

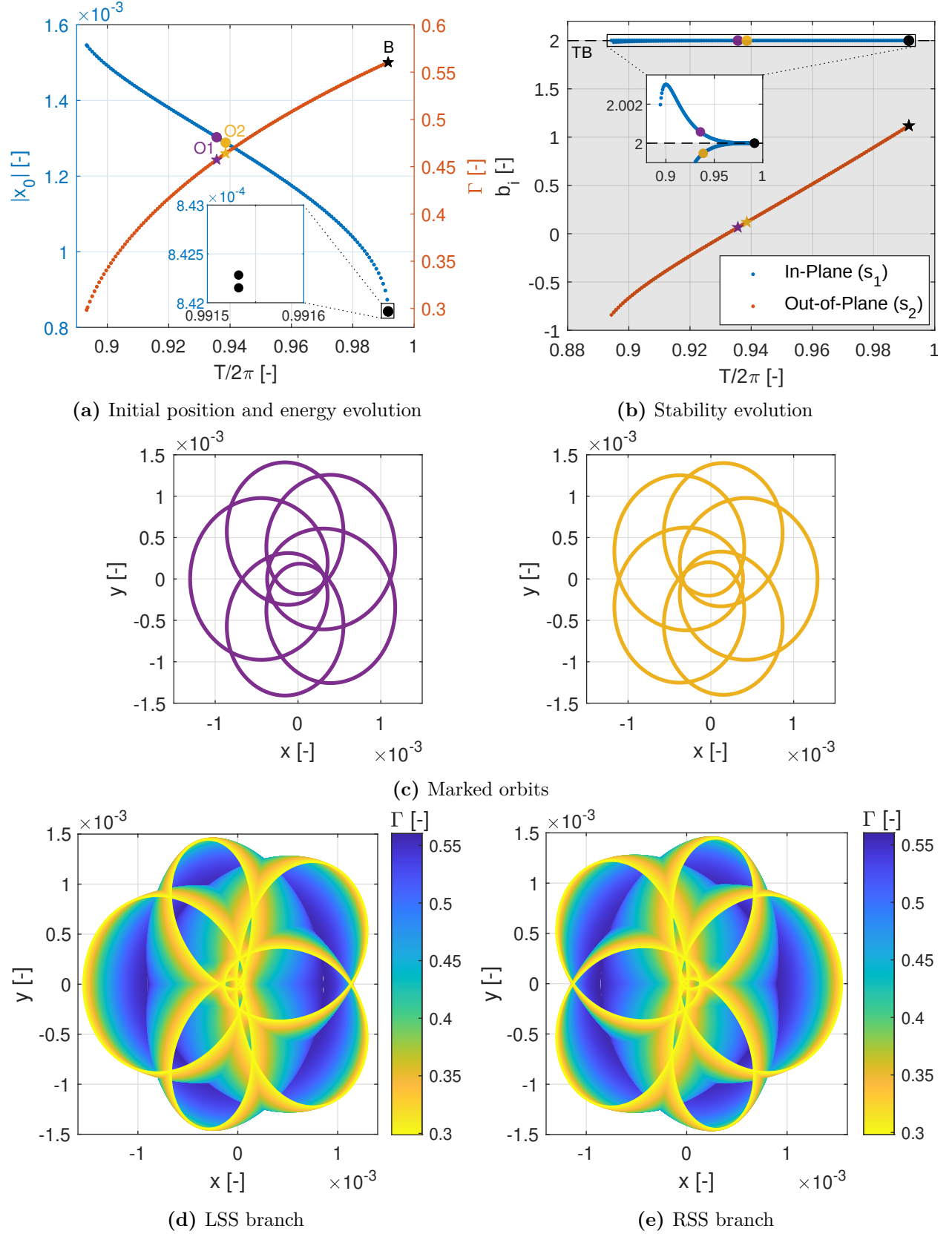
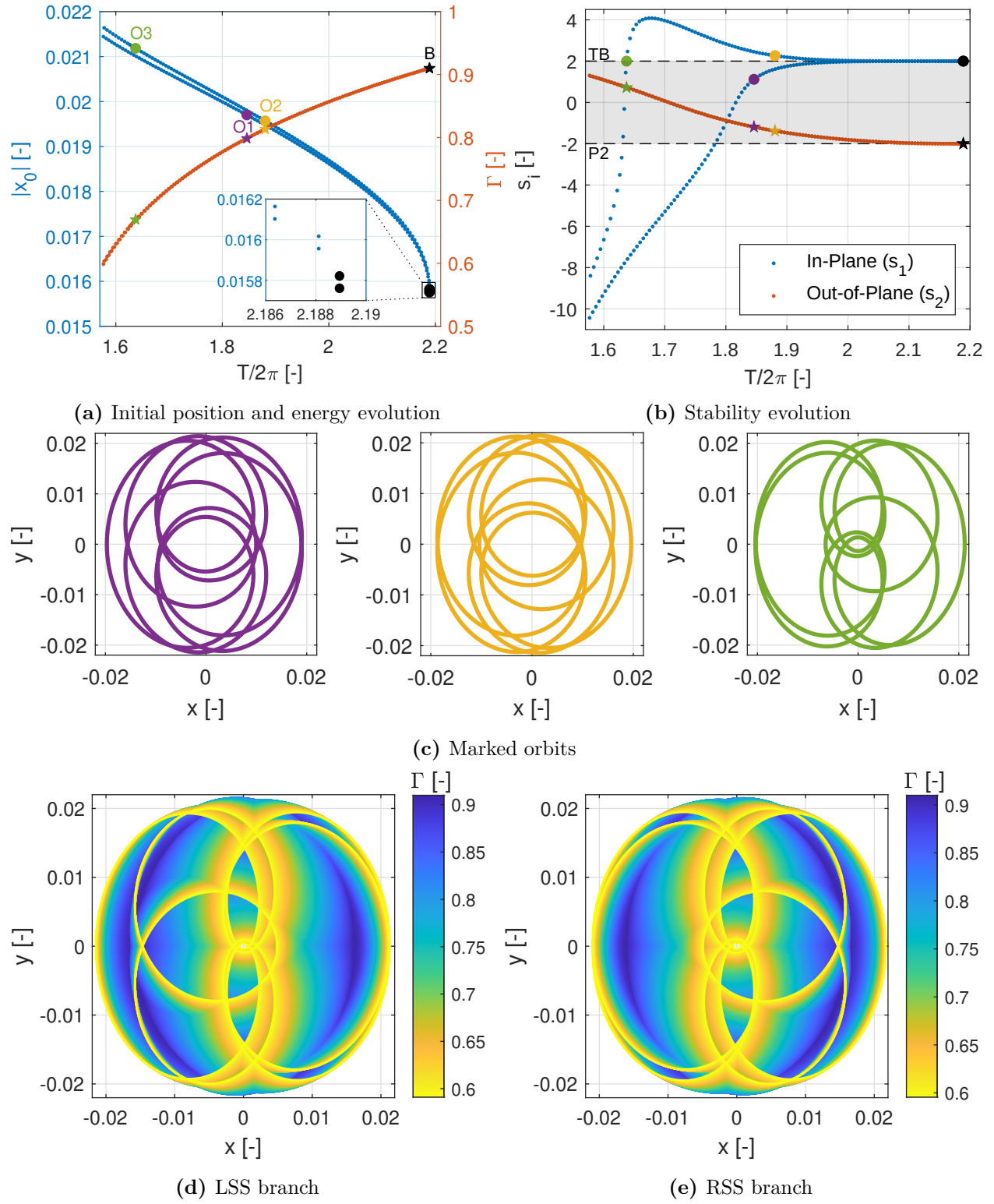


Figure 6.17: Low-energy P6QSO₂ family in the Mars–Phobos CR3BP

without a zoom. The second difference involves the in-plane stability index, which is, actually, the characteristic of interest in this analysis. First of all, for the Mars–Phobos family, the in-plane stability index remains all the time close to the value '2', in a way that without a zoom no divergence can be observed, contrary to what happened in the Jupiter–Europa case. However, the zoomed area shows a very interesting behavior, in which it can be seen how the LSS branch, which is unstable, has a maximum after which it starts approaching the 'TB' line. This critical point was missing in the Jupiter–Europa case, and it is needed if a tangent bifurcation is to appear. Nevertheless, the continuation of the family fails, due to the proximity of the second primary, before the mentioned tangent bifurcation can take place. This behavior is a sign that for a smaller μ the asymmetric family may exist. A more in depth analysis of the influence of the μ on the existence of the asymmetric families and the location on the bifurcating point will be done in the next chapter.

The final case is the low-energy case of multiplicity 7 and order 2. The symmetric family of this case is the P7QSO₄ family, illustrated in Figure 6.12. This family, again, possesses a tangent bifurcation as it could be expected, making the case of the P6QSO₂ an exception. Before talking about the tangent bifurcation, let analyze the characteristics of the family. As every other single-symmetric family, it has two branches, one with LSS orbits (Figure 6.13c) and the other with RSS QSOs (Figure 6.13d). Like it happened with the other family coming from the multiplicity 7 second-order bifurcation, the two perpendicular crossings are placed on different sides of the orbit, which is not the common behavior for this type of families. Again, the exterior perpendicular crossing is used for the initial condition in Figure 6.12a. Continuing with this figure, in the comparison between the two branches, the one with the biggest initial distance to the second primary is the RSS branch, exemplified by the yellow-colored 'O2' orbit. That branch is also the one that has the highest energy, even though the gap, as with the rest of these families, is very small. Now, Figure 6.18b is analyzed with special focus on the tangent bifurcation. As in the P4QSO₄ family, from the beginning of the family at 'B', one branch is unstable for the in-plane motion and the other is stable. In this case, the stable branch is the LSS and the unstable is the RSS. For this reason, this time the tangent bifurcation is placed on the RSS branch, when the branch becomes stable, at the 'O2' orbit. The rest of the stability behavior is the expected one, with both branches ending unstable with a negative value of the stability index. However, there are still some aspects that are worth mentioning. First, the gap on the in-plane stability index of both branches is the highest seen until now. In addition, even though in the P4QSO₂ family the tangent bifurcation already took place very far from the beginning of the family, in this family this is even more prominent, as the 'O2' orbit is placed almost at the end of the RSS branch. This will make the asymmetric family even more short than the P4QSO₄ family, as it will be seen next.

The final family of this chapter is the P7QSO₆, that is the one derived from the P7QSO₄ by a subcritical pitchfork bifurcation and that is shown in Figure 6.19. As anticipated, this family is really short. This is because of where the beginning of the family takes place. At the beginning of the family, at the 'B2' orbit (that is the same as the 'O2' orbit from Figure 6.12), the orbit is already close to the second primary. As the family evolves, this distance becomes even shorter until the continuation fails close to a collision orbit. Like it happened with the P4QSO₄ family, the shape of the orbits of this family evolves from the RSS swing 'B2' QSO to

**Figure 6.18:** Low-energy P7QSO₄ family in the Jupiter-Europa CR3BP

a final orbit where this swing motion is orientated more vertically, but where the influence of the RSS initial orbit is still noticeable. For this reason, the differences between Figures 5.26c and 5.26d, with the two branches of the H3BP P7QSO₄, and Figures 6.19c and 6.19d, with the family in the Jupiter–Europa CR3BP, are very clear, even beyond the fact of the different extension of the two families. The rest of the characteristics of the family can be observed in Figures 6.19a and 6.19b. Comparing the family to the symmetric P7QSO₄, (a), both the energy and the $|\rho|_{max}$ are almost superimposed (in the case of the $|\rho|_{max}$ the asymmetric family is only superimposed to the RSS branch of the symmetric family). Like it happened in the $m = 4$ low-energy case, the asymmetric family has slightly higher energy values and smaller $|\rho|_{max}$ values than the symmetric family or RSS branch, respectively. As for the linear stability of the family, (b), the evolution is similar to that of the H3BP counterpart, with the evident differences due to the different extension of the family, and the expected trend for a low-energy asymmetric family.

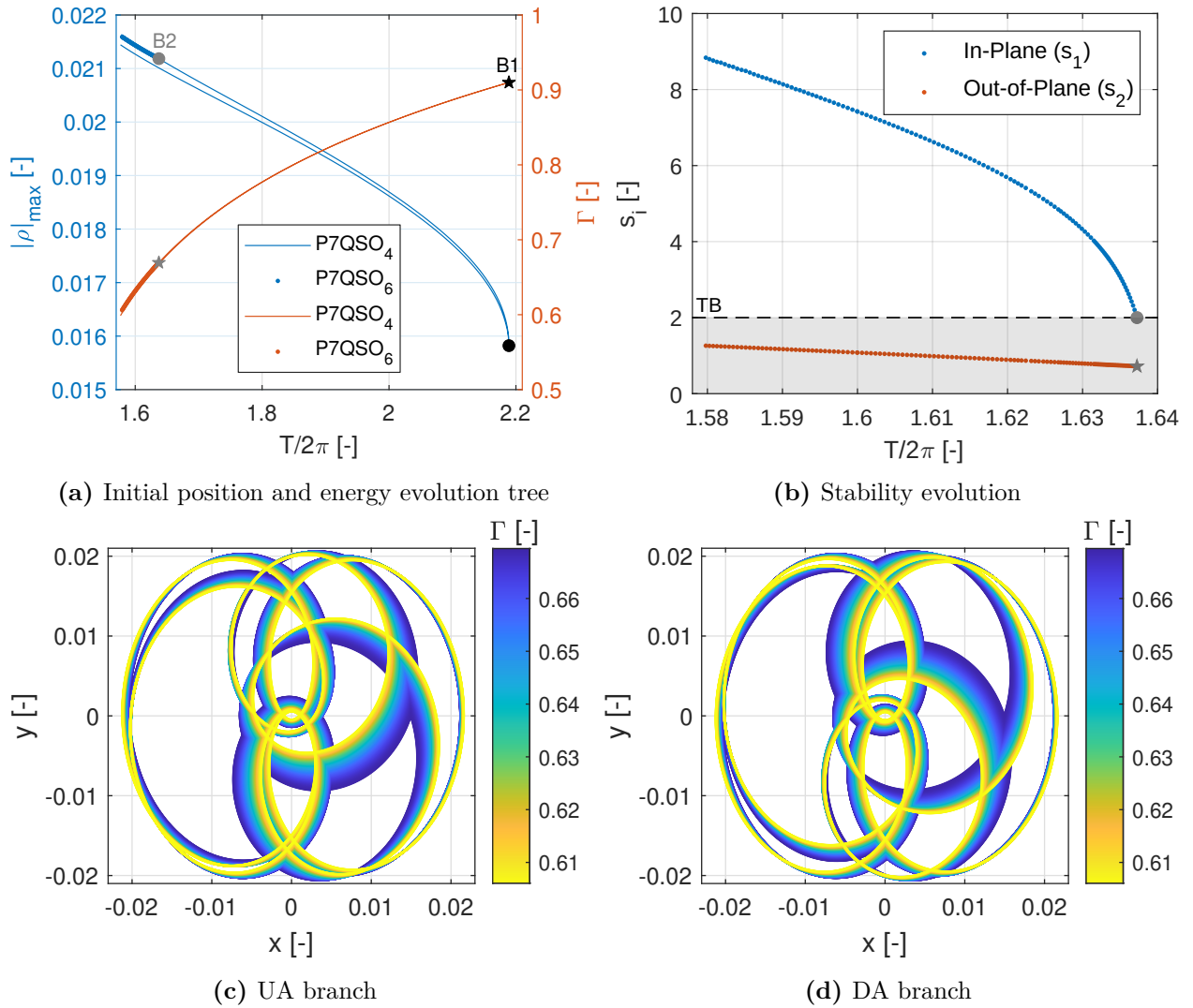


Figure 6.19: Low-energy asymmetric P7QSO₆ family in the Jupiter–Europa CR3BP

Chapter 7

Planar Resonant Quasi-Satellite Orbits

When the circular problem of the restricted three-body problem is considered, periodic orbits form families of orbits, so different orbits can be obtained starting from one single solution. However, in reality, the systems studied in this work move around elliptic orbits. So, in order to increase the level of fidelity, the ER3BP is used. Once the elliptic aspect of the primary bodies is introduced, only resonant orbits are periodic. This is the reason why this chapter is dedicated entirely to this type of orbits, for both steady QSOs and swing QSOs.

7.1 Resonant Steady QSOs

In the first place, resonant steady QSOs are analyzed in this section. These orbits belong to the steady QSO family in the H3BP and the CR3BP, and are isolated for the ER3BP, as is characteristic for this model. For obtaining the resonant orbits in the H3BP and the CR3BP, one just needs to compute the steady QSO family, as explained in the previous chapter, and select among them the orbit with the desired resonance. In order to obtain the exact resonant orbit, a natural parameter continuation is used with the period as the stepping parameter. This continuation is done starting from the closest orbit in period already closed. Nevertheless, for the ER3BP this procedure cannot be followed. Hence, a resonant orbit in the ER3BP is obtained by continuing the CR3BP orbit in eccentricity with a fixed period. As the steady QSOs are symmetric with respect to the x-axis for both models, this continuation is done using the perpendicular crossing algorithm. Another thing that has to be taken into account is the fact that, as the ER3BP is a non-autonomous system, the moment in which the initial conditions take place affects the orbit. In this sense and due to the mentioned symmetry with respect to the x-axis, the initial conditions have to be placed on either the periapsis or the apoapsis to obtain a periodic orbit. Depending on which of these moments is selected as initial condition, a different set of resonant orbits is obtained. In this work, the periapsis ($\nu = 0$) is selected. Finally, not only the moment selected as the initial time affects the final QSO obtained in the ER3BP, but also where the initial condition is placed within the orbit changes it. In this sense, among the two orthogonal crossings, the one placed between the two primaries, i.e. the one placed on the negative x-axis, is used.

Before obtaining and analyzing the resonant steady QSOs, the nomenclature used to

identify each resonant orbit is first explained below. Furthermore, since the set of resonant QSOs based on the literal definition is infinite, the next section also specifies the resonance ratios considered in this work.

7.1.1 Proposed Nomenclature and Practical Constraints

As for the nomenclature is concerned, a resonant steady QSO will be identified here with the resonant pair

$$N_{21} : N_{32}, \quad (7.1)$$

where N_{32} is the integer number of revolutions of m_3 around m_2 with respect to the LVLH frame ($\mathcal{L}$) and N_{21} is the integer number of revolutions of m_2 around m_1 with respect to an inertial frame. Note that the nomenclature in Eq. (7.1) must not be confused with the one employed to identify resonance conditions leading to period-multiplying bifurcations in the CR3BP (see, for instance [30]), where N_{21} does not play any role. In addition, resonant QSOs must not be confused with the families of resonant orbits in the CR3BP that come from resonances in the two-body problem.

In the CR3BP, a $N_{21} : N_{32}$ resonant *steady* QSO closes itself after m_3 has completed only one revolution around m_2 , irrespective of the angular location of m_2 around m_1 . In that time-span m_2 has completed N_{21}/N_{32} revolutions around m_1 . Accordingly, the dimensionless period and orbital frequency¹ of a resonant steady QSO with respect to the LVLH frame within the CR3BP are, respectively,

$$T_{steady}^C = 2\pi \frac{N_{21}}{N_{32}} \quad n_{steady}^C = \frac{N_{32}}{N_{21}}. \quad (7.2)$$

In the ER3BP, and unlike the CR3BP, an $N_{21} : N_{32}$ resonant QSO cannot close itself before m_2 has completed N_{21} revolutions around m_1 , and the dimensionless period of a resonant steady QSO within the ER3BP is given by

$$T_{steady}^E = 2\pi N_{21} = N_{32} T_{steady}^C.$$

The fact that this period is N_{32} times higher than the one of the CR3BP makes the convergence towards the elliptic problem the biggest computational challenge, and in some cases, it can lead to orbits with such long periods that they become impractical for real mission applications.

As for the results presented in this work, the maximum number of revolutions of m_3 around m_2 and the ratio N_{21}/N_{32} have been limited to

$$(N_{32})_{max} = 10 \quad \left(\frac{N_{21}}{N_{32}} \right)_{min} = \frac{3}{10}.$$

Since the period of a steady QSO is upper-bounded by approximately the 1:1 resonance, these assumptions lead to 22 resonant steady QSOs (see E). Extending the search beyond the first constraint would limit the applicability of the periodic orbits due to the excessive number of revolutions. As for the second limit, reducing the N_{21}/N_{32} ratio even further would yield periodic orbits far from the 1:1 resonance and falling in the category of perturbed 2-body orbits rather than QSOs.

¹The unit value corresponds to a 1:1 resonance

7.1.2 Results

The computation method described is applied to close resonant steady QSOs in the Mars–Phobos, Sun–Mars, Jupiter–Europa and Saturn–Titan systems, whose orbital parameters appear in Table 6.1 of the previous chapter. For conciseness, only the first nine resonance cases in the $N:N+1$ format are displayed in Figure 7.1, while the initial conditions for the remaining 13 resonance orbits are all included in E. From the orbits in the CR3BP and the ER3BP, it can be noticed how the periodic orbits involve more revolutions in the ER3BP. This is especially noticeable for the Sun–Mars system, that is the one with the higher eccentricity. As a consequence of this, each revolution of the orbit differs more from the others, as it can be seen in the figure.

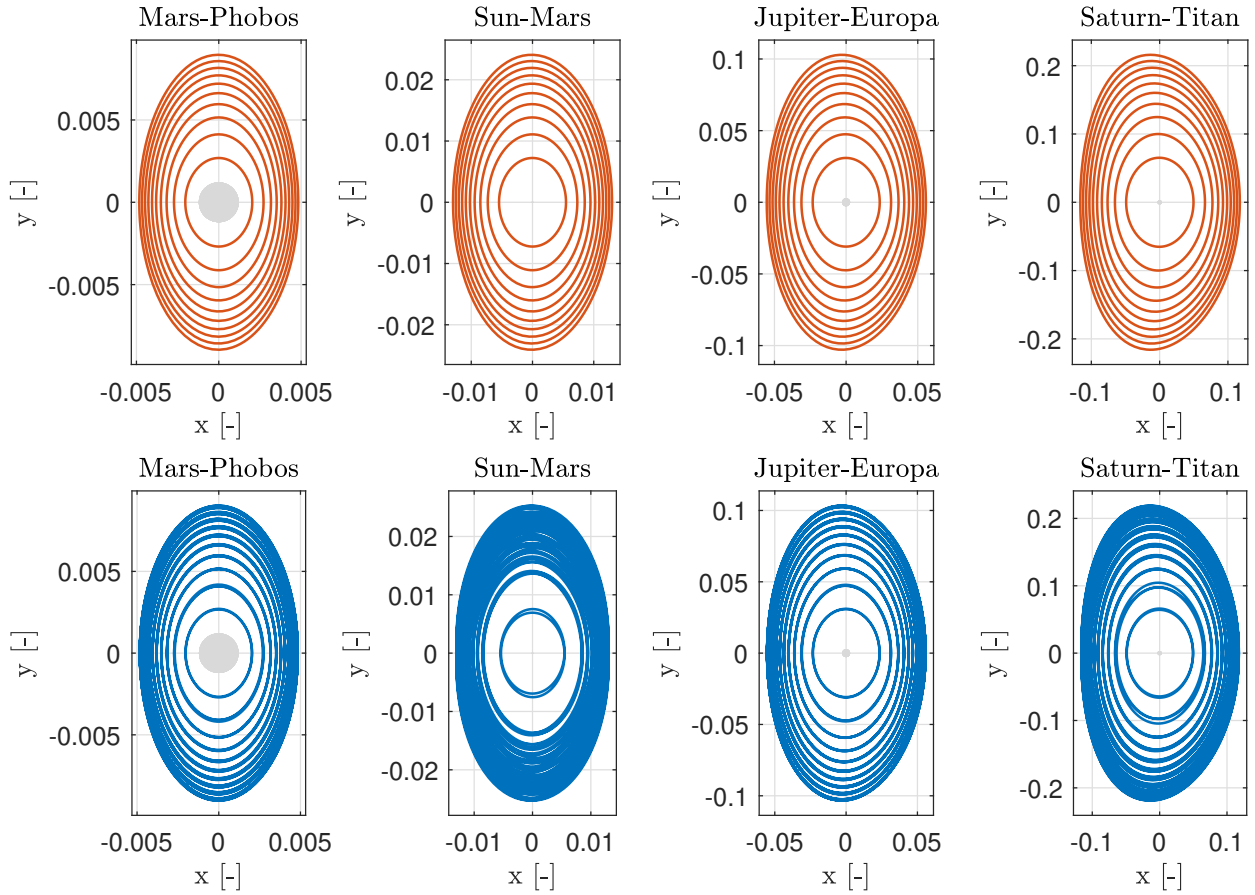


Figure 7.1: Resonant steady QSOs in different systems using the CR3BP (red) and ER3BP (blue) models, plotted in the $\mathcal{L}$ frame.

It is worth mentioning that, in the case of the Sun–Mars system, the 2:3 resonant steady QSO could not be closed for the ER3BP model and is therefore missing from Figure 7.1. When continuing this orbit in e for this system, a fold bifurcation was identified at $e = 0.0324$, marking a maximum in eccentricity for the branch (see [80] for more information about fold bifurcations in the ER3BP). This explains why the orbit could not be continued up to Sun–Mars eccentricity ($e = 0.0934$). Notably, if this branch is further continued past the fold bifurcation back to the CR3BP ($e = 0$), it connects to a swing QSO discussed in the next

subsection. This fold bifurcation will be studied with more detail later on.

7.2 Resonant Swing QSOs

In this section, the resonant swing QSOs are studied for the three models included in this work: the H3BP, the CR3BP and the ER3BP. When the guiding center motion of a QSO is not negligible with respect to the orbit size, the corresponding LVLH trajectory is a self-intersecting curve corresponding to what is here referred to as a *swing* QSO². In the H3BP and the CR3BP these orbits belong to different families, as observed in the previous chapters, while for the ER3BP they exist isolated for a fixed eccentricity. The same geometric classification made for the QSOs is used for the resonant ones with one extra subdivision, for the single-symmetric and asymmetric orbits. Depending on the ratio χ between the maximum and minimum distance to the second primary, they are referred to as ‘wide-’, when $\chi > 3.7$ and ‘narrow-’ when $\chi < 3.7$. In this way, left-wide-single-symmetric (LWSS), right-wide-single-symmetric (RWSS), left-narrow-single-symmetric (LNSS) and right-narrow-single-symmetric (RNSS) are distinguished among the single-symmetric QSOs; and up-wide-asymmetric (UWA), down-wide-asymmetric (DWA), up-narrow-asymmetric (UNA) and down-narrow-asymmetric (DNA) are differentiated for the asymmetric orbits. Note that down- and up- asymmetric orbits mirror each other not just in Hill’s approximation but also in the general CR3BP and ER3BP case. Therefore, from now on, this prefix will be omitted and these orbits will be referred to as WA and NA, respectively.

For obtaining the resonant periodic swing QSOs in the ER3BP, they can be first obtained for the planar CR3BP and continued into the ER3BP with the period fixed. For the CR3BP, two different approaches have been used to get the resonant swing orbits for each system. The first method is by computing the family to which they belong for the desired system and then selecting the correspondent resonance in the same way as for the steady QSOs. The second approach is by transitioning the resonant swing QSO from the H3BP to the CR3BP and subsequent continuation in μ with a period fixed to the desired system. For this continuation, for the symmetric orbits, among the two perpendicular crossings, the one on the negative axis is used preferentially when both are on opposite sides of the orbit. When both orthogonal crossings are placed on the same side of the x-axis, the most exterior one is selected preferentially. Regarding the asymmetric QSOs, in the H3BP, the preferential perpendicular crossing is the one placed on the positive side of the y-axis and the interior one if both are on the same side. When transitioning to the CR3BP and the ER3BP, this point is used as a starting point with the condition $\theta' = 0$. In the ER3BP, the initial point is still located on the periapsis like with the resonant steady QSOs.

7.2.1 Proposed Nomenclature

When studying the resonant swing QSOs, the added difficulty, compared to the case of steady QSOs, lies in the need to have the guiding center moving in resonance with the first primary

²Note that steady QSOs are characterized by a negligible but not zero guiding center motion as shown in [36] and [37] (see Eq. 27 in the latter reference).

orbital motion by performing an integer number of *swing cycles* by the time m_2 has completed its N_{21} revolutions around m_1 . A convenient way to characterize resonant swing QSOs is to specify the number of swing cycles in addition to the number of revolutions in LVLH axis. To this end, each resonant swing QSO will be identified with the *resonance triplet*

$$N_{21} : N_{32} : N_S,$$

where N_S is the integer number of swing cycles. In one swing cycle, the guiding center negative motion has only one crossing of the x-axis, except for the orbits that belong to a swing family derived from a period-septupling order 2 bifurcation. In this special case, the guiding center performs two revolutions in one swing cycle. Therefore, N_S also represents the number of revolutions of the guiding center, except for the orbits belonging to the families derived from a second-order bifurcation, that it is half of these revolutions. Note that, as in the case of steady QSOs, orbit closure occurs only for $N_{21} < N_{32}$.

