A projective-based formalism for symmetric modeling of electrical circuits

Ricardo Riaza[1]

Abstract We survey in this contribution some recent ideas involving the use of a homogeneous formalism to set up electrical circuit models. Broadly, the goal is to avoid any lack of generality in the modeling process by avoiding unnecessarily restrictive assumptions in the form of the characteristics of the circuit devices. We discuss how to use this approach in the framework of nodal analysis, aiming at the development of computationally efficient models. Except for some minor technicalities arising in the index analysis, the discussion is deliberately kept at a simple level.

1 Introduction

Choosing either a current-controlled or a voltage-controlled description for any given circuit device always entails both a lack of generality and of symmetry in symbolic circuit analysis. This is already the case in the most elementary stages of circuit theory, when one chooses between the forms $v = Ri$ and $i = Gv$ to write Ohm's law [16]. The same happens in the nonlinear context: choosing between any of the two nonlinear counterparts of such relations, namely $v = f(i)$ or $i = g(v)$ for whatever functions f or g, necessarily excludes some devices from the analysis. Focusing on specific forms of f or g which are known to admit a global inverse entails an obvious loss of generality. Similar remarks apply to capacitors, inductors and memristors, now involving the charge/flux variables in their characteristics.

A way to circumvent these limitations comes from choosing a homogeneous description for the devices (as a historical anecdote, worth noting is that homogeneous coordinates were created, in a completely different context, by A. F. Möbius in 1827 [14], exactly the year of publication of Ohm's book [16], what motivates an anachronic speculation about what the evolution of circuit theory might have

Ricardo Riaza
Departamento de Matemática Aplicada a las Tecnologías de la Información y las Comunicaciones & Information Processing and Telecommunications Center
Escuela Técnica Superior de Ingenieros de Telecomunicación, Universidad Politécnica de Madrid
Ciudad Universitaria s/n. 28040 - Madrid, Spain. ricardo.riaza@upm.es

This is the author's final copy of the paper published in M. van Beurden et al. (eds.): *Scientific Computing in Electrical Engineering* (Proceedings of the SCEE'2022 Conference), Mathematics in Industry, vol. 43, pp. 11-22, 2024.
https://doi.org/10.1007/978-3-031-54517-7_2

been should Ohm have used this formalism). In the linear resistive setting, such a homogeneous standpoint would amount to writing Ohm's law as

$$Pv - Qi = 0 \tag{1}$$

for two real parameters P, Q which do not vanish simultaneously. The assumptions $P \neq 0$ or $Q \neq 0$ imply that the resistance or the conductance, respectively, are well-defined as $R = Q/P$ or $G = P/Q$. The variables $(P : Q)$ are defined only up to a nonzero constant and therefore define a pair of homogeneous coordinates of a point lying on a *projective line* $\mathbb{RP}$, which seems to provide the natural mathematical context to accommodate all possible resistance values (including zero and infinity) in a comprehensive manner. The same ideas apply to linear circuits in sinusoidal steady state, working in this case in the complex domain.

Still in the linear context and from the same homogeneous perspective, a way to reduce model dimensionality without losing generality comes from writing Ohm's law in parametric form, that is,

$$i = Pu \tag{2a}$$
$$v = Qu, \tag{2b}$$

where u is an adimensional variable from which both the current and the voltage can be explicitly computed. In turn, this provides a route to extend the approach to the nonlinear context, namely by writing the characteristics of nonlinear (resistive) devices as

$$i = \psi(u) \tag{3a}$$
$$v = \zeta(u), \tag{3b}$$

for certain functions ψ, ζ. Such a global parametric description is well-defined for a broad class of devices, as discussed in [21, 22]. Current-controlled and voltage-controlled descriptions are of course included in (3), just by letting ψ or ζ be the identity map. However, (3) also accommodates, for example, hysteresis loops not admitting a global description in terms of either the current or the voltage. The form (3) is also of interest when, for the sake of generality, one wants to leave the nature of the device unspecified in the modeling process, at least up to a certain stage.

