Rational complex Bézier curves

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Abstract

In this paper we develop the formalism of rational complex Bézier curves. This framework is a simple extension of the CAD paradigm, since it describes arc of curves in terms of control polygons and weights, which are extended to complex values. One of the major advantages of this extension is that we may make use of two different groups of projective transformations. Besides the group of projective transformations of the real plane, we have the group of complex projective transformations. This allows us to apply useful transformations like the geometric inversion to curves in design. In addition to this, the use of the complex formulation allows to lower the degree of the curves in some cases. This can be checked using the resultant of two polynomials and provides a simple formula for determining whether a rational cubic curve is a conic or not. Examples of application of the formalism to classical curves are included.

Keywords: Rational Bézier curves, complex geometry.

1. Introduction

Rational spline parametrisations are the usual form for representing curves in CAGD [1]. In particular, using Bernstein polynomials

$$B_j^n(t) = \frac{n!}{j!(n-j)!} t^j (1-t)^{n-j}, \quad j = 0, \ldots, n$$

curves of degree n with just one piece are parametrised as Bézier curves,

$$c(t) = \frac{\displaystyle\sum_{j=0}^{n} \omega_j c_j B_j^n(t)}{\displaystyle\sum_{j=0}^{n} \omega_j B_j^n(t)}, \quad t \in [0,1], \tag{1}$$

in terms of a list of points $\{c_0, \ldots, c_n\}$ of $\mathbb{R}^p$ named *control polygon* and a list of coefficients $\{\omega_0, \ldots, \omega_n\}$ dubbed *weights*.

Alternatively, the rational curve in $\mathbb{R}^p$ may be viewed as a polynomial curve in $\mathbb{R}^{p+1}$,

$$\mathbf{c}(t) = \sum_{j=0}^{n} \mathbf{c}_j B_j^n(t), \quad t \in [0,1], \tag{2}$$

using a control polygon $\{\mathbf{c}_0, \ldots, \mathbf{c}_n\}$ formed by vectors $\mathbf{c}_j = (\omega_j, \omega_j c_j)$.

We can switch from the polynomial to the rational form dividing by the first component of (2) and projecting into $\mathbb{R}^p$. And conversely, the denominator of (1) provides the control vertices with the extra coordinate of the parametrisation in (2).

One of the key features of this representation is that Bernstein polynomials form a partition of unity,

$$1 = \sum_{j=0}^{n} B_j^n(t),$$

that is, a rational parametrisation is a barycentric combination of its control polygon, since the sum of their coefficients in (1) is one. This property does not hold in general if we use another basis of polynomials.

This property has many nice consequences. For instance, if we apply a projective transformation f to the curve, with $\mathbf{f}$ as associated linear map,

$$\mathbf{f}(\mathbf{c}(t)) = \sum_{j=0}^{n} \mathbf{f}(\mathbf{c}_j) B_j^n(t),$$

the transformed curve is another curve of degree n which has $\{\mathbf{f}(\mathbf{c}_0), \ldots, \mathbf{f}(\mathbf{c}_n)\}$ as vector control polygon. This is true since projective transformations preserve barycentric combinations [2].

Any regular linear transformation of $\mathbb{R}^{p+1}$ defines a projective transformation for $\mathbb{P}^p$ up to a non-zero constant. That is, two linear transformations of $\mathbb{R}^{p+1}$ $\lambda\mathbf{f}$ and $\mathbf{f}$ provide the same projective transformation f of $\mathbb{P}^p$ for $\lambda \in \mathbb{R}\backslash\{0\}$.

Focusing on the plane, we can construct projective transformations through linear transformations of $\mathbb{R}^3$, but there is another possibility:

We may identify $\mathbb{R}^2$ with the complex line $\mathbb{C}$ through the usual bijection

$$\begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{C} \\ (x,y) & \mapsto & x+iy. \end{array}$$

Control polygons of planar curves are readily written in complex form $\{z_0, \ldots, z_n\}$, with $z_k = c_k^x + i c_k^y$, $k = 0, \ldots, n$, for $c_k = ({c_k}^x, {c_k}^y)$.

Weights are real numbers, but their definition has been extended to complex numbers in several contexts [3, 4, 5].

