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Two quasi-optimal solutions for uniform load between two supports

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Two frameworks for the title problem are presented, for the half and full plane cases respectively, with semi-analytical layouts. It is shown that both are among the best layouts obtained by several methods, including Simulated Annealing (SA) search and Linear Programming (LP). Respectively, the heights are found to be 0.442 and 0.633 times the span L. The required volume of material is given by $V = kwL^2/f$ where w is the load intensity and f the allowable stress, being k respectively equal to 0.985 and 0.758. The corresponding results for the layouts on the basis of the one presented by W. Hemp in 1974 are $0.338L$, $0.676L$, $1.158wL^2/f$, and $0.788wL^2/f$. None of these frameworks meets the conditions of Michell theorem, hence there is not sound proof that they are absolute optima. The meaning of this fact is discussed and two future lines of research are proposed.

Keywords: layout optimization, trusses, Michell structures

1. Introduction

The plan of our exposition is as follows: Firstly, we introduce a minimal set of definitions and theorems for a complete and clear description of the class of structural problems we deal with. Secondly, we describe the "bridge class" of problems, showing the optimal solutions known up to date for the "minimal" problem of the class, and presenting the nearly optimal solutions for the Maxwell version of the title problem. Thirdly, we explain in detail how we get the full plane solution, and then we apply the same method for the half-plane case. Finally we discuss the significance of our results and suggest future lines of research.

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2. Definitions

DEFINITION 2.1 Maxwell problem: *any set of known, external forces in equilibrium. Each force must be determined in position, direction and magnitude.* (Cervera 1989)

DEFINITION 2.2 Maxwell structure: *any set of internal forces (tension or compression) in self-equilibrium that added to the external forces of a Maxwell problem satisfies that every subset of forces, internal or external, acting at the same point is in equilibrium. Each element of the set is defined by the internal force magnitude q (taking compression as negative) and its two application points, being ℓ the distance between them. (*Maxwell 1890, Cervera 1989)

In the following, we are only concerned with Maxwell problems and structures.

DEFINITION 2.3 Maxwell number of a structure: $\mathcal{M} = q_i \ell_i$. (Cervera 1989)

DEFINITION 2.4 Quantity of structure (or stress volumen): $\mathcal{Q} = |q_i|\ell_i$ (Prager 1974, Cervera 1989).

The second denomination arises from the fact that $|q|\ell = |\sigma|A\ell$, being A the cross-section area of the element and σ its stress. That is to say, $\mathcal{Q}$ can be viewed as the stress volume of the framework, named "structural volume" too. Obviously, for fully-stressed designs with allowable stress f, the geometrical volume is $\mathcal{Q}/f$ and the weight is $\rho\mathcal{Q}/f$, being ρ the specific weight of the material.

COROLLARY 2.5 *The Maxwell number is the difference between the quantity of structure in tension and in compression, $\mathcal{M} = \mathcal{Q}^+ - \mathcal{Q}^-$.*

LEMMA 2.6 (Maxwell lemma) *The Maxwell number of all the structures that solve a given Maxwell problem is constant.* (Maxwell 1890, Michell 1904, Hemp 1958, Cervera 1989)

THEOREM 2.7 (Michell theorem)

> *"A frame attains the limit of economy of material* [quantity of structure] *possible in any frame-structure under the same applied forces* [a given Maxwell problem]*, if the space occupied by it can be subjected to an appropriate small* [virtual or "adjoint"] *deformation, such that the strains in all the bars of the frame are increased by equal fractions of their lengths, not less than the fractional change of length of any element of the space"*
>
> *"If the space subjected to the deformations extends to infinity in all directions, the volume of the frame is a minimum relatively to all others, otherwise it will have been shown to be a minimum only relatively to those within the same assigned finite boundary."*
>
> Michell (1904) (see also Hemp 1958, Cervera 1989)

3. The 'bridge' class of structural problems

The "bridge problem" is the structural problem of equilibrate an uniform weight w over an horizontal length L with supports in the load line. Depending of the number S of suitable supports and its relative distances, we have different problems, so it is better to speak of the "bridge class" of problems. We are only concerned with Maxwell problems,