A systematic approach to linear and nonlinear circuit analysis stemming from these ideas is discussed in [20, 21]. One of the challenges of this approach is to accommodate it within a computationally efficient framework, for simulation purposes. Needless to say, nodal analysis is the key tool to set up automatically circuit models with a reduced dimensionality, something which is essential in large-scale circuit analysis and simulation. In this direction, the main purpose of this contribution is to discuss some features of nodal models of nonlinear circuits in which *some* branches are given a homogeneous description of the form (3), for any of the reasons discussed in the previous paragraph. This will be addressed in Sections 2 and 3: in particular, the latter section provides an index characterization of the resulting differential-algebraic circuit models. For simplicity, we will restrict the use of homogeneous descriptions to (some) resistive devices, by assuming that capacitors are voltage-controlled and inductors current-controlled. For space restrictions, other analytical aspects involving homogeneous descriptions of circuits will be discussed in less detail in Section 4. Finally, Section 5 compiles some concluding remarks.

2 Nodal models

Digraphs and the incidence matrix. We refer the reader to [1, 3–5] for background on graph and digraph theory and, in particular, for details on the claims which are here presented without proof. We assume throughout the document that the digraph underlying the circuit has n nodes (vertices) and m branches (edges), and also that it is connected and has no branches with just one incident node. Let us choose a reference node and map the remaining nodes and the branches onto the sets $\{1,\ldots,n-1\}$ and $\{1,\ldots,m\}$, respectively. In this setting, the entries of the *reduced incidence matrix* $A=(a_{jk})\in\mathbb{R}^{(n-1)\times m}$ are defined as $a_{jk}=1$ (resp. -1) if the k-th branch leaves (resp. enters) node j, and 0 otherwise. Any reduced incidence matrix of a digraph is totally unimodular (that is, the determinant of all square submatrices is either 1, -1 or 0), and the determinant of a square submatrix of order $n-1$ is ± 1 if and only if the branches corresponding to its columns define a spanning tree.

Nodal analysis. After choosing a reference node, Kirchhoff laws can be easily described in terms of the corresponding reduced incidence matrix as $Ai=0$, where i is the m-dimensional vector of branch currents, and $v=A^{\mathsf{T}}e$, where v and e stand for the vectors of branch voltages and node potentials, with dimensions m and $n-1$, respectively.

We will split the incidence matrix according to the nature of the circuit device lying on each branch: specifically, we will let A_c, A_l, A_g, A_v, A_i stand for the submatrices of A defined by the columns corresponding to capacitors, inductors, voltage-controlled resistors, voltage sources and current sources, respectively. Resistors with a homogeneous description will be later labeled with the subscript h. The same notational convention will apply to the components of the voltage and current vectors; that is, v_c, v_l, etc. will be defined from the components of the voltage vector which correspond to capacitors, inductors and so on.

As indicated in Section 1, we will assume for simplicity that capacitors are voltage-controlled by a smooth relation of the form $q_c=\eta(v_c)$, with $C(v_c)$ standing for the incremental capacitance $\eta'(v_c)$. Analogously, inductors will be assumed to be defined by a smooth current-controlled characteristic reading as $\varphi_l=\phi(i_l)$; we will let $L(i_l)=\phi'(i_l)$ denote the incremental inductance. Furthermore, $C(v_c)$ and $L(i_l)$ will be assumed to be nonsingular (invertible) for all values of v_c and i_l.

In the context of nodal analysis, it is very common to assume that resistors are voltage-controlled by a smooth relation of the form $i_g=\gamma(v_g)$. With this setup, the expressions provided above for Kirchhoff laws make it possible to write the nodal equations in the form

$$
\begin{aligned}
C(v_c)v_c' &= i_c && \text{(4a)}\\
L(i_l)i_l' &= A_l^{\mathsf{T}} e && \text{(4b)}\\
0 &= A_c i_c + A_v i_v + A_l i_l + A_g \gamma(A_g^{\mathsf{T}} e) + A_i i_s(t) && \text{(4c)}\\
0 &= v_c - A_c^{\mathsf{T}} e && \text{(4d)}\\
0 &= v_s(t) - A_v^{\mathsf{T}} e, && \text{(4e)}
\end{aligned}
$$

where $i_s(t)$ and $v_s(t)$ denote the excitations in the (assumed independent) current and voltage sources. By eliminating capacitor voltages and currents one easily gets the so-called Modified Nodal Analysis (MNA) model, namely

$$
\begin{aligned}
A_c C(A_c^{\mathsf{T}} e) A_c^{\mathsf{T}} e' &= -A_v i_v - A_l i_l - A_g \gamma(A_g^{\mathsf{T}} e) - A_i i_s(t) && \text{(5a)}\\
L(i_l)i_l' &= A_l^{\mathsf{T}} e && \text{(5b)}\\
0 &= v_s(t) - A_v^{\mathsf{T}} e, && \text{(5c)}
\end{aligned}
$$

widely used in nonlinear circuit simulation [7, 8, 12, 18, 19, 24, 25]. With some extra work, later results can be naturally extended to MNA models. However, we mostly restrict the analysis to models of the form (4) for the sake of simplicity.

