

UNIVERSIDAD POLITÉCNICA DE MADRID
Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio



Active control of three-dimensional separated
flow under unsteady and turbulent conditions

DOCTORAL THESIS

Submitted for the degree of Doctor by:

Himpu Marbona
Master of Science

Madrid, 2025



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Doctoral Degree in Aerospace Engineering

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Under the supervision of:

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‘No matter what he does, every person on earth plays a central role in the history of the world. And normally he doesn’t know it’
The Alchemist - Paulo Coelho

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Abstract

Flow separation over low-pressure turbine (LPT) airfoils is a critical issue that limits the operational envelope of high-bypass ratio turbofan engines. At high altitudes and under high load conditions, the reduced density and adverse pressure gradients make the suction side of LPT airfoils prone to boundary layer separation. Understanding the physics of separated flow under these conditions can provide valuable insights for improving performance through active flow control.

This thesis investigates the physical behavior of separated flow over a simplified wall-mounted bump geometry that mimics the separation on the suction side of an LPT airfoil. High-order numerical simulations are employed to examine the effects of harmonic inflow oscillation and a 2D synthetic jet on the separated flow under laminar and turbulent inflow conditions. Direct numerical simulations (DNS) is chosen over Large Eddy Simulations (LES) and Reynolds-Averaged Navier-Stokes (RANS) to eliminate any unphysical behavior caused by turbulence modeling. The DNS solves the compressible Navier-Stokes equations using the high-order Discontinuous Galerkin Spectral Element Method (DGSEM).

A novel mode decomposition technique is developed to analyze the flow and extract underlying physical processes that are capable of decomposing time-localized events and transient phenomena. The new method extends Variational Mode Decomposition (VMD), originally developed for the signal processing subfield of electrical engineering. It formulates a variational problem aimed at minimizing the resulting bandwidth of the decomposed modes. In the novel method, additional objectives are incorporated to impose weak orthogonality conditions. This novel approach, termed Orthogonalized Variational Mode Decomposition (OVMD), effectively prevents mode duplication and destructive interference while enhancing robustness. The method is adapted for multivariate/multidimensional signals, broadening its application to fluid dynamics. In conjunction with other common techniques in fluid dynamics, such as triple decomposition and power spectral density analysis, OVMD is used to analyze the transient process of separated flow.

The separated flow over a wall-mounted bump geometry is then studied under steady laminar inflow conditions and with a harmonic oscillation of the inflow stream. The harmonic oscillation models the passage of the wakes arriving at an LPT blade from the vanes of the previous stator stage. Three inflow oscillation frequencies and three amplitudes are examined, with frequencies comparable to wake-passing frequencies in practical LPTs and amplitudes ranging from 1% to 10% of the inflow total pressure. Three distinct flow scenarios are identified. In the first scenario, the flow behaves similarly to the steady-inflow case but with a slow temporal modulation of the amplitude of the KH vortices that follows the inflow changes. In the second scenario, KH vortex shedding is intermittently replaced by the formation and release of a large vortex cluster during part of the inflow period. The third scenario consists solely of the periodic formation and release of vortex clusters, resulting in a consistent reduction of the separated flow length over the entire period compared to the steady-inflow case.

Following these findings, two active control strategies that attempt to promote the dynamics

of Scenario 3 in the absence of inflow oscillations are analyzed. Active control is implemented through periodic blowing and suction boundary conditions around the summit of the bump. Two strategies are considered: Strategy I employs continuous periodic blowing and suction, while Strategy II introduces an inactive period or duty cycle between blowing and suction phases. Notably, Strategy II, successfully replicates Scenario 3, revealing an interesting novel strategy to reduce the overall length of the separated flow region and its detrimental impact on aerodynamics.

Resumen

La separación del flujo sobre los perfiles aerodinámicos de la turbina de baja presión (LPT) es un problema crítico que limita el rendimiento de los motores turbofán de alto índice de derivación. A grandes altitudes y bajo alta carga, la baja densidad y los gradientes de presión adversos favorecen la separación de la capa límite en el lado de succión de los álabes de la LPT. Comprender esta separación es clave para mejorar el rendimiento mediante el control activo del flujo.

Esta tesis investiga el comportamiento físico del flujo separado sobre una geometría simplificada de protuberancia montada en una pared, que imita la separación en el lado de succión de un perfil aerodinámico de una LPT. Se emplean simulaciones numéricas directas (DNS) con el Método de Elementos Espectrales Discontinuos de Galerkin (DGSEM) para analizar el efecto de la oscilación armónica de entrada y un chorro sintético 2D en condiciones de flujo laminar y turbulento. Se opta por DNS en lugar de simulaciones de Grandes Vórtices (LES) o simulaciones promediadas por Reynolds (RANS) para evitar errores de modelado de la turbulencia.

Se desarrolla una nueva técnica de descomposición modal para analizar el flujo y descomponer eventos transitorios y localizados en el tiempo. Este método amplía la *Variational Mode Decomposition* (VMD) al formular un problema variacional que minimiza el ancho de banda de los modos descompuestos. Además, incorpora condiciones de ortogonalidad débil para evitar la duplicación de modos y la interferencia destructiva, mejorando la robustez. Denominado *Orthogonalized Variational Mode Decomposition* (OVMD), este enfoque se adapta a señales multivariadas y multidimensionales, ampliando su aplicación a la dinámica de fluidos. Combinado con técnicas como la descomposición triple y el análisis de densidad espectral de potencia, OVMD permite un análisis detallado del proceso transitorio del flujo separado.

El flujo separado sobre la geometría de protuberancia montada en una pared se estudia bajo condiciones de flujo de entrada laminar estacionario y con una oscilación armónica del flujo de entrada. La oscilación armónica modela el paso de estelas que llegan a una pala de LPT desde los álabes de la etapa de estator anterior. Se examinan tres frecuencias de oscilación del flujo de entrada y tres amplitudes, con frecuencias comparables a las frecuencias de paso de estela en LPTs reales y amplitudes que varían entre el 1% y el 10% de la presión total del flujo de entrada. Se identifican tres escenarios de flujo distintos. En el primer escenario, el flujo se comporta de manera similar al caso de flujo estacionario, pero con una modulación temporal lenta de la amplitud de los vórtices KH que sigue los cambios del flujo de entrada. En el segundo escenario, la generación de vórtices KH es reemplazada intermitentemente por la formación y liberación de un gran cúmulo de vórtices durante parte del período del flujo de entrada. El tercer escenario consiste únicamente en la formación y liberación periódica de cúmulos de vórtices, lo que da lugar a una reducción constante de la longitud del flujo separado en todo el período en comparación con el caso de flujo estacionario.

A partir de estos hallazgos, se analizan dos estrategias de control activo que intentan promover la dinámica del Escenario 3 en ausencia de oscilaciones del flujo de entrada. El control activo

se implementa mediante condiciones de frontera de soplado y succión periódicos alrededor de la cima de la protuberancia. Se consideran dos estrategias: la Estrategia I emplea un soplado y succión periódicos continuos, mientras que la Estrategia II introduce un período inactivo o ciclo de trabajo entre las fases de soplado y succión. Cabe destacar que la Estrategia II logra replicar con éxito el Escenario 3, revelando una estrategia novedosa para reducir la longitud total de la región de flujo separado y su impacto negativo en la aerodinámica.

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Abbreviations and acronyms

ADMM Alternating Direction Method of Multipliers

AIAA American Institute of Aeronautics and Astronautics

AM Amplitude Modulated

APG Adverse Pressure Gradient

CFD Computational Fluid Dynamics

CWT Continuous Wavelet Transform

DGSEM Discontinuous Galerkin Spectral Element Method

DMD Dynamic Mode Decomposition

DNS Direct Numerical Simulation

DOF Degrees of Freedom

ECCOMAS European Community on Computational Methods in Applied Sciences

ECG Electrocardiogram

EMD Empirical Mode Decomposition

ETSIAE Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio

EUCASS European Conference for Aero-Space Sciences

FM Frequency Modulated

FPG Favourable Pressure Gradient

HHT Hilbert-Huang Transformation

HOHQ High Order Hex-Quad

IEEE Institute of Electrical and Electronics Engineers

IMF Intrinsic Mode Function

KH Kelvin-Helmholtz

LNS Linearized Navier-Stokes

LPT Low-pressure Turbine

LST Linear Stability Theory

MVMD Multivariate Variational Mode Decomposition

MPOVMD Multivariate Progressive Orthogonalized Variational Mode Decomposition

NS Navier-Stokes

NuMath Numerical Method group

OVMD Orthogonalized Variational Mode Decomposition

PDE Partial Differential Equation

PDF Probability Density Function

POD Proper Orthogonal Decomposition

PSD Power Spectra Density

RMS Root-Mean-Square

RVMD Reduced-order Variational Mode Decomposition

SSeCoID Stability and Sensitivity Methods for Flow Control and Industrial Design

STFT Short-Time Fourier Transform

SVMD Successive Variational Mode Decomposition

TF Time-Frequency

TKE Turbulence Kinetic Energy

UAV Unmanned Aerial Vehicle

UPM Universidad Politécnica de Madrid

VMD Variational Mode Decomposition

Chapter 1

Introduction

An increased concern is given nowadays towards more efficient and environmentally friendly technologies, driven by limited energy resources, rising carbon emissions, and the challenges of climate change. As a result, industry and academia need to continuously innovate and develop new methods, tools, and understanding to be able to improve the performance of their products. In the particular case of the aerospace sector, the technologies in use are in a very mature state and only marginal improvements can be obtained by further optimization and refinement of existing solutions with a high investment (Price, [2018](#)).

To address these challenges, the Stability and Sensitivity Methods for Flow Control and Industrial Design (SSeCoID) project, funded by the European Union's Horizon 2020 research and innovation program, is aimed at investigating novel methods and tools to advance aircraft design and development by deepening the understanding of underlying physical phenomena and improving overall performance. This thesis is part of the project and primarily focuses on controlling the unsteady separation that occurs in the turbine section of turbofan engines.

1.1 Motivation

High-bypass-ratio turbofan engines are employed by all medium to long-range aircraft in commercial transport. Among other contributors, their energy efficiency is conditioned by the aerodynamic efficiency of the low-pressure turbine (LPT), which presents substantial variations between different flight conditions. LPTs are the energy-harvesting component of turbofan engines and drive the main fan that produces more than 80% thrust (Howell et al., [2000](#)). It is located near the end of a turbofan engine. Due to this location, there is a significant drop in the Reynolds number of the incoming flow. In a mid-size turbofan engine at cruising altitude, this Reynolds can be as low as 100,000 based on the axial length of the blade (Hourmouziadis, [1989](#)) and can be even lower for a small turbofan (Bons et al., [2002](#)) or a turbofan in a high-altitude unmanned aerial vehicle (UAV) engine. At high-altitude cruise flights, reduced density and high-load operation typically led to flow separation on the suction side of the blades due to the low Reynolds number and the presence of a strong

adverse pressure gradient (Bons et al., 2002; Coull and Hodson, 2011; Howell et al., 2000).

At low Reynolds, the laminar boundary layer separates on the suction side of the LPT blades because it cannot withstand the sudden change to an adverse pressure gradient. This separation translates into relevant pressure losses and an overall thrust reduction for the same fuel consumption rate. To compensate for the losses, fuel consumption needs to be increased over the ideal situation. Hence, reducing the separated flow regions through active control is thus a way forward to improve their efficiency (Greenblatt and Wygnanski, 2000; Volino et al., 2011).

Hourmouziadis, 1989 and Mayle (1991) categorize boundary layer separation over a turbine blade into three states. The first state is laminar separation, characterized by the flow separating with a laminar boundary layer. This separated flow results in the formation of a long bubble and a substantial region of stagnant, or ‘dead air’. The second state involves separation with a turbulent boundary layer, generally creating a short bubble with a large entrainment region. The third state is an intermittent one. Fig. 1.1 illustrates these three states along with the loss coefficient.

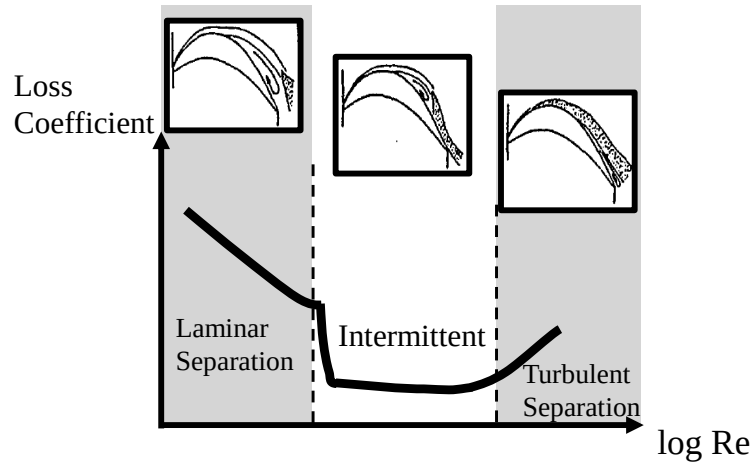


Figure 1.1: Loss coefficient as a function of Reynolds in different type of separations (Hourmouziadis, 1989; Mayle, 1991)

Following a laminar separation, the transition from laminar to turbulent flow is a key factor in the reattachment of the separated flow, which in turn influences not only the size and behavior of the separation bubbles (Dovgal et al., 1994; Gaster, 1967; Mayle, 1991; McAuliffe and Yaras, 2009) but also the structural loading of the LPT blades (e.g. Curtis et al. (1997)). The transition process in separated flows under steady conditions and low environmental disturbances is relatively well understood when the incoming flow and the aerodynamic surface are under steady conditions with relatively laminar flows. In cases such as leading-edge separation on airfoils and LPT blades with moderate loading, the transition is triggered by Kelvin-Helmholtz (KH) instability in the separated shear layer. This instability amplifies small disturbances present in the pre-separated boundary layer (Diwan and Ramesh, 2009; Dovgal et al., 1994), rapidly forming spanwise vortices and triggering transition through nonlinear interactions among these vortices (Marxen et al., 2013). Alternative or complementary

transition scenarios have been proposed for stronger adverse pressure gradients that involve the onset of absolute inflectional instability (Avanci et al., 2019; Hammond and Redekopp, 1998; Rist and Maucher, 2002), spanwise modulation of the recirculation region due to a self-excited global mode (Duck et al., 2000; Gallaire et al., 2007; Rodríguez and Theofilis, 2010; Rodríguez et al., 2013), or a combination of them (Passaggia et al., 2012; Rodríguez and Gennaro, 2019; Rodríguez et al., 2021).

The KH instability is a convective instability commonly characterized in separated boundary layers using the Strouhal number, based on the momentum thickness of the boundary layer at the separation point θ_s , with $St_\theta = 0.012$ (Dandois et al., 2007). Sigurdson (1995) highlights that determining the KH frequency is not straightforward due to challenges in calculating both the momentum thickness and the convection velocity. To address this, an alternative approximations and scaling methods based on the Reynolds number was proposed. An additional approach to characterizing the KH instability frequency involves scaling the piecewise linear velocity profile along the separated shear layer, as suggested by Diwan and Ramesh (2009).

Other instabilities observed in separated flows is the shedding instability (Dandois et al., 2007; Sigurdson, 1995). The literature suggests that the shedding instability may couple with the KH instability through vortex merging processes induced by interactions with the wall (Dandois et al., 2007; Hasan, 1992; Sigurdson, 1995). Consequently, the shedding instability has a distinct characteristic frequency that differs from the KH instability, as it originates from wall interactions.

Sigurdson, 1995 proposed a scaling for the shedding frequency based on the half-width of the shedding vortices h and the edge velocity of the boundary layer at separation U_s , with a Strouhal number defined as $St_h = f_{\text{shed}} h/U_s \approx 0.07 - 0.08$. This empirical formulation has proven problematic due to its dependency on geometric configurations and flow conditions. In more chaotic flows, the shedding instability can become indistinguishable from other eddies in instantaneous data, complicating accurate measurement of vortex size. A novel method capable of distinguishing these phenomena in chaotic flows through mode decomposition would greatly enhance our understanding of the underlying physics of separated flow.

Several concerns regarding separated flow over LPT blades motivate the research presented in this thesis. The first one is the development of a novel method capable of decomposing the underlying non-stationary physical phenomena. As mentioned earlier, current methods for determining the dominant frequency of such phenomena largely rely on measuring characteristic heights, velocities, and/or thicknesses. However, this approach becomes challenging for complex flows, increasing uncertainty in more intricate configurations, as the reference scaling is often derived from simplified test cases.

To provide significant benefits, the novel method must be capable of handling time-localized (transient) phenomena and delivering robust decompositions in the presence of noisy data. Existing methods in fluid dynamics, such as Proper Orthogonal Decomposition (POD) and Dynamic Mode Decomposition (DMD), struggle to effectively capture transient phenomena (Noack et al., 2016).

In the signal processing subfield of electrical engineering, several time-frequency (TF) methods

have emerged in recent years. The history of time-frequency analysis began in the 1980s with the introduction of the continuous wavelet transform (CWT) for geophysical applications by Grossmann and Morlet (1984) and Morlet et al. (1982a, 1982b). Its application to fluid dynamics was introduced by Farge (1992). An important development in TF analysis came with the introduction of Empirical Mode Decomposition (EMD) by Huang et al. (1998). The concept of decomposing a signal into several intrinsic mode functions (IMFs) has since become central to other methods (S. Chen et al., 2017; Dragomiretskiy and Zosso, 2014; Gilles, 2013; Nazari and Sakhaei, 2020). A recently introduced method for fluid dynamics applications is Variational Mode Decomposition (VMD), proposed by Dragomiretskiy and Zosso (2014). The potential improvements provided by this methodology for fluid dynamics are demonstrated by Liao et al. (2023). However, this method can be considered sensitive to input parameters and does not explicitly address the orthogonality and uniqueness of modes.

The second concern is the impact of temporal variations of the incoming flow on the overall separated flow, which has received less attention, especially in turbulent flow. Such variations occur naturally in LPTs as a result of the periodic passage of the wake of a blade row over the blades of the subsequent row and can modify the mean flow properties in a relevant manner. The transition process can be significantly altered in the presence of moderate to elevated levels of disturbances in the free flow. Incoming turbulence generates streaky structures in the attached boundary layer upstream of separation, called Klebanoff modes (Jacobs and Durbin, 2001), which prevent the formation of spanwise-homogeneous KH vortices. The interaction between the streaks and the inflectional instability leads to the formation of short-span KH structures which enhance momentum transfer in the normal direction of the wall and result in faster reattachment compared to cases with low free-stream disturbances (Coull and Hodson, 2011; Hosseinverdi and Fasel, 2019; McAuliffe and Yaras, 2009; Simoni et al., 2017).

Moreover, the periodic passage of wakes shed by upstream blade stages disrupts the flow conditions in downstream passages, generating inherent unsteadiness. This unsteadiness can potentially induce the formation of periodic, strong vortical structures. As a result, the transition to turbulence is driven by the interaction between periodic free-stream flow disturbances and the intrinsic instabilities within the separated flow (Hodson and Howell, 2005).

The third focus is on control strategies to improve LPT performance by reducing pressure losses. Reducing pressure losses by minimizing the size of the separated flow can be achieved through passive or active control strategies. In particular, active flow control strategies using synthetic jets have shown promising results for effectively controlling boundary layer separation (Dandois et al., 2007; Greenblatt, Paschal, Yao, and Harris, 2006; Postl et al., 2011; Seifert et al., 1993; Sigurdson, 1995). Furthermore, separation over the suction side of LPT blades is more likely to occur under off-design conditions. As such, implementing passive control may influence the optimal design conditions. The use of zero-net mass flux actuation is particularly relevant, as turbine blades often include cooling channels for structural reasons (Rolls-Royce, 1996).

Experimental studies on controlling separation with synthetic jets indicate that low-frequency actuation, associated with shear layer instability, is the most effective (Dandois et al., 2007; Kiya et al., 1997; Postl et al., 2011; Seifert et al., 1993; Sigurdson, 1995). Specifically,

Sigurdson (1995) presents a sensitivity map of drag reduction as a function of actuation frequency. This map reveals that the optimum frequency ranges from 2 to 5 times the shedding frequency. As previously mentioned, determining this frequency can be challenging. Moreover, the complex dynamics within this frequency range have not been thoroughly assessed.

Most studies on synthetic jets focus on steady blowing, suction, or periodic blowing-suction at a constant frequency. An alternative strategy explored by Kiya et al. (1997) involves double-frequency forcing. In this approach, the actuation signal combines two independent sinusoidal functions with different frequencies and a phase difference. Their study shows results comparable to single-frequency forcing, with minimal dependence on phase difference. An actuation strategy targeting specific coherent structures has not previously been explored or analyzed. Such a strategy could potentially offer advantages over conventional methods.

Assessing separated boundary layer flows in simplified academic setups typically involves applying localized adverse pressure gradients to test article sections. Two primary methods are commonly used for LPT blade research. The first method creates an artificial pressure gradient on a flat surface in a channel by using converging and diverging nozzles on the opposite side of the surface (Coull and Hodson, 2011; Postl et al., 2011; Rodríguez et al., 2021). The second method uses bump-shaped geometries to simulate the suction side of LPT blades or airfoils (Balin and Jansen, 2021; Greenblatt, Paschal, Yao, Harris, et al., 2006; Nowak et al., 2022; Prakash et al., 2024; Saavedra and Paniagua, 2018; Seifert and Pack, 2002; Uzun and Malik, 2022). In this thesis, the bump-shaped geometry is adopted based on the experimental work of Saavedra and Paniagua (2018) and Nowak et al. (2022). The details of the bump geometry are described further in the Chapter 3.

The study in this thesis is conducted as a numerical study. Numerical simulations of separated flow present greater challenges than those of attached flow configurations. Previous numerical studies have demonstrated the difficulties associated with low-fidelity methods in providing accurate results (Nowak et al., 2022). Additionally, the presence of turbulence modeling can suppress certain phenomena that affect the separated flow region, requiring the mesh to meet similar grid criteria as without turbulence modeling (Marbona, Rodríguez, Martínez-Cava, Valero, et al., 2024). To address these challenges, a high-fidelity approach using an in-house high-order method is employed in this thesis, with simulations performed without turbulence modeling, i.e., Direct Numerical Simulation (DNS).

1.2 Objectives and Scope

The primary objective of this thesis is to investigate the fundamental physics of transitional and turbulent separated flows in configurations and flow conditions representative of one airfoil (either stator or rotor) of low-pressure turbines, as well as to explore actuation methods for effective and efficient control or suppression of separated flow regions. The control strategies focus on manipulating coherent flow structures arising from hydrodynamic instability processes, which ultimately govern the time-averaged separation and reattachment locations.

The proposed approach involves exciting the appropriate instability mechanisms to enhance

entrainment and promote early reattachment of separated flow. However, the most effective control mechanism and the most efficient excitation method depend on various external factors, including operating conditions, the magnitude of the adverse pressure gradient, upstream turbulence intensity and structure, and, in practical LPTs, the impact of periodic wakes generated by the preceding row of blades.

Therefore, this thesis also aims to accurately identify and exploit the most suitable mechanisms under different conditions to achieve optimal control of the separated flow, thereby reducing losses. Achieving this understanding will require the development of a novel mode decomposition technique capable of effectively separating the dynamics of transitional and turbulent separated flows, both with and without control.

Efficiency losses can be mitigated by either preventing flow separation or reducing its extent. Achieving this requires a comprehensive understanding of the complex physical processes involved, including:

1. flow separation influenced by the interplay of adverse pressure gradients, geometric curvature, and unsteady upstream flow conditions
2. the intricate details of the laminar-to-turbulent transition
3. the dynamics of turbulent flow structures and the role of enhanced entrainment in promoting flow reattachment
4. the use of appropriate mode decomposition techniques that can effectively separate the dynamics of time-localized (transient) events
5. the practical implementation of effective flow actuation techniques

1.3 Outline

The content of this thesis is divided into three distinct parts:

- **Part I : High-order simulations of laminar and turbulent separated flow over a wall-mounted bump geometry.** This section presents the flow solver, validation cases, test case geometry and setups, as well as the characteristics of separated flow under laminar and turbulent inflow conditions. It encompasses Chapters 2 and 3.

Chapter 2 introduces the high-order solver HORSES3D and two specific validation cases of interest for the flows studied in the thesis. Two validation cases are conducted: disturbance analysis over Plane Poiseuille flow and turbulent channel flow simulation. These cases are chosen to demonstrate the high-order solver's capability to simulate complex flows beyond steady and laminar regimes, with comparisons made to either theoretical predictions or experimental results.

Chapter 3 focuses on the test article geometry (the wall-mounted bump), the computational domain, and the flow conditions considered in this thesis. Additionally, it examines the characteristics of separated flow under laminar and turbulent inflow

conditions over the test article. The steady laminar inflow discussed here serves as the baseline for Chapters 6 and 7, which investigate the impact of harmonic inflow oscillations and control actuation, respectively. Similarly, the turbulent inflow serves as the baseline for the actuation cases with turbulent inflow in Chapter 7.

- **Part II : Orthogonalized Variational Mode Decomposition.** This section introduces a novel mode decomposition technique, presenting its fundamental theory, numerical experiments, comparisons with other methods, and sensitivity analyses. It spans Chapters 4 and 5. The technique is primarily designed for general applications in the signal processing subfield. Furthermore, an example from fluid dynamics is included to demonstrate the improvements achieved by this method.

Chapter 4 focuses on the development of a univariate mode decomposition technique called Orthogonalized Variational Mode Decomposition (OVMD), an enhancement of the VMD method. This chapter introduces the fundamental theory and a concept to impose weak orthogonality on the resulting modes, addressing VMD's limitations in ensuring mode uniqueness and orthogonality. Additionally, a novel approach for estimating the filter bandwidth parameter is presented. Experiments using a synthetic signal comprising multiple sinusoidal and time-localized events are conducted to evaluate the efficacy of the novel method compared to VMD. Furthermore, a sensitivity analysis is performed to assess the robustness of the method across a range of input parameters for both clean and highly noisy input signals.

Chapter 5 extends the method to multivariate/multichannel data, which enables its application to flow fields. Prior to this, a modification to the algorithm is proposed, enabling the method to progressively decompose modes, starting from the most dominant. For handling multivariate data, an approach is introduced to separate the intrinsic mode function (IMF) into temporal and channel/spatial coefficients. This approach offers improved computational efficiency compared to common multivariate approach. Experiments on univariate and bivariate data are presented on the synthetic signal to evaluate the progressive algorithm and the multivariate extension. Additionally, the chapter explains an empirical approach for estimating the filter bandwidth parameter, reducing the number of input parameters required for the final algorithm. The complete algorithm, named Multivariate Progressive Orthogonalized Mode Decomposition (MPOVMD), is then tested on non-stationary AM/FM signals, including an interwave frequency modulation signal, an electrocardiogram signal, and a transient cylinder wake case, to demonstrate the method's suitability for general applications.

- **Part III : Impact of harmonic inflow oscillations and active actuation.** This section examines the physical phenomena and the effects of harmonic oscillations and active actuation on separated flow. It encompasses Chapters 6 and 7.

Chapter 6 investigates the impact of harmonic inflow oscillations on a low-pressure turbine blade, simulating the inherent unsteadiness in a multi-stage turbine configuration. This chapter details the harmonic inflow conditions and setup, results, mode decomposition, classification of cases based on transient processes, and a final discussion. The results analyze the physical phenomena through instantaneous flow fields, mean flow

fields, and coherent structures to understand the mean characteristics of the separated flow and the dynamic behavior of the separated flow under harmonic inflow. The simulated cases are categorized into three scenarios based on the transient process of coherent structures, streamwise acceleration parameter, recirculation length, and mode decomposition. Special attention is given to scenarios featuring a periodic formation and release of large vortex clusters that effectively reduce the mean recirculation length. A detailed discussion of the underlying physics and potential applications for flow control is also provided.

Chapter 7 investigates active control actuation under both steady laminar and turbulent inflow conditions. The actuation is applied through a synthetic jet boundary condition located just upstream of the separation point. Two strategies are analyzed in this chapter. The first strategy involves periodic blowing and suction. The mean and instantaneous flow fields are analyzed and compared with results from Sigurdson (1995) and Kiya et al. (1997). The second strategy combines periodic blowing and suction with inactive intervals. This strategy, proposed in this thesis, aims to promote specific physical processes learned in Chapter 6. The results of this strategy are compared with those of the first strategy. Mode decomposition is performed for the results of both strategies to reveal the coherent dynamics and transient processes. A comparison with the dynamics of the scenarios from Chapter 6 is also provided to assess the benefits and potential of the newly proposed strategy.

Chapter 8 presents a general discussion of the results obtained in Chapters 4-7. The final Chapter 9 concludes the key findings and contributions of this thesis to the future development of LPT blades and other applications. Additionally, this chapter proposes several potential research directions based on the findings of this study.

This thesis includes several appendices, provided to maintain the brevity of the chapters. The appendices contain derivations of the solver, methods, and additional results that support the conclusions.

1.4 List of Contributions

Based on the presented studies, the main contributions of the thesis can be highlighted:

- A novel mode decomposition technique called Orthogonalized Variational Mode Decomposition (OVMD) is introduced, which is capable of decomposing non-stationary signals and providing robust, accurate decompositions. This novel tool not only helps understand the underlying transient processes in complex flows but also offers insights that can drive the development of future technologies. Its potential for general applications further underscores the significance of this contribution.
- A detailed analysis of the impact of harmonic oscillations on separated flow reveals scenarios involving large vortex clusters, which may benefit the performance of LPT blades.

- A new actuation strategy is introduced, demonstrating potential by promoting the formation of large vortex clusters.

1.4.1 Journal Publications (Peer-reviewed)

- Marbona, H., Rodríguez, D., Martínez-Cava, A., & Valero, E. (2024). Impact of harmonic inflow variations on the size and dynamics of the separated flow over a bump. *Phys. Rev. Fluids*, 9, 053901. <https://doi.org/10.1103/PhysRevFluids.9.053901>
- Marbona, H., Rodríguez, D., Martínez-Cava, A., & Valero, E. Orthogonalized Variational Mode Decomposition. *Submitted to Signal Processing Journal*.
- Marbona, H., Rodríguez, D., Martínez-Cava, A., & Valero, E. A time-frequency modal decomposition for the analysis of non-stationary separated flow. *Submitted to Physics of Fluids*.
- Marbona, H., Rodríguez, D., Martínez-Cava, A., Valero, E., Vandenhoeck, R., Puri, K., & Paniagua, G. Large-eddy simulations of a turbulent boundary layer separation over a bump with strong pressure gradients and freestream turbulence. *To be submitted to AIAA Journal*.
- Ferrer, E., Rubio, G., Ntoukas, G., Laskowski, W., Mariño, O., Colombo, S., Mateo-Gabín, A., Marbona, H., Manrique de Lara, F., Huergo, D., Manzanero, J., Rueda-Ramírez, A., Kopriva, D., & Valero, E. (2023). HORSES3D: A high-order discontinuous Galerkin solver for flow simulations and multi-physics applications. *Computer Physics Communications*, 108700. <https://doi.org/10.1016/j.cpc.2023.108700>

1.4.2 Conferences Presentations and Proceedings

- Marbona, H., Rodríguez, D., Martínez-Cava, A., & Valero, E. (2022). Transition of separated flow over a bump under unsteady inflow conditions. In *ECCOMAS Congress 2022 - Oslo, Norway*. <https://doi.org/10.23967/eccomas.2022.277>
- Marbona, H., Rodríguez, D., Martínez-Cava, A., & Valero, E. (2022). The impact of harmonic inflow variations on the size and dynamics of the separated flow over a bump. In *the 75th Annual Meeting of the Division of Fluid Dynamics 2022 - Indianapolis, United States*.
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Part I:

**High-order simulations of laminar and turbulent
separated flow over a wall-mounted bump geometry**

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Chapter 2

Numerical Methods

A numerical approach to studying the properties of fluid flows offers several advantages over an experimental approach. The ability to easily determine flow properties at any spatial location and to adjust flow conditions with minimal effort are among its main benefits. In this method, the physics of the system is analyzed by approximating a mathematical model that represents the underlying physical phenomena. These nonlinear models are typically expressed as systems of non-linear integral equations or partial differential equations (PDEs). Due to the complexity of non-linear systems, exact solutions are generally unavailable and solutions must be iteratively approximated. This numerical approach to fluid flow analysis is commonly referred to as Computational Fluid Dynamics (CFD).

The techniques can be categorized based on several aspects, such as the method used to approximate the PDEs, the incorporation of ad hoc physical modeling (e.g., modeling turbulent stresses as increased viscosity) into the mathematical models, and the approaches to spatial discretization and time integration. Commercial Computational Fluid Dynamics solutions as used in the industry often do not provide enough fidelity to study the fundamental physics involved, and state-of-the-art academic high-fidelity simulations are required instead.

2.1 High-Order CFD Solver - HORSES3D

In this thesis, direct numerical simulations (DNS) are performed using the in-house discontinuous Galerkin spectral element method (DGSEM) code called HORSES3D by Ferrer et al. (2023). The code implements the DGSEM technique by Kopriva (2009) to solve PDEs. The detailed implementation for compressible Navier-Stokes as well as the theory behind are available in Laskowski (2022), Mateo Gabín (2024), and Rueda-Ramírez (2019). The extension of the solver is also available for multiphase, aeroacoustics, and incompressible Navier-Stokes applications (Manzanero Torrico, 2020; Mariño Sánchez, 2024; Ntoukas, 2023). The solver is a high-fidelity state-of-the-art academic software, developed within the Department of Applied Mathematics in Aerospace Engineering, in particular the NuMath group, to solve three-dimensional viscous and inviscid flows.