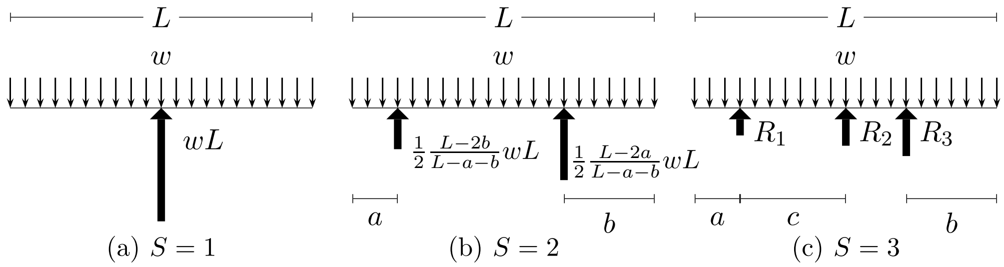


Figure 1. Maxwell problems in the bridge class

although these are only a subset of the class, see figure 1. For any Maxwell problem of this class the Maxwell number is zero and after Lemma 2.6 any solution will have the same quantity of structure in tension than in compression.

We will consider only the complete plane and the half-plane above the uniform load as the space Ω occupied for feasible solutions, following Michell approach.

For $S = 1$, the vertical reaction is equal to wL acting at mid span, so there is an unique problem, see figure 1(a). The optimal solutions for the two spaces are known after Michell solutions for one load, see figure 2. The two solutions fulfil the sufficient condition of the Michell theorem. This problem can be viewed as a superposition of elemental problems that belong to the named “three points” class, for which a general Michell solution is well known (e.g. Chan 1960). As the solution for every elemental problems fulfil the Michell theorem with the very same virtual displacement field, we can superpose these solutions to form the uniform load solution.

For $S = 2$, the sum of the two vertical reactions must be equal to wL, but its magnitudes depend of its relative position, see figure 1(b). Each couple of values a, b with $L - a - b > 0$ and $a \leq b$ defines a different Maxwell problem. This case covers a fairly large subset of real bridges. In this case, it is not useful to decompose the problem in elemental ones as the displacement fields of each of these are different.

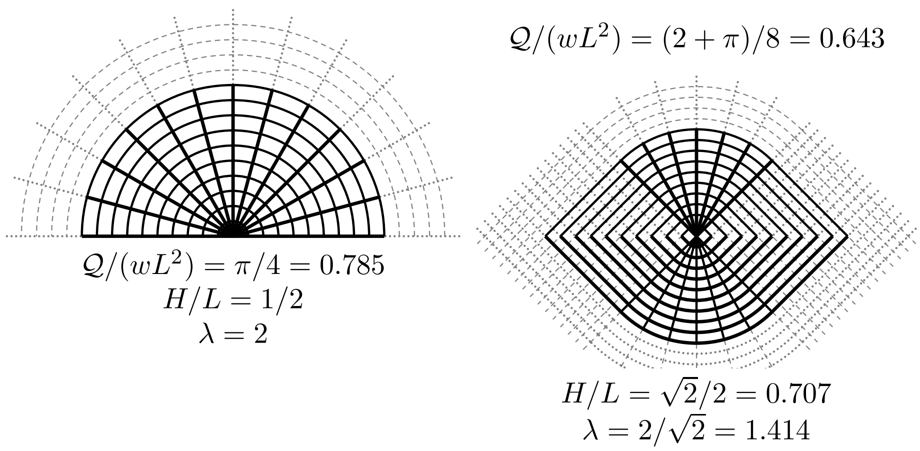
For $S = 3$, we have four degrees of freedom for defining a Maxwell problem: the positions of supports, a, b, c, and the magnitude of one of the reactions, see figure 1(c). As S increase so the number of Maxwell problems under consideration does.

Note that the optimal solutions for $S = 1$ can be used to assemble feasible solutions for $S > 1$ (maybe good ones) but the latter will not be optimal in the Michell sense. Thereafter, each value of S represents a new set of Maxwell problems that requires a specific search.

4. Quasi-optimal solutions for two supports

“In spite of a prolonged international research effort, Michell layouts have only been determined for a few simple loading conditions” (Rozvany 1984). This is true for the bridge class too, as we have optimal solution only for the case $S = 1$. Or, more precisely, we do not know any solution for the cases $S > 1$ that fulfils the Michell theorem conditions.

Here, we are mainly interested for the case $S = 2$, with $a = b = 0$, that is to say, with the supports at the edges of L. Hemp (1974) showed a non-optimal solution for a non-Maxwell version of this problem. The Hemp solution solves the problem for the upper half-plane with fixed supports, and hence with horizontal reactions whose magnitude depends of the shape of the solution and in consequence is unknown. Hemp shows that his solution is not optimal in the Michell sense because it fulfils the Michell theorem



(a) Half-plane solution (b) Full-plane solution

The solid lines stand for the layout of the solutions. The dashed and dotted lines stand for the principal deformations of the Michell field, $\varepsilon = \mu$ and $\varepsilon = -\mu$ respectively.