As seen before, a swing QSO belongs to a family of a certain multiplicity, defined by the type of bifurcation from which they derive from the steady QSO family. This multiplicity is implicit on the resonant triplet as it is

$$m = \frac{N_{32}}{N_S}. \quad (7.3)$$

Note that the limit case $N_{32} = N_S$ would yield a steady QSO ($m = 1$) as a particular case. For later use, the dimensionless swing period is defined as

$$T_S = 2\pi \frac{N_{21}}{N_S}.$$

Following reasoning similar to the previous section, the period of a swing QSO in the CR3BP and in the ER3BP yields

$$T_{swing}^C = T_S \quad T_{swing}^E = 2\pi N_{21} = N_S T_S.$$

Similar to the steady QSOs case, this nomenclature must not be confused with the one associated with the search of period-multiplying bifurcations of orbit families.

7.2.2 Feasible Resonant Triplets

A fundamental question addressed by this work is what specific resonance triplets correspond to one or more swing periodic orbit type. The question is far from trivial, and depends on the complex evolution of the orbit period of each swing family (in the CR3BP or Hill problem) from its corresponding steady QSO bifurcation point as its energy is varied. For applicability and robustness purposes of the computed orbits, the results presented in this work are confined to

$$\left(\frac{N_{32}}{N_S}\right)_{max} = 7 \quad (N_{21})_{max} = 11 \quad (N_S)_{max} = 6. \quad (7.4)$$

The first restriction limits the period-multiplying bifurcations up to a multiplicity of $m = 7$ (Eq. (7.3)), as in the rest of this work. Given the restrictions, each multiplicity can lead to a maximum number of resonant triplets (32 for $m = 3$, 34 for $m = 4$, 36 for $m = 5$, 38 for $m = 6$ and 39 for $m = 7$). However, not all triplets can lead to an existing periodic orbit, as it depends on the characteristics of each specific bifurcating family. All possible resonant triplets are listed in Tables 7.1-7.5 with the ones leading to at least one existing periodic orbit in Hill's approximation marked in bold. A number of 22 feasible triplets is obtained for $m = 3$, four for the case of $m = 4$, only one for $m = 5$ and seven for $m = 7$. No feasible triplets exist for the $m = 6$ case within the restrictions specified in Eq. (7.4). Note that, in general, a feasible resonance triplet can be met by more than one geometric type of periodic orbit as it will be shown.

Table 7.1: Different possibilities of resonant swing QSOs with $m = 3$.

$N_S = 1$	$N_S = 2$	$N_S = 3$	$N_S = 4$	$N_S = 5$	$N_S = 6$
1:3:1	1:6:2	1:9:3	1:12:4	1:15:5	1:18:6
2:3:1	3:6:2	2:9:3	3:12:4	2:15:5	5:18:6
-	5:6:2	4:9:3	5:12:4	3:15:5	7:18:6
-	-	5:9:3	7:12:4	4:15:5	11:18:6
-	-	7:9:3	9:12:4	6:15:5	-
-	-	8:9:3	11:12:4	7:15:5	-
-	-	-	-	8:15:5	-
-	-	-	-	9:15:5	-
-	-	-	-	11:15:5	-

Table 7.2: Different possibilities of resonant swing QSOs with $m = 4$.

$N_S = 1$	$N_S = 2$	$N_S = 3$	$N_S = 4$	$N_S = 5$	$N_S = 6$
1:4:1	1:8:2	1:12:3	1:16:4	1:20:5	1:24:6
2:4:1	3:8:2	2:12:3	3:16:4	2:20:5	5:24:6
3:4:1	5:8:2	4:12:3	5:16:4	3:20:5	7:24:6
-	7:8:2	5:12:3	7:16:4	4:20:5	11:24:6
-	-	7:12:3	9:16:4	6:20:5	-
-	-	8:12:3	11:16:4	7:20:5	-
-	-	10:12:3	-	8:20:5	-
-	-	11:12:3	-	9:20:5	-
-	-	-	-	11:20:5	-

Table 7.3: Different possibilities of resonant swing QSOs with $m = 5$.

$N_S = 1$	$N_S = 2$	$N_S = 3$	$N_S = 4$	$N_S = 5$	$N_S = 6$
1:5:1	1:10:2	1:15:3	1:20:4	1:25:5	1:30:6
2:5:1	3:10:2	2:15:3	3:20:4	2:25:5	5:30:6
3:5:1	5:10:2	4:15:3	5:20:4	3:25:5	7:30:6
4:5:1	7:10:2	5:15:3	7:20:4	4:25:5	11:30:6
-	9:10:2	7:15:3	9:20:4	6:25:5	-
-	-	8:15:3	11:20:4	7:25:5	-
-	-	10:15:3	-	8:25:5	-
-	-	11:15:3	-	9:25:5	-
-	-	-	-	11:25:5	-

Table 7.4: Different possibilities of resonant swing QSOs with $m = 6$.

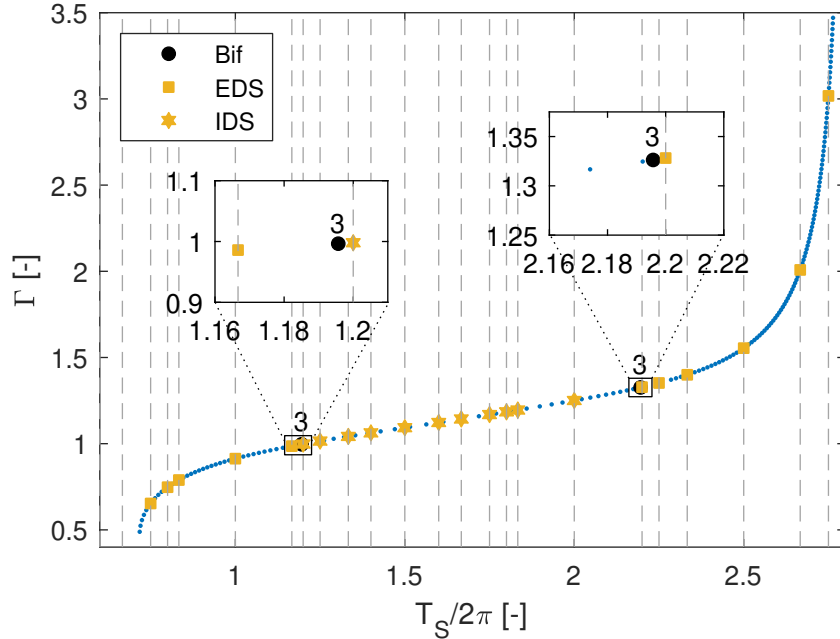
$N_S = 1$	$N_S = 2$	$N_S = 3$	$N_S = 4$	$N_S = 5$	$N_S = 6$
1:6:1	1:12:2	1:18:3	1:24:4	1:30:5	1:36:6
2:6:1	3:12:2	2:18:3	3:24:4	2:30:5	5:36:6
3:6:1	5:12:2	4:18:3	5:24:4	3:30:5	7:36:6
4:6:1	7:12:2	5:18:3	7:24:4	4:30:5	11:36:6
5:6:1	9:12:2	7:18:3	9:24:4	6:30:5	-
-	11:12:2	8:18:3	11:24:4	7:30:5	-
-	-	10:18:3	-	8:30:5	-
-	-	11:18:3	-	9:30:5	-
-	-	-	-	11:30:5	-

In order to confirm whether a resonant triplet exists or not, the evolution of each swing family that derives directly from the steady QSO family in the H3BP is used to determine all possible intersections with the relevant resonant periods. Working within Hill's approximation, to determine the existence of the resonant triplets is possible thanks to the similar evolution observed in the previous chapter between the H3BP and the desired systems in the CR3BP. In addition, when there were some differences between the H3BP and the CR3BP, none of them implied a larger extension on the CR3BP with respect to the period, but the opposite if anything. Therefore, all the possible resonant QSOs in the CR3BP should be included in the H3BP.

Table 7.5: Different possibilities of resonant swing QSOs with $m = 7$.

$N_S = 1$	$N_S = 2$	$N_S = 3$	$N_S = 4$	$N_S = 5$	$N_S = 6$
1:7:1	1:14:2	1:21:3	1:28:4	1:35:5	1:42:6
2:7:1	3:14:2	2:21:3	3:28:4	2:35:5	5:42:6
3:7:1	5:14:2	4:21:3	5:28:4	3:35:5	7:42:6
4:7:1	7:14:2	5:21:3	7:28:4	4:35:5	11:42:6
5:7:1	9:14:2	7:21:3	9:28:4	6:35:5	-
6:7:1	11:14:2	8:21:3	11:28:4	7:35:5	-
-	-	10:21:3	-	8:35:5	-
-	-	11:21:3	-	9:35:5	-
-	-	-	-	11:35:5	-

As seen before, there are several advantages of working with the H3BP when computing all the swing families rather than directly within the μ -dependent CR3BP. First, in Hill's approximation, not only do symmetric orbits exhibit a perpendicular crossing that can be exploited during the computation of the families, but asymmetric QSOs studied in this work also have a perpendicular crossing, albeit with respect to the y-axis. Secondly, all the swing families investigated in this work³ directly bifurcate from the family of steady QSOs in Hill's approximation through period-multiplying bifurcations. This scenario no longer holds true for asymmetric orbits in the general CR3BP, where these families bifurcate from the other swing QSOs families via tangent bifurcations.


Figure 7.2: Energy evolution with respect to the period for $m = 3$ symmetric family of swing QSOs (P3QSO) in the H3BP.

³The P5QSO_{1,T} in the H3BP or P5QSO_{1,S} in the CR3BP are not included in this work (same applies to the correspondent ones in the $m = 7$ case). The reason behind it is that they are secondary families in the H3BP and are not even connected to the swing families in the CR3BP. In addition, none of these families would produce any resonant QSO within the restrictions of this work due to their period range. This also happens with the P5QSO₁ and P7QSO₁, as seen next.

Figures 7.2, 7.3 and 7.4 summarize the evolution of the total energy vs. swing period of all symmetric families of swing orbits up to multiplicity $m = 7$, including the resonant orbits, whereas figure 7.5 represents the asymmetric ones. The resonant orbits, identified by the intersection with the vertical lines in the resonant periods, are marked according to their geometric type. Additionally, bifurcating orbits ('Bif'), in which swing families originate from the family of steady QSOs, are highlighted along with a number indicating its multiplicity and, if applicable, the order of the bifurcation in parentheses.

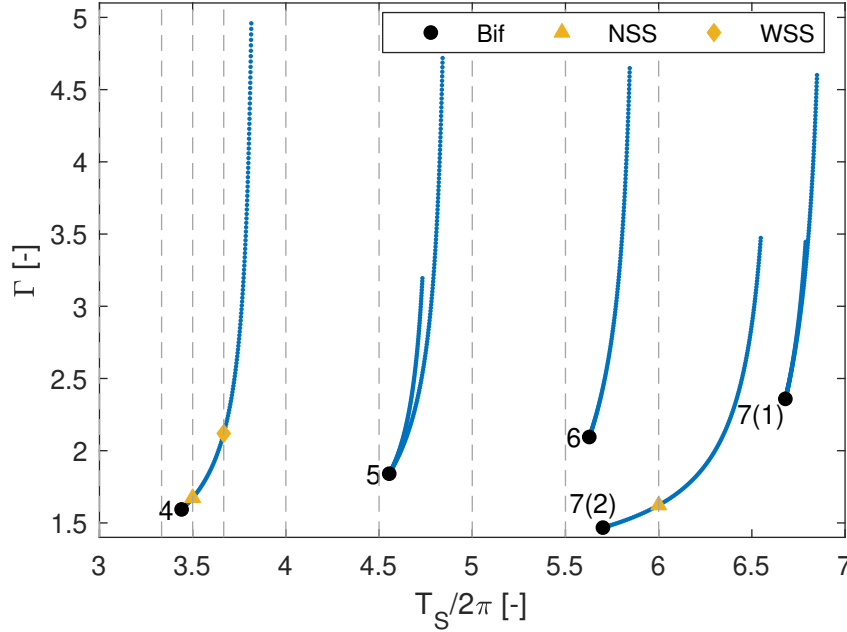


Figure 7.3: Energy evolution with respect to the period for high-energy symmetric families of swing QSOs in the H3BP.

The case of $m = 3$ (Figure 7.2) includes 22 resonant periodic orbits, distributed among EDS and IDS QSOs. In the case of $m = 5$, which also gave rise to doubly-symmetric, only two resonant periodic orbits are found: a $1 : 5 : 1$ EDS and a $1 : 5 : 1$ IDS along the low-energy $m = 5$ family (Figure 7.4). Regarding the case of $m = 7(1)$, which is the final one composed by EDS and IDS QSOs, it has no resonant orbit within the established restrictions. The cases of $m = 4, 6, 7(2)$ all give rise to both single-symmetric orbits and asymmetric orbits stemming from each of the bifurcation points. As for the symmetric branches, the $m = 4$ case leads to two high-energy (Figure 7.3) and two low-energy (Figure 7.4a) resonant orbits, while the $m = 7(2)$ case leads to 6 low-energy (Figure 7.4b) and only one high-energy (Figure 7.3) resonant orbit. No resonances are found for the $m = 6$ case. All these symmetric orbits belong to the NSS and WSS geometric classes. Regarding the asymmetric families, the same resonances as for their symmetric counterpart are found. This means a total of four resonant orbits found for the $m = 4$ case (two high-energy and two low-energy) and seven for the $m = 7(2)$ case (six low-energy and one high-energy case), as shown in Figure 7.5.

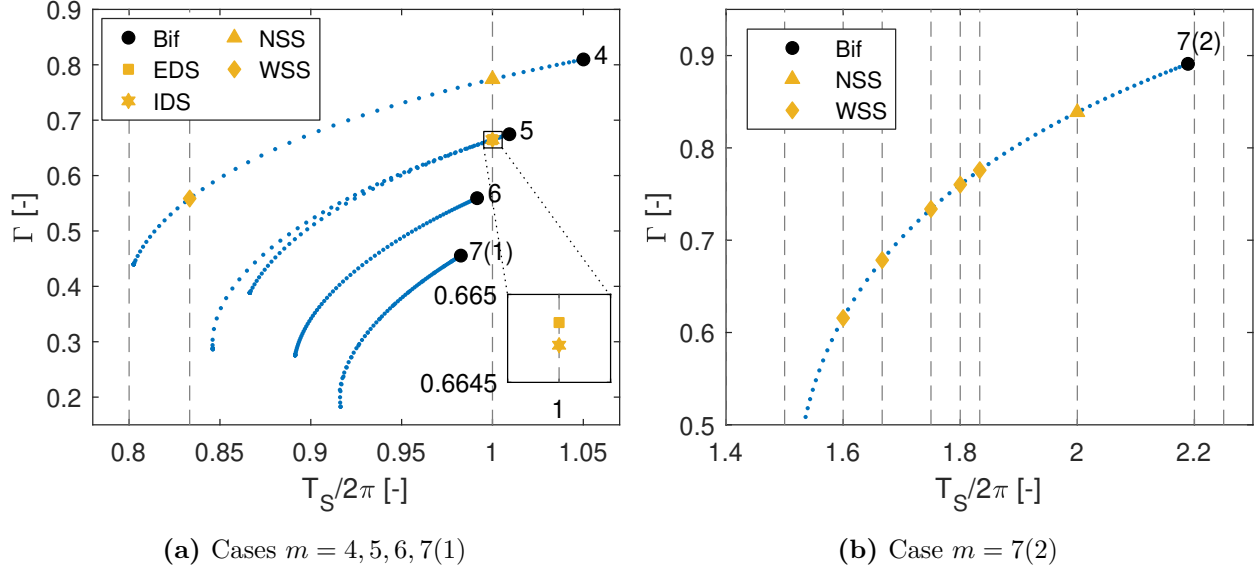


Figure 7.4: Energy evolution with respect to the period for low-energy symmetric families of swing QSOs in the H3BP.

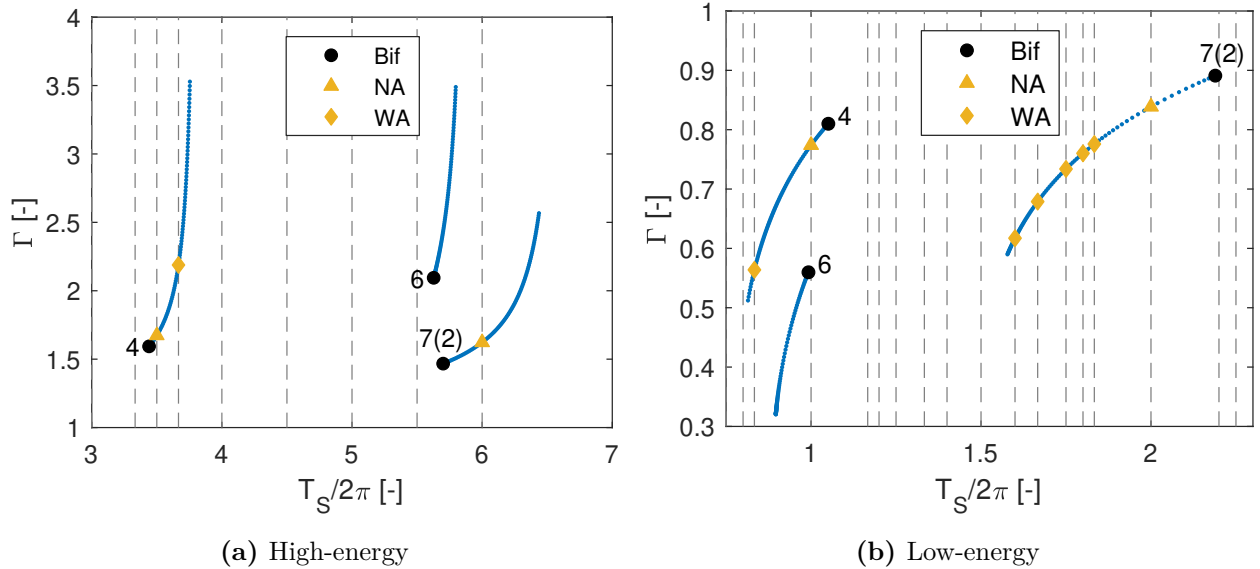


Figure 7.5: Energy evolution with respect to the period for asymmetric families of swing QSOs in the H3BP.

7.2.3 Mass Ratio Effect on the Existence of Resonant Asymmetric QSOs in the CR3BP

A final crucial question regarding the continuation of the orbits to the CR3BP and the ER3BP arises when closure on one of these models is not achieved. This can be due to numerical challenges (poor stability and large number of revolutions before closure) or the fact that a specific resonance ceases to exist when going from the H3BP to a μ -dependent model (see later on in more detail) or when going from the CR3BP to the ER3BP model. Although

validating the existence of these orbits within the ER3BP is, in general, beyond the scope of the present work, as it may involve phenomena such as fold bifurcations or broken bifurcations that can be difficult to identify, one specific case has been identified as a fold bifurcation, which will be further addressed later. Within the CR3BP, however, this validation can be easily accomplished by computing and examining the evolution of the families for each μ system when necessary.

As will be shown in the results section, all symmetric resonant QSOs found in the H3BP were successfully closed in all the studied models within the CR3BP. However, this is no longer the case for asymmetric orbits. As analyzed in the previous chapter, the origin of the asymmetric families in the H3BP and the CR3BP differs. As a consequence, the range of periods for which the asymmetric families exist in the CR3BP is more limited than in the H3BP model, so it is no surprise that some of the resonant asymmetric QSOs cease to exist for some CR3BP systems. Therefore, for these orbits, an analysis of their existence is conducted by computing the corresponding family of QSOs in the problematic systems. In fact, in all the cases where no orbit within a CR3BP system was obtained, its non-existence was confirmed through Figure 7.6, where the four families of asymmetric QSOs that possess resonant orbits are represented for each of the systems studied.

As previously mentioned, while asymmetric orbits bifurcate from the family of steady QSOs in the H3BP model, in the μ -dependent CR3BP model, they bifurcate from the corresponding family of symmetric QSOs via a tangent bifurcation. However, the position of the tangent bifurcation varies with the value of μ , as it can be observed in Figure 7.6, for all the four mentioned families. Each of the four plots that compose this figure shows the evolution of Γ with respect to the period for each of the studied families across the four systems considered: Mars–Phobos, Sun–Mars, Jupiter–Europa, and Saturn–Titan. In addition, the tangent bifurcation from which each family arises is indicated by the 'Bif' marker. Analyzing the position of the bifurcating orbit for each of the systems, it can be observed how for the two high-energy families, as the μ increases the bifurcating orbit takes place for a higher period, i.e. further in the symmetric family. Taking into account that the tangent bifurcation for these families established a minimum in period and that the maximum period achievable is more or less invariant, the consequence of this is that the period range reduces as μ increases. For the low-energy families, even the period behavior is reversed, the same situation is observed. In this case, a higher μ results in a smaller period for the bifurcating orbit. As this orbit corresponds to the maximum in period, the range of period is again reduced. In fact, for the two low-energy families studied in here, this effect is more pronounced than for the high-energy ones.

Moreover, the figure includes vertical lines indicating the possible resonant triplets, with the existent QSOs marked with a yellow symbol, following the same procedure seen before. This allows to observe which of the resonant orbits exists for each system. In the case of the P4QSO₃ family, there were two resonant QSOs in the H3BP (Figure 7.5a): 7:8:2 NA and 11:12:3 WA. As it can be observed in the figure, both of them exist for all of the 4 systems studied, even though for Saturn–Titan this takes place very near the bifurcating orbit. It is important to notice that as the lines for Mars–Phobos and Sun–Mars are very close in the plot (due to their close mass ratio), the markers for the resonant orbits of both systems are

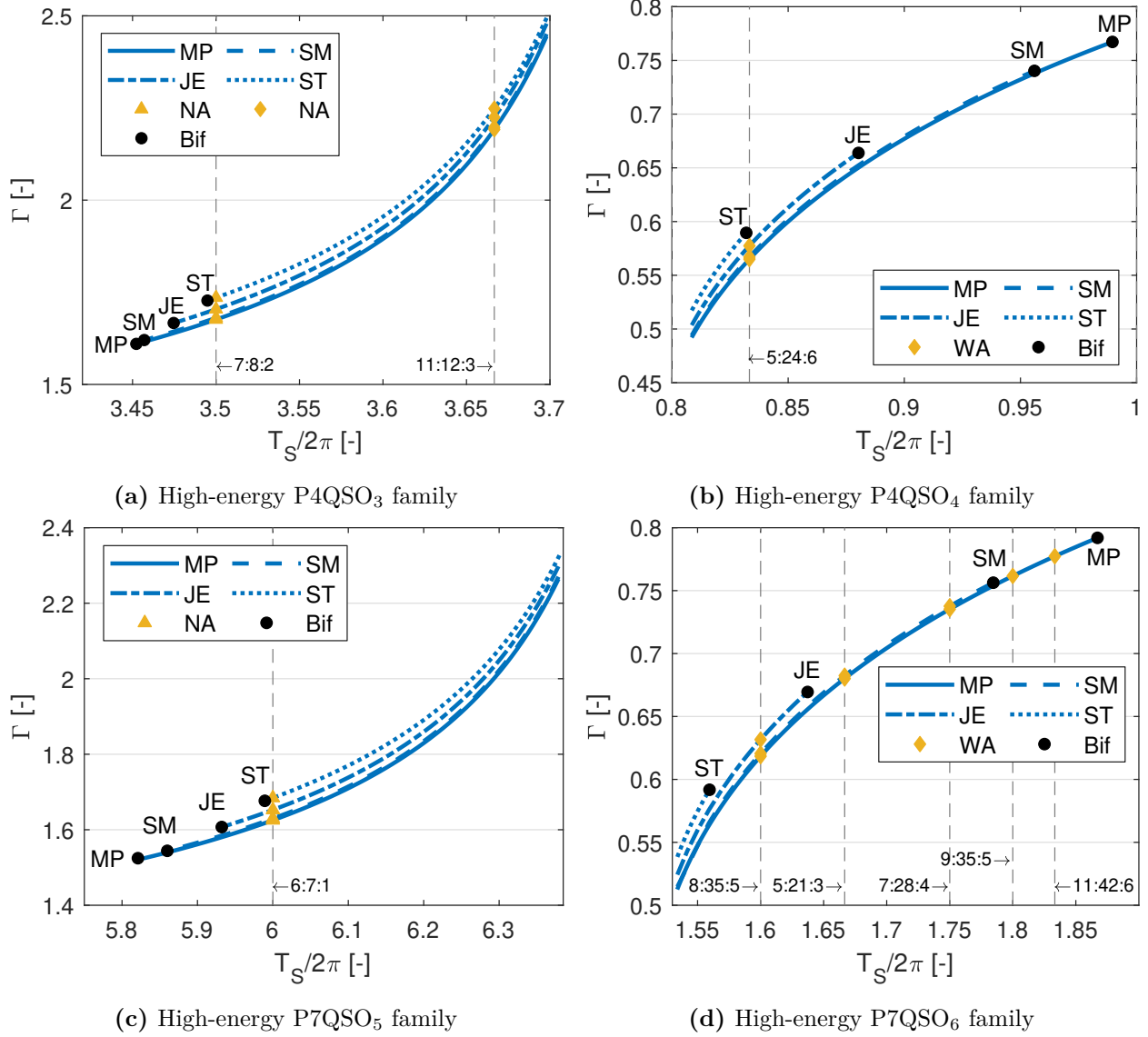


Figure 7.6: Energy evolution with respect to the period for asymmetric families of swing QSOs in the CR3BP for: Mars–Phobos, Sun–Mars, Jupiter–Europa and Saturn–Titan.

overlapped in the plot. The P4QSO₄ family possesses also two resonant QSOs in the H3BP (Figure 7.5b): 1:4:1 NA and 5:24:6 WA. However, this time the 1:4:1 NA QSO does not exist for any of the systems and the 5:24:6 WA exists for all the systems but Saturn–Titan. For the third case studied, the P7QSO₅ family, only the 6:7:1 NA existed in the H3BP (Figure 7.5a) and it maintains its existence for all the four models. Finally, the P7QSO₆ family is the most complex case, as in the H3BP it contains 6 different resonant QSOs (Figure 7.5b) that are, from higher period to lower: 2:7:1 NA, 11:42:6 WA, 9:35:5 WA, 7:28:4 WA, 5:21:3 WA and 8:35:5 WA. Out of them, 2:7:1 NA QSO does not exist for any system; 11:42:6 WA and 9:35:5 WA only exist for Mars–Phobos; 7:28:4 WA and 5:21:3 WA are achieved also for Sun–Mars; and finally 8:35:5 WA exists for all systems but Saturn–Titan. In this way, out of 6 resonant QSOs, none of them are found for the Saturn–Titan system.

7.2.4 Results

The methodology described is used to close resonant swing QSOs, within the established restrictions, for the H3BP and Mars–Phobos, Sun–Mars, Jupiter–Europa and Saturn–Titan cases for both the CR3BP and the ER3BP. Tables 7.6, 7.7 and 7.8 summarize all the resonant QSOs found in the Hill’s problem and the results in terms of convergence for the different systems in both the circular (C) and the elliptic (E) restricted three-body problems. Both “x” and “c” indicate convergence, but “c” applies to orbits that are not applicable due to a collision with the surface of the second primary. The empty spaces correspond to orbits that could not be continued, which, as discussed in Section 7.2.3, could be because the orbit does not exist for that specific model or due to numerical challenges. More details are given in the following sections devoted to the individual geometric classes of orbits. In addition, Appendix D contains the figures with all the resonant swing QSOs and Appendix E includes all the initial conditions of them.