The dimensionless compressible Navier-Stokes (NS) equations in conservative form take the form

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot (\mathbf{f}^a - \mathbf{f}^v) = \mathbf{s}, \quad (2.1)$$

where $\mathbf{q} = [\rho, \rho u, \rho v, \rho w, \rho E]^T$ are the conservative variables, E is the specific internal energy and $\mathbf{s}$ is a source or volumetric forcing term. Advective ($\mathbf{f}^a$) and viscous ($\mathbf{f}^v$) fluxes are expressed in primitive variables as equations (2.2) and (2.3), respectively. Appendix A describes the equations in detail.

$$\mathbf{f}_1^a = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho uv \\ \rho uw \\ u(\rho E + p) \end{bmatrix}, \quad \mathbf{f}_2^a = \begin{bmatrix} \rho v \\ \rho uv \\ p + \rho v^2 \\ \rho vw \\ v(\rho E + p) \end{bmatrix}, \quad \mathbf{f}_3^a = \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ p + \rho w^2 \\ w(\rho E + p) \end{bmatrix}, \quad (2.2)$$

$$\mathbf{f}_1^v = \frac{1}{Re} \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ v_i \tau_{1i} + \kappa \partial_x T \end{bmatrix}, \quad \mathbf{f}_2^v = \frac{1}{Re} \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ v_i \tau_{2i} + \kappa \partial_y T \end{bmatrix}, \quad \mathbf{f}_3^v = \frac{1}{Re} \begin{bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ v_i \tau_{3i} + \kappa \partial_z T \end{bmatrix}, \quad (2.3)$$

The equation of state for ideal gas takes the form

$$p = (\gamma - 1) \rho \left[E - \frac{u^2 + v^2 + w^2}{2} \right], \quad (2.4)$$

and Sutherland's law is used for the dynamic viscosity

$$\mu = \frac{1 + T_{suth}/T_{Ref}}{T + T_{suth}/T_{Ref}} T^{\frac{3}{2}}, \quad (2.5)$$

where $T_{suth} = 110.4$ K. The dimensionless thermal conductivity is expressed as

$$\kappa = \frac{\mu}{(\gamma - 1) Pr M}, \quad (2.6)$$

where Pr is the Prandtl number, assumed to be constant and equal to 0.72. The stress tensor, using Stokes hypothesis, is defined as

$$\boldsymbol{\tau} = \mu \left((\nabla \mathbf{v})^T + \nabla \mathbf{v} \right) - \frac{2}{3} \mu (\nabla \cdot \mathbf{v}) \mathbf{I} \quad (2.7)$$

In this thesis, simulations are performed using third-order polynomials with Gauss nodes, unless stated otherwise. Time integration is carried out using an explicit third-order Runge-Kutta scheme. A standard discontinuous Galerkin discretization of the inviscid fluxes is done using Roe's method for the Riemann problem and the Bassi-Rebay 1 scheme is used for the discretization of the viscous fluxes. Further details on the numerical implementation can be found in Ferrer et al. (2023) and references therein.

2.2 Validations

Two cases are simulated as part of this thesis to validate the implementation of the solver for flow conditions beyond steady and laminar regions. The first case involves linearized Navier-Stokes (LNS) simulations of an infinitesimally small disturbance in a plane Poiseuille flow beyond its critical Reynolds number. The second case is a turbulent channel flow simulation at $Re_\tau = 586$.

2.2.1 Linearized NS simulations of disturbance spatial evolution in Plane Poiseuille flow

Plane Poiseuille flow refers to the fluid motion between two parallel, spanwise infinite, and stationary plates. The flow is steady and fully developed. If the Reynolds number, based on the half-height distance between the plates (Re_h) is below 5772 — known as the critical Reynolds number (Re_{cr}) for this case (Schmid and Brandt, 2014; Thomas, 1953) — any small disturbance introduced into the flow will be damped and eventually vanish. However, beyond this critical Reynolds number, certain small disturbances, depending on their wavenumber, may grow and alter the general flow solution (base flow). In such cases, disturbances initially grow exponentially until they reach sufficient amplitudes, after which they transition to non-linear behavior and modify the overall flow description.

Linear Stability Theory (LST) can predict whether the introduced disturbances will grow or decay, along with their growth/decay rates, as long as the linear terms are dominant. In the local stability analysis, the LNS equations are reformulated into the Orr-Sommerfeld equation and solved as a generalized eigenvalue problem. As part of the validation process, a linearized form of the Navier-Stokes equations has been implemented in HORSES3D to compare the evolution of disturbances with solutions from LST.

Disturbance equations

The conservative variables ($\mathbf{q}$) can be decomposed into a combination of the base flow ($\bar{\mathbf{q}}$) and the disturbances variables ($\mathbf{q}'$) such that

$$\mathbf{q}(\mathbf{x}, t) = \bar{\mathbf{q}}(\mathbf{x}) + \epsilon \mathbf{q}'(\mathbf{x}, t), \quad (2.8)$$

where $\bar{\mathbf{q}}$ is the steady solution which is invariant in time, and $\epsilon \ll 1$ signifies small fluctuation. The NS-equations (2.1) can be separated into the base flow and disturbance term

$$\frac{\partial \bar{\mathbf{q}}}{\partial t} + \vec{\nabla} \cdot \left(\vec{\mathbf{f}}(\mathbf{q})^a - \tilde{\mathbf{f}}(\mathbf{q})^v \right) + \epsilon \frac{\partial \mathbf{q}'}{\partial t} + \epsilon \vec{\nabla} \cdot \left(\vec{\mathbf{f}}'(\mathbf{q})^a - \tilde{\mathbf{f}}'(\mathbf{q})^v \right) = \mathbf{s}. \quad (2.9)$$

The base flow term, in equation (2.9) can be taken out as it satisfies the NS-equations. Since it is time-invariant, the equation can be further simplified into:

$$O(1) \quad \vec{\nabla} \cdot \left(\vec{\mathbf{f}}(\mathbf{q})^a - \tilde{\mathbf{f}}(\mathbf{q})^v \right) = 0. \quad (2.10)$$

The remaining terms constitute the disturbance equations, including the forcing term or perturbation ($\mathbf{s}$), expressed as:

$$O(\epsilon \leq) \quad \frac{\partial \mathbf{q}'}{\partial t} + \vec{\nabla} \cdot (\vec{\mathbf{f}}'(\mathbf{q})^a - \vec{\mathbf{f}}'(\mathbf{q})^v) = \mathbf{s}. \quad (2.11)$$

The LNS equations can then be derived directly with these disturbance equations. The detailed linearization process and final expression for fluxes are given in the Appendix B for brevity.

LNS simulations of disturbances in Plane Poiseuille flow

A plane channel domain is constructed with a height of $2h$ and a streamwise length of $80h$ where h is the half-height of the channel and is used as the characteristic length, as shown in Fig. 2.1. The domain extends in the spanwise direction by $0.5h$. It is discretized using an unstructured quadrilateral mesh with 5,800 hexahedral elements, generated by the High Order Hex-Quad (HOHQ) mesh generator by Kopriva et al. (2024). The domain contains only one element in the spanwise direction. Both spanwise planes have periodic boundary conditions, while the top and bottom are horizontal plates with a no-slip wall boundary condition imposed. The inlet has a Dirichlet-type inflow condition, imposing a parabolic streamwise velocity profile, expressed as:

$$\begin{aligned} \rho_{inlet} &= \text{constant} - \text{specified} \\ \rho u_{inlet}(y) &= \rho_{inlet} U_{Ref} \left(1 - \frac{y^2}{h^2} \right) \\ \rho v_{inlet} &= 0 \\ \rho w_{inlet} &= 0 \\ \rho E_{inlet} &= \frac{p_{inlet}}{\gamma - 1} + \frac{\rho_{inlet}}{2} (u_{inlet})^2 \end{aligned} \quad (2.12)$$

The steady solution of this setup is then used as the baseline $\bar{\mathbf{q}}(\mathbf{x})$ for disturbance simulation.

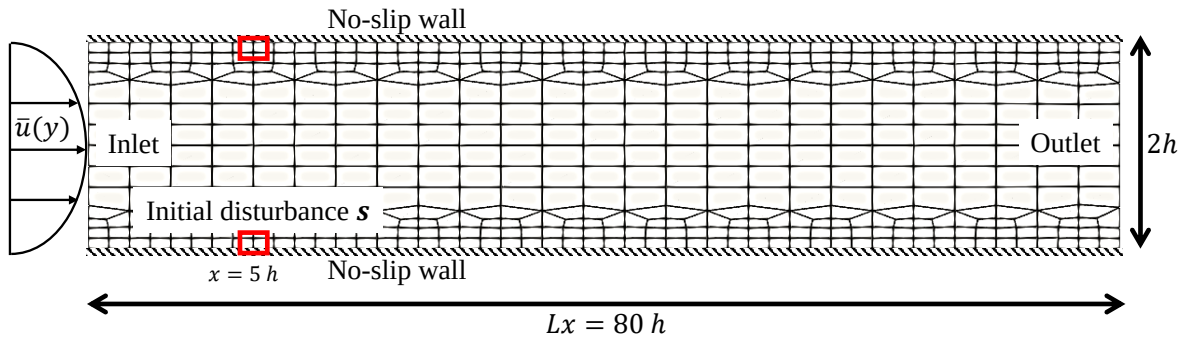


Figure 2.1: Illustration of 2D-channel domain for plane Poiseuille flow. The figure is cropped for visibility with aspect ratio of the figure is 1:1.

Small two-dimensional volumetric forcing is imposed on the top and bottom wall regions with a specified frequency f_d^* and amplitude A_d . The forcing is applied in a region around $x = 5h$ with a spatial structure designed to ensure a continuous profile, and is expressed as:

$$\mathbf{s}(\mathbf{x}, t) = \begin{bmatrix} 0 \\ A_d C_n(x, y) \sin(2\pi f_d^* t) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (2.13)$$

where $A_d = 0.1$ and C_n is the spatial structure function expressed as:

$$C_n(x, y) = \exp\left(-\left(\frac{x - x_c}{sx}\right)^2 - \left(\frac{y - y_{c1}}{sy}\right)^2\right) - \exp\left(-\left(\frac{x - x_c}{sx}\right)^2 - \left(\frac{y - y_{c2}}{sy}\right)^2\right) \quad (2.14)$$

with constant $x_c = 5h$, $y_{c1} = -0.9h$, $y_{c2} = 0.9h$, $sx = 0.05$, and $sy = 0.05$.

The disturbance simulations are performed using third-order polynomials that translates into 371,200 Gauss nodes with other setups as mentioned in 2.1. The simulations are characterized by the Reynolds number based on the half-height $Re_h = \rho U_{ref} h / \mu$, Mach number $M = U_{ref} / a$, where a is the speed of sound, and the disturbance normalized frequency f_d^* . Three cases are simulated:

- $Re_h = 10^4 - M = 0.1 - f_d^* = 0.5$,
- $Re_h = 10^4 - M = 0.2 - f_d^* = 0.4$, and
- $Re_h = 2 \times 10^4 - M = 0.2 - f_d^* = 0.5$.

One case corresponds to unstable (growing) disturbances, as shown in Fig. 2.2, while the other two cases result in the decay of the disturbances; one of these cases is shown in Fig. 2.3.

Comparison with LST

A comparison with LST results are done by expressing the result from LNS simulations in the same ansatz with LST. The evolution of disturbance variables can be expressed with ansatz

$$\mathbf{q}'(\mathbf{x}, t) = \tilde{\mathbf{q}}'(y) \exp(-i(\alpha x + \omega t)) + c.c., \quad (2.15)$$

where $\tilde{\mathbf{q}}'$ represents the variable profiles, α is a complex constant that states the growth/decay rate of disturbance in the streamwise direction, and ω is the frequency of disturbances. Following this ansatz, a comparison of $\tilde{\mathbf{q}}'$ and α from LNS simulations and LST can be performed.

Figures 2.4 and 2.5 present the comparison of normalized velocity profile and spatial growth/decay rate with the LST. The velocity profile shows excellent agreement with LST for all three cases. The spatial growth is given by the real and imaginary components of the constant alpha, expressed as $\alpha = \alpha_r + \alpha_i \sqrt{-1}$. Following the ansatz in Eq. (2.15), a negative

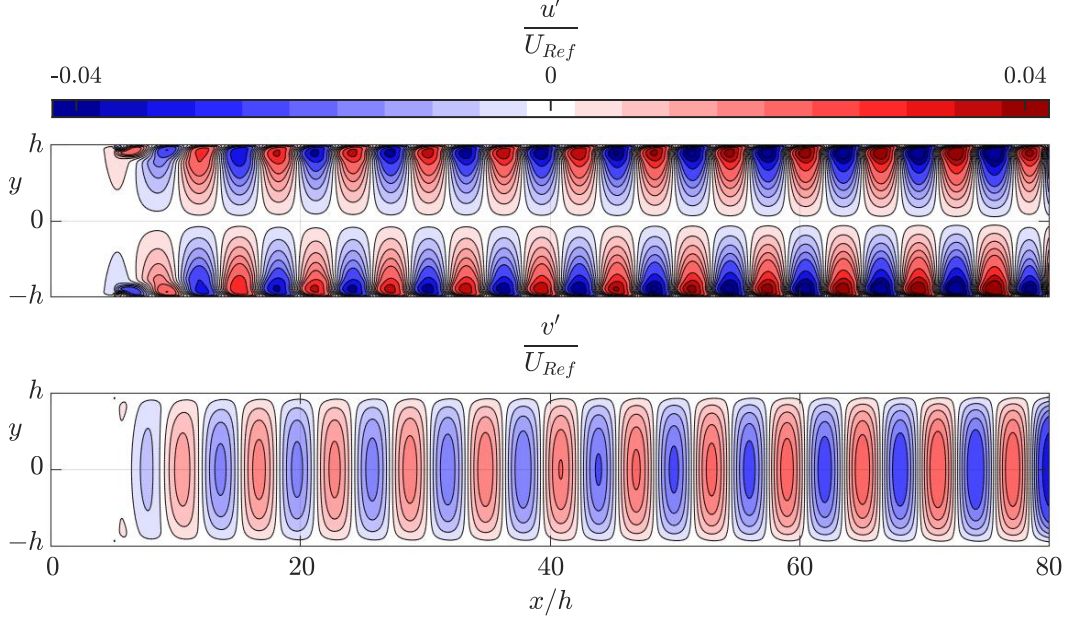


Figure 2.2: Streamwise and wall normal disturbance velocity at $t^* = 300$ for $Re_h = 10^4 - M = 0.2 - f_d^* = 0.4$ where disturbance grow

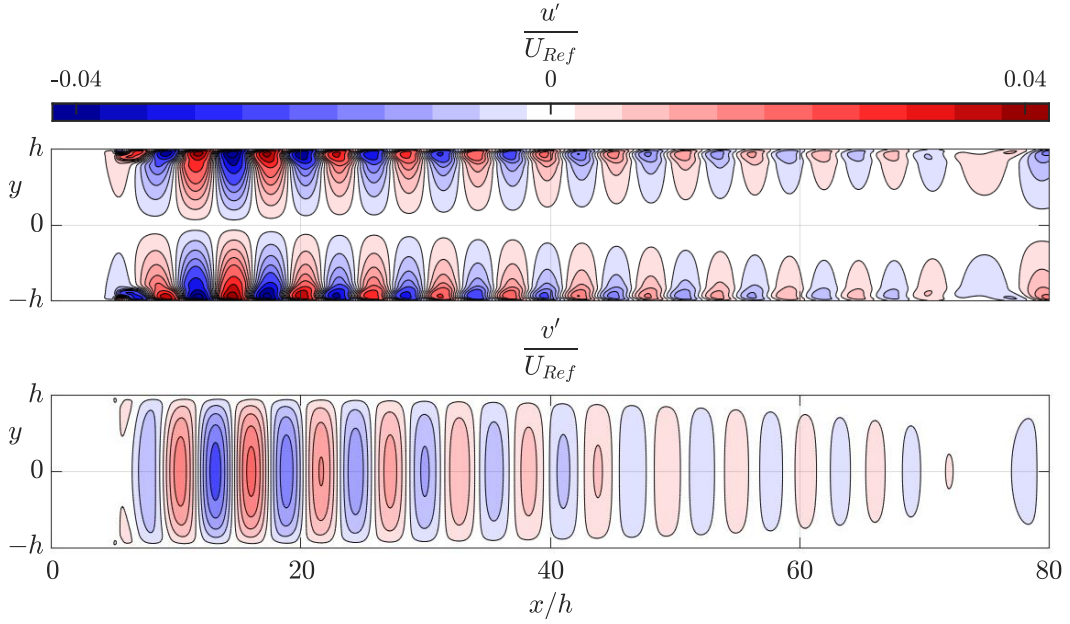


Figure 2.3: Streamwise and wall normal disturbance velocity at $t^* = 300$ for $Re_h = 2 \times 10^4 - M = 0.2 - f_d^* = 0.4$ where disturbance decay

value of α_i denotes disturbance growth while a positive value signifies decay. For all three cases, a good comparison is obtained with LST, with a slight difference observed for the case with $f_d^* = 0.5$. These test cases validate the HORSES3D solver's implementation of the PDEs with the LNS equations for unsteady flow.

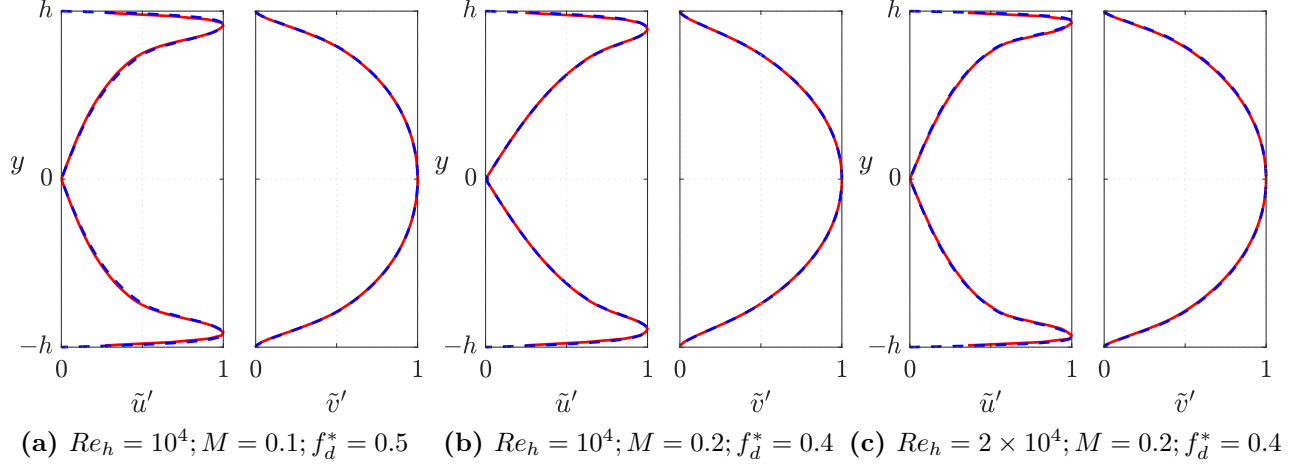


Figure 2.4: Streamwise and wall normal disturbance velocity profile comparison at $x/h = 40$ between the LNS simulations (—) and linear stability theory (---)

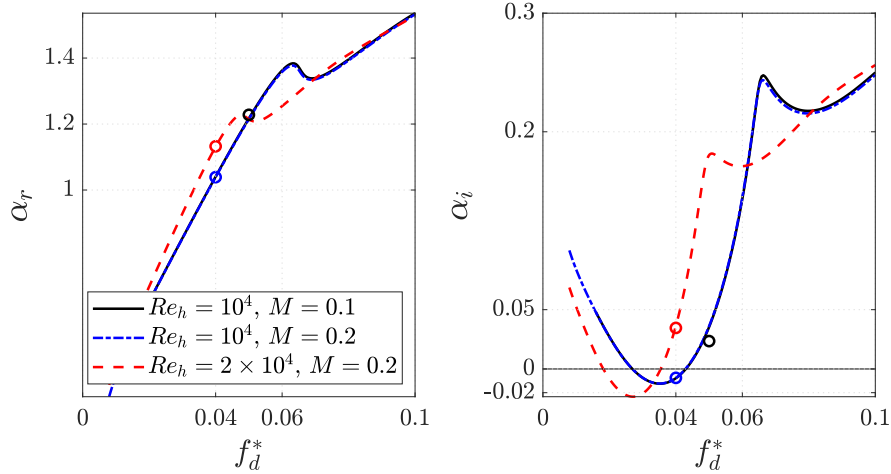


Figure 2.5: Comparison of the streamwise wavenumber α_r and growth rate $-\alpha_i$ between the LNS simulations ($\circ$) and linear stability theory (lines)

2.2.2 Turbulent channel at $Re_\tau \approx 586$

The second validation case examines the HORSES3D compressible NS solver capability for turbulent flow. A turbulent channel flow simulation is performed and compared with other DNS results from Hoyas and Jiménez (2006, 2008). The channel domain is different from previous validation cases as the flow is three-dimensional and requires spanwise resolution. The channel domain has a height of $2h = 0.17$ m, length $4\pi h$ and width $0.3\pi h$, Fig. 2.6. The height and width of the channel is set as such to be identical with the bump test case because the result of this validation case is used as the inflow for the turbulent flow of the bump test case. The domain is discretized into $Nx \times Ny \times Nz = 107 \times 20 \times 16$ elements, with the y and z axes being the normal direction of the wall and the spanwise direction. The elements have uniform distribution in streamwise and spanwise, while the wall normal direction has a non-uniform distribution with grid close to the walls have smaller distance than far from

walls.

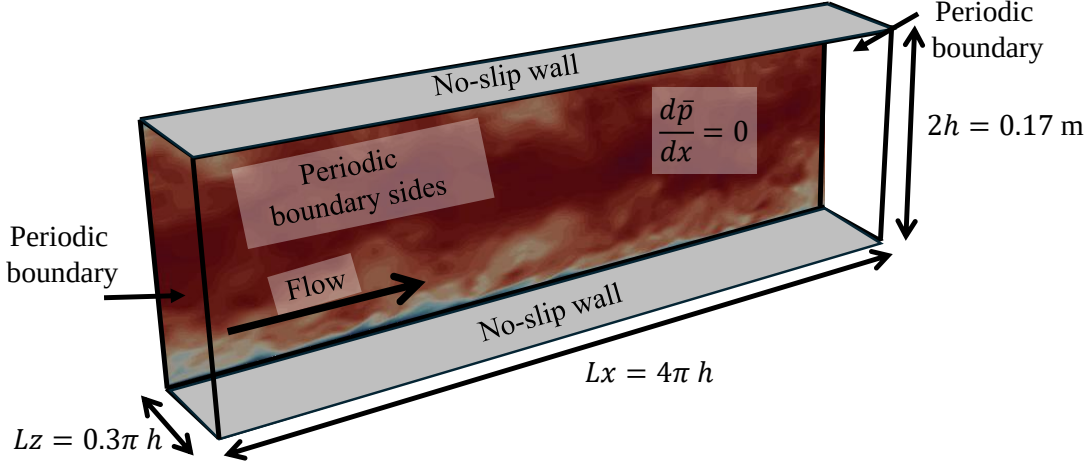


Figure 2.6: Description for turbulent channel setup, $N_x=107$ $N_y=20$ $N_z=16$

Periodic boundary conditions are applied at the spanwise sides and the inlet-outlet side. Maintaining a constant pressure gradient in the channel without decreasing the actual flow pressure is crucial for ensuring comparability and achieving statistically homogeneous flow throughout the domain at all simulation times. To achieve this, a volumetric source term is introduced to sustain the constant pressure gradient. The source term is defined as described in Brun et al. (2008).

An adequate definition of the initial condition is crucial in this validation case. An initial flow field composed of random fluctuations not only risks a decay in the disturbance over time, as demonstrated in Section 2.2.1, but can also generate coherent wakes that will take a long time to transition into fully developed turbulent flow. Computational time is an issue for the effective use of resources. For this reason, the initial condition is formulated as a combination of the baseflow steady solution of the plane Poiseuille flow, the least stable eigenmodes corresponding to Tollmien-Schlichting waves as obtained by LST, and a small component of white noise. With this initial condition, the flow transitions to fully developed turbulence after $t = 5 \times \frac{Lx}{U_{ref}}$, i.e., when the flow at mid-channel has passed through the domain five times for the current setup and flow conditions.

The simulation is performed with third-order polynomial which gives the number of degree of freedom (NDOF) to be 2,191,360 with Gaussian quadrature. The setup is as described in Section 2.1. Reynolds number based on the half-height is $Re_h = 8,500$ with Mach number 0.2. The fully developed turbulent channel flow has a Reynolds number based on the skin friction velocity of 586. The expression of this Reynolds is

$$Re_\tau = \frac{\rho u_\tau h}{\mu} \quad (2.16)$$

where the friction velocity u_τ is

$$u_\tau = \sqrt{\frac{\tau_{wall}}{\rho}} \quad (2.17)$$

The profile of the mean velocity, the root-mean-square (RMS) velocity fluctuation, and the Reynolds shear stress are given in Figs. 2.7, 2.8a, and 2.8b along with results from Hoyas and Jiménez (2006, 2008) and the ‘law of the wall’ (Spalding, 1961) for comparison. The results are normalized with the wall shear velocity as such

$$\bar{u}^+ = \frac{\bar{u} U_{Ref}}{u_\tau} \quad (2.18)$$

and

$$y^+ = \frac{\rho u_\tau y}{\mu}. \quad (2.19)$$

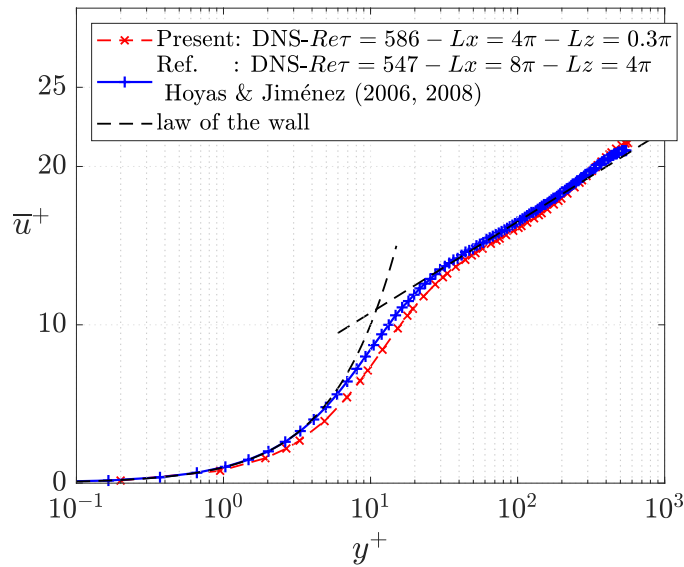


Figure 2.7: Mean velocity profile of turbulent channel flow at $Re_\tau \approx 580$

In Fig. 2.7 the node locations are indicated by symbols ($\times$). In general, the results show a good comparison with the reference DNS. It appears that a slight offset starting from the buffer region is more due to grid resolution.

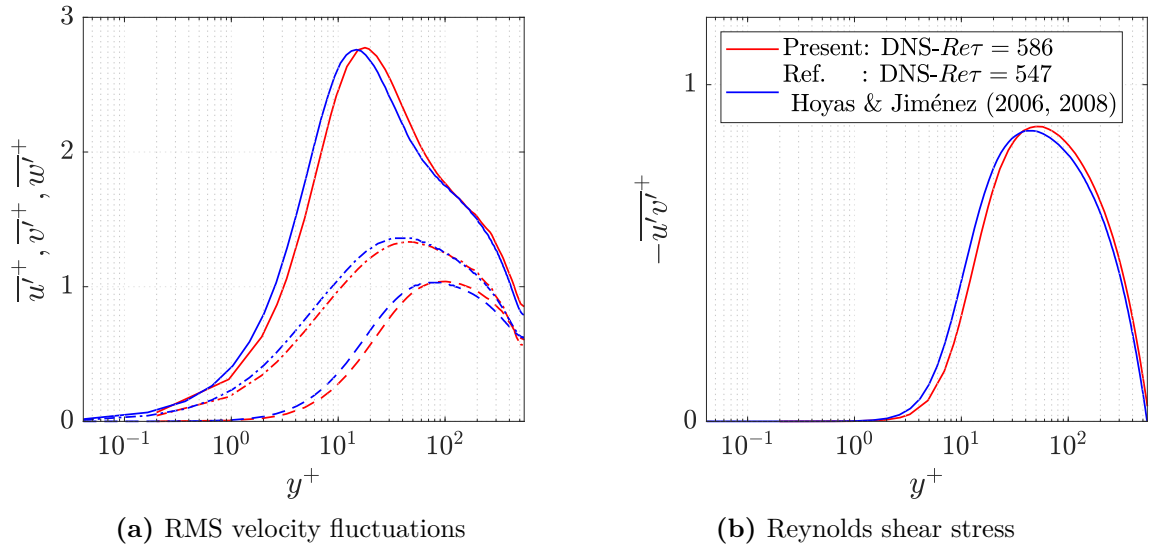


Figure 2.8: Root-mean-square velocity fluctuations normalized by the wall shear velocity (a) and Reynolds shear stress normalized by the shear wall velocity (b) of turbulent channel flow at $Re_\tau \approx 580$

Chapter 3

Laminar and Turbulent Boundary Layer Separation over a Bump

The separation of a boundary layer over a wall-mounted bump geometry has been extensively studied in the past, both in numerical and experimental works. This geometry simplifies the study of separated flow and its physical characteristics (Balin and Jansen, 2021; Matai and Durbin, 2019; Prakash et al., 2024; Uzun and Malik, 2022), facilitates the investigation of control strategies (Greenblatt, Paschal, Yao, and Harris, 2006; Nowak et al., 2022), and serves as a reference case for validating CFD codes or turbulence models (Greenblatt, Paschal, Yao, Harris, et al., 2006).

This chapter discusses the geometrical configuration of the bump, the computational domain, boundary conditions, mesh, convergence analysis, and other aspects common to the simulations used later in the analysis of the physics. The simulations in this chapter also provide baseline results for the cases presented in Chapters 6 and 7. In this chapter, two simulations are performed and analyzed: laminar and turbulent boundary layer separation over a wall-mounted bump geometry.

3.1 Laminar boundary layer separation over a bump

Part of this section has been published in Marbona, Rodríguez, Martínez-Cava, and Valero (2024). Impact of harmonic inflow variations on the size and dynamics of the separated flow over a bump. Phys. Rev. Fluids, 9, 053901.

3.1.1 Geometry and domain

The geometry of interest is shown in Fig. 3.1. This geometry is based on the experimental setup used by Saavedra and Paniagua (2018, 2021) and Nowak et al. (2022), consisting of a plane channel with a bump protruding from one of the walls (2.5 D setup). The bump geometry is defined by Bezier curves using 11 control points, to ensure continuity of the

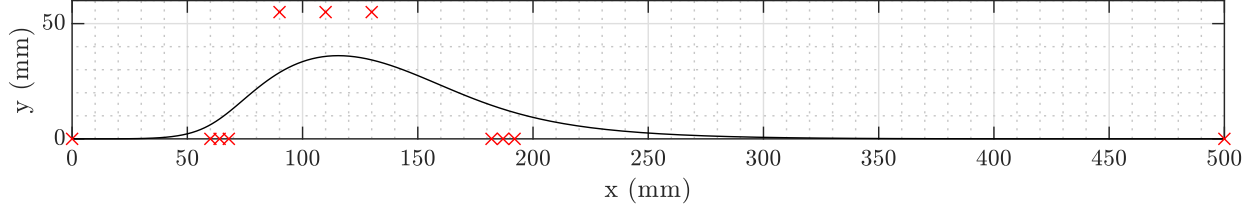


Figure 3.1: Definition of the wall-mounted bump geometry. The symbols $\times$ show the location of the control points defining the Bezier curves.

surface up to the second derivatives. The maximum height of the bump (summit) and the height of the channel are 0.036 m and 0.17 m, respectively, resulting in a throat width of 0.134 m and a blockage ratio of 21.2 %. This relatively large blockage ratio impacts the pressure gradient over the bump and especially near the bump summit. However, for the related geometry of the NASA hump (Greenblatt, Paschal, Yao, Harris, et al., 2006) concluded that reducing the blockage ratio indeed modifies the wall pressure distribution, but does not affect the locations of separation and reattachment or the root mean squared pressure distribution.

The flow variables are made dimensionless using L_{Ref} , T_{Ref} , p_{Ref} and the reference velocity U_{ref} for the steady inflow case, resulting in a unit-length Reynolds number $Re = 100,000$ or $Re_h = 8,500$ with half-height of the channel and Mach number 0.2.

The computational domain and a representative mesh used in this section and Chapter 6 are shown in Fig. 3.2. The channel is considered to be homogeneous in the spanwise direction. The span width of the computational domain is set as $L_z = 0.08$ m (more than twice the height of the bump), and periodic boundary conditions are imposed on the lateral walls. For comparison, the larger spanwise domain used by Wissink and Rodi (2003) is $L_z = 0.12L$, where their L is comparable to the length of the bump. In the present geometry, the bump length is $L \sim 0.25$ m, yielding a ratio $L_z/L \sim 0.3$, which is approximately three times larger than that of Wissink and Rodi. Furthermore, this spanwise domain is considered to be sufficiently large since it accommodates several of the major spanwise structures, as shown later in Figs. 6.2 and 6.3.