Figure 2. Optimal solutions for the $S = 1$ bridge problem

Table 1. Solutions for $S = 2$ and $a = b = 0$

Problem:	← Hemp (1974) →		←		Maxwell problem		→
Ω:	← half plane →			← half plane →		← full plane →	
Entry:	Hemp	SA,hemp[a]	Hemp,mh	AH,mh	SA,mh[a]	Hemp,mp	AH,mp
$\mathcal{Q}/(wL^2)$	0.788	0.79312	1.158	0.98602	0.97431	0.788	0.75800
H/L	0.338	0.32628	0.338	0.44189	0.45535	0.676	0.63340
$\lambda = L/H$	2.959	3.07548	2.959	2.27853	2.19611	1.479	1.57878

[a]Geometric data available at `http://habitat.aq.upm.es/gi/mve/dt/`.

conditions except in a bound, small region at mid-span.

It is easy to derive two feasible solutions for the Maxwell version after Hemp solution, with the method explained in Vázquez & Cervera (2011), see the "Hemp" entries in table 1. The two solutions presented here, "AH" entries ibidem, improve those, so we hope they could serve as new benchmarks for future researches as they have a closed, semi-analytical form.

5. The AH full-plane solution

The solution consists of an arch with oblique hangers above the load and its x-symmetrical figure below, see figure 3 and AH,mp entry in table 1. This solution was suggested by some results after standard simulated annealing search (see Press *et al.* 1988, pp. 346–351, for the algorithm; see Dhingra & Bennage 1995, Botello *et al.* 1999 for structural applications)—we use a SA version with geometric and connectivity variables but without any ground structure (Vázquez 1995). Anyway this solution was guessed by the authors (Vázquez 2011), and *not found* by the SA algorithm.

For the horizontal position χL, with $\chi \in [0, 1/2]$, the hanger will have direction $\alpha(\chi)$

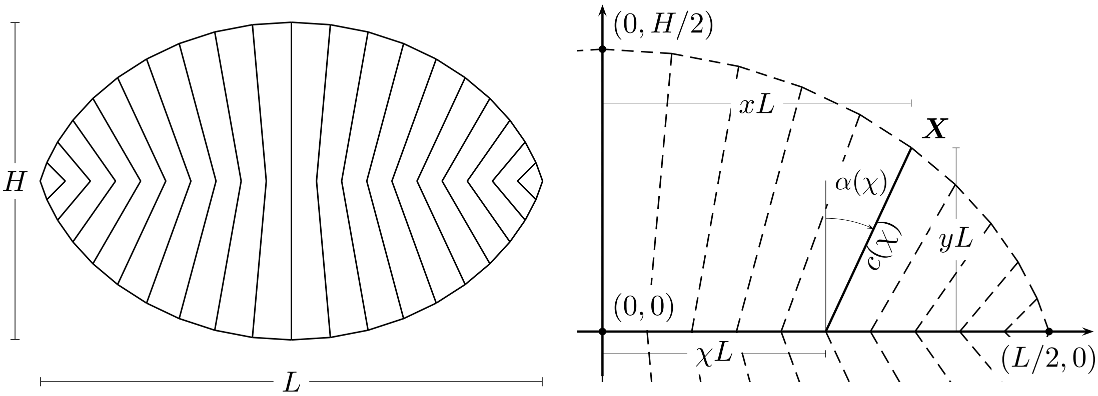


Figure 3. The solution AH,mp

and length $c(\chi)$, and its upper extreme defines a point of the arch:

$$\boldsymbol{X} = L(x, y)^T = (L\chi + c\sin(\alpha), c\cos(\alpha))^T = L(\chi + y\tan\alpha, y)^T,$$

With the notation $t(\chi) = \tan\alpha(\chi)$, $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}\chi$, the geometric conditions are:

$$\boldsymbol{X} = L(x, y)^T = L(\chi + y\,t, y)^T, \qquad \boldsymbol{X}' = L(x', y')^T = L(1 + y'\,t + y\,t', y')^T. \tag{1}$$

The equilibrium condition in the inferior extreme of the hanger defines its internal force, $\boldsymbol{P} = (P_\mathrm{h}, P_\mathrm{v})^T$ with $P_\mathrm{h} = wt/2$, and $P_\mathrm{v} = w/2$.