Homogeneous description of (some) resistors. The aforementioned voltage-control assumption on resistors may be unnecessarily restrictive in different practical situations. Even if this assumption may be reasonable for most resistors, it excludes resistive devices for which no global controlling variable exist and, more important, rules out the chance to leave the nature of the device unspecified up to a certain stage of the modeling process: for instance, the latter is important when one wants the model to account for *both* a voltage-controlled or a current-controlled device in a given circuit branch (think e.g. of an ideal switch). Along the lines presented in Section 1, a way to do so is to consider a second class of resistors (besides the voltage-controlled ones already introduced above) and give them a parametric description of the form

$$
\begin{aligned}
i_h &= \psi(u_h) && \text{(6a)}\\
v_h &= \zeta(u_h). && \text{(6b)}
\end{aligned}
$$

Notice the use of the subindex h for these resistors and be aware of the fact that ψ and ζ are now vector-valued (these maps were used for a single device in Section 1). Also worth noting is that the family described by (6) includes current-controlled resistors as a particular case, obtained by setting $\psi(u_h) = u_h$. Under a smoothness assumption on such resistors and provided that there are no coupling effects, we will let P_j and Q_j (with j indexing the set of homogeneous resistors) stand for the corresponding entries of ψ' and ζ', respectively. We further assume that, for each homogeneous resistor, at least one of the parameters P_j or Q_j is not zero.

In this framework, a nodal model with the structure depicted in (4) takes the form

$$
\begin{aligned}
C(v_c)v_c' &= i_c && \text{(7a)}\\
L(i_l)i_l' &= A_l^{\top} e && \text{(7b)}\\
0 &= A_c i_c + A_v i_v + A_l i_l + A_g \gamma(A_g^{\top} e) + A_h \psi(u_h) + A_i i_s(t) && \text{(7c)}\\
0 &= v_c - A_c^{\top} e && \text{(7d)}\\
0 &= v_s(t) - A_v^{\top} e && \text{(7e)}\\
0 &= \zeta(u_h) - A_h^{\top} e, && \text{(7f)}
\end{aligned}
$$

whereas the MNA reads as

$$
\begin{aligned}
A_c C(A_c^{\top} e) A_c^{\top} e' &= -A_v i_v - A_l i_l - A_g \gamma(A_g^{\top} e) - A_h \psi(u_h) - A_i i_s(t) && \text{(8a)}\\
L(i_l)i_l' &= A_l^{\top} e && \text{(8b)}\\
0 &= v_s(t) - A_v^{\top} e && \text{(8c)}\\
0 &= \zeta(u_h) - A_h^{\top} e. && \text{(8d)}
\end{aligned}
$$

3 Index analysis

The notion of the *index* is essential for the analysis and the numerical treatment of differential-algebraic systems such as (4), (5), (7) or (8) [11, 12, 19]. In particular, index-one systems are important because they admit (at least in a local sense) a state-space reduction in terms of the differential variables arising in the model, and also because they are better suited for numerical treatment than higher-index systems.

Under the assumption that $C(v_c)$ and $L(i_l)$ are nonsingular, the nodal models (4) and (7) display a *semiexplicit* form: this makes the index analysis a bit simpler than for the MNA counterparts (5) and (8). Indeed, for semiexplicit equations the index-one condition amounts to the nonsingularity of the matrix of partial derivatives of the functions defining the algebraic part of the system (namely, the equations which do not involve time derivatives) with respect to the algebraic variables (again, those for which no time derivative appears in the model).

Specifically, for the model (7), the algebraic variables are e, i_c, i_v and u_h, and the index-one condition is equivalent to the nonsingularity of the matrix

$$
\begin{pmatrix}
A_g G A_g^{\top} & A_c & A_v & A_h P \\
-A_c^{\top} & 0 & 0 & 0 \\
-A_v^{\top} & 0 & 0 & 0 \\
-A_h^{\top} & 0 & 0 & Q
\end{pmatrix}, \tag{9}
$$

where $G = \gamma'(v_g)$ is diagonal (that is, no coupling effects among resistors are allowed), and P and Q are diagonal matrices whose diagonal entries are defined by the corresponding parameters of the different homogeneous resistors. Note that in (9) we omit the dependence of G on $A_g^{\top} e$ and of P and Q on the homogeneous variables u_h.