Class	Res.	MP		SM		JE		ST	
		C	E	C	E	C	E	C	E
IDS	1 : 5 : 1	c	c	x	x	x	x	x	x
	2 : 3 : 1	x	x	x		x	x	x	x
	3 : 6 : 2	x	x	x	x	x	x	x	x
	4 : 9 : 3	x	x	x	x	x	x	x	x
	5 : 9 : 3	x	x	x	x	x	x	x	x
	5 : 12 : 4	x	x	x	x	x	x	x	x
	6 : 15 : 5	x	x	x	x	x	x	x	x
	7 : 12 : 4	x	x	x	x	x	x	x	x
	7 : 15 : 5	x	x	x	x	x	x	x	x
	8 : 15 : 5	x	x	x	x	x	x	x	x
	9 : 15 : 5	x	x	x	x	x	x	x	x
	11 : 18 : 6	x	x	x	x	x	x	x	x
EDS	1 : 3 : 1	c	c	x	x	x	x	x	x
	1 : 5 : 1	c	c	x	x	x	x	x	x
	3 : 12 : 4	c	c	x	x	c	c	x	x
	4 : 15 : 5	c	c	x	x	x	x	x	x
	5 : 6 : 2	x	x	x	x	x	x	x	x
	5 : 18 : 6	c	c	x	x	x	x	x	x
	7 : 9 : 3	x	x	x	x	x	x	x	x
	7 : 18 : 6	x	x	x	x	x	x	x	x
	8 : 9 : 3	x	x	x	x	x	x	x	x
	9 : 12 : 4	x	x	x	x	x	x	x	x
	11 : 12 : 4	x	x	x	x	x	x	x	x
	11 : 15 : 5	x	x	x	x	x	x	x	x

Table 7.6: Computed doubly-symmetric swing QSOs for different systems in the CR3BP (C) and ER3BP (E)

Class	Res.	MP		SM		JE		ST	
		C	E	C	E	C	E	C	E
LNSS	1 : 4 : 1	c	c	x	x	x	x	x	x
	2 : 7 : 1	c	c	x	x	x	x	x	x
	6 : 7 : 1	x	x	x	x	x	x	x	x
	7 : 8 : 2	x	x	x	x	x	x	x	x
RNSS	1 : 4 : 1	c	c	x	x	x	x	x	x
	2 : 7 : 1	c	c	x	x	x	x	x	x
	6 : 7 : 1	x	x	x	x	x	x	x	x
	7 : 8 : 2	x	x	x	x	x	x	x	x
LWSS	5 : 21 : 3	c	c	x	x	c	c	x	x
	5 : 24 : 6	c		x		c		c	
	7 : 28 : 4	c		x		x		x	
	8 : 35 : 5	c		x		c		c	
	9 : 35 : 5	c	c	x	x	x		x	
	11 : 12 : 3	x	x	x	x	x	x	x	x
	11 : 42 : 6	c	c	x	x	x		x	
RWSS	5 : 21 : 3	c		x		c	c	x	x
	5 : 24 : 6	c		x		c		c	
	7 : 28 : 4	c		x		x		x	
	8 : 35 : 5	c		x		c		c	
	9 : 35 : 5	c	c	x	x	x		x	
	11 : 12 : 3	x	x	x	x	x	x	x	x
	11 : 42 : 6	c	c	x	x	x		x	

Table 7.7: Computed single-symmetric swing QSOs for different systems in the CR3BP (C) and ER3BP (E)

Class	Res.	MP		SM		JE		ST	
		C	E	C	E	C	E	C	E
NA	1 : 4 : 1								
	2 : 7 : 1								
	6 : 7 : 1	x	x	x	x	x	x	x	x
	7 : 8 : 2	x	x	x	x	x	x	x	x
WA	5 : 21 : 3	c		x					
	5 : 24 : 6	c		x		c			
	7 : 28 : 4	c		x					
	8 : 35 : 5	c		x		c			
	9 : 35 : 5	c							
	11 : 12 : 3	x	x	x	x	x	x	x	x
	11 : 42 : 6	c							

Table 7.8: Computed asymmetric swing QSOs for different systems in the CR3BP (C) and ER3BP (E)

Interior-doubly-symmetric Swing QSOs

All the IDS swing QSOs obtained, except the 1:5:1 IDS, belong to the P3QSO family. Conversely, the 1:5:1 IDS belongs to one of the two families with a multiplicity of 5, specifically the P5QSO₂. It can be observed that the 2:3:1 IDS swing QSO cannot be closed in the ER3BP for the Sun–Mars system. Remarkably, this orbit connects to the 2:3 steady QSO through a family with a variable eccentricity and a fixed period thanks to a fold bifurcation, as mentioned in the previous section, which is the reason why it cannot be closed at the Sun–Mars eccentricity in the ER3BP. This family can be observed in Figure 7.7, where the steady QSO family and the intermediate branch of the P3QSO are also depicted, so the bifurcation tree can be observed. The e-Family limits are located in the 2:3:1 IDS swing and the 2:3 Steady propagated for three periods, both of them with $e = 0$. From either of these points the family grows in eccentricity until a bifurcation is found in 'FB'. This point corresponds to a fold bifurcation, where no intersected family has been located and the eccentricity has a maximum that is located at $e = 0.0324$. In addition to this, the two bifurcating orbits connecting the P3QSO and the QSO families in the CR3BP are marked as 'B1' and 'B2'. Finally, the QSO family is represented with 3 times its period, i.e. for three revolutions of each orbit, in order to be represented in the same plot as the other two, as its connection to the P3QSO family is through a period-tripling bifurcation.

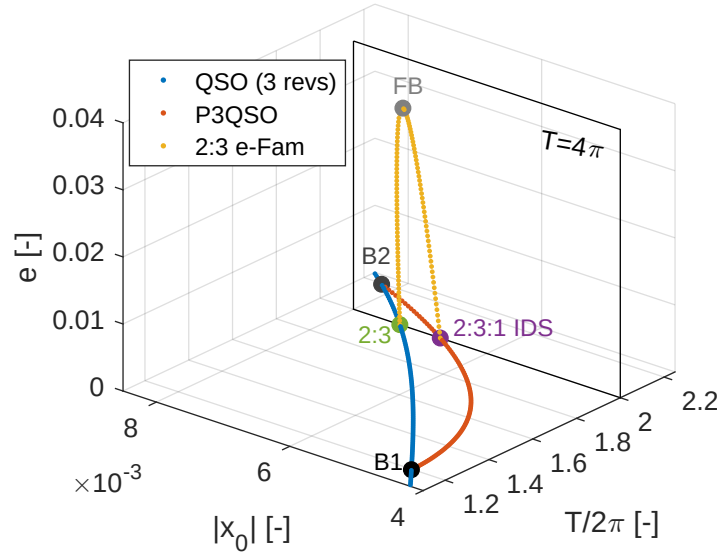


Figure 7.7: Bifurcation tree of the e-Family with the fold bifurcation that connects the 2:3 steady QSO and the 2:3:1 IDS for the Sun–Mars system.

In Tables 7.9 and 7.10 the stability and minimum height of each IDS QSO is shown. The stability is expressed by means of the maximum norm of the vector formed by the 6 eigenvalues, whereas the minimum height is measured from the surface of the second primary in kilometers, considering this body as a perfect sphere of the average radius of the body, collected in Table 6.1. In addition, Figures D.1, D.2, D.3 and D.4 of Appendix D show all the resonant orbits of this category for the 4 systems and the H3BP. Regarding their aspect it is important to mention that the 6:15:5 IDS swing QSO is highly similar to the 6:15 steady QSO, since it is located in close proximity to one of the bifurcation points of the P3QSO family. As

Res.	MP		S-M		JE		ST	
	C	E	C	E	C	E	C	E
1:5:1	c	c	$4.68 \cdot 10^5$	$4.33 \cdot 10^5$	4380	4370	$2.02 \cdot 10^4$	$2.00 \cdot 10^4$
2:3:1	12.1	12.4	$1.52 \cdot 10^6$		$1.76 \cdot 10^4$	$1.78 \cdot 10^4$	$7.07 \cdot 10^4$	$7.31 \cdot 10^4$
3:6:2	5.3	5.3	$1.08 \cdot 10^6$	$1.08 \cdot 10^6$	$1.21 \cdot 10^4$	$1.21 \cdot 10^4$	$4.95 \cdot 10^4$	$4.94 \cdot 10^4$
4:9:3	4.6	4.6	$1.03 \cdot 10^6$	$1.04 \cdot 10^6$	$1.15 \cdot 10^4$	$1.15 \cdot 10^4$	$4.73 \cdot 10^4$	$4.73 \cdot 10^4$
5:9:3	6.7	6.6	$1.17 \cdot 10^6$	$1.16 \cdot 10^6$	$1.32 \cdot 10^4$	$1.32 \cdot 10^4$	$5.38 \cdot 10^4$	$5.37 \cdot 10^4$
5:12:4	4.4	4.4	$1.02 \cdot 10^6$	$1.02 \cdot 10^6$	$1.13 \cdot 10^4$	$1.13 \cdot 10^4$	$4.68 \cdot 10^4$	$4.67 \cdot 10^4$
6:15:5	4.4	4.4	$1.02 \cdot 10^6$	$1.02 \cdot 10^6$	$1.13 \cdot 10^4$	$1.13 \cdot 10^4$	$4.67 \cdot 10^4$	$4.66 \cdot 10^4$
7:12:4	7.6	7.6	$1.23 \cdot 10^6$	$1.22 \cdot 10^6$	$1.40 \cdot 10^4$	$1.40 \cdot 10^4$	$5.69 \cdot 10^4$	$5.66 \cdot 10^4$
7:15:5	4.8	4.8	$1.05 \cdot 10^6$	$1.05 \cdot 10^6$	$1.17 \cdot 10^4$	$1.17 \cdot 10^4$	$4.80 \cdot 10^4$	$4.79 \cdot 10^4$
8:15:5	6.0	6.0	$1.13 \cdot 10^6$	$1.12 \cdot 10^6$	$1.27 \cdot 10^4$	$1.27 \cdot 10^4$	$5.18 \cdot 10^4$	$5.17 \cdot 10^4$
9:15:5	8.3	8.3	$1.28 \cdot 10^6$	$1.26 \cdot 10^6$	$1.46 \cdot 10^4$	$1.45 \cdot 10^4$	$5.91 \cdot 10^4$	$5.88 \cdot 10^4$
11:18:6	8.9	8.8	$1.31 \cdot 10^6$	$1.29 \cdot 10^6$	$1.50 \cdot 10^4$	$1.50 \cdot 10^4$	$6.07 \cdot 10^4$	$6.02 \cdot 10^4$

Table 7.9: Minimum height $h_{\min}$ [km] of IDS QSOs for different systems in the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
1:5:1	1.00	1.00	1.00	1.01	1.00	1.00	1.00	1.00
2:3:1	2.16	1.88	2.16		2.17	2.00	2.20	1.53
3:6:2	2.82	7.95	2.82	7.43	2.82	7.97	2.84	8.01
4:9:3	1.78	5.61	1.78	5.10	1.78	5.63	1.79	5.65
5:9:3	3.44	40.7	3.44	36.1	3.45	40.9	3.48	41.5
5:12:4	1.28	2.71	1.28	2.44	1.28	2.72	1.29	2.73
6:15:5	1.02	1.11	1.02	1.10	1.02	1.11	1.02	1.12
7:12:4	3.42	135	3.42	113	3.42	136	3.45	139
7:15:5	2.20	51.9	2.20	44.5	2.21	52.2	2.22	52.6
8:15:5	3.29	383	3.29	318	3.29	386	3.31	393
9:15:5	3.28	376	3.28	291	3.28	380	3.32	392
11:18:6	3.14	957	3.14	674	3.15	971	3.18	1010

Table 7.10: Maximum eigenvalue $\|\mathbf{A}\|_{\infty}$ of the Monodromy matrix for IDS QSOs in different systems under the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

for the distance from the second primary, the 1:5:1 IDS is infeasible for the Mars–Phobos system, since most part of the orbit is embedded into Phobos and, therefore, it appears in Table 7.9 with a “c” of collision. In the rest of systems, the 1:5:1 is also the closest one to the second primary, but with very far distances from this body. This is even more pronounced for the Sun–Mars system, in which all the distances are bigger than for the other systems. Mars–Phobos, instead, has very small distances, being the 2:3:1, that is the highest one, a value only slightly above 10. This can complicate the applicability of these orbits, specially regarding its irregular surface.

Finally, regarding their stability, in the CR3BP all the orbits in all the systems have a very limited instability. In this sense, the 1:5:1 IDS QSO stands out for being stable for all of the systems, not only for the CR3BP but also for the ER3BP (except for Sun–Mars where the eigenvalue is 1.01, so it is strictly unstable, even though it can be considered stable). This makes this orbit specially suitable for long-term operations. Unlike in the CR3BP, in the ER3BP, some of the orbits become very unstable. This is specially notable in the 11:18:6 IDS QSO for all the systems, even though for Sun–Mars the maximum eigenvalue is considerably lower than in the other systems. In addition, the 7:12:4, 8:15:5 and 9:15:5 are also very unstable for the ER3BP with values over 100. All of these orbits would be bad candidates as parking orbits, but could be used as transfer ones. In contrast, while for almost all the resonances the maximum eigenvalue is higher in the ER3BP than in the CR3BP, the opposite takes place for the 2:3:1 IDS QSO.⁴

Exterior-doubly-symmetric Swing QSOs

Similar to the IDS swing QSOs, all the EDS orbits obtained belong to the family of swing QSOs with a multiplicity of 3, except for the 1:5:1 EDS QSO, which belongs to the same family as the 1:5:1 IDS, but to the other branch.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
1:3:1	c	c	$6.98 \cdot 10^5$	$6.95 \cdot 10^5$	7270	7270	$3.13 \cdot 10^4$	$3.13 \cdot 10^4$
1:5:1	c	c	$4.87 \cdot 10^5$	$5.07 \cdot 10^5$	4610	4630	$2.11 \cdot 10^4$	$2.14 \cdot 10^4$
3:12:4	c	c	$6.21 \cdot 10^4$	$4.03 \cdot 10^4$	c	c	572	283
4:15:5	c	c	$1.80 \cdot 10^5$	$1.52 \cdot 10^5$	745	720	6270	5920
5:6:2	18.1	18.0	$1.92 \cdot 10^6$	$1.88 \cdot 10^6$	$2.26 \cdot 10^4$	$2.26 \cdot 10^4$	$9.05 \cdot 10^4$	$9.02 \cdot 10^4$
5:18:6	c	c	$2.62 \cdot 10^5$	$2.28 \cdot 10^5$	1780	$1.09 \cdot 10^4$	$1.02 \cdot 10^4$	9850
7:9:3	18.5	18.5	$1.95 \cdot 10^6$	$1.92 \cdot 10^6$	$2.30 \cdot 10^4$	$2.29 \cdot 10^4$	$9.14 \cdot 10^4$	$9.10 \cdot 10^4$
7:18:6	3.9	3.9	$9.88 \cdot 10^5$	$9.80 \cdot 10^5$	$1.09 \cdot 10^4$	$1.09 \cdot 10^4$	$4.52 \cdot 10^4$	$4.51 \cdot 10^4$
8:9:3	10.3	10.2	$1.41 \cdot 10^6$	$1.33 \cdot 10^6$	$1.62 \cdot 10^4$	$1.62 \cdot 10^4$	$6.66 \cdot 10^4$	$6.55 \cdot 10^4$
9:12:4	17.9	17.9	$1.91 \cdot 10^6$	$1.89 \cdot 10^6$	$2.24 \cdot 10^4$	$2.24 \cdot 10^4$	$8.93 \cdot 10^4$	$8.91 \cdot 10^4$
11:12:4	1.0	0.8	$8.00 \cdot 10^5$	$7.24 \cdot 10^5$	8610	8530	$3.76 \cdot 10^4$	$3.66 \cdot 10^4$
11:15:5	17.4	17.4	$1.87 \cdot 10^6$	$1.87 \cdot 10^6$	$2.20 \cdot 10^4$	$2.20 \cdot 10^4$	$8.76 \cdot 10^4$	$8.71 \cdot 10^4$

Table 7.11: Minimum height $h_{\min}$ [km] of EDS QSOs for different systems in the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

Again, both the minimum distance to the second primary and the maximum eigenvalue of each orbit are collected in Tables 7.12 and 7.11, whereas their trajectory in the rotating frame can be observed in Figures D.5, D.6 and D.7. Resembling the situation with the 6:15:5 IDS swing QSO, the 11:15:5 EDS swing QSO is close to the other bifurcation point of the P3QSO family, so it is similar to the 11:15 steady QSO. Moreover, for the 1:3:1, 1:5:1, 4:15:5, and 5:18:6 EDS swing QSOs, the orbits pass through the second primary in both the CR3BP

⁴It is important to mention that as the N_S of this orbit is one, the period in the CR3BP and the ER3BP is the same. This is essential in order to have similar values of stability in both cases, as otherwise the ER3BP orbit would be longer, and any instability in the CR3BP would be increased.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
1:3:1	3.39	3.39	3.39	3.27	3.39	3.39	3.38	3.34
1:5:1	1.03	1.04	1.03	1.14	1.03	1.03	1.03	1.06
3:12:4	109	$1.97 \cdot 10^9$	109	$1.94 \cdot 10^8$	109	$1.40 \cdot 10^8$	109	$1.46 \cdot 10^8$
4:15:5	34.8	$5.14 \cdot 10^7$	34.8	$7.43 \cdot 10^7$	34.8	$5.11 \cdot 10^7$	34.8	$5.30 \cdot 10^7$
5:6:2	10.2	104	10.2	108	10.1	102	9.64	93.3
5:18:6	20.2	$6.79 \cdot 10^7$	20.2	$1.01 \cdot 10^8$	20.2	$6.98 \cdot 10^7$	20.2	$6.98 \cdot 10^7$
7:9:3	2.28	11.8	2.28	11.2	2.27	11.7	2.20	10.6
7:18:6	1.16	2.42	1.16	2.77	1.16	1.16	1.15	2.40
8:9:3	117	$1.59 \cdot 10^6$	117	$1.76 \cdot 10^6$	115	$1.54 \cdot 10^6$	107	$1.24 \cdot 10^6$
9:12:4	1.34	3.17	1.34	2.42	1.33	3.14	1.30	2.77
11:12:4	1130	$9.26 \cdot 10^{11}$	1130	$8.97 \cdot 10^{11}$	1100	$1.20 \cdot 10^{12}$	970	$7.41 \cdot 10^{11}$
11:15:5	1.02	1.09	1.02	1.11	1.02	1.09	1.00	1.09

Table 7.12: Maximum eigenvalue $\|\lambda\|_\infty$ of the Monodromy matrix for EDS QSOs in different systems under the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

and the ER3BP for the Mars–Phobos system. Additionally, for the 3:12:4 EDS, this happens not only for Mars–Phobos but also for Jupiter–Europa (Figure D.5). In addition, this orbit is the closest to the secondary in both Sun–Mars and Saturn–Titan. In fact, for Saturn–Titan it has a distance of less than 300 km for the ER3BP, so it is an interesting candidate for a science orbit of reduced observation altitude. In Jupiter–Europa, however, the smallest distance (outside of the “c” cases) is the 4:15:5 QSO with around 700 km of altitude. Finally, for Mars–Phobos, again, the distances are very small, with the highest one of only 18.5 km for the 7:9:3 EDS QSO.

Regarding the stability, in general the EDS resonant orbits are more unstable than the IDS ones. This is specially noticeable for the 11:12:4 EDS in which even for the CR3BP the maximum eigenvalue reaches values of around 1000, while for the ER3BP they are of the order of 10^{12} . On the other side, the 1:5:1, 7:18:6, 9:12:4 and 11:15:5 EDS QSOs have a very limited instability even for the ER3BP model, so they are good candidates as parking orbits.

Left-narrow-single-symmetric Swing QSOs

Each closed LNSS QSO belongs to a distinct family. Specifically, the 1:4:1 and 7:8:2 LNSS QSOs are associated with the two symmetric families with $m = 4$, P4QSO₂ and P4QSO₁, respectively. Conversely, the 2:7:1 and 6:7:1 LNSS QSOs belong to two families with a multiplicity of 7 and order 2, in particular the P7QSO₄ and the P7QSO₃, respectively.

Similar to some of the EDS and IDS swing QSOs, the 1:4:1 and 2:7:1 LNSS QSOs are not feasible for the Mars–Phobos system due to their trajectory intersecting Phobos, as observed in Table 7.13 and figure D.8. As for the 7:8:2 LNSS orbit for Mars–Phobos, it is the orbit until now with the higher minimum distance to Phobos, therefore, the most practical for now. In the rest of systems, the 1:4:1 LNSS QSO is the one of this category with the smaller height

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
1:4:1	c	c	$4.65 \cdot 10^5$	$5.45 \cdot 10^5$	4360	4470	$2.03 \cdot 10^4$	$2.17 \cdot 10^4$
2:7:1	c	c	$4.93 \cdot 10^5$	$4.79 \cdot 10^5$	4690	4690	$2.14 \cdot 10^4$	$2.14 \cdot 10^4$
6:7:1	19.2	19.1	$1.99 \cdot 10^6$	$1.94 \cdot 10^6$	$2.35 \cdot 10^4$	$2.35 \cdot 10^4$	$9.42 \cdot 10^4$	$9.37 \cdot 10^4$
7:8:2	24.3	24.3	$2.32 \cdot 10^6$	$2.30 \cdot 10^6$	$2.78 \cdot 10^4$	$2.77 \cdot 10^4$	$1.11 \cdot 10^5$	$1.11 \cdot 10^5$

Table 7.13: Minimum height $h_{\min}$ [km] of LNSS QSOs for different systems in the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
1:4:1	1.02	1.33	1.05	1.65	1.13	1.28	1.19	1.47
2:7:1	1.00	1.03	1.00	1.18	1.00	1.02	1.00	1.07
6:7:1	1.98	1.11	2.01	1.86	2.13	1.57	2.12	1.04
7:8:2	1.17	1.38	1.17	1.57	1.14	1.30	1.06	1.15

Table 7.14: Maximum eigenvalue $\|\lambda\|_{\infty}$ of the Monodromy matrix for LNSS QSOs in different systems under the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

and the 7:8:2 the one with the bigger, but all of them being rather far from the secondary. Regarding the stability, all the LNSS resonant QSOs are near stable, for the four systems and for both the CR3BP and the ER3BP.

Right-narrow-single-symmetric Swing QSOs

The shape of the resonant RNSS QSOs is depicted in Figure D.9, whereas the minimum height and the maximum $|\lambda|$ are collected in Tables 7.15 and 7.16. All the data in these tables are very similar to those of the RNSS tables, with such small variations that the same considerations made for the LNSS swing QSOs are applied to the RNSS QSOs too. As for the small variations, there are no clear patterns that indicate whether RNSS or LNSS QSOs are more stable or have a smallest distance to the secondary, as the sense of the variations depends on the specific orbit. It can even vary for the same resonance with the system or the model used.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
1:4:1	c	c	$4.63 \cdot 10^5$	$5.43 \cdot 10^5$	4290	4410	$1.98 \cdot 10^4$	$2.11 \cdot 10^4$
2:7:1	c	c	$4.93 \cdot 10^5$	$4.79 \cdot 10^5$	4690	4680	$2.14 \cdot 10^4$	$2.13 \cdot 10^4$
6:7:1	19.2	19.1	$1.99 \cdot 10^6$	$1.95 \cdot 10^6$	$2.36 \cdot 10^4$	$2.36 \cdot 10^4$	$9.48 \cdot 10^4$	$9.44 \cdot 10^4$
7:8:2	24.3	24.3	$2.32 \cdot 10^6$	$2.30 \cdot 10^6$	$2.78 \cdot 10^4$	$2.78 \cdot 10^4$	$1.11 \cdot 10^5$	$1.11 \cdot 10^5$

Table 7.15: Minimum height $h_{\min}$ [km] of RNSS QSOs for different systems in the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
1:4:1	1.00	1.32	1.00	1.65	1.00	1.21	1.00	1.40
2:7:1	1.06	1.03	1.10	1.17	1.22	1.01	1.34	1.01
6:7:1	1.94	1.10	1.91	1.73	1.71	1.11	1.24	1.08
7:8:2	1.17	1.39	1.18	1.59	1.19	1.42	1.17	1.39

Table 7.16: Maximum eigenvalue $\|\lambda\|_\infty$ of the Monodromy matrix for RNSS QSOs in different systems under the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

Left-wide-single-symmetric Swing QSOs

These orbits present the same characteristics of narrow-single-symmetric orbits regarding symmetries. However, the main characteristic of these orbits is that the swing motion is more pronounced than in other swing QSOs, so they present the larger relation between y_{max} and y_{min} (thus named wide). The orbits in this category belong to three different families. The 5:24:6 and 11:12:3 LWSS swing QSOs are associated with distinct period-quadrupling symmetric families, P4QSO₂ and P4QSO₁, respectively, whereas the other five QSOs share membership in the P7QSO₄ family, which also encompasses the 2:7:1 LNSS QSO. Although all orbits were closed within the CR3BP for all systems, only the 5:21:3 and the 11:12:3 LWSS swing QSO achieve full convergence up to the ER3BP. These two orbits have the fewest revolutions around the second primary in this category, making them the most applicable for real missions.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
5:21:3	c	c	$9.47 \cdot 10^4$	$7.09 \cdot 10^4$	c	c	2090	1870
5:24:6	c		$3.41 \cdot 10^4$		c		c	
7:28:4	c		$1.77 \cdot 10^5$		706		6090	
8:35:5	c		$3.78 \cdot 10^4$		c		c	
9:35:5	c	c	$2.32 \cdot 10^5$	$2.05 \cdot 10^5$	1390		8740	
11:12:3	15.9	15.8	$1.77 \cdot 10^6$	$1.71 \cdot 10^6$	$2.07 \cdot 10^4$	$2.07 \cdot 10^4$	$8.33 \cdot 10^4$	$8.30 \cdot 10^4$
11:42:6	c	c	$2.70 \cdot 10^5$	$2.33 \cdot 10^5$	1880		$1.06 \cdot 10^4$	

Table 7.17: Minimum height h_{min} [km] of LWSS QSOs for different systems in the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

Tables 7.17 and 7.18 include the minimum height and maximum eigenvalue for each orbit of this category and Figures D.10 and D.11 in Appendix D show their trajectory. Once again, some orbits are unsuitable for specific systems due to their size. Specifically, the 7:28:4, 9:35:5, and 11:42:6 orbits are not feasible for the Mars–Phobos system; the 5:21:3 orbit encounters collisions in both Mars–Phobos and Jupiter–Europa systems; and both the 5:24:6 and 8:35:5 QSOs are exclusively viable for the Sun–Mars system. As it happened with the rest of categories, none of the orbits are close to Mars for the Sun–Mars system, with distances over 10000 km. For Jupiter–Europa, the 5:24:6 LWSS QSO is the closest one

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
5:21:3	24.5	$3.66 \cdot 10^5$	29.2	$4.66 \cdot 10^8$	54.8	$2.25 \cdot 10^5$	94.3	$4.39 \cdot 10^6$
5:24:6	1.00		1.00		1.00		1.32	
7:28:4	1.00		1.00		5.77		12.8	
8:35:5	237		262		400		619	
9:35:5	1.00	$7.24 \cdot 10^4$	1.00	$1.75 \cdot 10^8$	1.00		3.13	
11:12:3	25.1	$1.58 \cdot 10^4$	25.1	$1.73 \cdot 10^4$	24.4	$1.46 \cdot 10^4$	20.8	8980
11:42:6	1.00	$2.74 \cdot 10^4$	1.00	$4.37 \cdot 10^8$	1.00		1.00	

Table 7.18: Maximum eigenvalue $\|\lambda\|_\infty$ of the Monodromy matrix for LWSS QSOs in different systems under the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

without collision, with around 700 km of minimum height in the CR3BP, but unable to be closed for the ER3BP. Finally, for Saturn–Titan, the minimum height is around 2000 km for the 5:21:3 LWSS QSO, which is higher than for other categories.

Regarding the stability, in the CR3BP, there is a high range of value for the stability. The 5:24:6, 7:28:4, 9:35:5 and 11:42:6 LWSS QSOs are stable for at least Mars–Phobos and Sun–Mars in the CR3BP, whereas the 8:35:5 LWSS QSO is already quite unstable even for the circular model. Nevertheless, even the orbits that are stable in the CR3BP, when closed in the ER3BP are very unstable. In fact, all of the orbits closed for the ER3BP are very unstable. This limits the applicability of these orbits as transfer orbits.