The shape of the arch, $\boldsymbol{X}(\chi)$, is determined by its internal force, $(R_\mathrm{h}, R_\mathrm{v})^T$:

$$\frac{R_\mathrm{h}}{R_\mathrm{v}} = -\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{x'}{y'}. \tag{2}$$

The equilibrium in the joint of the hanger and the arch determines the variation of the internal force of the latter:

$$\mathrm{d}R_\mathrm{h} = -P_\mathrm{h}\,L\,\mathrm{d}\chi, \quad \mathrm{d}R_\mathrm{v} = P_\mathrm{v}\,L\,\mathrm{d}\chi. \tag{3}$$

Now, using the equilibrium condition of the half structure in the vertical axis of symmetry:

$$R_\mathrm{h}(0) = \frac{wL^2/8}{2y(0)L} = \frac{wL}{2}\frac{1}{4h} = \frac{wL}{2}\frac{\lambda}{4}, \tag{4}$$

being $h = 2y(0)$, a non-dimensional quantity, which is the inverse of the global slenderness $\lambda = L/H$. Finally, we get:

$$R_\mathrm{h}(\chi) = R_\mathrm{h}(0) - \int_0^\chi P_\mathrm{h}\,L\,\mathrm{d}u = \frac{wL}{2}\left(\frac{\lambda}{4} - \int_0^\chi t(u)\,\mathrm{d}u\right), \quad R_\mathrm{v}(\chi) = \int_0^\chi P_\mathrm{v}\,L\,\mathrm{d}u = \frac{wL}{2}\chi. \tag{5}$$

Let us now compute the quantity of structure of one quadrant, $\mathcal{Q}_\mathrm{c}$. As $\mathcal{Q} = \int |q|\,\mathrm{d}s$ and $|q|$ and $\mathrm{d}s$ are the modulus of parallel vectors, its product can be decomposed in horizontal and vertical parts, hence $\mathcal{Q} = \int |q_x|\,|x'|\mathrm{d}\chi + \int |q_y|\,|y'|\mathrm{d}\chi = \mathcal{Q}^{=} + \mathcal{Q}^{||}$. Furthermore, we can decompose each quantity in the hangers and arch parts:

$$\begin{aligned}
\mathcal{Q}_\mathrm{c} &= \mathcal{Q}_\mathrm{c}^{+=} + \mathcal{Q}_\mathrm{c}^{+||} + \mathcal{Q}_\mathrm{c}^{-=} + \mathcal{Q}_\mathrm{c}^{-||}, \\
\mathcal{Q}_\mathrm{c}^{+=} &= \frac{wL^2}{2}\int_0^{1/2} t^2\, y\,\mathrm{d}\chi, \\
\mathcal{Q}_\mathrm{c}^{+||} &= \frac{wL^2}{2}\int_0^{1/2} y\,\mathrm{d}\chi, \\
\mathcal{Q}_\mathrm{c}^{-=} &= \frac{wL^2}{2}\int_0^{1/2}\left(\frac{\lambda}{4} - \int_0^{\chi} t(u)\,\mathrm{d}u\right)\left(1 + t\,y' + t'\,y\right)\,\mathrm{d}\chi, \text{ and} \\
\mathcal{Q}_\mathrm{c}^{-||} &= \frac{wL^2}{2}\int_0^{1/2} \chi\,(-y')\,\mathrm{d}\chi.
\end{aligned} \tag{6}$$

Note that we do not need to use absolute value operator because all integrands above are positive $\forall\chi \in [0, \frac{1}{2}]$. For example, if the thrust component R_h changes of sign, it follows from (2) and $y' < 0$ that x' does as well, so that the product $R_\mathrm{h}x'$ will be positive ever.

The terms of the arch can be integrated by parts —noticing that $t(0) = 0$ and $y(1/2) = 0$— as:

$$\begin{aligned}
\mathcal{Q}_\mathrm{c}^{-=} &= \frac{wL^2}{2}\int_0^{1/2}\left(\frac{\lambda}{4} - \int_0^{\chi} t(u)\,\mathrm{d}u + t^2\,y\right)\,\mathrm{d}\chi, \text{ and} \\
\mathcal{Q}_\mathrm{c}^{-||} &= \frac{wL^2}{2}\int_0^{1/2} y\,\mathrm{d}\chi.
\end{aligned} \tag{7}$$

Finally, we can re-write (6) as:

$$\min_{y,t,\lambda} \mathcal{Q} = 2wL^2 \min_{y,t,\lambda}\int_0^{1/2}\left\{2y(t^2+1) + \frac{\lambda}{4} - \int_0^{\chi} t(u)\,\mathrm{d}u\right\}\mathrm{d}\chi. \tag{8}$$