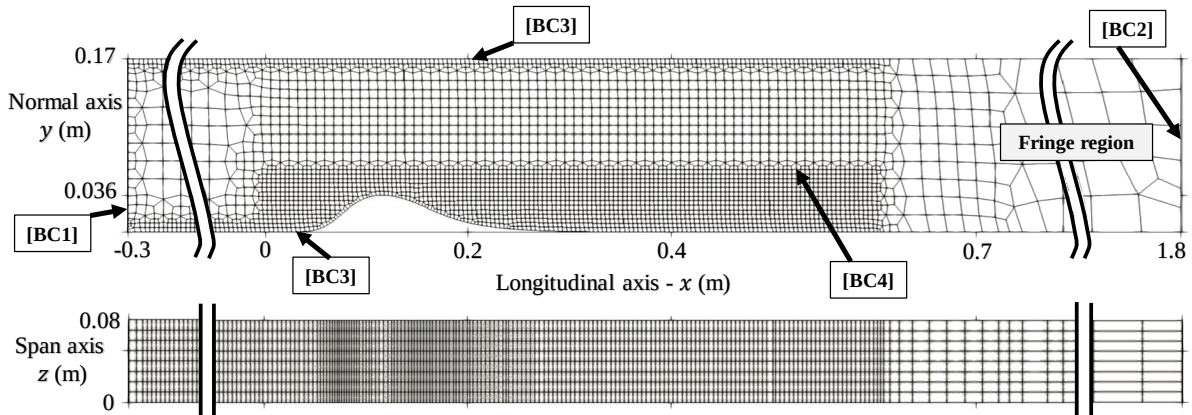


Figure 3.2: Computational domain and representative mesh for spectral element computations for study with laminar inflow in this section and Chapter 6.

The computational domain is extended in the streamwise direction with respect to the reference experiments and simulations to minimize the impact of the boundary conditions and to allow for the introduction of a Fringe region before the downstream boundary. The inlet section (denoted BC1) is located 0.3 m upstream of the start of the bump, that is, upstream of the first control point in the Bezier curves as shown in Fig. 3.1. This relatively long distance between the inlet section and the bump is intended to allow the development of the incoming boundary layer.

The HOHQ mesher (Kopriva et al., 2024) is used to generate a mesh suitable for spectral element computations. The bump curvature within the elements is represented using a 5th-order polynomial. Local mesh refinement is performed in the regions adjacent to the wall and around and downstream of the bump, where the strongest velocity gradients are expected to appear. Section 3.1.3 summarizes the mesh refinement studies. The mesh finally used for laminar boundary layer inflow in this section and in Chapter 6 consists of 31,640 rectangular elements with 8 elements in the spanwise z -direction, and a 3rd-order polynomial in all elements and directions.

3.1.2 Simulation setup and boundary conditions

Table 3.1 describes the boundary conditions. The flow is driven by a pressure gradient imposed through the inflow and outflow boundary conditions. The static pressure $p_{Ref} = 101,325$ Pa is imposed at the outlet and used as a reference pressure. Uniform values of total pressure p_t and temperature T_t are prescribed at the inlet. A reference case is defined in which the inlet total pressure is constant and equal to $p_t = 105319$ Pa and the inlet total temperature is $T_t = 291.2$ K; this case is referred to as *steady inflow* case in Chapter 6. Under these conditions, the mean velocity at the inlet section is $U_{Ref} = 68.158$ m/s, the static temperature at inflow (which is taken as reference temperature) is $T_{Ref} = 288.9$ K, and the resulting flow has (unit length) Reynolds number $Re = \rho_{Ref} U_{Ref} L_{Ref} / \mu_{Ref} \approx 100,000$ and Mach number $M = U_{Ref} / c_{Ref} = 0.2$, where ρ_{Ref} , μ_{Ref} and c_{Ref} are respectively the density, dynamic viscosity and speed of sound based on the reference temperature and pressure (T_{Ref} and p_{Ref}), and L_{Ref} is an arbitrary reference length which is chosen as $L_{Ref} = 1$ m.

Table 3.1: Boundary Conditions - Steady Laminar Inflow

[BC1] Inlet	Total pressure $p_t = 105,319$ Pa and total temperature $T_t = 291.2$ K
[BC2] Outlet	Static pressure: $p = 101\,325$ Pa
[BC3] Wall	No-slip, adiabatic
[BC4] Sides	Periodicity

Dimensionless velocity, time, and frequency are defined using U_{Ref} and unit length (L_{Ref}) as such

$$u^* = \frac{u}{U_{Ref}}, \quad t^* = t \frac{U_{Ref}}{L_{Ref}}, \quad f^* = f \frac{L_{Ref}}{U_{Ref}}. \quad (3.1)$$

The reference length, L_{Ref} , is defined as 1 m unless stated otherwise, and U_{Ref} represents the freestream velocity.

Early simulations showed the presence of non-physical disturbance reflections from the outlet boundary which impacted notably the dynamics of the separated flow. To minimize their upstream influence, a Fringe region is applied as in Nordström et al. (1999), P. Spalart (1989), and P. R. Spalart and Watmuff (1993). This approach imposes a forcing term into a bounded spatial region extending upstream from the outlet, to drive the flowfield towards prescribed values and effectively dampen the flow fluctuations before they reach the outlet boundary. The forcing term takes the form

$$\mathbf{s} = -\bar{\lambda} F\left(\frac{x - x_{start}}{\Delta_{rise}}\right) (\mathbf{q} - \mathbf{q}_{target}), \quad (3.2)$$

where $\bar{\lambda}$ is a constant controlling the strength of the forcing, $F(r)$ is a smooth function describing the spatial structure of the forcing term, x_{start} is the coordinate where the Fringe region starts, Δ_{rise} is the length of the Fringe region, $\mathbf{q}$ is the vector of fluid variables, written in conservative variables (as described in section 2.1), and $\mathbf{q}_{target}$ is the prescribed flow to be recovered at the outlet. Following P. R. Spalart and Watmuff (1993), the function $F(r)$ is defined as:

$$F(r) = \begin{cases} 0, & r \leq 0, \\ 1 / \left[1 + \exp\left(\frac{1}{r-1} + \frac{1}{r}\right) \right], & 0 < r < 1, \\ 1, & 1 \leq r. \end{cases} \quad (3.3)$$

The reference values of the density ρ_{Ref} and temperature T_{Ref} , and the streamwise velocity of the steady inflow are imposed for the target values $\mathbf{q}_{target}$, together with the vanishing of the wall-normal and transversal velocity components. The Fringe parameters used in the simulations have been determined from different tests (not reproduced here for the sake of brevity) and ensure that the flowfield does not fluctuate at the outlet and has negligible upstream effects. The Fringe forcing is very effective in dampening vortical disturbances and short Fringe regions are typically used for incompressible flow. Conversely, present computations consider compressible flow at relatively low Mach number and consequently additional disturbances of acoustic nature are introduced. Previous tests simulating plane channels under the same inflow conditions as the wall-mounted bump showed that Fringe regions comprising at least one acoustic wavelength were required to effectively damp the acoustic reflections and prevent acoustic feedback with the current numerical setup. The Fringe region used starts a sufficient distance from the bump summit, i.e. at $x_{start} = 0.7$ m and extends to the outlet located at $x_{outlet} = 1.7$ m. The parameter Δ_{rise} is set equal to $x_{outlet} - x_{start}$ in order to provide the lowest gradient possible, and $\bar{\lambda} = 400$.

3.1.3 Laminar separation over the bump with steady inflow

The baseline simulation of steady laminar inflow is performed with third-order polynomial (2,024,960 DOFs). Figure 3.3 shows the laminar separations over the bump geometry with steady inflow. The flow separates just upstream of the bump summit. Initially, the flow maintains the two-dimensional structure of the separated shear layer, Fig. 3.4. During

transition to turbulence, the separated shear layer sustains a self-excited shedding of spanwise-dominant Kelvin-Helmholtz (KH) vortices, followed by their breakdown into smaller three-dimensional structures and an abrupt transition to turbulence. In Fig. 3.4, the contour is colored by the streamwise vorticity (ω_x) to show the location where the flow starts to have three-dimensional structures. The entrainment provided by turbulence leads to mean flow reattachment.

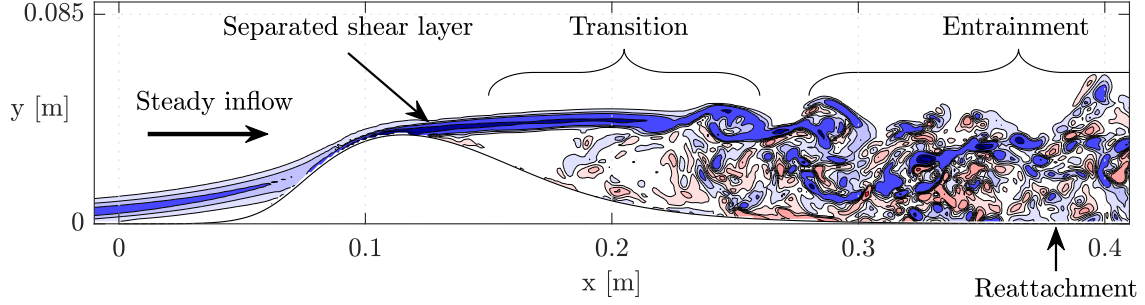


Figure 3.3: Spanwise vorticity field around a wall-mounted bump under steady inflow conditions.

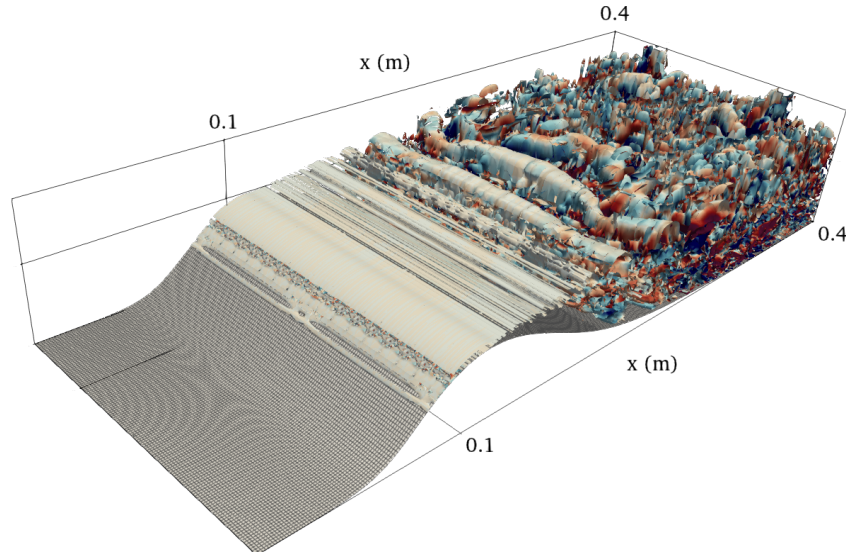


Figure 3.4: Q-criterion contour ($Q=+500$) colored with streamwise vorticity (ω_x) - steady laminar inflow

Mean Flowfield

The time-averaged characteristics of the streamwise velocity flow fields are presented in Fig. 3.5. The black line represents the dividing streamline starting from the mean separation point. The maximum reverse flow reaches approximately 20-25 % of the freestream velocity, which is well above the 7% criterion for three-dimensional centrifugal instability to become linearly unstable (Rodríguez et al., 2013). The initial region of separated flow exhibits relatively low mean and root-mean-square (rms) streamwise velocity. This region of ‘dead-air’ corresponds to the plateau observed in the surface pressure coefficient, as shown in Fig. 3.6.

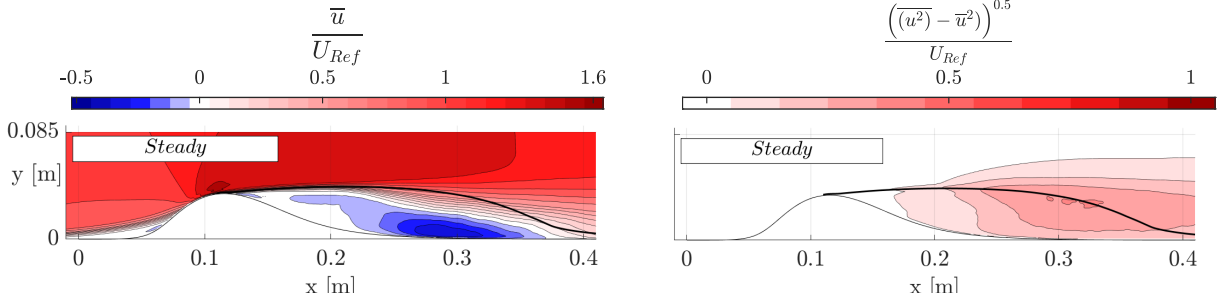


Figure 3.5: Time-averaged (left) and root-mean-square (right) streamwise velocity field with dividing streamline - steady laminar inflow

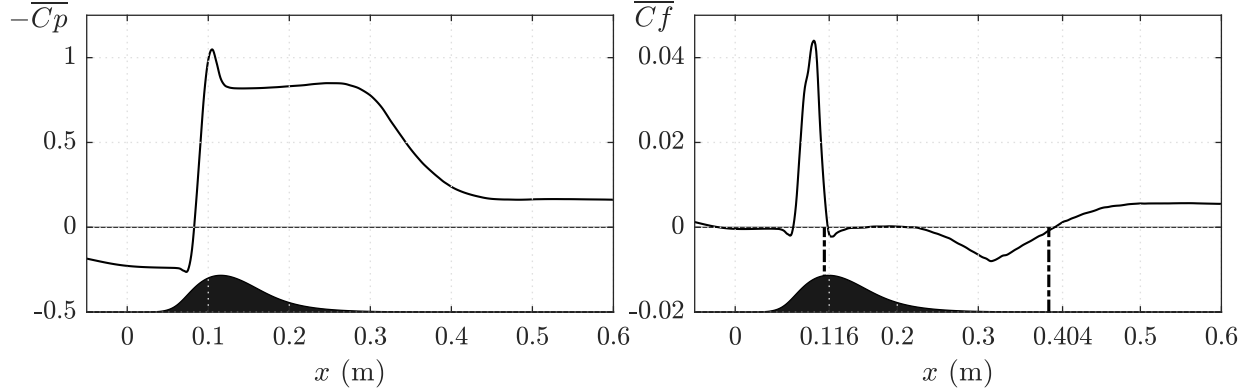


Figure 3.6: Mean surface pressure and skin friction coefficient - steady laminar inflow

In Fig.3.6, the surface pressure coefficient is expressed as

$$\overline{Cp}(x) = \frac{\bar{p}(x) - p_{Ref}}{p_{t\ Ref} - p_{Ref}} \quad (3.4)$$

and the skin friction coefficient is

$$\overline{Cf}(x) = \frac{\bar{\tau}_w(x)}{p_{t\ Ref} - p_{Ref}}, \quad (3.5)$$

where the reference total ($p_{t\ Ref}$) and static (p_{Ref}) pressure are taken at $(x, y, z) = (0, 0.15, 0.4)$ m, as illustrated in Fig. 3.9. The extreme negative value of the skin friction coefficient is located beneath the maximum reverse flow and the root-mean-square velocity.

The length of the separated flow region, defined as the streamwise distance from the time-averaged separation to reattachment points, is $L_{s,steady} = 0.2782$ m. These points are estimated from the time-averaged skin friction data. The Reynolds number based on the mean inflow velocity and $L_{s,steady}$ is $Re_{L_s} = 27,000$, which is comparable to the work by, e.g. Dellacasagrande et al. (2020), Embacher and Fasel (2014), and Kurelek et al. (2019). The Reynolds number based on the maximum height of the recirculation region, approximated here as the height of the bump, is $Re_h = 3,600$, which lies in the range of values reported by Gaster (1967) and other experiments cited by Diwan et al. (2006). The Reynolds number based on the momentum thickness of the boundary layer at the separation point is $Re_{\theta,s} = 27$. This

number is significantly smaller than most of the values reported in the literature for laminar separation bubbles. However, this discrepancy is explained by the use of a wall-mounted bump, as opposed to flat plates under an adverse pressure gradient or airfoils with small to moderate camber. The strong acceleration of the flow in the upstream portion of the bump leads to a substantial reduction of the boundary layer thickness. The rather low value of $Re_{\theta,s}$ is comparable to that in geometry-induced separation bubbles like those reported by Aniffa et al. (2023).

Convergence study

A series of simulations are performed to study the robustness of the results with respect to spatial discretization and numerical parameters. Spectral element methods allow the spatial discretization to be refined either by reducing the size of the elements (h -refinement) or by increasing the order of the polynomial used in the evaluation of derivatives within each element (p -refinement), Fig. 3.7 illustrates the h and p -refinement.

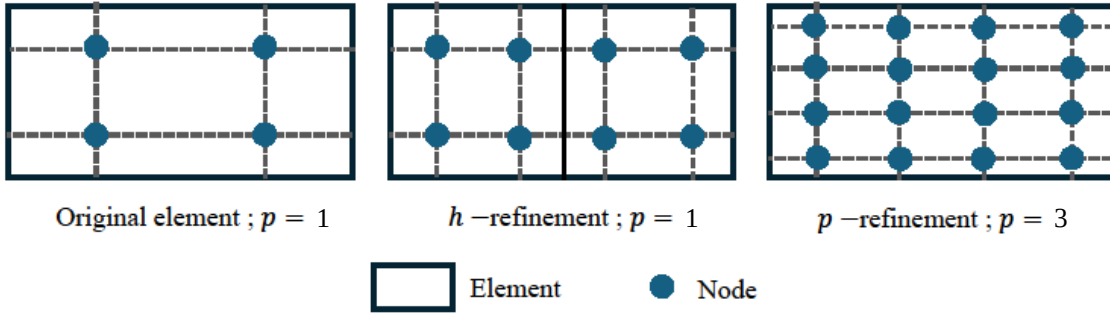


Figure 3.7: Illustration of h and p -refinement

A baseline mesh is defined based on a regular element size and different levels of zonal refinement, as shown in Fig. 3.2. To study the p -refinement, 2nd, 3rd and 4th order polynomials are used together with the baseline mesh. To study the h -refinement, two additional meshes were built with the element size respectively decreased (*fine mesh*) and increased (*coarse mesh*) by $\sim 20\%$ in all directions with respect to the baseline mesh. The number of spanwise elements is 6, 8, and 10 in the *coarse*, *baseline* and *fine* meshes, respectively. Additionally, increasing the polynomial order p also increases the effective spanwise resolution, enabling the assessment of results convergence with respect to spanwise discretization. The parameters of the 5 discretizations tested are given in Table 3.2, together with the number of spatial degrees of freedom (DOF) and the time step Δt^* used.

The mean length of the separated flow region L_s is compared for the different discretizations. The h -refinement study shows that L_s changes are below 2%, while the relative computational cost increases an order of magnitude for the finer mesh. The effect of the polynomial order p is found to be stronger. However, increasing the order from $p = 3$ to $p = 4$ results in a change of L_s below 3%.

The power spectra density (PSD) of the spanwise vorticity at different probe locations are

Table 3.2: Convergence study of spatial and temporal resolutions. h/h_{ref} : relative spatial mesh size; p : order of the polynomial discretizing each mesh element; DOF: degrees-of-freedom of the spatial discretization; Δt^* : time-step; L_s : mean length of the recirculation region; N_s : total number of time steps simulated after initial transient; N_{ss} : number of time steps per segment used in the computation of the PSD.

Discretization	h/h_{ref}	p	DOF	Δt^*	L_s [m]	Cost	N_s	N_{ss}
Baseline	1	3	2 024 960	3×10^{-5}	0.2782	1	2 410 000	33 332
Fine	0.8	3	5 012 480	5×10^{-6}	0.2818	13.82	2 400 000	200 000
Coarse	1.2	3	1 343 616	3×10^{-5}	0.2836	0.60	1 350 000	33 332
Base + low p	1	2	854 280	3×10^{-5}	0.2931	0.68	1 000 000	33 332
Base + high p	1	4	3 955 000	2×10^{-5}	0.2858	2.31	550 000	50 000

computed and shown in Fig. 3.8. Power spectral densities are estimated using Welch’s method Welch (1967). For each spatial discretization, different time-step sizes Δt^* are used and varying total numbers of snapshots N_s are stored. The PSD is computed by averaging the periodograms of segments of N_{ss} snapshots each with 50% overlap and a Hamming window. The segment length N_{ss} is chosen separately for each simulation to ensure that the frequency bin $\Delta f^* = 1$.

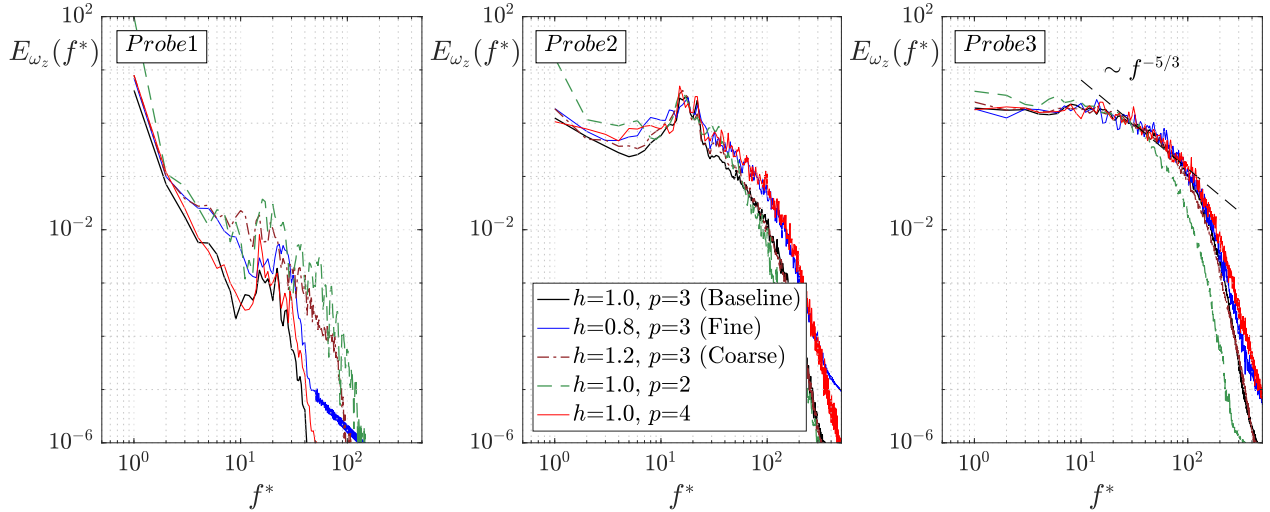


Figure 3.8: PSD of spanwise vorticity at selected probe locations for different spatial and temporal discretizations.

The location of the probes is illustrated in Fig. 3.9 and given in Table 3.3, together with other locations that are used in Chapter 6. Probes 1, 2, and 3 are located along the mean shear layer, the dividing streamline. Parameter ξ describe the relative distance of the probe to the separation point on the dividing streamline axis. The reference point and Probe 4 will be referred to in Chapter 6.

Probe 1 is located relatively close to the separation point, upstream of the region where the first KH vortices are observed. Probe 2 is in the region in which KH vortices are shed. Probe 3 is located downstream, where vortical structures of a broad range of scales are already

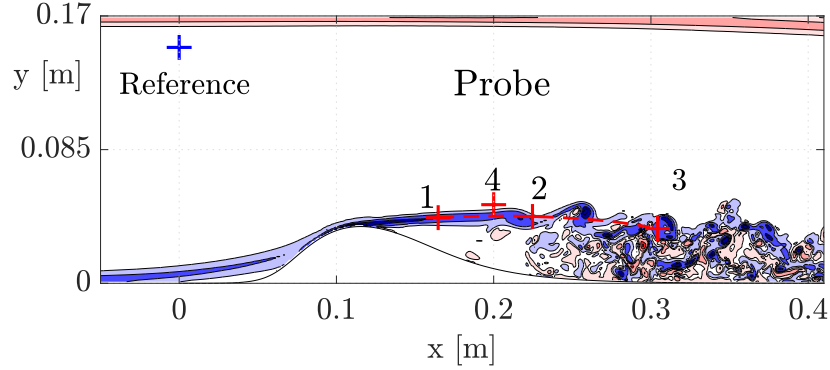


Figure 3.9: Location of the reference point and sampling probes.

Table 3.3: Coordinates of the reference point and sampling probes.

Probe	x [m]	y [m]	ξ [m]
Reference	0.0	0.15	-
1	0.165	0.0417	0.06
2	0.225	0.0425	0.12
3	0.304	0.0348	0.2
4	0.2	0.05	-

present. For all five spatial resolutions, the energy spectra are roughly the same except for the case with 2nd-order polynomial, coarser mesh, and at higher frequencies. These frequencies are associated with small scales and present very low energy levels, and the deviation of the spectra may indicate that the turbulent cascade is not fully resolved.

Similarly, the y^+ coordinate of the first internal node of the wall-adjacent element takes values as large as 2 at certain locations, indicating that the baseline discretization may not fully resolve the attached boundary layer there. However, the focus of present simulations is to resolve the flow structures dominating the separated flow dynamics, which correspond to comparatively larger scales than the wall viscous sub-layer, lower frequencies than those potentially under-resolved in the frequency spectra, and are nevertheless robustly captured by the baseline spatial discretization and temporal resolution (cf. Table 3.2). This convergence study shows that further spatial refinement can improve the resolution of finer turbulent scales, at the cost of significantly more expensive simulations, but does not result in significant changes in the physics of interest.

Frequency of the Kelvin-Helmholtz instability

Diwan and Ramesh, 2009 proposed a scaling of the frequency corresponding to the most amplified KH instability based on the linear instability analysis of piecewise linear velocity profiles resembling separated shear layers in the presence of a wall. This scaling combines the local (i.e., at individual streamwise locations) properties of the flow: the wall-normal coordinate and streamwise velocity of the inflection point (y_{in} and U_{in}), the vorticity thickness

δ_ω , and the most amplified frequency f , into two dimensionless parameters: the dimensionless frequency $f(y_{in}^2 + \delta_\omega^2)/\nu$ and a modified Reynolds number $\bar{R} = (U_{in}y_{in}/\nu) \sqrt{y_{in}/\delta_\omega}$. A linear dependence between the two numbers is postulated. In a practical flowfield, the separated shear layer evolves downstream, which results in increasing values of $\bar{R}$. As the scaling is due to an inviscid linear instability, its predictions are expected to hold up to limited $\bar{R}$ values. Diwan and Ramesh (2009) demonstrated that the scaling correctly predicts the KH frequency of several different experiments, up to $\bar{R} = 1400$.

To verify that the narrowband frequency peak of $f^* \approx 18$ identified in the PSD for the steady inflow case corresponds to KH instability, the scaling is applied to our simulation data and shown in Fig. 3.10. The theoretical and experimental results of Diwan and Ramesh (2009) are also shown for comparison. This comparison concludes that $f^* = 18$ indeed corresponds to the KH instability.

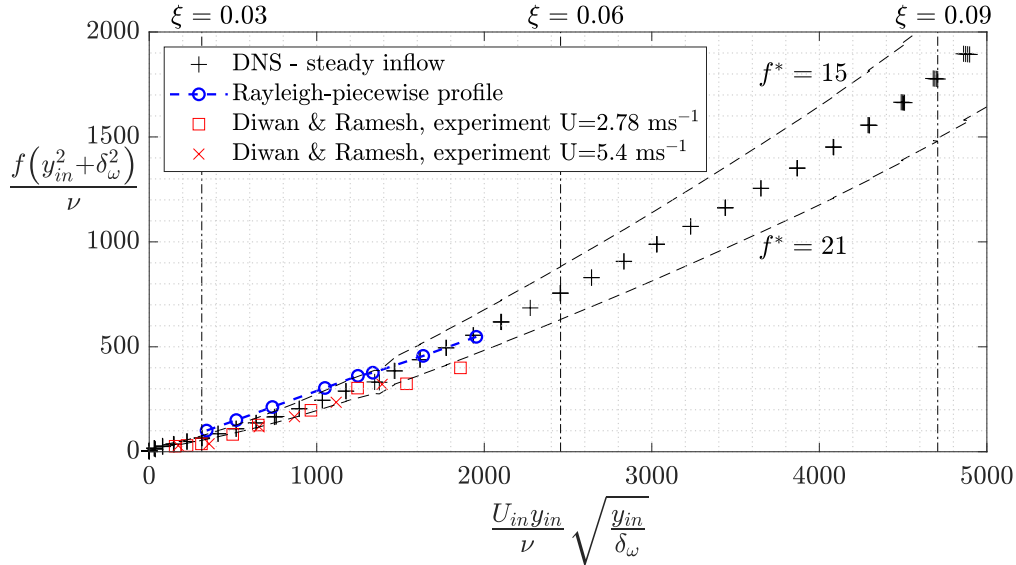


Figure 3.10: Scaling of the most amplified frequency along the separated shear layer.

3.2 Turbulent boundary layer separation over a Bump

The bump geometry features a concave-convex-concave curvature when observed from above. These curvature changes cause the incoming flow to transition from a favorable to an adverse pressure gradient. Under turbulent inflow conditions, strong favorable pressure gradients (FPG) can lead to relaminarization of the boundary layer. Balin and Jansen (2021) investigated this phenomenon using Direct Numerical Simulations, focusing on a Gaussian bump within a turbulent boundary layer at low Reynolds numbers and low subsonic speeds. The relaminarization process can be characterized using the streamwise pressure gradient parameter (dp^+)

$$dp^+ = \frac{\mu}{\sqrt{\rho}\tau_w^{3/2}} \frac{\partial p_{wall}}{\partial s} \quad (3.6)$$

and Launder’s acceleration parameter (K)

$$K = \frac{\nu}{U_{edge}^2} \frac{\partial U_{edge}}{\partial s}, \quad (3.7)$$

where ∂s denotes the gradient in the streamline direction or the wall surface to simplify the matter.

The onset of relaminarization typically occurs when the Launder’s parameter $K > 3.0 \times 10^{-6}$ (Launder, 1964). Regarding the streamwise pressure gradient, dp^+ , Patel (1965) observes that a value of $dp^+ < -0.018$ signifies a major departure in velocity profile within the inner wall region, implying the initiation of relaminarization. Meanwhile, P. R. Spalart (1986b) suggests a threshold value of $dp^+ < -0.025$. The pressure gradient, dp^+ , diverges as the shear stress approaches zero, indicating the proximity of either flow separation or reattachment. There is not a specific value to predict the onset of separation; Patel (1965) estimates, based on the experiment in Schubauer and Klebanoff (1951), that a value of $dp^+ > 0.007$ indicates a departure from the law of the wall in the adverse pressure gradient (APG) region.

The impact of relaminarization cannot be disregarded, as it can also generate low-speed streamwise streaks in the outer region of the boundary layer (Balin and Jansen, 2021; Cohen and Gloerfelt, 2018; Marbona, Rodríguez, Martínez-Cava, Valero, et al., 2024; Matai and Durbin, 2019). If relaminarization forces the flow into a more laminar state, turbulence in the outer region could generate Klebanoff modes, which may reduce the extent of the separation region (Balzer and Fasel, 2016; Hosseinverdi and Fasel, 2018). Klebanoff modes are characterized as streaky low-speed structures within the boundary layer and present characteristics similar to Görtler vortices.

3.2.1 Simulation setup and boundary conditions

The computational domain and mesh used for turbulent inflow in this section and in Chapter 7 are shown in Fig. 3.11. The size of the domain is identical to that of the steady laminar inflow study, Fig. 3.2. A more refined topology near the wall is needed for turbulent inflow to accommodate the small-scale structures near the wall upstream of the bump and around the separation region. This mesh has 192,000 hexahedral elements. The simulation imposed the result of the turbulent inflow validation case, Section 2.2.2, as the inlet velocity profile. The inlet boundary has identical size to the turbulent channel cross section in the validation case. Table 3.4 summarizes the boundary conditions. The setup results in $Re/L \approx 100,000$, similar to the laminar case. Third-order polynomial is assigned for spatial discretization which translates to 12,288,000 DOF.

3.2.2 Turbulent boundary layer separation

The instantaneous spanwise vorticity field is presented in Fig. 3.12. In turbulent settings, the separated shear layer breaks into smaller eddies sooner, generating more chaotic flow structures without distinct K-H vortices. A subtle presence of vortex shedding is visible. The

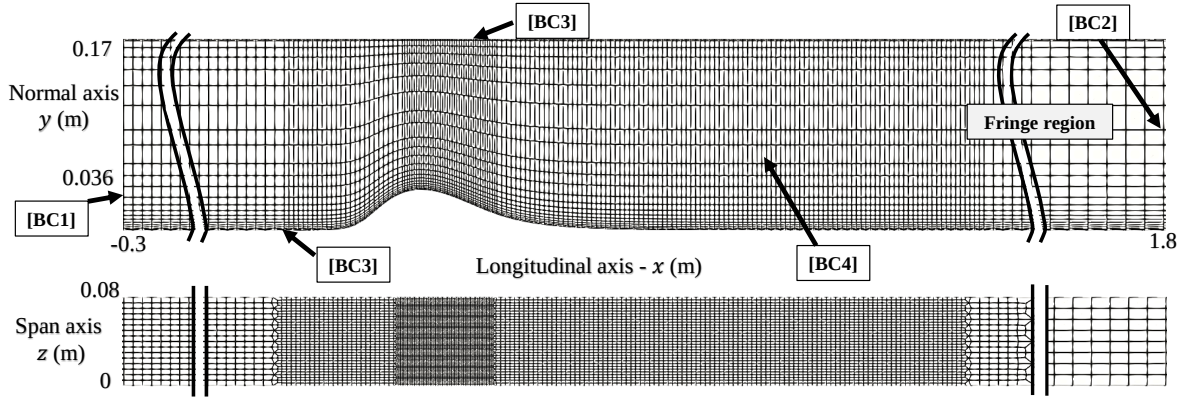


Figure 3.11: Computational domain and representative mesh for study with turbulent inflow in this section and Chapter 7

Table 3.4: Boundary conditions - turbulent inflow

[BC1] Inlet	From turbulent channel Section 2.2.2. $p_t = 108,186$ Pa and $T_t = 291.2$ K
[BC2] Outlet	Static pressure: $p = 101\,325$ Pa
[BC3] Wall	No-slip, adiabatic
[BC4] Sides	Periodicity

frequency of this phenomenon was characterized by Sigurdson (1995) with a dimensionless frequency $St_{shedding} = fh/U_s = 0.07 - 0.08$ (where U_s is the free-stream velocity at the separation point and h is the bump height) and has been shown to correlate well with a variety of geometry-induced separation bubbles.