Right-wide-single-symmetric Swing QSOs

Most of the comments made for the LWSS swing QSOs are also relevant for the RWSS QSOs. The biggest difference between these two categories is that while the 5:21:3 LWSS achieved full convergence up to the ER3BP that is not the case for the 5:21:3 RWSS. Neither for Mars–Phobos nor for Sun–Mars was convergence achieved for the ER3BP.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
5:21:3	c		$9.50 \cdot 10^4$		c	c	2230	1790
5:24:6	c		$3.30 \cdot 10^4$		c		c	
7:28:4	c		$1.77 \cdot 10^5$		720		6200	
8:35:5	c		$3.81 \cdot 10^4$		c		c	
9:35:5	c	c	$2.32 \cdot 10^5$	$2.05 \cdot 10^5$	1400		8820	
11:12:3	15.9	15.9	$1.78 \cdot 10^6$	$1.71 \cdot 10^6$	$2.11 \cdot 10^4$	$2.11 \cdot 10^4$	$8.69 \cdot 10^4$	$8.56 \cdot 10^4$
11:42:6	c	c	$2.70 \cdot 10^5$	$2.33 \cdot 10^5$	1890		$1.07 \cdot 10^4$	

Table 7.19: Minimum height $h_{\min}$ [km] of RWSS QSOs for different systems in the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

This time the QSOs are illustrated in Figures D.12 and D.13. As for the minimum height, Table 7.19, and the maximum λ , Table 7.20, again, there are small differences to those of

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
5:21:3	18.9		14.3		9.11	$2.25 \cdot 10^5$	41.4	$2.76 \cdot 10^6$
5:24:6	1.39		1.90		3.90		6.80	
7:28:4	1.00		1.00		5.59		11.8	
8:35:5	208		183		60.9		103	
9:35:5	1.00	$7.11 \cdot 10^4$	1.33	$1.74 \cdot 10^8$	3.52		6.05	
11:12:3	25.1	$1.59 \cdot 10^4$	25.1	$1.75 \cdot 10^4$	24.6	$1.49 \cdot 10^4$	21.1	9500
11:42:6	1.00	$2.69 \cdot 10^4$	1.42	$4.34 \cdot 10^8$	2.68		4.12	

Table 7.20: Maximum eigenvalue $\|\lambda\|_\infty$ of the Monodromy matrix for RWSS QSOs in different systems under the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

LWSS QSOs, but they still remain on the same order of magnitudes. The biggest differences in this sense, are: that the 5:24:6 RWSS is not stable for any of the systems, unlike 5:24:6 LWSS, even though the instability is small; and that the 8:35:5 LWSS QSO has not as big $\|\lambda\|_\infty$, specially for Jupiter–Europa and Mars–Phobos. Finally, there is not a clear pattern on which orbits are more stable or closer to the secondary between RWSS and LWSS, like it happened with the NSS orbits, as this changes depending on the orbit.

Narrow-asymmetric Swing QSOs

The orbits in this category belong to four different families. The 1:4:1 and 7:8:2 NA swing QSOs are associated with distinct asymmetric families with a multiplicity of 4, P4QSO₄ and P4QSO₃, whereas the 2:7:1 and 6:7:1 NA QSOs are members of the two asymmetric families with $m = 7$, P7QSO₆ and P7QSO₅, respectively. As detailed previously, these families of asymmetric orbits bifurcate directly from the steady QSOs family in H3BP. However, in the CR3BP these families bifurcate from the corresponding family of symmetric orbits. For instance, the 7:8:2 NA QSO belongs to the asymmetric family that in the CR3BP bifurcates from the family containing the 7:8:2 LNSS and the 7:8:2 RNSS. Moreover, the 6:7:1 NA QSO belongs, in the CR3BP, to the asymmetric family bifurcating from the family containing the 6:7:1 LNSS and 6:7:1 RNSS, as expected. These two families have full convergence for both the CR3BP and the ER3BP.

The case of the 1:4:1 NA and the 2:7:1 NA are quite different, as they do not exist for any of the systems studied. The reason behind this is that the corresponding family of asymmetric orbits for these systems within the CR3BP in each case only exists for lower values of the period, as observed in Figure 7.6. Consequently, when continuing the 1:4:1 NA orbit in μ for a fixed period, the orbit converges to the 1:4:1 LNSS for $\mu = 4.9 \cdot 10^{-9}$, establishing the maximum μ for which this orbit exists in the CR3BP. A similar phenomenon occurs with the 2:7:1 NA QSO, where while continuing the resonant orbit in μ , it converges to the 2:7:1 RNSS QSO for $\mu = 2.4 \cdot 10^{-11}$.

In order to determine the maximum μ for which an asymmetric orbit exists, the family obtained while continuing the asymmetric QSO in μ for a fixed period is compared to the

one obtained when continuing the symmetric QSO. An example of this is shown in here for the 1:4:1 NA orbit. As said, this orbit belongs to the P4QSO₄ family, that in the CR3BP, bifurcates from the LSS branch of the P4QSO₂ family. For this reason, a comparison is made between the 1:4:1 NA family in μ and the 1:4:1 LNSS family, to determine the moment from which both families match. As both families are obtained with a variation in μ keeping the period fixed, this variable cannot be used as a comparison, as it is always $T = 2\pi$. In addition, the initial conditions are placed on a different place within the orbit for symmetric and asymmetric orbits. Therefore, the only parameters that can be compared are the energy and the stability. As for the energy, the Jacobi constant is used instead of Γ . The reason behind this is that as with the variation of μ the JC value of reference of the Lagrange's points change, the use of Γ is unpractical. Moreover, as the energy of the LNSS and the NA families is almost identical, the difference between them is used, instead of plotting both of them separately. This can be observed in Figure 7.8 where the continuation is done until the μ value of the Mars–Phobos system, together with the in-plane stability index of both families. The figure shows how from a certain μ value, inferior to that of the Mars–Phobos system, the 1:4:1 NA family in mass ratio, converges to the 1:4:1 LNSS one. A similar analysis is done for every asymmetric orbit, but it is not shown in here for conciseness.

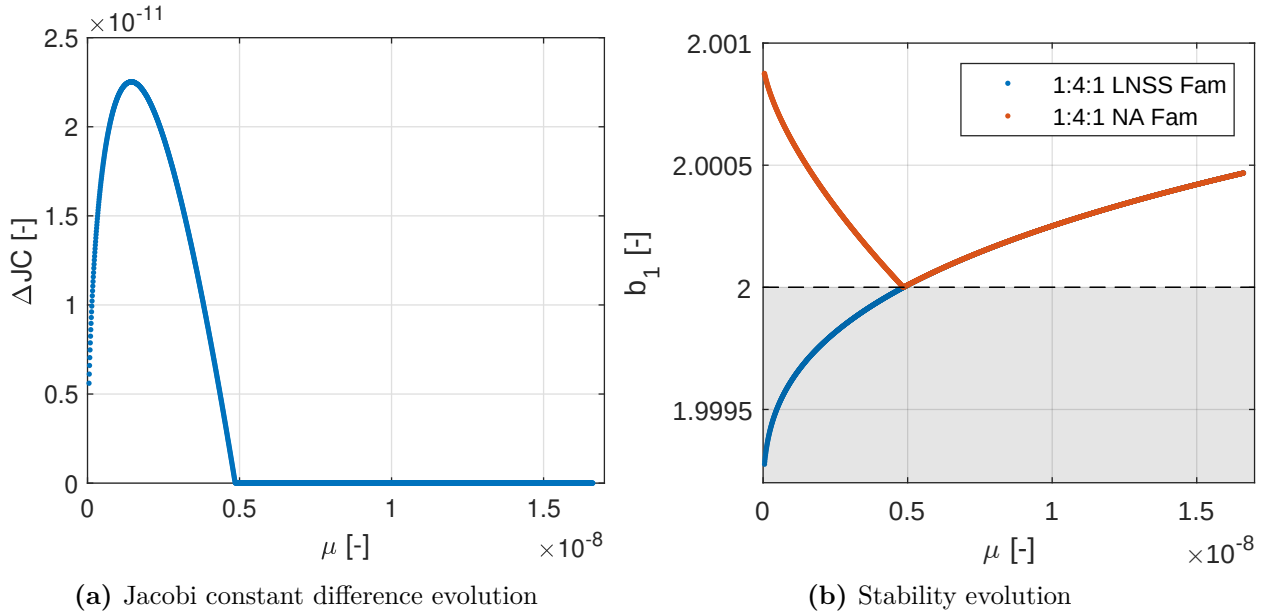


Figure 7.8: Comparison between 1:4:1 NA μ family and the 1:4:1 LNSS μ family.

Finally, the minimum distance and the stability of the orbits of this category can be observed in Tables 7.21 and 7.22, respectively. Their trajectories, additionally, are illustrated in Figure D.14. The orbits that were able to be closed have very similar characteristics than those correspondent of the LNSS and RNSS. The most remarkable characteristic is the stability of both resonances, specially the 7:8:2 that is even stable for the ER3BP in some systems.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
1:4:1								
2:7:1								
6:7:1	19.0	19.0	$1.98 \cdot 10^6$	$1.96 \cdot 10^6$	$2.34 \cdot 10^4$	$2.34 \cdot 10^4$	$9.43 \cdot 10^4$	$9.35 \cdot 10^4$
7:8:2	24.1	24.0	$2.31 \cdot 10^6$	$2.26 \cdot 10^6$	$2.76 \cdot 10^4$	$2.75 \cdot 10^4$	$1.10 \cdot 10^5$	$1.10 \cdot 10^5$

Table 7.21: Minimum height $h_{\min}$ [km] of NA QSOs for different systems in the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
1:4:1								
2:7:1								
6:7:1	1.00	1.91	1.00	6.41	1.00	2.04	1.00	3.00
7:8:2	1.00	1.00	1.00	1.04	1.00	1.00	1.00	1.01

Table 7.22: Maximum eigenvalue $\|\lambda\|_{\infty}$ of the Monodromy matrix for NA QSOs in different systems under the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

Wide-asymmetric Swing QSOs

Similar to the wide-single-symmetric family, the distinguishing feature of these orbits is their strong swing compared to other swing QSOs. The 5:24:6 and the 11:12:3 WA QSO belong to the same two asymmetric families of multiplicity 4 as the 1:4:1 and the 7:8:2 NA QSOs, respectively. The remaining five WA QSOs are all part of the same family as the 2:7:1 NA QSOs. Among these, only the 11:12:3 WA achieves full convergence up to ER3BP for all systems, while the others do not converge within the ER3BP for any of the cases. In the CR3BP, except for the 11:12:3 WA, the others do not exist for all systems, as seen in Figure 7.6.

The 5:24:6 WA orbit can only be closed for the H3BP, as it ceases to exist in the CR3BP for a μ smaller than that of Mars–Phobos due to a collision singularity with the second primary. For the five orbits belonging to the P7QSO₆ family, as μ increases, they converge to the corresponding RWSS for different values of the mass ratio, implying that they only exist for systems with a lower μ than this critical value. The critical values are $\mu = 6.0 \cdot 10^{-8}$ for the 11:42:6 WA QSO and $\mu = 1.9 \cdot 10^{-7}$ for the 9:35:5 WA QSO, making them exclusive to Mars–Phobos. The 7:28:4 and 5:21:3 WA QSOs not only exist for Mars–Phobos but also for Sun–Mars, with critical values of $\mu = 9.7 \cdot 10^{-7}$ and $\mu = 1.1 \cdot 10^{-5}$, respectively. Additionally, the 8:35:5 WA QSO exists in the CR3BP up to $\mu = 7.2 \cdot 10^{-5}$, extending its existence to the Jupiter–Europa system.

Finally, Tables 7.23 and 7.24 collect the minimum height and maximum λ for each of the orbits of this category, respectively. As for the trajectory in the rotating frame, it is shown in Figures D.15 and D.16. In line with most of the categories analyzed, certain studied orbits are impractical for some systems due to their relative size compared to the second

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
5:21:3	c		$9.20 \cdot 10^4$					
5:24:6	c		$2.27 \cdot 10^4$		c			
7:28:4	c		$1.73 \cdot 10^5$					
8:35:5	c		$3.74 \cdot 10^4$		c			
9:35:5	c							
11:12:3	10.6	10.4	$1.43 \cdot 10^6$	$1.35 \cdot 10^6$	$1.65 \cdot 10^4$	$1.64 \cdot 10^4$	$6.86 \cdot 10^4$	$6.74 \cdot 10^4$
11:42:6	c							

Table 7.23: Minimum height $h_{\min}$ [km] of WA QSOs for different systems in the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

Res.	MP		SM		JE		ST	
	C	E	C	E	C	E	C	E
5:21:3	24.7		22.8					
5:24:6	5.18		5.17		4.39			
7:28:4	3.53		2.63					
8:35:5	200		204.6		106			
9:35:5	1.81							
11:12:3	$7.11 \cdot 10^4$	$1.33 \cdot 10^5$	50.9	$1.59 \cdot 10^5$	49.1	$1.19 \cdot 10^5$	38.6	$5.81 \cdot 10^4$
11:42:6	1.36							

Table 7.24: Maximum eigenvalue $\|\lambda\|_{\infty}$ of the Monodromy matrix for WA QSOs in different systems under the CR3BP (C) and ER3BP (E). The data have been rounded to a maximum of three significant figures.

primary. Specifically, the 5:21:3, 7:28:4, 9:35:5 and 11:42:6 orbits would result in collisions with the surface for Mars–Phobos alone, while the 8:35:5 orbit would lead to collisions for both Mars–Phobos and Jupiter–Europa. When comparing both the minimum height and the maximum λ to those of the LWSS and RWSS categories, they have in general the same orders of magnitude. However, this time the tendency of the minimum height is to be smaller for WA than the symmetric cases. As for the stability, not a clear pattern can be observed.

Chapter 8

Discussion

In this chapter, the results obtained throughout the dissertation are analyzed and the most important insights are put together. Following the structure of the thesis, the chapter is subdivided into three sections. In the first section, the conclusions made on curvilinear coordinates are highlighted. In the second section, all the results obtained regarding the behavior of the families of QSO in the circular version of the restricted three-body problem are analyzed. Finally, the results obtained regarding resonant QSOs in the three models studied are presented, putting special emphasis on their applications to real missions.

8.1 Curvilinear Coordinates

One innovative aspect of this thesis is the use of curvilinear coordinates, particularly cylindrical coordinates, for the formulation of the co-orbital motion. For this reason, in Chapter 2 their performance with respect to Cartesian coordinates were analyzed for the task of uncertainty propagation. This comparison was done by means of a Cramer-von Mises test carried out for several orbits. In this comparison, the performance of the Yamanaka-Ankersen STM formulated as a solution of the Tshauer-Hemple equations for relative motion was compared with its cylindrical analogue formulated in this work. Their relative performance changed depending on the orbit, but in none of the cases studied, the Cartesian coordinates had a better performance than the curvilinear ones. Among the cases studied, the best comparative performance was found for the circular cases, both for a GEO orbit and a circular LEO. After that, for the case of a LEO orbit with $e = 0.1$, still curvilinear coordinates keep a good performance that is much better than the Cartesian one. When the eccentricity of this LEO is increased even more, the differences between the two types of coordinates reduce, but the best performance is still with the curvilinear coordinates up to $e = 0.8$, where both coordinates have the same performance.

The best performance of the curvilinear coordinates for the linearized motion can be explained by their intrinsic curvature that better adapts better to the curvature of the orbits that are close to a circular orbit. When the eccentricity is too high, the curvature of the cylindrical coordinates fails to match that of the orbits and the performance gets worse. However, as in this work all the systems studied in the ER3BP have an eccentricity lower

than 0.1, and taking into account the importance of the linear behavior for periodic orbits continuation, the use of curvilinear coordinates for both the ER3BP and the CR3BP seems only natural.

8.2 Families of QSO

In Chapters 5 and 6, different families of QSO were studied for both the H3BP (or Hill approximation of the CR3BP) and the CR3BP. In both models, steady QSOs form a unique family that can also be referred to as the DRO family. In the Hill's approximation, this family corresponds to the f family depicted by Hénon [25]. As it is well known, this family possesses none tangent bifurcation. However, when period-multiplying bifurcations are also considered, a total of 12 bifurcating orbits are found for the planar motion and 6 related to the out-of-plane one, considering up to a period-7 multiplying bifurcation. Each of these bifurcating orbits comes in pairs for each type of bifurcation found.

Focusing on the planar motion and the simpler H3BP model for now, a total of 17 families composed by swing QSOs bifurcate from these points. Based on their geometrical characteristics that the orbits of each family have, these families can be categorized in asymmetric (A), single-symmetric (SS) or doubly-symmetric (DS). In addition, for 16 of the 17 families, they can be subdivided into high-energy and low-energy. The only exception to this is the P3QSO family, the unique family of multiplicity three, as it possesses both a low-energy and a high-energy branch, plus an intermediate energy branch. This family has the special characteristic of crossing the steady family in two different points, which is the reason why it is unique instead of coming in pairs as the rest of the families. All of these characteristics of each of the H3BP families, along with some others, are collected in table 8.1.

To summarize, all the families presented in the table are listed. Apart from the P3QSO family, composed of doubly-symmetric orbits, there are four families of $m = 4$, two of $m = 5$, four of $m = 6$ and six families of $m = 7$. Out of the four families of $m = 4$, two of them are single-symmetric families coming from each of the two period-quadrupling bifurcating orbits, respectively. The other two are asymmetric families of QSOs, in which each comes from the same bifurcating orbits as the single-symmetric families. For the multiplicity 5, there is only one pair of families (coming from the one pair of period-quintupling bifurcations) composed by doubly-symmetric orbits. The case of multiplicity 6 follows the same distribution of families as the $m = 4$ case. Finally, the case of multiplicity 7 is the most complex one as there are two different types of bifurcating orbits, with a total of 4 of them rather than 2. The first type is the period-septupling bifurcation of order 1 or $m = 7(1)$, that represents the first pair of bifurcating orbits. From each of these orbits, one family arises composed by doubly-symmetric orbits, like in the $m = 5$ case. The second type of bifurcation corresponding to $m = 7(2)$ has again two different families arising from the same orbit, with a total of two single-symmetric families and two asymmetric ones, like the cases $m = 4$ and $m = 6$.

Apart from these 17 families, two extra families of planar orbits are studied in this work. These two families are secondary swing families as they come from bifurcations on already swing families. As these bifurcations are tangent ones, specifically pitchfork bifurcations, the multiplicity of these families remains the same as from the family that they arise from. The

Family	Energy	Sym.	Mix. br.	Special characteristics
P3QSO	N/A	DS	1	Two intersections with steady QSO family. Contains high and low energy branches. 2 EDS, 1 IDS and 1 mixed EDS branches.
P4QSO ₁	High	SS	-	
P4QSO ₂	Low	SS	2 (M)	
P4QSO ₃	High	A	-	
P4QSO ₄	Low	A	-	
P5QSO ₁	High	DS	-	It has an in-plane tangent bif. on EDS branch.
P5QSO ₂	Low	DS	2	
P6QSO ₁	High	SS	-	
P6QSO ₂	Low	SS	2 (M)	
P6QSO ₃	High	A	-	
P6QSO ₄	Low	A	-	
P7QSO ₁	High	DS	-	It has an in-plane tangent bif. on IDS branch.
P7QSO ₂	Low	DS	2	
P7QSO ₃	High	SS	-	The 2 x-orthogonal crossings are on the same side. Doubly swing motion.
P7QSO ₄	Low	SS	2 (M)	The 2 x-orthogonal crossings are on the same side. Doubly swing motion.
P7QSO ₅	High	A	-	The 2 y-orthogonal crossings are on the same side. Doubly swing motion.
P7QSO ₆	Low	A	-	The 2 y-orthogonal crossings are on the same side. Doubly swing motion.

Table 8.1: Main characteristics of swing QSO families in the H3BP, where DS means doubly-symmetric, SS single-symmetric and A asymmetric. In addition, 2 (M) indicates that both branches are mirrored.

first of the families is the P5QSO_{1,T} that bifurcates from the EDS branch of the P5QSO₁ family and that is composed by single-symmetric orbits. The second family, also a SS one, is the P7QSO_{1,T}, and this time derives from the IDS branch of the P7QSO₁.

Finally, from the six out-of-plane bifurcations found, six different families of symmetric 3D-QSO arise. Among them, there are two families of $m = 5$ composed of doubly-symmetric, z-asymmetric orbits; two $m = 6$ families of single-symmetric, z-symmetric QSOs; and two families of $m = 7$ which, like those of $m = 5$, are composed of doubly-symmetric, z-asymmetric QSOs.

Moving onto the CR3BP, in this work four different systems are considered: Mars–Phobos, Sun–Mars, Jupiter–Europa and Saturn–Titan. All of these systems are characterized by a small mass ratio, making them suitable for the application of the Hill approximation studied before. Nevertheless, even with this small mass ratio μ , some of the characteristics of the families change from the ones observed in the HR3BP. For this reason, a study of these

differences was conducted using the Jupiter–Europa system as the model.

The first of the differences has nothing to do with the specific behavior of the QSO, but it is something intrinsic of the dynamical systems. While in the planar H3BP there exists a double-symmetry, the CR3BP only possesses a symmetry with respect to the x-axis. For this reason, the orbits that were doubly-symmetric in the H3BP are no longer symmetric with respect to the y-axis. However, they are still referred to as DS QSO to keep track of them and as their characteristic doubly-symmetric shape is still conserved. In addition, thanks to the small μ , these orbits are close to be symmetric, specially for the Mars–Phobos system, in which the difference can be almost undetectable. As for the asymmetric orbits, in the H3BP they were y-symmetric, and again they are no longer in the CR3BP. This complicates the task of continuing this type of orbits, as a perpendicular crossing algorithm can no longer be used. This implies not only that the propagation time is longer, but also complicates targeting asymmetric orbits over symmetric ones.

The second difference involves the two secondary swing families found in the H3BP. While in the Hill approximation, these families come from a tangent bifurcation, in the CR3BP they become what is known as broken bifurcations. Hence, where in the H3BP there was an intersection between the secondary families and their primary swing families, where an orbit was shared between them, this intersection no longer exists on the CR3BP. In addition, in the CR3BP the distribution of the branches between the families changes. For these two reasons, the secondary families in the CR3BP are referred to as $P5QSO_{1,S}$ and $P7QSO_{1,S}$, respectively, to differentiate them from their H3BP counterpart. In this sense, the branch that corresponds to the EDS branch of the $P5QSO_1$ in the HR3BP, now, in the CR3BP, starts out as EDS but after the broken bifurcation it transitions to RSS orbits. As a consequence, the $P5QSO_{1,S}$ is no longer composed by two SS branches, but by an LSS branch and an EDS branch. In the case of the $P7QSO_1$ family, it is the IDS branch the one with the broken bifurcation and the one transitioning to RSS. Finally, the $P7QSO_{1,S}$ family has one LSS branch and one IDS.

The final and most important difference is the fact that the families of asymmetric orbits do not bifurcate directly from the steady QSO in the CR3BP. Instead, they bifurcate from the symmetric family composed by single-symmetric orbits with which they shared the bifurcation orbit in the H3BP. This completely changes the tree of QSOs, as there are no longer 17 planar families bifurcating, but 11. The other 6 families were the asymmetric ones. However, one of them, the $P6QSO_4$ low-energy family, is no longer found in the CR3BP for Jupiter–Europa. In fact, considering that the Mars–Phobos system is the closest system to the H3BP model as it has the smallest ratio, the existence of this asymmetric family was studied for this system also, leading to the same result. Nevertheless, the trend indicates the asymmetric family only exists for mass ratio values even smaller than that of the Mars–Phobos system. As for the remaining 5 asymmetric families, they are collected in the Table 8.2, together with the family from which they originate and the specific branch of this family in which the pitchfork tangent bifurcation takes place, in each case. In the case of the missing $P6QSO_4$, the trend indicates that it would bifurcate from the LSS branch of the $P6QSO_2$ family, when it actually existed.

The final question that is addressed in this work regarding the families of QSOs is what happens when the mass ratio increases more and the Hill approximation is no longer valid. For this reason, even though a full analysis of all the families in an additional system is

Family	Energy	Orig. fam.	Orig. br.
P4QSO ₃	High	P4QSO ₁	LSS
P4QSO ₄	Low	P4QSO ₂	LSS
P6QSO ₃	High	P6QSO ₁	RSS
P7QSO ₅	High	P7QSO ₃	RSS
P7QSO ₆	Low	P7QSO ₄	RSS

Table 8.2: Original families and branches of asymmetric swing QSO families in the CR3BP for Mars–Phobos, Sun–Mars, Jupiter–Europa and Saturn–Titan, where LSS indicates left-single-symmetric and RSS means right-single-symmetric.

out of the scope of this work, the steady QSO for the Earth–Moon system is also analyzed, due to the importance of this system. The mass ratio of the Earth–Moon system is several orders of magnitude higher than any of the systems included in this work. For this reason, the differences are notable even for the steady QSO family. Among these differences, the most important one is the fact that the in-plane stability index, responsible of the bifurcating orbits that lead to the swing QSO families, differs from that of the other systems. This divergence starts to be noticeable from $T = 0.8\pi$ and increases as the period grows. This implies that all the high-energy bifurcating orbits take place at a different location within the family for the Earth–Moon system than what observed before. In fact, the high-energy period-7 multiplying bifurcation of order one is no longer found. As for the high-energy period-sixtupling bifurcation, it takes place in an orbit that is already close to the primary, i.e. the Earth, which was not the case for the systems with lower mass ratio. For these reasons, the study made for the H3BP is no longer useful for a system as the Earth–Moon, and an independent study of the families is needed for a system with that high mass ratio.

Finally, it is important to remark the utility of the H3BP for the understanding and computation of the different QSO families in the CR3BP. The main advantage the H3BP provides is the fact that it eases the discovery and obtaining of some of the swing QSOs, specially the asymmetric QSO. In fact, the computation of the asymmetric families in the CR3BP was achieved through the transition of one of the asymmetric orbits from the H3BP into the desired CR3BP system. Once that orbit was transitioned, the rest of the family could be obtained through the continuation within the system. The reason behind this is the difficulty of reaching an asymmetric QSO from a symmetry one, due to the similarity of them. As mentioned before, as the y-axis symmetry does not exist for the CR3BP, there is no specific algorithm that could be used that targets exclusively asymmetric orbits, but it is the same one that can be used for the symmetric ones. Beyond the asymmetric orbits, the H3BP analysis helped with the computation of the secondary families, as otherwise their existence would have been more difficult to predict. Finally, the previous study done in the H3BP made the understanding of the geometrical characteristics of each of the families of swing QSO more clear, as it gave the possibility of a geometrical classification that otherwise would not have been possible.

8.3 Resonant QSO

In the ER3BP, only the orbits known as resonant orbits are periodic orbits because of their dependence on time. For this reason, as all the primaries included in this work move on an elliptic orbit, the study of resonant QSO is essential for the applicability of them. In this sense, in Chapter 7, the resonant QSO are studied. As families of periodic orbits do not exist for the ER3BP unless the eccentricity is allowed to change, the process for the computation of resonant QSOs in the ER3BP implies its previous computation in the CR3BP with the subsequent continuation in eccentricity until achieving the eccentricity of the corresponding system. Therefore, in this chapter, resonant orbits are not only studied for the ER3BP but also for the CR3BP. In addition, as the four different already mentioned systems are included in this analysis, the H3BP stands out, again, as a perfect starting point and, for this reason, it is included also in this study.

For the identification of each of the resonant QSO, a labeling based on resonant pairs for steady QSOs and resonant triplets for the swing ones is established. Following the applicability point of view, in both cases reasonable limits were imposed. In the case of steady QSO, the limits were a maximum of 10 revolutions around the second primary and a minimum period of $T = 0.6\pi$, leading to 22 resonant steady QSOs, all belonging to the same family.

The case of the swing QSOs is more complex, due to the amount of families involved, so the first limit was establishing a maximum multiplicity of 7, like in the rest of the work. Due to this complexity and aiming to obtain a broad enough sample of resonant orbits, the limits within the swing QSOs are less restricted than for the steady case, with a maximum of 6 revolutions of the guidance center and 11 revolutions of the primaries. These restrictions lead to a total of 68 different resonant orbits for the H3BP, covering a total of 34 different resonant triplets. According to the geometrical classification made, out of these 68 resonant QSOs, 24 of them are DS, 8 NSS, 14 WSS, 8 NA and 14 WA. Due to the symmetries of the H3BP, single-symmetric and asymmetric orbits share the same triplets, which is the reason why there is the same amount of NA resonant QSOs than NSS, or WA than WSS. In addition, inside each of these categories, each triplet corresponds to 2 orbits: left and right for the single-symmetric orbits and up and down for the asymmetric ones. However, during the work down and up orbits are treated as one, as one can always be obtained mirroring the other with respect to the x-axis, even in the CR3BP or ER3BP model. Among these triplets that belong to single-symmetric or asymmetric families of QSOs, 4 belong to the families with $m = 4$ and 7 belong to families with $m = 7(2)$. Therefore, there are no resonant orbits in families with $m = 6$. In the case of the DS category, only one of the triplets corresponds to two different orbits, one EDS and one IDS, and corresponds to a case of $m = 5$, specifically to the P5QSO₂ family. The rest of the triplets are equally divided into 11 IDS and 11 EDS, all belonging to the unique family with $m = 3$, which is the one with the highest number of resonant orbits. Like it happened with $m = 6$, the case $m = 7(1)$ leads to zero resonant QSOs with the restrictions established.

