

UNIVERSIDAD POLITÉCNICA DE MADRID
Escuela Técnica Superior de Ingenieros de Caminos, Canales y
Puertos



**Bifurcation of Hyperelastic Cylindrical
Structures under Swelling and Residual
Stresses with Application to Aneurysm
Formation**

DOCTORAL THESIS

Submitted for the degree of Doctor by:

MURTADHA JASIM HUSSEIN AL-CHLAIHAWI
M.Sc Structural Engineering

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Doctoral Degree in Engineering of Structures, Foundations and
Materials

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Abstract

This thesis presents an in-depth understanding of the mechanical behavior of thick-walled isotropic hyperelastic tubes undergoing residual stresses and thin-walled anisotropic ones capable of swelling, subjected to the combined effects of axial load and internal pressure. Particular emphasis is placed on their application to aneurysms within instability theory and in the context of bifurcation analysis.

The influence of swelling on prismatic and bending bifurcation modes of inflated thin-walled cylinders under axial loading is examined. Bifurcation criteria for a membrane cylinder subjected to axial loading, internal pressure, and swelling are provided. Orthotropic materials with two equivalent, symmetrically disposed preferred directions are considered. The matrix behavior is modeled as swellable isotropic with unidirectional reinforcement. Two fiber-reinforcement models—standard and exponential—are analyzed. The study explores the effects of axial stretch, fiber strength, and winding angle on bifurcations. Prismatic bifurcation is infeasible without fibers but may occur in the standard model. Bending bifurcation is more likely in swollen cylinders, while the exponential model prevents prismatic bifurcation.

Furthermore, A thin-walled hollow cylinder composed of a fibrous and swellable hyperelastic material is considered in this thesis. The fibers are arranged in two families and are assumed to be parallel within each fiber family. The two fiber families are also assumed to be mechanically equivalent and symmetrically disposed in the ground substance material. At each instant of homogeneous swelling, the material is considered to be incompressible. The interplay of swelling, fiber orientation, and the mechanical properties of the constituents on the initiation and axial propagation of bulging is studied.

Moreover, Bifurcation for a cylindrical membrane capable of swelling, subjected to axial loading and internal pressure, is analyzed. The material is modeled as an isotropic, absorbent matrix containing nonabsorbing fibers, symmetrically arranged in two helically distributed, mechanically equivalent families. Arterial wall tissue is modeled using this theoretical framework. The matrix is a swellable neo-Hookean material. The swollen membrane is inflated and axially stretched to preserve its circular cylindrical geometry. Prismatic, bulging, and bending bifurcation conditions are analyzed. It is shown that prismatic bifurcation is not significant for membranes with or without fibers, but bending and bulging are feasible for fiber-reinforced membranes, with results capturing bifurcation onset and possible coupling during post-bifurcation.

Also, Previous studies have shown that the mechanical response of incompressible hyperelastic materials is affected by residual stresses, which result from factors like growth and development in biological soft tissues. The effect of these initial stresses on mechanical behavior remains to be fully explored. Residual stresses in arterial wall tissue affect the location of instabilities, such as bulges. This study develops a new approach to assess material behavior during bifurcation instability in the presence of residual stresses. A finite element protocol is developed for bifurcation and post-bifurcation of residually stressed thick-walled hyper-elastic tubes subjected to axial stretches and internal pressure, incorporating three-dimensional residual stresses. Results indicate that bending bifurcation is likely for small axial stretches

but less probable with larger stretches, while bulging bifurcation occurs for all axial stretches. Pressures associated with pure bulging are higher than those for bulging induced by bending, suggesting that aneurysms can be delayed by avoiding bending bifurcation. Bulging from bending bifurcation occurs on one side of the tubes, reflecting abdominal aortic aneurysm (AAA) conditions.

Resumen

Esta tesis presenta una comprensión profunda del comportamiento mecánico de tubos hiperlásticos isotrópicos de paredes gruesas con tensiones residuales y tubos anisotrópicos de paredes delgadas capaces de hincharse, sujetos a carga axial y presión interna. Se enfatiza su aplicación a aneurismas dentro de la teoría de inestabilidad y el análisis de bifurcación.

Se examina la influencia del hinchamiento en los modos de bifurcación prismática y de flexión de cilindros delgados inflados bajo carga axial. Se establecen criterios de bifurcación para un cilindro de membrana sometido a carga axial, presión interna e hinchamiento. Se consideran materiales ortotrópicos con dos direcciones preferenciales equivalentes y dispuestas simétricamente. La matriz se modela como isotrópica hinchable con refuerzo unidireccional. Se analizan dos modelos de refuerzo con fibras: estándar y exponencial. Se estudian los efectos del estiramiento axial, la resistencia de las fibras y el ángulo de enrollamiento en las bifurcaciones. La bifurcación prismática no es viable sin fibras, pero puede ocurrir en el modelo estándar. La bifurcación por flexión es más probable en cilindros hinchados, mientras que el modelo exponencial impide la bifurcación prismática.

Además, se considera un cilindro hueco de pared delgada compuesto por un material hiperlástico fibroso e hinchable. Las fibras están dispuestas en dos familias paralelas y simétricas en el material base. En cada instante de hinchamiento homogéneo, el material se considera incompresible. Se estudia la interacción entre hinchamiento, orientación de las fibras y propiedades mecánicas en la iniciación y propagación axial del abultamiento.

Asimismo, se analiza la bifurcación de una membrana cilíndrica hinchable sometida a carga axial y presión interna. El material se modela como una matriz isotrópica absorbente con fibras no absorbentes, simétricamente dispuestas en dos familias helicoidales equivalentes. La matriz es un material neo-Hookeano hinchable. La membrana hinchada se infla y se estira axialmente para preservar su geometría cilíndrica. Se analizan las condiciones de bifurcación prismática, de abultamiento y de flexión. Se demuestra que la bifurcación prismática no es significativa para membranas con o sin fibras, pero la bifurcación por flexión y el abultamiento son factibles en membranas reforzadas con fibras, capturando la aparición de la bifurcación y su posible acoplamiento en la post-bifurcación.

Estudios previos han demostrado que la respuesta mecánica de materiales hiperlásticos incompresibles se ve afectada por tensiones residuales, derivadas del crecimiento de tejidos biológicos blandos. El efecto de estas tensiones en el comportamiento mecánico aún no ha sido completamente explorado. Las tensiones residuales en la pared arterial afectan la ubicación de inestabilidades como los abultamientos. Este estudio desarrolla un enfoque para evaluar el comportamiento del material en la inestabilidad de bifurcación con tensiones residuales. Se desarrolla un protocolo de elementos finitos para la bifurcación y post-bifurcación de tubos hiperlásticos de paredes gruesas con tensiones residuales, sometidos a estiramientos axiales y presión interna. Los resultados indican que la bifurcación por flexión es probable para pequeños estiramientos axiales, pero menos probable con estiramientos mayores, mientras que la bifurcación por abultamiento ocurre para todos los estiramientos axiales. Las presiones asociadas con el abultamiento puro son mayores que las inducidas por flexión, sugiriendo

que los aneurismas pueden retrasarse evitando la bifurcación por flexión. El abultamiento por flexión ocurre en un lado de los tubos, reflejando las condiciones del aneurisma aórtico abdominal (AAA).

Table of Contents

Acknowledgement	v
Abstract	vi
Resumen	viii
List of Figures	xiii
List of Tables	xxi
Abbreviations and acronyms	xxiv
1 Introduction	1
1.1 Residual Stresses in Biological Systems: A Thorough Examination	1
1.2 Hyperelastic Materials and Non-Linear Elasticity	2
1.3 Motivation	3
Predicting Aneurysm Formation and Growth	3
Developing Medical Equipment	3
Enhancing Surgical Results	3
Improving Computational Models	3
1.4 Objectives	4
1.5 Thesis Schema	4
1.6 Summary	5
2 On prismatic and bending bifurcations of fiber-reinforced elastic membranes under swelling with application to aortic aneurysms	7
2.1 Introduction	7
2.2 Basic equations	9
2.2.1 The boundary value problem	9
2.2.2 Prismatic and bending bifurcation criteria	11
2.3 Material Models	12
2.4 Prismatic mode	14
2.4.1 Prismatic bifurcation of the standard model	14
2.4.2 Prismatic bifurcation of the exponential model	19
2.5 Bending (composite) mode	20
2.5.1 Bending bifurcation of standard model	20
2.5.2 Bending bifurcation of exponential model	21
2.6 Conclusions	23
3 Bulging initiation and propagation in fiber-reinforced swellable Mooney–Rivlin	

membranes	25
3.1 Introduction	25
3.2 Geometric and mechanical conditions	26
3.2.1 Material Behavior	28
3.2.2 Bulging Bifurcation in the Membrane Approximation	30
3.3 Bulging for specific material models	30
3.3.1 Mooney-Rivlin Behavior	33
3.3.2 Fibers in neo-Hookean Matrix	34
Fiber behavior in terms of I_4 ($k_1 \neq 0$, $k_2 = 0$ in (3.18))	34
Fiber behavior in terms of I_5 ($k_1 = 0$, $k_2 \neq 0$ in (3.18))	36
Fiber behavior in terms of both I_4 and I_5	37
3.4 Axial propagation of bulging under swelling	38
3.5 Concluding remarks	41
4 Bifurcation of Fiber-Reinforced Cylindrical Membranes under Extension, Inflation, and Swelling	43
4.1 Introduction	43
4.2 Kinematics and Mechanical Behavior of the Cylindrical Membrane	45
4.3 Bifurcation Modes in the Membrane Approximation	49
4.3.1 Bulging bifurcation	50
4.3.2 Prismatic bifurcation	51
4.3.3 Bending bifurcation	52
4.4 Bifurcation for W_m , the ground substance without fibers	53
4.4.1 Swelling-independent ground substance stiffness and a fixed axial stretch	54
4.4.2 Unswollen ground substance and different axial stretches	54
4.5 Bifurcation for W_f , the fibers	56
4.5.1 The standard reinforcing model	58
4.5.2 The exponential form	59
4.6 Bifurcation for the ground substance matrix with fibers	60
4.6.1 Bulging and Bending	61
4.6.2 Ground substance with embedded fibers that may or may not be extended	62
4.7 Discussion and Outlook	63
5 Computational modeling of a residually stressed thick-walled cylinder under combined action of axial extension and inflation	71
5.1 Introduction	71
5.2 Mathematical description of the problem	73
5.2.1 Geometry	73
5.2.2 Residually stressed cylinders	74
5.2.3 Stable Inflation	76
5.2.4 Disruption of the stable inflation in the membrane treatment	77
5.3 Finite element analysis	79
5.3.1 Finite element model	80
5.3.2 Inflation of a pressurized Cylinder	80
5.3.3 Analysis procedure	81

5.4	Numerical results	84
5.4.1	Approach for triggering the two modes of instability	85
5.4.2	Comparison of stress distribution: Case 1 and Case 3 from the Table 5.1, taking $\lambda_z = 1.6$	87
5.4.3	Normalized inflation pressure versus axial stretch at the inflation point	88
5.4.4	Normalized inflation pressure versus azimuthal stretch during the for- mation and propagation of the bulging	88
5.4.5	Azimuthal and axial stretches scenarios in bending bifurcation	89
5.4.6	Influence of axial stretching on the stress distributions	90
5.4.7	Stress distribution for Case 1 and Case 3, taking $\lambda_z = 1.10$	90
5.4.8	Normalized inflation pressure versus axial stretch at the bulging initia- tion point	91
5.4.9	Role of increasing axial stretches in the post-bifurcation behavior . .	92
5.4.10	Development of the post-bifurcation configuration	93
5.5	Conclusions	94
6	Conclusions	103
6.1	Summary of Findings	103
	Referencias	107
	Annexes	121

List of Figures

2.1	Schematic representation of the three bifurcation modes for a membrane cylinder, bulging bifurcation (left), prismatic bifurcation (center), and bending bifurcation (right).	12
2.2	Plot of $\hat{W}_{\lambda\lambda}/\mu$ against λ for $\gamma = 0$ (no fibers), $\lambda_z = 1$, and the values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 . It follows that $\hat{W}_{\lambda\lambda}$ is positive and prismatic bifurcation is not possible.	15
2.3	Plot of $\hat{W}_{\lambda\lambda}/\gamma$ against λ associated with the standard fiber model when matrix contribution is negligible ($\mu = 0$) for the values of the fiber winding angle $\beta = 15^\circ, 30^\circ, 45^\circ, 60^\circ$, and 75° . (a) is for $\lambda_z = 1$; (b) is for $\lambda_z = 1.2$. In each curve, the value of λ associated with $\hat{W}_{\lambda\lambda} = 0$ gives the onset of prismatic bifurcation. It follows that for the fiber winding angles $\beta = 15^\circ, 30^\circ$ prismatic bifurcation is not possible since $\hat{W}_{\lambda\lambda}$ is positive.	17
2.4	(a): Plot of $\hat{W}_{\lambda\lambda}/\mu$ against λ for the standard fiber model when $v = 1$, $\beta = \pi/4$, and $\lambda_z = 1.2$ for selected values of the matrix stiffness to fiber stiffness ratio $\mu/\gamma = 0.001, 0.001377$, and 0.002 . (b): Plot of $\hat{W}_{\lambda\lambda}/\mu$ against λ for the standard fiber model when $\mu/\gamma = 0.001$, $\beta = \pi/4$ and $\lambda_z = 1.2$ for selected values of the swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 . Among the given curves in (b), only the one corresponding to $v = 1.2$ does not intersect the horizontal axis. Hence no bifurcation occurs for $v = 1.2$ for specified set of parameter values	18
2.5	The curves show results associated with prismatic bifurcation as given by (2.19) for the standard fiber model with $\mu/\gamma = 0.001$, $\lambda_z = 0.8, 1$, and 1.2 . and selected fiber winding angles. (a) is for $\beta = 45^\circ$; (b) is for $\beta = 60^\circ$; and (c) is for $\beta = 75^\circ$	19
2.6	Plot of $\hat{W}_{\lambda\lambda}/\mu$ against λ when $k_1 = 0.05\mu$, $k_2 = 0.01$, $\beta = \pi/4$ and $\lambda_z = 1$ for selected values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2	20
2.7	Values of g/μ as given by the left hand side of (2.24) against λ when $\lambda_z = 1$ for selected values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 and (a) $\gamma = 0$ (no fibers), and (b) $\gamma = 0.01\mu$ (slightly reinforced matrix) with $\beta = \pi/4$. Unstable configurations are associated with values $g/\mu < 0$	21
2.8	Normalized values of g as given by the left hand side of (3.24) versus λ and λ_z for $\mu = 0$ (negligible matrix stiffness), $v = 1$ and different angles β : (a) $\beta = 0^\circ$; (b) $\beta = 15^\circ$; (c) $\beta = 45^\circ$; (d) $\beta = 90^\circ$	22

2.9	Values of q (normalized using μ) in (2.26) as given by the left hand side of (2.24) against λ when $k_1 = 0.05\mu$, $k_2 = 0.1$, $\beta = \pi/4$ and $\lambda_z = 1$ for selected values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 . Stable configuration obey that $q/\mu < 0$	23
2.10	Values for λ versus λ_z in (a) obey the bending bifurcation condition $q = 0$ as given by (2.26) for the exponential material model with $k_1 = 0.05\mu$, $k_2 = 0.1$, $\beta = \pi/4$ and selected values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 . (b) Values λ versus v obeying the bending bifurcation condition $q = 0$ as given by (2.26) for the exponential material model with $k_1 = 0.05\mu$, $k_2 = 0.1$, $\beta = \pi/4$ and selected values of the axial stretch $\lambda_z = 1, 1.2, 1.4$, and 1.6	24
3.1	A cylindrical membrane. The material consists of a matrix and fibers. The fibers are arranged in two families of parallel fibers with equal mechanical properties, and these fiber families are in a symmetric and helical arrangement that guarantees the cylinder inflation described by equation (3.6) is an admissible deformation. The unit vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ describe the orientation of the fibers from the two families in the reference configuration	29
3.2	Equation (3.20) $f = f_{I_1} + f_{I_2} = 0$ gives the bulging condition for a Mooney-Rivlin material in the absence of fibers. (a) shows different curves of values f/μ_1^2 vs. λ for an axial stretch of $\lambda_z = 1.1$ and different ratios of μ_1 to μ_2 . In both figures, the blue curves show the results for $\mu_2/\mu_1 > 0$, the red curves show the response for $\mu_2/\mu_1 < 0$, and the black curve shows the neo-Hookean response ($\mu_2 = 0$). (b) depicts relations between the circumferential and the axial stretches λ and λ_z , respectively, which fulfill the bulging condition $f_{I_1} + f_{I_2} = 0$ for different ratios of μ_1 to μ_2	34
3.3	Bulging for a fiber-reinforced neo-Hookean material (3.29). (a) shows the function (3.20), which is $f = f_{I_1} + f_{I_4}$ (normalized by the square of the shear modulus μ_1), for an axial stretch $\lambda_z = 1.1$, $\beta = \pi/4$, $k_1 = \mu_1/100$, and for three swelling ratios $v = 1$ (no swelling), $v = 1.1$, and $v = 1.2$. The solid lines represent results for $\bar{k}_1 \rightarrow 0$ and the dashed lines represent results for $\bar{k}_1 = 0.4$. (b) shows different combinations for λ_z and λ that fulfill the condition $f_{I_1} + f_{I_4} = 0$ for $\beta = \pi/4$, $k_1 = \mu_1/100$, and for three swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$. The different line styles represent the results for different values of $\bar{k}_1$	35
3.4	Impact of the fiber orientation on bulging for a fiber-reinforced neo-Hookean material (3.29). (a) shows the function (3.20) $f = f_{I_1} + f_{I_4}$ (normalized by the square of the shear modulus μ_1) for an axial stretch $\lambda_z = 1.1$, $k_1 = \mu_1/100$, $v = 1$, $\bar{k}_1 = 0.4$ and for different values for β on different ranges for f/μ_1^2 . The polar diagram of (b) shows different combinations for β (circumferential orientation) and λ (radial values) that fulfill the condition $f_{I_1} + f_{I_4} = 0$ for $k_1 = \mu_1/100$, $\bar{k}_1 = 0.4$, and $\lambda_z = 1$ for the three swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$	36

3.5	Bulging for a fiber-reinforced neo-Hookean material (3.30).(a) shows the function (3.20) for $f = f_{I_1} + f_{I_5}$ (normalized by μ_1^2) for an axial stretch of $\lambda_z = 1.1$, $\beta = \pi/4$, $k_2 = \mu_1/100$, and for the three swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$. The solid lines represent results for $\bar{k}_2 \rightarrow 0$, and the dashed lines represent the results for $\bar{k}_2 = 0.001$	37
3.6	Impact of fiber orientation on bulging for a fiber-reinforced neo-Hookean material (3.30).(a) shows the function (3.20) for $f = f_{I_1} + f_{I_5}$ (normalized by the square of the shear modulus μ_1) for an axial stretch of $\lambda_z = 1.1$, $k_2 = \mu_1/100$, $v = 1$, $\bar{k}_2 = 0$, and for different values for β . The polar diagram of (b) shows different combinations for β (circumferential orientation) and λ (radial values) that fulfill the condition $f_{I_1} + f_{I_5} = 0$ for $\lambda_z = 1.00$, $k_2 = \mu_1/100$, $\bar{k}_1 = 0.0$, and for the two swelling ratios $v = 1$ and $v = 1.2$	37
3.7	This figure shows combinations for $k_1 \geq 0$ and $k_2 \geq 0$ that fulfill the bulging condition (3.20) for $\lambda = 2$ with f_{I_1} given by (3.21a), $f_{I_2} = 0$, and f_{I_4} and f_{I_5} given by (3.25) and (3.26), respectively, where k_1 and k_2 are normalized by μ_1 . These figures take $\lambda_z = 1.1$ and swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$ for the matrix. (a) is for $\beta = \pi/6$ and (b) is for $\beta = \pi/4$. There is a unique solution under these circumstances. On the other hand, (c) and (d) show solutions for $\beta = \pi/3$	38
3.8	Cylindrical membrane under a constant inner pressure: bulge in equilibrium after bifurcation. Two uniform cylinders are shown, one that is characterized by the pair of azimuthal and axial stretches $(\lambda_{\theta 1}, \lambda_{z 1})$ and other that is characterized by the pair $(\lambda_{\theta 2}, \lambda_{z 2})$. These cylinder are connected by a transition zone between these cylinders.	39
4.1	A cylinder that is subjected to both an inflation pressure P and an axial load N . The material consists of a swellable isotropic ground substance material that is reinforced by two families of fibers.	45
4.2	Bulging bifurcation of a circular cylinder.	49
4.3	Prismatic bifurcation of an initially circular cylinder.	51
4.4	Bending bifurcation of a circular cylinder.	53

- 4.5 Values (normalized) of $f = f_{I_1}$, $g = g_{I_1}$ and $h = h_{I_1}$ vs λ for $\lambda_z = 1$ and a ground substance material (without fibers) with different values of the swelling parameter v . Red curves give values of the bulging function $f = f_{I_1}$, black curves give values of the prismatic function $g = g_{I_1}$ and blue curves give values of the bending function $h = h_{I_1}$. Negative values of f and g and positive values of h are associated with bulging bifurcation, prismatic bifurcation and bending bifurcation, respectively, i.e. with unstable configurations of the initially perfect circular cylindrical membrane. The undeformed und unswollen configuration of the membrane is given by the point $(\lambda, \lambda_z) = (1, 1)$. Close to this configuration for $v = 1$ as well as greater than 1 (swelling) and $\lambda > 1$ (inflation of the membrane), values of h are positive, which means that bending is expected as soon as the membrane begins to be inflated. In addition, it is clear that the values of λ obeying $f = 0$ are also associated with values $h > 0$. This is an indication that while bulging does not give the onset of bifurcation (it is not the lower bound for bifurcation), during postbifurcation (after bending occurs) bulging might be expected. Values of g are positive; therefore, prismatic bifurcation is not expected. Nevertheless, prismatic bifurcation cannot be ruled out completely during post-bifurcation since, among other reasons, the g -based analysis only characterizes the first prismatic bifurcation mode. 55
- 4.6 Values (normalized) of $f = f_{I_1}$, $g = g_{I_1}$ and $h = h_{I_1}$ vs λ for an unswollen ground substance material ($v = 1$) without any fibers and different values of the axial stretch λ_z . Negative values of f and g and positive values of h are associated with bulging bifurcation, prismatic bifurcation and bending bifurcation, respectively. For $\lambda_z = 1$ and the associated value of λ obeying $f = f_{I_1} = 0$, it is clear that $h = h_{I_1} > 0$ and, whence, bending is the lower bifurcation mode. On the other hand, for $\lambda_z = 1.3$ and the associated value of λ obeying $f = f_{I_1} = 0$, it is clear that $h = h_{I_1} < 0$ and it follows that bulging is the lower bifurcation. 56
- 4.7 Points (λ, λ_z) obeying $f = f_{I_1} = 0$ (red curve) and $h = h_{I_1} = 0$ (blue curve) for a ground substance without fibers that remains unswollen $v = 1$. Both curves intersect for $(\lambda, \lambda_z) = (\lambda_{fh}, \lambda_{z,fh}) = (1.57, 1.18)$ so that bulging and bending bifurcation are initiated simultaneously. It follows that for $\lambda_z > 1.18$ the onset of bifurcation is given by bulging while for $\lambda_z < 1.18$ the lower bifurcation mode is bending. 57
- 4.8 Values (normalized) of $f = f_{I_4}$ (red curves), $g = g_{I_4}$ (black curves) and $h = h_{I_4}$ (blue curves) vs λ for different values of the axial stretch λ_z and selected values of the fiber winding angle β . The legend in panel (a) applies to all panels. More in particular, panel (a) is for $\beta = \pi/6$, while its magnification is given in panel (b); panel (c) is for $\beta = \pi/3$, while its magnification is given in panel (d); and panel (e) is for $\beta = \pi/2$. In the latter case, since the fibers are perpendicular to the axial direction, the results are independent of the axial stretch λ_z , so that there are only three curves, one for $f = f_{I_4}$ as well as one for $g = g_{I_4}$ and $h = h_{I_4}$ 64

4.9	Values of $f = f_{I_4}$ (red curves) and $h = h_{I_4}$ (blue curves) as a polar plot with respect to the angle β for different values of λ_z and $\lambda = 1.5$, panel (a) and its magnification panel (b), and $\lambda = 2$, panel (c) is its magnification panel (d). Left panels (a) and (c) show the general development of f/k_1^2 and h/k_1 , whereas right panels (b) and (d) highlight the behavior of these functions close to their zero values. The symmetry of the results is due to the fact that both fiber families are mechanically equivalent and are symmetrically disposed with respect to the axial direction.	65
4.10	For different values of λ_z , the top panels (a), for $\beta = \pi/6$, and (b), for $\beta = \pi/3$, show values of $\bar{k}_1$ vs λ obeying either $f = 0$ (bulging bifurcation, red curves) or $h = 0$ (bending bifurcation, blue curves). The bottom panels (c) and (d) give the corresponding values of λ , λ_z , and $\bar{k}_1$ that fulfill $f = 0$ given in panels (a) and (b), respectively.	66
4.11	Curves give values of circumferential stretch λ (radial value in the plot) and the fiber winding angle β obeying $h = 0$ for different values of the axial stretch λ_z . Panel (b) is a magnification of panel (a). In panel (a), it is easy to deduce that when $h = 0$ for $\beta = \pi/6$ greater values of λ are associated with greater values of λ_z , in agreement with the order shown by the blue curves in Figure 4.10a. Similarly, in panel (b), it is easy to deduce that when $h = 0$ for $\beta = \pi/3$ greater values of λ are not associated with greater values of λ_z , in agreement with the values of λ given by the blue curves shown in Figure 4.10b.	67
4.12	Values of v vs λ obeying either $f = 0$ (red curves in panels (a), (c) and (e)) or $h = 0$ (blue curves in panels (b), (d) and (f)) as well as their associated values of $P^* = PR/(\mu H)$ vs λ (black curves) for $\lambda_z = 1.0$, three selected values of q ($q = \frac{1}{3}$, $q = \frac{2}{3}$, and $q = 1$) and some values of β , in particular, $\beta = \frac{\pi}{6}$ (panels (a), (b)), $\beta = \frac{\pi}{3}$ (panels (c), (d)) and $\beta = \frac{\pi}{2}$ (panels (e), (f)). Results show for $v > 0.5$ that q does not have a qualitative effect on the bifurcation curves but it does have a qualitative effect on pressure curves.	68
4.13	This figure gives simultaneously the bulging and bending bifurcation curves shown in Figure 4.12 and, therefore, establishes accordingly the expected onset of bifurcation for $\lambda_z = 1$. Panels (b), (d) and (f) are magnification of panels (a), (c) and (e), respectively, near the region in which these curves intersect each other.	69
4.14	Values of pressure P^* vs λ that are stable with regard to both bulging and bending, taking $q = 2/3$, $k_1/\mu = 0.1$ and $\bar{k}_1 \rightarrow 0$. Top panels (a), (b) are for $\beta = \pi/6$; middle panels (c), (d) are for $\beta = \pi/3$; and bottom panels (e), (f) are for $\beta = \pi/2$. For the red curves the stable inflation process is interrupted by bulging, whereas for the blue curves the stable inflation is interrupted by bending bifurcation.	70
5.1	Post-bifurcation shape of the bending mode for $\lambda_z = 1$, $\bar{\alpha}_d = -40$, and $\bar{\alpha}_c = 0.5$	79

5.2	Plot of different components of the residual stress at the end of the first step. The panels (a) to (c) make use of $\bar{\alpha}_d = 40$ and $\bar{\alpha}_c = 0.5$; the panels (d) to (f) make us of $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$; the panel (g) makes use of $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = -0.5$. Note that the effect of the second term on the right side of $\sigma_{\theta\theta}$ (see Eqs. (5.9)) is negligible compared with the other term so, we only plot the planar term.	81
5.3	Finite element mesh of the cylinder model in Abaqus	82
5.4	Distribution of the von-Mises stresses in a contour plot of the cross-sectional area near the mid-span of the tube. The stresses are relatively large at the outer and inner walls, with the minimum in between.	82
5.5	Distribution of the axial normal stresses stresses (σ_{zz}) for the same location as in Fig. 5.4. Going from the inner to the outer wall, these axial stresses increase from negative values (compressive stresses) to positive values (tensile stresses).	83
5.6	Shear stresses $\sigma_{\theta RZ}$, which a distributed in an antisymmetric fashion. The largest absolute values can be found near the mean diameters	83
5.7	A curve for the load proportionality factor (LPF) vs. arc length. The maximum in this curve is associated with the initiation of an instability.	84
5.8	The profile of $\mathcal{M} = \sigma_{\theta\theta} - 2\sigma_{zz}$ versus arc length for the models in Case 1, with $\lambda_z = 1$ (solid line) and $\lambda_z = 1.4$ (dashed line), at a cross-section outside the mid-span of the tube, is presented. For the former case, the curve shows $\sigma_{\theta\theta} - 2\sigma_{zz} > 0$, indicating that bending may be the associated bifurcation mode. Conversely, for the latter case, the curve displays $\sigma_{\theta\theta} - 2\sigma_{zz} < 0$, suggesting that bending bifurcation is not expected under these conditions, and bulging is the associated bifurcation mode.	85
5.9	Normalized pressure versus arc length for Case 1 in Table 5.1, with $\lambda_z = 1$, is depicted. The solid line represents the model constrained to capture only the bulging mode, while the dashed line corresponds to the general condition model, which is unrestricted. The point where the two curves diverge is identified as the bending bifurcation point. The maximum points of both curves, beyond which the structure cannot withstand additional load, are designated as the bulging bifurcation points. The maximum point on the dashed line is associated with the formation of a one-sided bulge.	86
5.10	Varying location of the bulge, depending on the specifications for the residual stress parameters.	87
5.11	Normalized inflation pressure versus axial stretches at the bifurcation point. The curves represent all nine cases listed in Table 5.1.	88
5.12	The normalized inflation pressure versus hoop stretch λ_θ is plotted for $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$, comparing two locations on the tube: one point located inside the bulge and another point outside the bulge. The different curves correspond to varying axial stretches: $\lambda_z = 1.0$ ("1.00-OUT", "1.00-IN"), $\lambda_z = 1.2$ ("1.20-OUT", "1.20-IN"), $\lambda_z = 1.4$ ("1.40-OUT", "1.40-IN"), $\lambda_z = 1.6$ ("1.60-OUT", "1.60-IN"), and $\lambda_z = 1.8$ ("1.80-OUT", "1.80-IN").	89
5.13	Effect of increasing the axial stretches on the distributions of stress components for a bulging instability.	91
5.14	Different bulge locations of the cases with different residual stress parameters.	92

5.15	Different bulge locations of the cases with different residual stress parameters	93
5.16	Normalized inflation pressure versus axial stretches at the bifurcation point for selected residual stress distributions.	94
5.17	Normalized inflation pressure at bifurcation is plotted against axial stretch for the case where $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$. In this figure, for sufficiently small values of λ_z , three distinct curves are present. The solid line represents the pure bulging mode, which corresponds to zero azimuthal displacements. The dash-dotted and dotted lines correspond to the bending bifurcation and the one-sided bulge under general displacements (i.e., without any imposed constraints), respectively. As λ_z increases, the bending and one-sided bulge modes cease to occur.	95
5.18	Post-bifurcation behavior associated with bulging is presented for all nine combinations of $\bar{\alpha}_d$ and $\bar{\alpha}_c$ listed in Table 5.1, with an axial stretch of $\lambda_z = 1.2$. It can be observed that both the magnitude and the location of the instability vary depending on the different distributions of residual stresses.	96
5.19	Post-bifurcations associated with bending, taking the parameters $\bar{\alpha}_d = -40$, $\bar{\alpha}_c = 0.5$ ("Case 3") for the distribution of the residual stresses. The panels are for different axial stretch values, which correspond to the dash-dotted lines in Fig. 5.11.	99
5.20	Illustrated example of the formation history of a bulge, starting from its initiation in panel (a) until the final configuration in panel (d) after its radial propagation, taking the parameters $\bar{\alpha}_d = -40$, $\bar{\alpha}_c = 0.5$ for the residual stresses ("Case 3") and an axial stretch of $\lambda_z = 1.00$	100
5.21	Illustrated example of the formation history of a bending instability, starting from its initiation in panel (a) until the final configuration in panel (e), taking the parameters $\bar{\alpha}_d = -40$, $\bar{\alpha}_c = 0.5$ for the residual stresses ("Case 3") and an axial stretch of $\lambda_z = 1.00$. In the final configuration, one can observe a bulge that has propagated radially.	100
5.22	Illustrated example of the formation history of a bending instability, starting from its initiation in panel (a) until the final configuration in panel (e), taking the parameters $\bar{\alpha}_d = -40$, $\bar{\alpha}_c = 0.5$ for the residual stresses ("Case 3") and an axial stretch of $\lambda_z = 1.18$. In the final configuration, one can observe a one-sided bulge that has propagated radially.	101

List of Tables

2.1	Values of maximum axial stretch λ_{zmax} associated with different values of the fiber winding angle β for the fiber only material model (no matrix). For a given fiber winding angle β values of the axial stretch less than the associated maximum axial stretch λ_{zmax} give rise to prismatic bifurcation	16
3.1	Panel (a): Axial propagation of bulging under swelling for the Mooney Rivlin material (3.28). The table shows the pairs $(\lambda_{\theta 1}, \lambda_{z 1})$ and $(\lambda_{\theta 2}, \lambda_{z 2})$ for $\lambda_{z 1} = 1.1$ and different values for the swelling v that are determined from system of equation (3.31). The results are shown for $\mu_2 = 0$ (neo-Hookean specialization) and for $\mu_2 = -\mu_1/2$. Panel (b) studies axial propagation of bulging under swelling for a neo-Hookean matrix and fiber behavior in terms of I_4 (3.29). It shows the pairs $(\lambda_{\theta 1}, \lambda_{z 1})$ and $(\lambda_{\theta 2}, \lambda_{z 2})$ for $\lambda_{z 1} = 1.1$, for the material parameters $k_1 = 0.05\mu_1$ and $\bar{k}_1 = 0.1$, and for different values of the swelling v that are determined from system of equation (3.31). The results are consistent with those that have been obtained by Demirkoparan & Merodio (Demirkoparan and Merodio, 2020) and Alhayani et al. (Alhayani et al., 2014a). Panel (c) studies axial propagation of bulging under swelling for a neo-Hookean matrix and fiber behavior in terms of I_5 (3.30). It shows the pairs $(\lambda_{\theta 1}, \lambda_{z 1})$ and $(\lambda_{\theta 2}, \lambda_{z 2})$ for $\lambda_{z 1} = 1.1$, for the material parameters $k_2 = 0.01\mu_1$ and $\bar{k}_2 = 0.01$, and for different values of the swelling v that are determined from system of equation (3.31).	40
5.1	The combinations of parameters $\bar{\alpha}_c$ and $\bar{\alpha}_d$ define the magnitude and distribution of the residual stresses in (5.9). Case 7 corresponds to purely neo-Hookean material behavior, with no residual stresses present in the material.	98

Abbreviations and acronyms

UPM Universidad Politécnica de Madrid

Chapter 1

Introduction

1.1 Residual Stresses in Biological Systems: A Thorough Examination

The human cardiovascular system plays a key role in circulating oxygen and nutrients to the entire body (Xu and Shi, [2014](#); Taylor and Humphrey, [2009](#)), enabled by an intricate network of blood vessels. The blood vessel wall tissue consists of three distinct layers: intima, media, and adventitia (Camasão and Mantovani, [2021](#); Chen and Kassab, [2016](#); Lu et al., [2017](#)). These layers work together to provide mechanical strength to the vessel wall (Widmaier et al., [2022](#); Holzapfel and Ogden, [2018](#)), allowing it to withstand diverse physiological conditions. Central to the functionality of these biological structures is their ability to accommodate and respond to residual stress.

Residual stress is the stress retained within a material or structure even in the absence of external loads and naturally emerges in biological tissues like blood vessels during growth and development processes (Y.-c. Fung, [2013](#); Holzapfel and Ogden, [2003](#); Merodio and Ogden, [2016](#); Merodio et al., [2013](#); Shariff et al., [2017a](#); Valverde-González et al., [2023](#)). This stress is pivotal in upholding the structural integrity and optimal functionality of vessel walls. In blood vessels, residual stress aids in homogenizing stress distribution across the vessel wall (Chuong and Fung, [1986](#); Y. C. Fung, [1991](#)), a critical factor in averting mechanical failure and maintaining efficient blood flow.

In this context, research related to the present problem investigates the inflation of cylindrical structures, with particular emphasis on the effects of residual stresses within the material. Notably, studies such as (N. Jha, Reinoso, et al., [2019](#); Shariff et al., [2020](#); Bustamante and Merodio, [2011](#); Shariff et al., [2018](#); Ramírez-Torres et al., [2019](#)) have concentrated on developing constitutive models that account for bifurcation phenomena, incorporating considerations such as composites, viscoelasticity and the influence of electric fields.

Understanding the behavior of residually-stressed biological materials is especially significant in the study of pathological conditions such as aneurysms (Gasser, [2021](#); Alkhatib et al., [2024](#); Polzer et al., [2013](#)). An aneurysm is a localized abnormal bulge that occurs in blood

vessels in which the diameters become larger than the normal ones (Johnston et al., 1991; Jeong and Rhee, 2012; Alkhatib et al., 2024). It happens because of many factors that undermine the vascular wall. These include among others inflammations, hypertension, and hereditary (Nordon et al., 2011; Pinard et al., 2019). The deterioration of the wall exacerbates appreciably as a result of decreasing in collagen and elastin levels due to enzymatic activity, which leads to degeneration of these proteins (elastin and collagen are the two important proteins that strengthen the artery wall, which work as fiber-reinforcement materials, and provide it with elasticity). A change in the mechanical pattern of collagen fibers can lead to aneurysm formation, in other word bulging bifurcation (Rodríguez and Merodio, 2011). Consequently, this will result in a wall rupture, and in severe cases the death is inevitable consequence

1.2 Hyperelastic Materials and Non-Linear Elasticity

In modeling the behavior of blood vessels, hyperelastic material models are usually utilized (Holzapfel et al., 2000; Demirkoparan and Merodio, 2017; N.-K. Jha et al., 2019; Valverde-González et al., 2024) due to their ability to capture the nonlinear and large deformation properties of biological tissues. Hyperelastic materials are defined by a strain energy function, which governs the stress-strain relationship (see Puglisi and Saccomandi, 2015; Shariff et al., 2024; Topol, Nazari, et al., 2024; Moradalizadeh et al., 2024; Topol, Font, et al., 2024 for instance and references therein). Understanding this relationship is crucial for predicting material deformation under different loading conditions.

Blood vessels can be approximated as residually-stressed hollow thin/thick-walled structures (see Moradalizadeh et al., 2025; Al-Chlaihawi et al., 2025; Desena-Galarza et al., 2021; Asghari et al., 2023b; Topol, Asghari, et al., 2023; Melnikov et al., 2021; Shariff and Merodio, 2021; N. Jha, Merodio, and Reinoso, 2019; Shariff and Merodio, 2020 for example). This approximation simplifies the complex geometry and material behavior of the vessel walls, making it feasible to analyze and predict their mechanical response. When considering the combined effects of axial stretch and internal pressure, it is essential to develop precise models that account for residual stress, hyperelasticity, and fiber reinforcement (Merodio and Ogden, 2006a; Merodio, 2006; Merodio and Goicolea, 2007; El Hamdaoui et al., 2020; El Hamdaoui et al., 2015) to accurately represent the material's behavior.

Regarding the mathematical formulation of the current problem, remarkable studies over the past years have developed formulations to capture mechanical behaviors, considering many aspects of constitutive models (see Merodio and Ogden, 2003b; Ramírez-Torres et al., 2018; Shariff et al., 2021; P. Vinh and Merodio, 2013; Hamdaoui and Merodio, 2015; Shariff et al., 2017b; Shariff and Merodio, 2020; Merodio and Ogden, 2020a; Bustamante and Merodio, 2012; Topol, Al-Chlaihawi, et al., 2023; Soldatos et al., 2021; Merodio and Ogden, 2006b; Bustamante and Merodio, 2010 for example)

The study of non-linear elastic constitutive models is significant for understanding the mechanical behavior of materials that exhibit a non-linear stress-strain relationship (Bechtel and Lowe, 2014; Shariff et al., 2020; Merodio and Ogden, 2020; Merodio and Ogden, 2020b; Merodio and Ogden, 2005b; Merodio and Ogden, 2005a; Merodio and Ogden, 2002; Merodio

and Ogden, 2005c; Shariff et al., 2023). The deformation and stability of cylindrical tubular structures made from rubber-like materials under various conditions have been investigated within instability theory (Merodio and Ogden, 2023; Doblaré and Merodio, 2015; Merodio and Neff, 2006). Isotropic/anisotropic incompressible elastic materials (Merodio and Saccomandi, 2006; Merodio et al., 2007; Nam et al., 2016; P. C. Vinh et al., 2014; Merodio and Rajagopal, 2007; Dehghani et al., 2019; Rodríguez and Merodio, 2016; Kassianidis et al., 2008) have been used to capture the bifurcation and post-bifurcation of thin/thick-walled tubes separately undergoing residual stresses, swelling effects and finite torsion under the action of axial stretches and internal pressures. Several bifurcation modes can be identified, namely prismatic, pure bulging, and bulging after bending, each leading to different deformation patterns. These phenomena are extensively studied in the context of non-linear elasticity (Merodio and Saccomandi, 2011a; Merodio and Saccomandi, 2011b; Merodio and Saccomandi, 2011c; Merodio and Rosato, 2012; Doblare and Merodio, 2015).

1.3 Motivation

The main goal of this work is to enhance the understanding of the mechanical behavior of hyperelastic tubes, representing blood vessels, under the combined effects of axial stretch, internal pressure, residual stresses, and swelling. This study is particularly relevant to the biomechanics of aneurysm formation and progression, providing insights into the stability and failure mechanisms of arterial walls. This understanding is significant for several reasons:

Predicting Aneurysm Formation and Growth

Aneurysms are influenced by the mechanical properties of the vessel wall tissue. By understanding the interplay between residual stress and applied loads, it can more accurately predict the conditions under which aneurysms can initiate, develop, and expand.

Developing Medical Equipment

In medical practice, interventions frequently rely on devices like stents and grafts to treat aneurysms. A thorough grasp of the mechanical dynamics within the vessel wall is essential in shaping the design of these devices. This understanding guarantees that they offer sufficient support without worsening any existing damage.

Enhancing Surgical Results

Gaining insight into the mechanical behavior of blood vessels can inform surgical approaches, enabling the implementation of more precise and efficient treatments for aneurysms and other vascular conditions.

Improving Computational Models

Developing accurate computational models of blood vessels can help simulate various pathological and physiological conditions. These models can be employed for diagnosis, surgical preparation, and the formulation of innovative therapeutic approaches.