Our main result (Theorem 1 below) presents an index-one characterization of (7) in terms of the structure of spanning trees in the circuit. This approach can be traced back to Kirchhoff's seminal paper [10], and is of particular interest for nonpassive problems, namely, those in which some of the conductances G_i and/or the ratios P_j/Q_j or Q_j/P_j for homogeneous resistors (remember that, for each j, at least one of the parameters P_j and Q_j does not vanish) become zero or negative. Note that the subindices of the individual devices correspond here to the global numbering of the digraph edges and not to their position in the G, P and Q matrices. In the same direction, we assume that each spanning tree T is defined by the index set of its constituting branches or *twigs*; that is, every such T amounts to a subset of $\{1,\ldots,m\}$ with $n-1$ elements which specify a spanning tree. The complement of T in $\{1,\ldots,m\}$ is written as $\overline{T}$ and stands for the index set of the cotree branches or *chords*. Note also that the absence of proper trees implicitly defines a null sum in (10) below, ruling out such configurations from the index-one setting.

Theorem 1 *The determinant of the matrix (9) equals the polynomial*

$$\sum_{T\in\mathscr{T}_p} \prod_{\substack{i\in T_g \\ j\in T_h \\ k\in\overline{T}_h}} G_i P_j Q_k, \tag{10}$$

where $\mathscr{T}_p$ stands for the set of proper trees (namely, those including all voltage sources and capacitors, and neither current sources nor inductors), and T_g, T_h denote the indices of T corresponding to twigs with voltage-controlled resistors and with homogeneous resistors, respectively, whereas $\overline{T}_h$ denotes the set of chords with homogeneous resistors.

Therefore, provided that $C(v_c)$ and $L(i_l)$ are nonsingular, the model (7) is index one exactly for the values of the variables $v_g = A_g^{\mathsf{T}} e$, u_h which do not annihilate (10).

Proof. For notational simplicity, let us join together capacitors and voltage source under the subscript *cv*: we are therefore led to characterize the determinant of

$$\begin{pmatrix} A_g G A_g^{\mathsf{T}} & A_{cv} & A_h P \\ -A_{cv}^{\mathsf{T}} & 0 & 0 \\ -A_h^{\mathsf{T}} & 0 & Q \end{pmatrix}.$$

We will do so by looking at this matrix as the Schur complement [9] of the identity block in

$$\begin{pmatrix} 0 & A_{cv} & A_h P & A_g G \\ -A_{cv}^{\mathsf{T}} & 0 & 0 & 0 \\ -A_h^{\mathsf{T}} & 0 & Q & 0 \\ -A_g^{\mathsf{T}} & 0 & 0 & I \end{pmatrix}$$

and, in turn, the latter matrix as the one obtained after setting $P_0 = I$ and $Q_0 = 0$ in

$$\begin{pmatrix} 0 & A_{cv}P_0 & A_hP & A_gG \\ -A_{cv}^{\mathsf{T}} & Q_0 & 0 & 0 \\ -A_h^{\mathsf{T}} & 0 & Q & 0 \\ -A_g^{\mathsf{T}} & 0 & 0 & I \end{pmatrix}.$$

By denoting $\tilde{A} = (A_{cv}\ A_h\ A_g)$, $\tilde{P} = \mathrm{diag}(P_0, P, G)$, $\tilde{Q} = \mathrm{diag}(Q_0, Q, I)$, this matrix has the form

$$\begin{pmatrix} 0 & \tilde{A}\tilde{P} \\ -\tilde{A}^{\mathsf{T}} & \tilde{Q} \end{pmatrix}.$$