Moving on to the CR3BP model, most of the orbits previously found in the H3BP were also successfully computed. In fact, all symmetric resonant QSOs found in the H3BP were also found in the CR3BP for all the four systems. This is in line with the results on the

study of the QSO families in the CR3BP, where the only difference found in the symmetric families involved the families P5QSO₁ and P7QSO₁ that in the restrictions of this work lead to a non-resonant orbit. Nevertheless, the asymmetric case is very different, as three of the orbits could be computed in the CR3BP for all the four systems. This, again, aligns with the different origin of asymmetric families in the CR3BP with respect to the H3BP, discovered in the previous chapter. However, as some of the orbits could only be computed for some of the systems, a more extensive analysis on the effect that the mass ratio has on these families was computed. This analysis confirmed the previously observed trend that, as the mass ratio μ increases, the bifurcation of each asymmetric family occurs further along the corresponding symmetric one. This limits the extension in period of these families, reducing the number of resonant orbits found as the system's μ becomes larger.

The final aspect studied with respect to resonant QSO was the transition to ER3BP, which is even more complex than the one to the CR3BP. One of the difficulties associated to the ER3BP is the fact that as periodic orbits are isolated, each of the resonant QSOs needs to be computed particularly by the continuation in eccentricity from its CR3BP counterpart. This process is computationally more costly than when several resonant QSOs are obtained by the computation of one family in the circular problem. Another complication is due to the fact that in the ER3BP an integer number of revolutions of the primaries is needed. This usually implies that the period of the ER3BP resonant QSOs is longer than those of the H3BP and CR3BP orbits. In particular, for the swing resonant QSOs the relation is $T_{swing}^E = N_S T_{swing}^C$. Therefore, the numerical propagation error can be severely increased in some cases, preventing for computation of certain resonant QSOs for the ER3BP. Finally, for the Sun–Mars systems two of the resonant QSOs were connected by a family in eccentricity thanks to what is known as a fold bifurcation. In particular, the orbits connected are the 2:3 steady QSO and the 2:3:1 IDS QSO that belongs to the P3QSO family. Due to the mentioned fold bifurcation, the family is upper bounded by an eccentricity of $e = 0.0324$, which is inferior than Sun–Mars' eccentricity. Therefore, it prevents the transition of both orbits to the ER3BP.

Chapter 9

Conclusion

9.1 Concluding Remarks

This doctoral thesis has investigated quasi-satellite orbits (QSOs) within the framework of three dynamical models of increasing complexity: the Hill three-body problem (H3BP), the circular restricted three-body problem (CR3BP), and the elliptic restricted three-body problem (ER3BP). The primary objective has been to deepen the understanding of QSOs, with particular attention to asymmetric and resonant periodic orbits, which have been comparatively underexplored in the literature.

A key innovation introduced in this work is the use of curvilinear coordinates, specifically cylindrical coordinates, in the modeling of three-body dynamics. Through the derivation of a linearized solution for two-body relative motion analogous to the Yamanaka-Ankersen formulation, it has been shown that cylindrical coordinates yield improved linear behavior, especially in scenarios with limited orbital eccentricity. Building upon this foundation and making use of known numerical techniques, such as the predictor-corrector method proposed by Lara and Peláez in 2002 or the pseudo-arclength algorithm, custom continuation tools were developed exploiting curvilinear coordinates and applied within both the CR3BP and ER3BP dynamical models.

In the H3BP, a detailed characterization of planar periodic QSOs was conducted, leading to the identification of 17 swing QSO families bifurcating from the steady QSO family, up to seventh-order multiplicity. In addition, two secondary swing symmetric families were found to bifurcate from two of the previously identified swing families. These families were classified based on symmetry properties, bifurcation structure, and energy levels. This analysis revealed a rich dynamical structure and provided a solid foundation for extending the study to more complex models.

In transitioning to the CR3BP, two major differences with respect to the H3BP were identified. First, bifurcations connecting families in the H3BP do not always persist in the CR3BP. In particular, the two secondary bifurcations found between symmetric families in the H3BP are disconnected for the CR3BP, giving rise to a phenomenon referred to as "broken bifurcations". Second, asymmetric swing QSO families no longer bifurcate directly from the

steady family, but instead arise from single-symmetric swing families via pitchfork bifurcations. These findings highlight the utility of the H3BP when working with systems with small mass ratios μ , where it serves as a valuable precursor for identifying and understanding the complex structures presented in the CR3BP. However, for systems with higher mass ratios, such as the Earth-Moon system, this approach becomes less effective. In particular, it was shown that high-energy families for the Earth-Moon system followed markedly different trends.

In the last part of the investigation, resonant QSOs were studied in all three models, including the ER3BP. The analysis was performed for several planetary systems of interest: Mars–Phobos, Sun–Mars, Jupiter–Europa, and Saturn–Titan. The classification developed extends that of the circular case by incorporating additional geometric features and organizing the solutions via resonance triplets. By placing reasonable limits (from a mission design point of view) on the multiplicity of period-multiplying bifurcations ($m = 3, \dots, 7$), the maximum number of second primary revolutions (11 orbits), and guiding center cycles (6), all feasible resonant triplets have been identified for the different models. The most abundant feasible triplets were found in the $m = 3$ case, followed by $m = 7$. Only a handful of resonant triplets were feasible for $m = 4$, a single triplet was found for $m = 5$, and none for $m = 6$.

Both the primaries’ mass ratio μ and eccentricity e have been found to impact the existence of some specific resonant QSOs. An increase in the mass ratio was shown to delay the bifurcations giving rise to asymmetric families, leading to shorter asymmetric families and resulting in the disappearance of some resonances. As for the eccentricity, a fold bifurcation or turning point was identified in the ER3BP, notably linking the steady family with the P3QSO family in the Sun–Mars system. This turning point defines a maximum eccentricity beyond which the 2:3 steady QSO and the 2:3:1 IDS swing orbit cease to exist, thereby limiting the continuation of these solutions into the ER3BP for this system.

9.2 Future Work

The results presented in this thesis suggest several directions for future research. The most immediate extension concerns the study of three-dimensional (3D) QSOs, which has only been briefly addressed in this work and limited to the H3BP model. The study done here in the H3BP can be extended, including asymmetric 3D QSOs and studying the different tangent bifurcations found in the different families. In addition, a comprehensive continuation of this study in the CR3BP would allow for the investigation of how 3D QSO families are modified when incorporating additional dynamical complexity. Moreover, the study of resonant 3D QSOs remains entirely open and could be addressed in future work.

Another promising direction involves a more detailed analysis of QSOs in the Earth-Moon system. This system, characterized by a significantly higher mass ratio μ than those considered in this thesis, exhibits dynamical behaviors that can diverge from those captured accurately by the H3BP. While steady QSOs and some swing QSOs have been previously investigated in the literature for this system, the existence and structure of asymmetric and resonant QSOs remain, to the author’s knowledge, largely unexplored.

Furthermore, the evolution of QSO families in the ER3BP as a function of the eccentricity

e has not been studied systematically here, as this thesis focused on solutions computed for the fixed systems eccentricity. The only exception is the identification of a fold bifurcation in the Sun–Mars system connecting the 2:3 steady QSO and the 2:3:1 IDS swing QSO. A more detailed analysis of this bifurcation could reveal whether a nearby family for higher eccentricities exists, giving rise to a broken bifurcation. Similar investigations could be extended to other QSOs in several systems where the continuation into the ER3BP was unsuccessful, with special attention to changes in stability that could indicate potential new bifurcations.

Finally, the resonant QSOs identified in this work could be further developed for practical applications in mission design, particularly those with favorable dynamical properties. This would require transitioning the computed orbits from the ER3BP to higher-fidelity dynamical models that more accurately represent the conditions of the specific planetary systems under consideration.

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Appendix A

Conversion Between Cartesian and Pulsating Curvilinear Coordinates

The position vector of the third body with respect to the second one, in both the relative motion and the restricted three-body problem, can be written as

$$\mathbf{d} = \frac{\tilde{p}}{\gamma} ((r_\rho - \cos \theta) \mathbf{u}_\rho + \sin \theta \mathbf{u}_\theta + z \mathbf{k}) = \tilde{x} \mathbf{i} + \tilde{y} \mathbf{j} + \tilde{z} \mathbf{k}, \quad (\text{A.1})$$

with

$$\mathbf{u}_\rho = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \quad \mathbf{u}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$$

and

$$\left. \frac{d\mathbf{u}_\rho}{d\nu} \right|_{\mathcal{L}} = \theta' \mathbf{u}_\theta \quad \left. \frac{d\mathbf{u}_\theta}{d\nu} \right|_{\mathcal{L}} = -\theta' \mathbf{u}_\rho.$$

Its $\mathcal{L}$ -frame derivative with respect to τ in Cartesian coordinates is

$$\left. \frac{d\mathbf{d}}{d\tau} \right|_{\mathcal{L}} = \dot{\tilde{x}} \mathbf{i} + \dot{\tilde{y}} \mathbf{j} + \dot{\tilde{z}} \mathbf{k}, \quad (\text{A.2})$$

whereas the derivative with respect to ν in pulsating curvilinear coordinates results as

$$\begin{aligned} \left. \frac{d\mathbf{d}}{d\nu} \right|_{\mathcal{L}} &= \left(\frac{\tilde{p}e \sin \nu}{\gamma^2} (r_\rho - \cos \theta) + \frac{\tilde{p}}{\gamma} \rho' \right) \mathbf{u}_\rho \\ &+ \left(\frac{\tilde{p}e \sin \nu}{\gamma^2} \sin \theta + \frac{\tilde{p}}{\gamma} \theta' r_\rho \right) \mathbf{u}_\theta + \left(\frac{\tilde{p}e \sin \nu}{\gamma^2} z + \frac{\tilde{p}}{\gamma} z' \right) \mathbf{k}. \end{aligned} \quad (\text{A.3})$$

In addition, the derivatives with respect to τ and ν are related as

$$\left. \frac{d\mathbf{d}}{d\tau} \right|_{\mathcal{L}} = \frac{\gamma^2}{\tilde{p}^{3/2}} \left. \frac{d\mathbf{d}}{d\nu} \right|_{\mathcal{L}}. \quad (\text{A.4})$$

From Eqs. (A.1) and (A.4), and after taking into account Eqs. (A.2) and (A.3), the transformation from pulsating curvilinear to Cartesian is obtained

$$\left\{ \begin{array}{l} \tilde{x} = \frac{\tilde{p}}{\gamma} (r_\rho \cos \theta - 1) \\ \tilde{y} = \frac{\tilde{p}}{\gamma} r_\rho \sin \theta \\ \tilde{z} = \frac{\tilde{p}}{\gamma} z \\ \dot{\tilde{x}} = \frac{\gamma}{\tilde{p}^{1/2}} (\rho' \cos \theta - r_\rho \theta' \sin \theta) + \frac{e \sin \nu}{\tilde{p}^{1/2}} (r_\rho \cos \theta - 1) \\ \dot{\tilde{y}} = \frac{\gamma}{\tilde{p}^{1/2}} (\rho' \sin \theta + r_\rho \theta' \cos \theta) + \frac{e \sin \nu}{\tilde{p}^{1/2}} r_\rho \sin \theta \\ \dot{\tilde{z}} = \frac{\gamma}{\tilde{p}^{1/2}} z' + \frac{e \sin \nu}{\tilde{p}} z \end{array} \right. . \quad (\text{A.5})$$

This transformation can be inverted to obtain the reverse transformation

$$\left\{ \begin{array}{l} \rho = \frac{\gamma}{\tilde{p}} \sqrt{\left(\frac{\tilde{p}}{\gamma} + \tilde{x} \right)^2 + \tilde{y}^2 - 1} \\ \theta = \text{mod} \left(\text{atan2} \left(\tilde{y}, \frac{\tilde{p}}{\gamma} + \tilde{x} \right) + 2\pi, 2\pi \right) \\ z = \frac{\gamma}{\tilde{p}} \tilde{z} \\ \rho' = \frac{\tilde{p}^{1/2}}{\gamma} \left(\dot{\tilde{x}} \cos \theta + \dot{\tilde{y}} \sin \theta \right) - \frac{e \sin \nu}{\tilde{p}} (\tilde{x} \cos \theta + \tilde{y} \sin \theta) \\ \theta' = \frac{\frac{\tilde{p}^{1/2}}{\gamma} \left(-\dot{\tilde{x}} \sin \theta + \dot{\tilde{y}} \cos \theta \right) + \frac{e \sin \nu}{\tilde{p}} (\tilde{x} \sin \theta - \tilde{y} \cos \theta)}{r_\rho} \\ z' = \frac{\tilde{p}^{1/2}}{\gamma} \dot{\tilde{z}} - \frac{e \sin \nu}{\tilde{p}} \tilde{z} \end{array} \right. . \quad (\text{A.6})$$

Appendix B

Pulsating Curvilinear ER3BP Jacobian Matrix

The Jacobian matrix for the elliptic restricted three-body problem in pulsating curvilinear coordinates can be obtained by differentiating the equations of motion:

$$\begin{cases} \rho'' = r_\rho (2 + \theta') \theta' + \frac{1}{\gamma} \left[(1 - \mu) r_\rho \left(1 - \frac{1}{r^3} \right) + \mu d_\rho \left(1 - \frac{1}{d^3} \right) \right] \\ \theta'' = -\frac{2\rho' (1 + \theta')}{r_\rho} + \mu \frac{d_\theta}{r_\rho} \frac{1}{\gamma} \left(1 - \frac{1}{d^3} \right) \\ z'' = \frac{z}{\gamma} \left(-e \cos \nu - \frac{(1 - \mu)}{r^3} - \frac{\mu}{d^3} \right) \end{cases}$$

$$\left[\frac{\partial \mathbf{F}}{\partial \mathbf{s}} \right] = \begin{bmatrix} \frac{\partial \rho'}{\partial \rho} & \frac{\partial \rho'}{\partial \theta} & \frac{\partial \rho'}{\partial z} & \frac{\partial \rho'}{\partial \rho'} & \frac{\partial \rho'}{\partial \theta'} & \frac{\partial \rho'}{\partial z'} \\ \frac{\partial \theta'}{\partial \rho} & \frac{\partial \theta'}{\partial \theta} & \frac{\partial \theta'}{\partial z} & \frac{\partial \theta'}{\partial \rho'} & \frac{\partial \theta'}{\partial \theta'} & \frac{\partial \theta'}{\partial z'} \\ \frac{\partial z'}{\partial \rho} & \frac{\partial z'}{\partial \theta} & \frac{\partial z'}{\partial z} & \frac{\partial z'}{\partial \rho'} & \frac{\partial z'}{\partial \theta'} & \frac{\partial z'}{\partial z'} \\ \frac{\partial \rho''}{\partial \rho} & \frac{\partial \rho''}{\partial \theta} & \frac{\partial \rho''}{\partial z} & \frac{\partial \rho''}{\partial \rho'} & \frac{\partial \rho''}{\partial \theta'} & \frac{\partial \rho''}{\partial z'} \\ \frac{\partial \theta''}{\partial \rho} & \frac{\partial \theta''}{\partial \theta} & \frac{\partial \theta''}{\partial z} & \frac{\partial \theta''}{\partial \rho'} & \frac{\partial \theta''}{\partial \theta'} & \frac{\partial \theta''}{\partial z'} \\ \frac{\partial z''}{\partial \rho} & \frac{\partial z''}{\partial \theta} & \frac{\partial z''}{\partial z} & \frac{\partial z''}{\partial \rho'} & \frac{\partial z''}{\partial \theta'} & \frac{\partial z''}{\partial z'} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial \rho''}{\partial \rho} & \frac{\partial \rho''}{\partial \theta} & \frac{\partial \rho''}{\partial z} & 0 & \frac{\partial \rho''}{\partial \theta'} & 0 \\ \frac{\partial \theta''}{\partial \rho} & \frac{\partial \theta''}{\partial \theta} & \frac{\partial \theta''}{\partial z} & \frac{\partial \theta''}{\partial \rho'} & \frac{\partial \theta''}{\partial \theta'} & 0 \\ \frac{\partial z''}{\partial \rho} & \frac{\partial z''}{\partial \theta} & \frac{\partial z''}{\partial z} & 0 & 0 & 0 \end{bmatrix}$$

with

$$\frac{\partial \rho''}{\partial \rho} = (\omega_{\mathcal{H}}^2 - 1) + \frac{1}{\gamma} \left[(1 - \mu) \left(1 - \frac{1}{r^3} + 3 \frac{r_\rho^2}{r^5} \right) + \mu \left(1 - \frac{1}{d^3} + 3 \frac{d_\rho^2}{d^5} \right) \right]$$

$$\frac{\partial \rho''}{\partial \theta} = \frac{\mu d_\theta}{\gamma} \left(1 - \frac{1}{d^3} + 3r_\rho \frac{d_\rho}{d^5} \right)$$

$$\frac{\partial \rho''}{\partial z} = \frac{3z}{\gamma} \left[(1 - \mu) \frac{r_\rho}{r^5} + \mu \frac{d_\rho}{d^5} \right]$$

$$\frac{\partial \rho''}{\partial \theta'} = 2r_\rho \omega_{\mathcal{H}}$$

$$\frac{\partial \theta''}{\partial \rho} = \frac{2\rho' \omega_{\mathcal{H}}}{r_\rho^2} + \frac{\mu d_\theta}{\gamma r_\rho} \left[-\frac{1}{r_\rho} \left(1 - \frac{1}{d^3} \right) + 3 \frac{d_\rho}{d^5} \right]$$

$$\frac{\partial \theta''}{\partial \theta} = \frac{\mu}{\gamma r_\rho} \left[(r_\rho - d_\rho) \left(1 - \frac{1}{d^3} \right) + 3r_\rho \frac{d_\theta^2}{d^5} \right]$$

$$\frac{\partial \theta''}{\partial z} = \frac{3\mu z}{\gamma r_\rho} \frac{d_\theta}{d^5}$$

$$\frac{\partial \theta''}{\partial \rho'} = -\frac{2\omega_{\mathcal{H}}}{r_\rho}$$

$$\frac{\partial \theta''}{\partial \theta'} = -\frac{2\rho'}{r_\rho}$$

$$\frac{\partial z''}{\partial \rho} = \frac{3z}{\gamma} \left[(1 - \mu) \frac{r_\rho}{r^5} + \mu \frac{d_\rho}{d^5} \right]$$

$$\frac{\partial z''}{\partial \theta} = \frac{3\mu z r_\rho}{\gamma} \frac{d_\theta}{d^5}$$

$$\frac{\partial z''}{\partial z} = \frac{1}{\gamma} \left[-e \cos \nu - (1 - \mu) \left(\frac{1}{r^3} - 3 \frac{z^2}{r^5} \right) - \mu \left(\frac{1}{d^3} - 3 \frac{z^2}{d^5} \right) \right]$$

Appendix C

H3BP Jacobian Matrix

The Jacobian matrix for Hill's approximation of the three-body problem can be obtained by differentiating the equations of motion with respect to the state vector $\mathbf{s} = (\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta})^T$

$$\begin{cases} \ddot{\xi} = 2\dot{\eta} + 3\xi - \frac{\xi}{\varrho^3} \\ \ddot{\eta} = -2\dot{\xi} - \frac{\eta}{\varrho^3} \\ \ddot{\zeta} = -\zeta - \frac{\zeta}{\varrho^3} \end{cases}.$$

$$\left[\frac{\partial \mathbf{F}}{\partial \mathbf{s}} \right] = \begin{bmatrix} \frac{\partial \dot{\xi}}{\partial \xi} & \frac{\partial \dot{\xi}}{\partial \eta} & \frac{\partial \dot{\xi}}{\partial \zeta} & \frac{\partial \dot{\xi}}{\partial \dot{\xi}} & \frac{\partial \dot{\xi}}{\partial \dot{\eta}} & \frac{\partial \dot{\xi}}{\partial \dot{\zeta}} \\ \frac{\partial \dot{\eta}}{\partial \xi} & \frac{\partial \dot{\eta}}{\partial \eta} & \frac{\partial \dot{\eta}}{\partial \zeta} & \frac{\partial \dot{\eta}}{\partial \dot{\xi}} & \frac{\partial \dot{\eta}}{\partial \dot{\eta}} & \frac{\partial \dot{\eta}}{\partial \dot{\zeta}} \\ \frac{\partial \dot{\zeta}}{\partial \xi} & \frac{\partial \dot{\zeta}}{\partial \eta} & \frac{\partial \dot{\zeta}}{\partial \zeta} & \frac{\partial \dot{\zeta}}{\partial \dot{\xi}} & \frac{\partial \dot{\zeta}}{\partial \dot{\eta}} & \frac{\partial \dot{\zeta}}{\partial \dot{\zeta}} \\ \frac{\partial \ddot{\xi}}{\partial \xi} & \frac{\partial \ddot{\xi}}{\partial \eta} & \frac{\partial \ddot{\xi}}{\partial \zeta} & \frac{\partial \ddot{\xi}}{\partial \dot{\xi}} & \frac{\partial \ddot{\xi}}{\partial \dot{\eta}} & \frac{\partial \ddot{\xi}}{\partial \dot{\zeta}} \\ \frac{\partial \ddot{\eta}}{\partial \xi} & \frac{\partial \ddot{\eta}}{\partial \eta} & \frac{\partial \ddot{\eta}}{\partial \zeta} & \frac{\partial \ddot{\eta}}{\partial \dot{\xi}} & \frac{\partial \ddot{\eta}}{\partial \dot{\eta}} & \frac{\partial \ddot{\eta}}{\partial \dot{\zeta}} \\ \frac{\partial \ddot{\zeta}}{\partial \xi} & \frac{\partial \ddot{\zeta}}{\partial \eta} & \frac{\partial \ddot{\zeta}}{\partial \zeta} & \frac{\partial \ddot{\zeta}}{\partial \dot{\xi}} & \frac{\partial \ddot{\zeta}}{\partial \dot{\eta}} & \frac{\partial \ddot{\zeta}}{\partial \dot{\zeta}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial \ddot{\xi}}{\partial \xi} & \frac{\partial \ddot{\xi}}{\partial \eta} & \frac{\partial \ddot{\xi}}{\partial \zeta} & 0 & \frac{\partial \ddot{\xi}}{\partial \dot{\eta}} & 0 \\ \frac{\partial \ddot{\eta}}{\partial \xi} & \frac{\partial \ddot{\eta}}{\partial \eta} & \frac{\partial \ddot{\eta}}{\partial \zeta} & \frac{\partial \ddot{\eta}}{\partial \dot{\xi}} & 0 & 0 \\ \frac{\partial \ddot{\zeta}}{\partial \xi} & \frac{\partial \ddot{\zeta}}{\partial \eta} & \frac{\partial \ddot{\zeta}}{\partial \zeta} & 0 & 0 & 0 \end{bmatrix},$$

with

$$\frac{\partial \ddot{\xi}}{\partial \xi} = 3 - \frac{1}{\varrho^3} + \frac{3\xi^2}{\varrho^5}$$

$$\frac{\partial \ddot{\xi}}{\partial \eta} = \frac{3\xi\eta}{\varrho^5}$$

$$\frac{\partial \ddot{\xi}}{\partial \zeta} = \frac{3\xi\zeta}{\varrho^5}$$

$$\frac{\partial \ddot{\xi}}{\partial \dot{\eta}} = 2$$

$$\frac{\partial \ddot{\eta}}{\partial \xi} = \frac{3\xi\eta}{\varrho^5}$$

$$\frac{\partial \ddot{\eta}}{\partial \eta} = -\frac{1}{\varrho^3} + \frac{3\eta^2}{\varrho^5}$$

$$\frac{\partial \ddot{\eta}}{\partial \zeta} = \frac{3\eta\zeta}{\varrho^5}$$

$$\frac{\partial \ddot{\eta}}{\partial \dot{\xi}} = -2$$

$$\frac{\partial \ddot{\zeta}}{\partial \xi} = \frac{3\xi\zeta}{\varrho^5}$$

$$\frac{\partial \ddot{\zeta}}{\partial \eta} = \frac{3\eta\zeta}{\varrho^5}$$

$$\frac{\partial \ddot{\zeta}}{\partial \zeta} = -1 - \frac{1}{\varrho^3} + \frac{3\zeta^2}{\varrho^5}$$

Appendix D

Figures of Resonant Quasi-Satellite Orbits

D.1 Interior-doubly-symmetric Swing QSOs

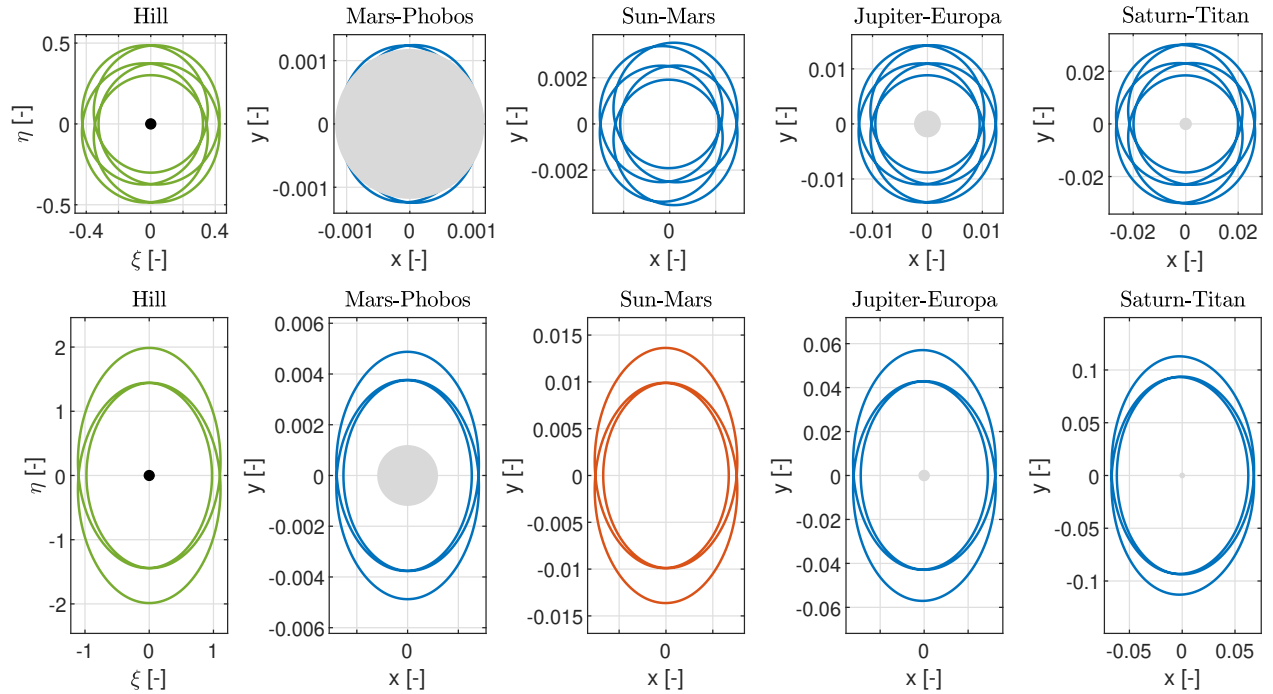


Figure D.1: IDS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 1:5:1, and 2:3:1 resonances.