Rational complex parametrisations,

$$c(t) = \frac{\displaystyle\sum_{j=0}^{n} \omega_j z_j B_j^n(t)}{\displaystyle\sum_{j=0}^{n} \omega_j B_j^n(t)}, \; t \in [0,1], \tag{3}$$

$z_j, \omega_j \in \mathbb{C}$, $j = 0, \ldots, n$, which can be viewed also as polynomial curves in $\mathbb{C}^2$,

$$\mathbf{c}(t) = \sum_{j=0}^{n} \mathbf{z}_j B_j^n(t), \quad \mathbf{z}_j = (\omega_j, \omega_j z_j), \quad j = 0, \ldots, n$$

become then a natural extension of rational parametrisations of plane curves.

Regular linear transformations of $\mathbb{C}^2$,

$$\mathbf{f}(z_0, z_1) = (a z_0 + b z_1, c z_0 + d z_1), \; a, b, c, d \in \mathbb{C}, \; ad - bc \neq 0,$$

provide complex projective transformations of the projective complex line, named Möbius transformations,

$$f(z) = \frac{c + dz}{a + bz}. \tag{4}$$

Since $c(t)$ is again a barycentric combination of the control polygon, we have for these curves a property of invariance under projective transformations,

$$\mathbf{f}(\mathbf{c}(t)) = \sum_{j=0}^{n} \mathbf{f}(\mathbf{z}_j) B_j^n(t). \tag{5}$$

On the other hand, we still have the invariance under real projective transformations,

$$f(x, y) = \left(\frac{a_1 + b_1 x + c_1 y}{a_0 + b_0 x + c_0 y}, \frac{a_2 + b_2 x + c_2 y}{a_0 + b_0 x + c_0 y} \right), \tag{6}$$

associated to regular linear transformations of $\mathbb{R}^3$,

$$\mathbf{f} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

This group of real projective transformations depends on a larger number of parameters than the group of complex projective transformations, but it does not comprise it.

For instance, the inversion transformation $f(z) = 1/z$,

$$f(z) = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2},$$

corresponding to a Möbius transformation with $a = 0 = d$, $b = c$, does not provide a real projective transformation of $\mathbb{R}^2$ of the form (6).

Hence both groups are different and none of them is a subgroup of the other.

This paper is organised in the following fashion. After this introduction, we define in the next section rational complex Bézier curves and show their properties. As an example of this construction, in Section 3 we produce arcs of circles. In order to relate real and complex parametrisations, in Section 4 we introduce the concept of reducible parametrisations and provide characterisations for it and we show in Section 5 how to simplify them in Bézier form. Having shown how to derive complex parametrisations from real ones, we go the way back in Section 6. Section 7 is devoted to examples of curves which can be parametrised as complex rational curves using inversion. Finally in the last section we derive some conclusions.

2. Rational complex Bézier curves

A rational complex Bézier curve of degree n is a planar curve which can be parametrised as

$$c(t) = \frac{\displaystyle\sum_{j=0}^{n} \omega_j z_j B_j^n(t)}{\displaystyle\sum_{j=0}^{n} \omega_j B_j^n(t)}, \quad t \in [0,1],$$

in terms of the Bernstein polynomials of degree n,

$$B_j^n(t) = \begin{pmatrix} n \\ j \end{pmatrix} t^j (1-t)^{n-j}, \quad j = 0, \ldots, n,$$

and two lists of complex numbers: the *control polygon* $\{z_0, \ldots, z_n\}$ and the complex *weights* $\{\omega_0, \ldots, \omega_n\}$ of the curve.

Most properties of real Bézier curves [1] are inherited by complex curves:

- z_0, z_n are the endpoints of the curve:

  $$c(0) = z_0, \qquad c(1) = z_n.$$

- The tangent vectors at the endpoints of the curve,

  $$c'(0) = \frac{n\omega_1}{\omega_0}(z_1 - z_0), \quad c'(1) = \frac{n\omega_{n-1}}{\omega_n}(z_n - z_{n-1}),$$

  consist of a rotation and a stretching of the first and last edges of the control polygon.