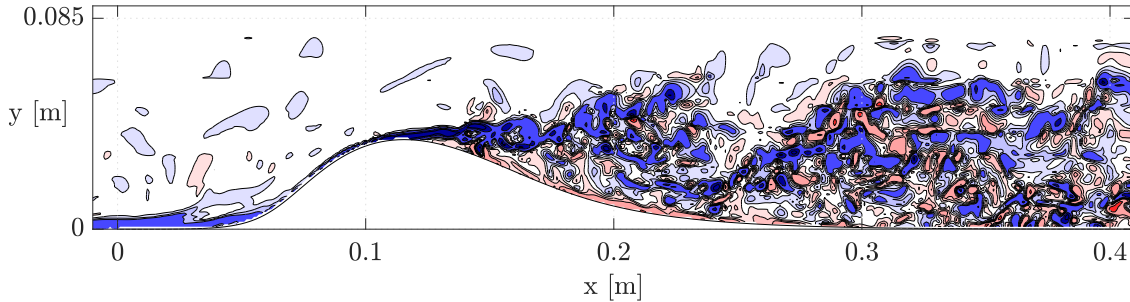


Figure 3.12: Spanwise vorticity field around a wall-mounted bump under turbulent inflow conditions.

The three-dimensional flow structures around the bump in turbulent settings are shown with the contour of the Qcriterion ($Q = +5000$) in Fig. 3.13. The Q-criterion is colored with streamwise vorticity (ω_x). Evident streamwise vortices are seen around the upstream concave region. In this region, the change of curvature creates a negative (favourable) pressure gradient with the mean pressure gradient parameter (dp^+) below the threshold value for relaminarization, Fig 3.14. Large-eddy simulations with identical bump geometry and

higher Reynolds number confirms that the presence of these streamwise vortices reduces the momentum thickness (Marbona, Rodríguez, Martínez-Cava, Valero, et al., 2024).

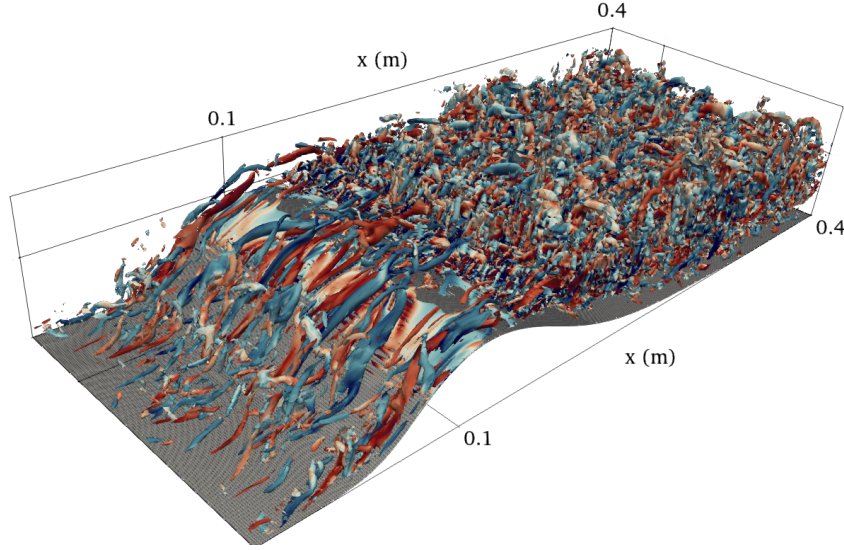


Figure 3.13: Qcriterion contour ($Q=+5000$) colored with streamwise vorticity (ω_x) - turbulent inflow

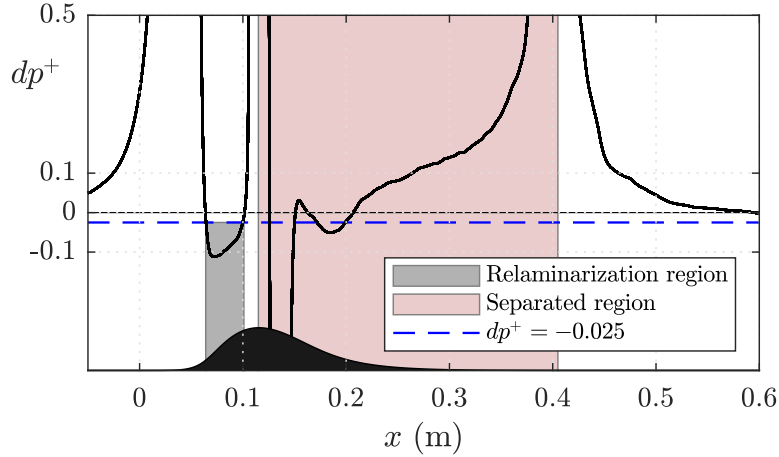


Figure 3.14: Time-averaged pressure gradient parameter (dp^+) with turbulent inflow

Meanflow field

The time-averaged and root-mean-square streamwise velocity field are shown in Fig. 3.15. The mean characteristics are different from the laminar case. The mean recirculation region is flattened towards the wall with no ‘dead air’ region. Moreover, the maximum reverse flow is in the same order magnitude with the laminar case and located more upstream. The maximum rms of streamwise velocity is spread evenly along the separated shear layer.

The bump mean pressure and skin friction coefficient are shown in Fig. 3.16. The presence of turbulence reduces the negative pressure and the maximum skin friction around the summit.

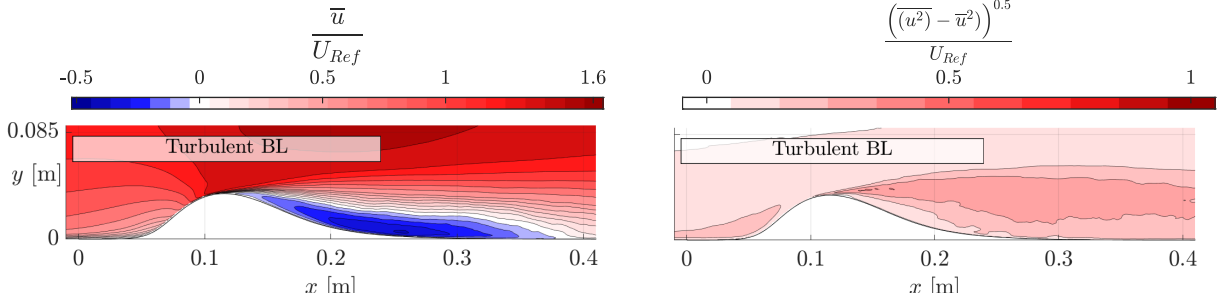


Figure 3.15: Time-averaged (left) and root-mean-square (right) streamwise velocity field - turbulent inflow

The pressure plateau is still present in the turbulent separation, although shorter than in the laminar setting. It ends around the region where the maximum reverse flow and minimum skin friction are located. Recovery of pressure occurs more gradually than in laminar separation. From the skin friction, the mean separation is slightly delayed to $x = 0.116$ m with mean reattachment location at $x = 0.404$ m. The mean separation region is $L_s = 0.2889$ m and the Reynolds based on this length is $Re_L = 28900$. The Reynolds number based on the momentum thickness at separation is $Re_{\theta_s} = 22$.

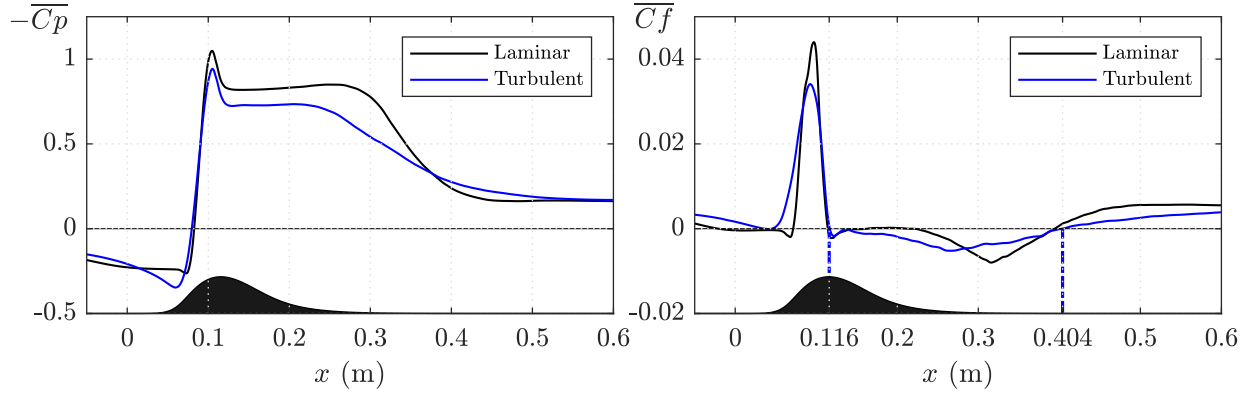


Figure 3.16: Pressure and Skin Friction coefficient

Part II:
Orthogonalized Variational Mode Decomposition

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Chapter 4

Orthogonalized Variational Mode Decomposition

Frequency analysis using the Fourier transform has been a cornerstone in engineering applications. It offers the capability to dissect data or signals according to their frequency spectrum, unveiling the predominant frequencies within properly sampled datasets. The requirement for stationary data is crucial, as the Fourier transform operates under the assumption that the spectrum remains uniform across time intervals, indicated by the convolution operation with infinite sinusoidal signals across various frequencies. Consequently, it is particularly suitable for stationary signals characterized by consistent modulation of amplitude and frequency (AM / FM). However, it does not provide any insight into the signal's time locality.

For non-stationary data, the Fourier transform might trivialize or smear frequencies that change over time. The depreciation of this frequency-modulated signal occurs because the Fourier transform averages the contribution of this frequency across the dataset, even though it only appears at certain periods (i.e, locally). In non-linear systems, this potentially disregards certain phenomena that might play important roles in transient and non-stationary dynamics.

In the 1980s, Morlet et al. (1982a, 1982b) introduced a method for time-frequency (TF) analysis by convolving the signal with a short-duration small wave, primarily for applications in geophysics. The term *wavelet* was then coined by Grossmann and Morlet (1984) with the introduction of the continuous wavelet transform (CWT). The wavelet transform offers time locality in its transformation, hence it is suitable to represent non-stationary signals. This method shares similar characteristics with the Short-Time Fourier Transform (STFT); however, it is capable of overcoming the limitation of fixed time-frequency resolution and frequency smearing in the STFT. Figure 4.1 presents a comparison of resolutions obtained from different analyses. Since its introduction, wavelet transform has been extensively applied in signal processing, image compression, and data analysis (Farge, 1992, Shapiro (1993), Walker (2006), Zerva et al. (2023), Marbona, Rodríguez, Martínez-Cava, Valero, et al. (2024)).

The potential application of wavelet transformation for decomposition of turbulent flows was acknowledged by Farge (1992). The use of CWT to track the dynamics of coherent structures and contributions in the energy spectrum was demonstrated. Moreover, a further mode

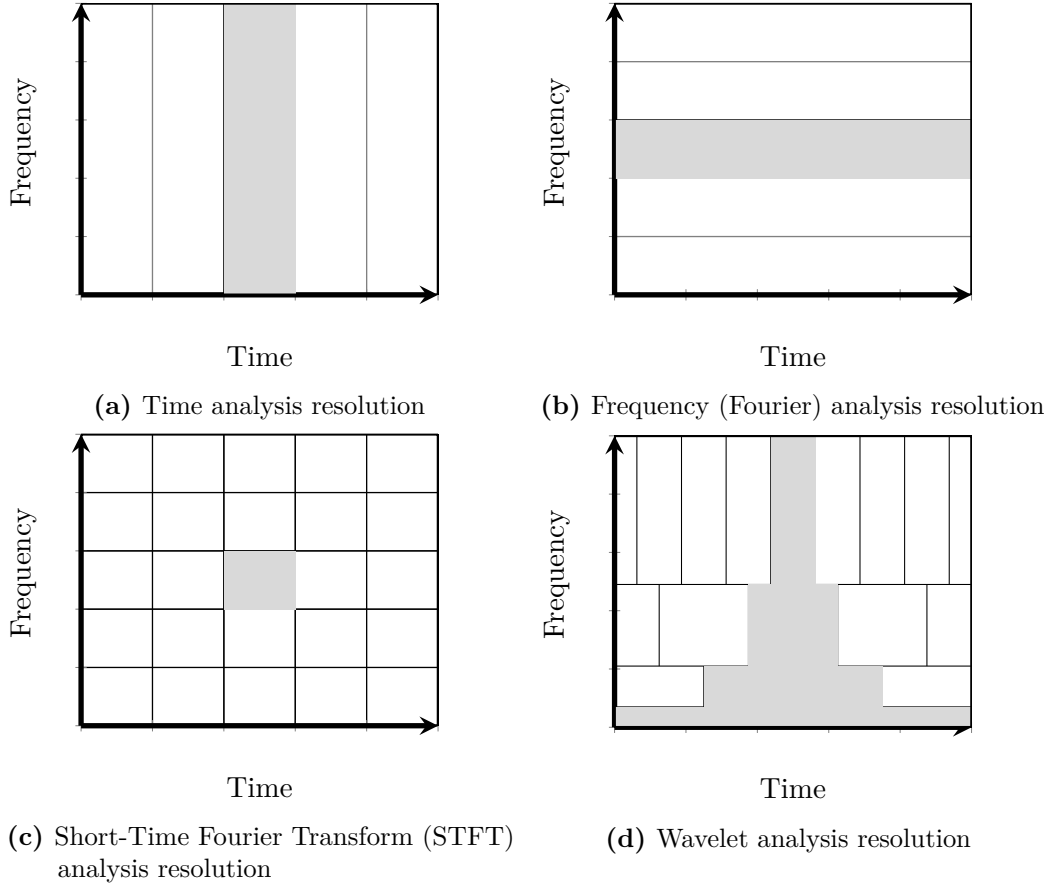


Figure 4.1: Resolution of time, frequency, short-time fourier and wavelet analysis

decomposition based on orthogonal projection with discrete wavelet transform was described in the manuscript.

A more comprehensive time-frequency (TF) analysis was introduced by Huang et al. (1998) through the Hilbert-Huang transformation (HHT) and the Empirical Mode Decomposition (EMD). The concept involves decomposing signals into different intrinsic mode functions (IMFs) empirically. These decomposed signals are represented as complex analytic signals by employing the Hilbert transform to characterize the signal phase and amplitude at specific time periods. The HHT potentially offers improved analysis capabilities for nonlinear systems. However, the method has weaknesses inherent in the EMD algorithm, due to its sensitivity to noise and sampling. The original algorithm relies on an ad hoc approach to identify local maxima and minima crossings, which can lead to significant variations in the resulting intrinsic mode function (IMF) decomposition when the input signal differs slightly.

To address this limitation, several approaches have been developed to generate intrinsic mode function (IMF) modes based on a solid mathematical framework. For instance, Daubechies et al. (2011) proposed the synchrosqueezed wavelet transform, a variational approach that incorporates thresholding within the wavelet transform process. Building on this, Gilles (2013) introduced an updated definition of IMFs and an adaptive wavelet-based method known as the Empirical Wavelet Transform (EWT). This method involves constructing a wavelet

filter bank tailored to the input signal's spectrum. Later, Dragomiretskiy and Zosso (2014) advanced the field by introducing Variational Mode Decomposition (VMD), a technique that decomposes a signal into its principal modes by solving a variational problem designed to minimize the cumulative bandwidth of modes while preserving their resemblance to the input signal.

This chapter focuses on the development of proposed extensions of VMD with added orthogonal objective constraints integrated into the minimization problem. The proposed orthogonalized variational mode decomposition (OVMD) employs alternating projections to iteratively transfer the component orthogonal to the mode while receiving the component parallel to the mode, ensuring efficient convergence and uniqueness, as will be explained later. Furthermore, it will be demonstrated that the novel method also enhances the robustness of the decomposed modes with respect to input parameters.

The subsequent sections are structured as follows: Section 4.1 introduces the method and categorizes signal components based on their temporal presence and characteristics. Section 4.2 provides an overview of the VMD method, as presented by Dragomiretskiy and Zosso (2014). Section 4.3 outlines the proposed method, incorporating orthogonality constraints and filter bandwidth parameter determination. Section 4.4 compares the VMD and OVMD methods using an artificial signal composed of distributed and localized event components. The sensitivity of the OVMD method to varying noise levels, sampling frequencies, data sizes, and the prescribed number of modes is analyzed and discussed to demonstrate the robustness of the method. Section 4.5 summarizes the improvement achieved by the orthogonalized variational mode decomposition method.

4.1 Signals and characteristics

In general, a modulated signal can be decomposed into a combination of sinusoidal carrier components (IMFs, $u_k(t)$), as expressed by

$$X(t) = \sum_{k=1}^K u_k(t) + r(t) = \sum_{k=1}^K A_k(t) \cos(\phi_k(t)) + r(t), \quad (4.1)$$

where A_k is the amplitude, ϕ_k is the angle variation over time t and r is the residual. The instantaneous frequency is expressed as $\omega_k(t) := \phi'_k > 0 \forall t$. In the IMF definition, the amplitude and frequency need to vary much slower than the phase for IMF mode to behave as a pure harmonic on an infinitely long data sample (Dragomiretskiy and Zosso, 2014; Gilles, 2013). The carrier components can be categorized based on their temporal presence and characteristics.

Temporal presence

- Stationary component: Characterized by their consistent presence across the length of the dataset. They are continuous and can be expressed as a combination of harmonic

sinusoidal components. In the frequency spectrum, a stationary signal component with constant amplitude and frequency is represented by a narrow-band peak.

- **Time-localized event:** Characterized by its discontinuous presence. It appears in specific lapses of the signal domain for a finite duration, as shown in Fig. 4.2. The magnitude of the signal rapidly increases and/or decreases during transitions between states, such as from active to inert and vice versa. In the signal spectrum, it appears as a broadband with multiple peaks of different magnitude, making it difficult to group and represent as a single IMF mode. A localized event signal is both amplitude and frequency modulated, but the inverse does not apply. Examples of this type of signal include ECG spikes, pulses, and wave packets.

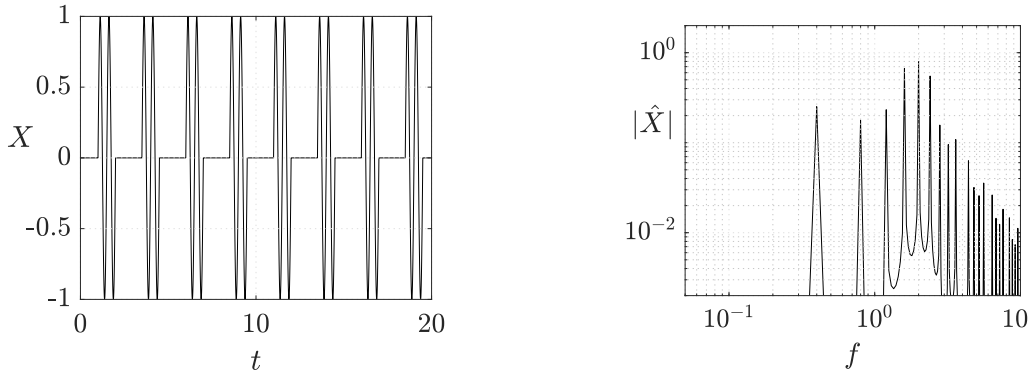


Figure 4.2: A time-localized events signal and its Fourier transform

Temporal characteristics

- **Carrier wave:** A pure sinusoidal signal with constant frequency and amplitude. The signal is characterized by a narrow bandwidth and is stationary by definition.
- **Frequency modulated (FM):** The frequency of the signal varies continuously over time, i.e. $\omega' \neq 0$. In the signal spectrum, FM generates multiple sidebands that are symmetrically spaced around the average carrier frequency, leading to a complex spectral distribution. A distinctive characteristic occurs when the frequency increases or decreases continuously along the span of the data, creating a frequency sweep or “chirp”. This results in several broad peaks with comparable magnitudes in the frequency spectrum, making it difficult to detect using the Fourier transform.
- **Amplitude modulated (AM):** The amplitude of the signal varies continuously over time, i.e. $A' \neq 0$. In the signal spectrum, AM creates sidebands around the carrier frequency, resulting in a broadened peak rather than a single sharp peak.
- **Amplitude-Frequency modulated (AM-FM):** A combination of amplitude modulation (AM) and frequency modulation (FM), where both the amplitude and frequency of the carrier signal are varied over time.

A detailed illustration of FM, AM, and AM/FM signals, along with their spectrum characteristics, is described in Dragomiretskiy and Zosso (2014).

4.2 Variational Mode Decomposition

Variational mode decomposition dissects a continuous real-valued input signal $X(t)$ into a discrete set of sub-signals $u_k(t)$ (modes). A set of constrained variational problems is formulated to decompose the signal into modes that are compact around a center pulsation frequency ω_k with minimal bandwidth. The bandwidth of mode k is determined by computing the complex analytic signal of each mode using the Hilbert transform for the positive frequencies (unilateral spectrum). It is then shifted to its center frequency by its convolution with an exponential function. The squared L_2 -norm is utilized to estimate the bandwidth. The constrained variational problem takes the form (Dragomiretskiy and Zosso, 2014):

$$\begin{aligned} \min_{\{u_k\}, \{\omega_k\}} & \left\{ \sum_{k=1}^K \left\| \partial_t \left[\left(\delta(t) + \frac{j}{\pi t} \right) * u_k(t) \right] e^{-j\omega_k t} \right\|_2^2 \right\} \\ \text{s.t.} & \sum_{k=1}^K u_k(t) = X(t), \end{aligned} \quad (4.2)$$

where $j = \sqrt{-1}$, $*$ denotes the convolution operation, and δ is the Dirac delta function.

The minimization problem is solved by constructing an augmented Lagrangian $\mathcal{L}$ that either minimizes the reconstruction error or imposes an exact reconstruction, and estimating the solution iteratively with the alternate direction method of multipliers (ADMM) (Gabay and Mercier, 1976). The augmented Lagrangian of VMD in spectral domain (see Eqs. (19)-(20) in Dragomiretskiy and Zosso (2014)):

$$\begin{aligned} \mathcal{L}(\{\hat{u}_k\}, \{\omega_k\}, \hat{\lambda}) &:= \sum_k^K \alpha_k \left\| j\omega [(1 + \text{sgn}(\omega + \omega_k)) \hat{u}_k(\omega + \omega_k)] \right\|_2^2 \\ &+ \left\| \hat{X}(\omega) - \sum_k \hat{u}_k(\omega) + \frac{\hat{\lambda}(\omega)}{2} \right\|_2^2, \end{aligned} \quad (4.3)$$

where $\{\hat{\cdot}\}$ denotes the Fourier domain and $\hat{\lambda}$ is the Lagrange multiplier.

In the augmented Lagrangian $\mathcal{L}$, the number of modes K is a pre-defined input for the optimization, whose estimation can be cumbersome for multimode noisy signals. It requires users to estimate, at post-processing, whether a computed mode represents more than one signal component (small K ; under-segmentation) or if multiple modes have similar center frequency and structure and hence are recovering parts of the same signal component (high K , over-segmentation). Some of these weaknesses are eliminated in the Successive VMD (SVMD) by Nazari and Sakhaei (2020), which automatically optimizes the number of modes. However, SVMD is sensitively more computationally expensive than VMD and tends to present larger representation errors.

Dragomiretskiy and Zosso, 2014 describe the VMD method as quasi-orthogonal when the modes are non-overlapping. However, its effectiveness in separating modes with frequency overlap depends on the prescribed number of modes K and the real-valued parameter α_k ,

which implicitly controls the resulting bandwidth. Furthermore, neither VMD nor SVMd explicitly deal with spikes in the frequency spectrum that are spread across multiple modes, nor do they isolate components affected by destructive interference within the modes.

4.3 Orthogonalized-Variational Mode Decomposition

Since VMD has a tendency to produce duplicate modes with destructive interference, particularly in over-segmentation, an enhancement is proposed by introducing weak orthogonal constraints to the variational problem. This method promotes orthogonality by allowing signal components to be transferred between modes, reducing interference, and preventing mode duplication.

4.3.1 Proposed method: Orthogonal Objective

Two additional objective functions are introduced in the augmented Lagrangian to promote orthogonality between each pair of modes. These objectives are applied conditionally (split) depending on whether a mode is “dominant” or “weaker” relative to another mode at each iteration step.

The determination of the “dominant” or “weaker” roles between modes k and i is made by comparing the maximum magnitude in the frequency spectrum of each mode. This operation is similar to a Heaviside step function and written as:

$$H_{ki} = \begin{cases} 1, & \max(|\hat{u}_k(\omega)|) \geq \max(|\hat{u}_i(\omega)|) \\ 0, & \text{otherwise} \end{cases}. \quad (4.4)$$

The subscript k represents the main mode considered in the loop, and i represent any other mode. A value of unity signifies that mode k is “dominant” over mode i . The transpose H_{ki} is the complementary to H_{ik} for $i \neq k$; if H_{ki} is one then H_{ik} is zero, and vice versa. Based on this characterization the split objective is then applied.

When $\hat{u}_k$ is the “weaker” mode, the objective is to minimize the sum of the total energy of mode k that correlates with other modes i . For practical reasons that will be discussed below, this is approximated in the frequency domain as:

$$\min_{\{\hat{u}_k\}} \left\{ \sum_k \sum_{i \neq k} \left\| H_{ik} \hat{\beta}_i^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_k(\omega) \right\|_2^2 \right\}. \quad (4.5)$$

Here, $\hat{\beta}_i$ is a Wiener filter centered around the center frequency of mode i :

$$\hat{\beta}_i(\omega) = \frac{1}{1 + 2 \alpha_i (\omega - \omega_i)^2}. \quad (4.6)$$

The parameter α_i in the Wiener filter controls the variance of the filter bandwidth and will be discussed later in 4.3.4.

The energy of the correlation between modes u_k and u_i is estimated in the frequency domain using the coherence / cross-spectral density

$$\hat{P}_{ik}(\omega) = \hat{P}_{ki}(\omega) = \frac{|\hat{u}_i(\omega)\hat{u}_k^*(\omega)|}{\|\hat{u}_i\|_2\|\hat{u}_k\|_2}, \quad (4.7)$$

where the symbol $\{\cdot^*\}$ denotes the complex conjugate, and the energy of mode u_k , as

$$\approx \|\hat{P}_{ik}^{1/2} \hat{u}_k(\omega)\|_2^2. \quad (4.8)$$

Coherence quantifies the similarity between two signals and helps to evaluate their orthogonality and correlation (Carter and Knapp, 1975; White and Boashash, 1990). When coherence values approach zero within a given frequency range ω , the signals are nearly uncorrelated (orthogonal). Conversely, values near one indicate a strong linear relationship between specific signal components (White and Boashash, 1990). In the implemented algorithm, the elements in $\hat{P}_{ik}$ are computed based on the latest results and treated as invariant in the variational problem. The symbol $\hat{P}_{ik}^{1/2}$ denotes the Cholesky decomposition of $\hat{P}_{ik}$ so that $\hat{P}_{ik} = \hat{P}_{ik}^{1/2} (\hat{P}_{ik}^{1/2})^*$.

Each element in the inner sum of the objective function in Eq.(4.5) thus aims at minimizing the energy of the portion of $\hat{u}_i$ that is correlated with $\hat{u}_k$ and falls within the frequency band corresponding to mode i (via filtering with $\hat{\beta}_i$). The objective will reach a minimum either when the cross-correlation of mode k with other modes is minimized (indicating that mode k remains intact) or when the energy of mode k becomes negligible. Effectively, this objective allows the transmission of the mode k component to other modes.

When $\hat{u}_k$ is the “dominant” mode, the objective is to minimize the energy interference aligned with mode k across modes that are “weaker” than mode k , including mode k itself. This objective is quantified by the absolute difference between the total energy of these individual modes and the cumulative energy from their summation, defined as:

$$\min_{\{\hat{u}_k\}, \{\omega_k\}} \left\{ \sum_k \left| \sum_i \left\| H_{ki} \hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_i(\omega) \right\|_2^2 - \left\| \sum_i H_{ki} \hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_i(\omega) \right\|_2^2 \right| \right\} \quad (4.9)$$

where β_k is a Wiener filter around the center frequency of mode k (ω_k) (same as (4.6) but with subscript k).

Minimization problems involving absolute values require evaluating solutions for both the positive and negative parts. In this case, the problem is reformulated into two expressions: one with a positive sign and one with a negative sign in front of (4.9) without the absolute value bars. Then, the solution yielding the smallest absolute value is selected. For this problem, extensive testing has demonstrated that the minimum value is consistently achieved with the positive part of the objective -specifically, Eq. (4.9) without the absolute value bars.

Effectively, the objective enables mode k to receive the signal components from other modes that are correlated with itself, achieving its minimum value when the cross-correlations between mode k and other modes i are minimized, thereby reducing interference.