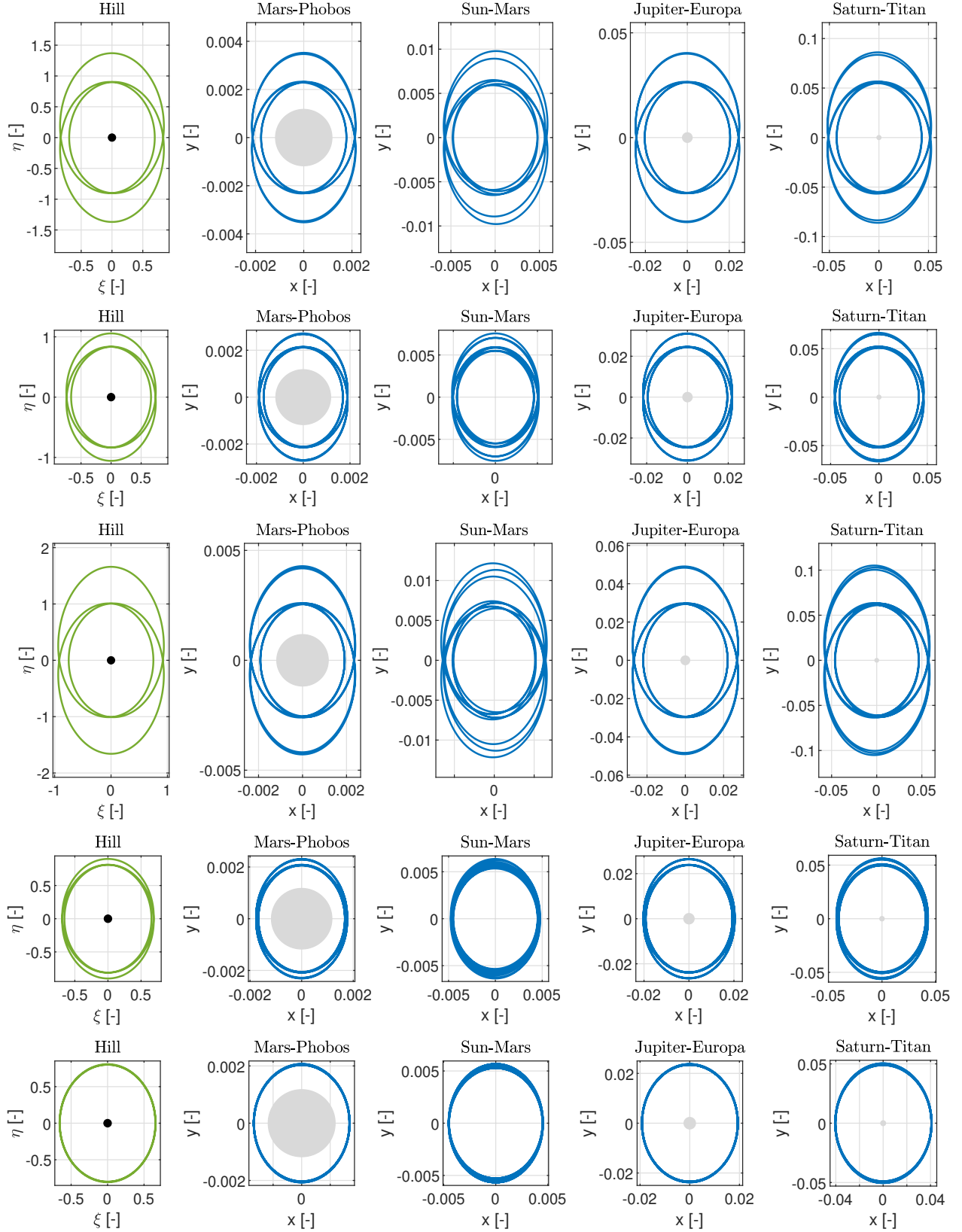


Figure D.2: IDS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 3:6:2 4:9:3, 5:9:3, 5:12:4 and 6:15:5 resonances.

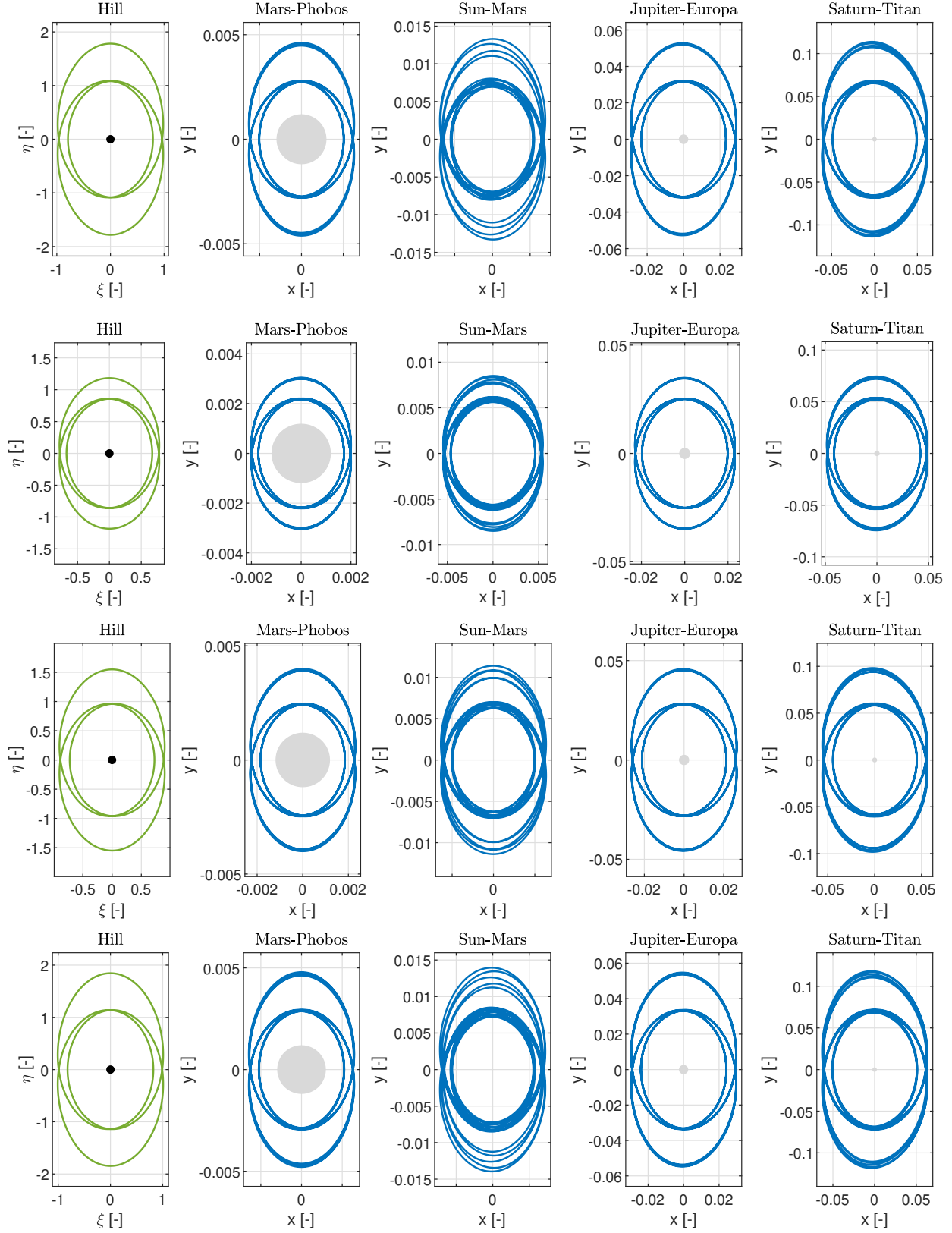


Figure D.3: IDS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 7:12:4, 7:15:5, 8:15:5 and 9:15:5 resonances.

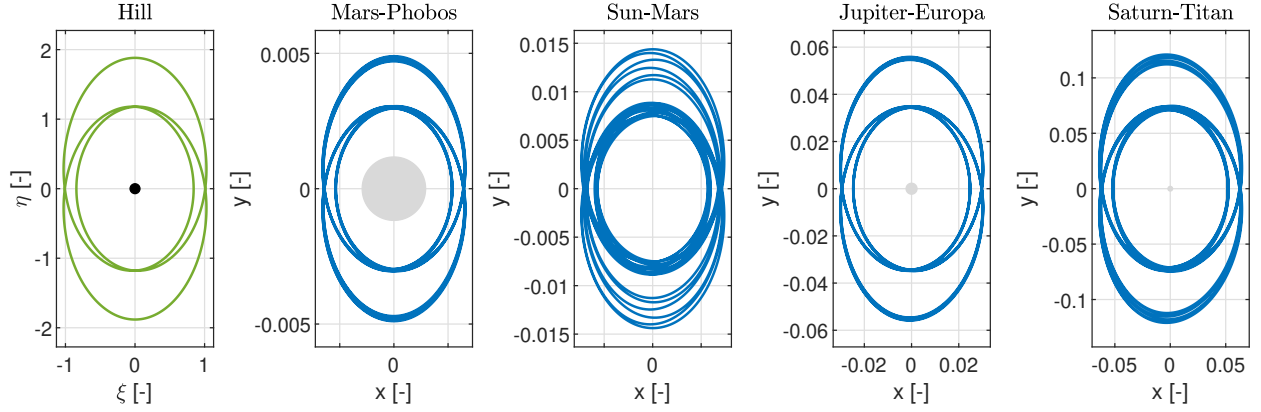


Figure D.4: IDS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red) 11:18:6 resonance, plotted in the $\mathcal{L}$ frame.

D.2 Exterior-doubly-symmetric Swing QSOs

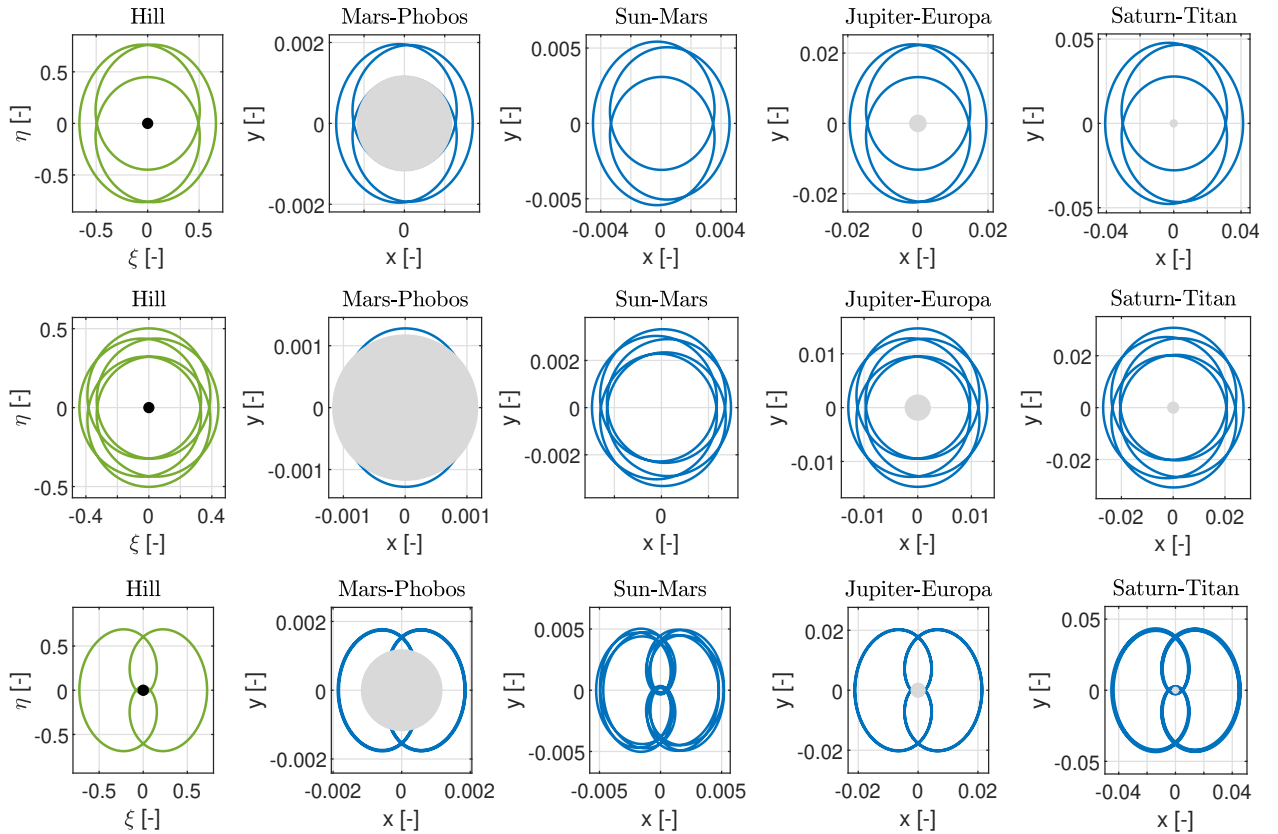


Figure D.5: EDS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 1:3:1, 1:5:1 and 3:12:4 resonances.

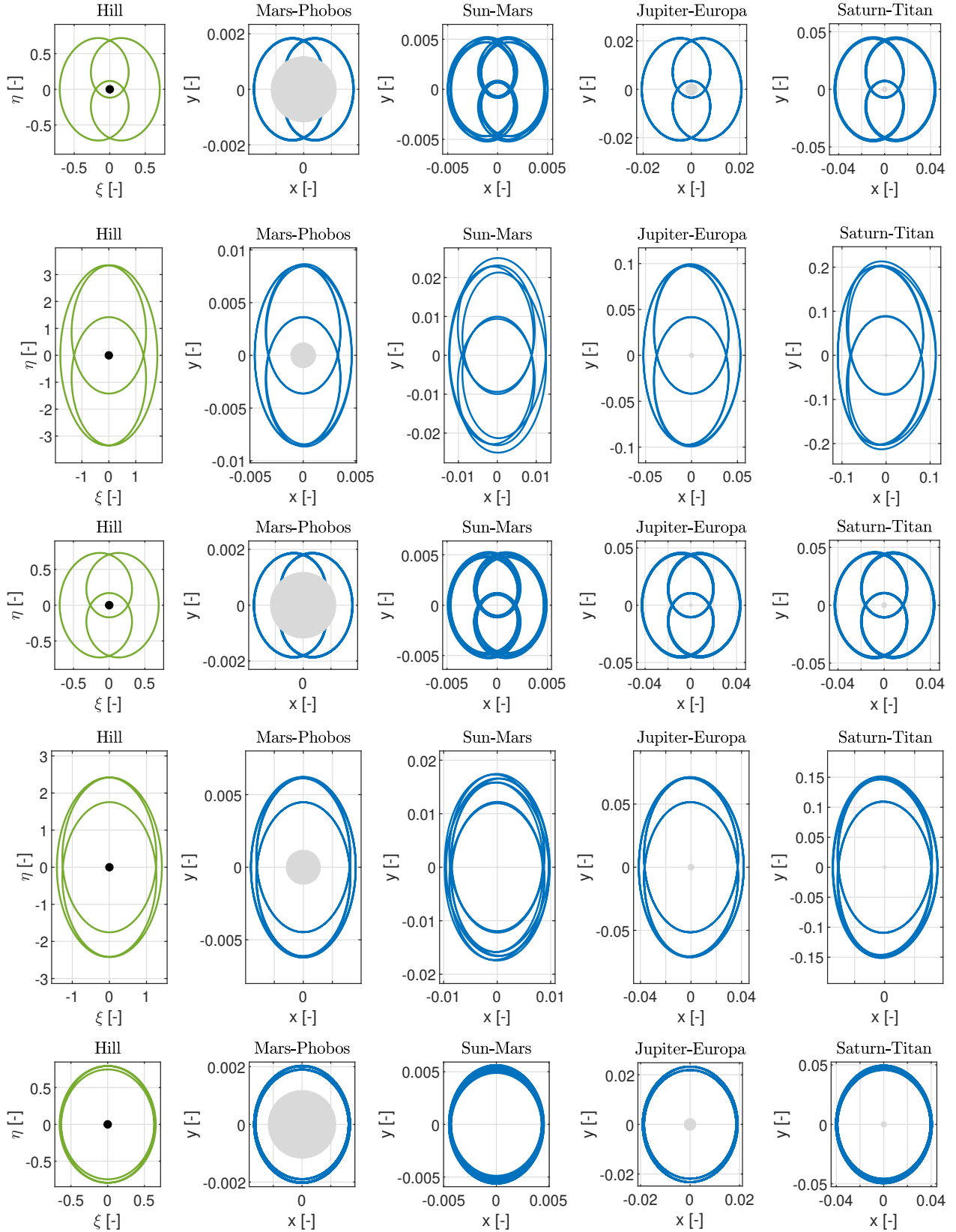


Figure D.6: EDS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 4:15:5, 5:6:2, 5:18:6, 7:9:3 and 7:18:6 resonances.

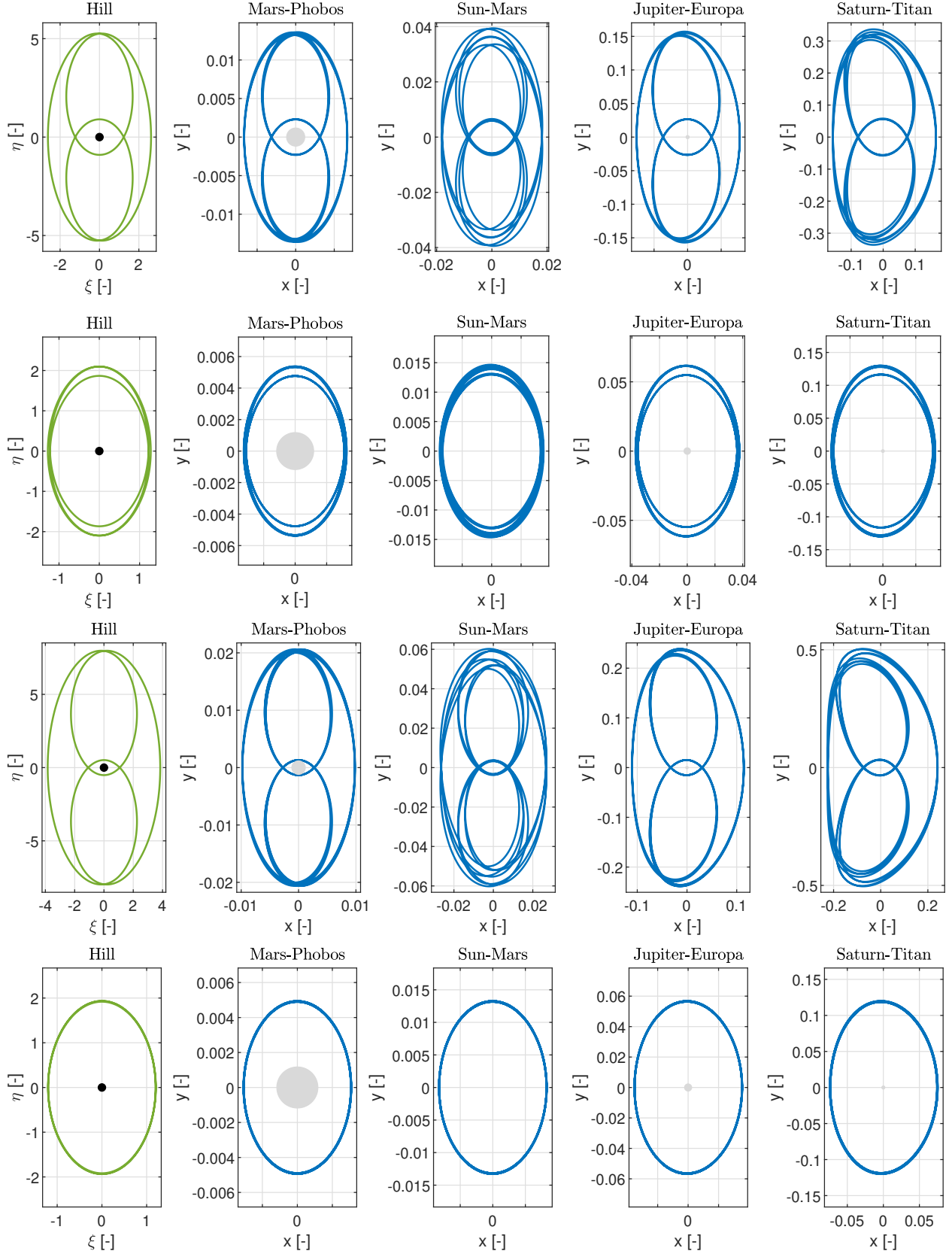


Figure D.7: EDS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 8:9:3, 9:12:4, 11:12:4 and 11:15:5 resonances.

D.3 Left-narrow-single-symmetric Swing QSOs

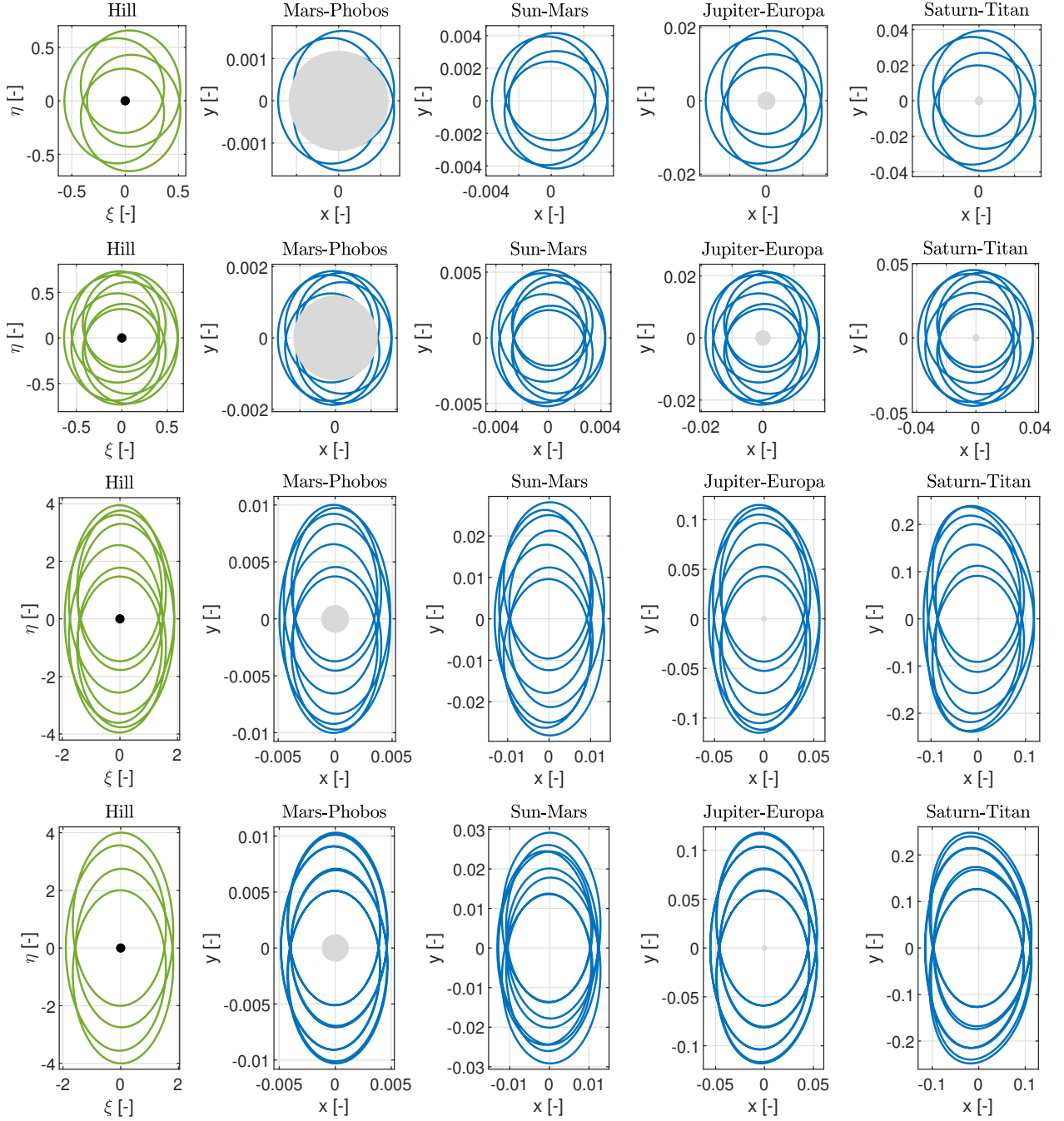


Figure D.8: LNSS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 1:4:1, 2:7:1, 6:7:1 and 7:8:2 resonances.

D.4 Right-narrow-single-symmetric Swing QSOs

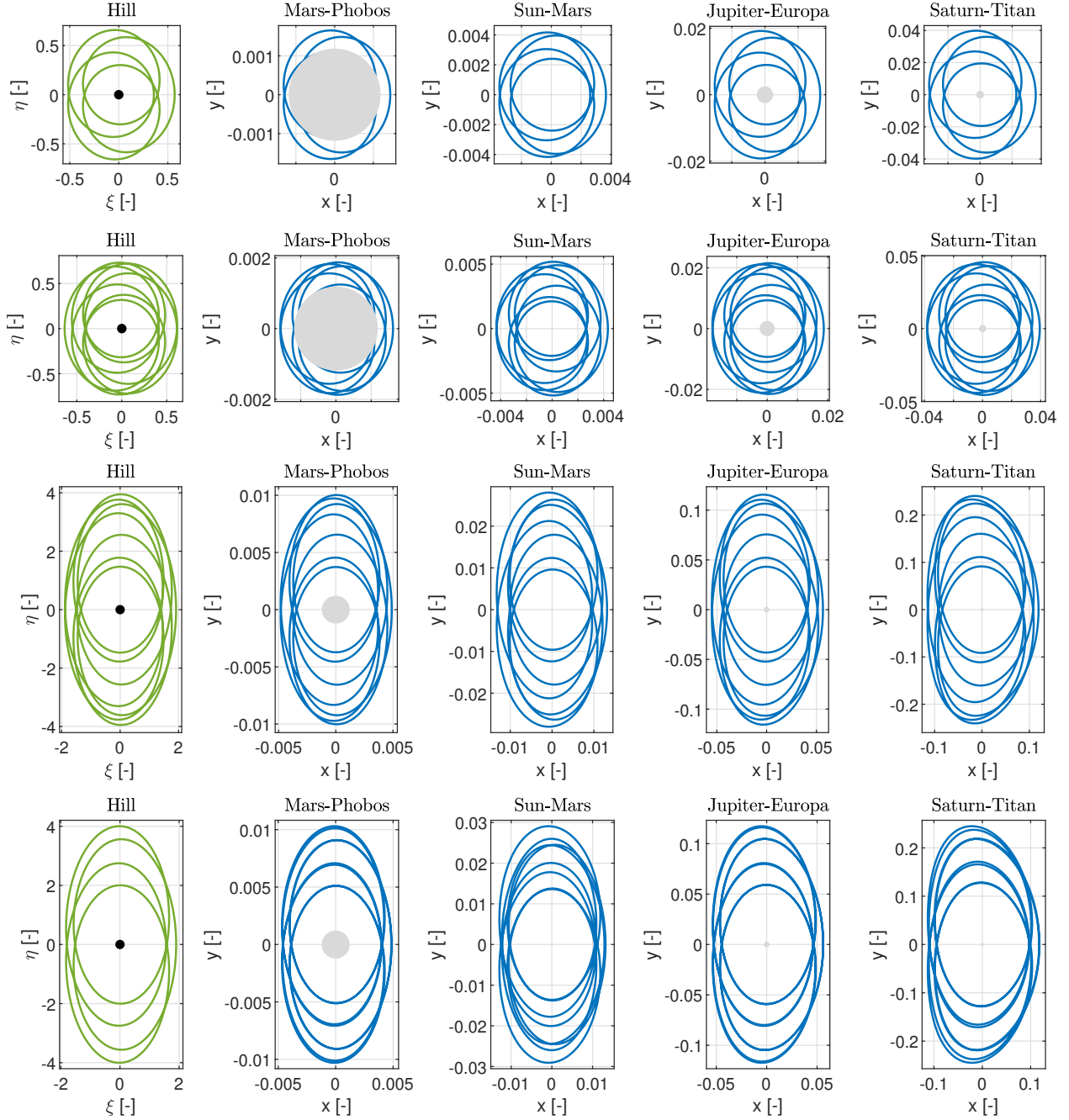


Figure D.9: RNSS swing QSOs in the H3BP (green) and ER3BP (blue), plotted in the $\mathcal{L}$ frame. From top to bottom: 1:4:1, 2:7:1, 6:7:1 and 7:8:2 resonance.