1.4 Objectives

The principal aims of this thesis are:

- To study the interplay of swelling, fiber orientation, and the mechanical properties of the constituents on the initiation as well as on the axial propagation of bulging in a thin-walled hollow cylinder, which is composed of a fibrous and swellable hyperelastic material. The fibers are arranged in two families, taken to be parallel within each fiber family, and are also assumed to be mechanically equivalent and symmetrically disposed in the ground substance material.
- To examine the influence of swelling on prismatic and bending bifurcation modes of inflated thin-walled cylinders under axial loading, where orthotropic materials with fiber reinforcements are used. In addition, to develop the bifurcation criteria for a membrane cylinder subjected to combined axial loading, internal pressure, and swelling. Moreover, to examine how axial stretch, fiber strength, and orientation influence bifurcation onset, aiding in the modeling of normal and diseased aortic tissue.
- To analyze bifurcation for a cylindrical membrane capable of swelling subjected to combined axial loading and internal pressure, where the material is conceptualized as isotropic and absorbent matrix and fibers are symmetrically arranged and mechanically equivalent. Also, to investigate prismatic, bulging, and bending bifurcation conditions in fiber-reinforced and non-reinforced membranes.
- To develop a new approach to assess material behavior during bifurcation instability in the presence of residual stresses, especially non-planar stresses, in thick-walled isotropic hyperelastic circular hollow tubes subjected to axial stretches and internal pressure.

1.5 Thesis Schema

This thesis is outlined as follows:

Chapter 2: This chapter investigates the prismatic and bending bifurcations of fiber-reinforced elastic membranes under swelling with application to aortic aneurysms.

Chapter 3: This chapter presents Bulging initiation and propagation in fiber-reinforced swellable Mooney–Rivlin membranes.

Chapter 4: This chapter describes the Bifurcation of Fiber-Reinforced Cylindrical Membranes under Extension, Inflation, and Swelling.

Chapter 5: This chapter exhibits the Computational modeling of a residually stressed thick-walled cylinder under combined action of axial extension and inflation.

Chapter 6: This chapter recaps the principal findings derived from the research conducted in this thesis., discusses the implications of the results, and suggests directions for future research.

1.6 Summary

This thesis provides a comprehensive understanding and in-depth analysis of the mechanical behavior of hyperelastic cylindrical tubes in biomechanics, with a specific application to aneurysms within the context of instability theory. It focuses on anisotropic, thin-walled tubes reinforced with fibers capable of swelling and isotropic, thick-walled tubes undergoing residual stresses, subjected to the combined effects of axial stretches and internal pressures. A thorough investigation of the bifurcation mechanism in three forms—prismatic, bulging, and bending—offers significant insights into the factors influencing aneurysm formation and propagation. The findings of this research have profound implications for predicting and treating aneurysms, with the ultimate goal of enriching biological systems research by enhancing finite element modeling techniques and investigating novel material models.

Chapter 2

On prismatic and bending bifurcations of fiber-reinforced elastic membranes under swelling with application to aortic aneurysms

“This chapter is based on the below article that is published corresponding this thesis: Al-Chlahawi, M. J., Topol, H., Demirkoparan, H., Merodio, J. (2023). On prismatic and bending bifurcations of fiber-reinforced elastic membranes under swelling with application to aortic aneurysms. *Mathematics and Mechanics of Solids* 28 (1), 108-123(Al-Chlahawi et al., [2023](#))

2.1 Introduction

Bulging bifurcation in pressurized tubes is a nonlinear phenomenon that has attracted a lot of attention recently since it has been related to the formation of abdominal aortic aneurysms. This pathology is often asymptomatic until rupture occurs and it has a mortality rate greater than 80% (Schermerhorn, [2009](#)). The analysis of other bifurcation modes has not received much attention in the context of biomechanics because only very recently has it been recognized that different bifurcation modes can be coupled (see Dehghani et al., [2019](#)) to understand modeling of aortic aneurysms. The coupling analysis of different instability modes involves the study of postbifurcation. The modes that were found to be coupled in the aortic aneurysms pathology are bending and bulging. With this in mind, we dedicate our effort here to analyze prismatic and bending bifurcations of swellable membranes. We pay attention to the onset of these bifurcation modes, a preliminary analysis necessary for further research into the modeling of cardiovascular diseases. Indeed, modeling and research of bulging may help to understand the formation of aortic aneurysms (permanent, irreversible, and localized dilatation of blood vessels), which are responsible for 1–2% of all deaths in industrialized countries (Lindsay and Dietz, [2011](#)).

Bifurcation from a membrane tube that is subjected to both axial loading and internal pressure

has been studied in (Dehghani et al., 2019; Biot, 1964; Haughton and Ogden, 1979; Kyriakides and Yu-Chung, 1991; Haughton and Merodio, 2009; Humphrey et al., 2009; R. W. Ogden, 2009; Merodio and Haughton, 2010; Rodríguez and Merodio, 2011; Alhayani et al., 2013; El Hamdaoui et al., 2014; El Hamdaoui and Merodio, 2015; Fu et al., 2016; Font et al., 2021), and the investigation of this problem is important in order to gain a deeper understanding of the dilatation of arteries. Nevertheless, it has to be pointed out that localized bulging can also occur if there is no pressure limit for tubes with open ends (Fu et al., 2008) such as those that are under study in this work. In addition to the inner pressure and the axial loading, the material may be subjected to changes in its volume due to swelling. In biological soft tissue, swelling is related to different factors such as edema, mechanical trauma, and inflammation (Barzó et al., 1997; D’lima et al., 2001. Tracey, 2002). Swelling of cardiovascular tissues has been treated in different articles (Lanir, 2012; Sorrentino et al., 2015). Bulging bifurcation of inflated circular cylinders of doubly fiber reinforced hyperelastic material under axial loading and swelling has been studied in (Demirkoparan and Merodio, 2017; Demirkoparan and Merodio, 2020). This work captured two interesting observations. The first observation is that swelling may lead to compressive hoop stresses near the inner radius of the tube, which could have potential benefits with respect to failure prevention. The second observation is that, compared with unswollen material, swelling involves greater values of the azimuthal stretch to reach bifurcation. These two observations may have implications in the formation of aneurysms in arterial wall tissue. In (Topol, Al-Chlahawi, et al., 2021) these considerations were continued and the authors also explored how swelling, the fiber orientation within the membrane wall, and the mechanical properties of the constituents are involved in the initiation of bulging as well as on its axial propagation. Indeed, bulging bifurcation has been studied in different contexts, but the same is not true for bending and prismatic bifurcation modes. For that reason, we pay attention to the existence of those instability modes in the context of soft tissue modeling, which may open further research in the analysis of cardiovascular diseases. A turning point in the surface tension in the azimuthal direction is a necessary condition for the formation of prismatic bifurcations (Haughton and Ogden, 1979). Prismatic bifurcation analyses that are based upon the application of small deformations, which are superimposed on finite deformation, have been studied for both cylindrical membranes (Haughton and Ogden, 1979) and for thick-walled tubes in (Haughton and Ogden, 1979). The different bifurcation modes, including prismatic and bending bifurcation, have been investigated in the modeling of arterial tissue that is affected by Marfan’s syndrome, a connective tissue disorder, in which the affected arterial wall shows a weakening behavior (Haughton and Merodio, 2009; Merodio and Haughton, 2010). The impact of residual stresses on the bifurcation of cylinders has been studied in (L. Dorfmann and Ogden, 2021; Melnikov et al., 2021). Although the focus of this work lies on the modeling of soft tissue behavior, the examination of prismatic bifurcation of hyperelastic tubes finds its use in different industrial applications. For example, the prismatic bifurcations of soft dielectric tubes has been studied in (Melnikov et al., 2021). Axial stress is reported to play a fundamental role in the compensatory adaptations by arteries (Humphrey et al., 2009). The normal and pathological functions of arteries can be captured by identifying the range of axial stretch values in healthy arteries. The azimuthal stretch has not attracted so much attention and there is no data regarding arterial wall mechanical modeling as opposed to axial stretch data. It is clear that azimuthal stretch in an artery is related to the axial stretch and the inflating pressure through the loading and the deformation

of the artery. In this work we show that both axial stretch and azimuthal stretch are coupled to establish the deformation conditions triggering the instabilities at hand. Furthermore, it is established here that greater values of swelling may prevent prismatic bifurcations. On the other hand, swelling triggers bending instabilities. The effect of the anisotropy depends on the reinforcing model together with the strength of the reinforcement as well as the deformation. One can conclude that the standard model promotes prismatic bifurcation and may prevent bending bifurcation while the exponential model does not allow prismatic bifurcation and may prevent bending bifurcation. This article is organized as follows. Section 2 presents the governing equations that describe the inflation of a thin-walled cylinder that is additionally subjected to both axial loading and swelling. The prismatic and bending bifurcation criteria for a membrane cylinder subjected to combined axial loading, internal pressure, and swelling is provided. Section 3 introduces two material models that define the mechanical behavior of the membrane. In both models, the mechanical behavior of the matrix is described by a swellable neo-Hookean model. The mechanical behavior of the fibers is given in a relatively standard form in the first model, which represents a special case for many fibers models that have been derived in order to describe the mechanical behavior of collagen fibers in soft tissue. In the second model, the strain energy density of the fibers has an exponential form, which found its popularity in the modeling of collagen fibers, especially in the context of arterial soft tissue. Prismatic bifurcation is studied in Section 4, and the bending mode of bifurcation is explored in Section 5. The final Section 6 gives a brief conclusion on the results.

2.2 Basic equations

2.2.1 The boundary value problem

Let us consider a circular cylindrical membrane under the action of combined axial loading and internal pressure (Haughton and Ogden, 1979). This section specializes these considerations for the case that in addition to axial loading and internal pressure, swelling is also included. Let the middle surface of a circular cylindrical membrane in its undeformed reference configuration be described by

$$\mathbf{X} = R\mathbf{E}_R(\Theta) + Z\mathbf{E}_Z, \quad 0 \leq \Theta \leq 2\pi, \quad -L/2 \leq Z \leq L/2 \quad (2.1)$$

where (R, Θ, Z) are cylindrical polar coordinates and $\mathbf{E}_R$, $\mathbf{E}_\Theta$ and $\mathbf{E}_Z$ are unit vectors in the indicated directions. The length L of the cylinder may be finite or infinite.

The unswollen membrane cylinder is exposed to a uniform swelling field and as a result local volume is changed by an amount v compared to its original volume. The material is taken to be incompressible at each stage of the swelling process, so that any change in the material volume results from a change in v ,

$$\det \mathbf{F} = v \quad (2.2)$$

In general, swelling may vary with the location in a tissue. Nevertheless, we assume that the difference in the amount of swelling throughout the tube are small, and we take swelling to

be uniform at each point of the tube. If the membrane remains unswollen, then $v = 1$. When local volume is increased, for example by some sort of diffusion mechanism or inflammatory responses, then $v > 1$ (swelling). If the local volume decreases due to diffusion of internal ingredients to outside of the membrane, then it will be the case that $0 < v < 1$ (deswelling). The swollen membrane is axially stretched by the axial force N , and it is inflated by the internal pressure p . The deformed configuration is described by

$$\mathbf{x} = r\mathbf{e}_r(\theta) + z\mathbf{e}_z, \quad a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad -\ell/2 \leq z \leq \ell/2 \quad (2.3)$$

where (r, θ, z) are cylindrical polar coordinates and $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are the unit vectors in a polar description of the deformed configuration. The deformed length of the cylinder is ℓ . The displacement u of a point is given by $u = x - X$. Referred to cylindrical coordinates, the associated deformation gradient tensor $\mathbf{F}$ has the components

$$\mathbf{F} = \text{diag}(v\lambda^{-1}\lambda_z^{-1}, \lambda, \lambda_z) \quad (2.4)$$

where $\lambda = \lambda_\theta = r/R > 0$ is the azimuthal principal stretch and $\lambda_z = \ell/L$ is the axial stretch. The incompressibility condition (2.2) for the swollen material implies that the radial principal stretch has the form $\lambda_r = v\lambda^{-1}\lambda_z^{-1}$. For the diagonal deformation gradient tensor $\mathbf{F}$ in (2.4) the usual right and left Cauchy–Green deformation tensors $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ are

$$\mathbf{C} = \mathbf{B} = \frac{v^2}{\lambda^2\lambda_z^2}\mathbf{e}_r \otimes \mathbf{e}_r + \lambda^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_z^2\mathbf{e}_z \otimes \mathbf{e}_z \quad (2.5)$$

For this deformation tensor (2.5), the invariant I_1 of $\mathbf{C}$ (or $\mathbf{B}$) has the form

$$\mathbf{I}_1 = \text{tr}\mathbf{C} = \frac{v^2}{\lambda^2\lambda_z^2} + \lambda^2 + \lambda_z^2 \quad (2.6)$$

We consider swellable membranes for which the principal Cauchy stresses σ_{ii} take the following form for the given deformation,

$$\sigma_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i} - q, \quad (\text{no summation}) \quad i = r, \theta, z, \quad (2.7)$$

where $W = W(v\lambda^{-1}\lambda_z^{-1}, \lambda, \lambda_z)$ is the strain energy function and q is an arbitrary stress value associated with the constraint $\det\mathbf{F} = v$. Such an equation is valid for materials that permit a symmetric inflation of the tube, including the herein considered case of fiber-reinforced materials with two preferred directions which are mechanically equivalent and are symmetrically disposed.

Using the membrane approximation $\sigma_{rr} = 0$, we can write

$$\sigma_{\theta\theta} = \lambda \frac{\partial \hat{W}}{\partial \lambda}, \quad \sigma_{zz} = \lambda_z \frac{\partial \hat{W}}{\partial \lambda_z}, \quad (2.8)$$

where the notation $\hat{W}(\lambda, \lambda_z) = W(\lambda^{-1}\lambda_z^{-1}, \lambda, \lambda_z)$ is adopted.

The relation between the inflating pressure and the mechanical behavior of the membrane can be presented in the form (R. W. Ogden, 1997)

$$p = \frac{H\hat{W}_\lambda}{R\lambda\lambda_z}, \quad (2.9)$$

where H is the undeformed thickness of the membrane (h in the deformed configuration), R is the midsurface radius of the cylinder, and $\hat{W}_\lambda = \partial\hat{W}/\partial\lambda$; similarly, we will use the notation $\hat{W}_{\lambda_z} = \partial\hat{W}/\partial\lambda_z$, $\hat{W}_{\lambda\lambda} = \partial^2\hat{W}/\partial\lambda^2$, etc for the different derivatives of $\hat{W}$. The axial load N per unit area of the undeformed cross-section of the membrane material is given by

$$N = \frac{rh\sigma_{zz}}{RH} = \hat{W}_{\lambda_z}. \quad (2.10)$$

2.2.2 Prismatic and bending bifurcation criteria

Using incremental displacements the bifurcation criterion for incompressible cylinders can be found in (Rodríguez and Merodio, 2011). Their result is in agreement with the analysis of Haughton and Ogden (Haughton and Ogden, 1979) and in recent papers the bulging condition has been used to analyze axial propagation of bulging for thin-walled cylinders under internal pressurization (Alhayani et al., 2014a). Now, swelling is considered and bifurcation conditions need to be obtained.

To investigate the possible bifurcation modes of the cylinder we consider incremental displacements with respect to the deformed configuration (under equilibrium), of the general form

$$\delta\mathbf{u} = \delta u_r(\theta, z)\mathbf{e}_r + \delta u_\theta(\theta, z)\mathbf{e}_\theta + \delta u_z(\theta, z)\mathbf{e}_z, \quad (2.11)$$

where δ is the increment.

Firstly we consider the possibility of prismatic bifurcation modes (see schematic representation of different bifurcation modes in Figure 2.2), for which the inflated cylinder bifurcates into a prismatic shape with a non-circular cross-section. In this case we consider incremental displacements of the form

$$\delta\mathbf{u} = \delta u_r(\theta)\mathbf{e}_r + \delta u_\theta(\theta)\mathbf{e}_\theta.$$

Following (Rodríguez and Merodio, 2011), after some manipulations, it is easy to obtain for a membrane cylinder that the bifurcation condition for the lowest (most likely) mode is given by

$$\hat{W}_{\lambda\lambda} = 0. \quad (2.12)$$

For the bending bifurcation mode it is not necessary to simplify $\delta \mathbf{u}$ any further. Again, following (Rodríguez and Merodio, 2011), after some manipulations, it is easy to obtain that the bending bifurcation condition (composite mode) is

$$\sigma_{\theta\theta} - 2\sigma_{zz} = 0. \quad (2.13)$$

It follows that if $\sigma_{\theta\theta} - 2\sigma_{zz} < 0$ then the membrane is stable while if $\sigma_{\theta\theta} - 2\sigma_{zz} > 0$ the membrane is unstable.

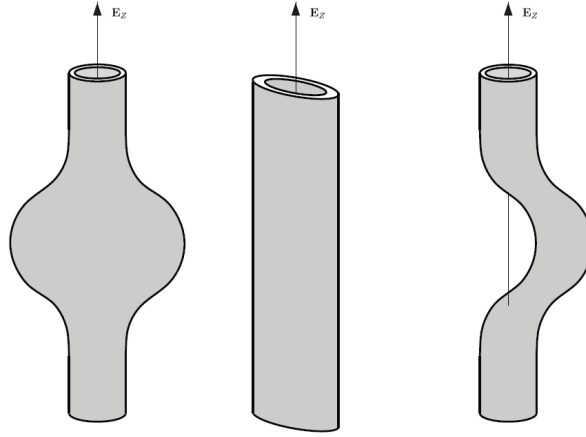


Figure 2.1: Schematic representation of the three bifurcation modes for a membrane cylinder, bulging bifurcation (left), prismatic bifurcation (center), and bending bifurcation (right).

For completeness, we mention that the bulging bifurcation criterion for a cylindrical membrane of radius R and finite or infinite length was derived for swelling materials in (Demirkoparan and Merodio, 2017). It follows that the bifurcation conditions for swellable materials are precisely the conditions given in (Rodríguez and Merodio, 2011) for incompressible materials, and for that reason we do not include further details. Nevertheless, it should be remarked that the bifurcation conditions given here include swelling in the strain energy function and its derivatives as opposed to the conditions obtained in (Rodríguez and Merodio, 2011). For instance, the swelling parameter v enters into the formulation as an argument of the principal Cauchy stresses σ_{ii} in (2.13).

2.3 Material Models

Multiple material models have been derived that account for fibrous soft tissue (Chagnon et al., 2015). To investigate the possible prismatic and bending bifurcations of the swellable cylinder we consider two material models. Both models consider two families of fibers in a swellable neo-Hookean type matrix. The fibers cannot swell and they are in a symmetric arrangement with respect to the circumferential and axial orientation of the membrane. The fibers are considered to be oriented perpendicularly to the radial direction of the membrane.

In the undeformed configuration, the fiber orientation of the two fiber families are defined by the two unit vectors

$$\mathbf{M}_1 = \cos \beta \mathbf{e}_Z + \sin \beta \mathbf{e}_\Theta \quad (2.14a)$$

$$\mathbf{M}_2 = \cos \beta \mathbf{e}_Z - \sin \beta \mathbf{e}_\Theta \quad (2.14b)$$

where the fiber winding angle $0 < \beta \leq \pi/2$ describes the fiber orientation in the membrane relative to the axial direction of the cylinder. These unit vectors are mapped into the deformed configuration via

$$\mathbf{m}_1 = \mathbf{F}\mathbf{M}_1 = \lambda_z \cos \beta \mathbf{e}_Z + \lambda \sin \beta \mathbf{e}_\Theta \quad (2.15a)$$

$$\mathbf{m}_2 = \mathbf{F}\mathbf{M}_2 = \lambda_z \cos \beta \mathbf{e}_Z - \lambda \sin \beta \mathbf{e}_\Theta \quad (2.15b)$$

Then because of the equal mechanical properties of the fibers and their symmetric arrangements, the invariants $I_4 = \mathbf{m}_1 \cdot \mathbf{m}_1$ and $I_6 = \mathbf{m}_2 \cdot \mathbf{m}_2$ are equal and given by

$$I_4 = I_6 = \lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta. \quad (2.16)$$

The first model that is considered will be referred to as the "*standard model*," and it has the form

$$W_1(I_1, I_4, I_6, v) = \frac{\mu}{2}(I_1 - 3v^{2/3}) + \frac{\gamma}{2}(I_4 - 1)^2 + \frac{\gamma}{2}(I_6 - 1)^2, \quad (2.17)$$

where μ and γ are material constants. The first term on the right-hand side of (2.17) accounts for the mechanical behavior of the matrix. It is a function of the first invariant I_1 of the left or right Cauchy–Green deformation tensor and the swelling parameter v . This term is called the Treloar model (Treloar, 1975) and it is a simple generalization of neo-Hookean model to include swelling (Demirkoparan and Pence, 2007b; Demirkoparan and Pence, 2008). The second and third terms on the right-hand side of (2.17) represent the reinforcing terms of the standard model (Demirkoparan and Pence, 2007b; A. Dorfmann et al., 2010; Merodio and Ogden, 2006a; Qiu and Pence, 1997; Zidi et al., 1999). These terms are functions of the invariants I_4 and I_6 , which can be interpreted as the squared magnitude of the fiber stretch of the respective fiber families. The parameter γ can be thought of as fiber stiffness parameter.

The second model to be analyzed is referred to as the "*exponential model*," and it is given by

$$W_2(I_1, I_4, I_6, v) = \frac{\mu}{2}(I_1 - 3v^{2/3}) + \frac{k_1}{2k_2} \sum_{i=4,6} \left(\exp \left[k_2(I_i - 1)^2 \right] - 1 \right), \quad (2.18)$$

where k_1 is a material parameter with units of stress, and k_2 is a dimensionless parameter. Both parameters are positive. For $v = 1$, model (18) has been applied in different modeling works

for collagenous soft tissue in arteries (Holzapfel et al., 2002). The well-known exponential form for the mechanical response of the fibers in (2.18) is motivated by crimped fibers that resist loading especially at large fiber extensions (Gasser et al., 2006; Holzapfel and Ogden, 2010a). The form (2.18) has been applied in (Topol, Al-Chlaihawi, et al., 2021) in order to explore bulging initiation and propagation in swellable and fiber-reinforced membranes. Swelling of the matrix may result in a change of both the ground substance itself as well as the constituents that are embedded in this ground substance. The shear modulus μ is taken to be independent of swelling in (2.17), whereas some works (Gou and Pence, 2016; Zamani et al., 2018) account for a change in the shear modulus with swelling. The swelling of the matrix may also change the mechanical properties of the fibers. For example, a volume change may mediate the remodeling of collagen fibers (Gou et al., 2020; Topol et al., 2018). Such consideration are beyond the scope of the present article and we assume that the mechanical properties of the matrix and the fibers remain independent from swelling. We also mention that although herein the applied forms for the invariant I_4 and I_6 in (2.16) are based on the assumption that all material constituents share a common natural configuration, in (Topol et al., 2019) these invariants were modified in order to account for natural fiber configurations that evolve during the inflation of a cylinder.

2.4 Prismatic mode

This section analyzes the prismatic mode of bifurcation (2.12) for both the standard model (2.17) and the exponential model (2.18).

2.4.1 Prismatic bifurcation of the standard model

The prismatic bifurcation condition (2.12) can be written as

$$\hat{W}_{\lambda\lambda} = \mu \left(1 + \frac{3v^2}{\lambda^4 \lambda_z^2} \right) + \gamma \left(-4 \sin^2 \beta + 12 \lambda^2 \sin^4 \beta + \lambda_z^2 \sin^2(2\beta) \right) = 0, \quad (2.19)$$

for the standard material model (2.17) by means of (2.6) and (2.16). For the set of material parameters, μ , γ , the angle β and the prescribed uniform swelling v equation (2.19) is equivalent to a fourth-order polynomial in λ_z and a sixth-order polynomial in λ .

To investigate the existence of prismatic bifurcation, it is instructive to start with considering two extreme cases in which the matrix and fiber constituents are considered separately. The first case is $\gamma = 0$, i.e., the mechanical behavior of the membrane is solely given by the matrix. This extreme case will be referred to as “*matrix only case*”. In such a case the prismatic bifurcation condition (2.19) reduces to

$$\hat{W}_{\lambda\lambda} = \mu \left(1 + \frac{3v^2}{\lambda^4 \lambda_z^2} \right) = 0. \quad (2.20)$$

Clearly, as $\hat{W}_{\lambda\lambda} > 0$ for all values of μ , v , λ and λ_z , the prismatic bifurcation is not feasible for the matrix only case. In particular (as swelling enters into the equation through the matrix

contribution), for the swellable matrix without any fibers prismatic bending cannot occur. Even though this is immediate, in Figure 2.2 we plot $\hat{W}_{\lambda\lambda}/\mu$ against the circumferential stretch λ when the axial stretch is held fixed at $\lambda_z = 1$ for selected values of the uniform prescribed swelling, namely $v = 0.8, 0.9, 1, 1.1$, and 1.2 . As it is expected, the curves in Figure 2.2 do not cross the horizontal axis ($\hat{W}_{\lambda\lambda}/\mu > 0$) so that prismatic bifurcation is not feasible. The black curve for $v = 1$ represents the response of a pure neo-Hookean membrane in the absence of swelling and it corresponds to the curve given in Figure 5 of the article by Haughton and Merodio (Haughton and Merodio, 2009). Furthermore, the result found here for swellable materials follows closely the results of (Haughton and Merodio, 2009) where it was shown that no prismatic bifurcation took place for a range of rubberlike materials when the cylinder was inflated.

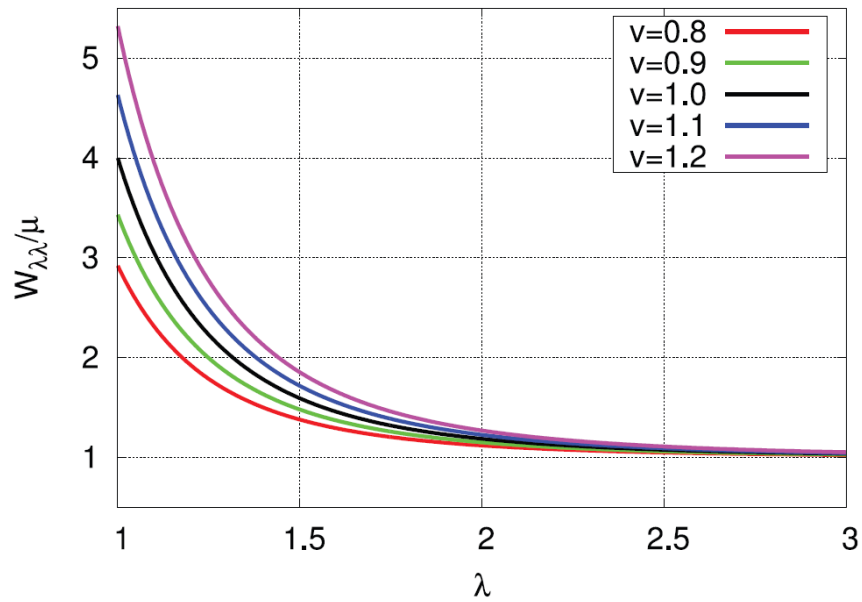


Figure 2.2: Plot of $\hat{W}_{\lambda\lambda}/\mu$ against λ for $\gamma = 0$ (no fibers), $\lambda_z = 1$, and the values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 . It follows that $\hat{W}_{\lambda\lambda}$ is positive and prismatic bifurcation is not possible.

The second extreme case is $\mu = 0$, for which the mechanical response of the matrix is negligible. This case will be referred to as “*fibers only case*”. One can argue that the matrix material must exist to hold fibers together, i.e. $\mu > 0$ has to be obeyed. Nevertheless, this case may be rationalized by assuming that compared with the stiffness of the fibers, the matrix contribution is negligible. In such a case, the prismatic bifurcation condition for the standard material model (2.19) reduces to

$$\hat{W}_{\lambda\lambda} = \gamma \left(-4 \sin^2 \beta + 12 \lambda^2 \sin^4 \beta + \lambda_z^2 \sin^2(2\beta) \right) = 0. \quad (2.21)$$

Equation (3.21) can be solved for either λ or λ_z to investigate the existence of prismatic bifurcation. If (2.21) is solved for λ , then

Table 2.1: Values of maximum axial stretch λ_{zmax} associated with different values of the fiber winding angle β for the fiber only material model (no matrix). For a given fiber winding angle β values of the axial stretch less than the associated maximum axial stretch λ_{zmax} give rise to prismatic bifurcation

β	$\lambda_{zmax} = \sec \beta$
	15° 1.03528
30°	1.1547
45°	1.4142
60°	2
75°	3.8637

$$\lambda = \pm \frac{\csc \beta}{\sqrt{3}} \sqrt{1 - \lambda_z^2 \cos^2 \beta}. \quad (2.22)$$

As $\beta \in (0, \frac{\pi}{2}]$, then $\cos \beta \in [0, 1)$. Consequently, for a given $\beta \in (0, \frac{\pi}{2}]$, real solutions of (2.22) for λ are feasible if and only if $1 - \lambda_z^2 \cos^2 \beta \geq 0$. The maximum value of λ_z , say λ_{zmax} , below which such solutions exist is given by $\lambda_{zmax} = \sec \beta$. Some values of λ_{zmax} are provided in Table 2.1 for selected values of the fiber winding angle β . Also note that when $\beta = \frac{\pi}{2}$ the prismatic bifurcation is feasible for every positive value of λ_z . It follows that the value λ_{zmax} increases as the value of β increases.

In Figure 2.3(a), we plot $\hat{W}_{\lambda\lambda}/\gamma$ against λ when the axial stretch is held fixed at $\lambda_z = 1$ for selected values of the fiber winding angle, $\beta = 15^\circ, 30^\circ, 45^\circ, 60^\circ$, and 75° . Note that when $\lambda_z = 1$, then, from equation (2.22) is immediate that $\lambda = 1/\sqrt{3}$ for any value of β . Hence, this particular pair $(\lambda, \lambda_z) = (1/\sqrt{3}, 1)$ results into a prismatic bifurcation for all values of the fiber winding angle β in the so-called fibers only model (no matrix) in the so-called fibers only model (no matrix).

In Figure 2.3(b), we also plot $\hat{W}_{\lambda\lambda}/\gamma$ against λ for $\lambda_z = 1.2$ and selected values of the fiber winding angle $\beta = 15^\circ, 30^\circ, 45^\circ, 60^\circ$, and 75° . As is expected, when $\beta = 15^\circ$ or $\beta = 30^\circ$, the prismatic bifurcation is not possible because $\lambda_z = 1.2$ is greater than the associated λ_{zmax} for these two β cases. On the other hand, for the other β values prismatic bifurcation is feasible. The λ values associated with the onset of prismatic bifurcation are 0.4320 ($\beta = 45^\circ$), 0.5333 ($\beta = 60^\circ$), and 0.5681 ($\beta = 75^\circ$).

Now, we can consider an intermediate case between the two extreme cases discussed previously: a swellable membrane with reinforcing fibers for which $\mu > 0$ and $\gamma > 0$. Based on the earlier analysis it is clear that when $\mu/\gamma \rightarrow \infty$ prismatic bifurcation of the membrane is not feasible. $\mu/\gamma \rightarrow 0$ prismatic bifurcation may be feasible, which also depends on the value of the fiber winding angle β . Therefore, one can explore the combination of material parameter values for which prismatic bifurcation occurs.

In Figure 2.4(a), we plot $\hat{W}_{\lambda\lambda}/\gamma$ against λ for $\lambda_z = 1.2$, selected values of μ/γ , $v = 1$ (no

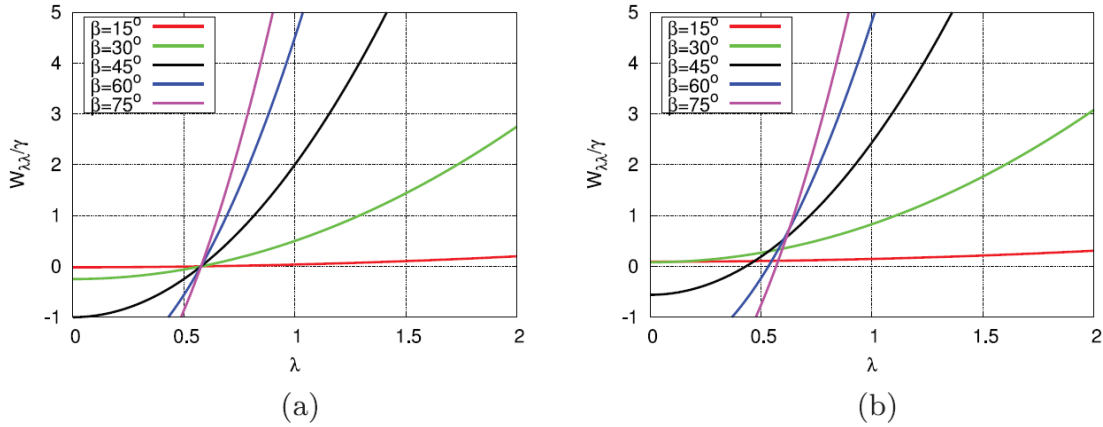


Figure 2.3: Plot of $\hat{W}_{\lambda\lambda}/\gamma$ against λ associated with the standard fiber model when matrix contribution is negligible ($\mu = 0$) for the values of the fiber winding angle $\beta = 15^\circ, 30^\circ, 45^\circ, 60^\circ$, and 75° . (a) is for $\lambda_z = 1$; (b) is for $\lambda_z = 1.2$. In each curve, the value of λ associated with $\hat{W}_{\lambda\lambda} = 0$ gives the onset of prismatic bifurcation. It follows that for the fiber winding angles $\beta = 15^\circ, 30^\circ$ prismatic bifurcation is not possible since $\hat{W}_{\lambda\lambda}$ is positive.

swelling) and the fiber winding angle $\beta = \pi/4$. Different values for the ratio μ/γ are selected to include transition from no prismatic bifurcation to prismatic bifurcation. For $\mu/\gamma = 0.002$ the values of $\hat{W}_{\lambda\lambda}/\mu$ are positive and prismatic bifurcation is not feasible. On the other hand, for $\mu/\gamma = 0.001$ there are two values of λ associated with prismatic bifurcation (two bifurcation points). These values are calculated numerically and are $\lambda = 0.285166$ and $\lambda = 0.398488$. When the ratio μ/γ is near 0.001377 there is a transition from no prismatic bifurcation to prismatic bifurcation. As prismatic bifurcation is observed for the ratio $\mu/\gamma = 0.001$ we further explore the effect of swelling for this specific ratio.

In Figure 2.4(b), we plot $\hat{W}_{\lambda\lambda}/\mu$ against λ when $\lambda_z = 1.2$, $\beta = \pi/4$ and $\mu/\gamma = 0.001$ for the values of the swelling parameter $v = 0.8, 0.9, 1, 1.1$, and 1.2 . For $v = 1.2$ prismatic bifurcation does not occur. Greater swelling provides stability against prismatic bifurcation. For each of the other values of v two values of λ are associated with prismatic bifurcation. It follows that swelling has a stabilizing effect against prismatic bifurcation.

Figure 2.5 gives a deeper insight into the combination of different parameter values that lead to prismatic bifurcation. In Figure 2.5 (a) values (v, λ) obeying the prismatic bifurcation condition (3.19) are shown for $\mu/\gamma = 0.001$, $\beta = 45^\circ$ and $\lambda_z = 0.8, 1$, and 1.2 . For a fixed swelling value v two values of λ meet this bifurcation condition as long as the swelling value is below some critical amount of swelling. This latter argument indicates that swelling has a stabilizing effect. The upper bound critical swelling values are calculated numerically as 2.9688, 2.3389, and 1.1748 for $\lambda_z = 0.8, 1$, and 1.2 , respectively. We turn our attention to the fiber winding angle. In Figure 2.5(b), the fiber winding angle is $\beta = 60^\circ$ and in Figure 2.5(c), the fiber angle is $\beta = 75^\circ$ whereas all the other parameter values are equal to those used in Figure 2.5(a). The respective upper bound critical swelling values are also calculated numerically for each value of λ_z and they are 3.3300, 3.5115, and 3.3211 for $\beta = 60^\circ$ and 3.6317, 4.3691, and 4.9964 for $\beta = 75^\circ$, respectively, for $\lambda_z = 0.8, 1$, and 1.2 . These critical

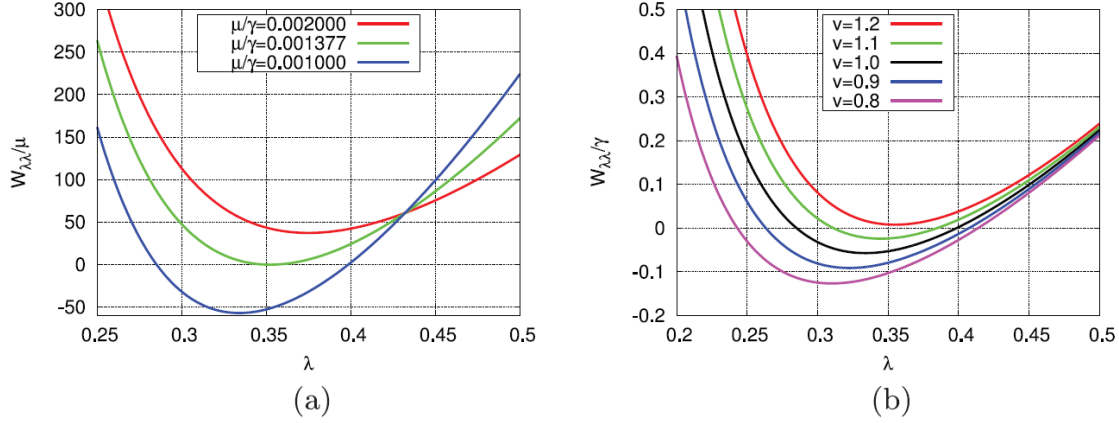


Figure 2.4: (a): Plot of $\hat{W}_{\lambda\lambda}/\mu$ against λ for the standard fiber model when $v = 1$, $\beta = \pi/4$, and $\lambda_z = 1.2$ for selected values of the matrix stiffness to fiber stiffness ratio $\mu/\gamma = 0.001, 0.001377$, and 0.002 . (b): Plot of $\hat{W}_{\lambda\lambda}/\mu$ against λ for the standard fiber model when $\mu/\gamma = 0.001$, $\beta = \pi/4$ and $\lambda_z = 1.2$ for selected values of the swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 . Among the given curves in (b), only the one corresponding to $v = 1.2$ does not intersect the horizontal axis. Hence no bifurcation occurs for $v = 1.2$ for specified set of parameter values

upper bound swelling values increase as β increases for a fixed λ_z . On the other hand, for this bifurcation mode the parameter β and the axial stretch λ_z have competing effects. Increasing the axial stretch tends to stabilize the tube for small values of β (see Figure 2.5(a)) while conversely, as β increases, greater values of λ_z introduce bifurcation points (see Figure 2.5(a) and (c)). We find for the given material and a value of λ_z that there are two bifurcation points and both points obey $0 \leq \lambda < 1$. One possible interpretation for the prismatic mode is that half of the tube (for example $0 \leq \theta < \pi$) contracts radially more than the other half (both are contracted). The result is that the cylinder does not depart from the cylindrical shape but the deformation is no longer cylindrically symmetric. This is also the typical (first) bifurcation mode associated with inflated spherical membranes. As pressure changes further, we expect the more-deformed half to expand and eventually match the other half, thereby returning to the cylindrical shape and symmetric deformation at the second bifurcation point. Nevertheless, if the situation under study involves inflated membranes under extension, the values of λ that have been found here are not expected under those circumstances and therefore prismatic bifurcations are not to be expected. The argument is the following. Greater contractions in the azimuthal direction, i.e., smaller values of λ , are associated with the case in which the membrane is in a state of uniaxial extension (corresponding to zero inflating pressure) and the material is not reinforced in the azimuthal direction. This is the case in which the membrane material is in a state of uniaxial extension with $\lambda = v/\sqrt{\lambda_z}$. Furthermore, the worst scenario is $v = 1$, for which $\lambda = 1/\sqrt{\lambda_z}$. Now, for values $1 \leq \lambda_z \leq 1.8$ (we just indicate a maximum value of λ_z , although it could be greater) and $1 \leq v$, clearly, the values of $\lambda = 1/\sqrt{\lambda_z}$ do not reach the necessary contraction (as given in Figure 2.5, for instance) to obtain prismatic bifurcations since for $\lambda_z = 1.8$, and it follows that $\lambda = 1/\sqrt{\lambda_z} = 0.707$.

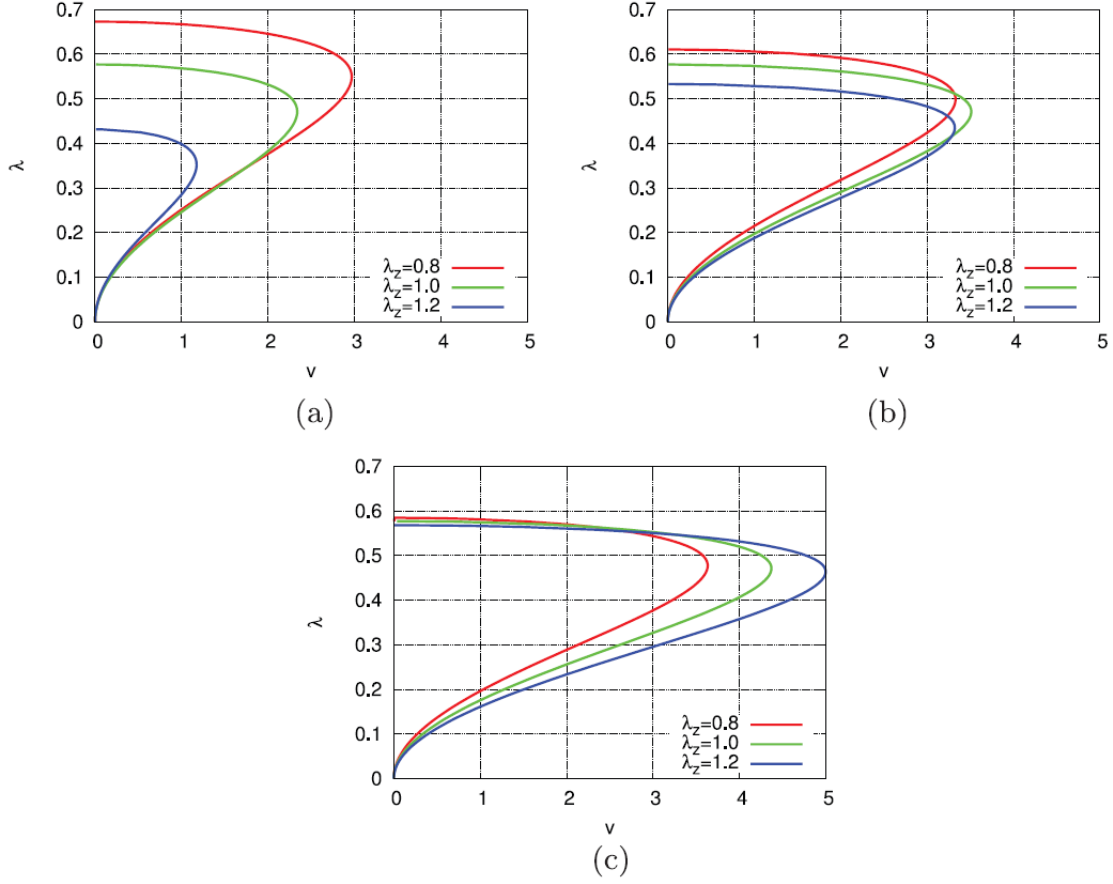


Figure 2.5: The curves show results associated with prismatic bifurcation as given by (2.19) for the standard fiber model with $\mu/\gamma = 0.001$, $\lambda_z = 0.8, 1$, and 1.2 . and selected fiber winding angles. (a) is for $\beta = 45^\circ$; (b) is for $\beta = 60^\circ$; and (c) is for $\beta = 75^\circ$.