The combined effect of Eqs. (4.5) and (4.9) is to promote that components of the signal $X(t)$ with dominant carrier frequencies are recovered as individual modes by transferring signal contents with similar carrier frequency from the weaker to the dominant modes. The modified augmented Lagrangian problem is constructed by adding these conditional objectives into Eq. (4.3), resulting in the following expression:

$$\begin{aligned} \mathcal{L}(\{\hat{u}_k\}, \{\omega_k\}, \hat{\lambda}) := & \sum_k^K \left[\alpha_k \left\| j\omega [(1 + \text{sgn}(\omega + \omega_k)) \hat{u}_k(\omega + \omega_k)] \right\|_2^2 + \sum_{i \neq k} \left\| H_{ik} \hat{\beta}_i^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_k(\omega) \right\|_2^2 \right. \\ & + \sum_i \left\| H_{ki} \hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_i(\omega) \right\|_2^2 - \left\| \sum_i H_{ki} \hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_i(\omega) \right\|_2^2 \Big] \quad (4.10) \\ & + \left\| \hat{X}(\omega) - \sum_k \hat{u}_k(\omega) + \frac{\hat{\lambda}(\omega)}{2} \right\|_2^2, \end{aligned}$$

4.3.2 Minimization with respect to $\hat{u}_k$

Following the same procedure as in ADMM algorithm (Gabay and Mercier, 1976) and VMD (Dragomiretskiy and Zosso, 2014), the minimization problem for updating the mode $\hat{u}_k$ in the subsequent iteration, denoted with $n + 1$, takes the form

$$\begin{aligned} \hat{u}_k^{n+1} = \arg \min_{\hat{u}_k, \omega_k \in X} & \left\{ \alpha_k \left\| j\omega [(1 + \text{sgn}(\omega + \omega_k)) \hat{u}_k(\omega + \omega_k)] \right\|_2^2 + \sum_{i \neq k} \left\| H_{ik} \hat{\beta}_i^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_k(\omega) \right\|_2^2 \right. \\ & + \sum_i \left\| H_{ki} \hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_i(\omega) \right\|_2^2 - \left\| \sum_i H_{ki} \hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_i(\omega) \right\|_2^2 \\ & \left. + \left\| \hat{X}(\omega) - \sum_i \hat{u}_i(\omega) + \frac{\hat{\lambda}(\omega)}{2} \right\|_2^2 \right\}. \quad (4.11) \end{aligned}$$

Now the frequency in the first term is shifted with the change in variable $\omega \leftarrow \omega - \omega_k$, and the contribution from negative frequencies is accounted for by leveraging the Hermitian symmetry of the spectrum, which involves doubling the positive frequency components (see Eqs.(19)-(22)

in Dragomiretskiy and Zosso (2014)), yielding

$$\begin{aligned} \hat{u}_k^{n+1} = \arg \min_{\hat{u}_k, \omega_k \in X} & \left\{ \int_0^\infty 4\alpha_k(\omega - \omega_k)^2 |\hat{u}_k(\omega)|^2 + 2 \sum_{i \neq k} H_{ik} \hat{\beta}_i \hat{P}_{ik} |\hat{u}_k(\omega)|^2 \right. \\ & + 2 \sum_i H_{ki} \hat{\beta}_k \hat{P}_{ik} |\hat{u}_i(\omega)|^2 - 2 \left| \sum_i H_{ki} \hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_i(\omega) \right|^2 \\ & \left. + 2 \left| \hat{X}(\omega) - \sum_i \hat{u}_i(\omega) + \frac{\hat{\lambda}(\omega)}{2} \right|^2 d\omega \right\}. \end{aligned} \quad (4.12)$$

The optimization problem Eq. (4.12) is solved by taking the first variation w.r.t $\hat{u}_k$:

$$\begin{aligned} & 8\alpha_k(\omega - \omega_k)^2 \hat{u}_k(\omega) + 4 \sum_{i \neq k} H_{ik} \hat{\beta}_i \hat{P}_{ik} \hat{u}_k(\omega) \\ & - 4 \sum_{i \neq k} H_{ki} \hat{\beta}_k \hat{P}_{ik}^{1/2} \hat{P}_{kk}^{1/2} \hat{u}_i(\omega) \\ & + 4 \left(\hat{u}_k(\omega) - \hat{X}(\omega) + \sum_{i \neq k} \hat{u}_i(\omega) - \frac{\hat{\lambda}(\omega)}{2} \right) = 0. \end{aligned} \quad (4.13)$$

Finally, isolating $\hat{u}_k$ in the previous expression, an expression for updating it based on the latest data is obtained:

$$\hat{u}_k^{n+1}(\omega) = \frac{\hat{X}(\omega) - \sum_{i \neq k} \hat{u}_i(\omega) + \sum_{i \neq k} H_{ki} \hat{\beta}_k \hat{P}_{ik}^{1/2} \hat{P}_{kk}^{1/2} \hat{u}_i(\omega) + \frac{\hat{\lambda}(\omega)}{2}}{1 + 2\alpha_k(\omega - \omega_k)^2 + \sum_{i \neq k} H_{ik} \hat{\beta}_i \hat{P}_{ik}}. \quad (4.14)$$

4.3.3 Minimization with respect to ω_k

In Eq.(4.12), the central frequency of the estimated mode k , ω_k , appears in the first term and $\hat{\beta}_k$. Hence, the minimization problem for ω_k takes the form:

$$\begin{aligned} \omega_k^{n+1} = \arg \min_{\omega_k} & \left\{ 4\alpha_k \int_0^\infty (\omega - \omega_k)^2 |\hat{u}_k(\omega)|^2 \right. \\ & \left. + 2 \sum_i H_{ki} \hat{\beta}_k \hat{P}_{ik} |\hat{u}_i(\omega)|^2 - 2 \left| \sum_i H_{ki} \hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{u}_i(\omega) \right|^2 d\omega \right\}. \end{aligned} \quad (4.15)$$

The first variation $\delta\beta_k/\delta\omega_k$ is approximately $2\alpha_k\hat{\beta}_k^2(\omega - \omega_k)$, where $\hat{\beta}_k$ is estimated using the frequency estimation from the previous iteration, ω_k^n . Taking the first variation of the term inside the curly brackets in Eq.(4.15) with respect to ω_k for positive frequencies, equating it to zero and solving for ω_k , an expression for updating the frequency is obtained:

$$\omega_k^{n+1} = \frac{\int_0^\infty \omega (|\hat{u}_k|^2 + O_{\omega_k}) d\omega}{\int_0^\infty |\hat{u}_k|^2 + O_{\omega_k} d\omega}, \quad (4.16)$$

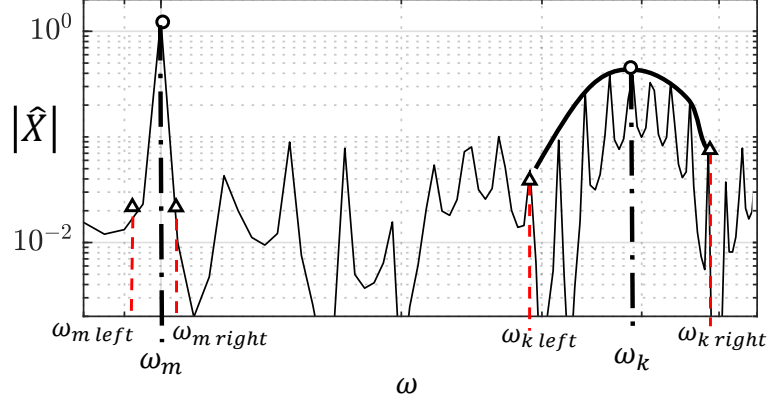


Figure 4.3: Illustration of bandwidth for a narrowband signal with center frequency ω_m and a broadband signal, ω_k .

where

$$O_{\omega_k} = \hat{\beta}_k^2 \left(\left| \sum_i H_{ki} \hat{P}_{ik}^{1/2} \hat{u}_i(\omega) \right|^2 - \sum_i H_{ki} \hat{P}_{ik} |\hat{u}_i(\omega)|^2 \right). \quad (4.17)$$

4.3.4 Proportional value of α_k with constant filter bandwidth

In the original formulation of VMD, the value of α_k is constant across all modes and is provided as input. This parameter controls the bandwidth of the Wiener filter and consequently controls the suppression of the contribution of frequency components that are far from the mode's central frequency. A high value of α indicates a narrower bandwidth with a smaller frequency variation. Assigning a constant value of α_k to all modes results in a relatively similar decay/variation in both low and high frequencies (Rehman and Aftab, 2019). In real-life scenarios, most signals exhibit varying frequency characteristics that depend on their scales (Brown and Thomas, 1977; Ganapathisubramani et al., 2012). A large-scale mode (low frequency) should have a smaller frequency variation than a lower mode (high frequency). To address this, a method for adapting α_k to the mode's central frequency ω_k and overall structure of the signal spectrum is proposed.

In the implementation, the value of α_k^{n+1} at iteration $n+1$ is updated based on the central frequency estimated in the precedent iteration, ω_k^n , and a normalized bandwidth, B :

$$\alpha_k^{n+1} = \frac{1}{2 B(\omega_k) \omega_k^n}. \quad (4.18)$$

Monitoring the signal spectrum, a representative bandwidth $B(\omega)$ can be estimated (see Fig. 4.3). Consider the narrowband peak centered at ω_m . The bandwidth around ω_m can be determined as the difference between the right (higher) and left (lower) cutoff frequencies, $\omega_{m \text{ right}}$ and $\omega_{m \text{ left}}$. These cutoff frequencies are defined based on a specified threshold in the energy fraction of $|\hat{X}(\omega_m)|$. The representative scaled bandwidth is then defined as

$$B(\omega_m) = \frac{(\omega_{m \text{ right}} - \omega_{m \text{ left}})}{\omega_{max}} \quad (4.19)$$

for a frequency interval including ω_m . To eliminate the dependency on sampling frequency and length, a normalization with the maximum (Nyquist) frequency ω_{max} is required. The same procedure can be applied to determine the representative bandwidth for a broadband feature in the spectrum, like the one centered at ω_k in Fig. 4.3. Further, it is possible to define B as a continuous function of ω for any given signal spectrum, which would provide a more refined value of α_k . However, determining the optimal filter bandwidth for all frequencies ω is beyond the scope of this study. Instead, we use the above definition to quickly estimate a constant filter bandwidth value that is applied for all central frequencies. This amounts to prescribing a constant value to $F = 1/(2B(\omega))$ in Eq. (4.18), resulting in

$$\alpha_k = F/\omega_k. \quad (4.20)$$

The robustness of this approach is evaluated in Section 4.4.3, where the energy sensitivity of modes to different noise levels, sampling frequencies, and sampling times is examined.

4.3.5 Complete OVMD algorithm

The orthogonalized VMD optimization problem is solved using the Alternating Direction Method of Multipliers (ADMM), as applied in the VMD (Dragomiretskiy and Zosso, 2014). Algorithm 1 performs the optimization process in the discrete Fourier space. The global iteration continues until the relative change in the representation energy between two consecutive iterations is reduced below a prescribed threshold ϵ or until a maximum number of iterations N_{max} is reached. Since the decomposition handles discrete and finite signals, addressing the signal boundaries is crucial for maintaining smooth characteristics in the mode decomposition at both the beginning and end of the data. The VMD method proposes a straightforward solution: mirroring the data at both the beginning and end, which is also adopted here. This effectively extends the signal's length in the Fourier domain (Dragomiretskiy and Zosso, 2014).

4.4 Experiments and Results

In this section, a synthetic signal constructed as the sum of different simple components is used as a benchmark for testing and comparing the efficacy of VMD and OVMD techniques in decomposing it into its constituent modes. The signal comprises stationary harmonic waves with constant frequency and amplitude, as well as time-localized components with a constant carrier frequency and amplitude, which define periodically repeating, short-time pulses. The amplitude and instantaneous frequency of the time-localized pulses change much faster than the phase, making their decomposition challenging (Gilles, 2013; Huang et al., 1998). Here, the goal is not to perfectly reconstruct this component but to demonstrate that it can be isolated with reasonable fidelity.

The synthetic signal is sampled for a time interval $t = [0, 20]$ with sampling frequency $f_s = 100$. The signal components are listed below and ranked based on their energy, with the last component being noise, and illustrated in Fig. 4.4. Note that the time axis range varies

Algorithm 1 Optimization of OVMD.

```

INITIALIZE( $\{\hat{u}_k^1\}, \{\omega_k^1\}, \{\hat{\lambda}^1\}, B, n \leftarrow 0$ )

COMPUTE  $\{\alpha_k^1\}$  : with (4.18)
COMPUTE  $\{\hat{\beta}_k^1\}$  : with (4.6)

while  $\sum_k \|\hat{u}_k^{n+1} - \hat{u}_k^n\|_2^2 / \|\hat{u}_k^n\|_2^2 > \epsilon$  and  $n < N_{max}$  do

     $n \leftarrow n + 1$ 

    for  $k = 1 : K$  do
        UPDATE  $\{\cdot\}$  using  $\{\cdot\}_{i \leq k}^{n+1}$  and  $\{\cdot\}_{i \geq k}^n$  (latest data) :
             $\{\hat{P}_{ik}^{n+1}\}$  : with (4.7)
             $\hat{u}_k^{n+1}(\omega)$  : with (4.14)
             $\omega_k^{n+1}$  : with (4.16)
    end for
    UPDATE :
         $\{\alpha_k^{n+1}\}$  : with (4.18)
         $\{\hat{\beta}_k^{n+1}(\omega)\}$  : with (4.6)

    DUAL ASCENT  $\{\hat{\lambda}^{n+1}\}$ :
         $\hat{\lambda}^{n+1} \leftarrow \hat{\lambda}^n + \tau (\hat{X}(\omega) - \sum_k \hat{u}_k^{n+1}(\omega))$ 
end while

```

between components to aid in the visualization of the effect of the sampling frequency. In the following, the frequency $f = \omega/(2\pi)$ is employed. The signal components are:

1. A carrier wave with a constant frequency $f_1 = 20$ and an amplitude of 1.
2. A carrier wave with a very low frequency $f_2 = 0.1$ and an amplitude of 0.75.
3. A carrier wave with a low frequency $f_3 = 1$ and an amplitude of 0.65.
4. A series of time-localized events with a constant frequency $f_4 = 4$ and an amplitude of 1. This wave starts oscillating at $t_{st4} = 1$ and continues for $\Delta t_4 = 1$. Afterwards, it deactivates during an interlude time $\Delta t_{d4} = 3.125$, and then the cycle repeats. Fig. 4.5 illustrates this signal. The amplitude waveform is formed by consecutive Heaviside step functions and represents an extreme case of time-localized events.
5. Another series of time-localized events with a constant frequency $f_5 = 16$ and an amplitude of 0.85. The start t_{st5} , duration $\Delta t_5 = 1$, and interlude $\Delta t_{d5} = 3.125$ times are identical to those of the previous signal component.
6. A carrier wave with a constant high frequency $f_6 = 40$, i.e. the first harmonic of f_1 , and an amplitude of 0.35. This frequency is close to the Nyquist frequency for the sampling rate used.

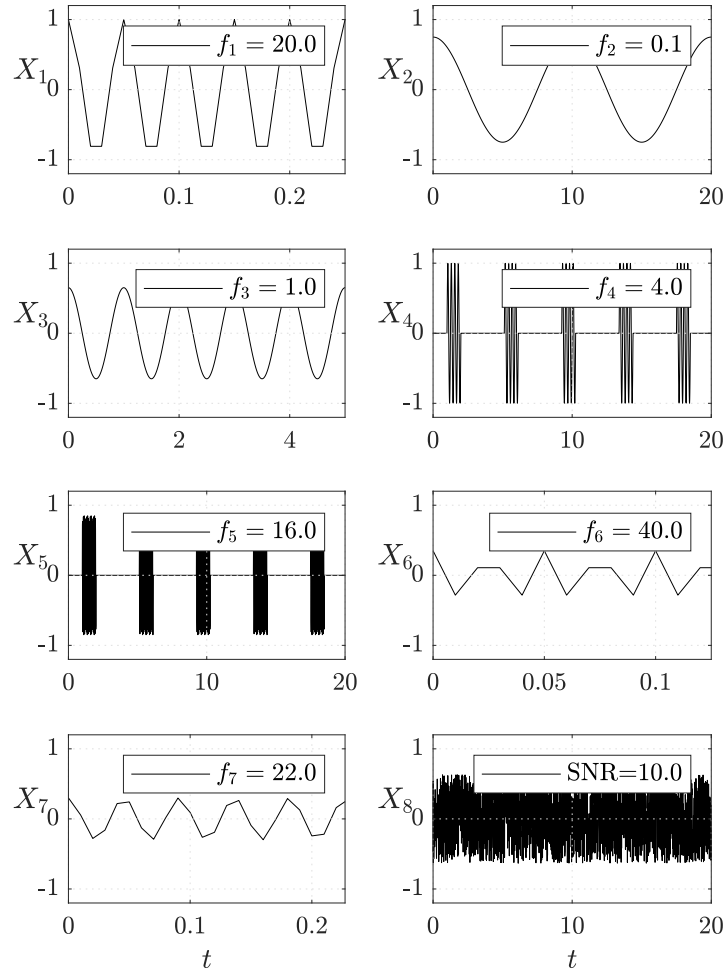


Figure 4.4: Components of the synthetic signal. The effects of the discrete sampling are highlighted.

7. A carrier wave with a constant frequency $f_7 = 22$ and an amplitude of 0.3. This frequency will mix with $f_1 = 20$ in the signal summation, hampering their recovery as separate modes.
8. A random white noise with a *Signal-to-Noise Ratio* (SNR) of 10. Signals with and without added noise are considered separately. The signal with noise is indicated by subscript- n ($_n$). The definition of SNR is expressed by:

$$SNR = 10 \log_{10} \frac{\int X(t)^2 dt}{\int (X_n(t) - X(t))^2 dt}. \quad (4.21)$$

Signal components are defined by Eqs.(4.22) and (4.23) for stationary and time-localized events, respectively. The energy ratio of each signal component relative to the total signal without noise is shown in Fig. 4.6. The summation of these components results in the complex signal illustrated in Fig. 4.7, with its frequency spectrum in Fig. 4.8.

$$X_i = \Re \{A_i \exp(j 2\pi f_i t)\} \quad (4.22)$$

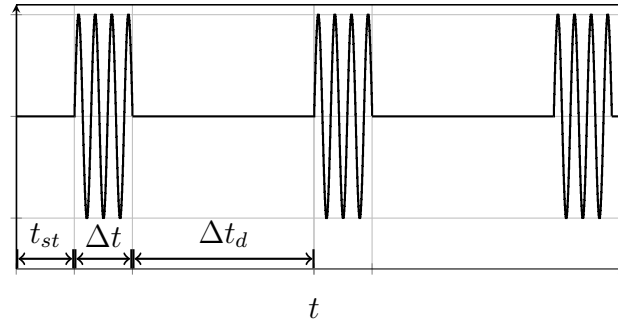


Figure 4.5: Illustration of signal with time-localized events.

$$X_i = \Re \{ A_i \exp(j 2\pi f_i (t - t_{st})) \} \quad (4.23)$$

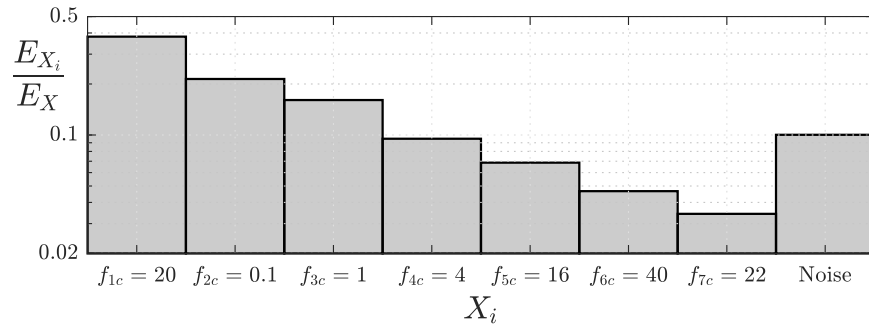


Figure 4.6: Energy ratio of signal components and noise SNR=10.

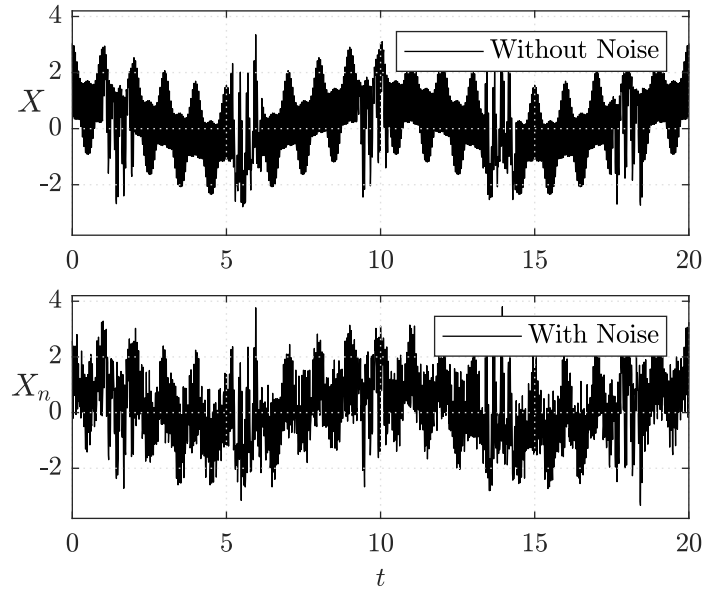


Figure 4.7: Synthetic signal with time-localized events.

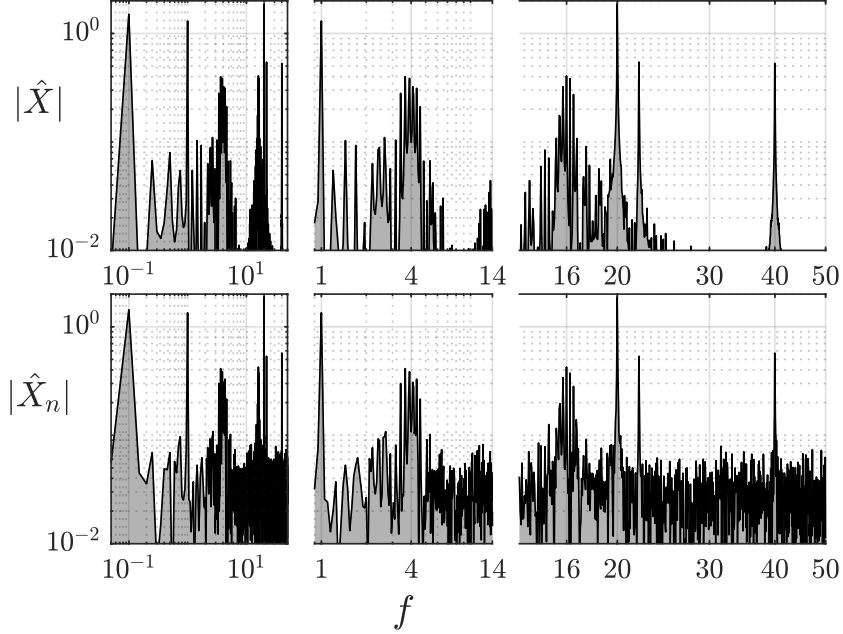


Figure 4.8: Frequency spectrum of synthetic signals with time-localized events - without (top) and with (bottom) noise.

4.4.1 Choice of the filter bandwidth B

In this section, we outline the choice of a constant filter bandwidth B for the determination of the parameter α_k , as defined in Section 4.3.4. The input signal comprises both narrowband and broadband spectral features, as shown in Fig. 4.9. Selecting a single value for the filter bandwidth B cannot provide an optimal fit for all frequencies; however, choosing a sensible value suffices to demonstrate the improvement of the OVMD over the VMD method. Although the value of B does not directly control the bandwidth of the output modes, a close estimation can aid in separating multiple modes and/or preventing the superposition of modes.

First, we estimate the upper bound for the constant filter bandwidth. Multiple narrowband components of the carrier wave are located close to each other, such as at frequencies pairs $f_1 = 20 / f_7 = 22$ and $f_2 = 0.1 / f_3 = 1$. A large bandwidth risks blending these components into a single mode or causing interferences between them. Predicting which components are more likely to merge during decomposition is not straightforward, as the magnitudes of the components, as well as contributions from other components, also affect the process. Therefore, selecting an upper bandwidth value based on the closest distance between two peaks is a conservative approach. For the present case, this would be the distance between f_2 and f_3 , giving an approximate upper bound $B_{upper} = 2\pi(f_3 - f_2)/\omega_{max} \approx 2\pi 0.9/\omega_{max}$.

Next, the lower bound value for the B is estimated based on the broadband components, which would require a larger filter bandwidth to prevent segmentation into multiple modes. However, this can pose challenges for the narrowband components. To estimate the lower bound, we use the widest distance between peaks within time-localized events. For instance, the time-localized component with carrier frequency $f_4 = 4$ is represented by a peaky broadband

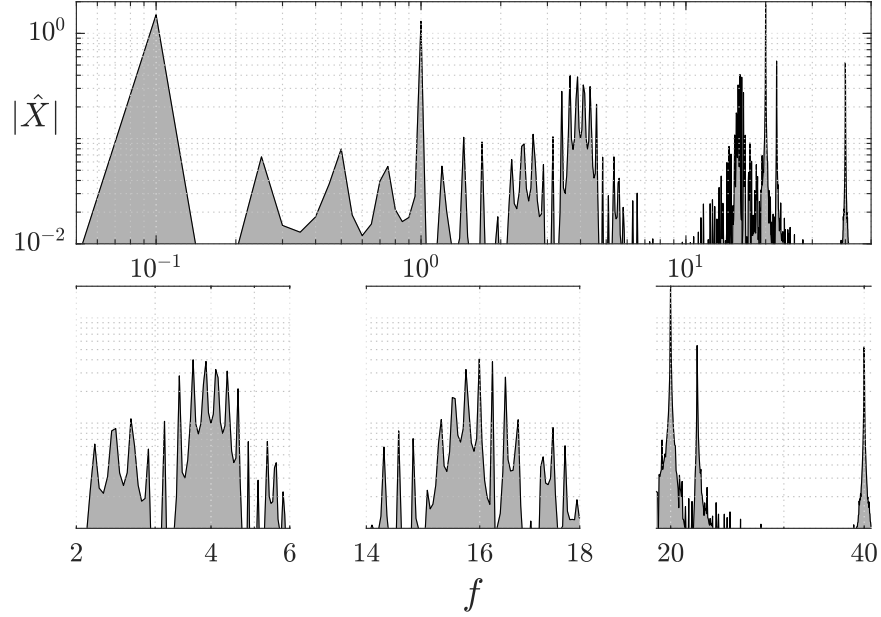


Figure 4.9: Frequency spectrum of the synthetic signal with time-localized events, without noise. The panels in the second row show zoom-ins of particular frequency ranges.

spectrum in Fig. 4.9. The separation between peak frequencies within this component is ≈ 0.25 , resulting in an approximate lower bound $B_{lower} = 2\pi \cdot 0.25 / \omega_{max}$.

Using the estimated upper and lower bounds for the filter bandwidth, we select three bandwidth values, $B = 2\pi\{0.25, 0.5, 1\} / \omega_{max}$, to evaluate and compare the OVMD method with the VMD method in decomposing a signal that contains both multiple narrowband and broadband components. These filter bandwidth values are then used to determine the parameter α_k using Eq.(4.20), with $F = \{100, 50, 25\}$, respectively.

4.4.2 Mode decomposition with over-segmentation

Over-segmentation denotes the situation when the number of waveform components in a signal (seven for the synthetic signal considered here) is lower than the prescribed number of modes, K . In most applications, the number of modes is unknown in advance. An ideal mode decomposition would recover the actual waveform components as independent modes while assigning negligible amplitudes to the excess ones, thus reducing the dependence on K . In noisy signals, the broadband noise component would be distributed over the additional modes. This makes the analysis of a complex signal under over-segmentation conditions a more comprehensive test for assessing the quality and robustness of the OVMD method. In this section, the number of modes $K = 10$ and bandwidth estimation leading to $\alpha_k = 50 / \omega_k$ are used. A sensitivity analysis examining different values of K , $F = (\alpha_k \omega_k)$, and noise level SNR will be presented in Section 4.4.3.

Fig. 4.10 presents the VMD decomposition of the clean signal (i.e. without added random noise). All 10 modes capture meaningful structures. Time-localized events are decomposed

into two modes: u_5 and u_7 represent X_4 , while u_9 and u_{10} represent X_5 . These modes are highly correlated due to mode sharing but also experience destructive interference around the state transitions. This issue may arise from the relatively narrow filter bandwidth selected, which allows the components to separate into modes with smaller total bandwidth. Additional mode duplication occurs in modes u_3 and u_8 that represent the X_3 component. This issue may be due to an overly broad filter bandwidth for this narrowband component and/or because the VMD will try to distribute the signal over all the available modes.

The OVMD method solves the issues of mode sharing and highly correlated modes, as shown in Fig. 4.11. The OVMD decomposition results in seven “dominant” modes (with relevant amplitude) and three “trivial” modes. The dominant modes correspond accurately to the signal components and are correctly ordered based on their energy. The sum of the decomposition effectively represents the original signal with a similar L_2 -norm error as the VMD method, but as opposed to it, a truncation on mode 7 would give very similar results.

The OVMD decomposition of the signal with added noise is consistent with that of the clean signal, as illustrated in Fig. 4.12. Seven dominant modes are generated, with the other three modes capturing most of the random noise component. The dominant modes still present some degree of contamination, which could be reduced with a refined method for the determination of α_k . A sensitivity analysis considering different noise intensities, along with a comparison to the VMD method, is covered in Section 4.4.3.

Further comparison between the VMD and OVMD methods is presented with the Fourier spectrum of the modes and the correlation matrix in Figs. 4.13 and 4.14, respectively. In Fig. 4.14, each matrix square contains a nested square displaying different parameters. The outer square (in greyscale) shows the covariance between mode i and mode j , normalized by the square root of their auto-covariances, as expressed by:

$$\rho_{ij} = \frac{\text{cov}(u_i, u_j)}{\sigma_i \sigma_j}. \quad (4.24)$$

A small value of ρ_{ij} implies that modes i and j are nearly orthogonal, while a value close to 1 means that the modes are nearly parallel, which usually indicates mode sharing and destructive interference. Relatively high values of ρ_{ij} can appear for non-dominant mode pairs, i.e. modes not capturing the main waveform components of the signal but small deviations or signal noise. This lack of orthogonality does not imply a flaw in the decomposition. A second parameter is introduced to discern this, namely the cross-covariance normalized with the maximum of the energy (auto-covariance) of the pair of modes:

$$C_{ij} = \frac{\text{cov}(u_i, u_j)}{\max(\sigma_i^2, \sigma_j^2)}, \quad (4.25)$$

where $\sigma_i^2 = \sum (u_i - \bar{u}_i)^2$. The inner square (from white to dark blue) shows the value of C_{ij} .

The horizontal and vertical axes in Fig. 4.14 represent the i and j modes used to assess the correlation coefficient at their intersection. For example, the correlation coefficients between modes 3 and 5 are located at square (3,5) for the VMD decomposition. Because ρ_{ij} and C_{ij} are symmetric, only the upper or lower triangular matrix needs to be shown. This is exploited in Fig. 4.14 to compare results from VMD and OVMD.

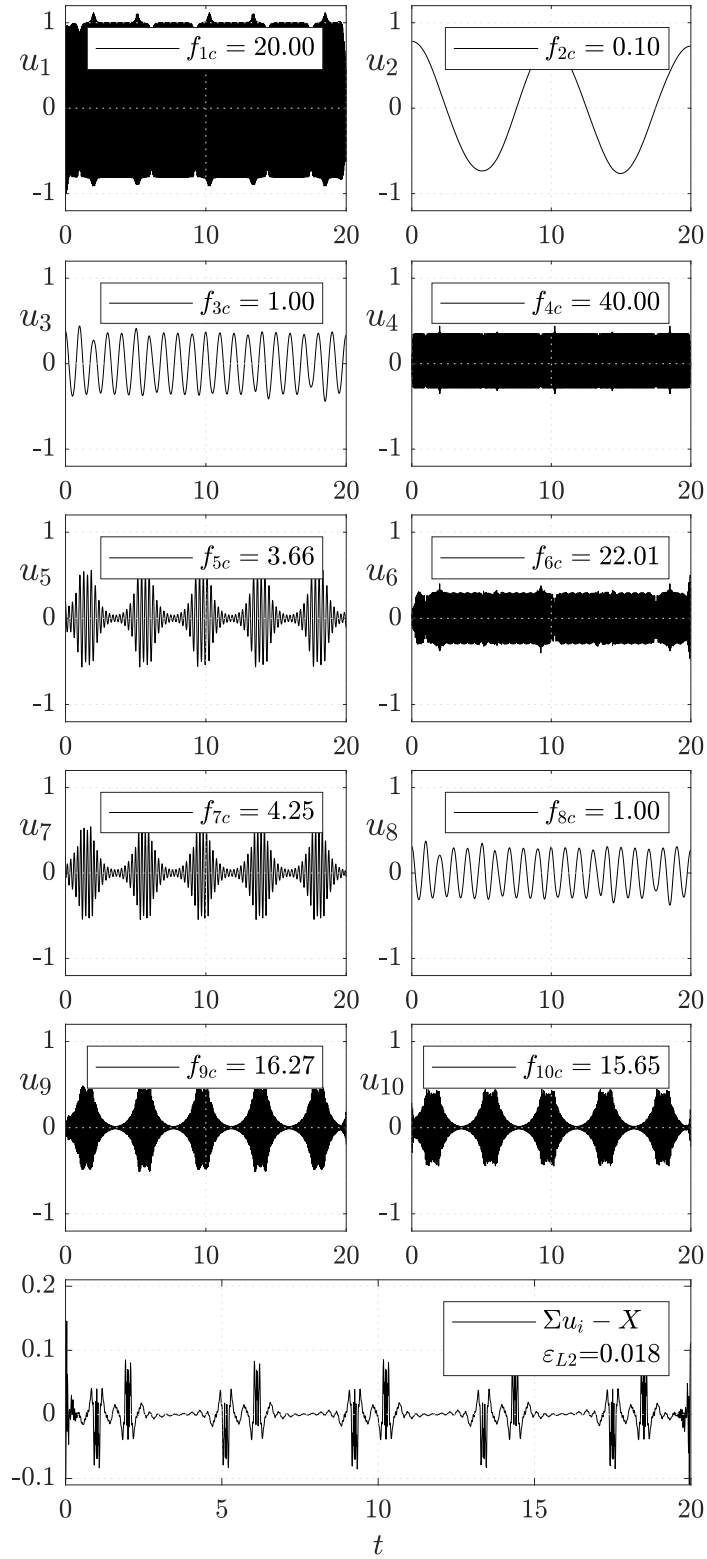


Figure 4.10: Over-segmentation in VMD of the synthetic signal without noise. $\alpha_k = 50/\omega_k$. The modes are arranged based on energy.

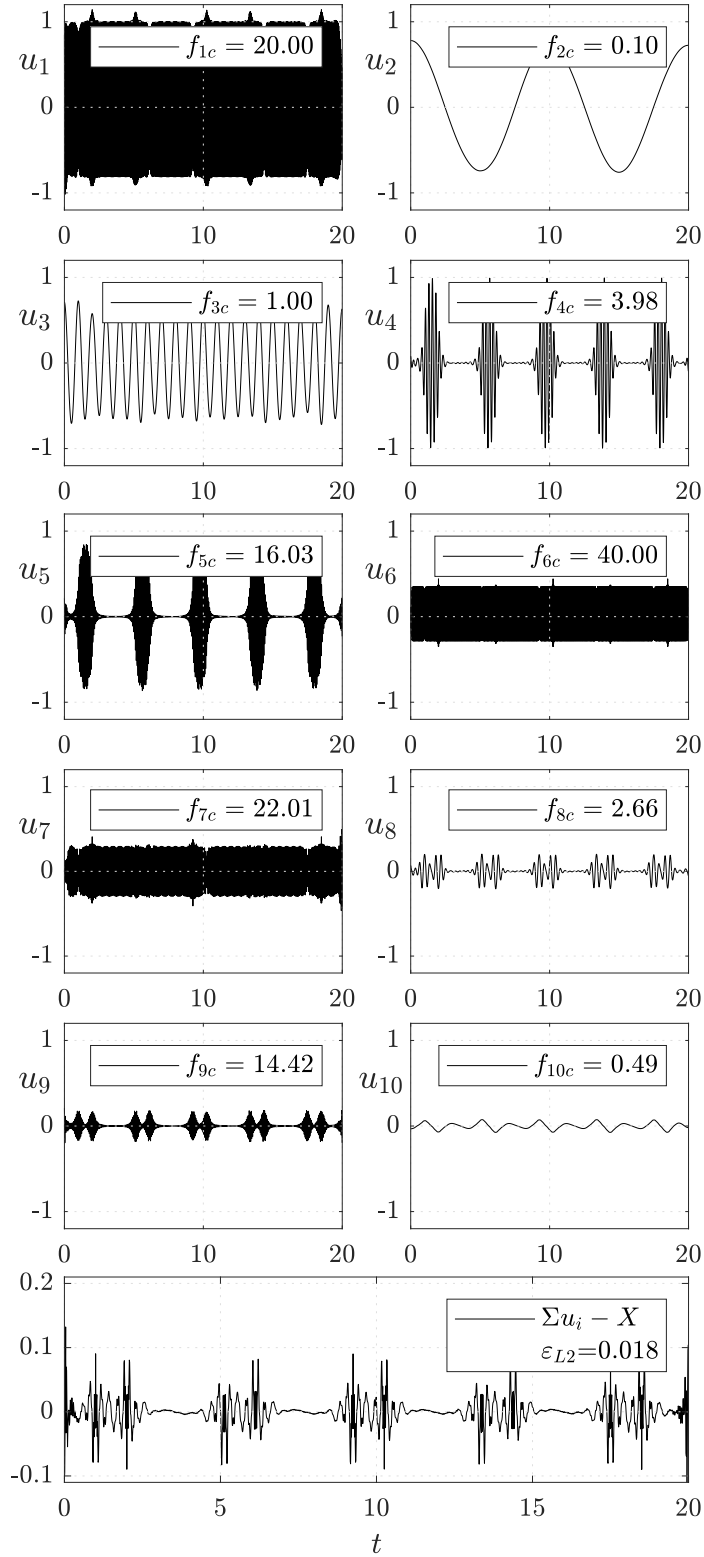


Figure 4.11: Over-segmentation in OVMD of the synthetic signal without noise. $\alpha_k = 50/\omega_k$. The modes are arranged based on energy.