By multiplying the first $n-1$ columns by -1 and performing an obvious permutation of columns, one can easily check that

$$\det \begin{pmatrix} 0 & \tilde{A}\tilde{P} \\ -\tilde{A}^{\mathsf{T}} & \tilde{Q} \end{pmatrix} = (-1)^{(n-1)(\tilde{m}+1)} \det \begin{pmatrix} \tilde{A}\tilde{P} & 0 \\ \tilde{Q} & \tilde{A}^{\mathsf{T}} \end{pmatrix},$$

where $\tilde{m}$ is the total number of voltage sources, capacitors and resistors. Now, the latter determinant can be computed by means of a generalized Laplace expansion (see e.g. [9]) along the first $n-1$ rows. In light of the properties of determinants of the square submatrices of the incidence matrix and the diagonal form of $\tilde{P}$, it is not difficult to see that

$$\det \begin{pmatrix} \tilde{A}\tilde{P} & 0 \\ \tilde{Q} & \tilde{A}^{\mathsf{T}} \end{pmatrix} = \sum_{T \in \mathscr{T}_{cvr}} \left((-1)^{\sigma(T)} \det A_T \left(\prod_{j \in T} \tilde{P}_j \right) \det \left(\tilde{Q}_{\overline{T}}\ \ \tilde{A}^{\mathsf{T}} \right) \right)$$

where $\tilde{Q}_{\overline{T}}$ is the submatrix of $\tilde{Q}$ defined by the columns indexed by $\overline{T}$ and $\sigma(T)$ is the exponent which corresponds to the spanning tree T in the Laplace expansion, namely

$$1 + \ldots + n - 1 + \sum_{j \in T} j.$$

Additionally, $\mathscr{T}_{cvr}$ denotes the family of spanning trees just including capacitors, voltage sources and resistors.

In turn, after a transposition and a permutation of rows one gets

$$\det \left(\tilde{Q}_{\overline{T}}\ \ \tilde{A}^{\mathsf{T}} \right) = (-1)^{(n-1)(\tilde{m}-n+1)} \det \begin{pmatrix} \tilde{A} \\ \tilde{Q}_{\overline{T}}^{\mathsf{T}} \end{pmatrix} = (-1)^{(n-1)(\tilde{m}-n+1)} (-1)^{\sigma(T)} \det A_T \prod_{k \in \overline{T}} \tilde{Q}_k,$$

where the last identity owes to the structure of the matrix $\tilde{Q}_{\overline{T}}$. Altogether, the different expressions obtained for the successive determinants yield

$$\det \begin{pmatrix} 0 & \tilde{A}\tilde{P} \\ -\tilde{A}^{\mathsf{T}} & \tilde{Q} \end{pmatrix} = \sum_{T \in \mathscr{T}_{cvr}} \left((-1)^{2[(n-1)(\tilde{m}+1)+\sigma(T)]-(n-1)n} (\det A_T)^2 \prod_{j \in T} \tilde{P}_j \prod_{k \in \overline{T}} \tilde{Q}_k \right).$$

Note that the exponent of -1 has the same parity as $n(n-1)$, which is necessarily an even number. The fact that $\det A_T = \pm 1$ then implies that the determinant above amounts to

$$\sum_{T \in \mathscr{T}_{cvr}} \left(\prod_{j \in T} \tilde{P}_j \prod_{k \in \overline{T}} \tilde{Q}_k \right).$$

Finally, we make use of the definition of $\tilde{P}$ and $\tilde{Q}$ to show that the latter expression yields (10). First, the conditions $P_0 = I$ and $Q_0 = 0$ for the entries which correspond to capacitors and voltage sources reduce the range of the sum to the family of proper trees (namely, those including *all* voltage sources and capacitors) since, otherwise, a voltage source or a capacitor in the cotree annihilates the corresponding term because of the condition $\tilde{Q}_k = 0$. For voltage-controlled resistors and by construction of $\tilde{P}$ and $\tilde{Q}$, each $\tilde{P}_i$-entry is simply the incremental conductance G_i, whereas the corresponding entry in $\tilde{Q}$ amounts to 1. This is responsible for the factors $\prod_{i \in T_g} G_i$ in (10). Finally, homogeneous resistors contribute the remaining factors in (10) simply because $\tilde{P}_j = P_j$ and $\tilde{Q}_k = Q_k$, again by the definition of $\tilde{P}$ and $\tilde{Q}$. This completes the proof. □