2.4.2 Prismatic bifurcation of the exponential model

The prismatic bifurcation condition (2.12) for the exponential model can be written as

$$\begin{aligned} & \mu \left(1 + \frac{3v^2}{\lambda^4 \lambda_z^2} \right) + 4k_1 \sin^2 \beta e^{k_2(I_4-1)^2} \\ & \times \left(2\lambda^2 \sin^2 \beta \left[2k_2(I_4-1)^2 + 1 \right] + (I_4-1) \right) = 0 \end{aligned} \quad (2.23)$$

with the help of (2.6). The invariant I_4 is given by (2.16). As it is clear from (2.23), if $k_1 = 0$, then it reduces to swellable neo-Hookean membrane without fibers (note that in the definition of (2.18), k_1 is required to be positive). This matrix only case has been studied in the previous Section 2.4.1. When $k_1 > 0$ the fibers provide additional strength to the material. It follows that a prismatic bifurcation is not possible. In order to provide an example, Figure 2.6 plots $\hat{W}_{\lambda\lambda}/\mu$ against λ when $\lambda_z = 1$ for selected values of the uniform prescribed swelling, namely $v = 0.8, 0.9, 1, 1.1$, and 1.2 , $k_1 = 0.05\mu$, $\beta = \pi/4$ and $k_2 = 0.01$. As it is expected, all curves are above the horizontal axis given by $\hat{W}_{\lambda\lambda}/\mu = 0$, and there is no prismatic bifurcation.

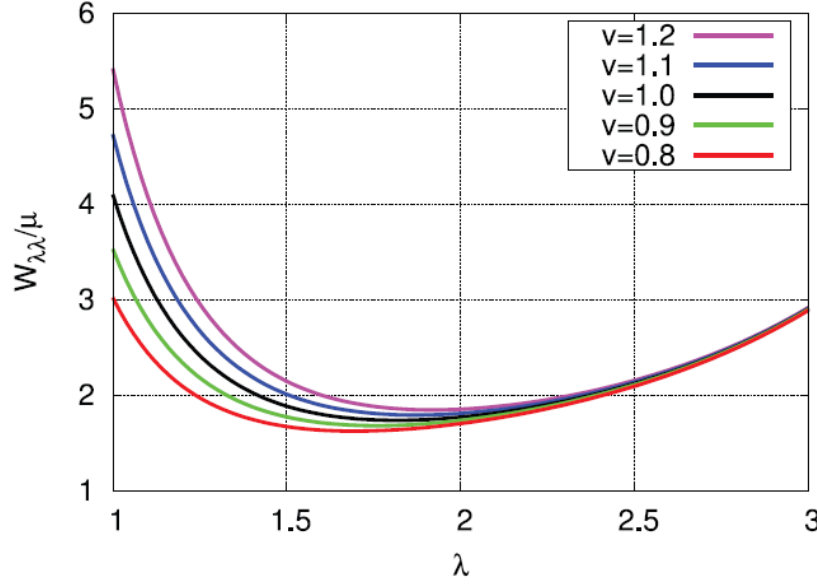


Figure 2.6: Plot of $\hat{W}_{\lambda\lambda}/\mu$ against λ when $k_1 = 0.05\mu$, $k_2 = 0.01$, $\beta = \pi/4$ and $\lambda_z = 1$ for selected values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2

2.5 Bending (composite) mode

In this section the bending mode of bifurcation (2.13) is analyzed for both the standard model (2.17) and the exponential model (2.18).

2.5.1 Bending bifurcation of standard model

For the standard material model (2.17) the bending bifurcation condition (2.13) can be written as

$$\begin{aligned}
 g = & \mu \left(\lambda^2 + \frac{v^2}{\lambda^2 \lambda_z^2} - 2\lambda_z^2 \right) + \gamma \left(8\lambda_z^2 \cos^2 \beta - 8\lambda_z^4 \cos^4 \beta - 4\lambda^2 \sin^2 \beta \right. \\
 & \left. + 4\lambda^4 \sin^4 \beta - \lambda^2 \lambda_z^2 \sin^2(2\beta) \right) = 0
 \end{aligned} \tag{2.24}$$

with the help of (2.6) and (2.16). In the absence of fibers (for $\gamma = 0$), we end up with a swellable neo Hookean membrane. The bending bifurcation condition for such a membrane without fibers is given by

$$\lambda^2 + \frac{v^2}{\lambda^2 \lambda_z^2} - 2\lambda_z^2 = 0. \tag{2.25}$$

Figure 2.7 shows values of g/μ as given by the left hand side of (3.24) against λ for $\lambda_z = 1$, selected values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 and $\gamma = 0$ in Figure 2.7 (a) (neo-Hookean matrix without fibers) and $\gamma = 0.01\mu$ (slightly reinforced

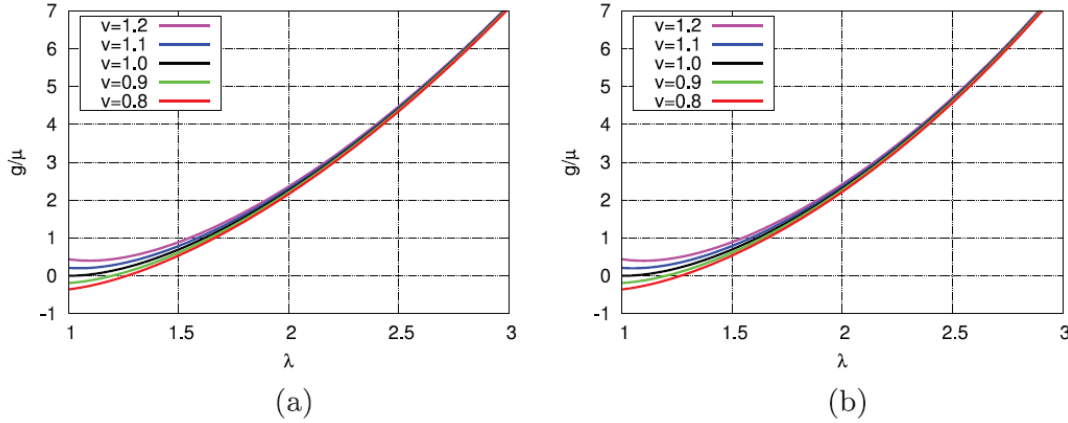


Figure 2.7: Values of g/μ as given by the left hand side of (2.24) against λ when $\lambda_z = 1$ for selected values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 and (a) $\gamma = 0$ (no fibers), and (b) $\gamma = 0.01\mu$ (slightly reinforced matrix) with $\beta = \pi/4$. Unstable configurations are associated with values $g/\mu < 0$

neo-Hookean matrix) with $\beta = \pi/4$ in Figure 2.7 (b). It becomes clear from Figure 2.7 (a) that bending bifurcation is feasible. Furthermore, if $v > 1$ then $g/\mu > 0$. Therefore, swelling promotes such a bifurcation, as opposed to the anisotropy whose effect is not obvious and can promote it or not depending on material constitutive parameters.

Figure 2.8 shows values of g/γ as given in (2.24) versus λ and λ_z for $\mu = 0$ (negligible matrix stiffness), $v = 1$ and different angles β . The regions obeying that $g < 0$ give the values (λ, λ_z) for which membranes are stable, conversely, the regions obeying that $g > 0$ give the values (λ, λ_z) for which membranes are unstable. In particular, for instance, yellow regions in the plot illustrate the transition from stable configurations to unstable ones. Figure 2.8 (a) is for $\beta = 0^\circ$, whereas Figure 2.8 (b) is for $\beta = 15^\circ$, $\beta = 45^\circ$ is for Figure 2.8 (c), and Figure 2.8 (d) is for $\beta = 90^\circ$.

2.5.2 Bending bifurcation of exponential model

For the exponential material model (2.18) the bending bifurcation condition (2.13) can be written as

$$q = \mu \left(\lambda^2 + \frac{v^2}{\lambda^2 \lambda_z^2} - 2\lambda_z^2 \right) - 4k_1(I_4 - 1)(2\lambda_z^2 \cos^2 \beta - \lambda^2 \sin^2 \beta) e^{k_2(I_4 - 1)^2} = 0 \quad (2.26)$$

with the help of (2.6). In equation (2.26), I_4 is the invariant given by (2.16). If $k_1 = 0$, then (2.26) reduces to the response for swellable neo-Hookean membrane without fibers (note that in the definition of (2.18), k_1 is required to be positive). The case of a matrix without fiber has been analyzed in Section 2.5.1. If $k_1 > 0$, then fibers contribute to the strain energy density function, and (2.26) is analyzed by using numerical techniques.

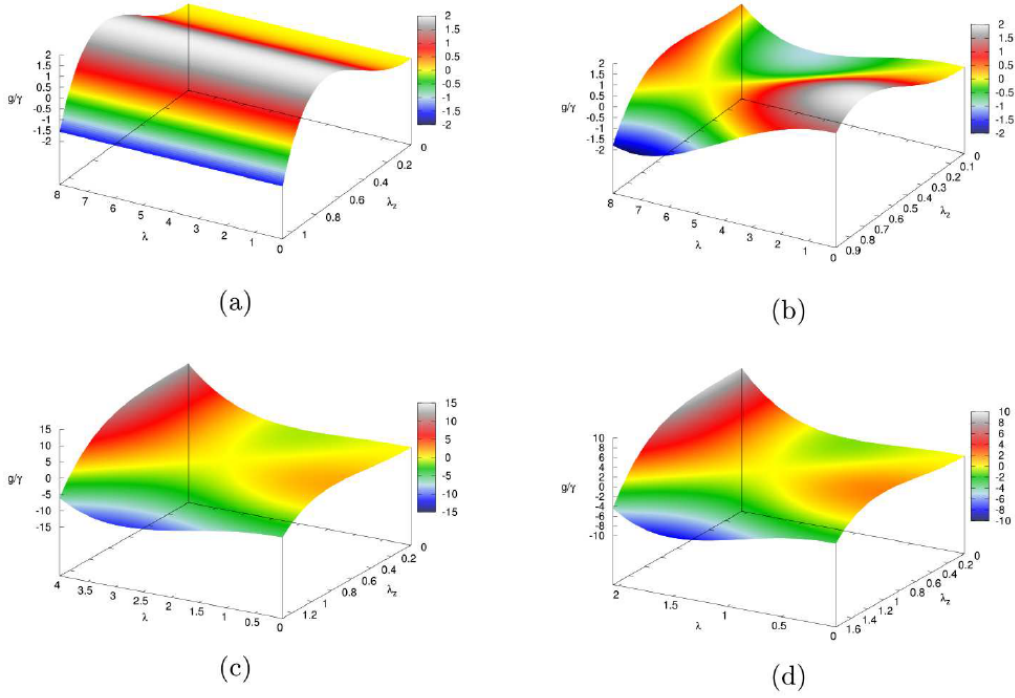


Figure 2.8: Normalized values of g as given by the left hand side of (3.24) versus λ and λ_z for $\mu = 0$ (negligible matrix stiffness), $v = 1$ and different angles β : (a) $\beta = 0^\circ$; (b) $\beta = 15^\circ$; (c) $\beta = 45^\circ$; (d) $\beta = 90^\circ$

In Figure 2.9 we plot several bending bifurcation curves which satisfy the condition (2.26) for selected values of swelling. When swelling values are less than one, the curves intersect the horizontal axis and there is a pair of values (λ, λ_z) for which bifurcation occurs. If swelling is greater than one, then $q/\mu > 0$ and the configuration is unstable with respect to bending bifurcation. One special case is for $v = 1$. There are two values of λ for which bending bifurcation occurs. These values are given numerically by $\lambda = 1$ and $\lambda = 1.02464$. For values of λ in between these two the tube is stable since $q < 0$. To gain more insight into the unstable deformation region of the cylinder, Figure 2.10 (a) shows curves λ versus λ_z that meet the bending bifurcation condition (2.24) for $k_1 = 0.05\mu$, $k_2 = 0.1$ and $\beta = \pi/4$ and selected values of the uniform prescribed swelling, namely $v = 0.8, 0.9, 1, 1.1$, and 1.2 . Each curve divides the plane in two regions. The region at the left of each curve, i.e. points (λ_z, λ) , obeys that $q > 0$ and therefore gives unstable configurations while the region at the right of each curve obeys that $q < 0$ and therefore gives stable configurations. For instance, the curve $v = 1.2$ for $\lambda_z = 1$ in Figure 9 is associated with values $q > 0$, and therefore they give pairs of values $(\lambda_z = 1, \lambda)$ that are in the region at the left of the curve $v = 1.2$ in Figure 2.10(a).

Similarly, Figure 2.10(b) plots λ versus v that meet the bending bifurcation condition for $k_1 = 0.05\mu$, $k_2 = 0.1$, $\beta = \pi/4$ and selected values of the axial stretch, namely, $\lambda_z = 1, 1.2, 1.4$, and 1.6 .

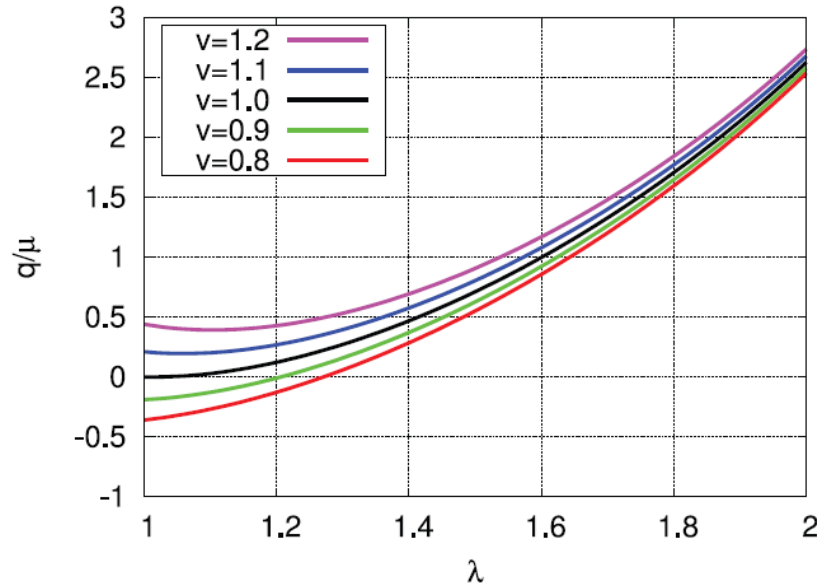


Figure 2.9: Values of q (normalized using μ) in (2.26) as given by the left hand side of (2.24) against λ when $k_1 = 0.05\mu$, $k_2 = 0.1$, $\beta = \pi/4$ and $\lambda_z = 1$ for selected values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 . Stable configuration obey that $q/\mu < 0$

2.6 Conclusions

Many articles that are devoted to the bifurcation of cylindrical tubes set their focus on bulging bifurcation of isotropic materials. This article has studied the prismatic and bending bifurcations of fiber-reinforced elastic membranes. Swelling has been considered, and the mechanical behavior of the fibers has been taken into consideration by means of two strain energy density functions that find their application in different models that account for the mechanical behavior of collagenous soft tissue. The results of the different analyses show that for an isotropic neo-Hookean type of membrane, without any fiber reinforcement, prismatic bifurcation is not feasible. On the other hand, our results have also shown that swelling promotes bending bifurcation. When fibers are taken into account, then the reinforcement functions for the fibers give different scenarios with respect to prismatic bifurcations. For the so-called standard reinforcement model, prismatic bifurcations may occur, and the occurrence of these bifurcations will depend on the specifications of the material parameters, the deformation as given by both axial and azimuthal stretches, and the fiber winding angles. For the exponential form for the strain energy density function for the fibers and for herein applied values for the parameters, prismatic bifurcation does not occur. We have further shown that certain specifications for the fiber winding angle and for the material parameters can trigger bending bifurcation in both fiber models. The present article finds that swelling may stabilize the cylinder to bending bifurcation. This result is similar to the results in (Demirkoparan and Merodio, 2017; Demirkoparan and Merodio, 2020), which have shown that swelling can prevent bulging bifurcation in fibrous cylinder. Owing to the interplay of countless biomechanical and physical processes in biological soft tissue, the mechanical behavior of

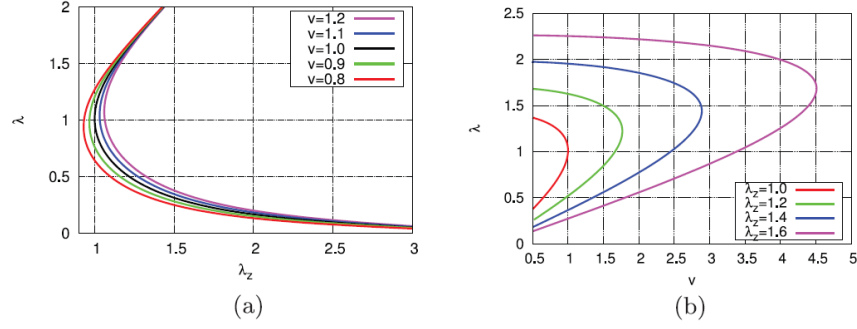


Figure 2.10: Values for λ versus λ_z in (a) obey the bending bifurcation condition $q = 0$ as given by (2.26) for the exponential material model with $k_1 = 0.05\mu$, $k_2 = 0.1$, $\beta = \pi/4$ and selected values of the uniform prescribed swelling $v = 0.8, 0.9, 1, 1.1$, and 1.2 . (b) Values λ versus v obeying the bending bifurcation condition $q = 0$ as given by (2.26) for the exponential material model with $k_1 = 0.05\mu$, $k_2 = 0.1$, $\beta = \pi/4$ and selected values of the axial stretch $\lambda_z = 1, 1.2, 1.4$, and 1.6

blood vessels is complex, so that the modeling the mechanical behavior of tissues usually relies on idealizations. Nevertheless, our preliminary results provide a deeper understanding on how the interplay of the mechanical behavior of the ground substance and the fibers contribute to the formation of instabilities and, therefore, to the failure of arteries. This understanding may help to identify the formation of tissue instabilities before failure takes place. Our work may be extended in different ways in order to refine the constitutive modeling relations. We study the bifurcation of membranes, whereas a comprehensive understanding of the formation and the development of aneurysms has to be studied in the context of thick-walled membranes. This may be necessary, for example, if one wishes to include residual stresses in order to model the development of an aneurysm (Melnikov et al., 2021; Merodio and Ogden, 2016). In our work, the fibers are taken to be parallel to share a common natural configuration with the matrix. This idealization simplifies the mathematical treatment on the one hand, and, on the other hand, the results may remain sufficiently accurate for slightly dispersed or pre-stretched fibers. Future research may consider the dispersion of the fibers. Different models have been developed that account for fiber dispersion, among them the angular integration method and the generalised structure tensor (Cortes and Elliott, 2016). The mechanical properties of the different constituents have been taken to be time-independent in this work, which is a reasonable approximation when the typical time scale of the bifurcation initiation is small in comparison with the typical timescales that are associated with different phenomena such as viscoelastic effects or changes in the microstructure. These changes can be mechano-regulated and therefore directly linked to the bifurcation process, for example, in the case of the strain stabilization of collagen to enzymatic degradation (Topol, Demirkoparan, and Pence, 2021b, 2021c). Different factors that account for an evolving microstructure as well as considering these factors in the modeling of fibrous materials, are discussed in the review (Wineman and Pence, 2021).

Chapter 3

Bulging initiation and propagation in fiber-reinforced swellable Mooney–Rivlin membranes

“This chapter is based on the below article that is published corresponding this thesis: Topol, H., Al-Chlahawi, M. J., Demirkoparan, H., Merodio, J. (2021). Bulging initiation and propagation in fiber-reinforced swellable mooney–rivlin membranes. *Journal of engineering mathematics*, 128, 1–15 (Topol, Al-Chlahawi, et al., [2021](#))

3.1 Introduction

Bulging of a pressurized cylinder has been studied in different works such as Fu et al. (Fu et al., [2016](#); Fu et al., [2008](#)) and Merodio and collaborators (Dehghani et al., [2019](#); El Hamdaoui and Merodio, [2015](#); Font et al., [2021](#); Merodio and Haughton, [2010](#); Rodríguez and Merodio, [2011](#)) in the context of volume-preserving material behavior. While Kyriakides and Chang (Kyriakides and Yu-Chung, [1991](#)) and Kyriakides (Kyriakides, [1993](#)) show that bulging can be the result of a limit load instability, Fu et al. (Fu et al., [2008](#)) illustrate that localized bulging can occur even when there is no maximum pressure. Formation of a localized bulge is a purely non-linear phenomenon of bifurcation. For isotropic elastomer tubes with one end closed, when air is forced into the tube from the other end, a necessary condition for bulging is that the pressure-change in volume response of a section of the constrained tube being inflated uniformly has an up–down behavior (see, e.g., (Kyriakides, [1993](#); Chater and Hutchinson, [1984](#))). The bulge then grows radially unless the mechanical response recovers to a second stable branch, i.e., unless the pressure-change in volume response has an up–down–up behavior which is associated with axial propagation of bulging. The propagation of bulging into axial direction can be subdivided into two stages. In the first stage, the inner pressure remains fixed during the ensuing propagation of the bulging instability mode beyond the onset of bifurcation until a suitable configuration is obtained. Then in the second stage, the inflation pressure has to be increased for axial propagation of bulging to continue. The bulging propagation pressure in the first stage corresponds to the one obtained using the

equal-area rule, i.e., the Maxwell (propagation) pressure. In the case of open-end hollow cylinders, axial propagation of bulging also involves the two periods captured for closed-end tubes (see Alhayani et al. (Alhayani et al., 2014a)). Nevertheless, the so-called propagation pressure (first stage) is not captured with the Maxwell condition and furthermore, bulging can be obtained with no pressure maximum. This article continues the considerations from (Demirkoparan and Merodio, 2020) and it focuses on these aspects in regard to swelling. Due to different factors, such as injuries, inflammation, and hormonal changes, biological soft tissue may swell. Swelling (and shrinking) of soft tissue is accompanied with changes in the mechanical properties (see, e.g., Guo et al. (X. Guo et al., 2007)) on swelling of arterial tissue). This change in the tissue volume has been taken into account in different mechanical modeling works, for example in the context of isotropic material behavior (see, e.g., Tsai et al. (Tsai et al., 2004), and Pence and Tsai (Pence and Tsai, 2005; Pence and Tsai, 2006) for homogeneous swelling and van der Sman (Van der Sman, 2015) for inhomogeneous swelling), in the context of anisotropic material behavior due to fibrous constituents in a swellable matrix (see, e.g., Demirkoparan and Pence (Demirkoparan and Pence, 2007b; Demirkoparan and Pence, 2007a; Demirkoparan and Pence, 2008)). The effect of residual stresses and swelling in soft tissue modeling has been studied in different works such as Lanir (Lanir, 2009; Lanir, 2012) and Sorrentino et al. (Sorrentino et al., 2015). The recent work by Topol et al. (Topol et al., 2018) models the interaction between soft tissue swelling of aground substance material and strain stabilization of collagenous fiber to enzymatic degradation. This effect has been further investigated in the finite element study by Gouetal. (Gou et al., 2020) for fiber remodeling homeostasis in swelling cervical soft tissue. The recent works by Demirkoparan and Merodio (Demirkoparan and Merodio, 2020; Demirkoparan and Merodio, 2017) study the swelling-induced bulging in hyperelastic fibrous tubes. In the framework of the present work, we study bulging initiation and propagation in a tube open at both ends made of a swellable Mooney–Rivlin material. In the swellable ground substance material two families of fibers symmetrically disposed and with the same mechanical properties are embedded. Within a fiber family, the fibers are parallel to each other. The strain energy density function for the fibers is considered in both exponential and quadratic form. The use of exponential form for fibrous component has gained some popularity in the modeling of collagen fibers in arterial tissue. This is due to the fact that the fibers may be wavy when unloaded (see, e.g., the review by Holzapfel and Ogden (Holzapfel and Ogden, 2010a)). This article studies the interplay of swelling, fiber orientation, and the mechanical properties of the constituents on bulging initiation and propagation, and it is organized as follows. In Section 3.2, the mathematical problem is described including the bulging condition for the materials at hand. In Section 3.3, the analysis of bulging initiation is carried out while Section 3.4 deals with axial propagation of bulging. Section 3.5 gives some final conclusions.

3.2 Geometric and mechanical conditions

Consider an incompressible elastic body in its undeformed reference configuration $\mathcal{B}_r$, in which the location of a material point is described by a position vector $\mathbf{X}$. In the deformed configuration $\mathcal{B}$, the location of the same material point is described by the position vector $\mathbf{x}$, so that the mapping for this deformation is described by the deformation gradient tensor

$\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$. The corresponding left and right Cauchy–Green deformation tensors are $\mathbf{B} = \mathbf{F} \mathbf{F}^T$ and $\mathbf{C} = \mathbf{F}^T \mathbf{F}$.

Now consider a cylindrical coordinate system that is defined by the three base unit vectors $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$ into the radial, circumferential and axial directions, respectively. In the undeformed reference configuration $\mathcal{B}_r$, the location of a hollow cylinder can be described in terms of the cylindrical coordinates $\{R, \Theta, Z\}$ as

$$A \leq R \leq B, \quad 0 \leq \Theta < 2\pi, \quad 0 \leq Z \leq L, \quad (3.1)$$

where A and B are the inner and outer radii, and L is the length of the cylinder, which may be finite or infinite. The position vector of a material point $\mathbf{X}$ in the undeformed configuration then reads

$$\mathbf{X} = R \mathbf{E}_R(\Theta) + Z \mathbf{E}_Z. \quad (3.2)$$

Prior to its bifurcation the cylinder is subjected to an axial force N , an inflation pressure P and a swelling field

$$v = \det(\mathbf{F}), \quad (3.3)$$

which is the ratio of the volume of the swollen material to the volume of the unswollen material. Condition (3.3) ensures that at each instant of the swelling the material remains incompressible. Any location of the deformed cylinder is described by the position vector

$$\mathbf{x} = r \mathbf{e}_r(\theta) + z \mathbf{e}_z, \quad 0 \leq \theta < 2\pi, \quad 0 \leq z < \ell, \quad (3.4)$$

where $(r, \theta = \Theta, z)$ are the cylindrical coordinates, $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the three base unit vectors of the deformed configuration, and ℓ is the length of the cylinder in the deformed configuration. The coordinates of the deformed configuration are

$$r = r(R), \quad \theta = \Theta, \quad z = \lambda_z Z, \quad (3.5)$$

where λ_z is the stretch ratio on the axial direction of the cylinder. The azimuthal stretch ratio corresponds to the ratio from the deformed to the undeformed radius, $\lambda_\theta = \lambda = r/R > 0$, and the radial stretch then becomes $\lambda_r = v \lambda^{-1} \lambda_z^{-1} = \partial r / \partial R$. From (3.3) it follows that $v = (\partial r / \partial R)(r/R) \lambda_z$.

Furthermore, in the cylindrical coordinate system the deformation gradient tensor and the left and right Cauchy–Green deformation tensors become

$$\mathbf{F} = \frac{v}{\lambda\lambda_z} \mathbf{e}_r \otimes \mathbf{E}_R + \lambda \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_z \mathbf{e}_z \otimes \mathbf{E}_Z \quad (3.6a)$$

$$\mathbf{B} = \mathbf{C} = \frac{v^2}{\lambda^2\lambda_z^2} \mathbf{e}_r \otimes \mathbf{e}_r + \lambda^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_z^2 \mathbf{e}_z \otimes \mathbf{e}_z \quad (3.6b)$$

respectively.

3.2.1 Material Behavior

The material of the cylinder consists of a swellable isotropic matrix and reinforcing fibers. From the mechanical modelling perspective, the material is treated in a homogenized fashion so that any location $\mathbf{X}$ of the cylinder contains both matrix and fiber material. In an incompressible and swellable solid, the Cauchy stress tensor can be expressed as

$$\boldsymbol{\sigma} = -q\mathbf{I} + \frac{1}{v} \frac{\partial W}{\partial \mathbf{F}} \mathbf{F}^T, \quad (3.7)$$

where $\mathbf{F}$ is given in (3.6a), $\mathbf{I}$ is the second order identity tensor and q is a scalar that results from the constraint (3.3). The mechanical behavior of a material is described in terms of a strain energy density function

$$W = W(I_1, I_2, I_4, I_5, I_6, I_7, v), \quad (3.8)$$

where I_1, I_2, I_4, I_5, I_6 , and I_7 are invariants defined in what follows.

The three principal invariants of $\mathbf{C}$ (and due to the symmetry, also of $\mathbf{B}$) in equation (3.6b) have the forms

$$I_1(\mathbf{C}) = \text{tr}(\mathbf{C}) = \frac{v^2}{\lambda^2\lambda_z^2} + \lambda^2 + \lambda_z^2 \quad (3.9a)$$

$$I_2(\mathbf{C}) = \frac{1}{2} \left[[\text{tr}(\mathbf{C})]^2 - \text{tr}(\mathbf{C}^2) \right] = \frac{v^2}{\lambda^2} + \frac{v^2}{\lambda_z^2} + \lambda^2\lambda_z^2 \quad (3.9b)$$

$$I_3(\mathbf{C}) = \det(\mathbf{C}) = v^2 \quad (3.9c)$$

In modeling of compressible solids, the third invariant represents a change of material volume during the deformation, while in our modeling the material remains incompressible at all stages of the deformation. Hence the volume change in the material is solely due to swelling. This motivates the swelling v instead of I_3 in the argument structure of the strain energy density function (3.8).

The fibers are arranged in two families of parallel fibers within each family with equal mechanical properties, and these fiber families are in a symmetric and helical arrangement that guarantees the cylinder inflation described by equation (3.6). The unit vectors

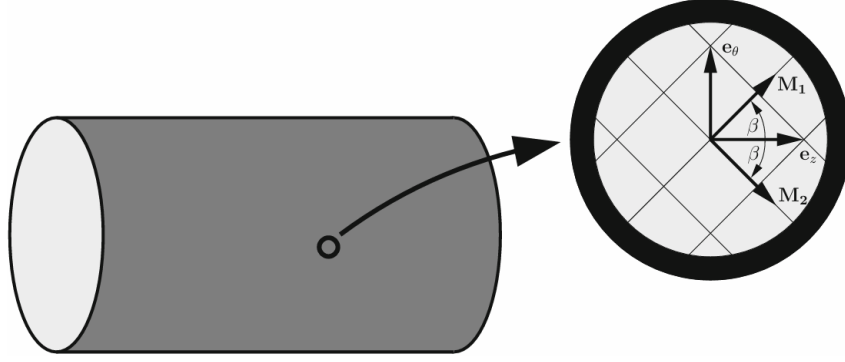


Figure 3.1: A cylindrical membrane. The material consists of a matrix and fibers. The fibers are arranged in two families of parallel fibers with equal mechanical properties, and these fiber families are in a symmetric and helical arrangement that guarantees the cylinder inflation described by equation (3.6) is an admissible deformation. The unit vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ describe the orientation of the fibers from the two families in the reference configuration

$$\mathbf{M}_1 = \cos \beta \mathbf{e}_z + \sin \beta \mathbf{e}_\theta, \quad \mathbf{M}_2 = \cos \beta \mathbf{e}_z - \sin \beta \mathbf{e}_\theta \quad (3.10)$$

describe the orientation of the two fiber families in the reference configuration using the angle β (see Figure 3.1). These fibers require further invariants to be introduced and used in the definition of the strain energy density function. In particular, the invariants associated with the orientations $\mathbf{M}_1$ and $\mathbf{M}_2$ are

$$I_4 = \mathbf{C} : \mathbf{M}_1 \otimes \mathbf{M}_1 = \lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta \quad (3.11a)$$

$$I_5 = \mathbf{C}^2 : \mathbf{M}_1 \otimes \mathbf{M}_1 = \lambda_z^4 \cos^2 \beta + \lambda^4 \sin^2 \beta \quad (3.11b)$$

$$I_6 = \mathbf{C} : \mathbf{M}_2 \otimes \mathbf{M}_2 = \lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta \quad (3.11c)$$

$$I_7 = \mathbf{C}^2 : \mathbf{M}_2 \otimes \mathbf{M}_2 = \lambda_z^4 \cos^2 \beta + \lambda^4 \sin^2 \beta \quad (3.11d)$$

where the invariants I_4 and I_5 are associated with the first fiber family (the one with orientation angle $\mathbf{M}_1$) and the invariants I_6 and I_7 are associated with the second fiber family (the one with orientation angle $\mathbf{M}_2$).

The invariants I_4 and I_6 can be interpreted as the squared magnitude of the fiber stretch ratios, while the invariants I_5 and I_7 are related to fiber shearing. Due to the symmetric arrangement of the fibers in the cylinder that results from (3.10) we obtain $I_4 = I_6$ and $I_5 = I_7$ (See equation (3.11)).

The expressions of the invariants in equation (3.9) and (3.11) motivate the reformulation of the strain energy density function (3.8)) in terms of the principal stretches as

$$W(I_1, I_2, I_4, I_5, I_6, I_7, v) = \bar{W}(\lambda, \lambda_z, v). \quad (3.12)$$

In order to simplify the notation, partial derivatives of $\bar{W}$ shall be indicated by the subscripts as $\bar{W}_{\lambda_z} = \partial \bar{W} / \partial \lambda_z$, $\bar{W}_{\lambda_z \lambda_z} = \partial^2 \bar{W} / \partial \lambda_z^2$, etc.

3.2.2 Bulging Bifurcation in the Membrane Approximation

In the absence of body forces, as it is the case in the present work, the Cauchy stress tensor needs to satisfy $\text{div } \sigma = \mathbf{0}$. Let us consider the cylinder in the membrane approximation in which it is taken a unique radius $R = (A+B)/2$, the midsurface radius, and $H = (B-A) \ll R$ is the thickness. In this membrane approximation, the relation between the inflation pressure P , the axial normal force N , and the deformation is given by (Demirkoparan and Merodio, 2020)

$$P = \frac{H}{R} \frac{\bar{W}_{\lambda}}{\lambda \lambda_z}, \quad N = \bar{W}_{\lambda_z}. \quad (3.13)$$

The radial normal stresses are negligible in the membrane approximation, hence we take $\sigma_{rr} = 0$. Then the circumferential and axial components $\sigma_{\theta\theta}$ and σ_{zz} of the Cauchy stress tensor (3.7) become

$$\sigma_{\theta\theta} = \lambda \frac{\partial \bar{W}}{\partial \lambda}, \quad \sigma_{zz} = \lambda_z \frac{\partial \bar{W}}{\partial \lambda_z}. \quad (3.14)$$

In order to investigate the possible bulging bifurcation mode of the cylinder we consider incremental displacements with respect to the deformed configuration (under equilibrium), of the form

$$\delta \mathbf{u} = \delta u_r(z) \mathbf{e}_r + \delta u_z(z) \mathbf{e}_z, \quad (3.15)$$

in which δ is the increment. The bulging condition for a cylinder membrane of radius R and length L is derived in (Rodríguez and Merodio, 2011) and given by

$$f(\bar{W}, \lambda, \lambda_z) + \left(\frac{2\pi R}{L} \right)^2 \lambda^2 \lambda_z \bar{W}_{\lambda_z} \bar{W}_{\lambda_z \lambda_z} = 0, \quad (3.16)$$

where

$$f(\bar{W}, \lambda, \lambda_z) = \lambda_z^2 \bar{W}_{\lambda_z \lambda_z} (\lambda^2 \bar{W}_{\lambda\lambda} - \lambda \bar{W}_{\lambda}) - (\lambda \lambda_z \bar{W}_{\lambda\lambda_z} - \lambda \bar{W}_{\lambda})^2 \quad (3.17)$$

The last term on the left side of (3.16) vanishes for a cylinder of infinite length, $L \rightarrow \infty$, which is the focus of this analysis.

3.3 Bulging for specific material models

Different hyperelastic models have been proposed that account for the mechanical behavior of fibrous biological soft tissue (see, e.g. Chagnon et al. (Chagnon et al., 2015)). Let us consider a strain energy density function in the form

$$W(I_1, I_2, I_4, I_5, v) = \frac{\mu_1}{2} (I_1 - 3v^{-2/3}) + \frac{\mu_2}{2} (I_2 - 3v^{-4/3}) + \frac{k_1}{\bar{k}_1} \left[\exp(\bar{k}_1 [I_4 - 1]^2) - 1 \right] + \frac{k_2}{\bar{k}_2} \left[\exp(\bar{k}_2 [I_5 - 1]^2) - 1 \right]. \quad (3.18)$$

The first two terms on the right side of (3.18) define the isotropic behavior of the ground substance material, in which the fibers are embedded, and they describe a swellable modification of the classical Mooney–Rivlin material (see, e.g., (Pence and Tsai, 2006)). The material parameters μ_1 and μ_2 are taken to be independent of the swelling, and we refer to works such as (Topol et al., 2018; Gou et al., 2020) that discuss the changes of the material stiffness with the amount of swelling. The last two terms on the right side of equation (3.18) account for the anisotropic material behavior due to the two fiber families. The expression given in (3.18) has already made use of the equal properties and the symmetric arrangement of the fibers that are associated with the first fiber family (and therefore with I_4, I_5) and with the second fiber family (and therefore with I_6, I_7), i.e. the following identities are already used in (3.18)

$$\frac{k_1}{\bar{k}_1} \left[\exp(\bar{k}_1 [I_4 - 1]^2) - 1 \right] = \frac{k_1}{2\bar{k}_1} \sum_{i=4,6} \left[\exp(\bar{k}_1 [I_i - 1]^2) - 1 \right] \quad (3.19a)$$

$$\frac{k_2}{\bar{k}_2} \left[\exp(\bar{k}_2 [I_5 - 1]^2) - 1 \right] = \frac{k_2}{2\bar{k}_2} \sum_{i=5,7} \left[\exp(\bar{k}_2 [I_i - 1]^2) - 1 \right] \quad (3.19b)$$

The material parameters k_1 and k_2 can be regarded as relative strength parameters for the fiber stiffness. Moreover $\bar{k}_1$ and $\bar{k}_2$ are also material parameters which may be regarded as fiber sensitivity parameters. The exponential terms account for fibers in an initially wavy form that have to be extended in order to fully bear loading. Specialization of the strain energy density function (3.18) have been applied in different works. For example, a non-swellable version of this material model has been applied in the work (Masson et al., 2010) for modeling of arteries.

For the material model (3.18) the bulging condition for an infinitely long cylinder (3.17) can be written as

$$f = f_{I_1} + f_{I_2} + f_{I_4} + f_{I_5} = 0. \quad (3.20)$$

The first two terms on the right side of (3.20) account for the behavior of the matrix. In particular, the first term on the right side of (3.20) results from the first term on the right side of (3.18), and the second term of (3.20) results from the second term on the right side of (3.18). In particular, for the material model (3.18), one can find that

$$f_{I_1} = \frac{\mu_1^2(-\lambda_z^4\lambda^8 + 4\lambda_z^4\lambda^2v^2 + 6\lambda_z^2\lambda^4v^2 + 3v^4)}{(\lambda_z^4\lambda^4)} \quad (3.21a)$$

$$f_{I_2} = \frac{\mu_2^2(-\lambda_z^6\lambda^8 + 2\lambda_z^4\lambda^4v^2 - \lambda_z^2v^4 + 12\lambda^2v^4)}{\lambda_z^2\lambda^4} \quad (3.21b)$$

Moreover the third term in (3.20) accounts for the exponential response in (3.18) which is in turn associated with the invariant I_4 (and I_6). One can also deduce that

$$\begin{aligned}
f_{I_4} = & \left[\lambda_z^2 \left\{ \lambda^2 \left[4k_1 \sin^2 \beta [I_4 - 1] + 8k_1 \lambda^2 \sin^4 \beta + 16\bar{k}_1 k_1 \lambda^2 \sin^4 \beta [I_4 - 1]^2 \right] \right. \right. \\
& \left. \left. - 4k_1 \lambda^2 \sin^2 \beta [I_4 - 1] \right\} \right. \\
& \times \left[4k_1 \cos^2 \beta [I_4 - 1] + 8k_1 \lambda_z^2 \cos^4 \beta + 16\bar{k}_1 k_1 \lambda_z^2 \cos^4 \beta [I_4 - 1]^2 \right] \\
& \left. - \left\{ \lambda_z \lambda \left[8k_1 \lambda_z \lambda \cos^2 \beta \sin^2 \beta + 16\bar{k}_1 k_1 \lambda_z \lambda \cos^2 \beta \sin^2 \beta [I_4 - 1]^2 \right] \right. \right. \\
& \left. \left. - 4k_1 \lambda^2 \sin^2 \beta [I_4 - 1] \right\}^2 \right] \exp \left(2\bar{k}_1 [I_4 - 1]^2 \right), \tag{3.22}
\end{aligned}$$

where I_4 is given in terms of the stretches in equation (3.11a). The last term in (3.20) accounts for the exponential response in (3.18) that is associated the invariant I_5 (and I_7),

$$\begin{aligned}
f_{I_5} = & \left[\lambda_z^2 \left\{ \lambda^2 \left[32k_2 \lambda^6 \sin^4 \beta + 24k_2 \lambda^2 \sin^2 \beta [I_5 - 1] + 64\bar{k}_2 k_2 \lambda^6 \sin^4 \beta [I_5 - 1]^2 \right] \right. \right. \\
& \left. \left. - 8k_2 \lambda^4 \sin^2 \beta [I_5 - 1] \right\} \right. \\
& \times \left[32k_2 \lambda_z^6 \cos^4 \beta + 24k_2 \lambda_z^2 \cos^2 \beta [I_5 - 1] + 64\bar{k}_2 k_2 \lambda_z^6 \cos^4 \beta [I_5 - 1]^2 \right] \\
& \left. - \left\{ \lambda_z \lambda \left[32k_2 \lambda_z^3 \lambda^3 \cos^2 \beta \sin^2 \beta + 64\bar{k}_2 k_2 \lambda_z^3 \lambda^3 \cos^2 \beta \sin^2 \beta [I_5 - 1]^2 \right] \right. \right. \\
& \left. \left. - 8k_2 \lambda^4 \sin^2 \beta [I_5 - 1] \right\}^2 \right] \exp \left(2\bar{k}_2 [I_5 - 1]^2 \right), \tag{3.23}
\end{aligned}$$

where I_5 is given in terms of the stretches in equation (3.11b)

Bulging in the limit $\bar{k}_1 \rightarrow 0$ and $\bar{k}_2 \rightarrow 0$: The exponential terms in the strain energy density function reduce to the standard forms in the limits

$$\lim_{\bar{k}_1 \rightarrow 0} \frac{k_1}{\bar{k}_1} \left[\exp \left(\bar{k}_1 [I_4 - 1]^2 \right) - 1 \right] = k_1 [I_4 - 1]^2 \tag{3.24a}$$

$$\lim_{\bar{k}_2 \rightarrow 0} \frac{k_2}{\bar{k}_2} \left[\exp \left(\bar{k}_2 [I_5 - 1]^2 \right) - 1 \right] = k_2 [I_5 - 1]^2 \tag{3.24b}$$

In these limits the contributions of the fibers to the bulging condition in (3.22) and (3.23) become

$$\begin{aligned}
 f_{I_4} = & \lambda_z^2 \{ \lambda^2 [4k_1 \sin^2 \beta [I_4 - 1] + 8k_1 \lambda^2 \sin^4 \beta] - 4k_1 \lambda^2 \sin^2 \beta [I_4 - 1] \} \\
 & \times [4k_1 \cos^2 \beta [I_4 - 1] + 8k_1 \lambda_z^2 \cos^4 \beta] \\
 & - [4k_1 \lambda^2 \sin^2 \beta [I_4 - 1] - 8k_1 \lambda_z^2 \lambda^2 \cos^2 \beta \sin^2 \beta]^2,
 \end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
 f_{I_5} = & \lambda_z^2 \{ \lambda^2 [32k_2 \lambda^6 \sin^4 \beta + 24k_2 \lambda^2 \sin^2 \beta [I_5 - 1]] - 8k_2 \lambda^4 \sin^2 \beta [I_5 - 1] \} \\
 & \times [32k_2 \lambda_z^6 \cos^4 \beta + 24k_2 \lambda_z^2 \cos^2 \beta [I_5 - 1]] \\
 & - [8k_2 \lambda^4 \sin^2 \beta [I_5 - 1] - 32k_2 \lambda_z^4 \lambda^4 \cos^2 \beta \sin^2 \beta]^2.
 \end{aligned} \tag{3.26}$$

The bulging condition function (3.20) with f_{I_1} and f_{I_2} in (3.21) and f_{I_4} and f_{I_5} in (3.25) and (3.26) corresponds to a strain energy density function in the form

$$W(I_1, I_2, I_4, I_5, v) = \frac{\mu_1}{2} (I_1 - 3v^{-2/3}) + \frac{\mu_2}{2} (I_2 - 3v^{-4/3}) + k_1 [I_4 - 1]^2 + k_2 [I_5 - 1]^2. \tag{3.27}$$

Different forms of (3.27) (for instance with $\mu_2 = k_2 = 0$, etc) have been used in the literature (see, e.g., El Hamdaoui et al. (El Hamdaoui et al., 2020; El Hamdaoui et al., 2014), Vinh et al. (P. C. Vinh and Merodio, 2013) and Goriely (Goriely, 2017)). In what follows bulging for different models is analyzed.

