete the gap in the literature regarding nonholonomic systems on Riemannian homogeneous spaces. The second part was dedicated to the development of the theory of virtual nonholonomic constraints on Riemannian homogeneous spaces. The primary objective is to establish the existence and uniqueness of a control law that preserves the constraint distribution. Additionally, we characterize the resulting closed-loop nonholonomic dynamics, derived under the unique control law, in terms of an affine connection.

A sphere rolling over another sphere in Section 6.4 and a blade moving on a sphere in Section 6.5 where studied to exemplify the theory. The results of this Chapter were published in Stratoglou, Simoes, Bloch, and Colombo, 2024a.

Finally, in Chapter 7 we explored the stabilizability of the virtual nonholonomic constraints. Specifically, we introduced a more general control law that drives the system into complying with the constraint. Moreover, if the system already complies with the constraint at some instant, we proved that this is the unique control law that maintain the trajectories at the constraint manifold. We highlighted the theory by means of examples involving the control of a flocking motion and a boat moving on a stream. This last chapter follows research that did in Simoes et al., 2024 and Stratoglou et al., 2025a.

8.2 Research not included

Additional research on related topics was conducted during this period and have not been included in this dissertation:

In *Virtual constraints on Riemannian homogeneous spaces*, E. Stratoglou, A. A. Simoes, A. Bloch, L. Colombo, IFAC-PapersOnLine, vol. 58 (6), 77-82 (2024) we studied virtual *holonomic* constraints on Riemannian homogeneous spaces.

8.3 Future work

Further topics can be examined based on the theory of virtual nonholonomic constraints developed in this dissertation. These include:

1. Following the geometric framework developed in Chapter 4, it would be interesting to impose the energy of the mechanical system as a virtual nonlinear nonholonomic constraint and check if it is possible to design a feedback controller similarly keeping the energy constant like in Moran-MacDonald et al., 2024 and Čelikovský et al., 2021.

2. Following Chapters 3 and 4, it would be interesting to study conditions under which the closed-loop system obtained from Theorem 4.2.2 is equivalent to a nonholonomic system in the same spirit of the approach followed in Ricardo and Respondek, 2010. Two control systems on a manifold Q of the form

$$\dot{q} = G(q) + u_a Y^a(q),$$

where G and Y^a are vector fields on Q, are S-equivalent if there exists a diffeomorphism $\phi : Q \to Q$ such that both their drift vector fields G and control vector fields Y^a are ϕ-related. Then, we may define a control system to be equivalent to a nonholonomic system if it is S-equivalent to a mechanical control system for which there exists a control law making its trajectories nonholonomic trajectories. Equivalence is a less restrictive condition than the relation with nonholonomic systems provided in Chapters 3 and 4. Hence, in principle, it is easier to impose a control law making a control system equivalent to a nonholonomic mechanical system. Though it is a weaker condition, equivalent systems still share the same qualitative behaviour such as stability properties, periodic orbits, etc.

3. One important step for further generalization is the obvious necessity of extending the range of applicability of our results to the cases in which the transversality assumptions are not met. In some of this cases, a control law might exist though it is possible that it is no longer unique. One of our objectives for a future work is to adapt these results to applications to bipedal robot locomotion. To this end, we will extend our results to the same setting of the type of virtual nonlinear constraints appearing in Griffin and Grizzle, 2015. These constraints are of the form $\phi(q_a, q_u, \dot{q}_a, \dot{q}_u) = q_a - h(q, \dot{q}_u)$, where $q = (q_a, q_u)$ are local coordinates of Q, with respect to which the actuated coordinate vector fields $\frac{\partial}{\partial q_a}$ are the control force vector fields Y^a, i.e, the controlled equations are of the type $\nabla_{\dot{q}_u}\dot{q}_u = Y^0(q, \dot{q})$, and $\nabla_{\dot{q}_a}\dot{q}_a = Y^0(q, \dot{q}) + u_a\frac{\partial}{\partial q_a}$. Under this assumption, the vector fields Y^a belong to $S(v_q)$ (see Section 4.1), since $\langle \frac{\partial \phi}{\partial \dot{q}_i} dq_i, Y^a\rangle = \frac{\partial \phi}{\partial \dot{q}_a} = 0$. Therefore, $\mathcal{F} \subseteq S(v_q)$. Hence, this set of virtual nonholonomic constraints does not fall under the assumptions of Theorem 4.2.2.

4. An interesting application we will consider in the future consists on the combination of virtual holonomic and nonholonomic constraints in a leader-follower framework. For instance, we would like to stabilize the motion of a USV following a prescribed path (virtual holonomic constraints) and also stabilize the parallel velocity constraints (virtual nonholonomic constraints) of another USV following the first one.

5. On mechanical systems involving in Lie groups is very common one or more components of the systems to break the symmetry, e.g. the potential energy. An interesting path to explore is to study mechanical systems with broken symmetry through the concept of virtual nonholonomic constraints.

6. Chapter 7 gave a new control law that guarantees the stability of the distribution associated with virtual nonholonomic constraints. An immediate extension is the development of a control law that stabilizes the constraint distribution for the case where a mechanical system evolves in a Lie group or a homogeneous space. An interesting question is to understand the qualitative behavior of the energy. In particular, we have checked numerically that in general the energy stabilizes around a specific value or converges to a bounded set of values. Understanding the limit value of the energy might give a clue to the nature of the closed-loop dynamics.

7. Under the assumptions of Theorem 3.1.8 & Theorem 4.2.2 there is a unique control law generating a closed loop dynamics. In future work, we are interested in studying dynamical and geometric properties of this dynamical system. In particular, study of equilibrium points and stability, existence of constants of motion and preservation of geometric structures, such as weaker notions of symplectic forms or Poisson brackets. Indeed, in the linear case when the control input distribution is orthogonal to the constraint distribution, we know that the closed-loop dynamics is standard nonholonomic dynamics and we know that the energy is a constant of motion, while the system's flow preserves an almost Poisson bracket (see Cantrijn et al., 1999b, de León et al., 2010).

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