Hence the functions $y(\chi)$, $t(\chi)$ and the slenderness λ that solve the last equation fulfilling the condition (2), determine the optimal layout. This last condition can be written as $\chi\,\mathrm{d}x = -\left(\lambda/4 - \int_0^{\chi} t(u)\,\mathrm{d}u\right)\mathrm{d}y$, and with $\mathrm{d}x = x'\mathrm{d}\chi$, $\mathrm{d}y = y'\mathrm{d}\chi$ and using (1), we can re-write it in the form:

$$\{\chi\} + \left\{\chi\,y\,t' + y'\left(\frac{\lambda}{4} - \int_0^{\chi} t(u)\,\mathrm{d}u + \chi\,t\right)\right\} = 0. \tag{9}$$

Noticing that this expression is the sum of two exact differentials, we integrate it explicitly to get:

$$\frac{1}{2}\chi^2 + y\left(\frac{\lambda}{4} - \int_0^{\chi} t(u)\,\mathrm{d}u + \chi t\right) = \frac{1}{8}, \tag{10}$$

with $2y(0) = 1/\lambda$, that is to say, we get a closed form for $y(\chi)$.

Table 2. Polynomial solutions for the full-plane domain.

$T(\chi)$	$\mathcal{Q} \div wL^2$	$h = 1/\lambda$	z_1	z_2	z_3	z_4	$\alpha_{1/2}$
$T_1 = z_1$	0,81650	0,61237	$-0,40825$				$0,00°$
$T_2 = z_1 + z_2\chi^2$	0,75822	0,63013	$-0,39674$	0,96323			$43,93°$
$T_3 = z_1 + z_2\chi^2 + z_3\chi^3$	0,75802	0,63406	$-0,39428$	0,78647	0,41997		$47,76°$
$T_4 = z_1 + z_2\chi^2 + z_3\chi^3 + z_4\chi^4$	0,75800	0,63340	$-0,39470$	0,89280	$-0,16636$	0,79663	$49,39°$
variational, numerical sol.	0.75800	0.63339					

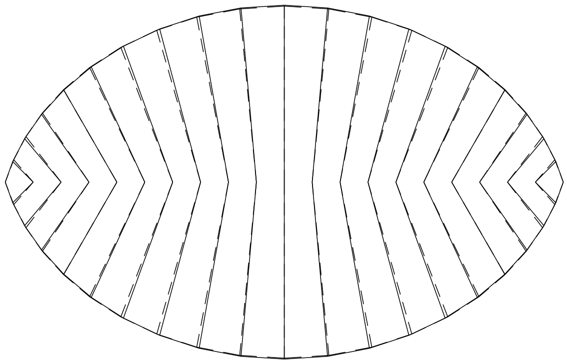


Figure 4. Polynomial solutions for the full-plane problem. The T_4 solution is drawn in solid line, while the T_2 solution in dashed line; it can be seen that the differences in shape are very small.

Finally, it is convenient the following change of notation:

$$T = \int_0^\chi t(u)\,\mathrm{d}u - \lambda/4, \quad T' = t = \tan\alpha, \quad \text{with } T(0) = -\lambda/4, \quad T'(0) = 0,$$

and then the problem (8) subject to (10) can be written as:

$$\min_T \mathcal{Q} = 2wL^2 \min_T \int_0^{1/2} \Big\{ 2y(T'^2 + 1) - T \Big\}\,\mathrm{d}\chi \qquad \text{with } y = \frac{1 - 4\chi^2}{8(\chi T' - T)}. \tag{11}$$

5.1. *Polynomial solutions*

For the sake of brevity, we solve (11) into a small function space, because a general, variational approach leads to a similar solution. With this aim, we explore function T in polynomial form.

Let $T(\chi) = z_i \zeta_i(\chi)$, being ζ_i selected functions such that $\zeta_i'(0) = 0$, and z_i a real coefficient. The equations $\partial\mathcal{Q}/\partial z_i = 0$ determine the values of $(z_1, \ldots)$ that minimize $\mathcal{Q}$ for each selected $(\zeta_1, \ldots)$ set. Some solutions are shown in table 2 and in the figure 4.

6. The AH half-plane solution

It was natural to get a solution for the half plane case following the same scheme that for the full plane case. Now we have that the internal force in the hanger is $P_{\mathrm{h}} = wt$, $P_{\mathrm{v}} = w$; and the internal force in the arch is:

$$R_{\mathrm{h}}(\chi) = wL\left(\frac{\lambda}{8} - \int_0^{\chi} t(u)\,\mathrm{d}u\right), \quad R_{\mathrm{v}}(\chi) = wL\chi, \tag{12}$$

but it is to be noted that now $y(0) = h = 1/\lambda$. The traction in the tie is $S(\chi) = R_{\mathrm{h}}(\chi)$. Thanks to previous analytical results, we was advised that both S and R_{h} could change the sign near the supports, and then the layout width would be greater than the span L. To take account of this fact we name χ_{cr} to the solution of $S(\chi) = 0$ in the following.