3.3.1 Mooney-Rivlin Behavior

Consider a swellable Mooney-Rivlin material having no fibers, which is described by the strain energy density function (3.18) with $k_1 = k_2 = 0$,

$$W(I_1, I_2, v) = \frac{\mu_1}{2} (I_1 - 3v^{-2/3}) + \frac{\mu_2}{2} (I_2 - 3v^{-4/3}). \tag{3.28}$$

This model has been applied in the modeling of the elastinous ground substance behavior of arterial tissue in works such as (Gundiah et al., 2007). While μ_1 is usually taken to be positive, the parameter μ_2 can take positive or negative values in the mechanical modeling. In this case, the bulging condition (3.20) reduces to $f = f_{I_1} + f_{I_2} = 0$. As it is clear from (3.21a) and (3.21b), the resulting equation is a polynomial of order 8 in λ and λ_z (and order 4 polynomial in v). Therefore numerical methods are utilized to analyze the bulging condition.

Some results are shown in Figure 3.2 for an unswollen material ($v = 1$). In both panels, the blue curves show the results for $\mu_2/\mu_1 > 0$, the red curves show the response for $\mu_2/\mu_1 < 0$, and the black curve shows the neo-Hookean response ($\mu_2 = 0$).

Figure 3.2 (a) shows different curves f/μ_1^2 vs. λ for an axial stretch of $\lambda_z = 1.1$ and different ratios of μ_1 to μ_2 . These curves take larger values for f when μ_2 increases. The curve for $\mu_2 = -\mu_1$ does not take positive values for f/μ_1^2 . Figure 3.2 (b) depicts relations between the circumferential and the axial stretches λ and λ_z which fulfill the bulging condition $f_{I_1} + f_{I_2} = 0$

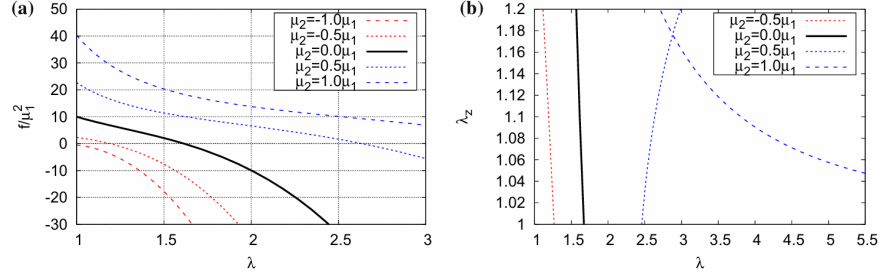


Figure 3.2: Equation (3.20) $f = f_{I_1} + f_{I_2} = 0$ gives the bulging condition for a Mooney-Rivlin material in the absence of fibers. (a) shows different curves of values f/μ_1^2 vs. λ for an axial stretch of $\lambda_z = 1.1$ and different ratios of μ_1 to μ_2 . In both figures, the blue curves show the results for $\mu_2/\mu_1 > 0$, the red curves show the response for $\mu_2/\mu_1 < 0$, and the black curve shows the neo-Hookean response ($\mu_2 = 0$). (b) depicts relations between the circumferential and the axial stretches λ and λ_z , respectively, which fulfill the bulging condition $f_{I_1} + f_{I_2} = 0$ for different ratios of μ_1 to μ_2 .

for different ratios of μ_1 to μ_2 . Notice that for $\mu_2 = -\mu_1$ and for the depicted values for λ_z and λ the bulging condition $f = 0$ is not fulfilled.

3.3.2 Fibers in neo-Hookean Matrix

In this section we consider two fiber families that are embedded in a swellable neo-Hookean ground material ($\mu_2 = 0$). Despite its simple form, the neo-Hookean model is often sufficient in describing the mechanical behavior of a ground material, in which the fibers are embedded. In the following parts we study the bulging behavior of fibers with a mechanical behavior described by either I_4 or I_5 .

Fiber behavior in terms of I_4 ($k_1 \neq 0$, $k_2 = 0$ in (3.18))

Let us consider a neo-Hookean matrix ($\mu_2 = 0$) with embedded fibers and $k_1 \neq 0$, $k_2 = 0$, for which the strain energy density (3.18) becomes

$$W(I_1, I_4, v) = \frac{\mu_1}{2} (I_1 - 3v^{-2/3}) + \frac{k_1}{\bar{k}_1} [\exp(\bar{k}_1 [I_4 - 1]^2) - 1]. \quad (3.29)$$

The contribution of the fibers to the strain energy density function is described by the exponential term and the invariant I_4 . This model has gained some popularity in the modeling of mechanical behavior of arteries (see, e.g., Holzapfel and Ogden, 2010a; Holzapfel et al., 2002). The exponential form is motivated by the observation that many biological materials have a common "exponential-shaped" stress-strain curve Shearer, 2015. Some aspects of the bulging bifurcation for (3.29) have been studied in Demirkoparan and Merodio, 2017. This article continues that investigation of the bulging bifurcation, which will then also serves as a reference for the other models analyzed in what follows in this article.

For the strain energy density function (3.29), (3.20) becomes $f_{I_1} + f_{I_4} = 0$, where f_{I_1} is given by equation (3.21a) and f_{I_4} is given by equation (3.22).

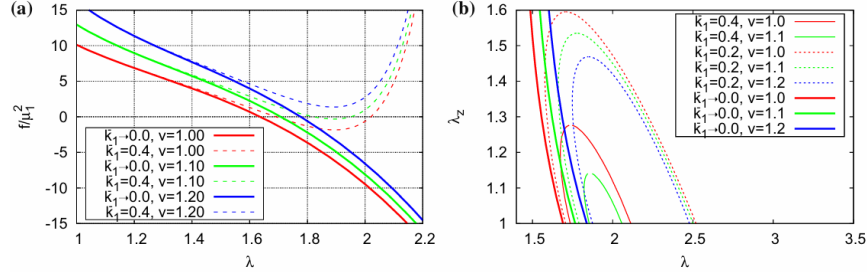


Figure 3.3: Bulging for a fiber-reinforced neo-Hookean material (3.29). (a) shows the function (3.20), which is $f = f_{I_1} + f_{I_4}$ (normalized by the square of the shear modulus μ_1), for an axial stretch $\lambda_z = 1.1$, $\beta = \pi/4$, $k_1 = \mu_1/100$, and for three swelling ratios $v = 1$ (no swelling), $v = 1.1$, and $v = 1.2$. The solid lines represent results for $\bar{k}_1 \rightarrow 0$ and the dashed lines represent results for $\bar{k}_1 = 0.4$. (b) shows different combinations for λ_z and λ that fulfill the condition $f_{I_1} + f_{I_4} = 0$ for $\beta = \pi/4$, $k_1 = \mu_1/100$, and for three swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$. The different line styles represent the results for different values of $\bar{k}_1$.

Some conditions for bulging are studied in the two panels of Figure 3.3, in which the results for the three swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$ are shown using different line colors.

Figure 3.3(a) shows the function (3.20), which is $f = f_{I_1} + f_{I_4}$ (normalized by μ_1^2), for an axial stretch $\lambda_z = 1.1$, $\beta = \pi/4$ and, $k_1 = \mu_1/100$. Increasing values for $\bar{k}_1$ or v lead to larger values of f . The solid lines represent results for $\bar{k}_1 \rightarrow 0$, which corresponds to the mechanical response of (3.27) for $\mu_2 = k_2 = 0$. The dashed lines represent the results for $\bar{k}_1 = 0.4$. When $\bar{k}_1 = 0.4$ bulging may occur for $v = 1$, while if $\bar{k}_1 = 0.4$ and $v = 1.2$ bulging will not occur because f remains positive.

Figure 3.3(b) shows different combinations for λ_z and λ that fulfill the condition $f_{I_1} + f_{I_4} = 0$ for $\beta = \pi/4$ and $k_1 = \mu_1/100$. The different line styles represent the results for different values of $\bar{k}_1$. This diagram shows that as $\bar{k}_1 \rightarrow 0$ the curves are decreasing so that for an axial stretch λ_z we find a unique value of the azimuthal stretch λ on the depicted range. On the other hand, when $\bar{k}_1 > 0$ a value of the axial stretch may have multiple corresponding values of the azimuthal stretch λ associated with bulging.

The examples of Figure 3.3 are restricted to a fixed value $\beta = \pi/4$ and the impact of the fiber orientation on the bulging for a fiber-reinforced neo-Hookean material (3.29) is studied in Figure 3.4.

Figure 3.4(a) shows the function (3.20) for $f = f_{I_1} + f_{I_4}$ (normalized by μ_1^2) for an axial stretch $\lambda_z = 1.1$, $k_1 = \mu_1/100$, $v = 1$, $\bar{k}_1 = 0.4$ and for different values for β on different ranges for f/μ_1^2 . In the case $\beta = 0$ the fibers are parallel to the axial direction of the cylinder, $\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{e}_z$, and in the case $\beta = \pi/2$ the fibers are oriented in the circumferential direction of the cylinder, $\mathbf{M}_1 = -\mathbf{M}_2 = \mathbf{e}_\theta$. The different curves show that if β is sufficiently close to $\pi/2$, then bulging will not occur. The polar diagram of Figure 3.4(b) shows different combinations for β (circumferential orientation) and λ (radial values) that fulfill the condition $f_{I_1} + f_{I_4} = 0$ for $k_1 = \mu_1/100$, $\bar{k}_1 = 0.4$, and $\lambda_z = 1$ and the three swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$. This diagram shows that the smallest values of λ that fulfill the bulging

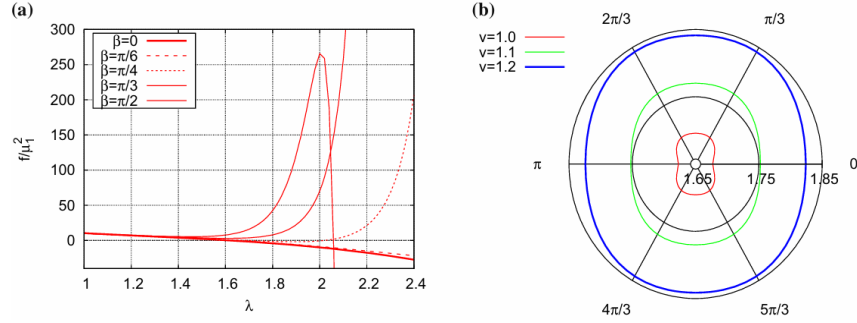


Figure 3.4: Impact of the fiber orientation on bulging for a fiber-reinforced neo-Hookean material (3.29). (a) shows the function (3.20) $f = f_{I_1} + f_{I_4}$ (normalized by the square of the shear modulus μ_1) for an axial stretch $\lambda_z = 1.1$, $k_1 = \mu_1/100$, $v = 1$, $\bar{k}_1 = 0.4$ and for different values for β on different ranges for f/μ_1^2 . The polar diagram of (b) shows different combinations for β (circumferential orientation) and λ (radial values) that fulfill the condition $f_{I_1} + f_{I_4} = 0$ for $k_1 = \mu_1/100$, $\bar{k}_1 = 0.4$, and $\lambda_z = 1$ for the three swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$.

condition $f_{I_1} + f_{I_4} = 0$ occur for $\beta = 0$ ($\beta = \pm\pi$), while the largest values of λ that fulfill this condition occur for $\beta = \pm\pi/2$. The figure shows symmetries with respect to horizontal and vertical axes of the polar diagram.

Fiber behavior in terms of I_5 ($k_1 = 0$, $k_2 \neq 0$ in (3.18))

Let us now consider a neo-Hookean matrix with embedded fibers and $k_1 = 0$, $k_2 \neq 0$, for which the strain energy density function (3.18) becomes

$$W(I_1, I_5, v) = \frac{\mu_1}{2} (I_1 - 3v^{-2/3}) + \frac{k_2}{\bar{k}_2} [\exp(\bar{k}_2 [I_5 - 1]^2) - 1]. \quad (3.30)$$

In this model the contribution of the fibers to the strain energy density function is described in exponential form by the invariant I_5 . For this strain energy density function the bulging condition (3.20) becomes $f = f_{I_1} + f_{I_5} = 0$, where f_{I_1} is given in equation (3.21a) and f_{I_5} is given in equation (3.22).

Figure 3.5 shows the bulging condition for three swelling ratios, which are highlighted by different line colors. In particular, Figure 3.5(a) shows the function (3.20) for $f = f_{I_1} + f_{I_5}$ (normalized by μ_1^2) for an axial stretch of $\lambda_z = 1.1$, $\beta = \pi/4$, and $k_1 = \mu_1/100$. The solid lines represent results for $\bar{k}_2 \rightarrow 0$, and the dashed lines represent the results for $\bar{k}_2 = 0.001$.

Figure 3.5(b) shows different combinations for λ_z and λ that fulfill the condition $f_{I_1} + f_{I_5} = 0$ for $\beta = \pi/4$, $k_2 = \mu_1/100$, $\bar{k}_2 \rightarrow 0$ and for the three swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$.

Figure 3.6 depicts the impact of fiber orientation on bulging for the fiber-reinforced neo-Hookean material (3.30).

Figure 3.6(a) shows the function (3.20) for $f = f_{I_1} + f_{I_5}$ (normalized by the square of the

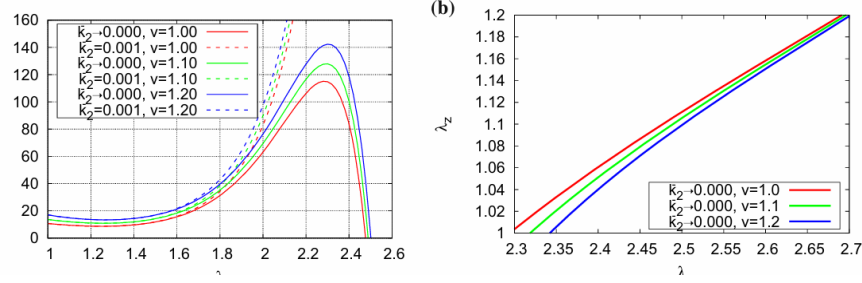


Figure 3.5: Bulging for a fiber-reinforced neo-Hookean material (3.30). (a) shows the function (3.20) for $f = f_{I_1} + f_{I_5}$ (normalized by μ_1^2) for an axial stretch of $\lambda_z = 1.1$, $\beta = \pi/4$, $k_2 = \mu_1/100$, and for the three swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$. The solid lines represent results for $\bar{k}_2 \rightarrow 0$, and the dashed lines represent the results for $\bar{k}_2 = 0.001$.

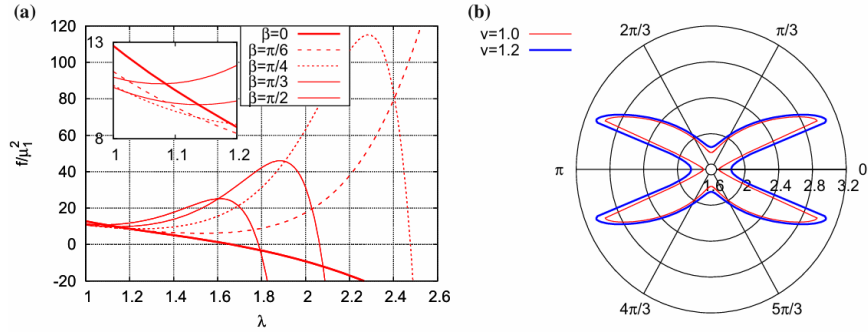


Figure 3.6: Impact of fiber orientation on bulging for a fiber-reinforced neo-Hookean material (3.30). (a) shows the function (3.20) for $f = f_{I_1} + f_{I_5}$ (normalized by the square of the shear modulus μ_1) for an axial stretch of $\lambda_z = 1.1$, $k_2 = \mu_1/100$, $v = 1$, $\bar{k}_2 = 0$, and for different values for β . The polar diagram of (b) shows different combinations for β (circumferential orientation) and λ (radial values) that fulfill the condition $f_{I_1} + f_{I_5} = 0$ for $\lambda_z = 1.00$, $k_2 = \mu_1/100$, $\bar{k}_1 = 0.0$, and for the two swelling ratios $v = 1$ and $v = 1.2$.

shear modulus μ_1) for an axial stretch of $\lambda_z = 1.1$, $k_2 = \mu_1/100$, $v = 1$, $\bar{k}_2 = 0$, and for different values β .

The polar diagram of Figure 3.6(b) shows different combinations for β (circumferential orientation) and λ (radial values) that fulfill the condition $f_{I_1} + f_{I_5} = 0$ for $\lambda_z = 1.00$, $k_2 = \mu_1/100$, $\bar{k}_2 = 0.0$, and two swelling ratios $v = 1$ and $v = 1.2$. This diagram shows the following for values $\pi/2 \geq \beta \geq 0$. As β increases from $\beta = 0$, the values of λ associated with bulging increase until β reaches an angle close to $\pi/6$ from which as β increases the values of λ associated with bulging decrease.

Fiber behavior in terms of both I_4 and I_5

As a last example of this section, we now consider a swellable neo-Hookean matrix containing two families of symmetric fibers, for which the strain energy density is given by (3.27) with $\mu_2 = 0$. In this case, the strain energy density function is formulated in terms of both I_4 and

I_5 . Figure 3.7 shows combinations for $k_1 \geq 0$ and $k_2 \geq 0$ that fulfill the bulging condition (3.20) for $\lambda = 2$ with f_{I_1} given by (3.21a), $f_{I_2} = 0$, and f_{I_4} and f_{I_5} given by (3.25) and (3.26), respectively, with $\lambda_z = 1.1$, three swelling values $v = 1$, $v = 1.1$, and $v = 1.2$ and different values of β . Fiber winding angle values $\beta = \pi/6$ and $\beta = \pi/4$ are used in Figure 3.7(a) and (b), respectively. Under these conditions there is only one solution. This is not the same for $\beta = \pi/3$. Under these circumstances, there are two positive roots fulfilling the bulging condition (3.20), which are shown Figures 3.7 (c) and (d).

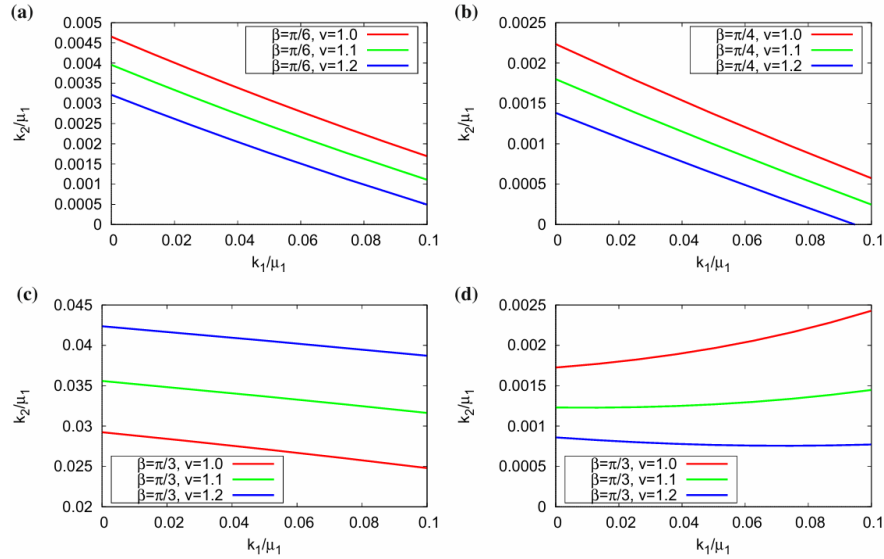


Figure 3.7: This figure shows combinations for $k_1 \geq 0$ and $k_2 \geq 0$ that fulfill the bulging condition (3.20) for $\lambda = 2$ with f_{I_1} given by (3.21a), $f_{I_2} = 0$, and f_{I_4} and f_{I_5} given by (3.25) and (3.26), respectively, where k_1 and k_2 are normalized by μ_1 . These figures take $\lambda_z = 1.1$ and swelling ratios $v = 1$, $v = 1.1$, and $v = 1.2$ for the matrix. (a) is for $\beta = \pi/6$ and (b) is for $\beta = \pi/4$. There is a unique solution under these circumstances. On the other hand, (c) and (d) show solutions for $\beta = \pi/3$.

3.4 Axial propagation of bulging under swelling

Axial, quasi-static propagation of a bulging-instability in thick-walled cylinders may be subdivided into two stages. In the first stage the inner pressure remains constant during the ensuing propagation of the bulging instability mode beyond the onset of bifurcation until a suitable configuration is obtained. In the second stage, in a subsequent motion, for an ongoing axial propagation of the bulging the inflation pressure has to be increased. These configurations can be described by two uniform cylinders, one that is characterized by the pair of azimuthal and axial stretches $(\lambda_{\theta 1}, \lambda_{z 1})$ and other that is characterized by the pair $(\lambda_{\theta 2}, \lambda_{z 2})$, and by a transition zone between these cylinders (see Figure 3.8). Furthermore, it shall be noticed that radial expansion of bulging is related to a decrease of pressure beyond the onset of bulging. These two pairs $(\lambda_{\theta 1}, \lambda_{z 1})$ $(\lambda_{\theta 2}, \lambda_{z 2})$ fulfill the conditions (Demirkoparan and Merodio, 2020)

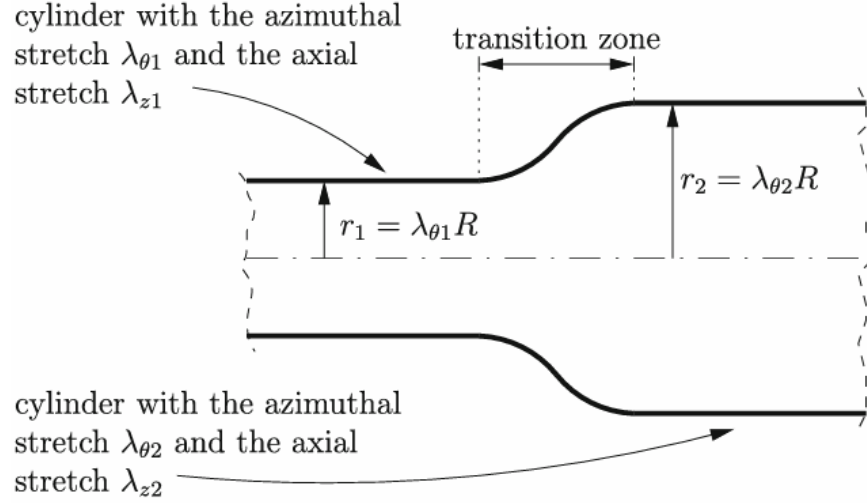


Figure 3.8: Cylindrical membrane under a constant inner pressure: bulge in equilibrium after bifurcation. Two uniform cylinders are shown, one that is characterized by the pair of azimuthal and axial stretches $(\lambda_{\theta 1}, \lambda_{z 1})$ and other that is characterized by the pair $(\lambda_{\theta 2}, \lambda_{z 2})$. These cylinder are connected by a transition zone between these cylinders.

$$\left. \frac{\partial \hat{W}}{\partial \lambda_z} \right|_{\lambda_z = \lambda_{z2}}^{\lambda_\theta = \lambda_{\theta 2}} - \frac{\lambda_{\theta 2}^2 - \lambda_{\theta 1}^2}{2\lambda_{\theta 1}\lambda_{z1}} \left. \frac{\partial \hat{W}}{\partial \lambda_\theta} \right|_{\lambda_z = \lambda_{z1}}^{\lambda_\theta = \lambda_{\theta 1}} - \left. \frac{\partial \hat{W}}{\partial \lambda_z} \right|_{\lambda_z = \lambda_{z1}}^{\lambda_\theta = \lambda_{\theta 1}} = 0 \quad (3.31a)$$

$$\left. \hat{W} \right|_{\lambda_z = \lambda_{z2}}^{\lambda_\theta = \lambda_{\theta 2}} - \lambda_{z2} \left. \frac{\partial \hat{W}}{\partial \lambda_z} \right|_{\lambda_z = \lambda_{z2}}^{\lambda_\theta = \lambda_{\theta 2}} - \left. \hat{W} \right|_{\lambda_z = \lambda_{z1}}^{\lambda_\theta = \lambda_{\theta 1}} + \lambda_{z1} \left. \frac{\partial \hat{W}}{\partial \lambda_z} \right|_{\lambda_z = \lambda_{z1}}^{\lambda_\theta = \lambda_{\theta 1}} = 0 \quad (3.31b)$$

$$\lambda_{\theta 2}\lambda_{z2} \left. \frac{\partial \hat{W}}{\partial \lambda_\theta} \right|_{\lambda_z = \lambda_{z1}}^{\lambda_\theta = \lambda_{\theta 1}} - \lambda_{\theta 1}\lambda_{z1} \left. \frac{\partial \hat{W}}{\partial \lambda_\theta} \right|_{\lambda_z = \lambda_{z2}}^{\lambda_\theta = \lambda_{\theta 2}} = 0 \quad (3.31c)$$

that can be derived after some manipulations from either (Fu et al., 2008) or (Alhayani et al., 2014a). While the works (Fu et al., 2008; Alhayani et al., 2014a) do not include swelling, the system of equation (3.31) has been studied for an swellable fiber-reinforced matrix in (Demirkoparan and Merodio, 2020), where the material has been taken to remain incompressible at all instants of swelling.

In what follows we study axial propagation of bulging by means of some numerical examples. Results are presented in the different parts of Table 3.1.

Consider a swellable Mooney-Rivlin material with a strain energy density given by equation (3.28). Table 3.1(a) shows results for axial propagation of bulging under swelling. This table shows the pairs $(\lambda_{\theta 1}, \lambda_{z 1})$ and $(\lambda_{\theta 2}, \lambda_{z 2})$ for $\lambda_{z 1} = 1.1$ and different values for the swelling v that are determined from system of equation (3.31). The results are shown for $\mu_2 = 0$ (neo-Hookean specialization) and for $\mu_2 = -\mu_1/2$.

We now consider a neo-Hookean matrix with embedded fibers and $k_1 \neq 0$, $k_2 = \mu_2 = 0$, for which the strain energy density function (3.18) takes the form (3.29). Table 3.1(b) shows

Table 3.1: Panel (a): Axial propagation of bulging under swelling for the Mooney Rivlin material (3.28). The table shows the pairs $(\lambda_{\theta 1}, \lambda_{z1})$ and $(\lambda_{\theta 2}, \lambda_{z2})$ for $\lambda_{z1} = 1.1$ and different values for the swelling v that are determined from system of equation (3.31). The results are shown for $\mu_2 = 0$ (neo-Hookean specialization) and for $\mu_2 = -\mu_1/2$. Panel (b) studies axial propagation of bulging under swelling for a neo-Hookean matrix and fiber behavior in terms of I_4 (3.29). It shows the pairs $(\lambda_{\theta 1}, \lambda_{z1})$ and $(\lambda_{\theta 2}, \lambda_{z2})$ for $\lambda_{z1} = 1.1$, for the material parameters $k_1 = 0.05\mu_1$ and $\bar{k}_1 = 0.1$, and for different values of the swelling v that are determined from system of equation (3.31). The results are consistent with those that have been obtained by Demirkoparan & Merodio (Demirkoparan and Merodio, 2020) and Alhayani et al. (Alhayani et al., 2014a). Panel (c) studies axial propagation of bulging under swelling for a neo-Hookean matrix and fiber behavior in terms of I_5 (3.30). It shows the pairs $(\lambda_{\theta 1}, \lambda_{z1})$ and $(\lambda_{\theta 2}, \lambda_{z2})$ for $\lambda_{z1} = 1.1$, for the material parameters $k_2 = 0.01\mu_1$ and $\bar{k}_2 = 0.01$, and for different values of the swelling v that are determined from system of equation (3.31).

a)						
v	$\mu_2 = 0$			$\mu_2 = -\mu_1/2$		
	λ_{z2}	$\lambda_{\theta 1}$	$\lambda_{\theta 2}$	λ_{z2}	$\lambda_{\theta 1}$	$\lambda_{\theta 2}$
1.00	1.1021	1.6120	1.6160	1.1012	1.1908	1.1921
1.10	1.1017	1.6836	1.6869	1.1031	1.2333	1.2363
1.20	1.1014	1.7520	1.7549	1.1012	1.2743	1.2754
b)						
v	$\mu_2 = 0$					
	λ_{z2}	$\lambda_{\theta 1}$	$\lambda_{\theta 2}$			
1.00	1.1021	1.6120	1.6160			
1.10	1.1017	1.6836	1.6869			
1.20	1.1014	1.7520	1.7549			
c)						
v	$\mu_2 = 0$					
	λ_{z2}	$\lambda_{\theta 1}$	$\lambda_{\theta 2}$			
1.00	1.1021	1.6120	1.6160			
1.10	1.1017	1.6836	1.6869			
1.20	1.1014	1.7520	1.7549			

the pairs $(\lambda_{\theta 1}, \lambda_{z 1})$ and $(\lambda_{\theta 2}, \lambda_{z 2})$ for $\lambda_{z 1} = 1.1$, and for the material parameters $k_1 = 0.05\mu_1$ and $\bar{k}_1 = 0.1$, and for different values of the swelling v that are determined from system of equation (3.31).

In a last example, let us now turn to study a neo-Hookean matrix with embedded fibers and $k_2 \neq 0$, $k_1 = \mu_2 = 0$, for which the strain energy density function (3.18) takes the form (3.30). Table 3.1(c) shows the pairs $(\lambda_{\theta 1}, \lambda_{z 1})$ and $(\lambda_{\theta 2}, \lambda_{z 2})$ for $\lambda_{z 1} = 1.1$, and for the material parameters $k_2 = 0.01\mu_1$ and $\bar{k}_2 = 0.01$, and for different values of the swelling v that are determined from system of equation (3.31).

3.5 Concluding remarks

This article studies the effect of swelling, as well as other factors, on bulging initiation and propagation of pressurized membranes under axial loading. The material is taken to be hyperelastic and reinforced by two fiber families in helical arrangements.

In our present modeling the fibers are taken to be parallel to each other within a fiber family. The article (Gasser et al., 2006) studies bulging bifurcation for a fiber-reinforced cylinder, in which the fiber dispersion is quantified by the so-called κ -model (Gasser et al., 2006). Modeling of fiber dispersion has been beyond the scope of this article, and for different modeling techniques that account for fiber dispersion we refer to the review (Holzapfel et al., 2019). The mechanical properties of the constituents have been taken to be time independent, and for example viscoelastic behavior has been neglected. In biological soft tissue, collagen fiber undergo remodeling processes that are related to different biochemical and physical effects such as their stretch history (Topol et al., 2015). The interplay of inflation behavior and their instabilities in combination with fiber remodeling has been studied in (Topol et al., 2017). In this work, the hollow cylinder has been studied in the membrane treatment. The recent article studies the inflation of thick-walled cylinders with mechano-sensitive fiber remodeling (Topol et al., 2019).

The herein presented results may have implications in the initiation and propagation of aneurysms in the different layers of arterial tissue.

Chapter 4

Bifurcation of Fiber-Reinforced Cylindrical Membranes under Extension, Inflation, and Swelling

“This chapter is based on the below article that is published corresponding this thesis: Heiko Topol, Murtadha J Al-Chlahawi, Hasan Demirkoparan, and José Merodio. Bifurcation of fiber-reinforced cylindrical membranes under extension, inflation, and swelling. *Journal of Applied and Computational Mechanics*, 9(1):113–128, 2023 (Topol, Al-Chlahawi, et al., [2023](#))

4.1 Introduction

Abdominal aortic aneurysms are reported to be responsible for 1-2% of all deaths in industrialized countries (Lindsay and Dietz, [2011](#)). In many cases, this pathology remains undiagnosed until rupture occurs and it has a mortality rate that is greater than 80% (Schermerhorn, [2009](#)). This has been a main motivation for researchers in different areas to develop models to capture and to understand the formation, development and rupture of aneurysms.

There are many factors that play an important role in the formation of such pathological dilatations of arteries, among them, one can include geometry, mechanical properties as well as numerous biochemical processes. Different research works study the formation and development of aneurysms from a mechanical point of view as a mechanism related to various bifurcation and post-bifurcation behaviors of extended and pressurized soft hollow cylinders. Indeed, the formation of instabilities depends on the material as well as on both the internal pressure of the cylinder and the axial loading (stretch), which has been shown in different theoretical and experimental works during the last decades (Corneliussen and Shield, [1961](#); Biot, [1965](#); Gonçalves et al., [2008](#); Haughton and Ogden, [1979](#); Humphrey et al., [2009](#); Pamplona et al., [2006](#); R. W. Ogden, [2009](#); Merodio and Haughton, [2010](#); Rodríguez and Merodio, [2011](#); Alhayani et al., [2013](#); El Hamdaoui et al., [2014](#); El Hamdaoui and Merodio, [2015](#); El Hamdaoui et al., [2018](#); Fu et al., [2016](#); Amabili et al., [2019](#)),

This article focuses on three bifurcation modes that may occur in the context of axially loaded

and inflated anisotropic and volume changing cylinders, namely bulging bifurcation, prismatic bifurcation, and bending (composite) bifurcation. In the context of hyperelastic materials the conditions for the existence of such instability modes have been derived, for example, by (R. W. Ogden, 2009). Bulging bifurcation has been investigated in various contexts (R. W. Ogden, 2009; Merodio and Haughton, 2010), but the same attention has not been paid to the other two bifurcation modes. Very recently it has been recognized that coupling of different bifurcation modes during postbifurcation is important to capture the development of abdominal aortic aneurysms (El Hamdaoui et al., 2018, (Fu et al., 2016)). With this in mind, we pay attention to the onset of different bifurcation modes for a swellable material reinforced with two preferred directions. Prismatic bifurcation analyses can be found for both cylindrical membranes (Gonçalves et al., 2008) and for thick-walled tubes (Dehghani et al., 2019). Similarly, different bifurcation modes have been studied in the modeling of arterial tissue that is affected by *Marfan's syndrome*, which is a connective tissue disorder (Pamplona et al., 2006; Font et al., 2021). Residual stresses have an influence on the bifurcation of extended and inflated cylinders (Hejazi et al., 2021; Haughton and Ogden, 1979). In particular, (Haughton and Merodio, 2009) presents a computational bending and bulging bifurcation analysis for residually stressed hyperelastic tubes under combined inflation and extension to model aneurysms in arterial tissue.

Many works on bifurcation consider a volume-preserving context, whereas different factors such as edema, mechanical trauma, and inflammations (L. Dorfmann and Ogden, 2021; Melnikov et al., 2021; Desena-Galarza et al., 2021) lead to swelling of biological materials. Swelling may not only increase the volume, but also the stiffness of the swollen material. Both, the change in the geometry and in the mechanical properties have an impact on bifurcation, i.e., swelling may stabilize the cylinder against the occurrence of one particular bifurcation mode while it may trigger the formation of other bifurcation modes. In hyperelastic materials, swelling has been taken into account in different modeling works for isotropic (Barzó et al., 1997; D'lima et al., 2001) and anisotropic materials (Tracey, 2002). It has been shown that collagen remodeling processes may be driven by changes in the tissue volume (see Tsai et al., 2004; Zamani et al., 2018). Our previous works have shown that changes in the material volume may have a beneficial effect with respect to preventing bulging bifurcation (Demirkoparan and Merodio, 2020; Gou et al., 2020). (Topol et al., 2018) then explore the role of swelling on the formation of instability modes, namely prismatic and bending bifurcation in some fibrous cylinders, which motivates this work. For most of the works that invoke swelling to study bifurcation of inflated and extended cylinders the material stiffness is taken to be independent of swelling (Demirkoparan and Merodio, 2020; Gou et al., 2020, (Topol et al., 2018)). Here we also pay attention to that aspect. In addition, many of the works on the formation of an instability have their focus on the individual analysis of bifurcation modes. This work compares the occurrence of bulging, prismatic, and bending bifurcation in fibrous and swelling cylinders with the aim to establish possible coupling of different modes during postbifurcation.

This article is structured as follows: Section 4.2 introduces the main kinematics of the problem at hand as well as the material models. Section 4.3 discusses the bifurcation modes which in Section 4.4 are examined for the material models under consideration. More in particular, Section 4.4 studies bifurcation for an isotropic cylinder (without fibers) based on

the assumption that the material stiffness is independent of swelling effects. On the other hand, in Section 4.5, bifurcation is studied for a fibrous material in which the mechanical properties of the ground substance are negligible. Finally, Section 4.6 illustrates the impact of swelling-dependent stiffness changes in a ground substance with embedded fibers on bifurcation. The last section, Section 4.7, concludes the article with a discussion of the results and a brief outlook.

4.2 Kinematics and Mechanical Behavior of the Cylindrical Membrane

In a reference (unloaded) configuration $\mathcal{B}_r$, an undeformed and hollow cylinder with inner radius A , outer radius B , and length L is described in terms of a cylindrical coordinate system (R, Θ, Z) with unit vectors $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$, respectively, as

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad -L/2 \leq Z \leq L/2 \quad (4.1)$$

Any point of this cylinder can be given by the position vector

$$\mathbf{X} = R\mathbf{E}_R(\Theta) + Z\mathbf{E}_Z \quad (4.2)$$

The cylinder is subject to both an inflation pressure P and an axial load N so that it preserves the shape of a circular tube (see Figure 4.1)). In the deformed configuration $\mathcal{B}$, the cylinder is described in terms of a cylindrical coordinate system (r, θ, z) with unit vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, respectively, as

$$a \leq R \leq b, \quad 0 \leq \theta \leq 2\pi, \quad -\ell/2 \leq z \leq \ell/2 \quad (4.3)$$

and any point of this cylinder can be given by the position vector

$$\mathbf{x} = r\mathbf{e}_r(\theta) + z\mathbf{e}_z \quad (4.4)$$

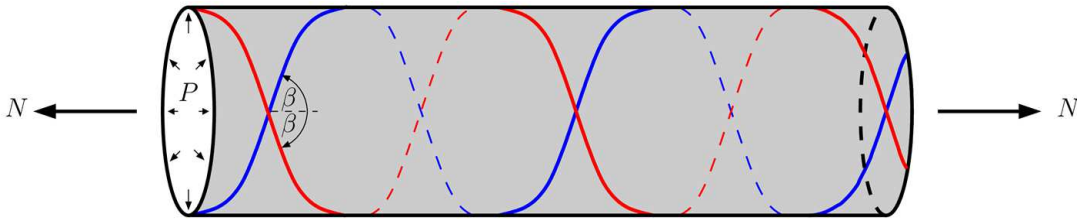


Figure 4.1: A cylinder that is subjected to both an inflation pressure P and an axial load N . The material consists of a swellable isotropic ground substance material that is reinforced by two families of fibers.

The material is capable of swelling, which is described by the ratio of the swollen material volume to the unswollen material volume by means of the parameter v . At each time during the swelling process the material is taken to be incompressible. Such an idealization is reasonable in arterial tissue, for instance, due to the high water content of the tissue. Our considerations focus on uniform swelling of the material without local variations. For such a deformation of the cylinder, the deformation gradient has the form

$$\mathbf{F} = \text{diag}(\lambda_r, \lambda_\theta, \lambda_z) \quad (4.5)$$

where λ_r is the radial stretch, $\lambda_\theta = r/R \equiv \lambda$ is the circumferential (azimuthal) stretch and $\lambda_z = \ell/L$ is the axial stretch. The condition $\det \mathbf{F} = v$ for the swollen material allows to write the radial stretch as $\lambda_r = v\lambda^{-1}\lambda_z^{-1}$.