  For instance, if $\omega_1/\omega_0 = r e^{i\phi}$, the tangent vector is a modification of the vector $\overrightarrow{z_0 z_1}$ by a factor nr and a rotation of an angle ϕ. And analogously at the other endpoint of the curve, where the stretching and rotation is given by ω_{n-1}/ω_n.

- The parametrisation does not change if we multiply all weights by a non-zero common factor $\lambda \in \mathbb{C}$, $\{\lambda\omega_0, \ldots, \lambda\omega_n\}$.

- Another rational parametrisation of degree n for the same curve is obtained by transforming the parameter with a real Möbius transformation of the interval $[0,1]$ onto itself,

$$t(u) = \frac{u}{(1-\rho)u+\rho}, \quad u \in [0,1], \tag{7}$$

for a non-zero real parameter ρ.

The control polygon remains the same, but the new weights are $\hat{\omega}_j = \rho^{n-j}\omega_j$, $j = 0, \ldots, n$.

- If $f(z) = 1/z$ then $f(c(t))$ is a rational complex curve with control polygon $\{\frac{1}{z_0}, \ldots, \frac{1}{z_n}\}$ and complex weights $\{\omega_0 z_0, \ldots, \omega_n z_n\}$.

- Degree elevation: If we multiply both the numerator and the denominator of $c(t)$ by a polynomial $p(t)$ of degree one

$$p(t) = \alpha(1-t) + \beta t, \quad \alpha, \beta \in \mathbb{C},$$

the parametrisation is formally of degree $n+1$ [6, 7] with new control polygon $\{z_0^1, \ldots, z_{n+1}^1\}$ and weights $\{\omega_0^1, \ldots, \omega_{n+1}^1\}$,

$$\begin{aligned} \omega_j^1 &= \alpha\frac{n+1-j}{n+1}\omega_j + \beta\frac{j}{n+1}\omega_{j-1}, \\ \omega_j^1 z_j^1 &= \alpha\frac{n+1-j}{n+1}\omega_j z_j + \beta\frac{j}{n+1}\omega_{j-1}z_{j-1} \end{aligned} \tag{8}$$

The possibility of having complex α and β increases the ways of performing degree elevation compared to the real case.

On the contrary, a complex curve is not contained in the convex envelope of its control polygon, unless its weights are real and positive. We lose this feature, but complex weights add flexibility for designing curves.

3. Arcs of circles

The image of a line segment by a Möbius transformation as in (4) is either a line segment or an arc of a circle when the denominator does not vanish at any point of the segment. Considering Möbius transformations that fix z_0 and z_1, the image of the line segment $z_0(1-t) + z_1 t$, $t \in [0,1]$ is

$$c(t) = \frac{\omega_0 z_0(1-t) + \omega_1 z_1 t}{\omega_0(1-t) + \omega_1 t},$$

where ω_0, ω_1 are non-zero complex numbers. The curve, $c(t)$, is an arc of a circle if ω_1/ω_0 is not a real number, so from now on this will be assumed.

As mentioned in Section 2, the velocities at both ends of the arc of the circle are

$$c'(0) = \frac{\omega_1}{\omega_0}(z_1 - z_0), \quad c'(1) = \frac{\omega_0}{\omega_1}(z_1 - z_0).$$

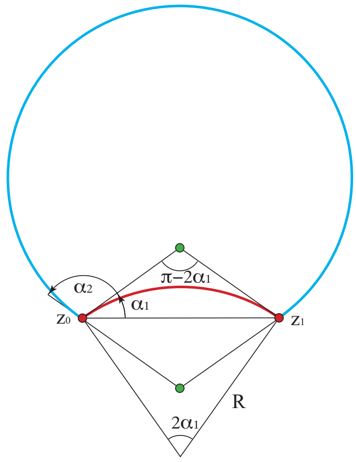

Figure 1: Two arcs of circle from z_0 to z_1

If $\omega_1/\omega_0 = re^{i\alpha}$ with $0 < |\alpha| < \pi$ and $r > 0$, then $c'(0)$ is a rotation of angle α and a stretching with factor r of the vector $z_1 - z_0$. Moreover, the parameter r can be obtained by choosing a particular point on the arc of the circle to be the image by the Möbius transformation of the midpoint of the segment (a Möbius transformation is completely determined by fixing the image of three different points). When $r = 1$, the midpoint of the segment is mapped to the midpoint of the arc of the circle. Thus performing a change of parameter as in (7), it can be obtained that $\omega_1/\omega_0 = e^{i\alpha}$ and the arcs of the circles in Figure 1 can be described by the rational complex curve