To compute the quantity of structure we must add the quantity of the tie between supports, which equilibrates the horizontal component of hangers and arch. As before, let us compute the quantity of structure of one half, $\mathcal{Q}_{\mathrm{h}}$. For the tie, we have:

$$\mathcal{Q}_{\mathrm{h}}^{\mathrm{tie}} = \int_0^{1/2} |S(\chi)|\, L\mathrm{d}\chi = wL^2\left(\int_0^{\chi_{cr}} - \int_{\chi_{cr}}^{1/2}\right)\left(\frac{\lambda}{8} - \int_0^{\chi} t(u)\,\mathrm{d}u\right)\mathrm{d}\chi. \tag{13}$$

The rest of the terms can be gotten from the corresponding terms $\mathcal{Q}_{\mathrm{c}}$ in (6) or (7), but multiplying by two and substituting $\lambda/4$ by $\lambda/8$. Therefore, with $T = \int_0^{\chi} t(u)\,\mathrm{d}u - \lambda/8$, the half plane layout is the solution of:

$$\min_T \mathcal{Q} = 4wL^2 \min_T \int_0^{1/2} y(T'^2 + 1)\mathrm{d}\chi - \int_0^{\chi_{cr}} T\mathrm{d}\chi \qquad \text{with } y = \frac{1 - 4\chi^2}{8(\chi T' - T)}. \tag{14}$$

Some polynomial solutions are shown in table 3 and in the figure 5. As for the full-plane case, the z_i constants are determined with differential conditions, but now we must introduce a new equation for χ_{cr}, $\partial\mathcal{Q}/\partial\chi_{cr} = 0$: this equation assures that the contribution of the tie, $\mathcal{Q}_{\mathrm{h}}^{\mathrm{tie}}$, will be maximum. However, note that $\chi_{cr} = 1/2$ in the case of the function T_1 because $T'(\chi) = 0$, that is to say, the hangers are vertical and $S(\chi)$ is constant.

It is to be outlined the difficulties we found in this case: e.g. note the carefully selection of the functions ζ_i in the table 3: whereas in the full plane case, we arrives at acceptable convergence with low degree polynomials, now we have not been able to arrive at similar precision with simple functions. This fact led us to suspect that the solution could be no optimal. Actually, solutions found by SA slightly improve those that are presented here—up to 1%, see the SA,mh entry in table 1 and the figure 6(a). Of course, the SA algorithm explores alternatives that could include local Michell nets near the supports, and its search domain is greater than the one provided by the AH scheme.

7. Discussion

The quantity of structure of the Hemp,mp solution is some 3.95% greater than that of the AH,mp solution, while the classical solution of parabolic arc with vertical hangers (and their symmetrical ones under the load line, T_1 in table 2) is about 3.5% greater than the Hemp one. However the meaningful difference among solutions is not that of

Table 3. Polynomial solutions for the half-plane domain.

$T(\chi)$	$\mathcal{Q} \div wL^2$	$h = 1/\lambda$	z_1	z_2	z_3 or z_5	z_7
$T_1 = z_1$	1.15470	0.43301	−0.28868			
$T_2 = z_1 + z_2\chi^2$	0.98515	0.44012	−0.28401	1.32299		
$T_3 = z_1 + z_2\chi^2 + z_3\chi^3$	0.98512	0.44081	−0.28357	1.26780	0.12405	
$T_4 = z_1 + z_2\chi^2 + z_5\chi^5 + z_7\chi^7$	0.98468	0.44189	−0.28288	1.20527	4.85375	−17.15594

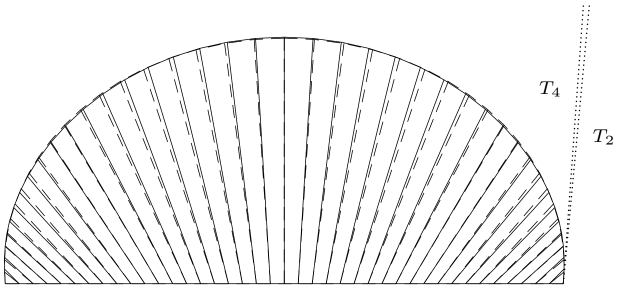
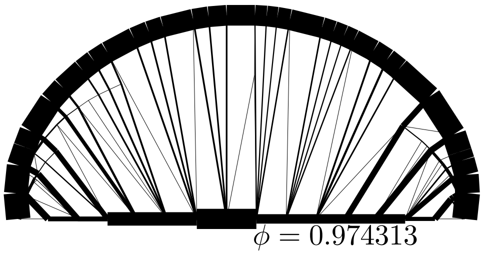


Figure 5. Polynomial solutions for the half-plane problem. The T_4 solution is drawn in solid line, while the T_2 solution, in dashed line; it can be seen that the differences in shape are small. The slope of the arches at the support is draw as dotted lines at the right.