For such a swelling solid the relation between the Cauchy stress tensor σ and the deformation can be expressed via

$$\sigma = -p\mathbf{I} + \frac{1}{v} \frac{\partial W}{\partial \mathbf{F}} \mathbf{F}^T \quad (4.6)$$

where $\mathbf{I}$ is the second order identity tensor and p is a scalar that results from the (incompressibility) constraint. The Cauchy stress tensor satisfies the condition $\text{div} \sigma = \mathbf{0}$ in the absence of body forces. There exist numerous material models that have been derived to describe the mechanical behavior of fibrous soft tissue (Chagnon et al., 2015). In these models, the material is treated in a homogenized sense so that each material point contains both ground substance and fiber constituents. We consider orthotropic materials with two preferred directions which are mechanically equivalent and are symmetrically disposed. The mechanical behavior of such materials is often defined in terms of a strain energy density function which may be decomposed into a contribution W_m of the ground substance material and into a contribution W_f of the embedded fibers, i.e.

$$W = W_m + W_f. \quad (4.7)$$

Here, the mechanical behavior of the isotropic ground substance is described by the Treloar model (Treloar, 1975),

$$W_m(I_1, v) = \frac{\mu v^{q-2/3}}{2} (I_1 - 3v^{2/3}) \quad (4.8)$$

where

$$I_1 = \text{tr} \mathbf{C} = \frac{v^2}{\lambda^2 \lambda_z^2} + \lambda^2 + \lambda_z^2 \quad (4.9)$$

is the first invariant of the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$, the parameter v accounts for swelling, and μ is the shear modulus for an unswollen material. The term $\mu v^{q-2/3}$

can be seen as a swelling-dependent shear modulus, in which the parameter q quantifies the change of the ground substance stiffness with swelling. If $q = 2/3$, then the ground substance is swelling-independent. The matrix stiffness decreases with swelling if $q < 2/3$, and it increases with swelling if $q > 2/3$. The model (4.8) is a generalization of the incompressible neo-Hookean model that accounts for swelling, and it has found its application in different modeling works (Gou et al., 2020; Topol et al., 2018; Demirkoparan and Pence, 2007b; Demirkoparan and Pence, 2008). Despite their relatively simple form, neo-Hookean type models are used to model arterial tissue. For example, it has been shown that such type of models can (sufficiently) describe the mechanical behavior of arterial elastin (Gundiah et al., 2009). Furthermore, the neo-Hookean form can be understood as a particular case of many constitutive equations used in the modeling of soft tissue constituents. For instance, Mooney-Rivlin materials are a generalization of the neo-Hookean model, and swelling of Mooney-Rivlin type models have been studied to capture bulging bifurcation in (Topol, Al-Chlahawi, et al., 2021). Different works that account for swelling in the examination of bifurcation neglect the role of the parameter q in (4.8), which, in addition, motivates the present work.

With respect to the fibers, it is assumed that they are embedded in the ground substance and are incapable of swelling. In the membrane, fibers are grouped in two (fiber) families with equal mechanical properties. The initial (undeformed) orientation of each fiber family is given by a unit vector in the reference configuration $\mathcal{B}_r$. Both families are symmetrically disposed with respect to the axial direction. It follows that the two unit vectors in $\mathcal{B}_r$ giving the directions of the two families of fibers can be written as

$$\mathbf{M}_1 = \cos \beta \mathbf{e}_Z + \sin \beta \mathbf{e}_\Theta \quad (4.10a)$$

$$\mathbf{M}_2 = \cos \beta \mathbf{e}_Z - \sin \beta \mathbf{e}_\Theta \quad (4.10b)$$

in which β , the fiber winding angle, gives the orientation of each fiber family with respect to the axial direction of the membrane. The mechanical behavior of the first fiber family is described in terms of the invariant I_4 , and the mechanical behavior of the second fiber family is described in terms of the invariant I_6 (which are sometimes referred to as “*pseudo-invariants*”), which are

$$I_4 = \mathbf{C} : \mathbf{M}_1 \otimes \mathbf{M}_1 = \lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta \quad (4.11a)$$

$$I_6 = \mathbf{C} : \mathbf{M}_2 \otimes \mathbf{M}_2 = \lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta \quad (4.11b)$$

Under the circumstances at hand, I_4 and I_6 are equal. These invariants are interpreted as the squared magnitude of the fiber stretch. The fibers bear tension, whereas they are taken to be incapable of carrying (fiber) compression. This means that I_4 and I_6 can just take values that are equal to one or greater than one. While in this article the elastic fiber behavior is described in terms of the invariants I_4 and I_6 , higher invariants may be introduced to account for different mechanical properties of the fibers and their interplay. We should also note that in this work we assume that the fibers are in a perfect parallel arrangement within each fiber

family. Modifications of I_4 and I_6 and higher invariants that account for fiber dispersion are discussed in (Gasser et al., 2006). In addition, in our modeling, all constituents are taken to share a common natural configuration, and, for convenience, this configuration is taken to be the reference configuration. Topol et al. (Topol et al., 2022) consider bulging bifurcation in a pressurized cylinder, in which the natural configuration of the constituents may differ.

We further consider a strain energy density for the fibers in the form of an exponential function as

$$W_f(I_4, I_6) = \frac{k_1}{2\bar{k}_1} \sum_{i=4,6} [\exp(\bar{k}_1(I_i - 1)^2) - 1] \quad (4.12)$$

where k_1 is a stiffness parameter and $\bar{k}_1$ can be regarded as a sensitivity parameter. These strain energy functions in exponential form are used to model arteries (Holzapfel et al., 2000; Holzapfel et al., 2004), and they are motivated by a peculiar mechanical behavior of fibers, namely, fibers are initially crimped when unloaded, then they unwind (or un-crimp) as they begin to be extended until, finally, the fibers fully bear (axial) loading as they are fully unwound (Holzapfel and Ogden, 2010a).

Since the fiber families are mechanically equivalent and symmetrically disposed, it follows that $I_4 = I_6$, and, furthermore, that

$$\frac{k_1}{\bar{k}_1} [\exp(\bar{k}_1 [I_4 - 1]^2) - 1] = \frac{k_1}{2\bar{k}_1} \sum_{i=4,6} [\exp(\bar{k}_1 [I_i - 1]^2) - 1], \quad (4.13)$$

which, in turn, allows to write the strain energy contribution (4.12) of the two families of fibers as

$$W_f(I_4) = \frac{k_1}{\bar{k}_1} [\exp(\bar{k}_1 [I_4 - 1]^2) - 1]. \quad (4.14)$$

Notice that the energy form (4.14) reduces to the so called standard reinforcing model in the limit

$$\lim_{\bar{k}_1 \rightarrow 0} W_f(I_4) = k_1 [I_4 - 1]^2. \quad (4.15)$$

In summary, the strain energy is considered to be $W(I_1, v, I_4) = W_m(I_1, v) + W_f(I_4)$. Both terms $W_m(I_1, v)$ and $W_f(I_4)$ will be analyzed independently as well as combined with regard to bifurcation. Due to the dependence of the invariants on the stretches λ and λ_z and on the swelling field we introduce the (convenient) notation $\hat{W}(v, \lambda, \lambda_z) = W(I_1, v, I_4)$. Similarly, and accordingly, one can also write $\hat{W}_m$ and $\hat{W}_f$. In the next section, firstly, bifurcation modes of the cylinder are introduced in an illustrative way.

4.3 Bifurcation Modes in the Membrane Approximation

We consider the cylinder in the membrane approximation, in which the cylinder is taken to have the midsurface radius $R = (A + B)/2$ and the wall thickness $H = (B - A) \ll R$. The cylinder is assumed to have an infinite length $L \rightarrow \infty$ or, alternatively, $R \ll L$. This membrane approximation permits to obtain analytical expressions for bifurcation modes. The results from these investigations give a basic understanding of bifurcation and serve as guidelines for future works dealing with thick-walled tubes, which may have to be carried out numerically.

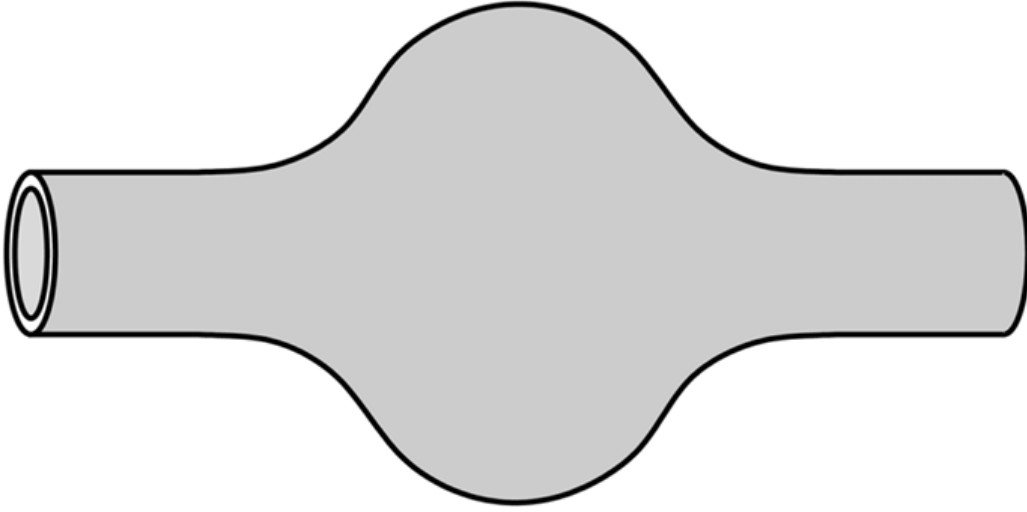


Figure 4.2: Bulging bifurcation of a circular cylinder.

In this membrane approximation, the relation between the inflation pressure P , the axial normal force N , and the deformation is given by (R. W. Ogden, 1997)

$$P = \frac{H}{R} \frac{\hat{W}_\lambda}{\lambda \lambda_z}, \quad N = \hat{W}_{\lambda_z} \quad (4.16)$$

where here and in the following we apply the notation $\hat{W}_\lambda = \partial \hat{W} / \partial \lambda$ in order to describe the partial derivatives of the strain energy density function $\hat{W}$. The radial normal stresses are negligible in the membrane approximation, i.e. $\sigma_{rr} = 0$ and the components $\sigma_{\theta\theta}$ and σ_{zz} of the Cauchy stress tensor (4.6) can be written as

$$\sigma_{\theta\theta} = v^{-1} \lambda \hat{W}_\lambda, \quad \sigma_{zz} = v^{-1} \lambda_z \hat{W}_{\lambda_z} \quad (4.17)$$

Note that the scalar p arising from the incompressibility constraint has already been eliminated in (4.17).

We study three bifurcation modes, namely, bulging bifurcation, prismatic bifurcation, and bending bifurcation. Using incremental displacements the bifurcation criterion for incompress-

ible and anisotropic cylinders can be found in (Rodríguez and Merodio, 2011). Their result is in agreement with the analysis of (Haughton and Ogden, 1979) for isotropic materials, and in recent papers the bulging condition has been used to analyze axial propagation of bulging for thin-walled cylinders under internal pressurization (Alhayani et al., 2014a). To investigate the possible bifurcation we consider incremental displacements with respect to the deformed configuration under equilibrium. In the general form, one considers (Rodríguez and Merodio, 2011)

$$\delta \mathbf{u} = \delta u_r(\theta, z) \mathbf{e}_r + \delta u_\theta(\theta, z) \mathbf{e}_\theta + \delta u_z(\theta, z) \mathbf{e}_z \quad (4.18)$$

in which δ is the increment and $\mathbf{u}$ is the displacement with components (u_r, u_θ, u_z) .

4.3.1 Bulging bifurcation

Figure 4.2 depicts bulging bifurcation in a circular cylinder. The origin of this analysis goes back a while ago (see Biot, 1965; Haughton and Ogden, 1979). Nevertheless this topic is being studied extensively by different research groups (Fu et al., 2016; Dehghani et al., 2019; Font et al., 2021; Hejazi et al., 2021). In order to investigate the possible bulging bifurcation mode of the cylinder, we particularize (4.18) and consider incremental displacements

$$\delta u = \delta u_r(z) \mathbf{e}_r + \delta u_z(z) \mathbf{e}_z. \quad (4.19)$$

The bulging condition for a cylindrical membrane of radius R and length L is (see for instance (Rodríguez and Merodio, 2011; Demirkoparan and Merodio, 2020; Demirkoparan and Merodio, 2017))

$$f(\hat{W}, \lambda, \lambda_z) + \left(\frac{2\pi R}{L} \right)^2 \lambda^2 \lambda_z \hat{W}_{\lambda_z} \hat{W}_{\lambda \lambda_z} = 0 \quad (4.20)$$

where

$$f(\hat{W}, \lambda, \lambda_z) = \lambda_z^2 \hat{W}_{\lambda \lambda_z} (\lambda^2 \hat{W}_{\lambda \lambda} - \lambda \hat{W}_\lambda) - (\lambda \lambda_z \hat{W}_{\lambda \lambda_z} - \lambda \hat{W}_\lambda)^2. \quad (4.21)$$

For a cylinder of infinite length, $L \rightarrow \infty$, the second term on the left side of ((4.20)) becomes negligible. Under these circumstances, for the material model at hand, the bulging condition ((4.20)) can be written as

$$f = f_{I_1} + f_{I_4} = 0, \quad (4.22)$$

in which the term f_{I_1} results from the mechanical behavior of the swellable neo-Hookean ground substance given in ((4.8)), while the term f_{I_4} represents the behavior of the fibers given in (4.14). More in particular, it is easy to obtain that

$$f_{I_1} = \frac{[\mu v^{q-2/3}]^2 (-\lambda_z^4 \lambda^8 + 4\lambda_z^4 \lambda^2 v^2 + 6\lambda_z^2 \lambda^4 v^2 + 3v^4)}{(\lambda_z^4 \lambda^4)} \quad (4.23a)$$

$$\begin{aligned} f_{I_4} = & \left[\lambda_z^2 \left\{ \lambda^2 [4k_1 \sin^2 \beta [I_4 - 1] + 8k_1 \lambda^2 \sin^4 \beta \right. \right. \\ & + 16\bar{k}_1 k_1 \lambda^2 \sin^4 \beta [I_4 - 1]^2] - 4k_1 \lambda^2 \sin^2 \beta [I_4 - 1] \left. \right\} \\ & \times [4k_1 \cos^2 \beta [I_4 - 1] + 8k_1 \lambda_z^2 \cos^4 \beta \\ & + 16\bar{k}_1 k_1 \lambda_z^2 \cos^4 \beta [I_4 - 1]^2] - \left\{ \lambda_z \lambda [8k_1 \lambda_z \lambda \cos^2 \beta \sin^2 \beta \right. \\ & + 16\bar{k}_1 k_1 \lambda_z \lambda \cos^2 \beta \sin^2 \beta [I_4 - 1]^2] \\ & \left. \left. - 4k_1 \lambda^2 \sin^2 \beta [I_4 - 1] \right\}^2 \right] \exp(2\bar{k}_1 [I_4 - 1]^2) \end{aligned} \quad (4.23b)$$

The fibers do not support compression which means that $f_{I_4} = 0$ for combinations of λ_z and λ giving rise to values of I_4 that are smaller than one.

Note that if $f < 0$, it follows that the membrane is unstable with regard to bulging, whereas if $f > 0$ the membrane is stable.

4.3.2 Prismatic bifurcation

In a cylindrical membrane, a sufficient condition for prismatic bifurcation to exist is a turning point in the curve giving values of azimuthal stress vs azimuthal stretch. Figure 4.3 depicts prismatic bifurcation in an initially circular cylinder. In the prismatic bifurcation mode, the inflated cylinder bifurcates into a prismatic shape with a non-circular cross-section.



Figure 4.3: Prismatic bifurcation of an initially circular cylinder.

In this case we particularize (4.18) and consider incremental displacements of the form

$$\delta \mathbf{u} = \delta u_r(\theta) \mathbf{e}_r + \delta u_\theta(\theta) \mathbf{e}_\theta. \quad (4.24)$$

After some manipulations, we obtain for a membrane cylinder that the bifurcation condition for the lowest (and typically most likely) mode is given by (Rodríguez and Merodio, 2011; Al-Chlahawi et al., 2023)

$$g = \hat{W}_{\lambda\lambda} = 0. \quad (4.25)$$

This condition has also been derived by (Haughton and Ogden, 1979) for thin-walled isotropic and non-swelling cylinders, while in (Haughton and Ogden, 1979) the analysis was broadened to consider thick-walled cylinders. If $g < 0$, it follows that the membrane is unstable with regard to bulging, whereas if $g > 0$ the membrane is stable.

For the material model at hand the prismatic bifurcation condition ((4.25)) can be written as

$$g = g_{I_1} + g_{I_4} = 0, \quad (4.26)$$

in which, as before, the term g_{I_1} represents the mechanical behavior of the ground substance ((4.8)), while the term g_{I_4} represents the mechanical behavior of the fibers (4.14). From ((4.8)), (4.14) and (4.26), we obtain the terms g_{I_1} and g_{I_4} in the form

$$g_{I_1} = \frac{\mu v^{q-2/3}}{2} \left[\frac{6v^2}{\lambda_\theta^4 \lambda_z^2} + 2 \right] \quad (4.27a)$$

$$\begin{aligned} g_{I_4} = & 8k_1 \lambda^2 \exp(\bar{k}_1(I_4 - 1)^2) \sin^4(\beta) \\ & + 4k_1 \exp(\bar{k}_1(I_4 - 1)^2) \sin^2(\beta)(I_4 - 1) \\ & + 16\bar{k}_1 k_1 \lambda^2 \exp(\bar{k}_1(I_4 - 1)^2) \sin^4(\beta)(I_4 - 1)^2 \end{aligned} \quad (4.27b)$$

For the ground substance, one can immediately notice that $g_{I_1} > 0$ for positive values of λ_z , λ and v . For the fibers, taking $k_1 > 0$ and $\bar{k}_1$, we notice that $g_{I_4} = 0$ for $\beta = 0$, i.e. when fibers are oriented in the axial direction. Further discussions regarding (4.27)b are given in Section 4.5.1.

4.3.3 Bending bifurcation

Figure 4.4 depicts bending bifurcation of a circular cylinder. To study bending bifurcation it is not necessary to simplify the form of $\delta \mathbf{u}$ given in (4.19) any further. It is straight forward to show (see Rodríguez and Merodio, 2011) that the bending bifurcation (or composite mode) occurs when

$$h = \sigma_{\theta\theta} - 2\sigma_{zz} = 0. \quad (4.28)$$

If $h < 0$ the membrane is stable, whereas if $h > 0$ the membrane is unstable. For the material model under consideration the bending bifurcation condition ((4.28)) can be written as

$$h = h_{I_1} + h_{I_4} = 0, \quad (4.29)$$

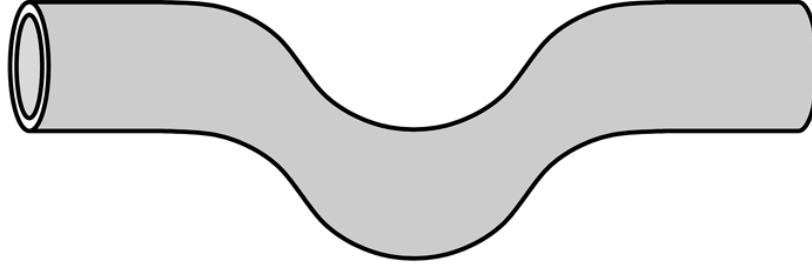


Figure 4.4: Bending bifurcation of a circular cylinder.

in which the term h_{I_1} is due to the mechanical behavior of the ground substance ((4.8)), and the term h_{I_4} results from the mechanical behavior of the fibers (4.14). It follows that

$$h_{I_1} = \frac{\mu v^{q-2/3} [\lambda_\theta^4 \lambda_z^2 - 2\lambda_\theta^2 \lambda_z^4 + v^2]}{\lambda_\theta^2 \lambda_z^2} \quad (4.30a)$$

$$\begin{aligned} h_{I_4} = & -4k_1 \exp(\bar{k}_1(I_4 - 1)^2) \\ & \times [2\lambda_z^4 \cos^4(\beta) - 2\lambda_z^2 \cos^2(\beta) + \lambda_\theta^2 \sin^2(\beta) \\ & - \lambda_\theta^4 \sin^4(\beta) + \lambda_\theta^2 \lambda_z^2 \cos^2(\beta) \sin^2(\beta)] \end{aligned} \quad (4.30b)$$

4.4 Bifurcation for W_m , the ground substance without fibers

Many articles on bifurcation of a cylindrical membrane have their focus on a single mode of bifurcation, whereas these modes of bifurcation may not occur as isolated events. One mode of bifurcation may join another mode during the inflation process, i.e, one mode may occur at an earlier stage of the inflation process, and another mode may then follow at a later stage of the inflation process. On the other hand, due to a change in the perfect cylindrical geometry, after occurrence of the first bifurcation mode, a second bifurcation mode may not occur at all. In order to understand the order of occurrence and coupling of bifurcation modes, we first study the bifurcation modes in the different constituents individually before we turn to the examination of bifurcation in fibrous materials. Let us consider a ground substance material without fibers for which the mechanical behavior of the material is solely determined by the strain energy density (4.8). It follows easily that

$$\hat{W} = \hat{W}_m = \frac{\mu v^{q-2/3}}{2} \left(\frac{v^2}{\lambda^2 \lambda_z^2} + \lambda^2 + \lambda_z^2 - 3v^{2/3} \right). \quad (4.31)$$

Moreover, the pressure inflation relation (4.16) takes the form

$$P = \frac{H}{R} \frac{\mu v^{q-2/3}}{\lambda^4 \lambda_z^3} [\lambda^4 \lambda_z^2 - v^2]. \quad (4.32)$$

To prevent negative values of (inflation) pressure P it follows from (4.32) that $\lambda^4 \lambda_z^2 \geq v^2$. For a ground substance without fibers, the bifurcation conditions (4.22), (4.26), and (4.29) reduce to $f = f_{I_1} = 0$, for bulging bifurcation, $g = g_{I_1} = 0$ for prismatic bifurcation, and $h = h_{I_1} = 0$ for bending bifurcation. It is clear that these conditions are independent of the swelling-dependent shear modulus $\mu v^{q-2/3}$. Indeed, this is true for the ground substance without fibers, while bifurcation of a ground substance with embedded fibers is affected by swelling-dependence ground substance stiffness. This will be shown in Section 4.6. Nevertheless, for illustrative purposes, we turn our attention to swelling-independent ground substance stiffness for which $q = 2/3$. This case is representative of other situations. We also establish the role of the axial stretch.

4.4.1 Swelling-independent ground substance stiffness and a fixed axial stretch

For $q = 2/3$, the shear modulus of the strain energy density of the ground substance (4.8) is independent of swelling. Under this circumstance and $\lambda_z = 1$, Figure 4.5 shows values of $f = f_{I_1}$, $g = g_{I_1}$ and $h = h_{I_1}$ vs λ for different values of the swelling parameter v . Let us focus on the bifurcation modes separately. Bifurcation is associated with either $f = 0$, $g = 0$ or $h = 0$. The undeformed and unswollen configuration of the membrane is given by the point $(\lambda, \lambda_z) = (1, 1)$. Values of λ obeying $f = 0$ (red curves) for bulging are greater for greater values of swelling. Blue curves for bending bifurcation show that a swollen ($v > 1$) and an unswollen ($v = 1$) material is unstable as inflation begins, i.e., $h > 0$ close to $(\lambda, \lambda_z) = (1, 1)$. On the other hand, deswelling ($0 < v < 1$) can stabilize the material with respect to the bifurcation modes under consideration for values close to $(\lambda, \lambda_z) = (1, 1)$ since $h < 0$. For a given value of v , the values of λ obeying $f = f_{I_1} = 0$ are associated with $h = h_{I_1} > 0$, i.e., comparing corresponding curves of f and h , it is easy to conclude that bending occurs at smaller values of λ (earlier stages of inflation) and it is the lower bound for bifurcation.

Black curves for prismatic bifurcation are associated with values $g = g_{I_1} > 0$ for the herein considered parameter values. Indeed, from (4.27)a one can immediately notice that $g_{I_1} > 0$ for positive, and therefore physically meaningful, values of λ_z , λ , and v . This result is in agreement with results given in (Al-Chlaihawi et al., 2023).

4.4.2 Unswollen ground substance and different axial stretches

In this section, we compare the different bifurcation modes for an unswollen material ($v = 1$) subject to several values of the axial stretch λ_z . We follow accordingly the procedure, arguments and results established in Section 4.4.1.

For $q = 2/3$ and $v = 1$, Figure 4.6 for different values of the axial stretch λ_z . Values of λ obeying $f = 0$ (red curves) for bulging are smaller for greater values of the axial stretch. Blue curves for bending bifurcation show that an extended membrane ($\lambda_z > 1$) is stable as inflation

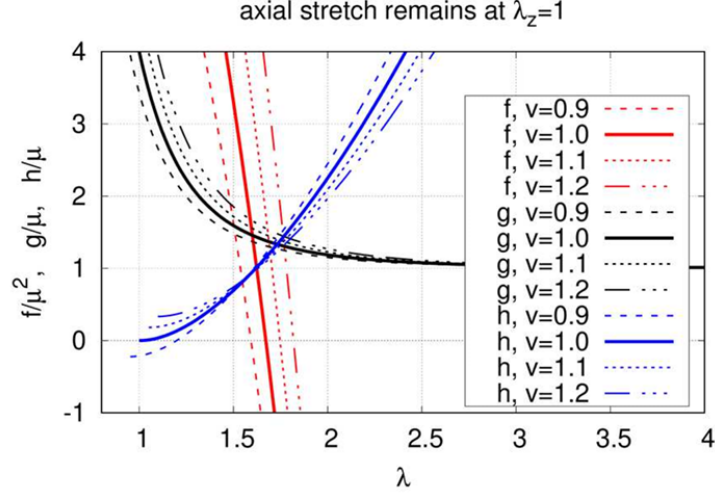


Figure 4.5: Values (normalized) of $f = f_{I_1}$, $g = g_{I_1}$ and $h = h_{I_1}$ vs λ for $\lambda_z = 1$ and a ground substance material (without fibers) with different values of the swelling parameter v . Red curves give values of the bulging function $f = f_{I_1}$, black curves give values of the prismatic function $g = g_{I_1}$ and blue curves give values of the bending function $h = h_{I_1}$. Negative values of f and g and positive values of h are associated with bulging bifurcation, prismatic bifurcation and bending bifurcation, respectively, i.e. with unstable configurations of the initially perfect circular cylindrical membrane. The undeformed and unswollen configuration of the membrane is given by the point $(\lambda, \lambda_z) = (1, 1)$. Close to this configuration for $v = 1$ as well as greater than 1 (swelling) and $\lambda > 1$ (inflation of the membrane), values of h are positive, which means that bending is expected as soon as the membrane begins to be inflated. In addition, it is clear that the values of λ obeying $f = 0$ are also associated with values $h > 0$. This is an indication that while bulging does not give the onset of bifurcation (it is not the lower bound for bifurcation), during postbifurcation (after bending occurs) bulging might be expected. Values of g are positive; therefore, prismatic bifurcation is not expected. Nevertheless, prismatic bifurcation cannot be ruled out completely during post-bifurcation since, among other reasons, the g -based analysis only characterizes the first prismatic bifurcation mode.

begins, i.e., $h < 0$ close to $(\lambda, \lambda_z) = (1, 1)$. In Figure 4.6, for a given λ_z , it is not always easy to deduce the lower bound for bifurcation since it is not easy to check the value of $h = h_{I_1}$ for the value of λ obeying $f = f_{I_1} = 0$. Clearly, for $\lambda_z = 1$ the lower bound is bending while for $\lambda_z = 1.3$ the lower bound is bulging. While values $f = 0$, red curves, shift to smaller values of the azimuthal stretch λ for greater values of the axial stretch λ_z , the opposite effect is observed for bending bifurcation, i.e., blue curves shift downwards and to the right side (values $h < 0$) for greater values of λ_z . Nevertheless, one can conclude that extension of the membrane stabilizes the material against bending bifurcation.

The observations from Figure 4.6 immediately leads us to establish the deformation conditions associated with the lower bifurcation mode, which can be either bulging or bending. This is obtained by establishing points (λ, λ_z) that simultaneously obey $f_{I_1} = 0$ and $h = h_{I_1} = 0$ for a given material. Figure 4.7 shows points (λ, λ_z) obeying $f_{I_1} = 0$ (red curve) and $h = h_{I_1} = 0$

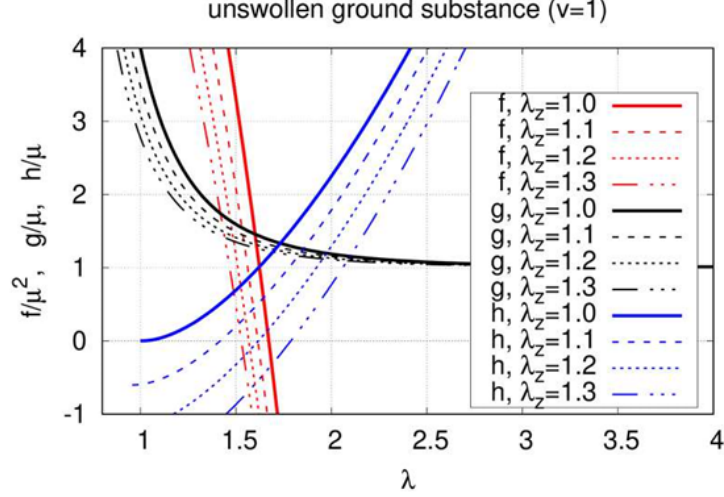


Figure 4.6: Values (normalized) of $f = f_{I_1}$, $g = g_{I_1}$ and $h = h_{I_1}$ vs λ for an unswollen ground substance material ($v = 1$) without any fibers and different values of the axial stretch λ_z . Negative values of f and g and positive values of h are associated with bulging bifurcation, prismatic bifurcation and bending bifurcation, respectively. For $\lambda_z = 1$ and the associated value of λ obeying $f = f_{I_1} = 0$, it is clear that $h = h_{I_1} > 0$ and, whence, bending is the lower bifurcation mode. On the other hand, for $\lambda_z = 1.3$ and the associated value of λ obeying $f = f_{I_1} = 0$, it is clear that $h = h_{I_1} < 0$ and it follows that bulging is the lower bifurcation.

(blue curve) for an unswollen material $v = 1$. When $(\lambda, \lambda_z) = (\lambda_{fh}, \lambda_{z,fh}) = (1.57, 1.18)$ both curves intersect and bulging and bending bifurcation are initiated simultaneously. It follows that for $\lambda_z < 1.18$ the lower bifurcation mode is bending while for $\lambda_z > 1.18$ the onset of bifurcation is given by bulging.

4.5 Bifurcation for W_f , the fibers

In this section we assume that the mechanical behavior given by the ground substance material is negligible compared to the mechanical behavior given by the fibers. One could think that this is a highly anisotropic material. The mechanical properties of the fibers dominate the properties of the matrix, i.e. $\mu/k_1 \rightarrow 0$. This is a useful case to understand the role of the fibers in the bifurcation of cylinders although it is a subtle case with limitations since the matrix response cannot be completely disregarded for a well-posed constitutive equation. Nevertheless, the purpose of this analysis is to illustrate the fibers' effect with regard to bifurcation and this is described in what follows.

The strain energy density function W for this material is completely determined via (4.14),

$$\hat{W} = \hat{W}_f = \frac{k_1}{\bar{k}_1} \left[\exp \left(\bar{k}_1 \left[\lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta - 1 \right]^2 \right) - 1 \right] \quad (4.33)$$

The pressure-inflation relation (4.16) for the fibers takes the form

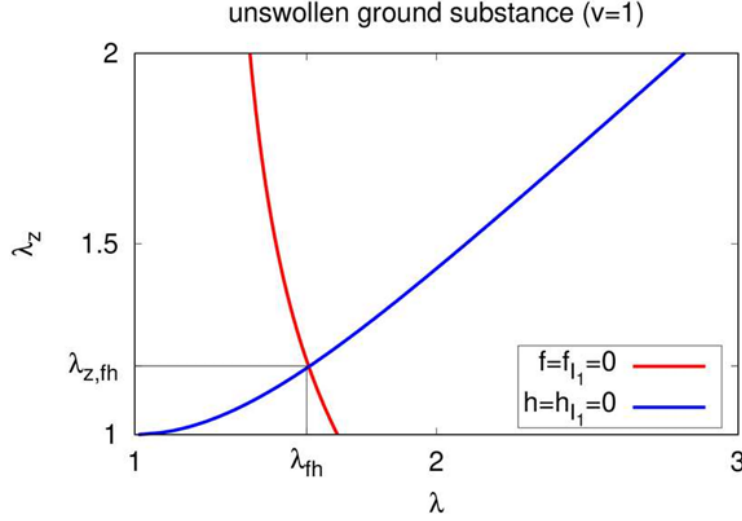


Figure 4.7: Points (λ, λ_z) obeying $f = f_{I_1} = 0$ (red curve) and $h = h_{I_1} = 0$ (blue curve) for a ground substance without fibers that remains unswollen $v = 1$. Both curves intersect for $(\lambda, \lambda_z) = (\lambda_{fh}, \lambda_{z,fh}) = (1.57, 1.18)$ so that bulging and bending bifurcation are initiated simultaneously. It follows that for $\lambda_z > 1.18$ the onset of bifurcation is given by bulging while for $\lambda_z < 1.18$ the lower bifurcation mode is bending.

$$P = 4 \frac{k_1}{\lambda_z} \frac{H}{R} \exp(\bar{k}_1 [\lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta - 1]^2) \sin^2(\beta) \times (\lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta - 1), \quad (4.34)$$

which is equal to zero for $\beta = 0$ or for $\lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta = I_4 = 1$. Furthermore, the bifurcation conditions given in (4.22), (4.26), and (4.29) reduce to $f = f_{I_4} = 0$, for bulging bifurcation, $g = g_{I_4} = 0$ for prismatic bifurcation, and $h = h_{I_4} = 0$ for bending bifurcation. It is clear that these conditions are independent of k_1 .

We recall that in the limit

$$\lim_{\bar{k}_1 \rightarrow 0} \frac{k_1}{\bar{k}_1} \left[\exp \left(\bar{k}_1 [\lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta - 1]^2 \right) - 1 \right] = k_1 [\lambda_z^2 \cos^2 \beta + \lambda^2 \sin^2 \beta - 1]^2, \quad (4.35)$$

i.e. the strain energy density for the fibers (4.33) approaches the so called standard reinforcing model (Demirkoparan and Pence, 2008; Qiu and Pence, 1997; Merodio and Ogden, 2003a; Topol et al., 2014).

Note that this standard form, the right hand side of (4.35), is not only the limiting case for the exponential form since it (the standard form) can be derived from logarithmic forms of strain energy densities for fibers, see, for instance (R. W. Ogden and Saccomandi, 2007). In what follows we analyze separately two cases: one obeying that $\bar{k}_1 \rightarrow 0$ for which the strain energy is the so called standard reinforcing model, and the exponential, which is used for any

other value of k_1 .

In the modeling of collagen fibers in arterial wall tissue, fibers are often taken to be incapable of supporting compression (Holzapfel and Ogden, 2010a). For that reason, fibers are considered to contribute if and only if $I_4 \geq 1$.

4.5.1 The standard reinforcing model

We first consider the specification $\bar{k}_1 \rightarrow 0$, for which the strain energy density (4.33) yields the so called standard reinforcing model (4.35). Figure 4.8 shows values (normalized) of $f = f_{I_4}$ (red curves), $g = g_{I_4}$ (black curves) and $h = h_{I_4}$ (blue curves) vs λ for different values of the axial stretch λ_z and selected values of the fiber winding angle β . More in particular, Figure 4.7a is for $\beta = \pi/6$, Figure 4.7c is for $\beta = \pi/3$ and Figure 4.7e is for $\beta = \pi/2$, while Figure 4.7b is just a magnification of Figure 4.7a near the point (1,0) and, similarly, Figure 4.7d is just a magnification of Figure 4.7c near the point (1,0). The labeled curve styles shown in Figure 4.7a applies to the other panels. In Figure 4.8e, for $\beta = \pi/2$, there is only one curve for $f = f_{I_4}$ as well as for $g = g_{I_4}$ and $h = h_{I_4}$, since the fibers are in the azimuthal direction and $f = f_{I_4}$, $g = g_{I_4}$ and $h = h_{I_4}$ are independent of λ_z . One can immediately notice that prismatic bifurcation is not expected to occur since $g > 0$.

Let us pay attention to the curves for $\lambda_z = 1.2$ in Figure 4.8b ($\beta = \pi/6$) and Figure 4.8d ($\beta = \pi/3$). The curves show that around the undeformed configuration $(\lambda, \lambda_z) = (1, 1)$ there are points (λ, λ_z) obeying that $h < 0$ (blue curves) while for these points f (red curves) is always positive. Furthermore, for $\lambda > 1$ blue curves change from negative to positive values while for these points the values of f (red curve) are positive. This means that the lower bifurcation mode is bending. This argument also applies to the other values of λ_z . It is also shown in Figure 4.8a and Figure 4.8c, that the f -curves have a maximum, which, comparing panels (a) and (c), decreases for greater values of β . In addition, Figure 7 (e) shows that f is negative while h is positive, which, in turn, means that as $\beta \rightarrow \pi/2$ bending and bulging may occur almost simultaneously. It follows that, as the membrane is inflated, greater values of β make the membrane more unstable with regard to both bulging and bending. Conversely, smaller values of β make the membrane more stable with regard to bulging and bending. It also follows for a given β that greater values of λ_z stabilize the membrane with respect to bending since the values of λ associated with $h = 0$ are greater as λ_z increases (see, for instance, panel (b)).

The different panels of Figure 4.8 show results for selected values of the fiber winding angle β . With the purpose of gaining a complete illustration of the analysis, now we broaden our attention and focus on the development of bifurcation conditions for all possible values of β . Results are shown in Figure 4.9 for different values of the axial stretch λ_z . Since $g > 0$ we just focus on f and h .

Figure 4.9 shows (normalized) values of $f = f_{I_4}$ (red curves) and $h = h_{I_4}$ (blue curves) as a polar plot with respect to the angle β for different values of the axial stretch λ_z and $\lambda = 1.5$, panel (a) and its magnification, panel (b), and $\lambda = 2$, panel (c) and its magnification, panel (d). The symmetry of the results is due to the fact that both fiber families are mechanically equivalent and are symmetrically disposed with respect to the axial direction. It follows that

it is sufficient to consider values $\pi/2 \geq \beta \geq 0$. More in particular, Figure 4.9 shows that close to and around $\beta = 0$, $f = f_{I_4} > 0$ (red curves) and $h = h_{I_4} < 0$ (blue curves), i.e., there is neither bulging nor bending. On the other hand, Figure 4.9 shows that close to and around $\beta = \pi/2$, $f = f_{I_4} < 0$ (red curves) and $h = h_{I_4} > 0$ (blue curves), i.e., the membrane is unstable with respect to both bulging and bending. Panel (b) for $\lambda = 1.5$ shows that $f = f_{I_4} = 0$ for values of β a little greater than $\pi/3$, while Panel (d) for $\lambda = 2$ shows that $f = f_{I_4} = 0$ for values of β a little smaller than $\pi/3$, i.e. greater values of λ require smaller values of β for the membrane to be stable regarding bulging. This means that smaller values of β stabilize the membrane with respect to bulging. A similar argument follows for the blue curves associated with bending bifurcation although numbers are not given. The values (and interpretation) of these plots are in agreement with the ones given in Figure 4.8.

This analysis has disentangled deformed configurations given by points (λ, λ_z) , around the undeformed configuration $(\lambda, \lambda_z) = (1, 1)$, that have been understood in terms of bending and bulging. This limiting case of just fibers has shown the effect of the fibers for different values of β although this case is just an extreme idealization. To interpret the results further and understand the limitations and the subtle scenario of this fiber-only case let us consider that the tube has been subjected first to a given uniaxial tension so that $\lambda = 1/\sqrt{\lambda_z}$. The value of λ_z is then fixed and the tube inflated, with λ increasing. This path of deformation is acceptable if and only if $I_4 \geq 1$, which also depends on β . Indeed, for $\beta = \pi/6$ it is easy to show that this path of deformation is admissible for any value of λ_z . For greater values of β , for instance $\beta = \pi/3$, and some values of λ_z , the value of λ given by $\lambda = 1/\sqrt{\lambda_z}$ is associated, in turn, with $I_4 < 1$, which is not permitted. Our analysis, then, considers for the λ_z at hand a greater value of λ until $I_4 = 1$, which is associated with the value $P = 0$. To eliminate completely the matrix response and simultaneously consider that $I_4 \geq 1$ leaves that scenario, which only considers some deformations. This fiber kinematics behaviour also applies to the next section and will be exploited later on to study and understand bifurcation for a matrix reinforced with fibers.

4.5.2 The exponential form

We now turn to study the bifurcation conditions for the exponential form (4.33), i.e., for values of $\bar{k}_1$ that are greater than zero. From (4.27)b, it is easy to show that $g_{I_4} > 0$, and we do not focus on prismatic bifurcations. Figure 4.10 illustrates the effect of $\bar{k}_1$ on both bulging and bending bifurcation. In particular, curves in Figure 4.10a, $\beta = \pi/6$, and Figure 4.10b, $\beta = \pi/3$, give, for different values of the axial stretch λ_z , points $(\lambda, \bar{k}_1)$ that obey $f = f_{I_4} = 0$ (bulging, red curves) and points $(\lambda, \bar{k}_1)$ that obey $h = h_{I_4} = 0$ (bending, blue curves).

Blue curves are vertical lines. The reason is the following. The bending bifurcation condition (4.30)b for $h = h_{I_4} = 0$ becomes

$$\begin{aligned} h_{I_4} &= 2\lambda_z^4 \cos^4(\beta) - 2\lambda_z^2 \cos^2(\beta) + \lambda^2 \sin^2(\beta) \\ &\quad - \lambda^4 \sin^4(\beta) + \lambda^2 \lambda_z^2 \cos^2(\beta) \sin^2(\beta) = 0, \end{aligned} \tag{4.36}$$

which depends on the stretches λ and λ_z as well as the angle β , but it is independent of the

parameter $\bar{k}_1$. Furthermore, it follows from (4.30)b that h is negative for $\beta = 0$ while it is positive for $\beta = \pi/2$ under fiber extension ($\lambda > 1$). Hence, smaller values of β stabilize the membrane with respect to bending in agreement with the result established for the standard reinforcing model.