Some particular cases merit additional remarks: in the absence of homogeneous resistors, (10) essentially amounts to Maxwell's characterization of the admittance nodal matrix [13], since only the conductances within the twigs of proper trees are involved. By contrast, when all resistors are given a homogeneous description, then (10) amounts to

$$\sum_{T \in \mathscr{T}_p} \prod_{\substack{j \in T_h \\ k \in \overline{T}_h}} P_j Q_k, \tag{11}$$

just formulated in terms of the homogeneous parameters P, Q. In particular, from (11) one can also accommodate the case in which all the parameters P_j are nonzero, an assumption which allows for a local current-controlled description of all resistors: after dividing the (11) by the product of all P_j-parameters, this expression then amounts to the sum of chord resistance products extended over the set of proper trees. The example that follows should be of help in clarifying the expressions arising in all these contexts. Let us also emphasize that, in strictly locally passive problems (in which all incremental conductances are positive and, for each j, P_j and Q_j do not vanish and have the same sign), all terms in (10) have the same sign and therefore the sum is not zero, provided that there exists at least one proper tree. The latter condition is equivalent to the absence of VC-cycles and IL-cutsets, a topological condition which is well-known to characterize index-one configurations in strictly locally passive circuits [7, 25].

Example. For illustration, consider the simple circuit depicted in Figure 1. To keep the terms of the discussion as simple as possible, assume that both the capacitor and the inductor are linear, with nonzero capacitance and inductance C and L, respectively.

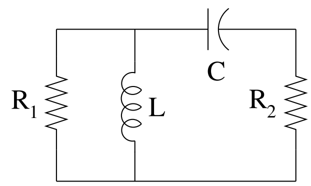

Fig. 1 An elementary circuit example.

As indicated above, in nodal analysis it is very often the case that resistors are assumed to be voltage-controlled. In our setting, this would mean that they are governed by the (assumed differentiable) characteristics $i_1 = \gamma_1(v_1)$ and $i_2 = \gamma_2(v_2)$. By directing the branches top-down or (the one accommodating the capacitor) towards the right, the nodal equations (4) take the form

$$Cv_c' = i_c \tag{12a}$$
$$Li_l' = e_1 \tag{12b}$$
$$0 = i_c + i_l + \gamma_1(e_1) \tag{12c}$$
$$0 = -i_c + \gamma_2(e_2) \tag{12d}$$
$$0 = v_c - e_1 + e_2, \tag{12e}$$

with the subindices 1 and 2 in the node potentials corresponding to the NW and NE nodes. The determinant of the matrix of partial derivatives of (12c)-(12e) with respect to the algebraic variables e_1, e_2, i_c is easily seen to be $G_1 + G_2$, with $G_j = \gamma_j'(e_j)$ for $j = 1, 2$. Notice that $G_1 + G_2$ corresponds to the sum of conductances in the circuit proper trees, shown in Figure 2.

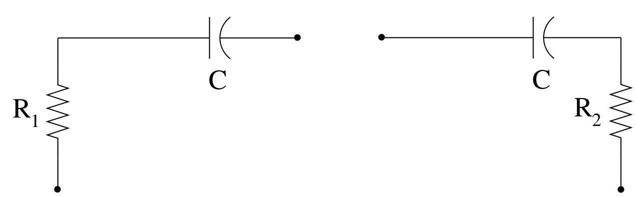

Fig. 2 Proper trees.

In this voltage-controlled setting, we may therefore guarantee that the circuit is index one if and only if $G_1 + G_2 \neq 0$. However, this assumption excludes some cases which may be of interest in practice. To avoid lacking generality, we may resort to a homogeneous description of the resistors and write $i_j = \psi_j(u_j)$, $v_j = \zeta_j(u_j)$ for $j = 1, 2$. Now the model (7) is defined by (12a)-(12b) together with the algebraic equations

$$0 = i_c + i_l + \psi_1(u_1) \tag{13a}$$
$$0 = -i_c + \psi_2(u_2) \tag{13b}$$
$$0 = v_c - e_1 + e_2 \tag{13c}$$
$$0 = -e_1 + \zeta_1(u_1) \tag{13d}$$
$$0 = -e_2 + \zeta_2(u_2) \tag{13e}$$

and some elementary computations show that the corresponding determinant is now

$$P_1Q_2 + Q_1P_2, \tag{14}$$

where $P_j = \psi_j{}'(u_j)$, $Q_j = \zeta_j{}'(u_j)$ for $j = 1, 2$.