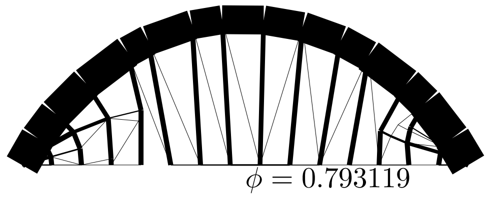
the quantity of structure (or volume), but the differences in shape, height or slenderness. Furthermore, the AH solutions are not orthogonal nets, being far from Michell layouts.

Anyway, there are several reasons for speaking of "quasi-optimal" solution about the AH ones:

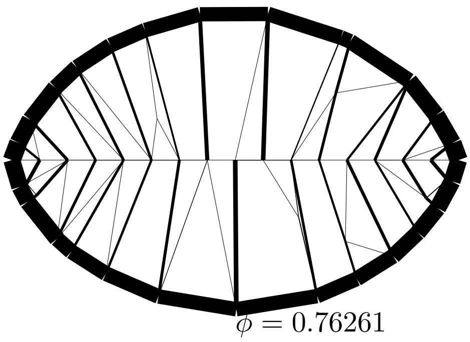
- There are some real bridges whose layouts resemble the AH ones, at least in the use of oblique hangers (e.g. the Apollo bridge in Bratislava, see Gabler 2006—it was the only European project among the five finalists for the 2006 Outstanding Civil Engineering Achievement Award of the American Society of Civil Engineers). In fact we was tempted to title this paper as "Two new solutions..." but actually the idea behind these layouts is not new: practical designers concerned with material economy re-discover it from time to time. The named "network arch" layout has a close relationship with the AH layout too (Tveit 2007). Furthermore, the oblique hanger motif arises in realistic optimization examples with code specifications (e.g. Hasançebi 2007, figure 4). The important point is that this motif agrees with a classical rule of thumb: "lead the load toward the supports as straight as possible".
- The results of a very parsimonious SA search confirm the AH solutions, see figures 6(a) and 6(c). It is to be noted that the SA algorithm can be trapped in search regions with Michell-like nets around the supports being its results sometimes better —half plane— sometimes worse —full plane— than our analytical results: this fact suggest that the way towards optimum region is very arduous, especially in the full plane case.
- Up to now several orthogonal nets were tested but all of them was worse than AH,mp solution. Furthermore, these nets were used as initial solutions for a geometric SA search (with fixed connectivity) with no better results.
- The best results using Linear Programming (e.g. Vanderbeit 2001:267–269) among those obtained by Hernando (2011) do not improve the AH,mp solution. As it can be viewed in figure 6(d), these solutions resemble the Hemp solution but with oblique hangers as in the AH solutions.



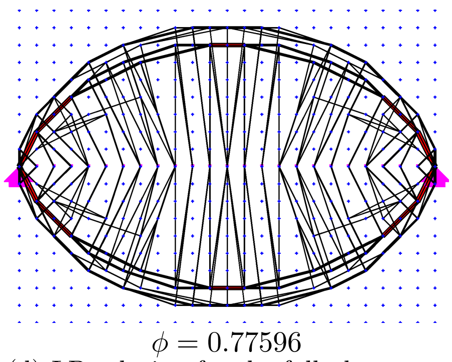
(a) SA solution for the half-plane case



(b) SA solution for the Hemp problem



(c) SA solution for the full-plane case



(d) LP solution for the full-plane case

Figure 6. Approximations to the benchmark solutions with SA and LP algorithms

For the sake of completeness, we made a SA search for the original problem stated by Hemp, see SA,hemp entry in table 1 and the figure 6(b). As the SA confirms the Hemp solution (without improvement), we can say that the latter is a quasi-optimal solution in some sense nearly as the AH solutions are.