The top panels (a) and (b) of Figure 4.10 show that for sufficiently large values of $\bar{k}_1$ the condition $f = f_{I_4} = 0$ is not satisfied, while for sufficiently small values of $\bar{k}_1$ the condition $f = f_{I_4} = 0$ is satisfied by just one point $(\lambda, \bar{k}_1)$. On the other hand for a given value of $\bar{k}_1$ in between these two cases, $f = f_{I_4} = 0$ is satisfied by two values of λ . As the membrane is inflated panels (a) and (b) show that the lower bifurcation mode is bending. Furthermore, during postbifurcation, it may be followed by bulging bifurcation for sufficiently small values of $\bar{k}_1$. This might be expected although it is not certain since the membrane is not a perfect circular cylinder. Nevertheless, for sufficiently large values of $\bar{k}_1$ bulging is not expected even after bending. Furthermore, a simple comparison of the curves $f = f_{I_4} = 0$ in panels (a) and (b) establishes that smaller values of β stabilize the membrane with respect to bulging in agreement with the result established for the standard reinforcing model. In addition, an interesting observation for $h = h_{I_4} = 0$ (blue curves) by comparison of Figure 4.10a and Figure 4.10b is the shift in the order of these curves for the different values of the axial stretch λ_z due to the change of the value of β . In Figure 4.10a the blue curves are ordered accordingly with the greater values of λ_z , i.e. greater values of λ are associated with greater values of λ_z , which does not occur in Figure 4.10b. This motivates the analysis of $h = 0$ with the winding angle.

The blue curves in Figure 4.11 give values of the circumferential stretch λ (radial value) and the fiber winding angle β that obey the bending bifurcation condition $h = h_{I_4} = 0$ for different values of the axial stretch λ_z . Figure 4.11b is a magnification of Figure 4.11a. Results for $\lambda = 1$ are associated with the trivial solution for an unloaded (undeformed) tube. The values of λ given by the blue curves in Figure 4.10a can be deduced from Figure 4.11a when $\beta = \pi/6$. Similarly, the values of λ given by the blue curves in Figure 4.10b can be deduced from Figure 4.11b when $\beta = \pi/3$.

4.6 Bifurcation for the ground substance matrix with fibers

Previous sections deal with bifurcation for the ground substance and for the fibers separately. In this section we turn to consider a ground substance with embedded fibers to examine bifurcation phenomena that occur when both constituents define the elastic behavior of the cylinder. We consider deformations for which the inflation pressure is held by both the fibers and the matrix, i.e. the fibers are extended since otherwise the analysis involves just the matrix (which was given previously in Section 4.4.1) and simultaneously the matrix also supports pressure. The following results are restricted to bulging and bending since prismatic bifurcations have been shown not to play a major role.

For the material at hand, the parameter q describes the change in the ground substance stiffness with swelling v , and, therefore, it is associated with the ratio of the ground substance

stiffness to the fiber stiffness. For swelling, $v > 1$, greater values of $v^{q-2/3}$ are associated with $q > 2/3$ (the ground substance behavior becomes relevant compared with the one given by fibers) while smaller values of $v^{q-2/3}$ are associated with $q < 2/3$ (the ground substance behavior is less relevant than the one associated with the fibers). Conversely, for deswelling, $v < 1$, greater values of $v^{q-2/3}$ are associated with $q < 2/3$ while smaller values of $v^{q-2/3}$ are associated with $q > 2/3$. The effect of this stiffness change on the occurrence of bifurcation is illustrated with the next analysis in which the strain energy density function W in (4.7) is composed of both a ground substance with an elastic behavior W_m given by (4.8) and embedded fibers with the strain energy density function W_f given by (4.14). For the elastic parameters we take $k_1/\mu = 0.1$ and $\bar{k}_1 \rightarrow 0$ (standard reinforcing model). Values of the dimensionless (and normalized) pressure $P^* = PR/(\mu H)$ are also shown.

4.6.1 Bulging and Bending

Figure 4.12 shows values of v vs λ obeying either $f = 0$ (red curves in panels (a), (c) and (e)) or $h = 0$ (blue curves in panels (b), (d) and (f)) as well as their associated values of P^* vs λ (black curves) for $\lambda_z = 1.0$, three selected values of q ($q = \frac{1}{3}$, $q = \frac{2}{3}$, and $q = 1$) and some values of β , in particular, $\beta = \frac{\pi}{6}$ (panels (a), (b)), $\beta = \frac{\pi}{3}$ (panels (c), (d)) and $\beta = \frac{\pi}{2}$ (panels (e), (f)). Values of $v < 0.5$ are shown for completeness although these are extreme cases and in what follows we focus on values $v > 0.5$. In general, for $v > 0.5$, the bulging bifurcation curves ($f = 0$) are monotonically increasing. On the other hand, the bending bifurcation curves ($h = 0$) are monotonically decreasing. The parameter q shows a quantitative effect rather than a qualitative one on the bifurcation curves since they keep close to each other in all panels. On the other hand, the associated pressure curves show a qualitative effect due to q and, in particular, they are not monotonically increasing or decreasing. A variety of results depending on v , β and λ_z are shown in the different panels. Nevertheless, panels (c) and (e) in Figure 4.12 (for sufficiently large values of β) show that for sufficiently large values of λ (around $\lambda = 1.7$) pressure curves are monotonically increasing and these values are associated with both $v > 1$ and $v < 1$. With regard to bending, for sufficiently large values of v , i.e. $v > 0.8$, pressure curves are monotonically decreasing.

We now turn our attention to establish the lower bound for bifurcation in an extended and pressurized cylinder membrane capable of swelling. Figure 4.13 gives simultaneously the bulging and bending bifurcation curves shown in Figure 4.12 and, therefore, establishes accordingly the expected onset of bifurcation for $\lambda_z = 1$. **The panels** (b), (d) and (f) are magnification of panels (a), (c) and (e), respectively, near the region in which these curves intersect each other. In general bulging gives the onset of bifurcation under two situations: 1. for sufficiently large values of swelling $v > 1$ as well as 2. for sufficiently large values of deswelling $v < 1$, and in both cases the value of v depends on the axial stretch. During postbifurcation bulging might be coupled with bending, which occurs in an abdominal aortic aneurysm (AAA).

Our analysis has considered that both matrix and fibers hold pressure. This means that fibers are extended. To interpret further the analysis and the results let us turn our attention to the effect that fiber extension has on pressure vs λ curves, which is the subject of next section.

4.6.2 Ground substance with embedded fibers that may or may not be extended

The previous examples analyze bifurcation modes in a cylinder, in which both the ground substance and the fibers contribute to the elastic behavior. These examples examine deformations (combinations of azimuthal and axial stretches), as well as values of the swelling field and the parameter q associated with bulging or bending. A cylinder may inflate or deflate due to a change in the pressure or due to swelling. Similarly, as a result of remodeling processes, the stiffness of the fibers and their orientation may change. As a result fibers may or may not be extended and here we analyze this effect in the context of our previous results. From (4.11)a one can conclude that $I_4 = 1$ if

$$\lambda = \lambda_o = \sqrt{\frac{1 - \lambda_z^2 \cos^2(\beta)}{\sin^2(\beta)}}. \quad (4.37)$$

Consider a tube which is subjected to an inner pressure P for a given axial stretch λ_z , a value of v , $q = 2/3$ (matrix stiffness is independent of swelling), a value of β , $k_1/\mu = 0.1$ and $\bar{k}_1 \rightarrow 0$ (standard reinforcing model). Let us analyze the relation between the inflation pressure and the deformation λ given by (4.16). We consider also deformations for which the fibers do not contribute since they are not extended and just the matrix holds pressure. Under these circumstances, Figure 4.14 shows values of the normalized pressure P^* vs λ that are stable with regard to both bulging and bending bifurcation, i.e., $f > 0$ and $h < 0$, for different values of the parameters β , λ_z and v . In particular, in all panels each curve is associated with a value of v . The curve at the left of each panel is associated with the smallest value of v (indicated at the top of each panel), and greater values of v are associated with the other curves to the right. More in particular, left panels (a), (c) and (e) are for $\beta = \pi/6, \pi/3$ and $\pi/2$, respectively, $\lambda_z = 1$ and swelling values between $v = 0.05$ (left bold curve) and $v = 0.95$ (right bold curve) with increments of $\Delta v = 0.05$. Similarly, right panels (b), (d) and (f) are for $\beta = \pi/6, \pi/3$ and $\pi/2$, respectively, $\lambda_z = 1.2$ and swelling values between $v = 0.1$ (left bold curves) and $v = 1.60$ or $v = 1.90$ (right bold curves) with increments of $\Delta v = 0.1$. The value of λ_o for each panel is obtained easily using (4.37). It follows that in panel (b) fibers are extended for all values of λ . On the other hand, in the other panels a vertical dashed line indicates the value of λ_o . If $\lambda < \lambda_o$ (left of the dashed line) then the fibers are wavy or crimped and loads are only held by the matrix. If $\lambda > \lambda_o$, then the fibers are recruited and both matrix and fibers bear loads. In the left panels, $\lambda_o = 1$ and it is independent of the fiber winding angle β . In panel (d) $\lambda_o = 0.923760431$ and in panel (f) $\lambda_o = 1$.

In general, for sufficiently small values of λ_z and sufficiently large values of β (for instance, in panels (c) to (f)) if the material is deswollen and the inflation pressure is relatively small, then, the crimped or wavy fibers are not extended since this is associated with small values of λ . If the inflation pressure and swelling are sufficiently large, then, fibers resist the inflation pressure. For $\lambda > \lambda_o$ fibers are extended and curves in the panels show a change in their slopes.

4.7 Discussion and Outlook

This article has shown that the occurrence and the order of the bifurcation modes in a symmetrically reinforced swellable membrane tube depend on different factors, including the geometry of the examined problem, the elastic properties of the constituents, and the arrangement of the fibers. An understanding of the circumstances under which geometrical instabilities occur may help to understand the formation and propagation of aneurysms, i.e. it may help to predict such formation and its rupture.

For the swellable neo-Hookean type ground substance and for the exponential form of the fibers strain energy density function prismatic bifurcation is not expected and attention has been paid to bulging and bending. Nevertheless, it should be pointed out that our considerations for bifurcation have been restricted to fibers that cannot support compression and also to internal inflation. If, for example, an external pressure exceeds the inner pressure in the cylinder, then, the onset of instability might be given by the prismatic bifurcation (see, e.g. Al-Chlahawi et al., 2023).

It has been shown that swelling may stabilize the material (Demirkoparan and Merodio, 2020, (Demirkoparan and Merodio, 2020) with respect to bulging bifurcation, whereas deswelling stabilizes the material with respect to bending bifurcation. Both deswelling and axial stretch may change the lower bound for bifurcation during extension and inflation of cylinders. Indeed, our results show that to fully understand and capture bifurcation of extended and inflated cylinders, it is necessary to compare the mechanical properties and deformations associated with the onset of the different bifurcation modes.

In passing, we also mention that material models are mostly restricted to a limited number of phenomena and a certain time frame. In our considerations, the mechanical properties of the material have been taken to be time-independent. This assumption is reasonable when the considered time-scale is small in comparison to the typical time-scale for remodeling or viscoelastic effects. Material models that account of a deformation-dependent remodeling of the fibrous constituents have been studied in (Topol et al., 2017). Such deformation-mediated remodeling process may be caused by external loading, but it may also be induced by swelling (Gou et al., 2020; Topol et al., 2018). A factor that needs to be considered into account in the relation between stress and strain for the development of fibers remodeling is the maximum lifespans of the individual fibers and differences in the natural configurations of the different constituents (Topol, Demirkoparan, and Pence, 2021c; Topol, Demirkoparan, and Pence, 2021b). A comprehensive review on the mechano-regulated remodeling of collagen fibers has been presented in (Topol, Demirkoparan, and Pence, 2021a). It is clear that future integration of these variables in bifurcation analyses needs the framework established in this analysis to further model aneurysms.

Our considerations take place in a membrane treatment of the cylinder, which permits analytical work in the study of geometrical instabilities. A natural extension of this work is a deeper examination of the different bifurcation modes and postbifurcation in thick-walled cylinders, which most of the times requires the application of numerical methods. The analytical membrane analysis is a guide for the thick-walled cylinder case, which may be carried out numerically.

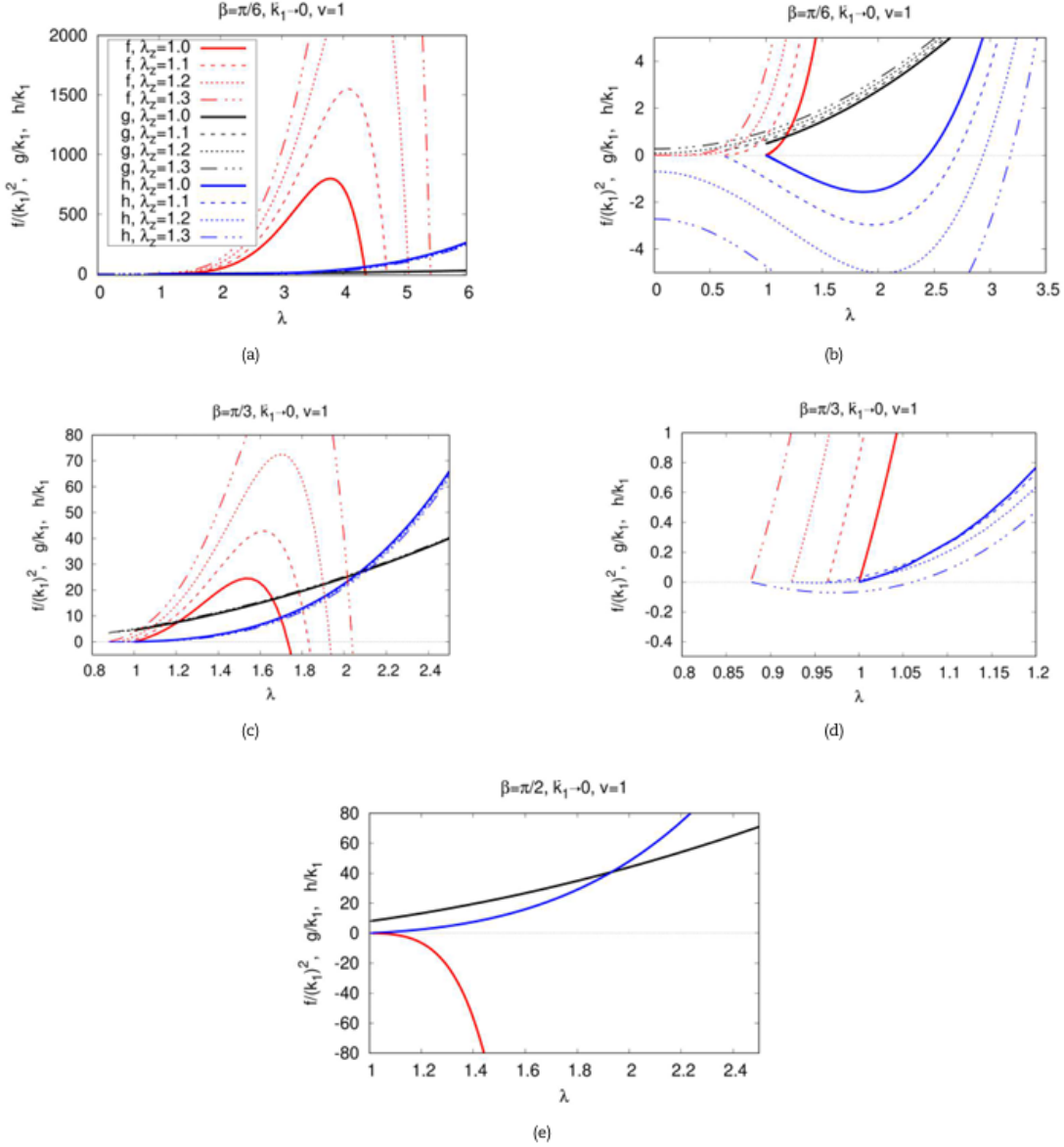


Figure 4.8: Values (normalized) of $f = f_{I_4}$ (red curves), $g = g_{I_4}$ (black curves) and $h = h_{I_4}$ (blue curves) vs λ for different values of the axial stretch λ_z and selected values of the fiber winding angle β . The legend in panel (a) applies to all panels. More in particular, panel (a) is for $\beta = \pi/6$, while its magnification is given in panel (b); panel (c) is for $\beta = \pi/3$, while its magnification is given in panel (d); and panel (e) is for $\beta = \pi/2$. In the latter case, since the fibers are perpendicular to the axial direction, the results are independent of the axial stretch λ_z , so that there are only three curves, one for $f = f_{I_4}$ as well as one for $g = g_{I_4}$ and $h = h_{I_4}$.

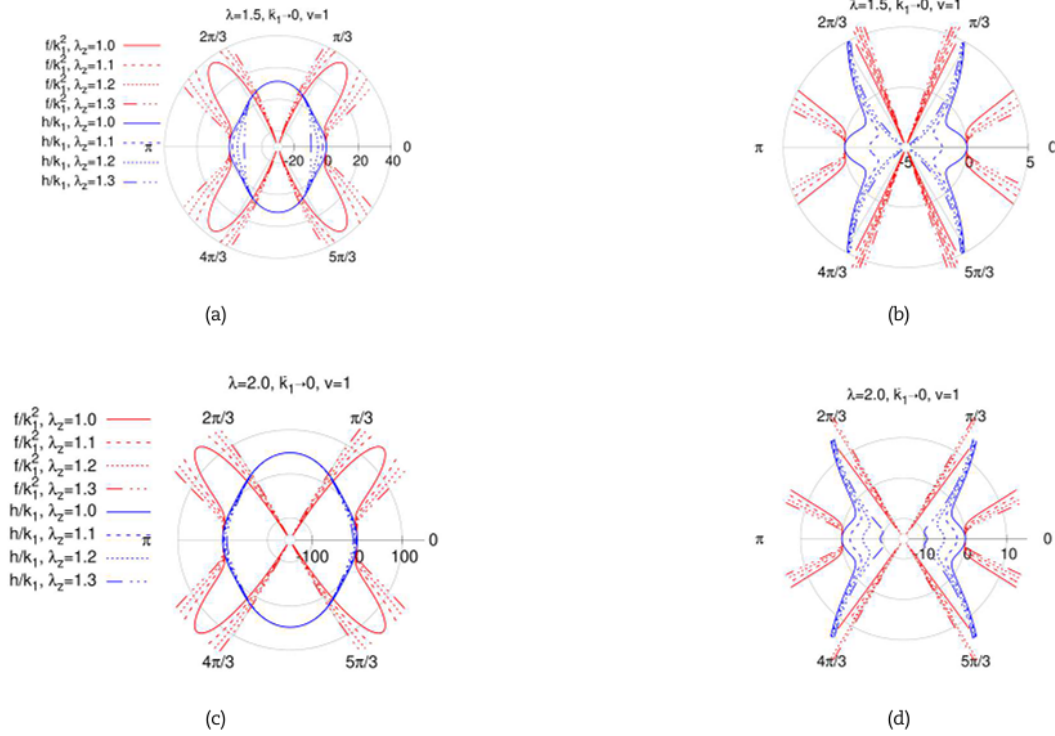


Figure 4.9: Values of $f = f_{I_4}$ (red curves) and $h = h_{I_4}$ (blue curves) as a polar plot with respect to the angle β for different values of λ_z and $\lambda = 1.5$, panel (a) and its magnification panel (b), and $\lambda = 2$, panel (c) is its magnification panel (d). Left panels (a) and (c) show the general development of f/k_1^2 and h/k_1 , whereas right panels (b) and (d) highlight the behavior of these functions close to their zero values. The symmetry of the results is due to the fact that both fiber families are mechanically equivalent and are symmetrically disposed with respect to the axial direction.

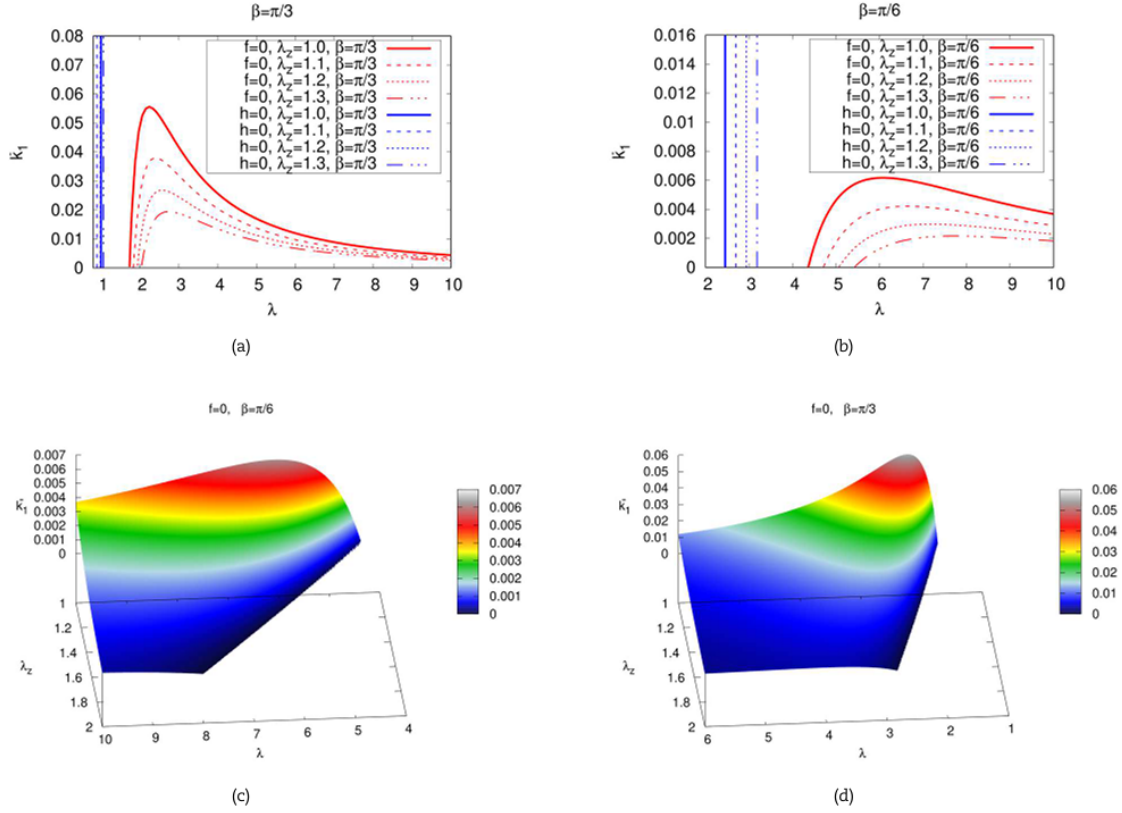


Figure 4.10: For different values of λ_z , the top panels (a), for $\beta = \pi/6$, and (b), for $\beta = \pi/3$, show values of $\bar{k}_1$ vs λ obeying either $f = 0$ (bulging bifurcation, red curves) or $h = 0$ (bending bifurcation, blue curves). The bottom panels (c) and (d) give the corresponding values of λ , λ_z , and $\bar{k}_1$ that fulfill $f = 0$ given in panels (a) and (b), respectively.

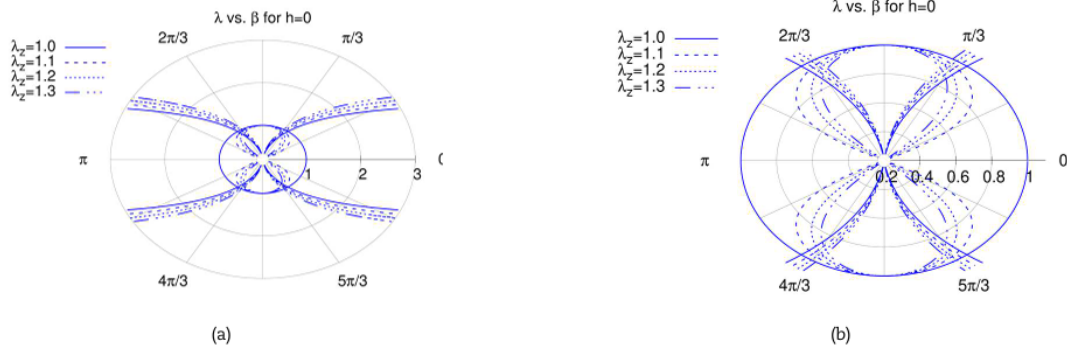


Figure 4.11: Curves give values of circumferential stretch λ (radial value in the plot) and the fiber winding angle β obeying $h = 0$ for different values of the axial stretch λ_z . Panel (b) is a magnification of panel (a). In panel (a), it is easy to deduce that when $h = 0$ for $\beta = \pi/6$ greater values of λ are associated with greater values of λ_z , in agreement with the order shown by the blue curves in Figure 4.10a. Similarly, in panel (b), it is easy to deduce that when $h = 0$ for $\beta = \pi/3$ greater values of λ are not associated with greater values of λ_z , in agreement with the values of λ given by the blue curves shown in Figure 4.10b.

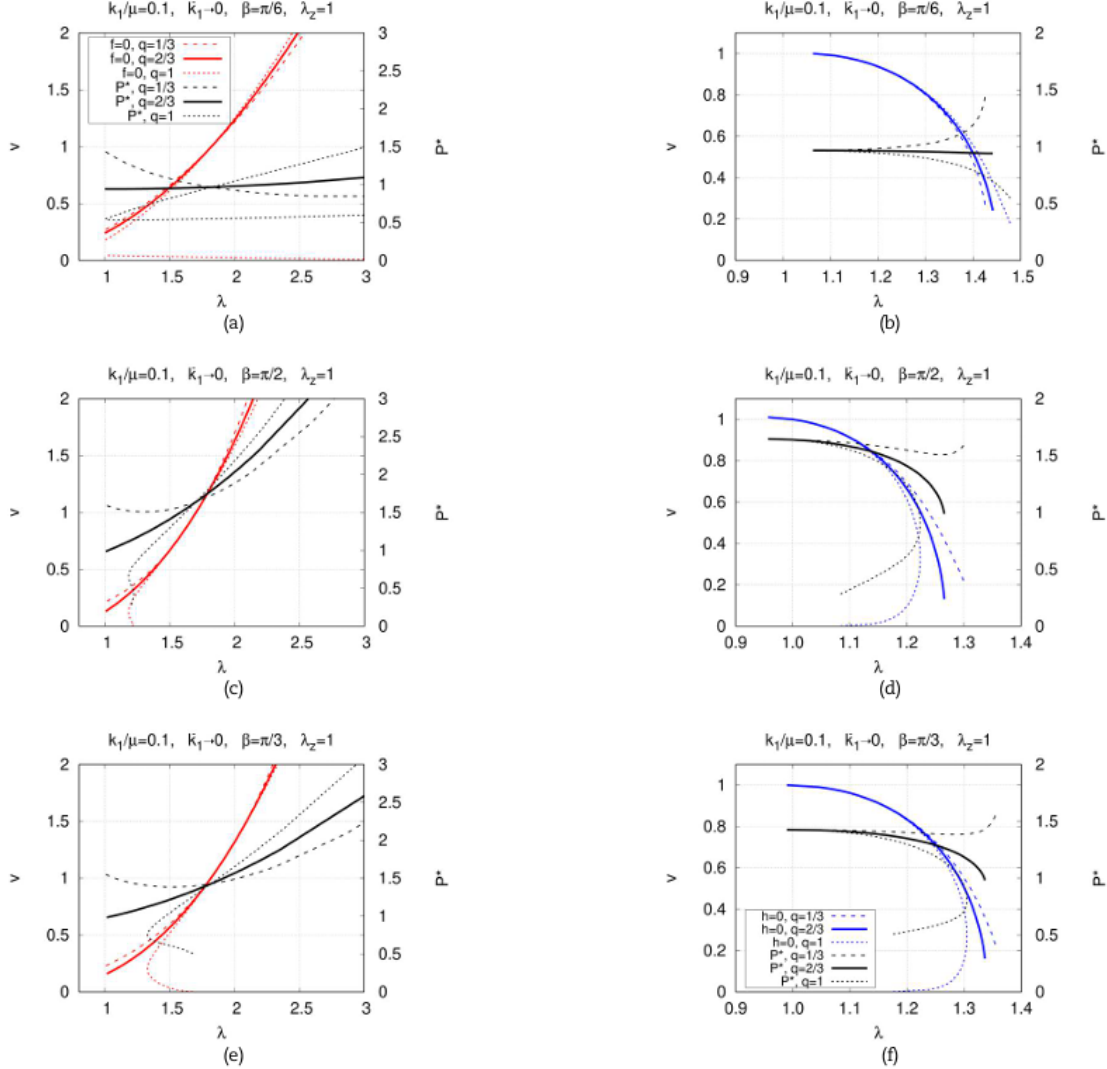


Figure 4.12: Values of v vs λ obeying either $f = 0$ (red curves in panels (a), (c) and (e)) or $h = 0$ (blue curves in panels (b), (d) and (f)) as well as their associated values of $P^* = PR/(\mu H)$ vs λ (black curves) for $\lambda_z = 1.0$, three selected values of q ($q = \frac{1}{3}$, $q = \frac{2}{3}$, and $q = 1$) and some values of β , in particular, $\beta = \frac{\pi}{6}$ (panels (a), (b)), $\beta = \frac{\pi}{3}$ (panels (c), (d)) and $\beta = \frac{\pi}{2}$ (panels (e), (f)). Results show for $v > 0.5$ that q does not have a qualitative effect on the bifurcation curves but it does have a qualitative effect on pressure curves.

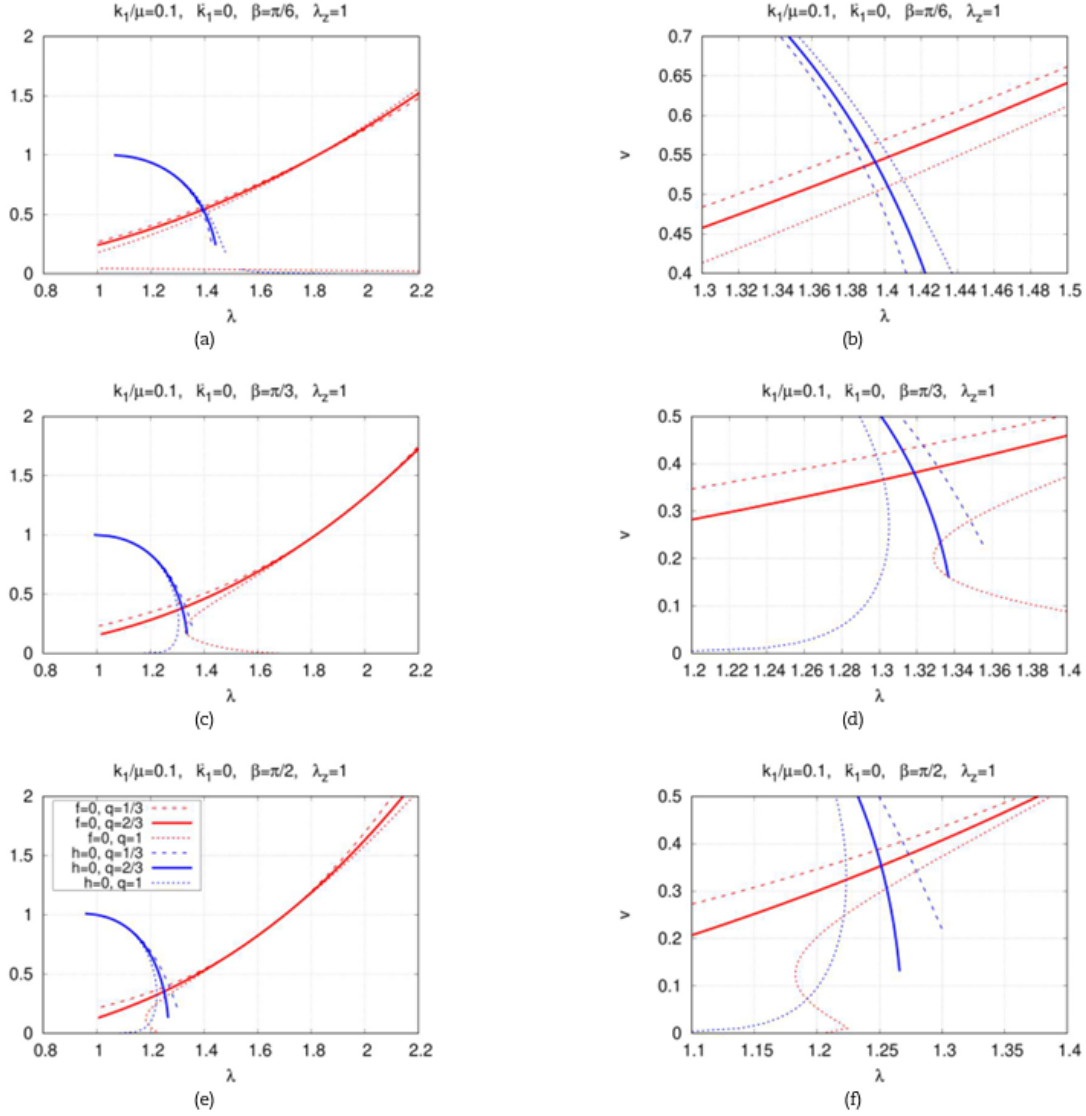


Figure 4.13: This figure gives simultaneously the bulging and bending bifurcation curves shown in Figure 4.12 and, therefore, establishes accordingly the expected onset of bifurcation for $\lambda_z = 1$. Panels (b), (d) and (f) are magnification of panels (a), (c) and (e), respectively, near the region in which these curves intersect each other.

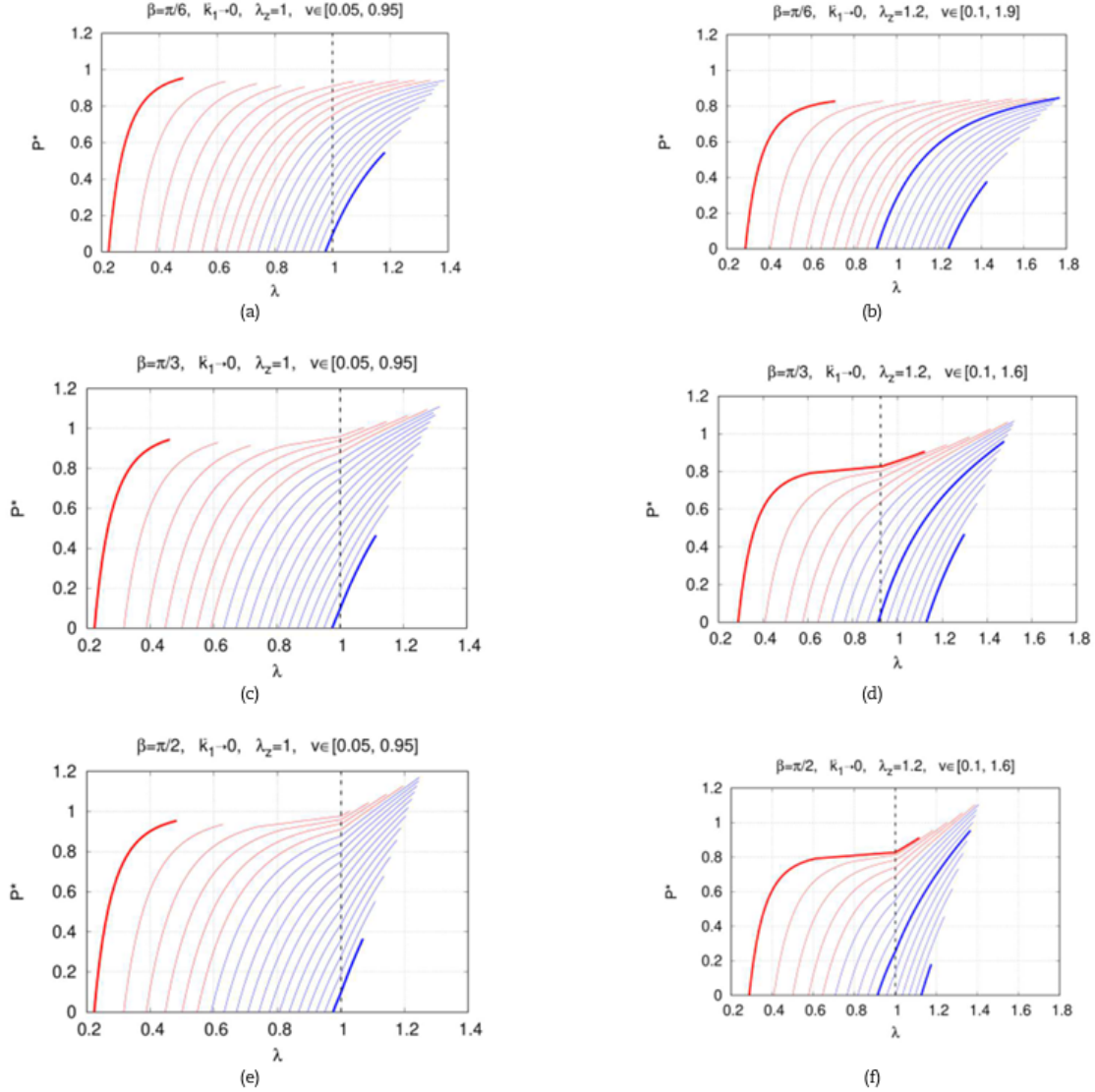


Figure 4.14: Values of pressure P^* vs λ that are stable with regard to both bulging and bending, taking $q = 2/3$, $k_1/\mu = 0.1$ and $\bar{k}_1 \rightarrow 0$. Top panels (a), (b) are for $\beta = \pi/6$; middle panels (c), (d) are for $\beta = \pi/3$; and bottom panels (e), (f) are for $\beta = \pi/2$. For the red curves the stable inflation process is interrupted by bulging, whereas for the blue curves the stable inflation is interrupted by bending bifurcation.

Chapter 5

Computational modeling of a residually stressed thick-walled cylinder under combined action of axial extension and inflation

“This chapter is based on the below article that is published corresponding this thesis: Murtadha J Al-Chlahawi, Dariel Desena-Galarza, Heiko Topol, José Merodio (2025). Computational modeling of a residually stressed thick-walled cylinder under the combined action of axial extension and inflation. Finite Elements in Analysis and Design 244, 104309(Al-Chlahawi et al., [2025](#))

5.1 Introduction

Aneurysms are localized abnormal bulges that occur in blood vessels. It has been reported that abdominal aortic aneurysms are involved in 1-2% of all deaths in industrialized countries (Lindsay and Dietz, [2011](#)). The initiation of such a bulge is associated with various factors, including hypertension (Kim et al., [2021](#)), inflammation (Skotsimara et al., [2022](#)), and genetics (Bakker and Ruigrok, [2021](#)). The weakening of the arterial wall may be affected by the degradation of collagen and elastin, for example, due to the enzymatic activity in the tissue (Yokoyama et al., [2012](#); Topol, Demirkoparan, and Pence, [2021a](#)).

Changes in the mechanical pattern of collagen fibers can lead to aneurysm formation, or bulging bifurcation (Rodríguez and Merodio, [2011](#)). Consequently, this may result in a wall rupture with a relatively high mortality rate (Schermerhorn, [2009](#)).

The aneurysm-related mortality rate motivated the biomechanics community to gain a deeper understanding of the formation of bulging instabilities of inflated and axially stretched tubular structures. Experimental investigations on bulging bifurcation have been conducted, for instance, in the works (Gonçalves et al., [2008](#); Hejazi et al., [2021](#); Hejazi and Phani, [2022](#); Kyriakides and Yu-Chung, [1991](#); Pamplona et al., [2006](#); Vangerko and Treloar, [1978](#); Wang

et al., 2019). Nevertheless, for practical reasons, the number of varying parameters in experimental investigations is usually limited.

Various works present mathematical descriptions of instabilities in inflated tubular membranes. Notable early modeling efforts include (Corneliussen and Shield, 1961) and (Haughton and Ogden, 1979; Haughton and Ogden, 1979). There have been significant attempts to examine bifurcation phenomena in many applications. Studies on instabilities in inflated cylindrical tubes under material volume changes, associated with swelling and inflammation effects, include (Demirkoparan and Merodio, 2017; Topol, Al-Chlaihawi, et al., 2023), i.e., it has been shown that material volume changes may either stabilize the material to certain types of instability modes or trigger them. The initiation and propagation of a bulge have been explored in isotropic solids and fibrous materials relevant to modeling arterial collagenous soft tissue (Demirkoparan and Merodio, 2020; Merodio and Haughton, 2010; Moradalizadeh et al., 2024; Rodríguez and Merodio, 2011; Fu et al., 2012; Topol, Font, et al., 2024).

Analytical solutions provide some basic insight into the physical interpretation of bulging phenomena. However, these solutions are often limited to idealizations in terms of geometry and material property distribution. Therefore, it is necessary to apply alternative solutions that consider the analysis of post-critical behavior for the problem at hand. Numerical investigations addressing bifurcation and post-bifurcation behaviors of thin- and thick-walled circular cylindrical tubes subjected to combined axial load and internal pressure have been conducted, for instance, by Merodio and co-workers in (Alhayani et al., 2014a; Alhayani et al., 2014b; Dehghani et al., 2019; Font et al., 2021; Desena-Galarza et al., 2021; Z. Guo et al., 2022).

Inflated and axially stretched cylindrical structures may encounter various instabilities such as necking, beading, bending, prismatic bifurcation, and helical buckling (Rodríguez and Merodio, 2011; Seddighi and Han, 2021; Fu et al., 2021; Yu and Fu, 2022; N. Jha et al., 2023). While many studies focus on a single mode of instability, some works account for multiple modes and their relevance. For instance, (Rodríguez and Merodio, 2011) presents criteria for the initiation of bulging, bending, and prismatic bifurcation. The initiation of these instability modes under material volume changes has been explored in (Al-Chlaihawi et al., 2023; Topol, Al-Chlaihawi, et al., 2021). Furthermore, (Topol, Nazari, et al., 2024) explores the interruption of a stable inflation process by either jump-inflation instability or bulging instability under strain-dependent damage processes.

The unloaded state of the material may not necessarily be the stress-free state. The natural configurations of the constituents may differ due to various factors often accompanying remodeling processes in soft tissue (Topol, Demirkoparan, and Pence, 2021a). For example, fibers may be pre-stretched and tend to contract to the surrounding tissue. This effect is considered in studies like (Topol et al., 2022; Topol, Asghari, et al., 2023) that investigate the initiation and propagation of bulging for both parallel and dispersed fibers. Residual stresses, which also occur in arterial wall tissue due to growth and development processes (Hoger, 1986), can be accounted for using various techniques. The opening angle technique assumes an open cylinder sector as the zero-stress state and introduces residual stresses by closing this sector (Bustamante and Holzapfel, 2010; Waffenschmidt and Menzel, 2014; Sigaeva et al., 2019). Another method, used in this article, introduces residual stresses via a residual stress

tensor that quantifies the stress as a function of location in the cylinder wall. This method has been used to study bulging in works such as (Ahamed et al., 2016; Dehghani et al., 2019; Desena-Galarza et al., 2021; Holzapfel and Ogden, 2010b; Merodio et al., 2013; Merodio and Ogden, 2016; Melnikov et al., 2021; Melnikov et al., 2022; Melnikov and Merodio, 2023). Various works, such as (Asghari et al., 2023b; Asghari et al., 2023b; Asghari et al., 2024; Asghari et al., 2024; Asghari et al., 2023a), apply sensitivity analysis methods to understand the role of residual stresses, fiber pre-strain, and remodeling effects in inflated and axially stretched cylinders.