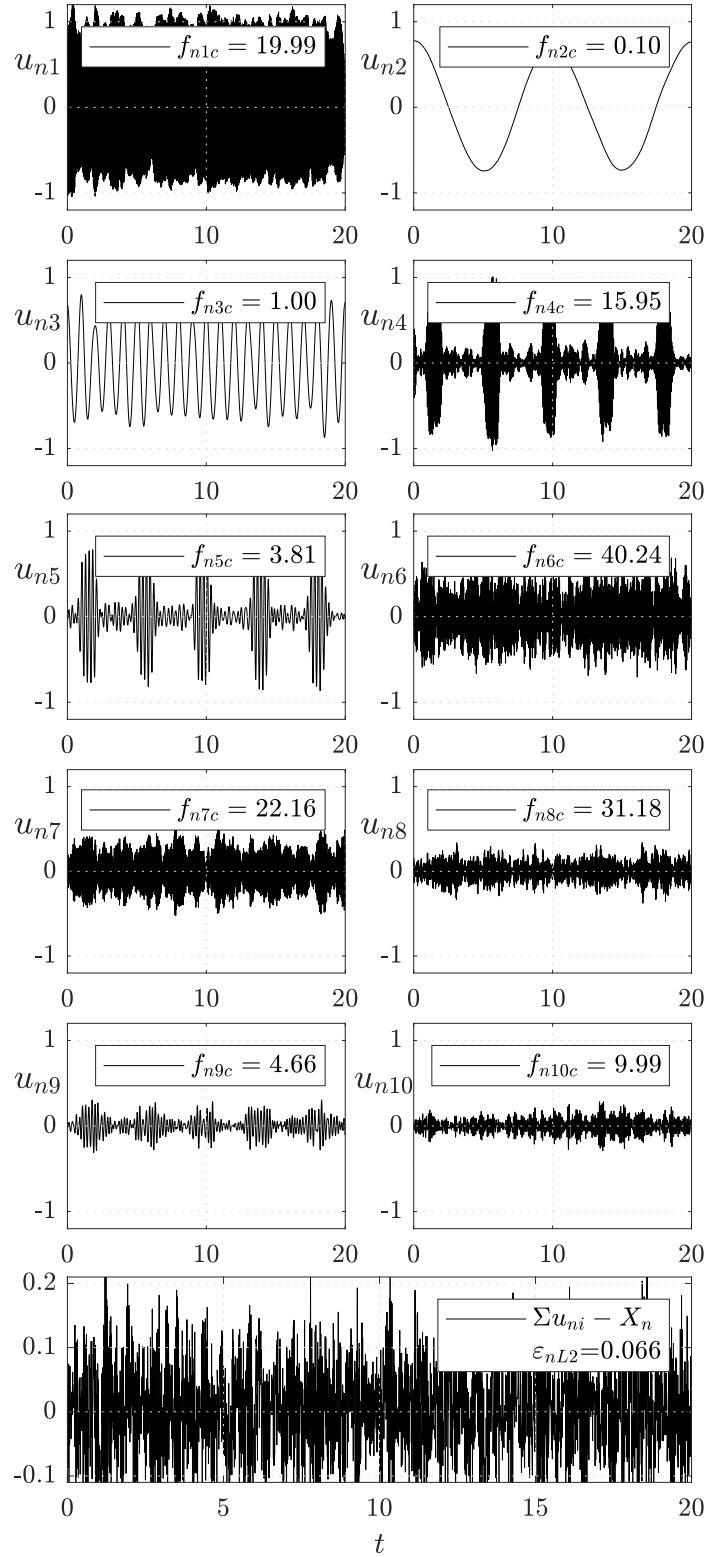


Figure 4.12: Over-segmentation in OVMD of the synthetic signal with noise. $\alpha_k = 50/\omega_k$.

In the VMD decomposition of the clean signal (Fig. 4.13a), mode duplication is evident from

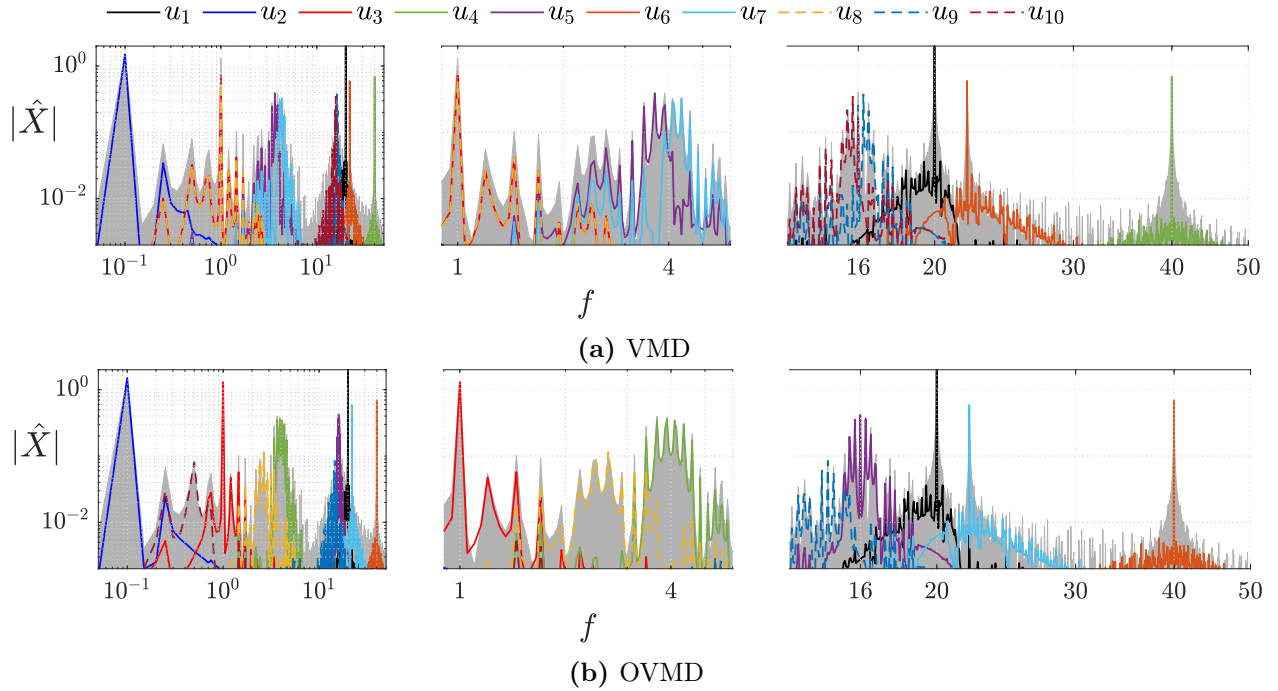


Figure 4.13: Frequency spectra of the modal decompositions of the synthetic signal without noise. $\alpha_k = 50/\omega_k$.

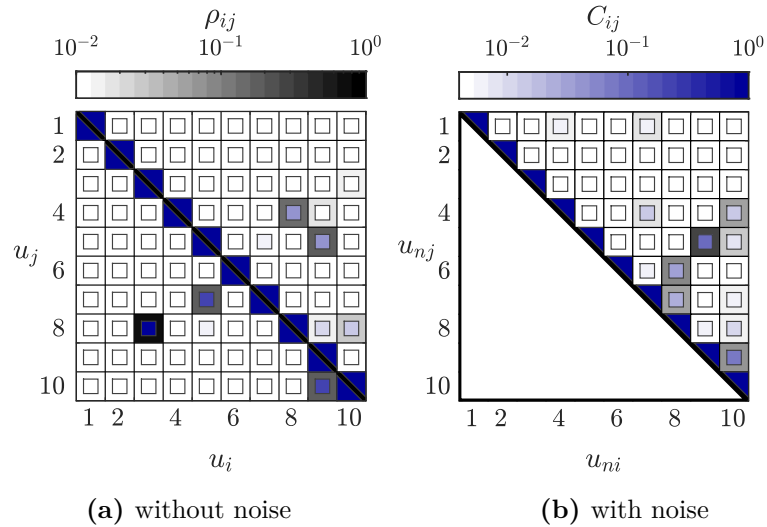


Figure 4.14: Orthogonality matrix. VMD (lower triangular); OVMD (upper triangular)

the closely aligned spectral peaks in the mode spectra, particularly for the modes recovering time-localized events ($u_5 - u_7$ and $u_9 - u_{10}$), but also for the component with $f = 1$, which is represented by two modes ($u_3 - u_8$). The modes with closely aligned spectral peaks exhibit high parallelism, as shown in Fig. 4.14a.

In contrast, the OVMD eliminates these closely aligned spectral peaks, resulting in highly orthogonal modes for the first seven dominant modes that correspond to the actual signal

components (Figs. 4.13b and 4.14a-upper triangular part). The other three “trivial” modes exhibit some parallelism with certain dominant modes. For clean signal decomposition, the number of trivial modes in an over-segmentation condition could be reduced by implementing an adaptive algorithm for the K parameter, allowing K to be reduced throughout the optimization process. For the noisy signal decomposition with OVMD, the orthogonality matrix (Fig. 4.14b - upper triangular part) shows that the first seven dominant modes (u_{n1} through u_{n7}) are highly orthogonal to each other and exhibit only minimal alignment with the extra modes (u_{n8} through u_{n10}).

Imposing orthogonality constraints on the variational problem improves the convergence rate of the optimization algorithm. For the clean signal decomposition presented, the VMD method requires 438 iterations to decompose the signal into $K = 10$ modes, whereas the OVMD needs only 80 iterations. Both methods use identical tolerance $\epsilon = 10^{-5}$ (see Algorithm 1). Performance is highly dependent on the noise present in the signal and the number of modes K , with the cost of one iteration in the OVMD method being approximately five times that of the VMD. Still comparing methods based solely on computational cost can be misleading, as the results from the OVMD method are superior to those of the VMD method.

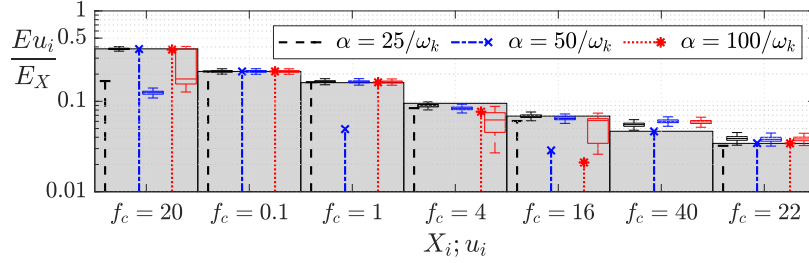
4.4.3 Sensitivity analyses

This section studies the sensitivity of the modes obtained using the OVMD method in response to varying prescribed values of constant filter bandwidth B , number of modes K , and noise level SNR , in terms of the modal energy. A comparison with results from the VMD method is also given. Energy, $Eu_i = \sigma_{u_i}^2$, is selected as a monitor function because it quantifies the completeness of the mode relative to its corresponding theoretical signal component. The synthetic signal described in the previous section is used, both with and without random white noise. Due to the randomness of the generated noise, the decomposition of the noisy signal is repeated $N_{ensemble} = 1000$ times with different instances of random noise to build statistics. Random noise is characterized by its SNR value and has a uniform distribution, with the highest frequency being the Nyquist frequency ($f_{max} = 0.5f_s$). As in the previous section, $t_s = 20$, $f_s = 100$.

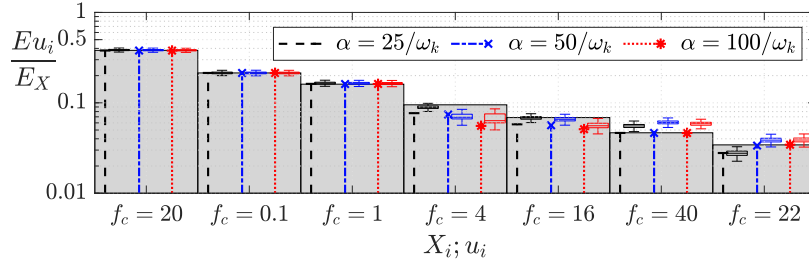
The energy of the modes is expressed as a ratio to the energy of the clean signal, $E_X = \sigma_X^2$ (without the noise component). For the noisy signal, the statistics are summarized using boxplots. A boxplot is a graphical tool that effectively represents the distribution of numerical data. The box represents the interquartile range (IQR), with a line within the box indicating the median value. The whiskers extending from the top and bottom of the box show the range of extreme values and illustrate the data.

Figs. 4.15, 4.16, and 4.17 present the sensitivity results concerning the filter bandwidth, the number of modes K , and the white noise level, respectively. The vertical axis represents the energy ratio, while the horizontal axis displays the seven modes corresponding to the synthetic signal components, arranged in order of magnitude. It is considered that, for a mode to represent a signal component, its center frequency must be within $\pm 5\%$ of the component’s center frequency, f_c . If multiple modes fall within a similar center frequency range, the

more energetic mode is selected as the representative. The energies of the original signal components are shown by the gray bars, serving as a reference for the decomposed modes. The energies of the decomposition of the clean signal are marked with symbols at the ends of dashed, dashed-dotted, or dotted vertical lines for clarity. Next to these, the corresponding energies for the noisy signal are depicted using boxplots, following the same color coding.



(a) VMD



(b) OVMD

Figure 4.15: Energy sensitivity comparison between VMD and OVMD in decomposing the synthetic signal, with noise (boxplot) and without noise (vertical lines), for different values of $F = (\alpha_k \omega_k)$, $K = 10$ and $SNR = 10$.

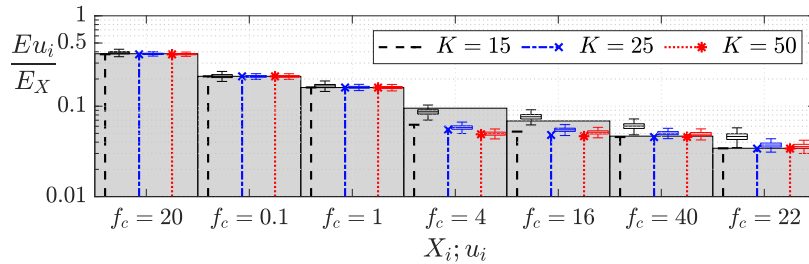


Figure 4.16: Energy sensitivity of OVMD modes for different K , $\alpha_k = 50/\omega_k$ and $SNR = 10$.

Sensitivity to the filter bandwidth

Evaluations with three selected values for the $F = (\alpha_k \omega_k)$ factor were performed, with results presented in Fig. 4.15 for decompositions using the VMD (a) and OVMD (b) methods. The input signal was decomposed with a prescribed $K = 10$, following the setup in Sec. 4.4.2.

The clean signal is considered first. For VMD, Fig. 4.15a, the modal energy shows no clear trend in relation with the values of F . A higher F factor (based on the smaller filter bandwidth

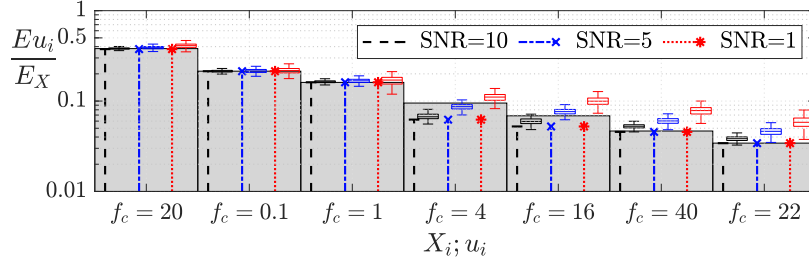


Figure 4.17: Energy sensitivity of OVMD modes for different values of SNR , $\alpha_k = 50/\omega_k$ and $K = 15$.

B) aids in accurately generating isolating narrowband components ($f_c = 20$ and 22), but for other narrowband components ($f_c = 1$ and 40), the trend remains unpredictable. Similarly, the recovery of the broadband components ($f_c = 4$ and 16) displays irregular behavior. In contrast, the OVMD results exhibit a predictable behavior with a clear trend. A higher F leads to a better agreement of obtained narrowband modes with the theoretical energy levels, reducing blending with other frequencies (e.g., $f_c = 22$). It also lowers the energy level of broadband modes ($f_c = 4$ and 16) due to mode-sharing and/or a higher residual.

For the noisy signal, the VMD results remain irregular, with no consistent trend for narrowband components as F varies. However, a trend is observed for broadband components, which aligns with the OVMD behavior for the clean signal. A wider range of energy statistics is expected for broadband components as F increases. The OVMD results with noise continue to show consistency with the clean signal decomposition and demonstrate a tighter range of energy statistics compared to VMD.

These findings indicate that the OVMD method, using a constant filter bandwidth for the selection of a proportional value of α_k , is superior to VMD in delivering accurate and robust decompositions for signals composed of multiple narrowband and broadband components. Providing a dynamic filter bandwidth as a function of frequency could further improve OVMD quality but would likely not enhance VMD results, given its lack of consistent behavior.

Sensitivity to the number of modes K

A sensitivity assessment with increasing K provides insight into how selecting a very large number of modes may impact the completeness of the dominant modes. Three additional cases with $K = 15, 25$, and 50 are chosen to create increasingly extreme cases of over-segmentation. The results of the OVMD method are presented in Fig. 4.16. For clean signals, a small decline in performance is expected in the time-localized event components, $f_c = 4$ and $f_c = 16$. These modes experience a reduction in the energy content as K increases with up to 40% energy loss when $K = 50$ for $f_c = 4$. This energy loss has asymptotic behavior and is mainly contributed by the low filter bandwidth selected and the over-segmentation condition. An optimized value of the filter bandwidth for these particular frequencies and/or an adaptive determination of K would improve the results. Overall, OVMD shows robust and consistent results as the seven dominant modes maintain similar energy levels.

The application of OVMD to noisy signals with $SNR = 10$ demonstrates that decompositions of small energy and/or high-frequency components can be improved to match the clean signal decomposition by prescribing a higher K , as seen in Fig. 4.16. A higher K enhances the optimization process, allowing the noise components to be segmented into separate modes and preventing it from contaminating the dominant modes; the result is a more accurate energy estimation with a narrower mode spread. The statistics for $K = 25$ and $K = 50$ do not show significant differences across the dominant modes, suggesting that the algorithm requires only a certain number of modes to effectively decompose the noisy signal. Prescribing a larger K only segments the background noise over more modes.

Sensitivity to signal noise level

Three different amplitudes of the random noise component of the synthetic signal, characterized by SNR are tested within OVMD, Fig. 4.17. The levels of SNR are 10, 5, and 1, where a lower SNR indicates more intense noise. For reference, the figure repeats the results for the clean signal. The decomposition shows excellent agreement in the energy content for the modes that recover stationary wave components, with only marginal differences from the corresponding theoretical energy for $f_c = 22$, which is attributed to the selected filter bandwidth of $\alpha_k = 50/\omega_k$.

As indicated before, 1000 repetitions with different random noise instances were used to generate statistics. The results indicate reasonably good performance at $SNR = 10$, but a significant deterioration occurs at lower SNR levels, especially for low energy and / or high frequency signal components. The presence of noise tends to affect more the recovery of the latter components because the optimization process struggles in identifying the modes accurately, often incorporating extra noisy signals that can either increase or decrease the mode's energy. This can produce a very broad (higher energy) or very narrow (lower energy) frequency band, which also depends on the selected value of F .

4.5 Summary

This chapter introduces two improvements to the Variational Mode Decomposition (VMD) introduced by Dragomiretskiy and Zosso (2014). The first modification introduces a set of constraints into the optimization problem to promote orthogonality between dominant modes. A component's projection into other modes, along with the Wiener filtering process, facilitates the determination of the transfer component at each iteration. These new constraints enhance convergence and mode uniqueness, particularly in conditions of over-segmentation. Consequently, the new method is named Orthogonalized Variational Mode Decomposition (OVMD).

The second modification involves an approach to estimate an adequate value for the weight parameter α_k , which roughly determines the frequency sidebands of the modes. First, a representative bandwidth B is identified based on the input signal spectrum. Then, α_k is

computed as being inversely proportional to the product of the bandwidth B and the mode central frequency ω_k . This second modification can be applied to the original VMD method, but present results show that it has a stronger positive impact when coupled with the mode orthogonalization. Further, more sophisticated methods for the selection of α_k can also be employed with both VMD and OVMD.

The OVMD provides a more accurate decomposition than the VMD for a synthetic signal consisting of multiple stationary and time-localized event waves. Under conditions of over-segmentation, OVMD successfully produces main modes that closely resemble the signal components, leaving the extra modes with minimal amplitudes or capturing the noise in the signal. Sensitivity assessments indicate that the new approach yields consistent results with different values of the inverse proportionality factor $F = (\alpha_k \omega_k)$ (estimated based on the signal spectrum), presence of significant noise, and extreme over-segmentation.

In very noisy conditions ($SNR = 1$), OVMD still struggles to accurately decompose low-energy and / or high-frequency components. Additionally, up to 40% of the component information may be lost when reconstructing time-localized events with rapid state transitions.

Chapter 5

Multivariate Progressive Orthogonalized Variational Mode Decomposition

In this chapter, an extension of orthogonalized variational mode decomposition (OVMD) is proposed to advance the initial development of the OVMD for its application to multivariate data, particularly flow field data. Extending the approach to decompose multidimensional or multivariate datasets is crucial to extend the application of the OVMD to the engineering (Liao et al., 2023; Padilla-Montero et al., 2024; Xu et al., 2019; Yang et al., 2024) and medical fields (Cichy and Pantazis, 2017; Das and Pachori, 2021).

The OVMD method described in the previous chapter is capable of decomposing signals with distributed and localized event components. The introduced parts that transmit and receive components that are not parallel to the mode and the estimation of the inverse bandwidth based on the Fourier transform contribute to the main improvement. However, two weaknesses are identified in the previous chapter: the high loss of energy content for localized event components and the need to prescribe the number of modes K , preferably a high value to give an over-segmentation condition. These weaknesses can be overcome by limiting the trivial modes in the decomposition by adjusting the prescribed K to the minimum. However, this value is determined arbitrarily without knowing the outcome.

5.1 Multivariate signals decomposition

Multivariate or multichannel signals consist of independent signals, with each channel representing a specific node/location and variable. These signals can correspond to field data at different points in the space (i.e., distinct channels). Extending univariate signal decomposition algorithms to multivariate/multichannel datasets requires a method to align mode frequencies across all channels. Frequency alignment is crucial, as the concept of multivariate decomposition relies on the common frequency content across all channels. Several multivari-

ate methods use this concept as an extension of univariate mode decomposition techniques (Rehman and Mandic, 2010, Rehman and Aftab, 2019, Q. Chen et al., 2021, and Cicone and Pellegrino, 2022).

Some extensions of univariate decomposition to multivariate decomposition are known to lack a solid mathematical framework, such as multivariate EMD in Rehman and Mandic (2010) and fast iterative filtering (FIF) by Cicone and Pellegrino (2022), which are based on the mean characteristics of the multivariate signals. A mathematical foundation for multivariate decomposition is provided by Lilly and Olhede (2012), who describe how a set of S real-valued AM-FM signal vectors $\mathbf{x}(t)$ can be decomposed into a set of unique vector amplitudes A and vector phases ϕ , obtained from the analytic signal $\mathbf{x}_+(t)$ transformation through the Hilbert transform, as expressed by the following equation:

$$\begin{aligned}\mathbf{x}_+(t) &\equiv 2\mathcal{A}\mathbf{x}(t) \equiv \mathbf{x}(t) + j\mathcal{H}\mathbf{x}(t) \\ \mathbf{x}_+(t) &= \begin{bmatrix} A_1(t)e^{j\phi_1(t)} \\ A_2(t)e^{j\phi_2(t)} \\ \vdots \\ A_S(t)e^{j\phi_S(t)} \end{bmatrix}.\end{aligned}\tag{5.1}$$

The above definition of multivariate decomposition represents the commonly used mathematical framework, as seen in multivariate VMD (MVMD) by Rehman and Aftab (2019) and multivariate chirp mode decomposition by Q. Chen et al. (2021). This framework enables modes to be grouped based on common modulated oscillations spanning multiple channels. The framework effectively enables modes that exhibit phase lags and different amplitudes across different channels to be represented in the same mode $\mathbf{u}_i(t)$. In the multivariate VMD implementation, the approach adds another loop to the algorithm, and a slowdown in the decomposition process is observed as the number of channels increases (Rehman and Aftab, 2019). Here, we propose dividing the mode u_i into spatial topology and temporal coefficients, as done in proper orthogonal decomposition (POD) (Berkooz et al., 1993, Chatterjee (2000)).

This chapter focuses on the improved algorithm of the OVMD and its extension for multivariate datasets. Section 5.2 introduces a modification to the algorithm to overcome the weaknesses of the original OVMD algorithm. The new algorithm sequentially adds the number of modes once the previous modes converge, making it a progressive algorithm. Section 5.3 extends the OVMD method to multivariate datasets. Section 5.4 presents experimental results for decomposing artificial signals as univariate and multivariate data using the progressive algorithm. The improvement in the quality of decomposition and computational cost are analyzed and discussed to demonstrate the suitability of the method for decomposing significantly large multichannel datasets. Section 5.5 introduces the concept of empirical filter bandwidth, which helps determine a suitable bandwidth at each discrete frequency in the signal spectrum. Section 5.6 provides a comprehensive example of the complete method for non-stationary signals, including the transient evolution of a cylinder wake. Section 5.7 summarizes the improvement achieved by the multivariate progressive orthogonalized variational mode decomposition method along with its limitations.

5.2 Progressive Orthogonalized Variational Mode Decomposition

In this section, a modification to the OVMD algorithm is proposed to address the over-segmentation issue, reduce overall computational cost, and enhance the algorithm's ability to progressively decompose the signal into the most energetic modes. The OVMD algorithm, derived from the original VMD, has shown that prescribing a high value of K , which leads to over-segmentation, results in the creation of trivial modes that partially cancel each other out. This mode duplicity not only diminishes the energy of time-localized event components, but also increases computational time. By truncating the number of modes when the energy ratio of the sum of the decomposed modes, $E_{\sum u}$, exceeds a certain threshold value, E_{th} , over-segmentation can be prevented.

Moreover, the OVMD method is generally more computationally expensive than the VMD method. Although it exhibits better convergence than VMD, the cost of a single iteration in OVMD is equivalent to that of five iterations in VMD. In the original OVMD algorithm, the optimization process is simultaneous, where all K modes are optimized together, and none of the modes are close to their final structure at the beginning of the process. While OVMD can prevent mode duplication and destructive interference, this optimization can be improved by adopting a sequential approach: optimizing one mode at a time until it converges, then incrementally increasing the number of modes K until the stopping criterion is met. This is known as progressive optimization.

In most applications, only the principal modes are needed for further analysis. Thus, it is crucial for sequential optimization to start with the most energetic mode, minimizing the number of modes at the end of optimization and enabling the process to capture only the most energetic modes. In sequential optimization, this can be facilitated by estimating the initial center frequency of each mode based on the location of the maximum magnitude in the residual spectrum, expressed as:

$$\omega_{k_{It}}^{n+1} = \arg \max_{\omega} \left| \hat{X}(\omega) - \sum_k^{k_{It}-1} \hat{u}_k(\omega) \right|. \quad (5.2)$$

With this approach, the signal is decomposed starting with the most probable energetic mode. However, this does not always occur, as the energy of a mode is influenced not only by its maximum magnitude but also by its bandwidth and other lower-magnitude components.

This new algorithm introduces an additional outer loop to progressively decompose the signal from the first principal mode to the K_{max} -th mode ($k_{It} = 1 : K_{max}$), where K_{max} is the maximum prescribed number of modes to determine before the algorithm stops. In each outer iteration, the algorithm performs the original OVMD optimization to determine the k_{It} modes that minimize the objective function and stops when K_{max} or E_{th} is reached. Thus, the number of modes in the dataset is determined successively. This progressive approach is similar to the method used in SVMD by Nazari and Sakhaei (2020). Algorithm 2 provides the complete steps of the progressive optimization of OVMD using ADMM.

Algorithm 2 Optimization of Progressive OVMD.

```

INITIALIZE( $\{\hat{u}_k^1\}, \{\hat{\lambda}^1\}, B, n \leftarrow 0, k_{It} \leftarrow 0$ )

while  $k_{It} < K_{max}$  and  $E_{\sum u} < E_{th}$  do

     $k_{It} \leftarrow k_{It} + 1$ 

    INITIALIZE  $\omega_{k_{It}}^{n+1}$ :
     $\omega_{k_{It}}^{n+1} \leftarrow \arg \max_{\omega} \left| \hat{X}(\omega) - \sum_i^{k_{It}-1} \hat{u}_i(\omega) \right|$ 

    COMPUTE  $\alpha_{k_{It}}^{n+1}$  : (4.18)
    COMPUTE  $\hat{\beta}_{k_{It}}^{n+1}(\omega)$  : (4.6)

    while  $\sum_k \|\hat{u}_k^{n+1} - \hat{u}_k^n\|_2^2 / \|\hat{u}_k^n\|_2^2 > \epsilon$  and  $n < N_{max}$  do

         $n \leftarrow n + 1$ 

        for  $k = 1 : k_{It}$  do
            UPDATE  $\{\cdot\}$  using  $\{\cdot\}_{i \leq k}^{n+1}$  and  $\{\cdot\}_{i \geq k}^n$  (latest data) :
                 $\{\hat{P}_{ik}^{n+1}\}$ : with (4.7)
                 $\hat{u}_k^{n+1}(\omega)$ : with (4.14)
                 $\omega_k^{n+1}$  : with (4.16)
                 $\alpha_k^{n+1}$  : (4.18)
                 $\hat{\beta}_k^{n+1}(\omega)$ : (4.6)
        end for

        DUAL ASCENT  $\{\hat{\lambda}^{n+1}\}$ 
    end while
end while

```

5.3 Multivariate Progressive Ortogonalized Variational Mode Decomposition

In this section, a novel multivariate extension is presented for the OVMD method that allows multivariate data to be decomposed with any number of channels S . For a multivariate method to be considered competitive, its performance must not deteriorate as the number of channels S increases while the modes still maintain the intended quality of the decomposition, in this case its time-frequency orthogonality characteristics. The direct multivariate extension of the VMD shows a significant increase in computational cost when the number of channels S is greater than 8 (Rehman and Aftab, 2019). To overcome these disadvantages, an alternative definition of multivariate mode is employed.

5.3.1 Proposed Method: Spatial Topology and Time Coefficient

The proposed method defines a multivariate mode k , $\mathbf{u}_k(t)$, as the product of a temporal coefficient $c_k(t)$ and a channel/spatial topology weight coefficient ψ_k . The decomposition of the multivariate dataset $\mathbf{X}(t)$ is expressed as:

$$\begin{aligned}\mathbf{X}(t) &= \sum_k^K \mathbf{u}_k(t) + \mathbf{r}(t), \\ \mathbf{X}[s](t) &= \sum_k^K \psi_k[s] c_k(t) + \mathbf{r}[s](t),\end{aligned}\tag{5.3}$$

where $[s]$ denotes the discrete channel s . The discrete real-valued spatial topology of mode k , ψ_k , has a size of S channels, with its values being independent across channels. It has no temporal dependency, i.e., $\psi'_k = 0$, and its L_2 -norm is equal to unity:

$$\|\psi_k\|_2 = 1.\tag{5.4}$$

The continuous real-valued temporal coefficient of mode k , $c_k(t)$, represents the common frequency modulation across all channels and is invariant to the channel.

As a consequence, a mode cannot represent phase lag or phase offset between channels unless the offset is $\pm\pi$. If a common frequency modulation exists in more than two channels with a phase offset ranging from 0 to 2π , the modulation will be represented as two separate modes with a phase offset between their temporal coefficients of $\pm\pi/2$.

Proof: Let $\mathbf{X}[s](t) := \cos(\omega_1 t + \theta[s])$, where $\mathbf{X}$ is a multivariate signal with S discrete channels indexed by s , each defined over a time interval t . Here, ω_1 is a constant frequency for all channels and θ is a phase vector for the discrete channels s , with a size of S . Using trigonometric identities and the above definition of a mode, the decomposition of $\mathbf{X}[s](t)$ is:

$$\begin{aligned}\mathbf{X}[s](t) &= \cos(\omega_1 t + \theta[s]) = \cos(\theta[s]) \cos(\omega_1 t) - \sin(\theta[s]) \sin(\omega_1 t) \\ &= \psi_1 c_1(t) + \psi_2 c_2(t),\end{aligned}\tag{5.5}$$

where ψ_1 and ψ_2 are the spatial topology vector of dimension S for modes 1 and 2, respectively, and c_1 and c_2 are the temporal coefficient functions.