The polynomial shown in (14) is homogeneous of degree one in each pair of variables (P_j, Q_j): in greater generality, so is (11). What we want to illustrate is that the expression just obtained accounts for all possible forms in the characteristics of resistors. For instance, the voltage-control assumption supporting (12) above essentially amounts to assuming $Q_1 \neq 0 \neq Q_2$, what makes it possible to divide (14) by Q_1Q_2 (this is known as a *dehomogenization* of (14)) to get the aforementioned expression $G_1 + G_2$, with $G_j = P_j/Q_j$. But other cases are also of interest: for instance, the assumption that only the first resistor is voltage-controlled yields, by the same token, the expression

$$G_1Q_2 + P_2,$$

which exemplifies the form (10) obtained in Theorem 1. The tree on the left of Figure 2 accounts for the term G_1Q_2 (since resistor 1 is a twig and resistor 2 is a chord) and, likewise, the tree on the right just yields the second term, namely P_2. Another case of interest results from the assumption that both resistors are current-controlled: here, after dividing (14) by P_1P_2 one gets the expression $R_1 + R_2$, with $R_j = Q_j/P_j$; note that the R_j's arise as the *chord* resistances from each spanning tree, that is, R_1 comes from the tree on the right of Figure 2 and R_2 from the one on the left. Worth remarking is that when both resistors are current-controlled, the condition $R_1 + R_2 = 0$ prevents the model from being index one, something which remains somehow hidden if devices are assumed to be voltage-controlled, as in the (so to speak) standard approach to nodal analysis. We encourage the reader to elaborate on the main idea by examining other cases, e.g. by checking that, when resistor 1 is voltage-controlled and resistor 2 current-controlled, (14) amounts to $G_1R_2 + 1$.

The essential idea behind this elementary example is that by means of dehomogenization techniques one gets the particular conditions characterizing the index in specific contexts from the general form shown in (14): we refer the reader to [20] for further uses of such techniques.

4 Other applications of homogeneous models

In this section we discuss, in less detail, other applications of the homogeneous formalism. The first one involves DC circuits, in which operating points are computed by open-circuiting capacitors and short-circuiting inductors. Using the model (7), the equations describing operating points are easily seen to be

$$
\begin{aligned}
0 &= A_l i_l + A_v i_v + A_g \gamma(A_g^{\mathsf{T}} e) + A_h \psi(u_h) + A_i I_s && \text{(15a)}\\
0 &= -A_l^{\mathsf{T}} e && \text{(15b)}\\
0 &= v_c - A_c^{\mathsf{T}} e && \text{(15c)}\\
0 &= V_s - A_v^{\mathsf{T}} e && \text{(15d)}\\
0 &= \zeta(u_h) - A_h^{\mathsf{T}} e, && \text{(15e)}
\end{aligned}
$$

where I_s and V_s stand for the DC excitation terms in the sources. Provided that a DC operating point does exist, the nonsingularity of the matrix of partial derivatives of the right-hand side of (15) makes it unique, as a straightforward consequence of the inverse function theorem. This matrix of partial derivatives reads as

$$
\begin{pmatrix}
A_g G A_g^{\mathsf{T}} & A_l & A_v & A_h P & 0 \\
-A_l^{\mathsf{T}} & 0 & 0 & 0 & 0 \\
-A_c^{\mathsf{T}} & 0 & 0 & 0 & I_c \\
-A_v^{\mathsf{T}} & 0 & 0 & 0 & 0 \\
-A_h^{\mathsf{T}} & 0 & 0 & Q & 0
\end{pmatrix},
$$

which is easily seen to be nonsingular if and only if so is

$$
\begin{pmatrix}
A_g G A_g^{\mathsf{T}} & A_l & A_v & A_h P \\
-A_l^{\mathsf{T}} & 0 & 0 & 0 \\
-A_v^{\mathsf{T}} & 0 & 0 & 0 \\
-A_h^{\mathsf{T}} & 0 & 0 & Q
\end{pmatrix}.
$$

Notice that this matrix has the same form as (9), and therefore the characterization of index-one configurations stated in Theorem 1 should have a dual result characterizing nondegenerate operating points in the homogeneous framework.

Other uses of the homogeneous formalism. From a different perspective, the intrinsic symmetry provided by the homogeneous description of devices can be extended, at least for linear circuits, to handle both voltage and current sources in a unifying framework. This idea can be pursued further as to accommodate Thévenin and Norton equivalent circuits within a comprehensive setting [23].

Projective-based techniques in circuit modeling also find applications in power system modeling, fault diagnosis and the qualitative analysis of nonlinear circuits: see [17, 20, 22] and references therein. The very idea behind (2) suggests that homogeneous descriptions should be useful in the modeling of switching circuits, a topic which is in the scope of future research. Possibly, other application fields may benefit from this formalism.