The role of residual stresses on the inflation behavior and the initiation of bifurcation in pressurized cylindrical is not fully understood. This article introduces a study that seeks to determine the role of the residual stress distribution on material behavior during such loading processes. Efforts have been dedicated to developing a methodology to capture bifurcation and post-bifurcation in cylinders under combined inflation and extension subjected to residual stresses using numerical analyses. These approaches employ finite element analysis with the Modified Riks Method in commercial software **ABAQUS**, implementing the material model through the **ABAQUS** subroutine (UMAT). This study aims to provide a creative framework for engineering analysis and design, incorporating the methodology into the finite element code of **ABAQUS** for application to comparable problems.

The article is organized as follows: Section 5.2 summarized the mathematical formulation for the inflation of a residually stressed, inflated, and axially stretched cylinder. Section 5.3 introduces a finite element approach for modeling the problem under consideration that accounts for a non-planar distribution of the residual stresses. In Section 5.4, some numerical results are presented and studied, with a focus on distinguishing between two bifurcation modes, bulging (axisymmetric) bifurcation and bending (asymmetric) bifurcation. This section also explores the development of deformation and its relevance to aneurysm formation. The final section concludes the paper by summarizing the key findings in Section 5.4.

5.2 Mathematical description of the problem

5.2.1 Geometry

Consider a circular cylinder with inner radius A , outer radius B , and length L in its reference configuration $\mathcal{B}_r$. In terms of cylindrical coordinates (R, Θ, Z) , the geometry of the cylinder can be described as

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L. \quad (5.1)$$

The location of any material point of the cylinder in its reference configuration $\mathcal{B}_r$ can be described by the position vector $\mathbf{X}$. In the deformed configuration $\mathcal{B}$, during the stable part of the inflation, the deformed cylinder has inner radius a , outer radius b , and length ℓ . In terms of the cylindrical coordinates (r, θ, z) , the geometry of the cylinder can be described as

$$a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq \ell. \quad (5.2)$$

The location of any material point in the deformed configuration $\mathcal{B}$ can be described by the position vector $\mathbf{x}$. The deformation of the material is given in terms of the deformation

gradient tensor $\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$. The associated left and right Cauchy-Green deformation tensors are, respectively, $B = FF^T$ and $C = F^T F$. The deformation gradient tensor can be expressed in terms of the three principal stretches in the cylinder wall, the radial stretch λ_r , the azimuthal stretch $\lambda_\theta = r/R$, and the axial stretch $\lambda_z = \ell/L$ as follows,

$$F = \text{diag}(\lambda_r, \lambda_\theta, \lambda_z). \quad (5.3)$$

The material is taken to be incompressible, which is primarily attributed to its high water content in biological soft tissue. For this assumption, the determinant of the deformation gradient tensor becomes equal to one,

$$\det F = 1. \quad (5.4)$$

The incompressible behavior is motivated by the large water content in biological soft tissue. This incompressibility constraint (5.4) allows us to express the radial stretch in terms of the other two principal stretches,

$$\lambda_r = \lambda_\theta^{-1} \lambda_z^{-1}. \quad (5.5)$$

Equations (5.4) and (5.5) are valid for incompressible materials that do not undergo material volume changes. Such material volume changes in an incompressible modeling setting may be relevant when, for example, swelling or deswelling effects are taken into consideration (Topol, Demirkoparan, and Pence, 2021a). Swelling effects in incompressible hyperelastic solids are taken into consideration in various modeling works (see, e.g. Al Chlahawi et al. (Al-Chlahawi et al., 2023), Gou et al. (Gou et al., 2020), Moradalizadeh et al. (Moradalizadeh et al., 2024), Topol et al. (Topol, Al-Chlahawi, et al., 2021; Topol, Al-Chlahawi, et al., 2023), and Yang & O'Connell (Yang and O'Connell, 2018)).

5.2.2 Residually stressed cylinders

In our modeling, all body forces are neglected, resulting in the Cauchy stress tensor $\boldsymbol{\sigma}$ and the nominal stress tensor $\mathbf{S}$ satisfying the following equilibrium equations:

$$\text{div} \boldsymbol{\sigma} = \mathbf{0}, \quad \text{Div} \mathbf{S} = \mathbf{0}, \quad (5.6)$$

at every point in the body. Here, “div” and “Div” represent the divergence operators with respect to the current configuration $\mathcal{B}$ and the reference configuration $\mathcal{B}_r$, respectively.

In this article, we assume that the body is initially subjected to residual stress, represented by a second-order tensor $\boldsymbol{\sigma}_0$, which exists in the absence of body forces and surface tractions on the boundary of the solid in the reference configuration $\mathcal{B}_r$ (Hoger, 1993; Agosti et al., 2018). Since the residual stress is defined in the reference configuration $\mathcal{B}_r$ and is assumed to be unaffected by rotations in the deformed configuration $\mathcal{B}$, the residual stress tensor is inherently objective (Merodio et al., 2013).

Due to the absence of traction on the surface, the residual stress tensor fulfills the condition (Merodio et al., 2013)

$$\boldsymbol{\sigma}_0 \mathbf{N} = \mathbf{0} \quad \text{on} \quad \partial \mathcal{B}_r, \quad (5.7)$$

where N is an outward normal unit vector. The residual stress tensor is symmetric $\boldsymbol{\sigma}_0^T = \boldsymbol{\sigma}_0$, and it fulfills the equilibrium equation

$$\text{Div} \boldsymbol{\sigma}_0 = \mathbf{0}. \quad (5.8)$$

The studies in (Dehghani et al., 2019; Ahamed et al., 2016; Asghari et al., 2023b) on the modeling of residually stressed tubular structures, applicable to arterial soft tissue, adopt a plane residual stress tensor, where only the two diagonal components, σ_{0RR} and $\sigma_{0\Theta\Theta}$, are non-zero. The work in (Desena-Galarza et al., 2021) introduces a three-dimensional residual stress field that is symmetric along the axial direction with respect to the middle surface of the tube. It follows that as loading is applied the deformation of the tube is symmetric with respect to the middle cross section of the tube giving results for that particular scenario. Now, we build upon these phenomenological models outlined in (Dehghani et al., 2019) and (Desena-Galarza et al., 2021) and include a general 3-D residual stress tensor that generalizes the previous approaches, introducing a novel ansatz in which the components of the residual stress tensor $\boldsymbol{\sigma}_0$ are considered in the form

$$\sigma_{0RR} = \alpha_c (R - A)(R - B), \quad (5.9a)$$

$$\sigma_{0R\Theta} = 0, \quad (5.9b)$$

$$\sigma_{0RZ} = \frac{\alpha_d}{R} \left[(R - A)(R - B)(5Z^4 - 9Z^2L^2 + 4ZL^3) \right], \quad (5.9c)$$

$$\begin{aligned} \sigma_{0\Theta\Theta} = & \alpha_c [(R - A)(R - B) + R(R - A) + R(R - B)] \\ & + \alpha_d [(R - A)(R - B)(20Z^3 - 18ZL^2 + 4L^3)], \end{aligned} \quad (5.9d)$$

$$\sigma_{0\Theta Z} = 0, \quad (5.9e)$$

$$\sigma_{0ZZ} = -\frac{\alpha_d}{R} \left[(2R - A - B)(Z^2) (Z^3 - 3ZL^2 + 2L^3) \right]. \quad (5.9f)$$

which depend on the radial and axial coordinates R and Z , respectively, and which fulfill the conditions (5.7) and (5.8). In equation (5.9), α_c and α_d are two parameters that define the magnitude of the residual stress field, with units of stress per squared length and quintupled length, respectively. When $\alpha_d = 0$, equation (5.9) reduces to the plane residual stress tensor, with two non-zero components, σ_{0RR} and $\sigma_{0\Theta\Theta}$, which depend solely on the radial coordinate R (Dehghani et al., 2019; Ahamed et al., 2016; Asghari et al., 2023b).

The residual parameters can also be expressed in dimensionless form by defining $\alpha_c = \mu \bar{\alpha}_c / (2RT)$ and $\alpha_d = \mu \bar{\alpha}_d / (2L^5)$, respectively.

In the mechanical modeling of hyperelastic solids, the behavior of a solid is usually described in terms of a strain energy density function W . We consider an isotropic solid that is subjected

to residual stress, for which W is taken in the form (Asghari et al., 2023b; Asghari et al., 2023b)

$$W := \frac{\mu}{2}(I_1 - 3) + \frac{1}{2}(I_5 - \text{tr}\boldsymbol{\sigma}_0). \quad (5.10)$$

The first term on the right-hand side of (5.10) represents the neo-Hookean isotropic behavior of the material, where μ denotes the shear modulus, and $I_1 = \text{tr}(\mathbf{C})$ is the first invariant of the right Cauchy-Green deformation tensor $\mathbf{C}$. This neo-Hookean form is commonly employed to model isotropic soft tissue or its components, for example, the isotropic ground material in collagenous soft tissue (Gasser et al., 2006). Despite its simplicity, experimental evidence demonstrates that this model effectively captures the mechanical behavior of the isotropic components in soft tissue (Gundiah et al., 2007).

Alternative models applicable to isotropic soft tissue behavior include exponential forms or models that combine features of both neo-Hookean and exponential approaches (Moradalizadeh et al., 2024). The last term on the right-hand side of (5.10) accounts for residual stresses, defined in terms of the invariant $I_5 = \text{tr}(\boldsymbol{\sigma}_0 \mathbf{C})$ (see, e.g., (Asghari et al., 2023b; Merodio and Ogden, 2016)). This invariant has units of stress, unlike the dimensionless invariant I_1 , or the higher-order invariants often used in modeling anisotropic soft tissue behavior (Merodio and Ogden, 2020b).

Higher-order invariants related to residual stress may be introduced to account for more complex distributions of residual stresses (Asghari et al., 2023b; Font et al., 2021; Dehghani et al., 2019; Merodio et al., 2013; Merodio and Ogden, 2016; Melnikov et al., 2021; Melnikov et al., 2022; Melnikov and Merodio, 2023).

If the parameters $\alpha_c = \alpha_d = 0$ in (5.9), then all components of the residual stress tensor $\boldsymbol{\sigma}_0$ vanish, and the strain energy density function (5.10) reduces to a purely neo-Hookean behavior.

5.2.3 Stable Inflation

The stable part of the inflation of a pressurized and axially stretched circular cylinder is described by (Holzapfel and Ogden, 2010a)

$$P = \int_{\lambda_B}^{\lambda_A} \frac{1}{\lambda^2 \lambda_z - 1} \frac{\partial \bar{W}}{\partial \lambda} d\lambda, \quad (5.11)$$

where P is the inflation pressure, $\bar{W} = W$ is the strain energy density function of the overall material, here defined in terms of the principal stretches, and λ_A and λ_B are the azimuthal stretches at the inner wall ($R = A$) and the outer wall ($R = B$), respectively. The so-called reduced axial forces F are the forces that are applied in the axial direction of the closed ends cylinder in addition that pressure force Holzapfel and Ogden, 2010a,

$$\begin{aligned} F &= \pi A^2 [\lambda^2 \lambda_z - 1]^2 \\ &\times \int_{\lambda_B}^{\lambda_A} \frac{1}{[\lambda^2 \lambda_z - 1]^2} \left[2\lambda_z \frac{\partial \bar{W}}{\partial \lambda_z} - \lambda \frac{\partial \bar{W}}{\partial \lambda} \right] \lambda d\lambda. \end{aligned} \quad (5.12)$$

In the so-called membrane approximation, which assumes that the wall-thickness $H = B - A$ of the cylinder is significantly smaller than its radius R , the pressure-inflation relation (5.11) and the reduced axial force (5.12) can be approximated by

$$P \approx \frac{1}{\lambda \lambda_z} \frac{\partial \bar{W}}{\partial \lambda} \frac{H}{A}, \quad F \approx \pi A^2 \left[2 \frac{\partial \bar{W}}{\partial \lambda_z} - \lambda \lambda_z^{-1} \frac{\partial \bar{W}}{\partial \lambda} \right] \frac{H}{A}. \quad (5.13)$$

When a cylinder is inflated and subjected to additional loads, it may experience a stable inflation phase, which can be disrupted by various types of instabilities such as bulging, necking, beading, bending, prismatic bifurcation, inflation-jump instabilities, and helical buckling (Rodríguez and Merodio, 2011; Seddighi and Han, 2021; Fu et al., 2021; Yu and Fu, 2022; N. Jha et al., 2023; Topol, Nazari, et al., 2024). In biomechanics research, particular attention is given to the initiation of bulging.

5.2.4 Disruption of the stable inflation in the membrane treatment

The membrane approximation of the tube geometry allows for some basic insight into the factors that are involved in the initiation of different instability modes. This section discussed these conditions before the numerical analysis, the main scope of this work, is carried out in Sections 5.3 and 5.4.

The stable phase of inflation in a pressurized and axially stretched cylinder can be disrupted by various types of instability modes. Particular attention has been given to the formation of bulging in pressurized tubes, often coupled with bending and prismatic bifurcation. These investigations are crucial for understanding the formation of aneurysms (Tutino et al., 2021) and other related diseases (Merodio and Haughton, 2010). To study the potential bifurcation modes, the incremental displacements with respect to the deformed configuration (under equilibrium) are considered in the general form (Rodríguez and Merodio, 2011)

$$\delta \mathbf{u} = \delta u_r(r, \theta, z) \mathbf{e}_r + \delta u_\theta(r, \theta, z) \mathbf{e}_\theta + \delta u_z(r, \theta, z) \mathbf{e}_z. \quad (5.14)$$

where $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the three base unit vectors of the deformed configuration, and δu_r , δu_θ , δu_z are the components of the incremental displacements in the principal directions.

To investigate bulging (the axisymmetric bifurcation mode), the incremental displacement condition (5.14) is simplified to

$$\delta \mathbf{u} = \delta u_r(z) \mathbf{e}_r + \delta u_z(z) \mathbf{e}_z, \quad (5.15)$$

where the azimuthal component of the incremental displacement is zero, $\delta u_\theta = 0$, and the other two components depend only on the axial coordinate z .

The force equilibrium in the axial direction is expressed as

$$\frac{\partial \delta(\sigma_{zz} h r)}{\partial z} d\theta dz = P r d\theta dz \frac{\partial \delta u_r}{\partial z}, \quad (5.16)$$

where $h = b - a$ represents the wall thickness, r is the radius of the deformed cylinder in the bulging mode, and σ_{zz} is the axial normal stress component of the Cauchy stress tensor. The

left side of (5.16) results from the force increment on an infinitesimal part of the cylinder wall. The right side of (5.16) represents the resulting force due to the pressure in the axial direction. Similarly, the force equilibrium in the radial direction for this finite volume element is derived as follows

$$\delta(\sigma_{\theta\theta}h) d\theta dz - \delta(Pr) d\theta dz = \frac{\partial}{\partial z} \left[\sigma_{zz}hr d\theta \frac{\partial \delta u_r}{\partial z} \right] dz, \quad (5.17)$$

where $\sigma_{\theta\theta}$ is the azimuthal normal component of the Cauchy stress tensor.

If the balance of forces in the radial and axial directions and the normal stresses in an infinitesimal volume element are determined, then the following equation can be obtained that gives the necessary condition for the initiation of bulging, (see, e.g. Rodríguez and J. Merodio (Rodríguez and Merodio, 2011) for details)

$$f(\bar{W}, \lambda_\theta, \lambda_z) + \left[\frac{2\pi R}{L} \right]^2 \lambda_\theta^2 \lambda_z \frac{\partial \bar{W}}{\partial \lambda_z} \frac{\partial^2 \bar{W}}{\partial \lambda_z^2} = 0, \quad (5.18)$$

where $\bar{W}(\lambda_\theta, \lambda_z)$ is the strain energy density function of the solid. The first term in (5.18),

$$\begin{aligned} f(\bar{W}, \lambda_\theta, \lambda_z) &= \lambda_z^2 \frac{\partial^2 \bar{W}}{\partial \lambda_z^2} \left[\lambda_\theta^2 \frac{\partial^2 \bar{W}}{\partial \lambda_\theta^2} - \lambda_\theta \frac{\partial \bar{W}}{\partial \lambda_\theta} \right] \\ &\quad - \left[\lambda_\theta \lambda_z \frac{\partial^2 \bar{W}}{\partial \lambda_\theta \partial \lambda_z} - \lambda_\theta \frac{\partial \bar{W}}{\partial \lambda_\theta} \right]^2, \end{aligned} \quad (5.19)$$

accounts for a tube of infinite length. If $f(\bar{W}, \lambda_\theta, \lambda_z) > 0$, the inflation remains stable; if $f(\bar{W}, \lambda_\theta, \lambda_z) \leq 0$ then bifurcation occurs, in particular, if $f(\bar{W}, \lambda_\theta, \lambda_z) = 0$ a bulge/neck is initiated. Equation (5.19) has been derived in the given or similar forms, for example, in the works by Haughton & Ogden (Haughton and Ogden, 1979), Fu et al. (Fu et al., 2012), Guo et al. (Z. Guo et al., 2022), and Alhayani et al. (Alhayani et al., 2014a).

When we consider the possibility of prismatic bifurcation mode, for which the inflated cylinder bifurcates into a prismatic shape with a non-circular cross-section, the incremental displacements in (5.17) reduce to the form

$$\delta \mathbf{u} = \delta u_r(\theta) \mathbf{e}_r + \delta u_\theta(\theta) \mathbf{e}_\theta, \quad (5.20)$$

for which the axial component of the increment displacement is equal to zero, $\delta u_z = 0$, and the two non-zero components $\delta u_r(\theta)$ and $\delta u_\theta(\theta)$ just depend on θ . Following (Rodríguez and Merodio, 2011), after some manipulations, it is easy to obtain for a membrane cylinder that the bifurcation condition for the lowest (most likely) mode is given by (Rodríguez and Merodio, 2011)

$$\frac{\partial^2 \bar{W}}{\partial \lambda^2} = 0. \quad (5.21)$$

Blood vessels, such as retinal arteries and veins, often exhibit tortuous paths. This phenomenon has been linked to conditions like hypertension, diabetes, and retinopathy (Hayreh et al., 1989; Taarnhøj et al., 2008). Studies have demonstrated that a reduction in axial loading in

arteries can result in the development of tortuous shapes (Jackson et al., 2005), causing a deviation from the ideal circular cylindrical geometry. Specific arteries, such as the superficial femoral artery, undergo significant bending, and saccular aneurysms may develop (Fortier et al., 2014).

In abdominal aortic aneurysms (AAAs), a bulge (abnormal enlargement) frequently develops asymmetrically, which cannot be fully explained by an axisymmetric bulging instability analysis. Therefore, it is essential to study various factors associated with arterial bending, an instability mode that can be considered a combination of the prismatic mode and the bulging mode (Dehghani et al., 2019).

For the bending bifurcation, following (Rodríguez and Merodio, 2011), after some manipulations, it can be shown that the bending (asymmetric) bifurcation condition (composite mode) can be expressed as (Rodríguez and Merodio, 2011)

$$\mathcal{M} = \sigma_{\theta\theta} - 2\sigma_{zz} = 0. \quad (5.22)$$

It follows that if $\mathcal{M} < 0$, the membrane is stable with respect to bending, whereas if $\mathcal{M} > 0$, the membrane is unstable. Figure 5.1 provides a schematic illustration of a tube subjected to bending.

The article (Dehghani et al., 2019) reports that bulging is more likely to occur compared to bending. However, the works (Al-Chlahawi et al., 2023; Topol, Al-Chlahawi, et al., 2021) have shown that the relevance of a bifurcation mode depends on numerous factors, including the geometry, mechanical properties, and loading, as well as material volume changes that can be linked to swelling and inflammation of the tissue.

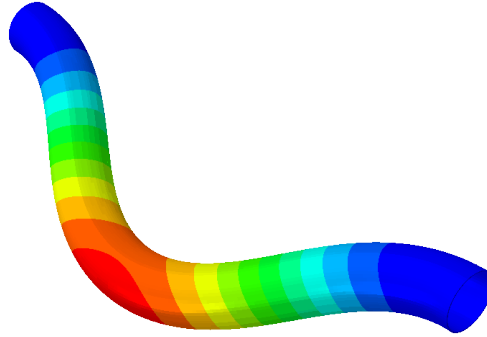


Figure 5.1: Post-bifurcation shape of the bending mode for $\lambda_z = 1$, $\bar{\alpha}_d = -40$, and $\bar{\alpha}_c = 0.5$.

The bifurcation criteria for membrane tubes are only used to guide the numerical bifurcation analyses for thick-walled tubes that are developed in the next Sections. The membrane analysis is not used in the numerical computations that follow.

5.3 Finite element analysis

The stable inflation, as well as the bulging and bending of axially stretched and inflated tubular structures, has often been investigated in membrane approximation. This approximation

provides some basic insights and frequently serves as guidelines for problems beyond the idealizations, and it often requires the use of numerical methods.

Numerical techniques play an important role in many engineering applications, as they allow us to understand and analyze complex engineering problems.

In this work, we employ finite element methods to investigate bifurcation in the inflated cylinder. The finite element modeling protocol developed in this study is presented in this section. It includes a discussion of the general methodology, material model considerations, the type of elements used, the mesh size, and the boundary and loading conditions. The simulations are performed using the commercial software package **Abaqus** for professional purposes.

5.3.1 Finite element model

A finite element model is constructed in **Abaqus** to simulate the inflation of circular cylindrical tubes subjected to axial pre-stretch, incorporating the three-dimensional of the residual stresses. In the simulation, a full three-dimensional model is employed, as axisymmetric simplifications are not applicable due to the presence of the residual stress field (see Alhayani et al., 2013). In passing, it should be noted that the elastic response of a material in which residual stresses are present is necessarily anisotropic, i.e. it cannot be isotropic.

In our simulation, we consider a cylindrical geometry with a mean diameter of $D = 10$ mm, a wall thickness of $T = 0.50$ mm, and a length of $L = 150$ mm. Thus, the geometric ratios are $L/D = 15$ and $T/D = 0.05$.

For this geometry, Figure 5.2 depicts the distribution of the residual stress components (5.9), taking different values for $\bar{\alpha}_d$ and $\bar{\alpha}_c$. Figures 5.2 (a) to (c) make use of $\bar{\alpha}_d = 40$ and $\bar{\alpha}_c = 0.5$; Figures 5.2 (d) to (f) make use of $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$; Figure 5.2 (g) makes use of $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = -0.5$. Even though there are no reliable data on residual stresses in aneurysms, the residual stress field at hand (see Figure 5.2 (b)) follows (qualitatively) results from the opening angle method, i.e., it is consistent with residual stresses arising from the opening angle method (see Holzapfel and Ogden, 2010a holzapfel2010constitutive for experimental values values carried out on arteries without aneurysms and (Sokolis et al., 2023; Sokolis et al., 2017) for residual strains in aortic aneurysms).

5.3.2 Inflation of a pressurized Cylinder

In the finite element discretization, we apply 8-node linear brick elements (C3D8RH) from the element library of **Abaqus**. A built-in hybrid formulation of **Abaqus** is applied for incompressible materials in order to address the strain-locking problem (Systèmes, 2014). Figure 5.3 depicts an example of the meshing. In our work, meshes of different sizes were applied to the model, depending on loading parameters such as the axial pre-stretch of the cylinder. Additionally, a sensitivity study was conducted. To capture the different instability effects, 64 elements were used in the circumferential direction and five elements throughout the wall thickness. In the axial direction of the cylinder, between 200 and 350 elements were used.

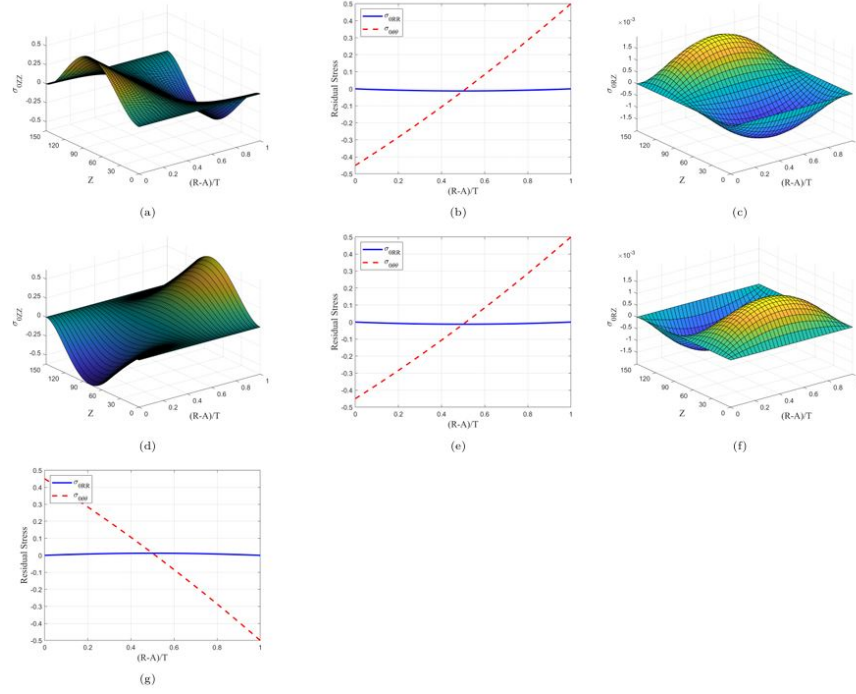


Figure 5.2: Plot of different components of the residual stress at the end of the first step. The panels (a) to (c) make use of $\bar{\alpha}_d = 40$ and $\bar{\alpha}_c = 0.5$; the panels (d) to (f) make use of $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$; the panel (g) makes use of $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = -0.5$. Note that the effect of the second term on the right side of $\sigma_{0\Theta\Theta}$ (see Eqs. (5.9)) is negligible compared with the other term so, we only plot the planar term.

The initiation of an instability mode in an inflated and axially stretched cylinder is highly sensitive to the presence of imperfections (also see, e.g. Ye et al., 2019). These imperfections affect the mechanical behavior of the cylinder under loading. The Riks method is used to solve instability problems in which the equilibrium path in a load-displacement space is relatively smooth (it does not branch). Usually, an imperfection is included in the model to avoid a numerical instability and capture the bifurcation. That imperfection is constructed according to the bifurcation modes at hand. In our case, there are two bifurcation modes: bulging and bending. To construct the imperfection in the model the bifurcation mode is used. The mode amplitude (the imperfection) implemented in the model is very small, scaled to 0.001 of the diameter. Several combined analyses need to be considered with different imperfections to verify that the type and size of imperfection do not affect the results.

5.3.3 Analysis procedure

In this work, the mechanical behavior of the material including the residual stresses has been implemented in **Abaqus** using the user-defined material subroutine (**UMAT**) via **SDVINI**.

The numerical simulation and analysis are conducted in a three-step procedure. In the first step, the residual stress field described by (5.9) is implemented into the mechanical model. To ensure the accuracy of the numerical models, various iterations are carried out to obtain

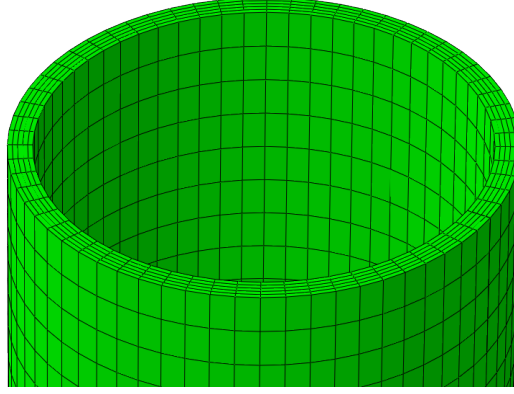


Figure 5.3: Finite element mesh of the cylinder model in Abaqus.

the desired results, which depend on the magnitude of the residual stress field in (5.9). The parameters that describe the distribution of residual stress are carefully selected in order to prevent the use of negligibly small or excessive behavior. Based on a sensitivity analysis, the residual stress parameters $\bar{\alpha}_d \in \{-40, 0, 40\}$ and $\bar{\alpha}_c \in \{-0.50, 0, 0.5\}$ will be applied.

Some results are presented in Figure 5.4–5.6, using $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$:

- Figure 5.4 shows the von Mises stress distribution in a contour plot of the cross-sectional area near the mid-span of the tube. The stresses are relatively high at both the outer and inner walls, with a minimum between them.
- Figure 5.5 illustrates the distribution of axial normal stresses (σ_{zz}) at the same location. Moving from the inner to the outer wall, these axial stresses increase from negative (compressive) to positive (tensile) values.
- Figure 5.6 displays the distribution of shear stresses $\sigma_{\theta RZ}$, which exhibit an antisymmetric pattern. The highest absolute values are located near the mean diameters.

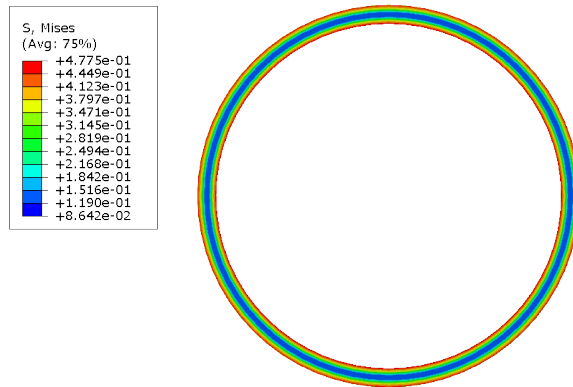


Figure 5.4: Distribution of the von-Mises stresses in a contour plot of the cross-sectional area near the mid-span of the tube. The stresses are relatively large at the outer and inner walls, with the minimum in between.

In the second stage, axial pre-stretch is imposed at one end of the tube, while the other end



Figure 5.5: Distribution of the axial normal stresses stresses (σ_{zz}) for the same location as in Fig. 5.4. Going from the inner to the outer wall, these axial stresses increase from negative values (compressive stresses) to positive values (tensile stresses).

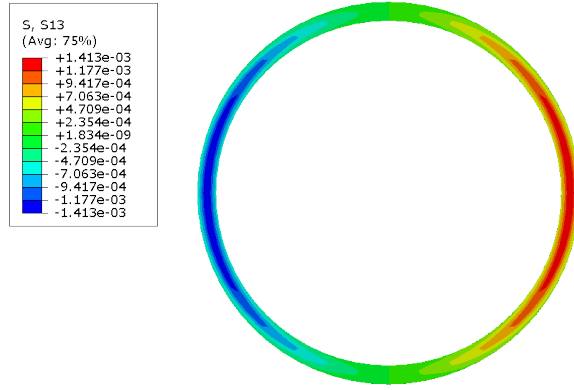


Figure 5.6: Shear stresses σ_{0RZ} , which a distributed in an antisymmetric fashion. The largest absolute values can be found near the mean diameters

is fixed in the axial direction with zero displacement applied.

Subsequently, an internal pressure of $p = 10\mu T/D$ is applied to the inner surface of the cylindrical tubes to capture both bifurcation and post-bifurcation modes, simulating arterial behavior under combined extension and inflation, including the effects of residual stresses. The modified Riks method is used for these simulations, as discussed in (Alhayani et al., 2013). Figure 5.7 presents a typical load proportionality factor (LPF) versus arc length curve (Systèmes, 2014). As the arc length increases, the applied internal pressure decreases, enabling easy identification of the peak point. Consequently, both the bifurcation and post-bifurcation patterns are captured.

To induce bulging instability, hoop displacement will be constrained to ensure that pure bulging bifurcation is the only mode that occurs. Conversely, to allow for additional instability modes, such as bending bifurcation, the hoop constraint will be subsequently removed.

The onset of bulging bifurcation can be identified by the following procedure. Two points at different axial locations on the bifurcated tube will be selected: one point at a location

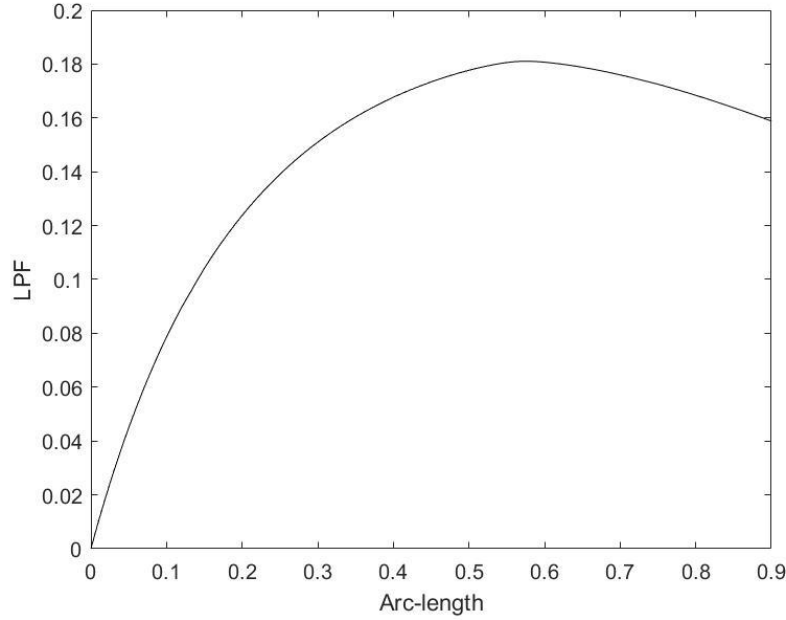


Figure 5.7: A curve for the load proportionality factor (LPF) vs. arc length. The maximum in this curve is associated with the initiation of an instability.

where bulging bifurcation is expected, and the other point at a position far from the bulging region. In the absence of bifurcation, the hoop displacements at these two points will be equal. However, when bulging bifurcation occurs, the hoop displacements at these points will differ, and the load proportionality factor (LPF) versus hoop stretch curves will diverge at a specific point, which will denote the onset of an instability mode. This underlying mechanism will be further discussed in Section 5.4.

For bending bifurcation, a membrane analysis will be conducted with the goal of identifying the onset of instability. The condition $\mathcal{M} = \sigma_{\theta\theta} - 2\sigma_{zz} > 0$ will be checked once pressure is applied. Additionally, by analyzing the development of the azimuthal and radial displacements, as well as by monitoring any lateral movement, the onset of bending will be identified. Numerical results from (Merodio and Haughton, 2010) suggest different asymmetric modes, with the first (and most likely mode) under quasi-static conditions being accepted in this study. The bending mode of bifurcation is illustrated in Fig. 5.1.

5.4 Numerical results

Nine parameter combinations for the residual stress parameters $\bar{\alpha}_c$ and $\bar{\alpha}_d$ have been selected based on the sensitivity analysis presented in Section 5.3.3. These combinations are referred to as "Case 1", "Case 2", ..., and "Case 9," and are summarized in Table 5.1.

This section explores the combined influence of axial stretches ($1 < \lambda_z < 1.8$), applied internal pressure, and non-planar residual stresses on the behavior of the numerical models. For each case with varying residual stress parameters, as outlined in Table 5.1, numerical results are

provided. The stress components are normalized with respect to the material parameter $\mu/2$.

Section 5.2.4 summarizes the conditions for which bulging (axisymmetric) bifurcation and bending (asymmetric) bifurcation are initiated. The numerical results suggest that the initiation of an instability mode is either associated with pure bulging or bending mode (related to the first mode). The present analysis provides a detailed examination to differentiate between these two potential modes of bifurcation.

5.4.1 Approach for triggering the two modes of instability

Before conducting the finite element analysis of stress distribution and deformation, as well as examining the impact of residual stresses on the development and location of bulging along the tube, it is necessary to outline the mechanisms that distinguish the two modes of instability. The bending stress criterion (5.22) from membrane analysis will be applied to the thin-walled tube, and the relationship between $\mathcal{M}$ and the arc length is illustrated.

Accordingly, if the curve for the specified relationship is positive (i.e., $\mathcal{M} > 0$), the bending mode can be identified. Conversely, if the curve is negative (i.e., $\mathcal{M} < 0$), the bulging mode will manifest at the point of bifurcation. Figure 5.8 illustrates this relationship for Case 3 in Table 5.1, with $\lambda_z = 1$ (solid line, representing the bending mode) and $\lambda_z = 1.4$ (dashed line, representing the bulging mode).

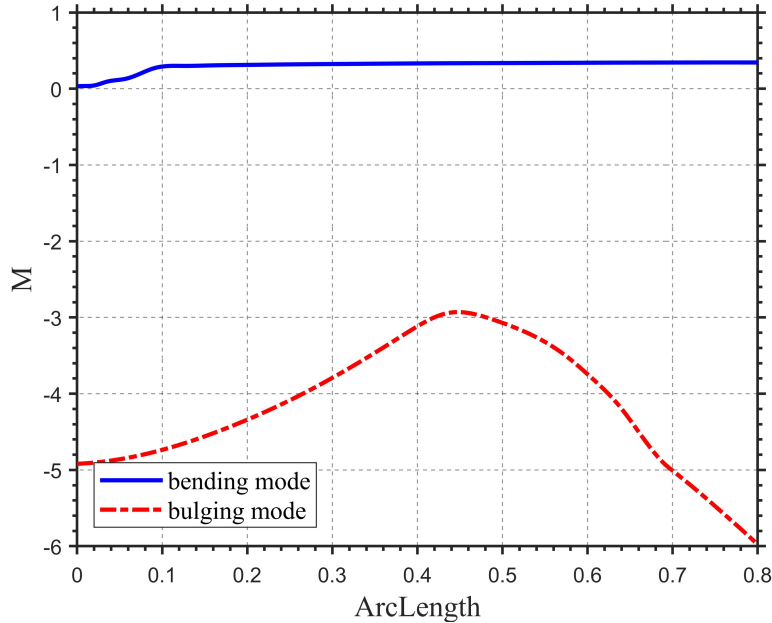


Figure 5.8: The profile of $\mathcal{M} = \sigma_{\theta\theta} - 2\sigma_{zz}$ versus arc length for the models in Case 1, with $\lambda_z = 1$ (solid line) and $\lambda_z = 1.4$ (dashed line), at a cross-section outside the mid-span of the tube, is presented. For the former case, the curve shows $\sigma_{\theta\theta} - 2\sigma_{zz} > 0$, indicating that bending may be the associated bifurcation mode. Conversely, for the latter case, the curve displays $\sigma_{\theta\theta} - 2\sigma_{zz} < 0$, suggesting that bending bifurcation is not expected under these conditions, and bulging is the associated bifurcation mode.

Additionally, to distinguish between the two instability modes and to identify the point at which bending may occur, curves of normalized pressure versus arc length for "Case 3" with two selected axial stretch values are plotted. As shown in Fig. 5.9, the two curves follow a similar path until a certain point, after which they begin to diverge, indicating the development of bending instability.

It can be further observed that the bulging curve increases monotonically, while the bending curve deviates to the right, following a new path until reaching a maximum point where the model can no longer sustain additional pressure. At this stage, the instability mode (i.e., bulging bifurcation on one side of the model) is triggered.

Based on the results, the two bifurcation modes are described as follows: For pure bulging bifurcation, the numerical results suggest that the bulging mode is feasible across the entire range of axial stretches $1 < \lambda_z < 1.8$. In contrast, the bending mode is more likely to initiate at lower axial stretch values and becomes more prominent as the axial stretch approaches $\lambda_z = 1$.

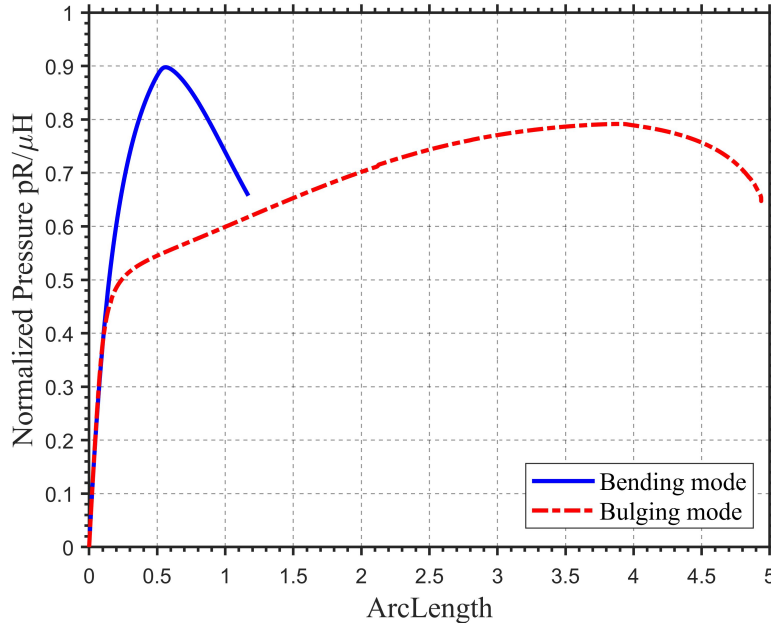


Figure 5.9: Normalized pressure versus arc length for Case 1 in Table 5.1, with $\lambda_z = 1$, is depicted. The solid line represents the model constrained to capture only the bulging mode, while the dashed line corresponds to the general condition model, which is unrestricted. The point where the two curves diverge is identified as the bending bifurcation point. The maximum points of both curves, beyond which the structure cannot withstand additional load, are designated as the bulging bifurcation points. The maximum point on the dashed line is associated with the formation of a one-sided bulge.

5.4.2 Comparison of stress distribution: Case 1 and Case 3 from the Table 5.1, taking $\lambda_z = 1.6$

In this section, we investigate Case 1 and Case 3 from Table 5.1, taking an axial stretch of $\lambda_z = 1.6$. Based on the findings from the previous section, bulging bifurcation is identified as the relevant instability mode. Bulging may occur at different axial positions along the tube, which is primarily influenced by the magnitude of residual stresses.

The results shown in Fig. 5.10 demonstrate that bulging may occur at the midpoint of the tube's length (Fig. 5.10(c)) when $\bar{\alpha}_d$ is negative, or at one of the tube's ends (Fig. 5.10(d)) when $\bar{\alpha}_d$ is positive, irrespective of the sign of $\bar{\alpha}_c$. This observation is further supported by the plotted stress components, specifically $\sigma_{\theta\theta}$ and σ_{zz} , along the tube's length, where the bulge forms at the locations corresponding to the maximum values in the stress distribution.

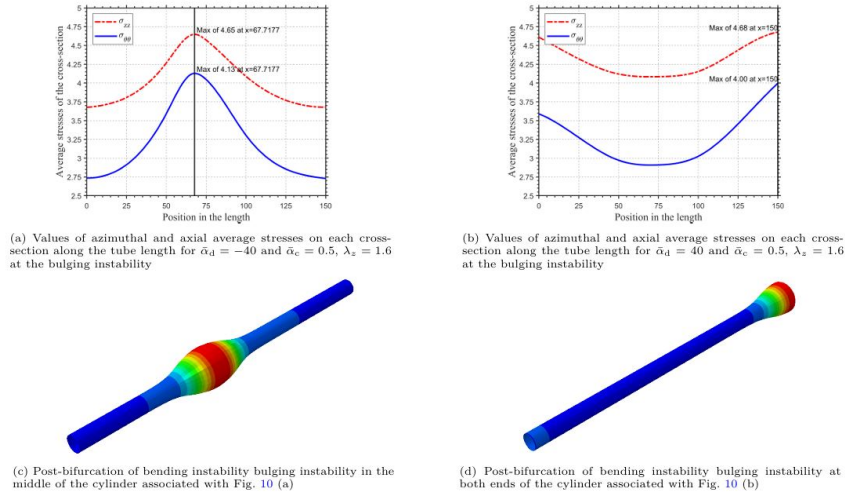


Figure 5.10: Varying location of the bulge, depending on the specifications for the residual stress parameters.