The dataset is represented by two modes, regardless of the size of S , with temporal coefficients having a phase difference of $\pm\pi/2$. In this approach, the spatial topology ψ_k represents not only the amplitude ratio across all channels, but also the phase offset of the temporal coefficient.

The modified augmented Langrangian problem for the multivariate dataset in the Fourier

domain is:

$$\begin{aligned}
\mathcal{L}(\{\hat{c}_k\}, \{\psi_k\}, \{\omega_k\}, \hat{\lambda}) &:= \left\| \hat{\mathbf{X}}(\omega) - \sum_k \psi_k \hat{c}_k(\omega) + \frac{\hat{\lambda}(\omega)}{2} \right\|_F^2 \\
&+ \sum_k^K \left[\alpha_k \left\| \psi_k \left(j(\omega - \omega_k) ((1 + \text{sgn}(\omega)) \hat{c}_k(\omega)) \right) \right\|_F^2 \right. \\
&\quad + \sum_{i \neq k} \left\| H_{ik} \psi_k \left(\hat{\beta}_i^{1/2} \hat{P}_{ik}^{1/2} \hat{c}_k(\omega) \right) \right\|_F^2 \\
&\quad + \sum_i \left\| H_{ki} \psi_i \left(\hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{c}_i(\omega) \right) \right\|_F^2 \\
&\quad \left. - \left\| \sum_i H_{ki} \psi_i \left(\hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{c}_i(\omega) \right) \right\|_F^2 \right].
\end{aligned} \tag{5.6}$$

where $\|\cdot\|_F^2$ denotes the squared of Frobenius norm and $\hat{\lambda}$ is the Langrange multiplier. The normalized cross-spectral density between the multivariate modes i and k , $\hat{P}_{ik}$, is given by:

$$\begin{aligned}
\hat{P}_{ik}(\omega) &= \frac{\sum_s |\hat{\mathbf{u}}_i[s](\omega) \hat{\mathbf{u}}_k^*[s](\omega)|}{\|\hat{\mathbf{u}}_i\|_F \|\hat{\mathbf{u}}_k\|_F} \\
&= \frac{\|\psi_i \psi_k\|_2 |\hat{c}_i(\omega) \hat{c}_k^*(\omega)|}{\|\hat{c}_i\|_2 \|\hat{c}_k\|_2}.
\end{aligned} \tag{5.7}$$

The Frobenius norm of the mode k , denoted by $\|\hat{\mathbf{u}}_k\|_F$ is equivalent to the L_2 -norm of the time coefficient k , $\|\hat{c}_k\|_2$ as given by the relation in (5.4). This characteristic also simplifies the minimization problem, as it now applies only to the variables $\{\hat{c}_k\}$ and $\{\omega_k\}$.

5.3.2 Minimization with respect to $\hat{c}_k$

From (5.6), the equivalent minimization problem for updating the temporal coefficient $\hat{c}_k$ in the subsequent iteration, denoted with $n+1$ is:

$$\begin{aligned}
\hat{c}_k^{n+1} &= \arg \min_{\hat{c}_k, c_k \in \mathbf{X}} \left\{ \int_0^\infty \sum_s \left[4\alpha_k \psi_k^2[s] \left((\omega - \omega_k)^2 |\hat{c}_k(\omega)|^2 \right) \right. \right. \\
&\quad + 2 \sum_{i \neq k} H_{ik} \psi_k^2[s] \left(\hat{\beta}_i \hat{P}_{ik} |\hat{c}_k(\omega)|^2 \right) \\
&\quad + 2 \sum_i H_{ki} \psi_i^2[s] \left(\hat{\beta}_k \hat{P}_{ik} |\hat{c}_i(\omega)|^2 \right) \\
&\quad \left. - 2 \left| \sum_i H_{ki} \psi_i[s] \left(\hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{c}_i(\omega) \right) \right|^2 \right. \\
&\quad \left. + 2 \left| \hat{\mathbf{X}}[s](\omega) - \sum_i \psi_i[s] \hat{c}_i(\omega) + \frac{\hat{\lambda}[s](\omega)}{2} \right|^2 \right] d\omega \Big\}.
\end{aligned} \tag{5.8}$$

The detailed step from Eq. (5.6) to Eq. (5.8) follows the same development carried out in (4.11) to (4.12). The optimization problem (5.8) is solved by taking the first variation w.r.t $\hat{c}_k$:

$$\begin{aligned}
 & 8\alpha_k(\omega - \omega_k)^2 \hat{c}_k(\omega) + 4 \sum_{i \neq k} H_{ik} \hat{\beta}_i \hat{P}_{ik} \hat{c}_k(\omega) \\
 & - 4 \sum_s \sum_{i \neq k} H_{ki} (\psi_i[s] \psi_k[s]) \left(\hat{\beta}_k \hat{P}_{ik}^{1/2} \hat{P}_{kk}^{1/2} \hat{c}_i(\omega) \right) + 4\hat{c}_k(\omega) \\
 & + 4 \sum_s \psi_k[s] \left(\sum_{i \neq k} \psi_i[s] \hat{c}_i(\omega) - \hat{\mathbf{X}}[s](\omega) - \frac{\hat{\lambda}[s](\omega)}{2} \right) = 0.
 \end{aligned} \tag{5.9}$$

By isolating the temporal mode- $\hat{c}_k$, the expression for updating it based on the latest data is given by (the parentheses (ω) are omitted for brevity):

$$\hat{c}_k^{n+1} = \frac{\sum_s \psi_k[s] \left(\hat{\mathbf{G}}_k[s] + \sum_{i \neq k} H_{ki} \psi_i[s] \hat{\beta}_i \hat{P}_{ik}^{1/2} \hat{P}_{kk}^{1/2} \hat{c}_i + \frac{\hat{\lambda}[s]}{2} \right)}{1 + 2\alpha_k(\omega - \omega_k)^2 + \sum_{i \neq k} H_{ik} \hat{\beta}_i \hat{P}_{ik}} \tag{5.10}$$

where $\hat{\mathbf{G}}_k[s]$ represents the residual plus mode k , which is described as:

$$\hat{\mathbf{G}}_k[s] = \hat{\mathbf{X}}[s] - \sum_{i \neq k} \psi_i[s] \hat{c}_i. \tag{5.11}$$

5.3.3 Minimization with respect to ω_k

In (5.6), the central frequency of the mode k , ω_k , appears in the main objective and the second orthogonal objectives through the variable $\hat{\beta}_k$. As a result, the modified equation for minimization of ω_k in the multivariate OVMD is:

$$\begin{aligned}
 \omega_k^{n+1} = \arg \min_{\omega_k} \Bigg\{ & \sum_s \left[\int_0^\infty 4\alpha_k \psi_k^2[s] \left((\omega - \omega_k)^2 |\hat{c}_k(\omega)|^2 \right) \right. \\
 & + 2 \sum_i H_{ki} \psi_i^2[s] \left(\hat{\beta}_k \hat{P}_{ik} |\hat{c}_i(\omega)|^2 \right) \\
 & \left. - 2 \left| \sum_i H_{ki} \psi_i[s] \left(\hat{\beta}_k^{1/2} \hat{P}_{ik}^{1/2} \hat{c}_i(\omega) \right) \right|^2 \right] d\omega \Bigg\}.
 \end{aligned} \tag{5.12}$$

The minimization problem can be solved by setting the first derivative w.r.t ω_k equal to zero and solving for ω_k . The expression for updating the center frequency of temporal mode k is:

$$\omega_k^{n+1} = \frac{\int_0^\infty \omega (|\hat{c}_k|^2 + O_{\omega_k}) d\omega}{\int_0^\infty |\hat{c}_k|^2 + O_{\omega_k} d\omega}, \tag{5.13}$$

where

$$O_{\omega_k} = \hat{\beta}_k^2 \left(\sum_s \left| \sum_i H_{ki} \psi_i[s] \hat{P}_{ik}^{1/2} \hat{c}_i \right|^2 - \sum_i H_{ki} \hat{P}_{ik} |\hat{c}_i|^2 \right). \quad (5.14)$$

In the implementation, the first term of the expression becomes expensive to compute as the number of channels S increases. A cheaper estimation of this variable can be obtained by:

$$O_{\omega_k} \approx \hat{\beta}_k^2 \left(\left| \sum_i H_{ki} \hat{P}_{ik}^{1/2} \hat{c}_i \right|^2 - \sum_i H_{ki} \hat{P}_{ik} |\hat{c}_i|^2 \right). \quad (5.15)$$

The first term in the estimation will sum the cumulative signal at every channel before the square operation then use (5.4) to simplify the expression. A computational cost comparison between different estimations of the variable O_{ω_k} will be provided in Section 5.4.2.

5.3.4 Updating the topology ψ_k

Updating the spatial topology ψ_k can be done directly from the updated $\hat{c}_k$, without solving the minimization problem. With definition (5.3), the equivalent mode- k , $\hat{\mathbf{u}}_k$, can be expressed as:

$$\psi_k \hat{c}_k(\omega) = \hat{\mathbf{X}}(\omega) - \sum_{i \neq k}^K \psi_i \hat{c}_i(\omega) - \hat{\mathbf{r}}(\omega) \quad (5.16)$$

The residual vector $\hat{\mathbf{r}}$ will have minimum components that align with the temporal mode, $\hat{c}_k$. Hence, the spatial topology can be updated and normalized as follows:

$$\psi_k^{n+1} = \frac{\Re \left\{ \left(\hat{\mathbf{X}}(\omega) - \sum_{i \neq k}^K \psi_i \hat{c}_i(\omega) \right)^* \hat{c}_k(\omega) \right\}}{\left\| \Re \left\{ \left(\hat{\mathbf{X}}(\omega) - \sum_{i \neq k}^K \psi_i \hat{c}_i(\omega) \right)^* \hat{c}_k(\omega) \right\} \right\|_2}. \quad (5.17)$$

5.3.5 Complete multivariate algorithm

The multivariate POVMD is solved using the Alternating Direction Method of Multipliers (ADMM) with the progressive algorithm approach. Algorithm 3 describes the process in the discrete Fourier space.

5.4 Experiments and Results

In this section, the newly proposed algorithms and its multivariate extension are tested on univariate and multivariate synthetic signals. The synthetic signals considered are identical to those used in the previous experiment described in Chapter 4.4.

Algorithm 3 Optimization of Multivariate POVMD.

```

INITIALIZE( $\{\hat{c}_k^1\}, \{\psi_k^1\}, \{\hat{\lambda}^1\}, B, n \leftarrow 0$ )

while  $k_{It} < K_{max}$  and  $E_{\sum u} < E_{th}$  do

     $k_{It} \leftarrow k_{It} + 1$ 

    INITIALIZE  $\omega_{kIt}^{n+1}$ :
     $\omega_{kIt}^{n+1} \leftarrow \arg \max_{\omega} \left| \hat{\mathbf{X}}(\omega) - \sum_i^{k_{it}-1} \psi_i \cdot \hat{c}_i(\omega) \right|$ 

    COMPUTE  $\alpha_{kIt}^{n+1}$  : (4.18)
    COMPUTE  $\hat{\beta}_{kIt}^{n+1}(\omega)$  : (4.6)

    while  $\sum_k \|\hat{u}_k^{n+1} - \hat{u}_k^n\|_F^2 / \|\hat{u}_k^n\|_F^2 > \epsilon$  and  $n < N_{max}$  do

         $n \leftarrow n + 1$ 

        for  $k = k_{It} : -1 : 1$  do
            UPDATE  $\{\cdot\}$  using  $\{\cdot\}_{i \leq k}^{n+1}$  and  $\{\cdot\}_{i \geq k}^n$  (latest data) :
                 $\hat{\mathbf{G}}_k^{n+1}(\omega)$ : (5.11)
                 $\hat{c}_k^{n+1}(\omega)$  : (5.10)
                 $\psi_k^{n+1}$  : (5.17)
                 $\omega_k^{n+1}$  : (5.13)
                 $\alpha_k^{n+1}$  : (4.18)
                 $\hat{\beta}_k^{n+1}(\omega)$ : (4.6)
            end for

            DUAL ASCENT  $\{\hat{\lambda}^{n+1}\}$ 
             $\hat{\lambda}^{n+1} \leftarrow \hat{\lambda}^n + \tau \left( \hat{\mathbf{X}}(\omega) - \sum_k \psi_k \cdot \hat{c}_k(\omega) \right)$ 
        end while
    end while
    
```

5.4.1 Univariate decomposition

Two univariate synthetic signals are constructed. The first signal is a clean signal constructed by adding the first seven signal components. The second signal is with the added noise component (X_8). These synthetic signals are identical to those in Section 4.4.

The progressive OVMD (POVMD) algorithm decomposes the signals into their corresponding modes, with the assigned maximum number of modes set to $K_{max} = 10$ and $\alpha_k = 50/\omega_k$, consistent with Section 4.4. The POVMD algorithm has two stopping criteria: K_{max} and the energy ratio threshold E_{th} . The energy ratio threshold is set to 98% and 95% for the clean and noisy synthetic signals, respectively. These thresholds imply that the optimization

process will stop searching for additional modes if the energy ratio of the cumulative modes exceeds these values, provided K_{max} has not been reached.

The corresponding decompositions of the clean and noisy univariate signals are presented in Figs. 5.1 and 5.2, respectively. The decompositions terminate before reaching K_{max} with no duplication or sharing of modes. The decompositions of clean and noisy signals yield seven dominant modes that resemble the principal components of the signals. No trivial modes are present in the decompositions. As a result, an improvement is observed when comparing the decomposition of localized event components with the previous algorithm 1, particularly for the clean signal. The cumulative modes from POVMD yield a slightly higher L_2 error (ϵ_{L2}) than the decomposition in Section 4.4. This is mainly due to the lower number of modes K and can be considered insignificant as the difference is minimal.

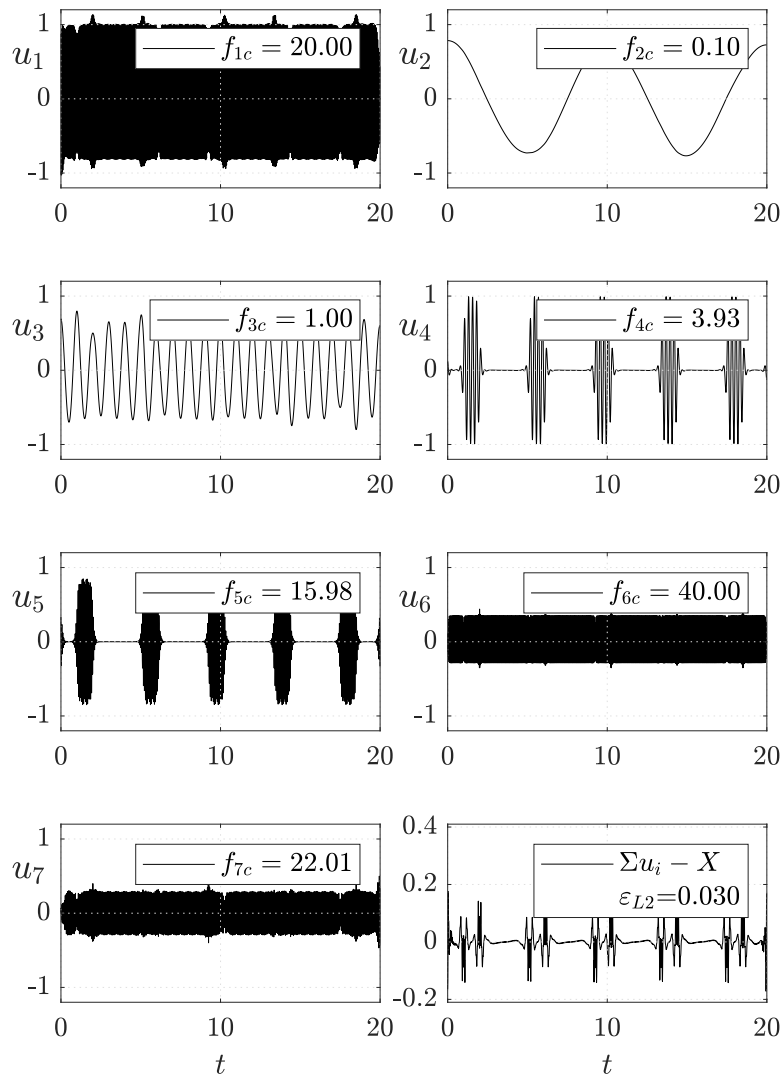


Figure 5.1: Mode decomposition using POVMD of the synthetic signal without noise. $\alpha_k = 50/\omega_k$.

The progressive algorithm also improves the computational cost of the signal decomposition. To compare the computational cost between the two algorithms, we consider the number

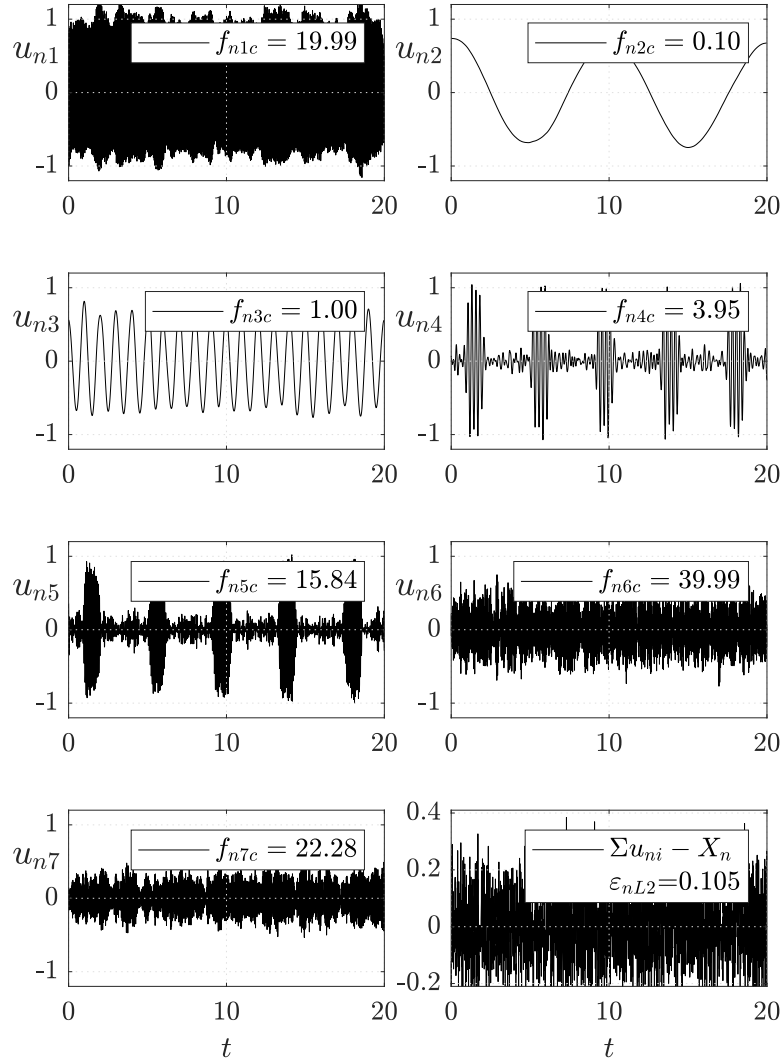


Figure 5.2: Mode decomposition using POVMD of the synthetic signal with noise. $\alpha_k = 50/\omega_k$.

of times the modes need to be updated. In the previous algorithm, for each iteration, the number of updated modes is equal to the number of prescribed modes K . Thus, the total number of mode updates equals the number of iterations $N_{iteration}$ multiplied by the number of modes K . The previous algorithm required around 80 and 191 iterations to decompose the clean and noisy signals, equivalent to 800 and 1910 total updates with $K = 10$, respectively. In POVMD, the number of mode updates required was 100 and 111. The faster convergence is primarily attributed to the lower number of final modes and the sequential approach in identifying successive modes, where only minor adjustments to the previously converged modes are needed to satisfy the cumulative minimization objectives.

The performance of POVMD in decomposing signals into their dominant modes is evaluated considering under-segmentation conditions. Two cases of under-segmentation are considered, with the number of final modes K_{max} set to 4, 6, and 7. POVMD is applied to decompose both clean and noisy signals into corresponding modes, and the energy ratios of the modes associated with each waveform component are presented in Fig.5.3.

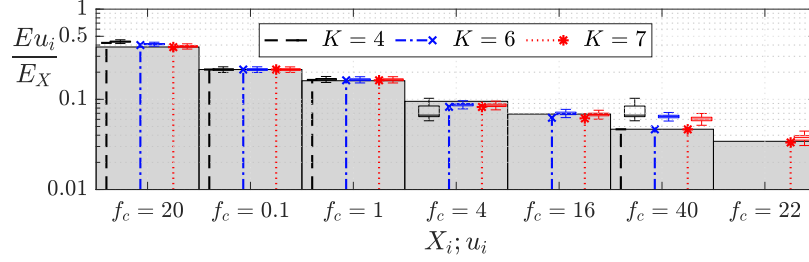


Figure 5.3: Energy sensitivity of modes with different number of final modes K ; without and with noise (boxplot) - $N_{ensemble}=1000$;

The energy ratio of each mode ($E_{u_i} = \sigma_{u_i}^2$), relative to the total energy of the clean signal ($E_X = \sigma_X^2$), is used to quantify how well each mode captures the corresponding theoretical signal. For the clean signal decomposition, the energy ratio of each mode is represented by a vertical line, with the line's endpoint indicating the energy ratio value. For the noisy signal, the decomposition is repeated $N_{ensemble} = 1000$ times with different random white noise to generate statistical data. The energy ratios of each mode are then presented as a boxplot.

The clean signal decomposition demonstrates that POVMD can sequentially recover modes, starting with the most energetically dominant mode and then extracting subsequent modes from the residual. The emergence of modes following the most energetic one is not predetermined. An initial estimation of the subsequent ω_{kIt} based on the location of the maximum magnitude in the residual spectrum, (5.2), helps the algorithm to identify the next mode. However, this location does not directly correspond to the mode's energy, as explained in Section 5.2. This phenomenon is evident in the clean signal decomposition with $K = 4$, Fig. 5.3. The final mode that appears corresponds to $f_c = 40$, not $f_c = 4$, despite the latter having higher energy. From the frequency spectrum of the signal in Fig. 4.8, it is clear that the signal exhibits a greater magnitude at $f = 40$ than at $f = 4$. Consequently, in POVMD, $f_c = 40$ is detected first, provided no contamination occurs.

In under-segmentation conditions, contamination is likely for closely aligned modes, such as $f_c = 20$ and $f_c = 22$. A fraction of the less dominant mode that has not yet been decomposed may appear in the more dominant mode, thereby increasing the energy content of the latter (e.g., $f_c = 20$) beyond its theoretical value when $K < 7$. This contamination reduces the residual spectrum's magnitude around the component affecting the dominant mode. As a result, other modes, $f_c = 40$, $f_c = 4$, and $f_c = 16$, are detected earlier.

In noisy signal decomposition, the algorithm retains its ability to extract dominant modes with high energy content. However, due to varying white noise components, the algorithm may sometimes identify an unexpected set of modes. This behavior can be observed as an increased number of boxplots compared to the expected K modes, such as when $K = 4$. In such cases, the last mode, whether $f_c = 40$ or $f_c = 4$, is not properly decomposed and may either mix with or be contaminated by another mode. As K increases, this issue becomes less frequent, and the dominant modes progressively converge to their intended energy levels.

The progressive algorithm effectively maintains a low correlation between the extracted modes, even under under-segmentation conditions. The correlation matrix shown in Fig.

5.4 illustrates the zero-lag cross-correlation between modes using covariance. The upper triangular matrix corresponds to the decomposition with $K = 4$, while the lower triangular matrix represents the decomposition with $K = 7$ for the signals shown in Fig. 4.7.

The horizontal and vertical axes represent the modes i and j , respectively, used to assess the correlation coefficient at their intersection. Under under-segmentation conditions ($K = 4$), the four modes exhibit high orthogonality in both noisy and noise-free signals. For over-segmentation ($K = 7$), the first seven dominant modes also remain nearly orthogonal. The highest correlation occurs for mode u_{n7} , which shows minimal alignment with u_{n1} and u_{n5} in the noisy signal. This mode is the weakest component, contaminated by neighboring modes and shared noise, as illustrated in Fig. 5.2.

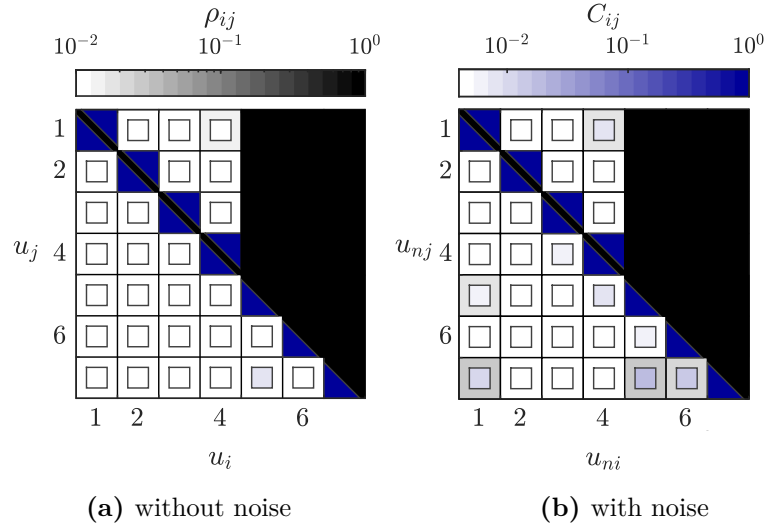


Figure 5.4: Orthogonality matrix. $K = 7$ (lower triangular); $K = 4$ (upper triangular)

5.4.2 Multivariate decomposition

In this section, the multivariate extension capability of POVMD is demonstrated for a bivariate signal. Bivariate signals are constructed from the selected signal components, as previously discussed, with specified amplitudes and phase offsets (θ) relative to Fig. 4.4. The expressions for the composed bivariate signal are:

$$\begin{aligned} \mathbf{X}[1](t) &= 0.8X_1 + X_3(\theta = 0) + 0.8X_4 + X_5 + X_8 \\ \mathbf{X}[2](t) &= 0.6X_1 + 1.3X_2 + X_3(\theta = 0.4\pi) - 0.6X_4 + X_6. \end{aligned} \quad (5.18)$$

Both channels share signal components X_1 and X_4 but with different amplitude ratios, and X_3 with a phase offset $\theta = 0.4\pi$ in the second channel $\mathbf{X}[2]$. The first channel $\mathbf{X}[1]$ includes additional components X_5 and X_8 , where X_8 represents random white noise. The second channel contains additional components X_2 and X_6 . As a result, the composite signal for the first channel is noisy, while the second channel remains clean, as shown in Fig. 5.5.

The decomposition of the bivariate signal is presented in Fig. 5.6, with modes ranked by energy. Modes dominated by noisy components are omitted for brevity. Seven principal

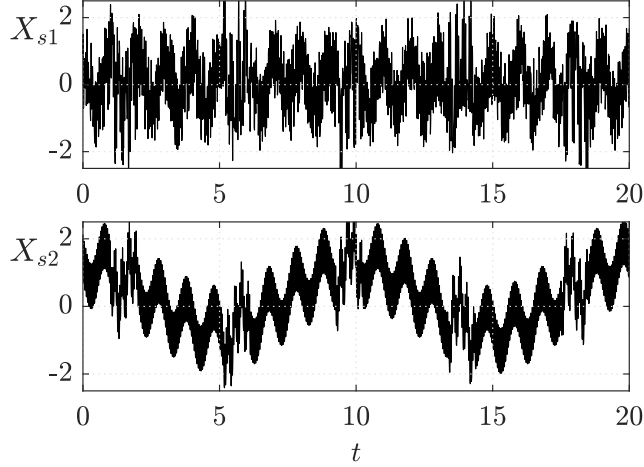


Figure 5.5: Bivariate signal input.

modes are observed, each comprising a temporal coefficient (common across channels) and a spatial topology (providing time-invariant weights for each channel). The product of the temporal coefficient and spatial topology reconstructs the mode $\mathbf{u}$, and the summation of all modes $\mathbf{u}$ resembles the input signal, as described in (5.3).

The first mode corresponds to the signal component X_1 , with time-invariant weights of 0.8 and 0.6 for the first and second channels, respectively. These weights match the specified amplitude ratios because of the normalization process, which ensures that the L_2 -norm of the spatial topology equals unity. Consequently, the temporal coefficient retains the magnitude of the signal component X_1 .

The second mode represents the signal component X_2 , which appears only in the second channel. As a result, the spatial topology has a value of one for the second channel and zero for the first channel after normalization. To reflect the specified amplitude coefficient of 1.3, the temporal coefficient is scaled accordingly.

Modes 3 and 4 are coupled and represent the signal component X_3 , which appears on both channels with a phase difference θ . As described in (5.5), a shared modulation between channels with different phases produces two modes with temporal coefficients that exhibit quadrature phase offset characteristics ($\pm\pi/2$). The actual mode can be reconstructed as:

$$\mathbf{u}_{X_3}(t) = \psi_3 c_3(t) + \psi_4 c_4(t). \quad (5.19)$$

Modes 5 and 6 represent the time-localized components X_5 and X_4 , respectively. The component X_4 appears on both channels with different amplitudes and relative phases. Despite a $\pi/2$ phase lag between channels, X_4 is captured within a single mode.

It is worth noting that the presence of a clean signal in one channel improves the quality of the decomposition, as evidenced by comparing the temporal coefficient of mode 5 with its representation in the univariate test (Section 5.4.1). Conversely, mode 7, representing X_6 (present only in the second channel), illustrates the impact of noise. In the noisy first channel, white noise with content near the center frequency of X_6 creates interference (common

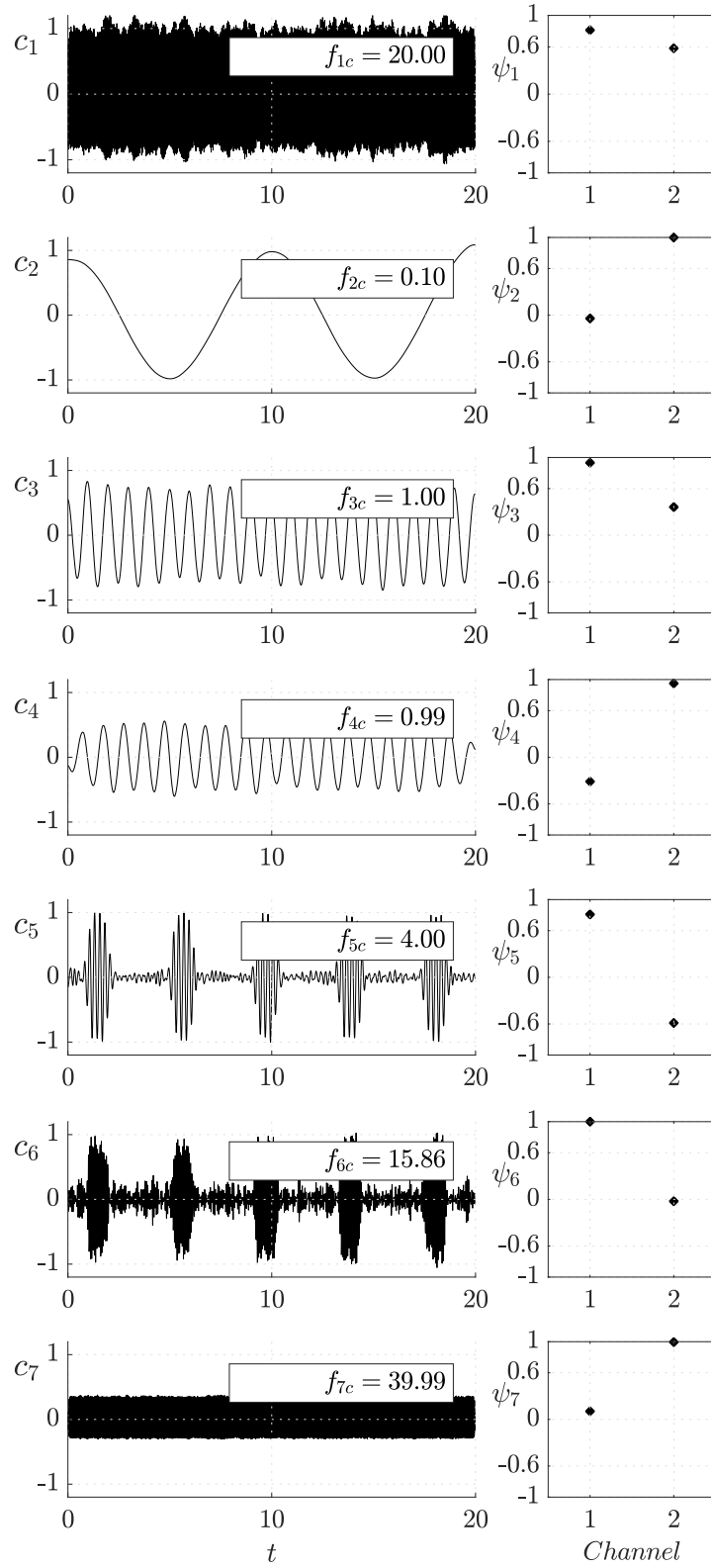


Figure 5.6: Principal modes in multivariate POVMD of the bivariate synthetic signal. $\alpha_k = 50/\omega_k$.

frequency modulation). The decomposition attempts to match the noisy modulation in channel 1 with the X_6 component in channel 2, leading to minimal noise contamination in the temporal coefficient and a near-zero spatial topology value for channel 1.