$$c(t) = \frac{z_0(1-t) + e^{i\alpha}z_1 t}{(1-t) + e^{i\alpha}t}.$$

The radius of the circle is:

$$R = \frac{|z_1 - z_0|}{2|\sin\alpha|},$$

and its center,

$$z_0 + \frac{ie^{i\alpha}}{2\sin\alpha}(z_0 - z_1).$$

In Figure 1 we may see two arcs, for $\alpha_1 = \pi/6$, $\alpha_2 = 5\pi/6$ and the same complex control polygon, but different weights. The green control points belong to control polygons for arcs as curves of degree two, whereas as complex curves we just need z_0 and z_1.

Therefore, any circular arc can be represented by a complex rational Bézier curve of degree one, whereas its degree is two as a real rational Bézier curve. Hence, when written in complex notation the numerator and denominator of the rational expression have a common factor (the rational function is reducible).

In the next sections we are going to show how to identify common factors in complex rational curves written in Bézier form.

4. Degree elevation and irreducibility

There is a simple way of checking whether a rational complex parametrisation of a planar curve, $c(t) = p(t)/q(t)$, cannot be further simplified, that is, if $c(t)$ is irreducible.

A complex rational parametrisation is irreducible if and only if the polynomials $p(t)$ and $q(t)$ are relatively prime. This means it cannot be further simplified by cancellation of common complex polynomial factors and the degree of the parametrisation is the lowest possible.

This is specially useful, since the simplest way of casting a rational parametrisation in complex form

$$c(t) = \left(\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)}\right) \to \frac{z(t)}{w(t)}, \quad z(t) := x(t) + iy(t),$$

often produces polynomials $z(t)$, $w(t)$ which may be simplified by cancellation of common polynomial factors. As in the case of the parametrisation of an arc of a circle, which is of degree two as a real parametrisation, but of degree one as a complex one.

In order to know if a parametrisation is irreducible, we may use the Sylvester resultant $R(p, q)$ of two polynomials p, q, respectively of degree $m > 0$ and $n > 0$ [8],

$$p(t) = \sum_{j=0}^{m} a_j t^j, \qquad q(t) = \sum_{j=0}^{n} b_j t^j,$$

$$R(p,q) := \begin{array}{c} \left|\begin{array}{ccccc} a_m & \cdots & a_0 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & a_m & \cdots & a_0 \\ \hline b_n & \cdots & b_0 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & b_n & \cdots & b_0 \end{array}\right| \end{array} \begin{array}{c} \uparrow \\ n \text{ rows} \\ \downarrow \\ \uparrow \\ m \text{ rows} \\ \downarrow \end{array} . \quad (9)$$

The resultant of two polynomials is related to the complex roots of both polynomials by a classical result [8]:

Theorem 1. *Let* $p(t) = \sum_{j=0}^{m} a_j t^j$, $q(t) = \sum_{j=0}^{n} b_j t^j$ *be two polynomials with respective complex roots* $\{x_1, \ldots, x_m\}$, $\{y_1, \ldots, y_n\}$, *then*

$$R(p, q) = (a_m)^n (b_n)^m \prod_{1 \le j \le m} \prod_{1 \le k \le n} (x_j - y_k).$$

As a consequence, the resultant of two polynomials vanishes if they share at least one root $x_J = y_K$. In this case, they share a common factor $t - x_J$ and their quotient is reducible:

Corollary 1. *The quotient of two polynomials* $p(t)$ *and* $q(t)$, $c(t) = p(t)/q(t)$ *is irreducible if and only if* $R(p, q) \neq 0$.