Figure 5.10 shows that the stress distribution is not symmetric about the mid-span of the tube. In Fig. 5.10(a), both the axial and hoop stresses increase until they reach their maxima near the center of the cylinder, after which they then decrease again. In contrast, Fig. 5.10(b) shows that both the axial and hoop stresses decrease until they approach a minimum near the center, after which they increase again to approach the global maximum.

The post-bifurcation patterns that correspond to the top panels of Fig. 5.10 are depicted in panels below: Fig. 5.10(c) corresponds to Fig. 5.10(a), and Fig. 5.10(d) corresponds to Fig. 5.10(b). These observations show the significant influence of residual stresses on the location of an occurrence of bulging along the tube length. Changing the sign of $\bar{\alpha}_d$ shifts the bulge position between the center and the end of the tube, while the sign of $\bar{\alpha}_c$ has no noticeable effect on this behavior.

5.4.3 Normalized inflation pressure versus axial stretch at the inflation point

Figure 5.11 depicts the relationship between the normalized inflation pressure and the axial stretches at the location of the occurrence of bifurcation. The magnitude of pressure decreases with increasing axial stretch for all considered cases of the residual stress parameters (see also, for example, (Topol, Nazari, et al., 2024; Moradalizadeh et al., 2024)). In the models that include only planar residual stress ($\bar{\alpha}_d = 0$) a slower rate of pressure decrease compared to models that include both residual stress parameters ($\bar{\alpha}_d, \bar{\alpha}_c$) can be observed. A difference in the pressure magnitudes can be observed for the models that include residual stresses. For instance, the differences between Case 1 and Case 3, as well as between Case 2 and Case 4, are relatively large, although all of these cases take non-zero values for both residual stress parameters $\bar{\alpha}_d$ and $\bar{\alpha}_c$. The observations from this section show how the presence of initial stresses affects the inflation behavior of the inflated and axially stretched tube.

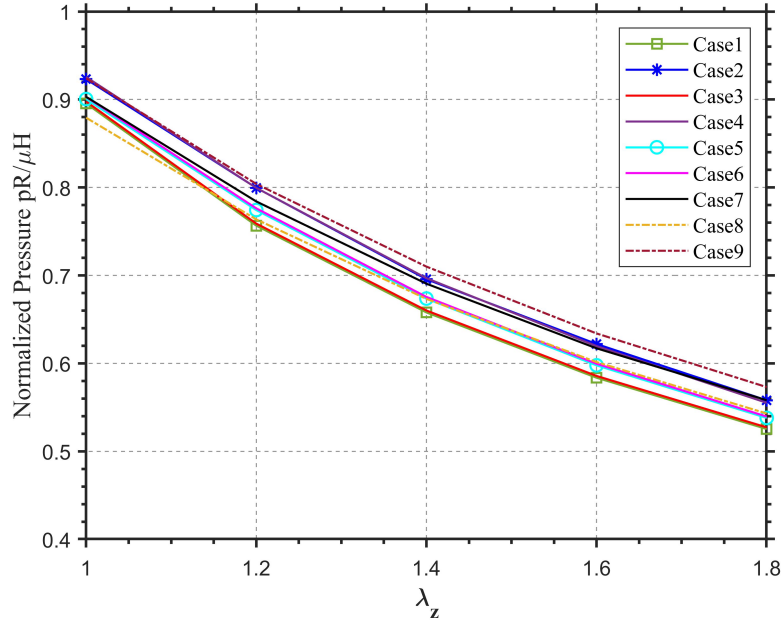


Figure 5.11: Normalized inflation pressure versus axial stretches at the bifurcation point. The curves represent all nine cases listed in Table 5.1.

5.4.4 Normalized inflation pressure versus azimuthal stretch during the formation and propagation of the bulging

In this section, we investigate the relationship between the azimuthal stretch (λ_θ) and the normalized inflation pressure in a tube to elucidate the azimuthal stretch pattern during the formation and propagation of the bulging. Consequently, we select two points located at the mid-thickness of the tube for analysis: one point inside the bulge and the other one outside the bulge.

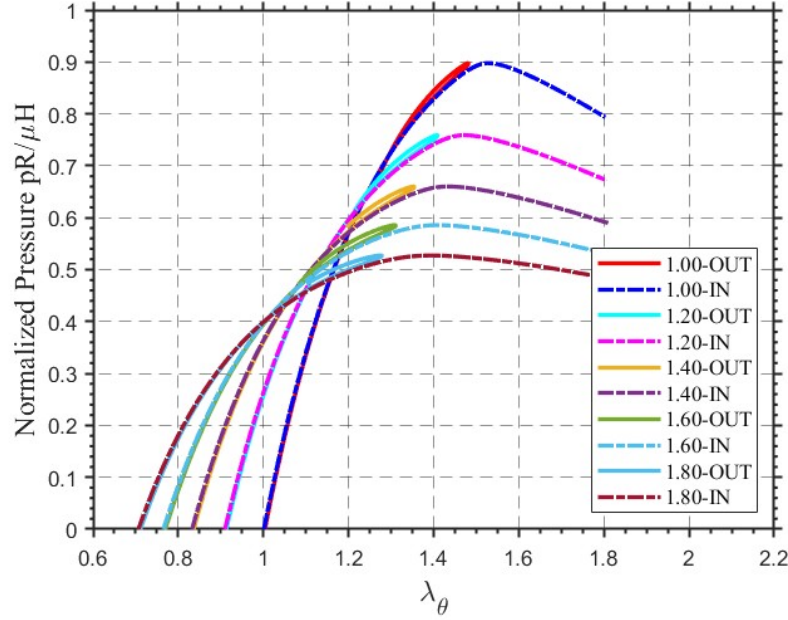


Figure 5.12: The normalized inflation pressure versus hoop stretch λ_θ is plotted for $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$, comparing two locations on the tube: one point located inside the bulge and another point outside the bulge. The different curves correspond to varying axial stretches: $\lambda_z = 1.0$ ("1.00-OUT", "1.00-IN"), $\lambda_z = 1.2$ ("1.20-OUT", "1.20-IN"), $\lambda_z = 1.4$ ("1.40-OUT", "1.40-IN"), $\lambda_z = 1.6$ ("1.60-OUT", "1.60-IN"), and $\lambda_z = 1.8$ ("1.80-OUT", "1.80-IN").

At the first point, the hoop stretches increase to a certain extent, and the pressure reaches its maximum value, beyond which the tube cannot sustain additional pressure, leading to failure. This results in a decrease in the load proportionality factor (LPF) as circumferential stretches continue to increase, indicating further inflation at that location despite a reduction in pressure. In contrast, the second point behaves similarly to the first up to the maximum pressure point; however, beyond this point, both the LPF and circumferential stretches decrease, indicating deflation.

Figure 5.12 illustrates this mechanism for two points located at different axial positions, with $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$, for $\lambda_z = 1.0, 1.2, 1.4, 1.6, 1.8$. The dashed line represents the first point, while the solid line represents the second point throughout the pressure application history.

5.4.5 Azimuthal and axial stretches scenarios in bending bifurcation

The relationship between circumferential stretch and pressure in models exhibiting bending bifurcation shows a similar trend to the azimuthal stretches λ_θ and normalized pressures discussed in Section 5.4.4 and illustrated in Fig. 5.12 for bulging bifurcation. Likewise, the effect of axial stretch on stress distribution in bending bifurcation is consistent with the

scenario described in Section 5.4.6 and depicted in Fig. 5.13 for bulging bifurcation. For brevity, the detailed results concerning these relationships in bending bifurcation have been omitted from this article.

5.4.6 Influence of axial stretching on the stress distributions

This section investigates the relationship between axial stretching and stress distributions in the axial and the azimuthal directions. Figure 5.13 presents results depicting normalized pressure against arc length for Case 1 and Case 3, considering two specific values of axial stretch ($\lambda_z = 1.2$ and $\lambda_z = 1.6$).

In Fig. 5.13 (a) for $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$ (Case 3), it can be observed that both the azimuthal and axial stresses increase with an increase of axial stress. The maxima of these stresses are located in the axial center of the tube.

In Fig. 5.13 (b) for $\bar{\alpha}_d = 40$ and $\bar{\alpha}_c = 0.5$ (Case 1), it can be observed that both the azimuthal and axial stresses increase with an increase of axial stress. The local maxima of these stresses are located at the ends of the tube, and the stress minima are located near the axial center.

5.4.7 Stress distribution for Case 1 and Case 3, taking $\lambda_z = 1.10$

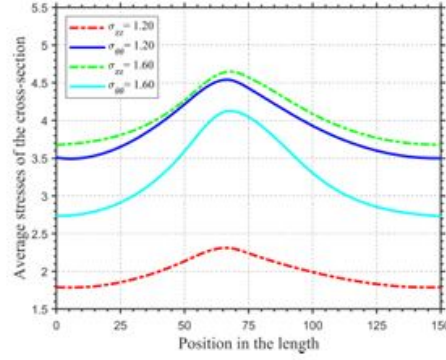
In this section, we analyze the outcomes of bending bifurcation by examining scenarios analogous to those previously investigated for bulging bifurcation in Section 5.4.2. Figures 5.14 and 5.15 illustrate the stress distributions for Case 3 and Case 1, respectively, from Table 5.1, under an axial stretch of $\lambda_z = 1.10$.

Figure 5.14(a) presents the azimuthal and axial average stresses along each cross-section of the tube for $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$ (Case 3) at the point of bending instability. Similarly, Fig. 5.15(a) shows the stress component distributions at the same instability point for $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$ (Case 1). These normal stresses exhibit minimal variation along the length of the tube.

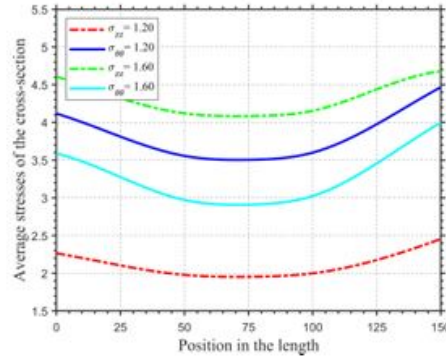
Figure 5.14(b) displays the azimuthal and axial average stresses along each cross-section of the tube at the bulging instability point for $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$ (Case 3). In contrast, Fig. 5.15(b) presents the stress component distributions at the bulging instability point for $\bar{\alpha}_d = 40$ and $\bar{\alpha}_c = 0.5$ (Case 1).

In Fig. 5.14(b), these stresses attain their maximum near the center of the tube and their minimum near the tube ends. The corresponding post-bifurcation configurations of the loaded tube are depicted in panels (c) and (d) of Fig. 5.14. Conversely, in Fig. 5.15(b), the stresses reach their minimum near the center of the tube and their maximum at one end, with the post-bifurcation configurations displayed in panels (c) and (d) of Fig. 5.15.

The results from Figs. 5.14 and 5.15 show that during the occurrence of a bending instability, the distributions of the considered stress components in both cases are nearly identical (see Figs. 5.14(a) and 5.15(a)). This illustrates the onset of a stability mode, with both cases following a similar trajectory until this point. However, a difference in the stress distributions occurs during the bulging instability phase. As observed in Figs. 5.14(b) and 5.15(b), the



(a) Values of azimuthal and axial average stresses on each cross-section along the tube length for $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$, (Case 3), taking two values for the axial stretches, $\lambda_z = 1.2$ and $\lambda_z = 1.6$.



(b) Values of azimuthal and axial average stresses on each cross-section along the tube length for $\bar{\alpha}_d = 40$ and $\bar{\alpha}_c = 0.5$, (Case 1), taking two values for the axial stretches, $\lambda_z = 1.2$ and $\lambda_z = 1.6$.

Figure 5.13: Effect of increasing the axial stretches on the distributions of stress components for a bulging instability.

bulge occurs at the maximum of the stress distributions, which is affected by the residual stress parameters.

5.4.8 Normalized inflation pressure versus axial stretch at the bulging initiation point

Figure 5.16 illustrates the relationship between normalized pressure and axial stretches at the point of bulging or bending bifurcation for selected cases with residual stress. Several key observations can be made from Fig. 5.16: First, the normalized pressure values associated with bending bifurcation increase with axial stretch up to specific limits, beyond which the instability transitions from bending to bulging bifurcation (i.e., the pressure inducing bending bifurcation equals that which causes one-sided bulging bifurcation). These transition limits vary among the cases listed in Table 5.1, with the range of limits lying between $1.18 < \lambda_z < 1.2$. This transition is clearly depicted in Fig. 5.17 for Case 3, where the shift from bending to bulging occurs at $\lambda_z = 1.18$.

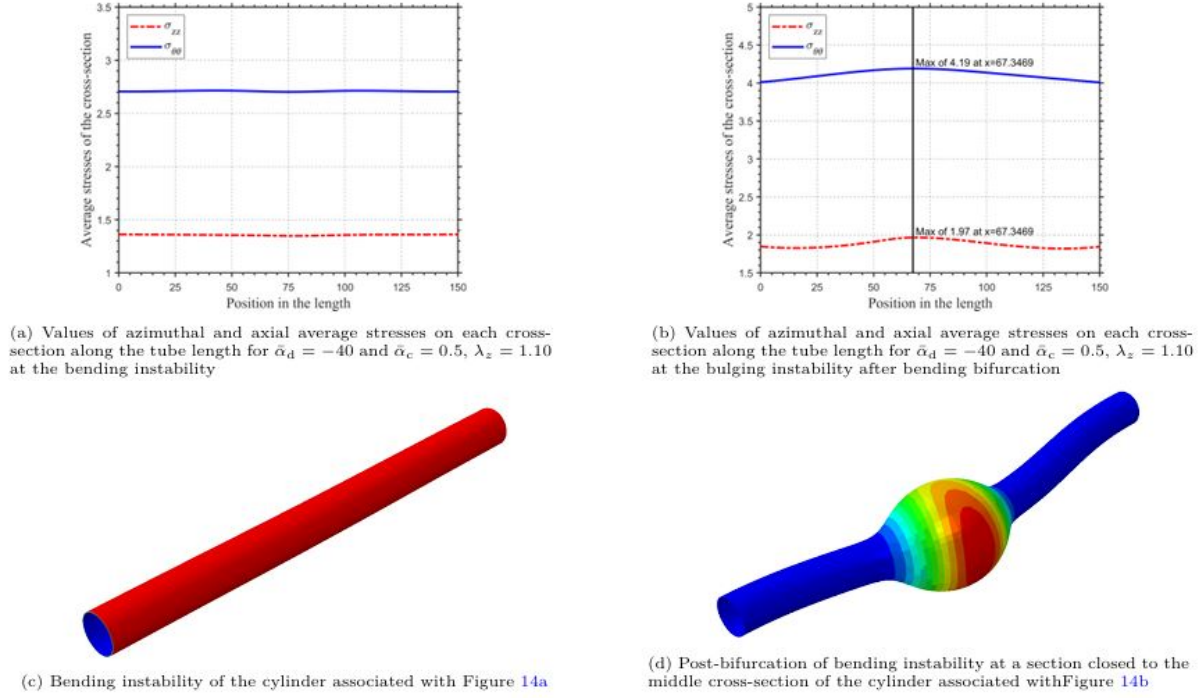


Figure 5.14: Different bulge locations of the cases with different residual stress parameters.

The model is capable of sustaining additional pressure after the onset of bending bifurcation up to a certain point, where a one-sided bulging bifurcation initiates, forming an irregular shape similar to that observed in abdominal aortic aneurysms (AAA), as discussed earlier.

The normalized pressure values associated with the pure bulging mode are higher than those related to bulging triggered by bending bifurcation. This suggests that delaying aneurysm formation is possible by avoiding bending bifurcation. However, for large values of λ_z , these two pressures converge, as seen in Fig. 5.17 after the transition point at $\lambda_z = 1.18$.

5.4.9 Role of increasing axial stretches in the post-bifurcation behavior

In this section, we study the post-bifurcation configuration that is obtained after the initiation of an instability. Figure 5.18 compares the instability configurations for all nine cases of residual stress distributions that are listed in Table 5.1, taking an axial stretch of $\lambda_z = 1.2$. Note that in the case where $\bar{\alpha}_d = 0$ (indicating planar residual stress), the bulge occurs in the center of the tube length, which highlights the role that non-planar residual stresses possess.

Figure 5.19 presents the post-bifurcation configurations for a cylinder subjected to residual stresses corresponding to $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$ (Case 3), for increasing axial stretches from $\lambda_z = 1.00$ in panel (a) to $\lambda_z = 1.2$ in panel (k), in increments of $\Delta\lambda_z = 0.02$. It can be concluded that for relatively small axial stretches, the bulge develops on one side in an irregular

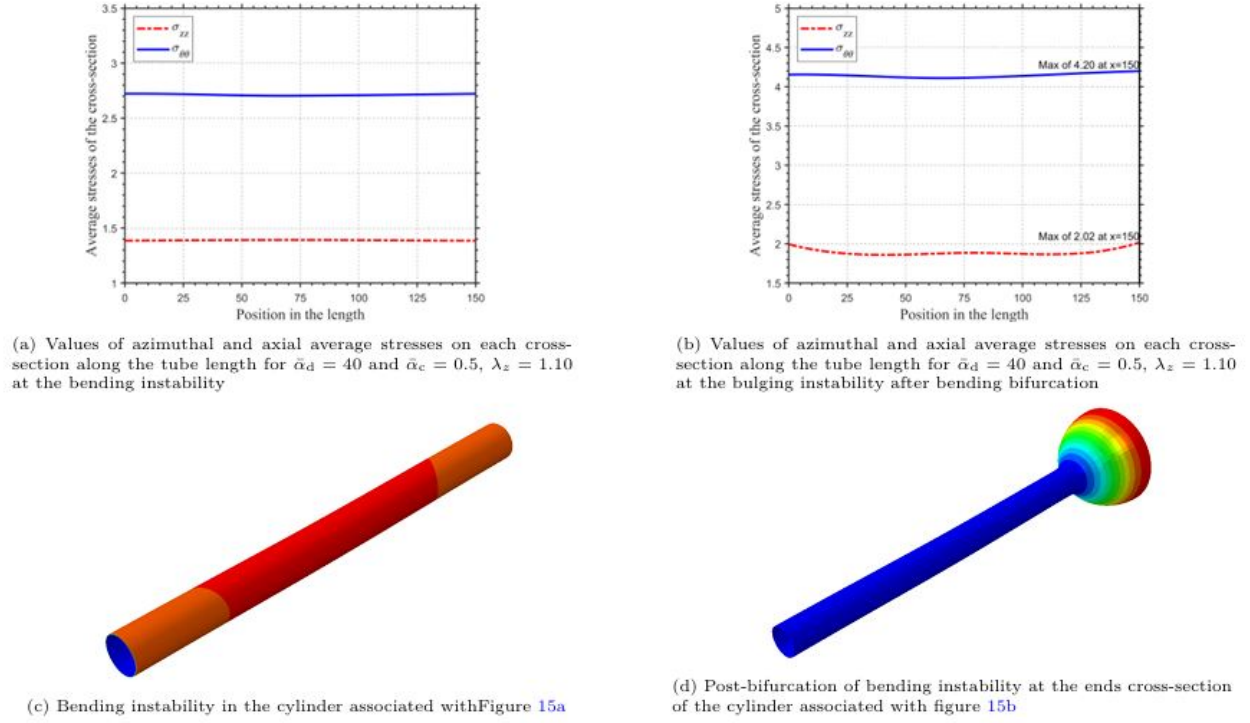


Figure 5.15: Different bulge locations of the cases with different residual stress parameters

shape, which is characteristic of an abdominal aortic aneurysm (AAA) (Desena-Galarza et al., 2021). As the axial stretch increases, the instability configuration approaches one associated with pure bulging.

5.4.10 Development of the post-bifurcation configuration

The final examples illustrate the development of instability from its initiation to the fully developed post-bifurcation configuration.

Figure 5.20 presents an example of the formation history of a bulge during the bulging bifurcation, detailing the process from initiation to the final configuration for the model incorporating both residual stress parameters (Case 3 with $\bar{\alpha}_d$ and $\bar{\alpha}_c$) at $\lambda_z = 1.0$.

Figure 5.21 depicts the transitional configurations during the formation and propagation of a one-sided bulge. Both Figs. 5.20 and 5.21 illustrate different sequences of configurations for two models with identical residual stress parameters, axial stretch values, and loading conditions. As described in Section 5.3, lateral movement is restricted in order to capture the bulging behavior shown in Fig. 5.20, whereas lateral bending is permitted in the cylindrical model to capture the bending mode depicted in Fig. 5.21. This distinction explains the differing configuration patterns observed in Figs. 5.14 and 5.15, despite the models sharing identical residual stress parameters and axial stretch values.

Figure 5.22 illustrates the transitional configurations for the model with residual stress

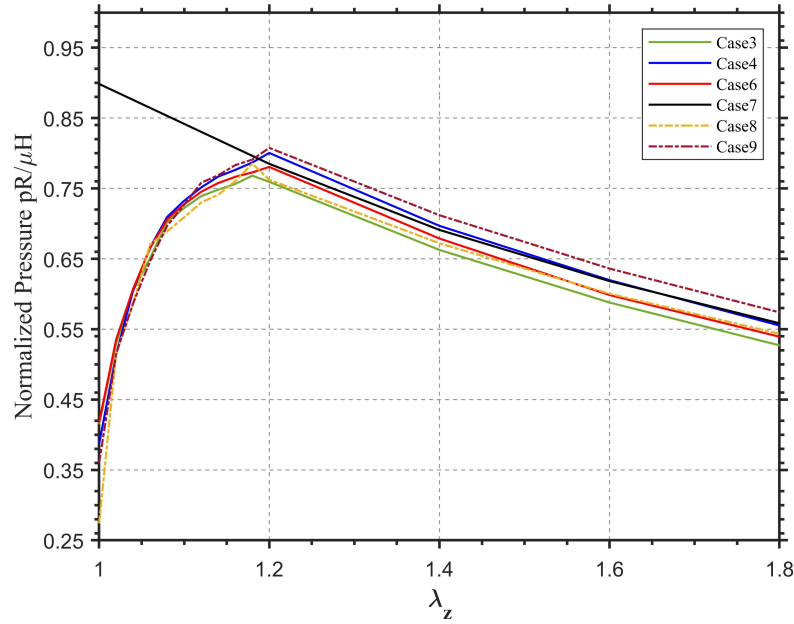


Figure 5.16: Normalized inflation pressure versus axial stretches at the bifurcation point for selected residual stress distributions.

parameters (Case 3 with $\bar{\alpha}_d$ and $\bar{\alpha}_c$) at $\lambda_z = 1.18$, from the initiation to the progression of the bulge in the bending mode. It can be observed that the bulge transitions from a one-sided configuration, as shown in Fig. 5.21, to an almost axisymmetric bulge, as demonstrated in Fig. 5.20. This observation underscores the conclusion presented in Section 5.4.8 and illustrated in Fig. 5.17, where, for sufficiently large values of λ_z , the tube no longer deflects, resulting in the formation of an axisymmetric bulge.

5.5 Conclusions

Residual stresses occur in biological soft tissues, and they have a significant influence on the mechanical behavior of biological systems. This article focuses on residual stresses in tubular structures that apply to the modeling of arterial wall tissues at bifurcation points. The role of residual stresses and their consequences remain the focus of ongoing investigation (Laubrie et al., 2020; Giudici et al., 2020). Research that addresses the behavior of these materials under non-planar residual stresses remains limited. To address this gap in research, it is necessary to study the boundary value problem for incompressible hyperelastic circular cylindrical tubes subjected to combined axial stretch and internal pressure. Such investigations must account for a certain magnitude and distribution of the residual stress parameters and axial stretch levels.

This article provides a study that aims to understand the role of the presence of residual stresses on the mechanical behavior of inflated and axially stretched tubes. A certain focus is laid on the bulging and bending bifurcation phenomena under the three-dimensional distribution of the residual stresses, which appears to be crucial to understanding the formation of aneurysms

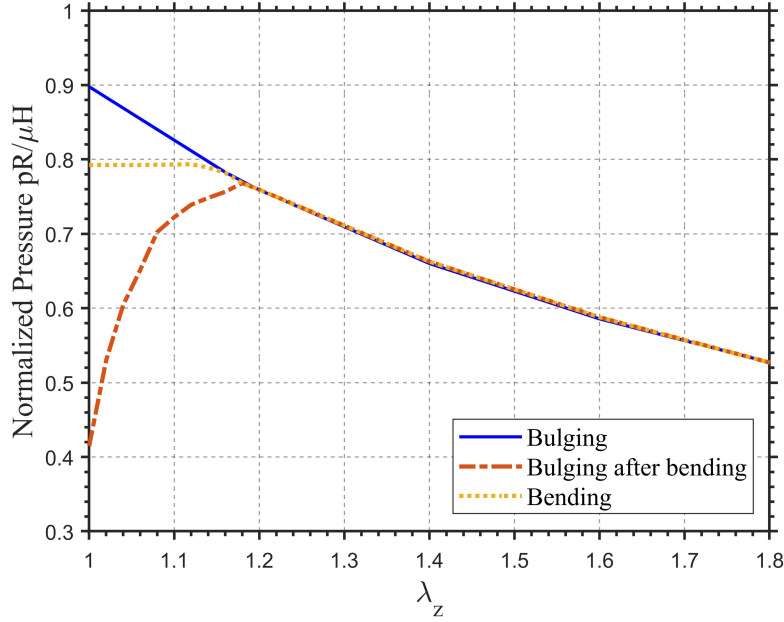


Figure 5.17: Normalized inflation pressure at bifurcation is plotted against axial stretch for the case where $\bar{\alpha}_d = -40$ and $\bar{\alpha}_c = 0.5$. In this figure, for sufficiently small values of λ_z , three distinct curves are present. The solid line represents the pure bulging mode, which corresponds to zero azimuthal displacements. The dash-dotted and dotted lines correspond to the bending bifurcation and the one-sided bulge under general displacements (i.e., without any imposed constraints), respectively. As λ_z increases, the bending and one-sided bulge modes cease to occur.

in blood vessels.

The results of the numerical analysis indicate that both bending and a superposition of bending and bulging are likely for relatively small values for the axial stress of the cylinder, and it becomes less likely as the axial stretch values increase. Moreover, the inflation pressure values that are required to initiate pure bulging modes are higher than those pressure values that bulging is triggered by bending. These observations suggest that avoiding bending bifurcation could potentially delay the onset of aneurysms. The distribution of residual stresses also plays a role in the location of bulges along the tube. It has been shown that the specification of the two α_d and α_c that define the distribution of the residual stresses in the cylinder affect whether a bulge is initiated near the center or at the ends of the tube. It has also been shown that bulging that is triggered by bending tends to occur on just one side of the tube, which reflects observations from the occurrence of abdominal aortic aneurysms (AAA). The asymmetrical development of the bulge aligns with the predictions of the adopted methodology.

In this study, a finite element model has been developed that allows the user to capture the non-planar residual stress distribution and the mechanical response of the cylinder under a combination of inflation pressure and axial stretch. The observations from these simulations may help to predict the formation of instabilities in arterial tissue so that countermeasures can

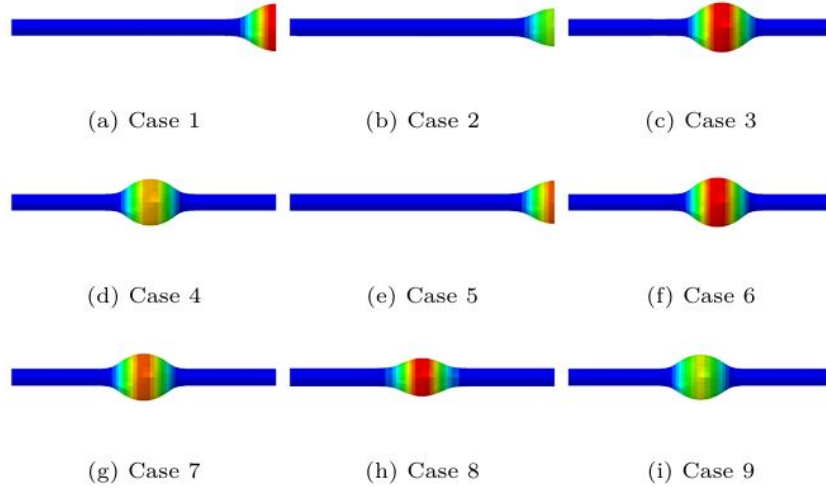


Figure 5.18: Post-bifurcation behavior associated with bulging is presented for all nine combinations of $\bar{\alpha}_d$ and $\bar{\alpha}_c$ listed in Table 5.1, with an axial stretch of $\lambda_z = 1.2$. It can be observed that both the magnitude and the location of the instability vary depending on the different distributions of residual stresses.

be taken to prevent to occurrence of, for example, abdominal aortic aneurysms. Maximum values of (average) azimuthal stress and (average) axial stress on the cross-sections along the tube are expected at the locations just prior to bulging. This is consistent with the results given in (Desena-Galarza et al., 2021) and further generalizes those results.

The results from this study may provide a starting point for further investigation of the mechanical behavior of arterial wall tissues under the formation of various types of instabilities.

It should be noted that mechanical models often idealize the problem under investigation. While our considerations have been restricted to purely elastic material behavior, at larger deformations, soft tissue may undergo damage or softening processes that have a significant influence on the initiation of unstable behavior of arterial tissue (Holzapfel and Ogden, 2020). Arterial tissue consists of multiple layers with distinct collagen fiber properties and arrangements (Holzapfel and Ogden, 2010a). The role of fiber properties and arrangements on the initiation of bulging has been discussed in studies such as (Topol, Asghari, et al., 2023; Topol et al., 2022). Nevertheless, it has been shown that even for purely isotropic materials the initiation of instability may depend on multiple factors that include the interplay of material geometry, mechanical behavior, and volume-changing effects (Moradalizadeh et al., 2024).

Arterial tissue is anisotropic due to the, for instance, collagen fibers, which are embedded in the ground material. In the mechanical modeling, the mechanical behavior of the fibers is taken into consideration by decomposing the strain energy function for the material into the isotropic contribution and the contribution of the different fiber families. The fibers may possess a natural configuration that may differ from that of the ground material (Topol, Asghari, et al., 2023; Topol et al., 2022) and therefore tend to contract the surrounding ground

material. This work restricts its considerations to an isotropic material that is residually stressed, without explicitly accounting for the source of such initial stresses.

Arterial soft tissue consists of three concentric layers with different properties (Gasser et al., 2006). Many works idealize the arterial wall in the modeling by considering just two or one of the layers. In the present modeling, the arterial tissue has been modeled using a single wall layer.

Many studies on the occurrence of bifurcation phenomena ignore the time-dependent nature of the mechanical soft tissue properties, which, evolve due to various reasons. These property changes are often the result of growth, remodeling, and healing processes (Gierig et al., 2021; Humphrey, 2021; Topol, Demirkoparan, and Pence, 2021a; Topol and Pence, 2024; Topol, Demirkoparan, and Pence, 2021b).

Table 5.1: The combinations of parameters $\bar{\alpha}_c$ and $\bar{\alpha}_d$ define the magnitude and distribution of the residual stresses in (5.9). Case 7 corresponds to purely neo-Hookean material behavior, with no residual stresses present in the material.

Case	$\bar{\alpha}_d$	$\bar{\alpha}_c$
Case 1	40	0.5
Case 2	40	-0.5
Case 3	-40	0.5
Case 4	-40	-0.5
Case 5	40	0
Case 6	-40	0
Case 7	0	0
Case 8	0	0.5
Case 9	0	-0.5

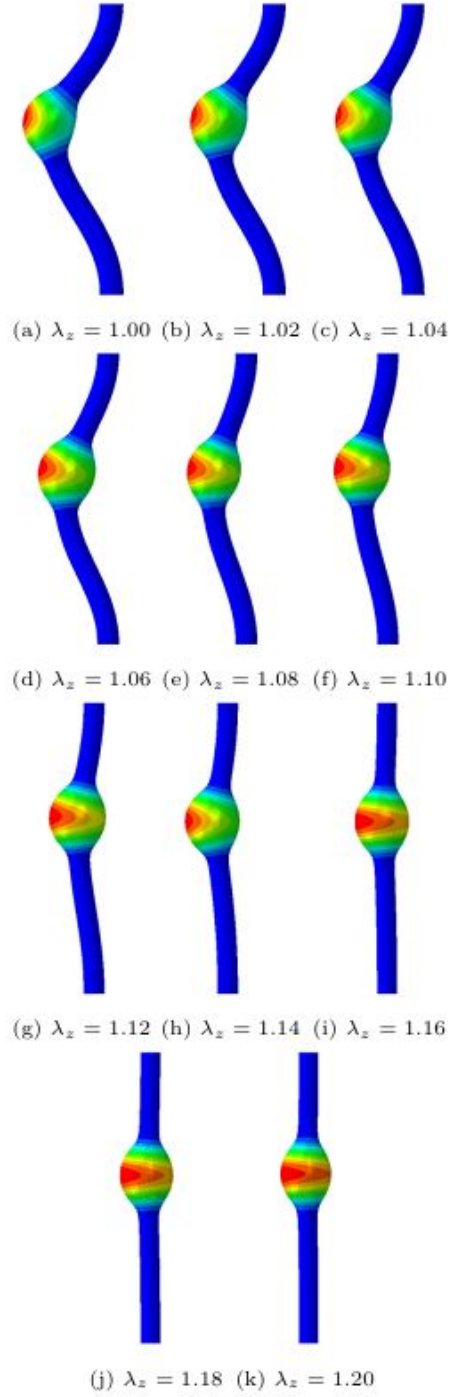


Figure 5.19: Post-bifurcations associated with bending, taking the parameters $\bar{\alpha}_d = -40$, $\bar{\alpha}_c = 0.5$ ("Case 3") for the distribution of the residual stresses. The panels are for different axial stretch values, which correspond to the dash-dotted lines in Fig. 5.11.

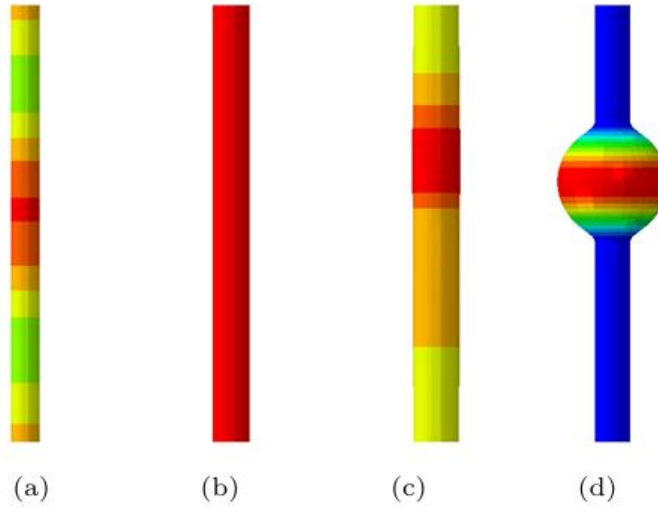


Figure 5.20: Illustrated example of the formation history of a bulge, starting from its initiation in panel (a) until the final configuration in panel (d) after its radial propagation, taking the parameters $\bar{\alpha}_d = -40$, $\bar{\alpha}_c = 0.5$ for the residual stresses ("Case 3") and an axial stretch of $\lambda_z = 1.00$.

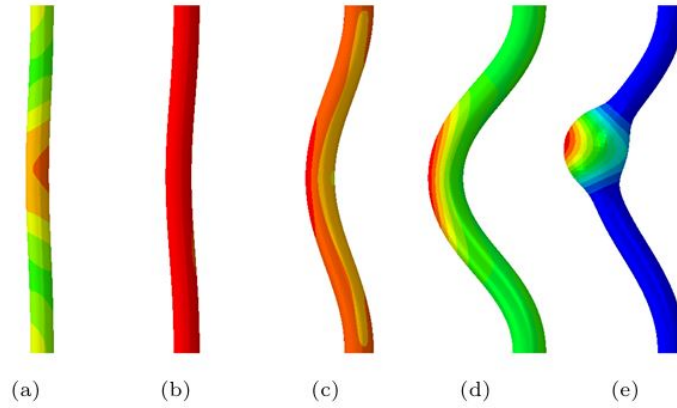


Figure 5.21: Illustrated example of the formation history of a bending instability, starting from its initiation in panel (a) until the final configuration in panel (e), taking the parameters $\bar{\alpha}_d = -40$, $\bar{\alpha}_c = 0.5$ for the residual stresses ("Case 3") and an axial stretch of $\lambda_z = 1.00$. In the final configuration, one can observe a bulge that has propagated radially.

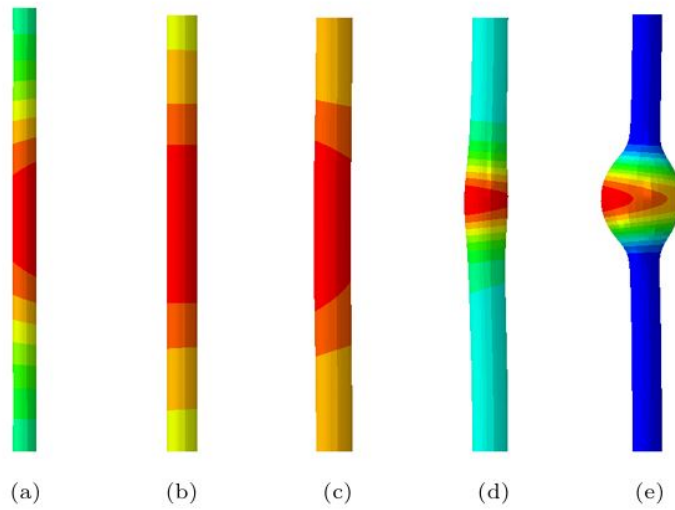


Figure 5.22: Illustrated example of the formation history of a bending instability, starting from its initiation in panel (a) until the final configuration in panel (e), taking the parameters $\bar{\alpha}_d = -40$, $\bar{\alpha}_c = 0.5$ for the residual stresses ("Case 3") and an axial stretch of $\lambda_z = 1.18$. In the final configuration, one can observe a one-sided bulge that has propagated radially.

Chapter 6

Conclusions

6.1 Summary of Findings

This thesis delves into the intricate relationship between various input parameters and the resulting mechanical behavior of circular cylindrical tubes within the field of applied biomechanics, with a specific focus on internal pressure, axial stretches, and residual stresses. Also, this work investigates several models (neo-Hookean and Mooney–Rivlin) with an emphasis on three bifurcation modes (prismatic, bulging and bending) with applications to aneurysms. The research project studies the effect of fiber reinforcement on the mechanical response of the utilized model.

In chapter 2: The mechanical instabilities of fiber-reinforced elastic membranes subjected to swelling, axial loading, and internal pressure have been investigated, with a focus on biomedical applications such as aortic aneurysms. The study presents a thorough analysis of bifurcation criteria, emphasizing two primary bifurcation modes: prismatic and bending.

The results of the different analyses show that for an isotropic neo-Hookean type of membrane, without any fiber reinforcement, prismatic bifurcation is not feasible. On the other hand, our results have also shown that swelling promotes bending bifurcation. When fibers are taken into account, then the reinforcement functions for the fibers give different scenarios with respect to prismatic bifurcations. In the standard reinforcement model, prismatic bifurcations may occur depending on material parameters, deformation (axial and azimuthal stretches), and fiber winding angles. For the exponential form for the strain energy density function for the fibers and for herein applied values for the parameters, prismatic bifurcation does not occur.

The results have further shown that certain specifications for the fiber winding angle and for the material parameters can trigger bending bifurcation in both fiber models. The present article finds that swelling may stabilize the cylinder to bending bifurcation. Owing to the interplay of countless biomechanical and physical processes in biological soft tissue, the mechanical behavior of blood vessels is complex, so that the modeling the mechanical behavior of tissues usually relies on idealizations. Nevertheless, our preliminary results provide a deeper understanding on how the interplay of the mechanical behavior of the ground substance and

the fibers contribute to the formation of instabilities and, therefore, to the failure of arteries.

In chapter 3: The effect of swelling, as well as other factors, on bulging initiation and propagation of pressurized membranes under axial loading has been studied. The material is taken to be hyperelastic and reinforced by two fiber families in helical arrangements and the fibers are taken to be parallel to each other within a fiber family.

In chapter 4: a detailed analysis of bifurcation phenomena in cylindrical membranes, incorporating the effects of swelling, axial loading, and internal pressure has been presented. It builds on previous work by introducing new material models and examining their implications for bifurcation behavior.

This work has shown that the occurrence and the order of the bifurcation modes in a symmetrically reinforced swellable membrane tube depend on different factors, including the geometry of the examined problem, the elastic properties of the constituents, and the arrangement of the fibers.

Also, for the swellable neo-Hookean-type ground substance and the exponential form of the fiber strain energy density function, prismatic bifurcation is not expected and attention has been paid to bulging and bending.

Moreover, it has been shown that swelling may stabilize material (Demirkoparan and Merodio, 2020) with respect to bulging bifurcation, whereas deswelling stabilizes the material with respect to bending bifurcation. Both deswelling and axial stretch may change the lower bound for bifurcation during extension and inflation of cylinders.

In chapter 5: The study has shed light on the mechanical behavior of thick-walled cylindrical tubes under the influence of residual stresses, focusing on the bifurcation phenomena that occur during axial extension and inflation. The research underscores the importance of considering three-dimensional residual stresses in predicting the loci of bulging, which is critical in understanding the formation of aneurysms.

The results elucidated that bending bifurcation is probable at small axial pre-stretch values and becomes less likely as these values increase. Moreover, in bulging bifurcation, the pressures associated with the pure bulging modes are larger than those of bulging bifurcations caused by bending modes. This reveals a significant conclusion: aneurysm formation can be delayed by avoiding bending bifurcation.

The study showed that the distribution of residual stresses significantly affects the location of bulges along the tube, with different stress parameters (e.g., α_d and α_c) influencing whether the bulge appears at the center or near the ends of the tube. Furthermore, the bulging triggered by bending bifurcation happens on one side of the tubes capturing the state of abdominal aortic aneurysm (AAA). Moreover, the location of the bulge was unsymmetrically developed as it was expected according to the methodology that has been adopted,

The developed finite element model, incorporating a user-defined material subroutine in ABAQUS, proves to be an effective tool for capturing the complex interactions of residual stress fields and mechanical responses in hyperelastic cylindrical tubes. The insights obtained from this work could inform better predictive models and therapeutic strategies for managing

conditions like abdominal aortic aneurysms.

By integrating these findings, the study offers a robust framework for further exploration into the mechanics of arterial wall tissues and similar biological systems, paving the way for advancements in biomedical engineering and clinical applications.

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Annexes