D.5 Left-wide-single-symmetric Swing QSOs

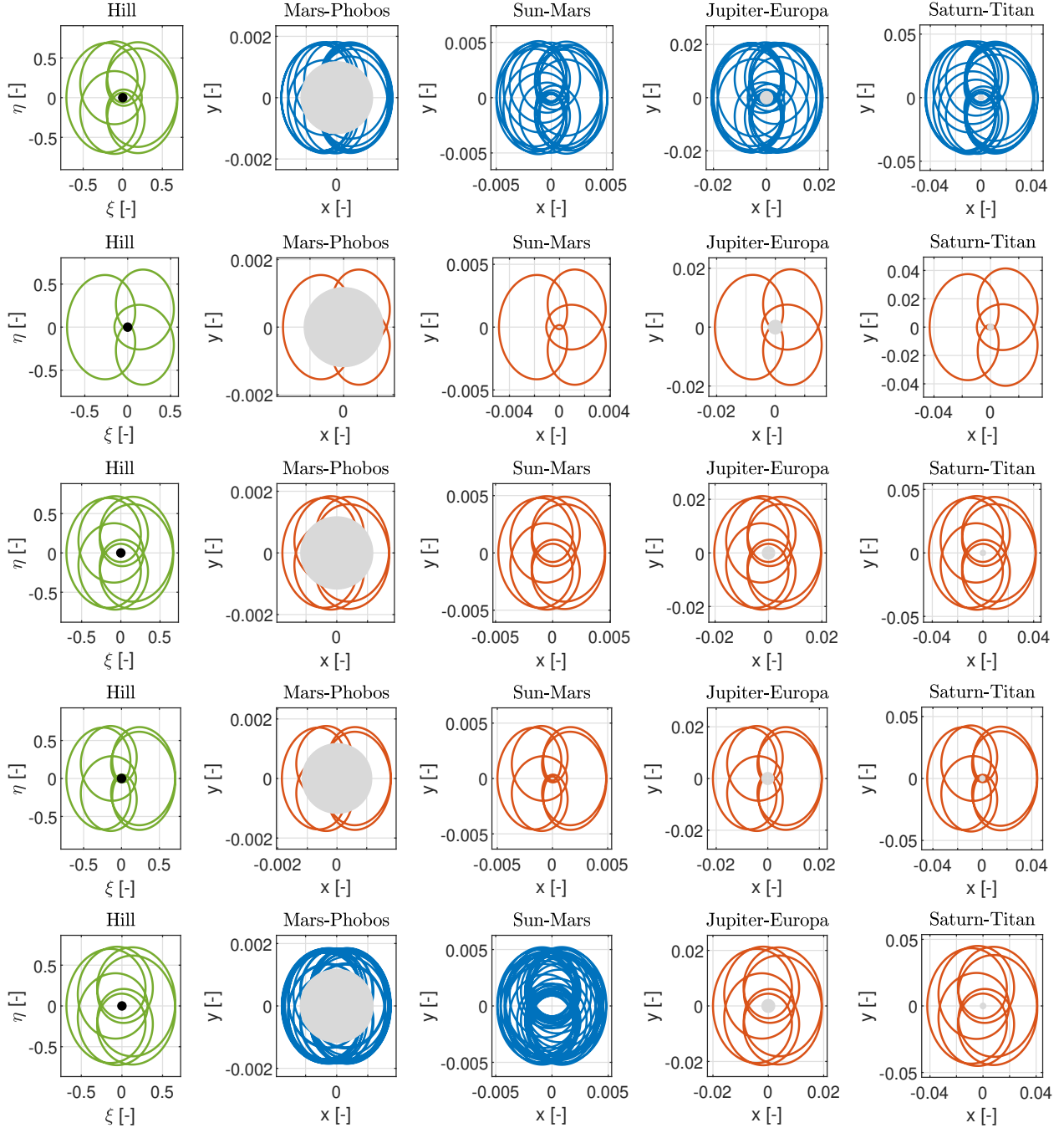


Figure D.10: LWSS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 5:21:3, 5:24:6, 7:28:4, 8:35:5 and 9:35:5 resonances.

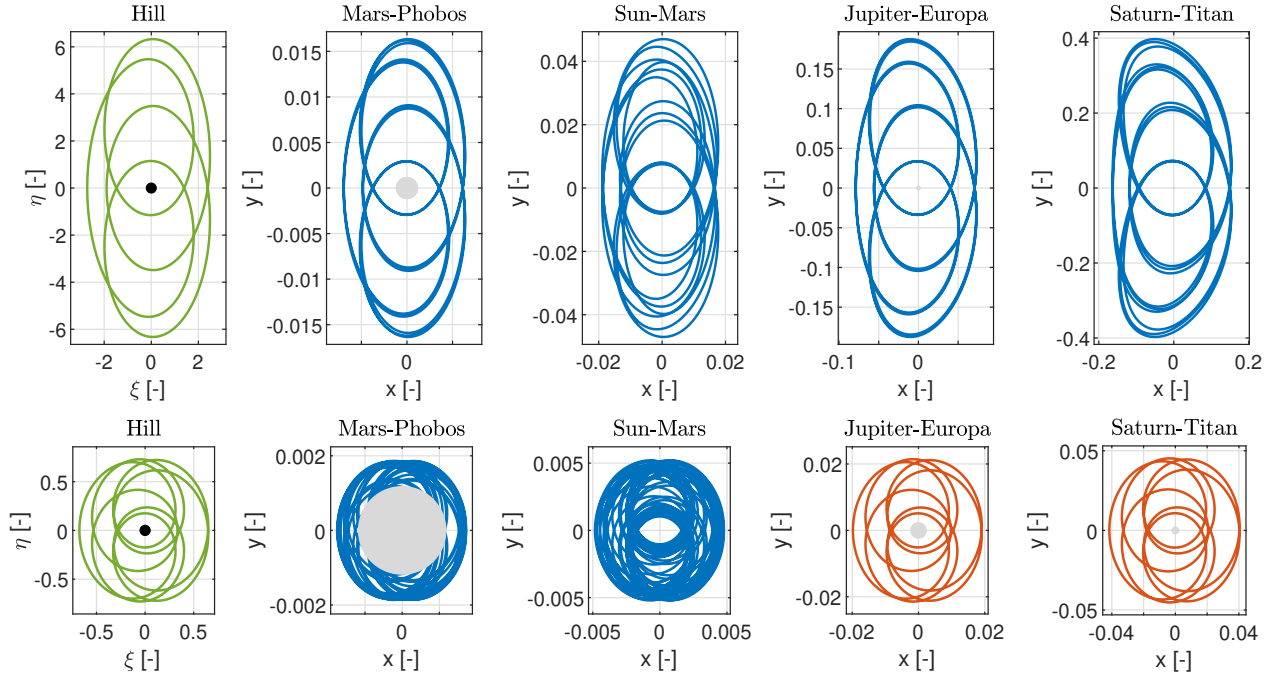


Figure D.11: LWSS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 11:12:3 and 11:42:6 resonances.

D.6 Right-wide-single-symmetric Swing QSOs

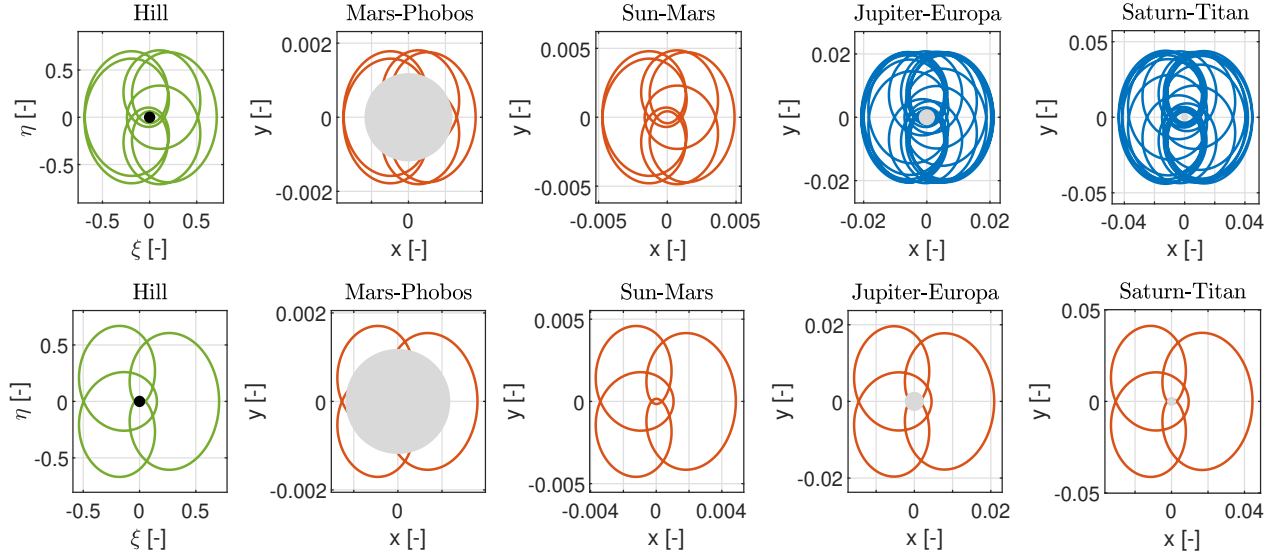


Figure D.12: RWSS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 5:21:3 and 5:24:6 resonances.

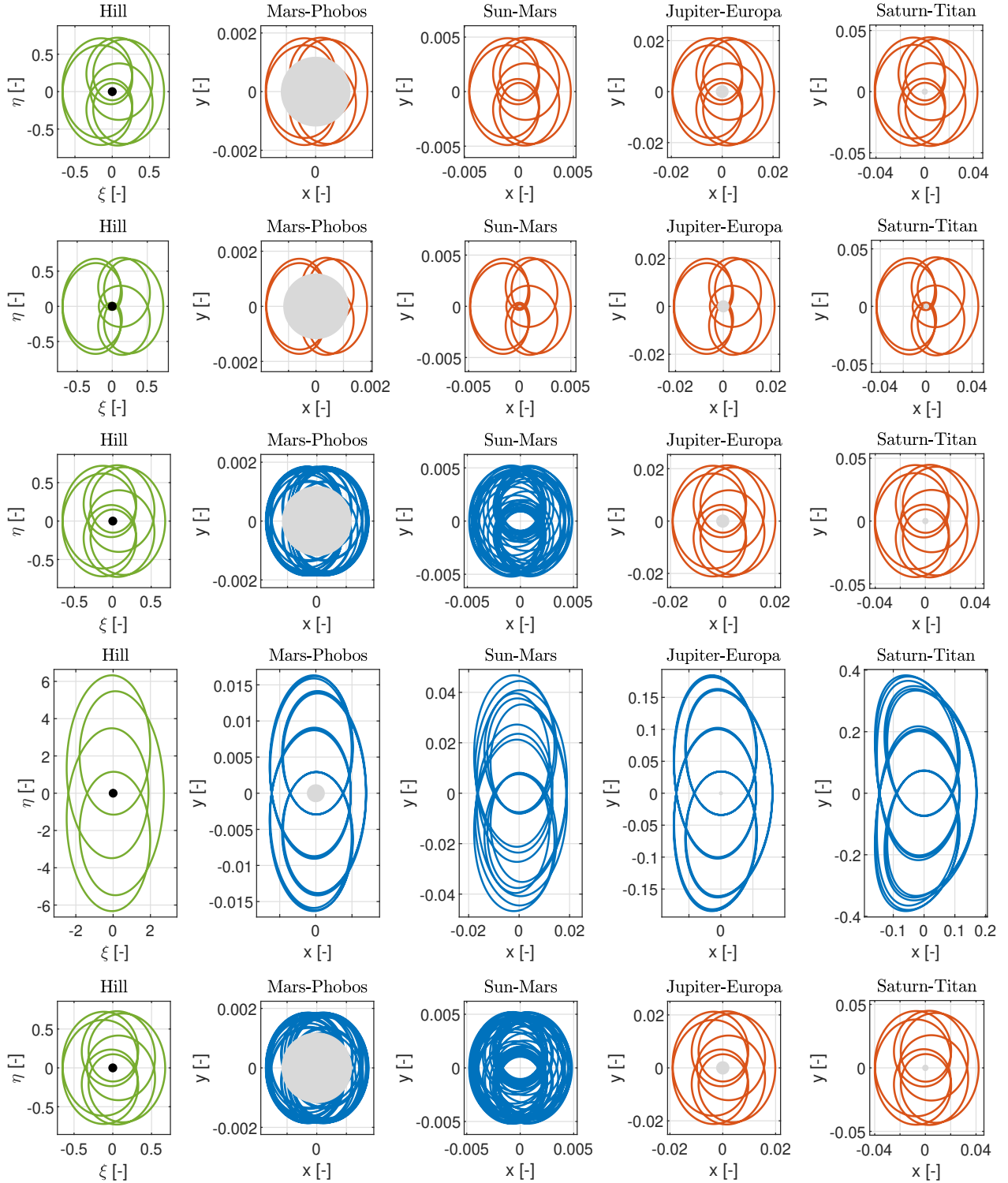


Figure D.13: RWSS swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 7:28:4, 8:35:5, 9:35:5, 11:12:3 and 11:42:6 resonances.

D.7 Narrow-asymmetric Swing QSOs

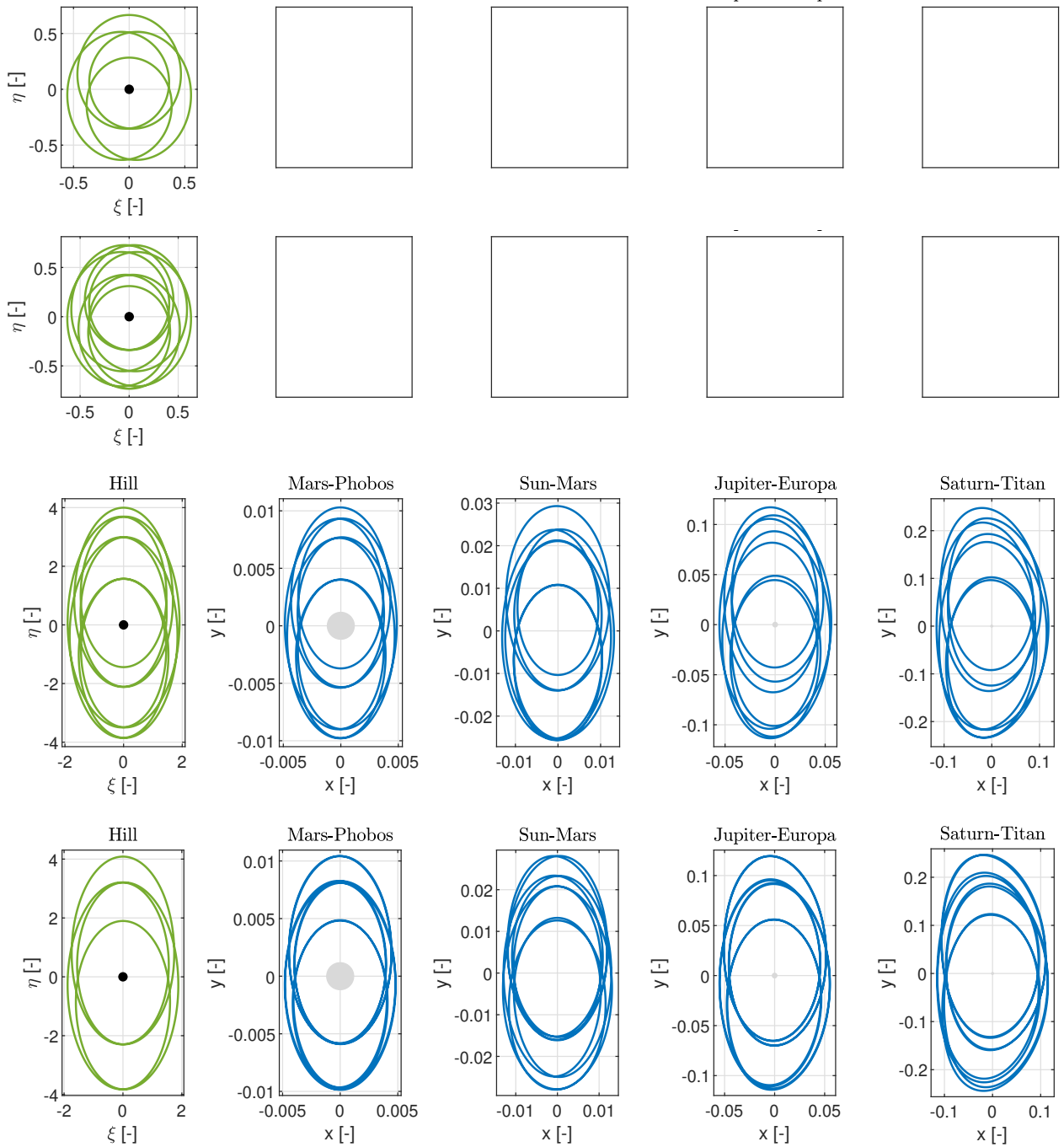


Figure D.14: NA swing QSOs in the H3BP (green) and ER3BP (blue), plotted in the $\mathcal{L}$ frame. From top to bottom: 1:4:1, 2:7:1, 6:7:1 and 7:8:2 resonances.

D.8 Wide-asymmetric Swing QSOs



Figure D.15: WA swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 5:21:3, 5:24:6, 7:28:4, 8:35:5 and 9:35:5 resonances.

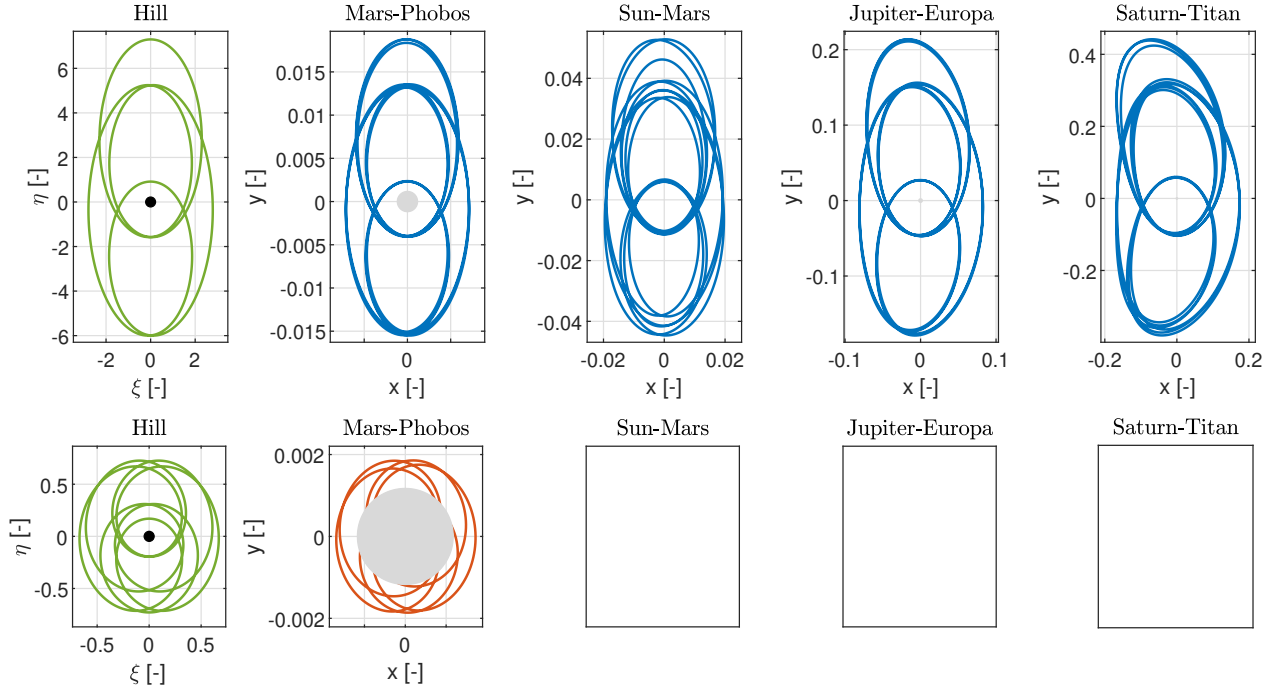


Figure D.16: WA swing QSOs in the H3BP (green), ER3BP (blue) and CR3BP (red), plotted in the $\mathcal{L}$ frame. From top to bottom: 11:12:3 and 11:42:6 resonances.

Appendix E

Initial Conditions of Quasi-Satellite Orbits

E.1 Steady QSOs

E.1.1 H3BP

Res.	ξ_0	$\dot{\eta}_0$
1:2	-0.79704264475925	2.19615537549956
1:3	-0.56614331750784	2.03606181798287
2:3	-1.06588835216964	2.52237853252648
2:5	-0.65705028140846	2.08094196532085
3:4	-1.24244387331954	2.78610605763434
3:5	-0.94970813479911	2.36789010874616
3:7	-0.69635491595878	2.10832143766013
3:8	-0.62289295591360	2.06082975333103
3:10	-0.52063068744336	2.02529715709564
4:5	-1.37856526812331	3.00716438789700
4:7	-0.90406754621451	2.31239745209144
4:9	-0.71837542121151	2.12547177452354
5:6	-1.49149842464224	3.19922311710374
5:7	-1.16103137099557	2.66076468674383
5:8	-0.99146257219825	2.42135619272517
5:9	-0.87951212913303	2.28391954694529
6:7	-1.58915529343717	3.37025262946208
7:8	-1.67586547927868	3.52523543403980
7:9	-1.31414099473864	2.90093590413639
7:10	-1.13111010907782	2.61623018152424
8:9	-1.75427834231616	3.66749992952298
9:10	-1.82614498818691	3.79938920930938

E.1.2 Mars–Phobos

Res.	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:2	-0.00203311409969	0.00561617149021	-0.00206739080331	0.00561785686032
1:3	-0.00144424541115	0.00520324553422	-0.00146705261075	0.00518125589598
2:3	-0.00271877690266	0.00645572984534	-0.00273351769707	0.00645153171425
2:5	-0.00167609236635	0.00531934422301	-0.00170258400240	0.00530687501138
3:4	-0.00316912012037	0.00713474325908	-0.00321080406305	0.00715991278426
3:5	-0.00242246163687	0.00605814289142	-0.00247714032852	0.00608460621416
3:7	-0.00177633149671	0.00538995190880	-0.00180480607964	0.00538139083803
3:8	-0.00158897945069	0.00526740872552	-0.00161377539673	0.00525154289473
3:10	-0.00132816676458	0.00517505417311	-0.00134874804632	0.00514832873964
4:5	-0.00351636371596	0.00770427240694	-0.00356543398954	0.00773957945102
4:7	-0.00230606102551	0.00591534103977	-0.00235033617442	0.00593069252652
4:9	-0.00183249029221	0.00543414844052	-0.00186212404820	0.00542778204829
5:6	-0.00380447592007	0.00819939177764	-0.00385886955352	0.00824202214387
5:7	-0.00296145312518	0.00681197398892	-0.00299720641338	0.00682949505089
5:8	-0.00252895327064	0.00619573269480	-0.00262536087656	0.00626164555884
5:9	-0.00224343635898	0.00584205596828	-0.00228458624627	0.00585358784249
6:7	-0.00405362868958	0.00864054342309	-0.00411237656884	0.00868905248341
7:8	-0.00427486134441	0.00904050566467	-0.00433736182412	0.00909397910242
7:9	-0.00335201459506	0.00743054231990	-0.00339779460333	0.00746129902769
7:10	-0.00288513289830	0.00669731741316	-0.00291796527174	0.00671134801552
8:9	-0.00447492957818	0.00940781523581	-0.00454079894571	0.00946566697053
9:10	-0.00465829894518	0.00974848241010	-0.00472715500284	0.00981014897502

E.1.3 Sun–Mars

Res.	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:2	-0.00546327973042	0.01515612806123	-0.00606928578541	0.01529217473349
1:3	-0.00388142450138	0.01402568978071	-0.00432416316369	0.01373581782546
2:3	-0.00730522383292	0.01744618027522	-	-
2:5	-0.00450424708998	0.01434505982658	-0.00496707959332	0.01422691274726
3:4	-0.00851526212267	0.01929970677147	-0.00925733818065	0.01983562440922
3:5	-0.00650916847335	0.01636167626812	-0.00744440236593	0.01694876062646
3:7	-0.00477351326417	0.01453830459888	-0.00527467194730	0.01448954616205
3:8	-0.00427023625048	0.01420260717442	-0.00469514106404	0.01402515803086
3:10	-0.00356957655388	0.01394658743847	-0.00394615155106	0.01357670764910
4:5	-0.00944843885873	0.02085612118237	-0.01032011123254	0.02158021105491
4:7	-0.00619647746556	0.01597221474410	-0.00697455724341	0.01636179476853
4:9	-0.00492436732192	0.01465912231441	-0.00544711257222	0.01464972087634
5:6	-0.01022279980988	0.02221060594785	-0.01118913090342	0.02307245208406
5:7	-0.00795724704391	0.01841837238887	-0.00860525521293	0.01882260208730
5:8	-0.00679525014045	0.01673693916063	-0.00799902314953	0.01765369708276
5:9	-0.00602825037079	0.01577233935608	-0.00675387341130	0.01609047498997
6:7	-0.01089250494320	0.02341859465853	-0.01193622111541	0.02439155000120
7:8	-0.01148719730611	0.02451473624102	-0.01259734055252	0.02558166060374
7:9	-0.00900675324187	0.02010785617601	-0.00982032627102	0.02074708620267
7:10	-0.00775218226392	0.01810541556198	-0.00836938610057	0.01846606782775
8:9	-0.01202501990497	0.02552218110274	-0.01320803178429	0.02669167932939
9:10	-0.01251796699906	0.02645723131910	-0.01374285495579	0.02768366969028

E.1.4 Jupiter–Europa

Res.	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:2	-0.02332421498911	0.06617791534837	-0.02355853309057	0.06620375002327
1:3	-0.01658275945134	0.06087142254685	-0.01673896182889	0.06072434772545
2:3	-0.03117491873203	0.07674619383174	-0.03129080633914	0.07673966605607
2:5	-0.01923758893085	0.06240488351371	-0.01941971218934	0.06232641987980
3:4	-0.03633707101836	0.08533755505618	-0.03662143517889	0.08554377222173
3:5	-0.02778118079346	0.07174010237707	-0.02815343214927	0.07195248775416
3:7	-0.02038508327675	0.06331078166044	-0.02058045281107	0.06326099173421
3:8	-0.01824022050062	0.06173013038875	-0.01841118471008	0.06162672465078
3:10	-0.01525302351084	0.06045693924377	-0.01539517231244	0.06027476730817
4:5	-0.04032105894469	0.09259604312973	-0.04065531994265	0.09288377209320
4:7	-0.02644851765791	0.06994416666411	-0.02675018836866	0.07007232104052
4:9	-0.02102791028370	0.06387397233098	-0.02123105791952	0.06384032296051
5:6	-0.04362863029337	0.09894864411905	-0.04399883761533	0.09929700876172
5:7	-0.03395595994161	0.08124584437202	-0.03420003991442	0.08139180654465
5:8	-0.02900061675231	0.07347114600329	-0.02967968109597	0.07400444180682
5:9	-0.02573161688682	0.06902242771760	-0.02601218934993	0.06912180749850
6:7	-0.04649004797189	0.10464313638701	-0.04688965315612	0.10504140690610
7:8	-0.04903141881291	0.10983433561532	-0.04945631718010	0.11027562131007
7:9	-0.03843508536235	0.08910106596224	-0.03874712821435	0.08935189615700
7:10	-0.03308116753787	0.07979587782793	-0.03330497430947	0.07991414855172
8:9	-0.05132998203582	0.11462574048194	-0.05177875404452	0.11510754315164
9:10	-0.05343683639945	0.11909030042470	-0.05390490059607	0.11960511808713

E.1.5 Saturn–Titan

Res.	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:2	-0.04899080563432	0.14367130598134	-0.05059971545321	0.14408604393503
1:3	-0.03486850501353	0.13096528367648	-0.03597773156881	0.13008029203726
2:3	-0.06543336009793	0.16848700179996	-0.06580977323768	0.16804931051163
2:5	-0.04043187353842	0.13473539051596	-0.04168489267826	0.13436392138324
3:4	-0.07625307869902	0.18881327240227	-0.07819743456940	0.19063216581836
3:5	-0.05832418265154	0.15672169726533	-0.06086696610183	0.15862124167899
3:7	-0.04283557832622	0.13690064890628	-0.04418043061146	0.13674321613188
3:8	-0.03834222853041	0.13310259306717	-0.03951469771695	0.13254712325743
3:10	-0.03208045379117	0.12984781850387	-0.03307919866427	0.12870837348605
4:5	-0.08460750946570	0.20614854157568	-0.08688708281765	0.20865342587435
4:7	-0.05553322805796	0.15250924748697	-0.05759915588308	0.15372657989267
4:9	-0.04418195585005	0.13823770073667	-0.04558042179727	0.13820189823361
5:6	-0.09154451445738	0.22145111820532	-0.09406611954566	0.22448508774394
5:7	-0.07126143198534	0.17910835890530	-0.07294170194677	0.18044322051813
5:8	-0.06087831677265	0.16078512096580	-0.06501934943643	0.16478475520690
5:9	-0.05403195385890	0.15034771368797	-0.05595399330799	0.15133476719039
6:7	-0.09754523391531	0.23527450435889	-0.10026428697938	0.23875604173794
7:8	-0.10287343903348	0.24796434443041	-0.10575986123852	0.25183685941595
7:9	-0.08065231179546	0.19778202861831	-0.08278252983291	0.19997312459840
7:10	-0.06942793956931	0.17568018156293	-0.07098603917145	0.17680987079216
8:9	-0.10769086934928	0.25975187312303	-0.11075879961051	0.26404497977974
9:10	-0.11210464295227	0.27080026244540	-0.11529111635029	0.27539118535665