The resultant of two polynomials is written in (9) in terms of their coefficients in the monomial basis $\{1, t, t^2, \ldots\}$. Since we are interested in applying it to CAGD, it is useful to write it in the Bernstein basis, and this has already been done in [9]:

Theorem 2. *The resultant of two polynomials* $p(t) = \sum_{j=0}^{m} a_j B_j^m(t)$, $q(t) = \sum_{j=0}^{n} b_j B_j^n(t)$ *in Bernstein form may be written as*

$$R(p,q) := D \left|\begin{array}{ccccc} a_0\binom{m}{0} & \cdots & a_m\binom{m}{m} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & a_0\binom{m}{0} & \cdots & a_m\binom{m}{m} \\ \hline b_0\binom{n}{0} & \cdots & b_n\binom{n}{n} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & b_0\binom{n}{0} & \cdots & b_n\binom{n}{n} \end{array}\right| \begin{array}{c} \uparrow \\ n \text{ rows} \\ \downarrow \\ \uparrow \\ m \text{ rows} \\ \downarrow \end{array} , \quad (10)$$

$$D^{-1} = \prod_{j=0}^{m+n-1} \binom{m+n-1}{j},$$

and this provides this straightforward result:

Corollary 2. *The quotient* $c(t) = p(t)/q(t)$ *of two polynomials* $p(t) = \sum_{j=0}^{m} a_j B_j^m(t)$, $q(t) = \sum_{j=0}^{n} b_j B_j^n(t)$ *in Bernstein form is irreducible if and only if*

$$\left|\begin{array}{ccccc} a_0\binom{m}{0} & \cdots & a_m\binom{m}{m} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & a_0\binom{m}{0} & \cdots & a_m\binom{m}{m} \\ \hline b_0\binom{n}{0} & \cdots & b_n\binom{n}{n} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & b_0\binom{n}{0} & \cdots & b_n\binom{n}{n} \end{array}\right| \begin{array}{c} \uparrow \\ n \text{ rows} \\ \downarrow \\ \uparrow \\ m \text{ rows} \\ \downarrow \end{array} \neq 0.$$

As we see the condition of non-vanishing resultant is pretty similar in both bases, except for the use of coefficients $\tilde{a}_j := a_j\binom{m}{j}$, $\tilde{b}_k := b_k\binom{n}{k}$ instead of a_j, b_k.

We shall call *reduced* the coefficients with the tilde. In order to tell reduced lists from standard one, we shall use double braces $\{\!\{, \}\!\}$ for the reduced ones. Of course, for polynomials of degree one, reduced coefficients are the same as the standard ones.

Using this reduced notation, the resultant is written as

$$\left|\begin{array}{ccccc} \tilde{a}_0 & \cdots & \tilde{a}_m & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \tilde{a}_0 & \cdots & \tilde{a}_m \\ \hline \tilde{b}_0 & \cdots & \tilde{b}_n & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \tilde{b}_0 & \cdots & \tilde{b}_n \end{array}\right| \begin{array}{c} \uparrow \\ n \text{ rows} \\ \downarrow \\ \\ \uparrow \\ m \text{ rows} \\ \downarrow \end{array},$$

and we may write a characterisation of irreducility of complex Bézier parametrisations in terms of their weights and control polygons:

Corollary 3. *A rational Bézier parametrisation with complex control polygon $\{z_0, \ldots, z_n\}$ and complex weights $\{\omega_0, \ldots, \omega_n\}$ is irreducible if and only if*

$$\left|\begin{array}{ccccc} \tilde{\omega}_0 z_0 & \cdots & \tilde{\omega}_n z_n & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \tilde{\omega}_0 z_0 & \cdots & \tilde{\omega}_n z_n \\ \hline \tilde{\omega}_0 & \cdots & \tilde{\omega}_n & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \tilde{\omega}_0 & \cdots & \tilde{\omega}_n \end{array}\right| \neq 0.$$

Example 1. *The arc of a circle described by the real control polygon $\{(1,0),(1,1),(0,1)\}$ and weights $\{1/2,1/2,1\}$, which produces the usual parametrisation*

$$c(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right), \quad t \in [0,1],$$

can be seen to be reducible.