In MVMD, the computational cost increases almost quadratically with the number of channels when the count exceeds eight (Rehman and Aftab, 2019). This characteristic limits its practicality in flow dynamics applications, where a large number of channels often need to be decomposed (Padilla-Montero et al., 2024, Das and Pachori (2021)). It is expected that a linear relationship between computational time and the number of channels is the optimal relationship to be achieved.

Multichannel signals are constructed to evaluate the performance of MPOVMD in the decomposition of multivariate datasets using previously defined synthetic signal components. The synthetic signal for each channel s is expressed as:

$$\begin{aligned} \mathbf{X}[s](t) = & 0.8X_1 + 0.8X_4 + X_5 + X_8 \\ & + \left(\frac{i-1}{S-1} \right) (-0.2X_1 + 1.3X_2 - 1.4X_4 - X_5 + X_6 - X_8) \\ & + X_3 \left(\theta = \frac{i-1}{S-1} 0.4\pi \right), \end{aligned} \quad (5.20)$$

where S is the total number of channels. The first and last channel are identical to the first and second channels of the bivariate signal (Fig. 5.2), while the intermediate channels are linearly interpolated. The noise component X_8 is present in all channels except the last. The signals tested range from 2 to 4096 channels.

The decompositions are repeated 20 times for each multivariate signal with $K_{max} = 15$, using MATLAB R2022b environment on a single-core Intel® Xeon® Gold 6230 processor. The fastest computational time is recorded as the representative time. Two expressions are tested for computing O_{ω_k} : the actual expression as in Eq.(5.14) and a faster estimation as in Eq.(5.15). The computational times for both approaches are plotted together in Fig. 5.7. In general, the faster estimation of O_{ω_k} reduces computational time by 30% while producing identical modes and the same L_2 -norm error. In MPOVMD, computational time maintains linear scaling even for very large channel counts.

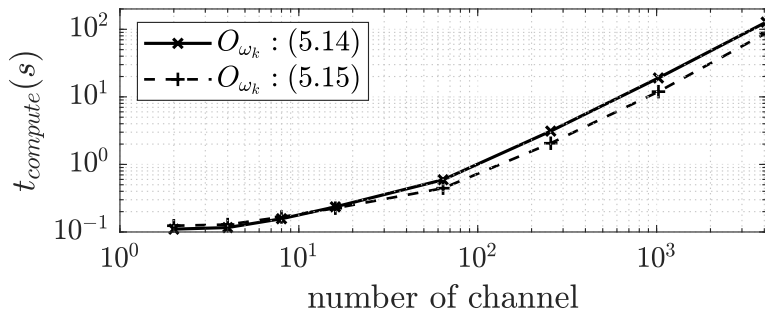


Figure 5.7: Computational time scaling for the multivariate signals decomposition, (s) is the time unit second. The variable O_{ω_k} is computed with different approaches.

5.5 Empirical Filter Bandwidth

The orthogonal objectives introduced in the OVMD result in a more robust decomposition with respect to a range of filter bandwidth parameters (α). A more precise estimation of the filter bandwidth will help in the decompositions. In this section, an attempt to create an empirical approach to estimate the filter bandwidth for every frequency range is explained. The key criterion of the algorithm is to prevent the misdetection of broadband bandwidth as narrowband.

Details of the algorithm are presented in the Appendix C. The general steps are the following:

1. Compute the amplitude of the Fourier spectrum (unilateral).
2. Extraction of local maxima $\{\hat{X}_{loc}\}$ and its index location $\{i_{loc}\}$ in the range of $\{\omega\}$
3. Detect possible broadband segment and only keep the maximum magnitude within the segment range.
4. Insert local minima between the two consecutive maxima $\{\hat{X}_{loc}\}$ when the ratio of the larger consecutive maxima to the smaller is greater than r_{tonal} , Fig. 4.3. In this thesis, a threshold value of $r_{tonal} = 5$ is selected on the basis of rigorous tests.
5. For every discrete frequency in $\{\omega\}$, determine the cut-off frequencies on the right (ω_{right}) and left (ω_{left}) using the local maxima-minima data in $\{X_{loc}\}$.
6. Estimate the representative scaled filter bandwidth for every discrete frequency $-\omega[i]$

$$B[i] = 2 \frac{\min(\omega_{right}[i] - \omega[i], \omega[i] - \omega_{left})}{\omega_{max}}. \quad (5.21)$$

The representative inverse filter bandwidth F for the univariate synthetic signal used in Section 5.4.1 is shown in Fig. 5.8 with blue lines representing the center frequency of the signal components. The Algorithm 4 (Appendix C) able to assign proper bandwidth value for the narrowband ($f_c = \{0.1, 1, 20, 22, 40\}$) and broadband ($f_c = \{4, 16\}$) signal component. The energy sensitivity study with the univariate signal is given in the Appendix.

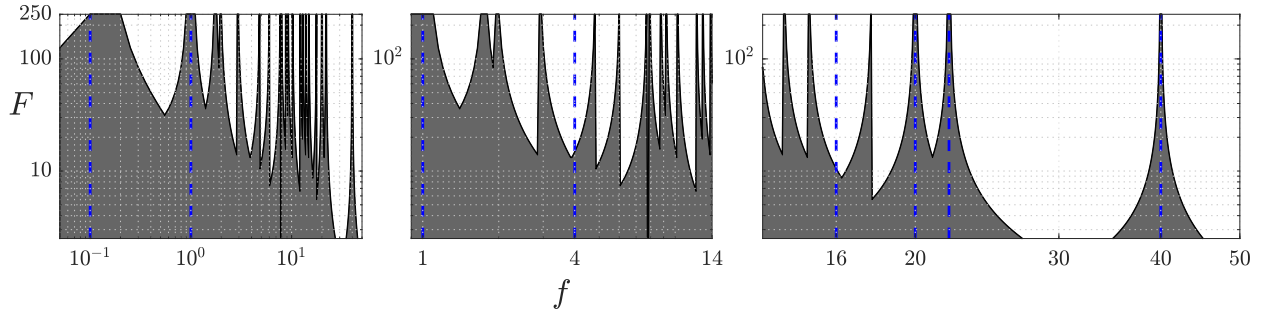


Figure 5.8: Representative inverse filter bandwidth $F(\omega)$ distribution from empirical estimation for the synthetic signal without noise. The blue lines represent the center frequency of signal components

5.6 Non-stationary AM/FM signals

In this section, three non-stationary AM/FM signals are decomposed with the POVMD method with empirical bandwidth. The first two signals are univariate signals that have been used in Gilles (2013) and Dragomiretskiy and Zosso (2014) to demonstrate the empirical wavelet decomposition and variational mode decomposition, respectively. The last signal is multivariate data of the transient cylinder case known in fluid dynamics and has been used to demonstrate the Dynamic Mode Decomposition (DMD) method in Noack (2016) and the reduced VMD method in Liao et al. (2023).

5.6.1 Intrawave frequency modulation signal (Sig_3)

The first signal is proposed by Hou and Shi (2011) and has intrawave frequency modulation. It is known as Sig_3 in Gilles (2013) and Dragomiretskiy and Zosso (2014). The signal is defined in Hou and Shi (2011) as

$$X_{Sig_3}(t) = \frac{1}{1.2 + \cos(2\pi t)} + \frac{\cos(32\pi t + 0.2 \cos(64\pi t))}{1.5 + \sin(2\pi t)}. \quad (5.22)$$

The signal and its components are shown in Fig. 5.9. The first signal component features a bell-shaped localized event with low-frequency content, while the second component presents the challenge of a non-linear high-frequency modulation.

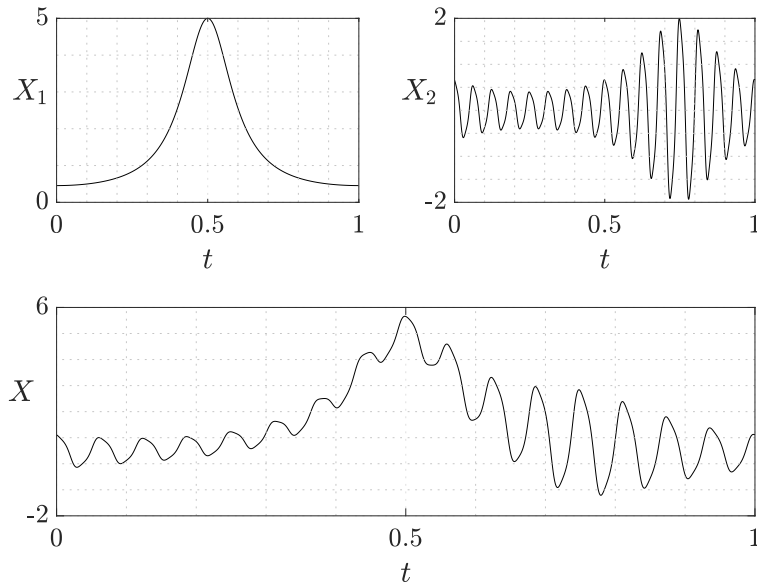


Figure 5.9: Two components of Sig_3 and the signal (bottom)

The POVMD decomposition using the empirical bandwidth approach is presented in Fig. 5.10. The parameters K_{max} and E_{th} are 6 and 0.99, respectively. Three main modes are produced. The first mode is a constant mean value of the signal. The second mode captures

the bell-shaped component and provides a good estimate of the first signal component offset by the mean, with no ripple observed as in the original VMD (Dragomiretskiy and Zosso, 2014). The last mode successfully reconstructs most of the key features of the second component, correctly locating the peak in the spectrum. A small fraction of this mode is separated as a trivial mode u_4 . The final dominant mode represents a high-frequency harmonic of the second signal component.

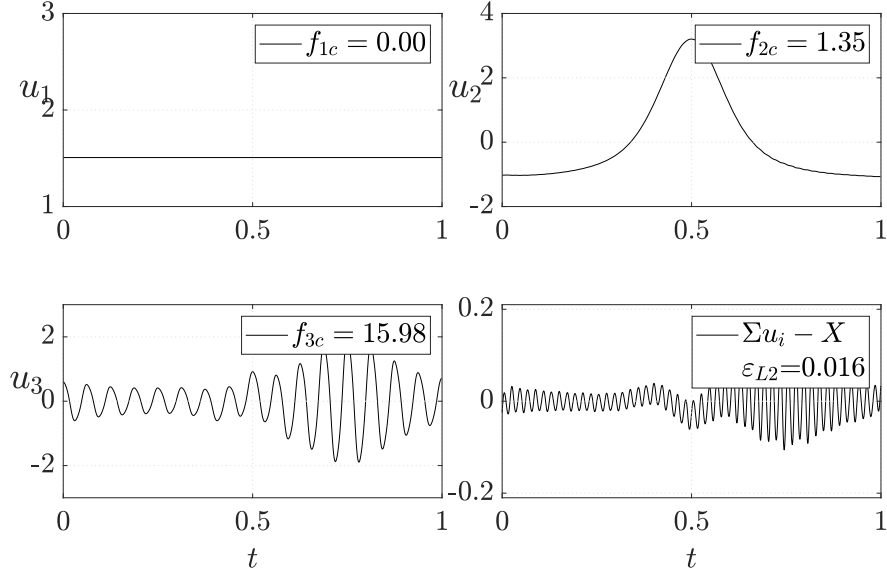


Figure 5.10: POVMD decomposition of Sig_3 .

5.6.2 ECG Data (Sig_4)

The second signal from Gilles (2013), referred to as Sig_4 , is a real electrocardiogram (ECG) signal containing multiple components. These components consist of both distributed and localized events spanning a broad range of frequencies, as shown in Fig. 5.11. The temporal length of the signal is set to 8 seconds, with a sampling frequency of $f_s = 512$ Hz.

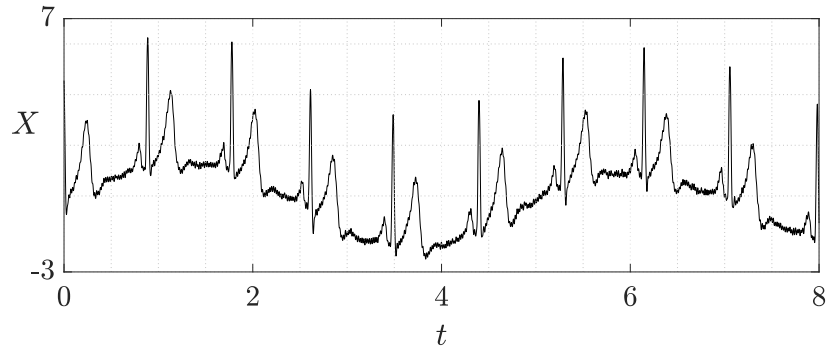


Figure 5.11: Signal Sig_4 (ECG data)

The parameters K_{max} and E_{th} are 10 and 0.94, respectively. The POVMD decomposes the signal into four modes, Fig. 5.12 is time domain and Fig. 5.13 is the spectrum. The first mode represents low-frequency fluctuations. The sudden spikes in the signal are captured in mode u_2 and u_3 , which are the modes with time-localized events. The last mode is a constant frequency component. Most of the peaks in the spectrum are not closely aligned. A full assessment of the quality of the decomposition would require evaluation by a cardiologist. However, the OVMD's ability to decompose the signal into seven main modes provides a clear advantage over the result with the original VMD in Dragomiretskiy and Zosso (2014)).

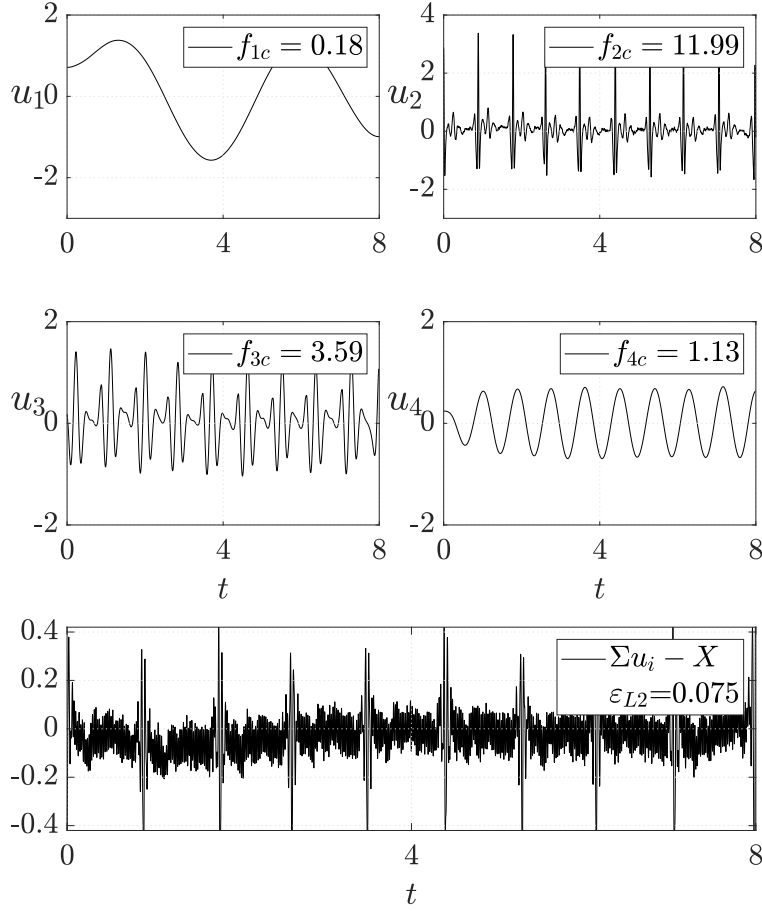


Figure 5.12: OVMD decomposition of *Sig4* with $K = 8$.

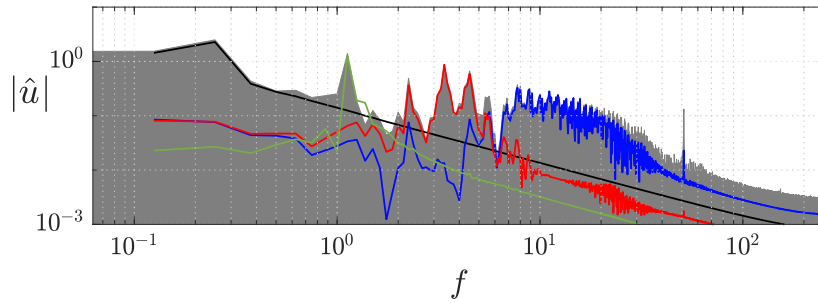


Figure 5.13: OVMD decomposition spectrum of *Sig4* with $K = 8$.

5.6.3 Transient Evolution of a Cylinder Wake (Liao et al., 2023; Noack et al., 2016)

In this section, multivariate data are used to demonstrate the improvement provided by MPOVMD for fluid dynamic applications. Low-dimensional modeling through modal decomposition is widely utilized in fluid dynamics to study the underlying physics. Some methods, such as Proper Orthogonal Decomposition (POD), maintain orthonormal properties between modes and achieve the least residual (Noack, 2016), but they struggle to represent dynamic changes in flow caused by frequency mixing. To address this, the spectral POD (Sieber et al., 2016) proposed to separate phenomena occurring at multiple frequencies and scales.

Another method, Dynamic Mode Decomposition (DMD), is capable of separating frequency mixing, but cannot maintain orthonormality (Noack, 2016). Recently, Liao et al. (2023) extended the Variational Mode Decomposition (VMD) method to develop the reduced-order VMD (RVMD). Although RVMD offers advantages over other methods, its results still depend on the arbitrary number of modes K and the parameter α . Moreover, none of the existing methods explicitly minimizes interference between modes in the presence of localized events.

The case considered is a transient cylinder wake, which is widely used to evaluate mode decomposition techniques such as POD (Noack et al., 2016; Siegel et al., 2008), DMD (Bagheri, 2013; Noack et al., 2016), and RVMD (Liao et al., 2023). The flow over a cylinder is steady and stable at very low Reynolds numbers. However, as the Reynolds number, defined as $Re_D \equiv UD/\nu$, exceeds 47, the steady solution transitions to a stable limit cycle solution (Boppa and Gajjar, 2011; Jackson, 1987; Nishioka and Sato, 1978), accompanied by the emergence of the periodic von Kármán vortex street.

A high-order simulation of flow past a cylinder is performed using HORSES3D. The domain consists of a cylinder with a diameter D of 1 m, with its center located at the origin. The domain spans $x \in [-20, 60]$ m in the longitudinal direction and $y \in [-20, 20]$ m in the transverse direction. The Reynolds number Re_D is set to 100, consistent with the studies in Noack (2016) and Liao et al. (2023). The Mach number is 0.2. The domain is discretized into 996 hexahedral elements and simulated using a fourth-order polynomial approximation, resulting in 124,500 degrees of freedom (DOF).

The simulation is initialized with a homogeneous freestream velocity in the domain, with no transverse or directional velocity components. The evolution of spanwise vorticity at $t^* = 150$ and $t^* = 350$ is shown in Fig. 5.14. Initially, the solution evolves into a steady symmetric state, as shown in Fig. 5.14a, before transitioning to a limit cycle between $t^* \approx 220$ and $t^* \approx 300$. After the transition, the dynamics are dominated by the periodic shedding of the von Kármán vortex street, as illustrated in Fig. 5.14b. The definition of normalized time is consistent with that used in Chapter 3, with $L_{Ref} = D$.

A collection of streamwise and transverse velocity snapshots is used to study the dynamics of the transient process. The snapshots span $x \in [-2, 16]$ m and $y \in [-4, 4]$ m, discretized into 181 and 81 data points, respectively. The sampling begins at $t^* = 150$ and continues until $t^* = 350$ with a sampling rate of $f_s^* = 8$. The dynamic evolution of streamwise and transverse velocities is analyzed by subtracting the mean flowfield, calculated over $t^* = 150$ to $t^* = 350$,

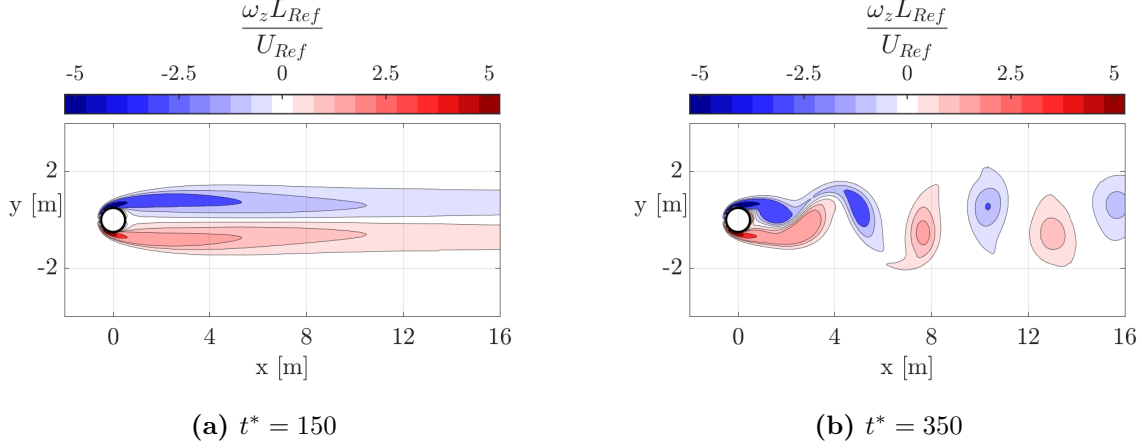


Figure 5.14: Evolution of spanwise vorticity around Cylinder with $Re_D = 100$

from the data:

$$\tilde{\mathbf{X}}(t^*) = \mathbf{X}(t^*) - \bar{\mathbf{X}}, \quad (5.23)$$

where $\mathbf{X}$ represents the multivariate data comprising the flow field (i.e., spatial topology) for both streamwise and transverse velocities, and $\bar{\mathbf{X}}$ denotes the temporal mean. The MPOVMD method is applied to decompose the data with $K_{max} = 15$ and $E_{th} = 0.99$.

The decomposition converges after 13 modes, denoted by $\tilde{\mathbf{u}}$, capturing 99% of the input energy while maintaining an L_2 error of approximately 7%. The energy ratio is presented in Fig. 5.15, while the temporal coefficients and the spanwise vorticity structures of each mode are shown in Figs. 5.16 and 5.17. The spanwise vorticity is reconstructed *a posteriori* from the streamwise and transverse velocity fields using the relation:

$$\psi_{\omega_z} = \frac{\partial \psi_{u_y}}{\partial x} - \frac{\partial \psi_{u_x}}{\partial y}. \quad (5.24)$$

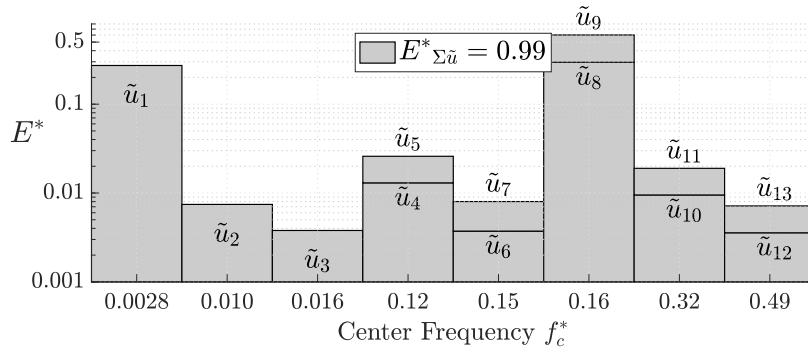


Figure 5.15: Energy Ratio

The decomposed modes can be categorized into three types. The first three low-frequency modes correspond to the shift phenomenon and represent the difference between the steady flow at the beginning of the simulation and the mean flow of the limit cycle post-transition

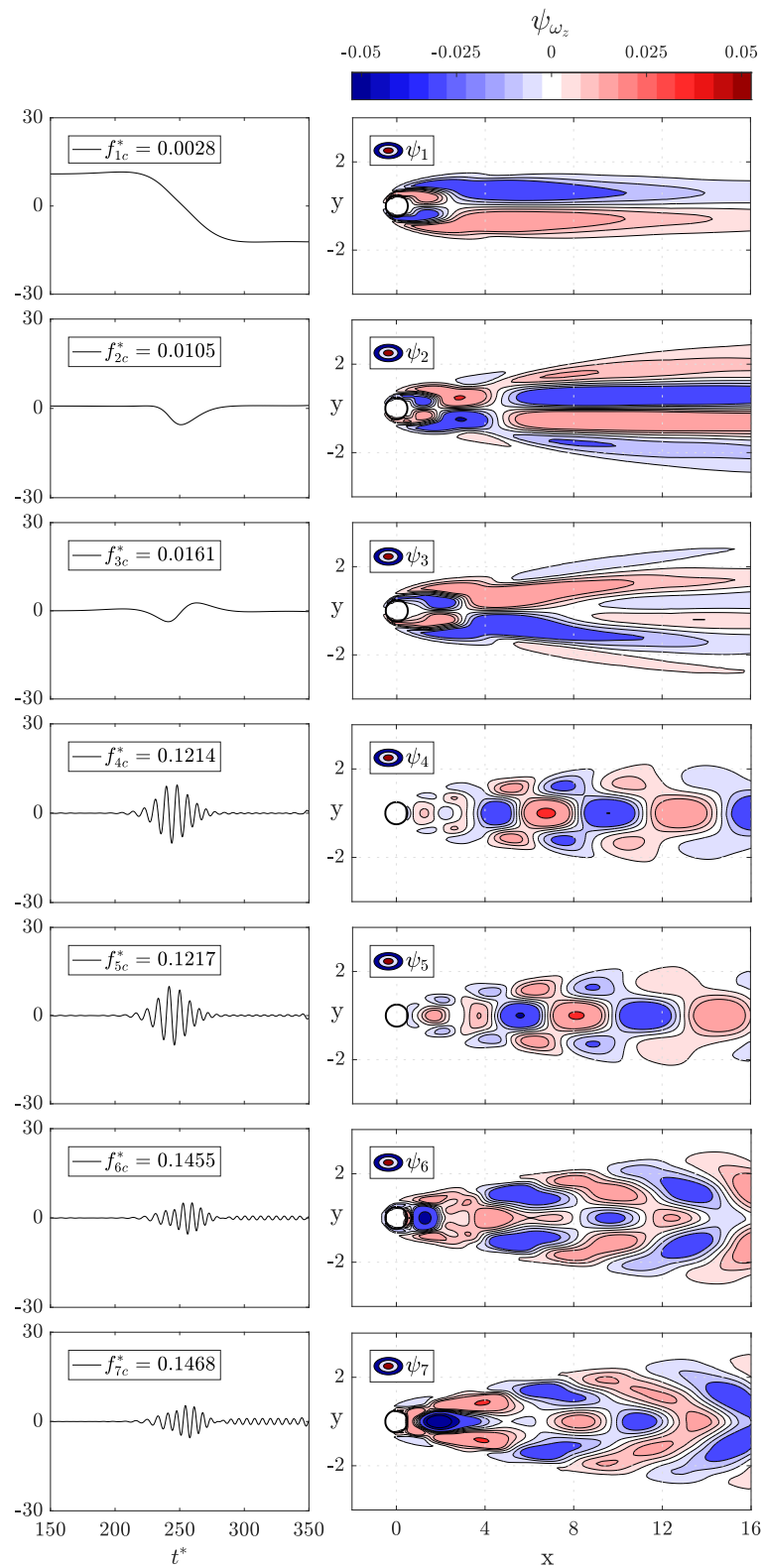


Figure 5.16: OVMD modes 1-7, represented by spanwise vorticity topology, for the transient cylinder $Re = 100$

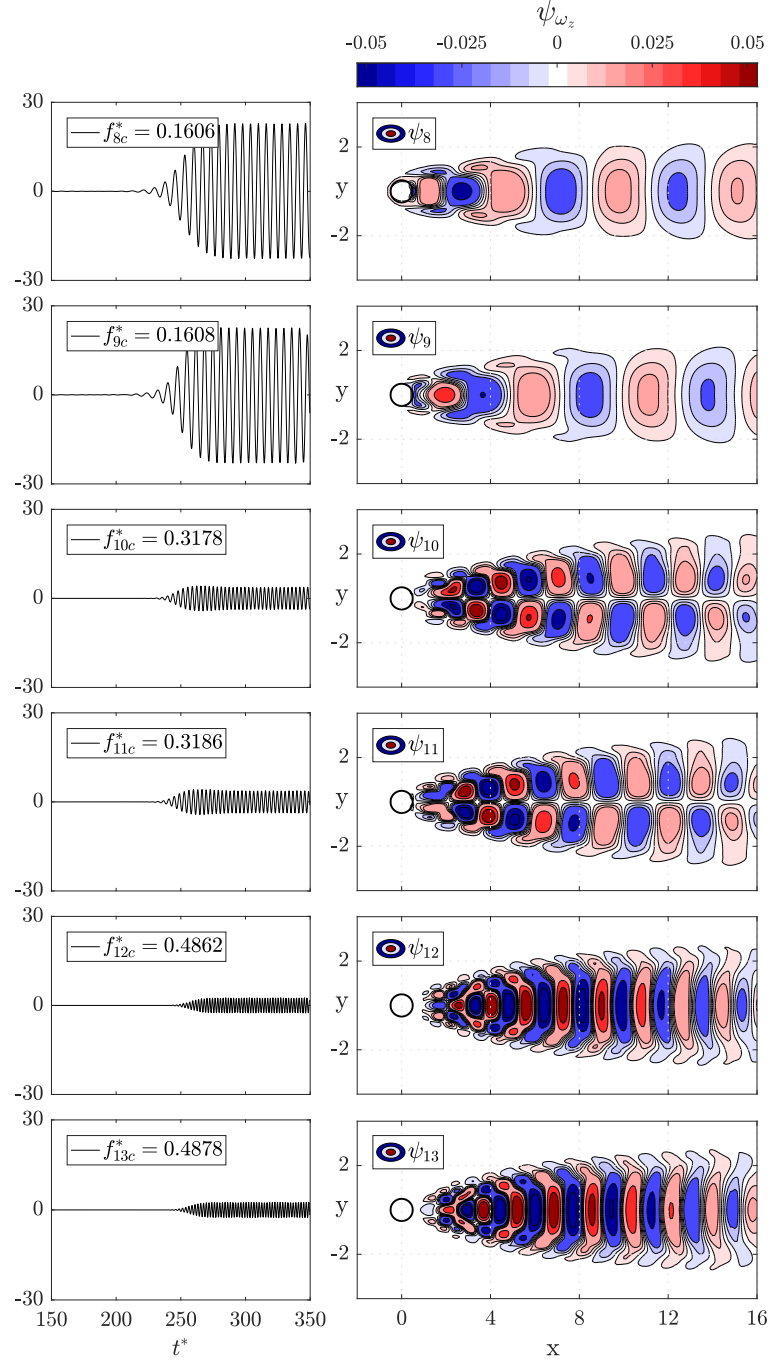


Figure 5.17: OVMD modes 8-13, represented by spanwise vorticity topology, for the transient cylinder $Re = 100$

(mode 1). The shift includes two antisymmetric low-energy modes that appear only during the transition stage (modes 2 and 3). These modes are absent in the RVMD but appear in the POD and DMD analyses reported by Noack et al. (2016).

Next, four symmetric modes with high frequency represent the transition to the limit cycle stage and the onset of the von Kármán vortex street. Two of these modes (modes 4 and 5)

are present in previous studies, while the other two low-energy modes are not. However, a merging of these modes with modes 4 and 5 is observed in Noack et al. (2016).

The final six modes, Fig. 5.17, represent the post-transition dynamics, characterizing the von Kármán vortex street. These include the first harmonic pair (symmetric, modes 8 and 9), the second harmonic pair (antisymmetric, modes 10 and 11), and the third harmonic pair (symmetric, modes 12 and 13). The center frequency of the first harmonic is 0.16, which corresponds to the Strouhal number $St_D = fD/U_{ref}$, aligning well with the theoretical frequency for this Reynolds number (Jiang and Cheng, 2017).

5.7 Summary

This chapter extends the Orthogonalized Variational Mode Decomposition (OVMD) technique for progressive decomposition and multivariate data. The modification introduces a progressive algorithm capable of sequentially decomposing modes. In the approach, the modes that appear earlier in the decomposition are more likely to have higher energy content compared to those that appear later. The extension adapts OVMD to handle multivariate or multi-channel data, enabling the decomposition of such datasets effectively. The resulting complete method is named Multivariate Progressive Orthogonalized Variational Mode Decomposition (MPOVMD).

In progressive OVMD, the frequency spectrum of the residual input signal guides the determination of the initial center frequency for each successive mode. The first center frequency is set to the frequency with the maximum residual magnitude. From this initialization, the algorithm optimizes and extracts the mode, removing its contribution before proceeding to the next successive mode. As a result, the decomposition process prioritizes the extraction of dominant modes earlier, effectively isolating the principal components.

The multivariate extension of OVMD utilizes an approach commonly employed in the Proper Orthogonal Decomposition method. This approach identifies common modulations across all channels, where each mode is composed of a spatial topology and a temporal coefficient. The spatial topology is a vector of real-valued, discrete, and time-invariant coefficients that are applied to each channel. In flow field data, these coefficients can be structured by their spatial location to form a spatial structure or an equivalent spatial mode for spatiotemporal decomposition. The temporal coefficient, on the other hand, is a time-variant component applied uniformly across all channels. Together, the product of the spatial topology and temporal coefficient represents the mode, $\mathbf{u}$. The method demonstrates excellent scalability with a very high number of channels, maintaining a linear relationship between computational time and the number of channels.

Furthermore, an empirical algorithm for determining the filter bandwidth is developed and tested to further enhanced the OVMD method. The complete novel decomposition technique is evaluated using non-stationary signals, including real data and multivariate signals, to demonstrate its effectiveness and capability.

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