E.2 Swing QSOs

E.2.1 H3BP

Symmetric Resonance	CR3BP	
	ξ_0	$\dot{\eta}_0$
1:5:1 IDS	-0.32382381506633	2.24501073437046
2:3:1 IDS	-0.97716364840151	2.44850205523003
3:6:2 IDS	-0.69406874079327	2.17447909337136
4:9:3 IDS	-0.66410845737864	2.12526197867405
5:9:3 IDS	-0.75087469778889	2.23034353702370
5:12:4 IDS	-0.65711861911485	2.09871167437477
6:15:5 IDS	-0.65516536553453	2.08126180431543
7:12:4 IDS	-0.79190701181407	2.26667390276233
7:15:5 IDS	-0.67333891419936	2.14512022165445
8:15:5 IDS	-0.72442055148143	2.20610397681447
9:15:5 IDS	-0.82116691089266	2.29291692543751
11:18:6 IDS	-0.84273087198380	2.31272770746125
1:3:1 EDS	-0.66336502150148	1.98985451565062
1:5:1 EDS	-0.44140912132909	1.91450190373556
3:12:4 EDS	-0.72416549946526	1.68354220793484
4:15:5 EDS	-0.70349550349119	1.79873931105017
5:6:2 EDS	-1.80184408988783	3.63979578881358
5:18:6 EDS	-0.69338714674539	1.84784374256675
7:9:3 EDS	-1.41491235542710	3.02466400808847
7:18:6 EDS	-0.65475532408708	2.06876410457611
8:9:3 EDS	-2.61951344372882	5.07051486591962
9:12:4 EDS	-1.27741831731141	2.82568177867984
11:12:4 EDS	-3.84148700507324	7.31575755342637
11:15:5 EDS	-1.20573747232516	2.72766480794969
1:4:1 LNSS	-0.66336502150148	1.98985451565062
2:7:1 LNSS	-0.63561008038315	1.91321769875571
6:7:1 LNSS	-1.91481999837931	3.83894039033950
7:8:2 LNSS	-1.90008603962666	3.84649078690934

Symmetric Resonance	CR3BP	
	ξ_0	$\dot{\eta}_0$
1:4:1 RNSS	0.56990571006806	-1.87240038966146
2:7:1 RNSS	0.63561008038315	-1.91321769875571
6:7:1 RNSS	1.91481999837931	-3.83894039033950
7:8:2 RNSS	1.90008603962704	-3.84649078690994
5:21:3 LWSS	-0.71206836946093	1.71416628087540
5:24:6 LWSS	-0.70646647170224	1.55509924779773
7:28:4 LWSS	-0.69389178764583	1.78178905145163
8:35:5 LWSS	-0.72783759699297	1.63533963456323
9:35:5 LWSS	-0.68307680939814	1.81384157477284
11:12:3 LWSS	-2.72015398700810	5.27032165273052
11:42:6 LWSS	-0.67575972714289	1.83294105148134
5:21:3 RWSS	0.71206836946093	-1.71416628087540
5:24:6 RWSS	0.70646647170224	-1.55509924779773
7:28:4 RWSS	0.69389178764583	-1.78178905145163
8:35:5 RWSS	0.72783759699297	-1.63533963456323
9:35:5 RWSS	0.68307680939814	-1.81384157477284
11:12:3 RWSS	2.72015398700810	-5.27032165273052
11:42:6 RWSS	0.67575972714289	-1.83294105148134

Asymmetric Resonance	CR3BP	
	η_0	$\dot{\xi}_0$
1:4:1 NA	0.28378411587450	2.46369982918650
2:7:1 NA	0.31103292749003	2.39416089556351
6:7:1 NA	1.44018798538296	2.02053385161064
7:8:2 NA	1.89778802852181	1.99182567186960
5:21:3 WA	0.06094409113805	5.60601367228688
5:24:6 WA	0.67988274289761	1.02688561797944
7:28:4 WA	0.11174043711419	4.09231783496871
8:35:5 WA	0.02607066668901	8.66361659896592
9:35:5 WA	0.14559948418591	3.56353504552658
11:12:3 WA	0.91533710987191	2.70679823370164
11:42:6 WA	0.16962250593277	3.28953710321003

E.2.2 Mars–Phobos

Symmetric Resonance	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:5:1 IDS	-0.00082613275367	0.00573330470048	-0.00083844050207	0.00567702722972
2:3:1 IDS	-0.00249274417809	0.00626504372047	-0.00256617581713	0.00630518674748
3:6:2 IDS	-0.00177051057607	0.00555898187256	-0.00179977314025	0.00554603987613
4:9:3 IDS	-0.00169407414268	0.00543271369998	-0.00172147257514	0.00541861459249
5:9:3 IDS	-0.00191546955523	0.00570275639235	-0.00194885784574	0.00569485611253
5:12:4 IDS	-0.0016762552626	0.00536475423479	-0.00170312135206	0.00535126193124
6:15:5 IDS	-0.00167128396936	0.00532013135902	-0.00169786007426	0.00530737131964
7:12:4 IDS	-0.00202017240801	0.00579638484395	-0.00205722212341	0.00579311537632
7:15:5 IDS	-0.00171761880311	0.00548361027463	-0.00174558668491	0.00546958029667
8:15:5 IDS	-0.00184796270663	0.00564032883215	-0.00187930866218	0.00562984501133
9:15:5 IDS	-0.00209482928311	0.00586403422584	-0.00213492139521	0.00586452651330
11:18:6 IDS	-0.00214984600601	0.00591510417085	-0.00219243178518	0.00591862064761
1:3:1 EDS	-0.00169224614567	0.00508668199442	-0.00171677518082	0.00507993786986
1:5:1 EDS	-0.00112614358223	0.00489081760289	-0.00114259755674	0.00486193972148
3:12:4 EDS	-0.00184697966017	0.00430483023453	-0.00185789121452	0.00431375050244
4:15:5 EDS	-0.00179443641796	0.00459893412366	-0.00180983709466	0.00460381354340
5:6:2 EDS	-0.0045944002764	0.0093360890721	-0.0046606118154	0.009401449699
5:18:6 EDS	-0.0017687236126	0.004724266929	-0.00178664078472	0.00472681079933
7:9:3 EDS	-0.00360855197696	0.00774950006466	-0.00365546259768	0.00778464153397
7:18:6 EDS	-0.00167024675800	0.00528818803598	-0.00169660160116	0.00527607424582
8:9:3 EDS	-0.00667687854961	0.0130372380069	-0.00677925142815	0.0131566747136
9:12:4 EDS	-0.00325816615320	0.00723675964720	-0.00329818831510	0.00726112852708
11:12:4 EDS	-0.00978658315703	0.0188781004639	-0.00994110574835	0.0190713418146
11:15:5 EDS	-0.00307547429294	0.00698424217252	-0.00311329015150	0.00700466338964
1:4:1 LNSS	-0.00145365414181	0.00478537374363	-0.00145749412256	0.00478808444438
2:7:1 LNSS	-0.00162146013989	0.00489042059309	-0.00164326270301	0.00488378364962
6:7:1 LNSS	-0.00488232257853	0.00985029594888	-0.00495575231214	0.00992623968939
7:8:2 LNSS	-0.0048454473661	0.0098700797237	-0.0049117868575	0.0099330212440

Symmetric Resonance	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:4:1 RNSS	0.00145455162361	-0.00476942871613	0.00145839378190	-0.00477213400828
2:7:1 RNSS	0.00162203184691	-0.00487267911352	0.00164387085818	-0.00486582378145
6:7:1 RNSS	0.00488889104906	-0.00974014452067	-0.00495575231214	0.00992623968939
7:8:2 RNSS	0.00485056404617	-0.00975888745910	0.00491699046361	-0.00981949796604
5:21:3 LWSS	-0.00181622782097	0.00438277474051	-0.00183919989151	0.00439032601373
5:24:6 LWSS	-0.00180143095943	0.00397714738519	-	-
7:28:4 LWSS	-0.00176996900630	0.00455540474312	-	-
8:35:5 LWSS	-0.00185633769118	0.00418139497487	-	-
9:35:5 LWSS	-0.00174243063985	0.00463717790009	-0.00168785478422	0.00464745803894
11:12:3 LWSS	-0.00693378563958	0.01355602501629	-0.00703514625775	0.01367153857393
11:42:6 LWSS	0.00463717790009	0.00468588574164	-0.00168022560971	0.00469485703682
5:21:3 RWSS	0.00181742872925	-0.00436457496428	-	-
5:24:6 RWSS	0.00180364123728	-0.00395848277249	-	-
7:28:4 RWSS	0.00177093297923	-0.00453702204362	-	-
8:35:5 RWSS	0.00185778869181	-0.00416370343233	-	-
9:35:5 RWSS	0.00174328280437	-0.00461881162280	0.00169023193654	-0.00462930565614
11:12:3 RWSS	0.00694695179609	-0.01333956756430	0.00704853507601	-0.01344989101901
11:42:6 RWSS	0.00172458141285	-0.00466756749833	0.00168241359370	-0.00467667802296

Asymmetric Resonance	CR3BP		
	ρ_0	θ_0	ρ'_0
1:4:1 NA	-	-	-
2:7:1 NA	-	-	-
6:7:1 NA	-3.9425313085e-06	0.00367466294787	0.00515533828672
7:8:2 NA	-5.1697590752e-06	0.00484227725531	0.00508208940258
5:21:3 WA	8.394487607741e-08	0.000155523985190	0.01430236132731
5:24:6 WA	-1.03804445409e-07	4.24036255312e-05	0.02776954475260
7:28:4 WA	6.39620396583e-07	0.000285358220541	0.01043646679190
8:35:5 WA	2.83969417111e-06	8.16655887334e-05	0.01989367142368
9:35:5 WA	1.77268173130e-06	0.00037249300487	0.00907903072688
11:12:3 WA	-7.2848963539e-06	0.00233550233548	0.00690630856522
11:42:6 WA	3.52071802235e-06	0.00043551982585	0.00836430398866

Asymmetric Resonance	ER3BP		
	ρ_0	θ_0	ρ'_0
1:4:1 NA	-	-	-
2:7:1 NA	-	-	-
6:7:1 NA	-4.55768152429e-06	0.00376350720416	0.00513177785992
7:8:2 NA	-5.228486459e-06	0.0049340318086	0.0050828333906
5:21:3 WA	-	-	-
5:24:6 WA	-	-	-
7:28:4 WA	-	-	-
8:35:5 WA	-	-	-
9:35:5 WA	-	-	-
11:12:3 WA	-7.27983216697e-06	0.00240146314866	0.00683044393766
11:42:6 WA	-	-	-

E.2.3 Sun–Mars

Symmetric Resonance	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:5:1 IDS	-0.00222047520987	0.01543661598051	-0.00236940511019	0.01477019971600
2:3:1 IDS	-0.00669912014105	0.01692348148256	-	-
3:6:2 IDS	-0.00475790902729	0.01499375701188	-0.00529944295415	0.01489086400029
4:9:3 IDS	-0.00455245851263	0.01465114536738	-0.00505793978401	0.01451840560524
5:9:3 IDS	-0.00514768719678	0.01538593425654	-0.00577034373134	0.01538936261709
5:12:4 IDS	-0.00450462798986	0.01446745922757	-0.00499262295045	0.01433908253474
6:15:5 IDS	-0.00449132067954	0.01434704402040	-0.00496707958908	0.01422691274650
7:12:4 IDS	-0.00542920129568	0.01564190740558	-0.00612548111412	0.01574478433099
7:15:5 IDS	-0.00461572211169	0.01478901632892	-0.00513116599955	0.01466093796794
8:15:5 IDS	-0.00496616683048	0.01521544950471	-0.00554827260626	0.0151642878366
9:15:5 IDS	-0.00562990334438	0.01582693545852	-0.00639007466028	0.01601418247994
11:18:6 IDS	-0.00577778856152	0.01596662471438	-0.00659415192887	0.01622610395939
1:3:1 EDS	-0.00454786257977	0.01371843736287	-0.00498295265872	0.01368332637580
1:5:1 EDS	-0.00302696035498	0.01317544117390	-0.00325697282763	0.01286054040893
3:12:4 EDS	-0.00496204151309	0.01161517703269	-0.00517318245236	-0.01175460834348
4:15:5 EDS	-0.00482167365896	0.01240666093869	-0.00511830979040	0.01251056092970
5:6:2 EDS	-0.01233756241783	0.02532429238926	-0.01355168561163	0.02665610282778
5:18:6 EDS	-0.00475290370244	0.01274379873504	-0.00509608896343	0.01282530064407
7:9:3 EDS	-0.00969372470137	0.02098026974025	-0.0105212152071	0.02168552690164
7:18:6 EDS	-0.00562990334453	0.01582693545829	-0.0402791279702	0.1339313125747
8:9:3 EDS	-0.01791830906888	0.03550890673279	-0.15679316488045	0.37942772516268
9:12:4 EDS	-0.00875373518838	0.01957868055018	-0.00942105149455	0.02003881290153
11:12:4 EDS	-0.02623920306941	0.05173364651263	-0.02918145487194	0.05584600733809
11:15:5 EDS	-0.00826351959869	0.0188886736524	-0.07381467957210	0.18402129105083
1:4:1 LNSS	-0.00390585317009	0.01290097117701	-0.00403160063239	0.01290852617401
2:7:1 LNSS	-0.00435770478831	0.01318751619348	-0.00477432972294	0.01314164744209
6:7:1 LNSS	-0.01311001359290	0.02673475554270	-0.01445799435549	0.028274477236923
7:8:2 LNSS	-0.01301400978083	0.02679021991180	-0.01421094670925	0.02805371908537

Symmetric Resonance	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:4:1 RNSS	0.00391233937703	-0.01278573428924	0.00404001558845	-0.01279028724144
2:7:1 RNSS	0.00436183655892	-0.01305929547540	0.00478004364577	-0.01300167273639
6:7:1 RNSS	0.01315748206817	-0.02593860743980	0.01450757428685	-0.02733610809463
7:8:2 RNSS	0.01305098429770	-0.02598655866051	0.01425187674690	-0.02713092439649
5:21:3 LWSS	-0.00487988424505	0.01182385148936	-0.00514522845018	0.01192932054279
5:24:6 LWSS	-0.00483783099368	0.01073439095913	-	-
7:28:4 LWSS	-0.00475606155508	0.01228835870778	-	-
8:35:5 LWSS	-0.00498714586431	0.01128130639191	-	-
9:35:5 LWSS	-0.00468228200149	0.01250815413021	-0.00456846394340	0.01261077039257
11:12:3 LWSS	-0.01860945779414	0.03694550588273	-0.02047428370482	0.03928532517570
11:42:6 LWSS	-0.00463232253786	0.01263898633935	-0.00455125578706	0.01272048042523
5:21:3 RWSS	0.00488856335325	-0.01169231829593	-	-
5:24:6 RWSS	0.00485380496871	-0.01059949871282	-	-
7:28:4 RWSS	0.00476302829278	-0.01215550342632	-	-
8:35:5 RWSS	0.00499763243186	-0.01115344651515	-	-
9:35:5 RWSS	0.00468844068455	-0.01237541755499	0.00457691719293	-0.01248056167907
11:12:3 RWSS	0.01870460314248	-0.03538084500528	0.02058016539732	-0.03744867809650
11:42:6 RWSS	0.00463801936786	-0.01250659694363	0.00455924935135	-0.01258988578195

Asymmetric Resonance	CR3BP		
	ρ_0	θ_0	ρ'_0
1:4:1 NA	-	-	-
2:7:1 NA	-	-	-
6:7:1 NA	-2.84900982773e-05	0.00987967645716	0.01385894813376
7:8:2 NA	-3.73674744711e-05	0.01301974708562	0.01366200106967
5:21:3 WA	6.270226164e-07	0.00041855697265	0.03842731152684
5:24:6 WA	-7.539740900e-07	0.00011424440239	0.07456977524690
7:28:4 WA	5.3148656793e-06	0.00077244870747	0.02795287834998
8:35:5 WA	6.8115807923e-06	0.00021548044516	0.05399886369027
9:35:5 WA	-	-	-
11:12:3 WA	-5.26472594581e-05	0.00627936340385	0.01856569235446
11:42:6 WA	-	-	-

Asymmetric Resonance	ER3BP		
	ρ_0	θ_0	ρ'_0
1:4:1 NA	-	-	-
2:7:1 NA	-	-	-
6:7:1 NA	-3.35408348731e-05	0.01141807078701	0.01360824550153
7:8:2 NA	-3.90170992706e-05	0.01459821515983	0.01378269645630
5:21:3 WA	-	-	-
5:24:6 WA	-	-	-
7:28:4 WA	-	-	-
8:35:5 WA	-	-	-
9:35:5 WA	-	-	-
11:12:3 WA	-5.28879806524e-05	0.00740380310614	0.01750284962430
11:42:6 WA	-	-	-

E.2.4 Jupiter–Europa

Symmetric Resonance	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:5:1 IDS	-0.00949275341668	0.06658429178925	-0.00957467906122	0.06620432829018
2:3:1 IDS	-0.02860874530415	0.07426880946690	-0.02909180124165	0.07457325807631
3:6:2 IDS	-0.02031661237204	0.06528287256379	-0.02051652650160	0.06520291132962
4:9:3 IDS	-0.01943939704921	0.06374441259949	-0.01962707099069	0.06365510480266
5:9:3 IDS	-0.02198475197949	0.06709119446605	-0.02221215626438	0.06704899423118
5:12:4 IDS	-0.01923676374183	0.06293592642772	-0.01942105234588	0.06285047623583
6:15:5 IDS	-0.01918131060963	0.06241040969786	-0.01936360841756	0.06232989873984
7:12:4 IDS	-0.02318929997009	0.0682847629608	-0.02344114993593	0.06827703848105
7:15:5 IDS	-0.01970898747535	0.06435823618000	-0.01990038027396	0.06426983306810
8:15:5 IDS	-0.02120779800298	0.06630054117524	-0.02142153772813	0.06623903335860
9:15:5 IDS	-0.02404751591209	0.06914939150819	-0.02431957686544	0.06916959310656
11:18:6 IDS	-0.02467951505137	0.06980236396866	-0.02496803492809	0.06984494205174
1:3:1 EDS	-0.01942815857827	0.05969810840447	-0.01959315782961	0.05965919082612
1:5:1 EDS	-0.01294093135048	0.05699945855627	-0.01304847396704	0.05681225711613
3:12:4 EDS	-0.02116157537412	0.05066492055169	-0.02123600528805	0.05072996836916
4:15:5 EDS	-0.02058035240433	0.05407218893726	-0.02068500721655	0.05411085271201
5:6:2 EDS	-0.05246207017748	0.11361640822746	-0.05290930281200	0.11414494859689
5:18:6 EDS	-0.02029380897303	0.05551975456577	-0.02041529391896	0.05554315757342
7:9:3 EDS	-0.04130321899699	0.09316871255122	-0.04162244684668	0.09345313254232
7:18:6 EDS	-0.02404751591214	0.06914939150822	-0.01935113236080	0.06196010263691
8:9:3 EDS	-0.07589596983274	0.1628313123240	-0.07658055362484	0.1638414104581
9:12:4 EDS	-0.03732716584659	0.08662851337807	-0.03760169307366	0.08683036793055
11:12:4 EDS	-0.11039894919138	0.24504360599376	-0.11140881665986	0.24679983485710
11:15:5 EDS	-0.03525114418227	0.08341553187986	-0.03551230752935	0.08358817997347
1:4:1 LNSS	-0.01666789349328	0.05602740434974	-0.01669208026852	0.05604781105983
2:7:1 LNSS	-0.01861762030187	0.05735051819351	-0.01876415814641	0.05731132210283
6:7:1 LNSS	-0.05572199445502	0.1203120248052	-0.05621914505484	0.12093263120176
7:8:2 LNSS	-0.05535606921386	0.12056758700649	-0.05580433084015	0.121085202684

Symmetric Resonance	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:4:1 RNSS	0.01678667342654	-0.05391697233617	0.01681079574249	-0.05393737964262
2:7:1 RNSS	0.01869327401014	-0.05500207250731	0.01884268371985	-0.05494587882629
6:7:1 RNSS	0.05658956643226	-0.10571361132784	0.05709091698624	-0.10612130650938
7:8:2 RNSS	0.05602995261025	-0.1058414567816	0.05648499078570	-0.10617608170902
5:21:3 LWSS	-0.02082123512838	0.05153925251945	0.02056252234931	-0.04919448055346
5:24:6 LWSS	-0.02059151142802	0.04689567004109	-	-
7:28:4 LWSS	-0.02030320315146	0.05353689121013	-	-
8:35:5 LWSS	-0.02126764534259	0.04919016657846	-	-
9:35:5 LWSS	-0.01999302027811	0.05447662104304	-	-
11:12:3 LWSS	-0.07885140787544	0.17000936863905	-0.07953081068938	0.17100258087854
11:42:6 LWSS	-0.01978237561575	0.05503394330213	-	-
5:21:3 RWSS	0.02098017821318	-0.04913000669557	0.02056252234931	-0.04919448055346
5:24:6 RWSS	0.02088406488474	-0.04442502812367	-	-
7:28:4 RWSS	0.02043078194729	-0.05110340515317	-	-
8:35:5 RWSS	0.02145969503851	-0.04684828873170	-	-
9:35:5 RWSS	0.02010579846567	-0.05204531770217	-	-
11:12:3 RWSS	0.08058914950605	-0.14127203412025	0.08128588004983	-0.14185664231995
11:42:6 RWSS	0.01988669446608	-0.05260901193476	-	-

Asymmetric Resonance	CR3BP		
	ρ_0	θ_0	ρ'_0
1:4:1 NA	-	-	-
2:7:1 NA	-	-	-
6:7:1 NA	-0.0005200302950760	0.04241377002428	0.05926327838147
7:8:2 NA	-0.0006873815877970	0.05603027503675	0.05840936930705
5:21:3 WA	-	-	-
5:24:6 WA	-1.54036454272e-05	0.00051279631957	0.3113672220463
7:28:4 WA	-	-	-
8:35:5 WA	1.16851890529e-06	0.00077081364385	0.2533394224675
9:35:5 WA	-	-	-
11:12:3 WA	-0.0009630252429097	0.02699345745167	0.07932766851222
11:42:6 WA	-	-	-

Asymmetric Resonance	ER3BP		
	ρ_0	θ_0	ρ'_0
1:4:1 NA	-	-	-
2:7:1 NA	-	-	-
6:7:1 NA	-0.0006375793408281	0.04323428725872	0.05905371963456
7:8:2 NA	-0.0006921386634135	0.05666047889895	0.05841463535799
5:21:3 WA	-	-	-
5:24:6 WA	-	-	-
7:28:4 WA	-	-	-
8:35:5 WA	-	-	-
9:35:5 WA	-	-	-
11:12:3 WA	-0.000962577330919	0.02744686591200	0.07880330472718
11:42:6 WA	-	-	-

E.2.5 Saturn–Titan

Symmetric Resonance	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:5:1 IDS	-0.01998860497962	0.14193300557437	-0.02047525174653	0.13968235467222
2:3:1 IDS	-0.06000708023422	0.16235926391803	-0.06376454547226	0.16559871660495
3:6:2 IDS	-0.04264778178660	0.14111813265961	-0.04403974059526	0.14076544136999
4:9:3 IDS	-0.04082072493550	0.13764934782851	-0.04212834754892	0.13721372595737
5:9:3 IDS	-0.04614287319597	0.14533721954050	-0.04772333998032	0.14528511534499
5:12:4 IDS	-0.04040587025897	0.13587586502136	-0.04168737876704	0.13545996580300
6:15:5 IDS	-0.04029661340369	0.13473703465805	-0.04156060542245	0.13435081874147
7:12:4 IDS	-0.04866894462456	0.14816116135794	-0.05042045277498	0.14838616706965
7:15:5 IDS	-0.04137980160645	0.13901746114623	-0.04271366242689	0.13859508479348
8:15:5 IDS	-0.04451391277195	0.14347922646705	-0.04599880879406	0.14327218198622
9:15:5 IDS	-0.05046811337293	0.15021280222630	-0.05236180113872	0.15066454139819
11:18:6 IDS	-0.05179238888687	0.15176321150892	-0.05380290342581	0.15239875185661
1:3:1 EDS	-0.04084124245139	0.12895581046269	-0.04195304829091	0.12883037716743
1:5:1 EDS	-0.02722610518703	0.12208544880638	-0.02785739043109	0.12109524567281
3:12:4 EDS	-0.04438693886973	0.10980168540414	-0.04491369149108	0.11026567969444
4:15:5 EDS	-0.04321846120139	0.11705774767448	-0.04395482561660	0.11738666443345
5:6:2 EDS	-0.10927140816405	0.25651051362983	-0.11230941814018	0.26107496713322
5:18:6 EDS	-0.04263658071656	0.12012639371328	-0.04348743413265	0.12037464944146
7:9:3 EDS	-0.08632804182541	0.20721407980683	-0.08848537919576	0.20963311298995
7:18:6 EDS	-0.05046811337289	0.15021280222628	-0.04152488928879	0.13357151489141
8:9:3 EDS	-0.15679316488045	0.37942772516268	-0.16140496083355	0.38869140818015
9:12:4 EDS	-0.07811313953933	0.19164421088803	-0.07995506535777	0.19336381974299
11:12:4 EDS	-0.22424027189467	0.59655975824951	-0.23085619472917	0.61426709790952
11:15:5 EDS	-0.07381467957210	0.18402129105083	-0.07555709512922	0.18549264117575
1:4:1 LNSS	-0.03498574961773	-0.03498574961773	-0.03518790604981	0.12076758270811
2:7:1 LNSS	-0.03914338033909	0.12376609914515	-0.04015734654309	0.12361573924824
6:7:1 LNSS	-0.11588143949536	0.27269227001244	-0.11927070442097	0.27811484353308
7:8:2 LNSS	-0.11491430778398	0.27261411974107	-0.11794553089634	0.27713133359377

Symmetric Resonance	CR3BP		ER3BP	
	ρ_0	θ'_0	ρ_0	θ'_0
1:4:1 RNSS	0.01678667342654	-0.05391697233617	0.03572583839098	-0.11136945759361
2:7:1 RNSS	0.03947928445700	-0.11332553778563	0.04053437582986	-0.11292142522023
6:7:1 RNSS	0.12313104323276	-0.20997765681491	0.12313104323276	-0.20997765681491
7:8:2 RNSS	0.11780705249635	-0.20746130873333	0.12092394460776	-0.20925804264512
5:21:3 LWSS	-0.04370201802970	0.11159384343839	-0.04450497267267	0.11197206027100
5:24:6 LWSS	-0.04307296187775	0.10183660602903	-	-
7:28:4 LWSS	-0.04264484775857	0.11583843719403	-	-
8:35:5 LWSS	-0.04460535766383	0.10654940229723	-	-
9:35:5 LWSS	-0.04200692579315	0.11781692100372	-	-
11:12:3 LWSS	-0.16284621856832	0.39809216336770	-0.16737512036342	0.40722098623685
11:42:6 LWSS	-0.04157176298363	0.11898349277680	-	-
5:21:3 RWSS	0.04440838216447	-0.10088011318187	0.04327190370660	-0.10143288853694
5:24:6 RWSS	0.04437357112768	-0.09085149606044	-	-
7:28:4 RWSS	0.04321169977798	-0.10501654585989	-	-
8:35:5 RWSS	0.04545899429107	-0.09613678352479	-	-
9:35:5 RWSS	0.04250794335330	-0.10700494430739	-	-
11:12:3 RWSS	0.17044418109310	-0.26979057443329	0.17520787477672	-0.27276186089084
11:42:6 RWSS	0.04203515329507	-0.10820012624464	-	-

Asymmetric Resonance	CR3BP		
	ρ_0	θ_0	ρ'_0
1:4:1 NA	-	-	-
2:7:1 NA	-	-	-
6:7:1 NA	-0.0022434901447191	0.09150821863087	0.12426064382621
7:8:2 NA	-0.0031927090317225	0.12330496399563	0.12224376613486
5:21:3 WA	-	-	-
5:24:6 WA	-	-	-
7:28:4 WA	-	-	-
8:35:5 WA	-	-	-
9:35:5 WA	-	-	-
11:12:3 WA	-0.0042468702915339	0.05860829721671	0.16554853952802
11:12:3 WA	-	-	-

Asymmetric Resonance	ER3BP		
	ρ_0	θ_0	ρ'_0
1:4:1 NA	-	-	-
2:7:1 NA	-	-	-
6:7:1 NA	-0.0025855818650702	0.09493174780860	0.12350977403923
7:8:2 NA	-0.0032348514126783	0.12749329628065	0.12256365362637
5:21:3 WA	-	-	-
5:24:6 WA	-	-	-
7:28:4 WA	-	-	-
8:35:5 WA	-	-	-
9:35:5 WA	-	-	-
11:12:3 WA	-0.0042455982937670	0.06177394135359	0.16227388722678
11:42:6 WA	-	-	-