The complex control polygon $\{1, 1+i, i\}$ and its weighted complex control polygon $\{1/2, (1+i)/2, i\}$,

$$\begin{vmatrix} \omega_0 z_0 & 2\omega_1 z_1 & \omega_2 z_2 & 0 \\ 0 & \omega_0 z_0 & 2\omega_1 z_1 & \omega_2 z_2 \\ \omega_0 & 2\omega_1 & \omega_2 & 0 \\ 0 & \omega_0 & 2\omega_1 & \omega_2 \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 1+i & i & 0 \\ 0 & \frac{1}{2} & 1+i & i \\ \frac{1}{2} & 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 & 1 \end{vmatrix} = 0.$$

are seen to produce a vanishing resultant.

The case of conic sections which are written as rational cubic Bézier curves has attracted some attention [10, 11]. In this case rational parametrisations of degree two are expressed as formal cubics through degree elevation [6], which involves multiplying numerator and denominator of the parametrisation by a linear polynomial. In such case, the parametrisation is reducible and we have:

Theorem 3. *A rational Bézier curve described by its complex control polygon $\{z_0, z_1, z_2, z_3\}$ and weights $\{\omega_0, \omega_1, \omega_2, \omega_3\}$ is a conic arc if and only if*

$$\begin{vmatrix} \omega_0 z_0 & 3\omega_1 z_1 & 3\omega_2 z_2 & \omega_3 z_3 & 0 & 0 \\ 0 & \omega_0 z_0 & 3\omega_1 z_1 & 3\omega_2 z_2 & \omega_3 z_3 & 0 \\ 0 & 0 & \omega_0 z_0 & 3\omega_1 z_1 & 3\omega_2 z_2 & \omega_3 z_3 \\ \omega_0 & 3\omega_1 & 3\omega_2 & \omega_3 & 0 & 0 \\ 0 & \omega_0 & 3\omega_1 & 3\omega_2 & \omega_3 & 0 \\ 0 & 0 & \omega_0 & 3\omega_1 & 3\omega_2 & \omega_3 \end{vmatrix} = 0.$$

Example 2. *The cubic Bézier curve with control polygon $\{(1,0),(1,4/5),(1/2,1),(0,1)\}$ and weights $\{2,5/3,4/3,1\}$ is in fact an arc of a conic.*

The complex control polygon for the curve is $\{1, 1+4i/5, 1/2+i, i\}$.

We may compute the Sylvester resultant of the parametrisation,

$$\begin{vmatrix} 2 & 5+4i & 2+4i & i & 0 & 0 \\ 0 & 2 & 5+4i & 2+4i & i & 0 \\ 0 & 0 & 2 & 5+4i & 2+4i & i \\ 2 & 5 & 4 & 1 & 0 & 0 \\ 0 & 2 & 5 & 4 & 1 & 0 \\ 0 & 0 & 2 & 5 & 4 & 1 \end{vmatrix}$$

and check that it vanishes, so that the parametrisation corresponds in fact to a conic arc.

Finally, multiplication of polynomials in Bernstein form is simpler and very similar to multiplication in monomial basis, using the reduced form, just taking into account the powers of $(1-t)$ and t. We shall make use of it in the following section.

If we multiply two polynomials with reduced coefficients $\{\!\{\tilde{a}_0, \ldots, \tilde{a}_m\}\!\}$ and $\{\!\{\tilde{b}_0, \ldots, \tilde{b}_n\}\!\}$, we get as a resulting list of reduced coefficients for the product,

$$\{\!\{\tilde{c}_0, \ldots, \tilde{c}_{m+n}\}\!\} = \{\!\{\tilde{a}_0, \ldots, \tilde{a}_m\}\!\} * \{\!\{\tilde{b}_0, \ldots, \tilde{b}_n\}\!\},$$
$$\tilde{c}_J = \sum_{j+k=J} \tilde{a}_j \tilde{b}_k, \; J = 0, \ldots, m+n, \quad (11)$$

which is a sort of convolution of both lists.