It can be noted, as well, the high sensitivity of the layout to the reactions costs: if we make a comparison between the cost for the Hemp problem and layout, —Hemp entry in table 1— (0.788), with that of the same layout applied to the half plane and Maxwell version of the problem —Hemp,mh entry ibidem— (1.158), that is no better than the classical parabolic arch with vertical hangers and horizontal tie —T_1 entry on table 3— (1.155), we can see their importance, that firmly supports our restriction to well defined problems and strictly comparable designs, as the solutions to Maxwell problems are.

For the Maxwell versions of the bridge problem analysed, the situation is analogous to that of the two load case (see Vázquez & Cervera 2011): for the full plane case the best solution known does not fulfil the Michell theorem, while for the half plane case the best SA solutions strongly suggest that a Michell layout could exist.

After all two main group of questions arise:

(1) Is the Michell theorem a sufficient condition only? Are there Maxwell problems for which the displacement field required by the theorem does not exist? Can those problems be characterised in any way?

While Hemp (1958) remarked the importance of searching a necessary condition because the Michell theorem was only a sufficient one (the same opinion than that of Prager 1965, p. 326), Chan (1975, p. 313) said that the Michell conditions "are now known to be also necessary", a point of view adopted by several authors since then.

A more precise assertion by Rozvany (1984, p. 170) stated that "since the Prager-Shield condition gives a strain requirement (usually inequality) also for vanishing members, its fulfilment for the entire structural universe [or ground structure] constitutes a necessary and sufficient condition for convex specific cost functions. The same problem is usually *non-convex* if it is expressed in terms of the unknown geometrical parameters."

The major problem with the revised literature claiming that the Michell criteria is a necessary one is about the "strain requirement" and "its fulfilment" mentioned by Rozvany, see above. For example Prager (1973, 1974) simply supposes that the conjectured, optimal solution furnishes him with a displacement *field* over the full domain of a fairly large *ground structure* (*basic truss*) and on this basis then he gets a necessary condition that this *field* must fulfil. The conjecture about the *field* lies on collapse mechanisms or displacement patterns (and so on) of the conjectured solution. But actually, what does any solution give to us? Obtained or conjectured solutions can define only a displacement *set* and the corresponding compatible strain *set* for the subdomain where the internal forces, volume or material of the solution lies (Vázquez & Cervera 2011), being the rest of the considered domain with undetermined displacements and strains, that is to say, generally solutions give *no field* that can be applied to all the domain, including any *ground structure* members that the solution does not use. Prager (1974), for example, does not show how to compute the collapse mechanism of a solution extended to the *ground structure*. And it is a *field of this kind* that is required by the Michell theorem.

We are advised that the displacement *set* given by any solution could be extended to the full domain of the ground structure, but there are many possibilities (in fact infinity in a general case). Hence, the designer is at the same position that when he or she confronts the Michell theorem: a complete displacement *set* is to be found (in Prager words: "suitable chosen displacements"), and if a selected *set* does not fulfil the Prager criterion, it can exist other that does although unknown. So the Prager criterion performs as the Michell theorem: if you found the *set* and this fulfils the criterion, you are lucky: your solution is optimal. But if it is not the case, you know nothing until you found a better solution. And of course you must restart to work with the new solution. (We assert this independently of the technical discussion about the *meaning* of the necessary and sufficient character of the Farkas' theorem used by Prager in his proofs.) For example, Prager & Rozvany (1977) analysed with care the displacement *field* to prove the existence of the optimal solution and to show its geometry variation as the geometry of loads does, but unfortunately only with very simple problems and they did not attain to realise the difficult in a general case.

We consider that the matter is a very important one (Vázquez & Cervera 2011). And as we do not know any sound proof about the necessary existence of a displacement *field* for every Maxwell problem that fulfils the Michell theorem, the question about if the latter is a necessary one remains open in our view.

The consideration of the continuum, the existence of strain discontinuities, and so on, simply add new problems to the list that any proof about a necessary condition has to do with.

(2) Can the solutions presented here be improved taking some regions from the Hemp solutions but considering oblique hangers? Would those improved solutions fulfil

the Michell theorem for this Maxwell problem? Does a generalised formulation of the Michell theorem exist that allows to form non-orthogonal nets with the contour of the framework and other special lines (as the load one)?

Our next steps will be to look for answers to these questions. The research will include to search quasi-optimal solutions for other instances of the $S = 2$ bridge problem—trying to check if the distinction between the half plane case (with a possible Michell solution) and the full plane one (without it) is a kind of invariant—, and to formulate the problem with a variational approach in such a detailed form that we will able to characterise precisely the region of the function space where the quasi-optimal solutions lie, looking for any reasons that prove if they are actually absolute optima.

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