5 Concluding remarks

To sum up, homogeneous description of devices are of interest when trying to avoid any lack of generality in the circuit modeling process. Additionally, one may retain the computational advantages of the nodal approach by using homogeneous descriptions only for specific devices which for whatever reason demand so. Needless to say, the idea can be naturally extended to reactive devices and also to memristors. Controlled sources and coupling effects may also be accommodated in this framework along the lines suggested in [21]. Future work may extend the index analysis here presented to MNA models, as well as to index-two configurations and also to other families of circuit models, including distributed models involving partial differential-algebraic equations [2, 6, 15, 24, 25].

References

1. R. B. Bapat, *Graphs and Matrices,* Springer-Verlag, 2010.
2. A. Bartel, S. Baumanns and S. Schöps, Structural analysis of electrical circuits including magnetoquasistatic devices, *App. Numer. Math.* **61** (2011) 1257-1270.
3. B. Bollobás, *Modern Graph Theory*, Springer-Verlag, 1998.
4. J. A. Bondy and U. S. R. Murty, *Graph Theory,* Springer-Verlag, 2008.
5. W.-K. Chen, *Graph Theory and its Engineering Applications,* World Scientific, 1997.
6. M. Diab and C. Tischendorf, Splitting methods for linear circuit DAEs of index 1 in port-Hamiltonian form, *in* M. van Beurden, N. Budko and W. Schilders (eds), *Proc. SCEE'2020*, 211-219, Springer, 2021.
7. D. Estévez-Schwarz and C. Tischendorf, Structural analysis of electric circuits and consequences for MNA, *Int. J. Circuit Theory Appl.* **28** (2000) 131-162.
8. M. Günther and U. Feldmann, CAD-based electric-circuit modeling in industry, *Surv. Math. Ind.* **8** (1999) 97-129, 131-157.
9. R. A. Horn and Ch. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.
10. G. Kirchhoff, Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird, *Annalen der Physik und Chemie* **72** (1847) 497-508. English translation: *IRE Tr. Circuit Theory* **5** (1958) 4-7.
11. P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations. Analysis and Numerical Solution*, EMS, 2006.
12. R. Lamour, R. März and C. Tischendorf, *Differential-Algebraic Equations: A Projector Based Analysis*, Springer, 2013.
13. J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Clarendon Press, 1892.
14. A. F. Möbius, *Der Barycentrische Calcul,* J. A. Barth, 1827.
15. N. Nedialkov, J. D. Pryce and L. Scholz, An energy-based, always index ≤ 1 and structurally amenable electrical circuit model, *SIAM J. Sci. Computing* **44** (2022) B1122-B1147.
16. G. S. Ohm, *Die Galvanische Kette, Mathematisch Bearbeitet,* T. H. Riemann, 1827. English translation: Van Nostrand Co., 1891; Kraus (reprint), 1969.

17. A. Penin, *Analysis of Electrical Circuits with Variable Load Regime Parameters,* Springer, 2015.
18. T. Reis, Circuit synthesis of passive descriptor systems. A modified nodal approach, *Int. J. Cir. Theory Appl.* **38** (2010) 44-68.
19. R. Riaza, *Differential-Algebraic Systems*, World Scientific, 2008.
20. R. Riaza, Circuit theory in projective space and homogeneous circuit models, *IEEE Trans. Circuits Syst. I* **66** (2019) 463-476.
21. R. Riaza, Homogeneous models of nonlinear circuits, *IEEE Trans. Circuits Syst. I* **67** (2020) 2002-2015.
22. R. Riaza, Associate submersions and qualitative properties of nonlinear circuits with implicit characteristics, *Intl. J. Bif. Chaos* **30** (2020) 2050033.
23. R. Riaza, A comprehensive framework for the Thévenin-Norton theorem using homogeneous circuit models, preprint, https://doi.org/10.36227/techrxiv.21347313.v1, 2022.
24. V. Shashkov, I. Cortés García and H. Egger, MONA: A magnetic oriented nodal analysis for electric circuits, *Int. J. Cir. Theory Appl.* **50** (2022) 2997-3012.
25. C. Tischendorf, Coupled systems of differential algebraic and partial differential equations in circuit and device simulation. Modeling and numerical analysis, Habilitationsschrift, Inst. Math., Humboltd University of Berlin, 2003.