Example 3. *Multiplication of polynomials with reduced coefficients $\{\!\{\tilde{a}_0, \tilde{a}_1, \tilde{a}_2\}\!\}$ and $\{\!\{\tilde{b}_0, \tilde{b}_1, \tilde{b}_2\}\!\}$:*

The reduced coefficients for the product $\{\!\{\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4\}\!\}$ are

$$\tilde{c}_0 = \tilde{a}_0\tilde{b}_0, \; \tilde{c}_1 = \tilde{a}_0\tilde{b}_1 + \tilde{a}_1\tilde{b}_0, \; \tilde{c}_2 = \tilde{a}_0\tilde{b}_2 + \tilde{a}_1\tilde{b}_1 + \tilde{a}_2\tilde{b}_0,$$

$$\tilde{c}_3 = \tilde{a}_1\tilde{b}_2 + \tilde{a}_2\tilde{b}_1, \quad \tilde{c}_4 = \tilde{a}_2\tilde{b}_2.$$

5. Factorisation

Once we know a complex parametrisation is reducible, it would be nice to have a procedure for factoring out the common terms in numerator and denominator, both of them written in the Bernstein basis. The common factor is the greatest common divisor of both polynomials. We would also need a procedure for dividing polynomials in the Bernstein basis.

For singling out the greatest common divisor of two polynomials, we may use Euclid's algorithm for polynomials in Bernstein's basis.

In [12] there are versions of the algorithms for dividing polynomials and computing their greatest common divisor in their Bernstein form:

Proposition 1. *If $p(t) = \sum_{j=0}^{m} a_j B_j^m(t)$ and $q(t) = \sum_{j=0}^{n} b_j B_j^n(t)$ are two polynomials with $m \geq n \geq 0$, such that $a_m = p(1) \neq 0$, $b_n = q(1) \neq 0$, then there exist two uniquely determined polynomials $f(t)$ (quotient) and $r(t)$ (remainder), respectively of degrees $m-n$ and $n-1$, such that*

$$p(t) = q(t)f(t) + (1-t)^{m-n+1}r(t).$$

Notice that this division algorithm provides an actual *remainder* of degree m, since, though $r(t)$ has formally degree $n-1$, it is multiplied by a factor of degree $m-n+1$. In this sense, the algorithm just eliminates the contribution of polinomials t^m, $t^{m-1}(1-t)$,..., $t^n(1-t)^{n-m}$.

The division algorithm works even if $a_m = 0$, but we have to factor powers of t beforehand.

Another division algorithm can be produced for eliminating the contribution of high powers of $(1-t)$ instead, requiring $a_0 = p(0) \neq 0$, $b_0 = q(0) \neq 0$.

These requirements on the polynomials are not a big issue, since common $(1-t)$ and t factors are easily introduced and factored:

Lemma 1. *If a polynomial $p(t) = \sum_{j=0}^{n} a_j B_j^n(t)$ of degree n has reduced coefficients $\{\!\{\tilde{a}_0, \ldots, \tilde{a}_n\}\!\}$, then the polynomial $q(t) = (1-t)^J p(t) t^K$, has reduced coefficients $\{\!\{\underbrace{0,\ldots,0}_{K \text{ times}}, \tilde{a}_0, \ldots, \tilde{a}_n, \underbrace{0,\ldots,0}_{J \text{ times}}\}\!\}$.*

On the contrary, if we want to factor out powers of t and $(1-t)$:

Lemma 2. *If a polynomial $p(t)$ of degree n has reduced coefficients $\{\!\{0, \ldots, 0, \tilde{a}_J, \ldots, \tilde{a}_K, 0, \ldots, 0\}\!\}$, $0 \leq J < K \leq n$, then it can be expressed as $p(t) = (1-t)^{n-K} q(t) t^J$, where the polynomial $q(t)$ has reduced coefficients $\{\!\{\tilde{a}_J, \ldots, \tilde{a}_K\}\!\}$.*

Both lemmas are readily proven, taking into account the missing or added powers of t and $(1-t)$. The use of reduced coefficients $\tilde{a}_j$, $\tilde{b}_k$ is useful for avoiding cumbersome binomial coefficients on changing the degree of polynomials.

Example 4. *The polynomial $p(t) = 2(1-t)^2t^2 + (1-t)t^3$ with reduced coefficients $\{\!\{0, 0, 2, 1, 0\}\!\}$.*

It can be written as $p(t) = (1-t)t^2 q(t)$, $q(t) = 2(1-t)+t$, where the coefficients of $q(t)$ are $\{2, 1\}$.

Example 5. *Divide $p(t) = 4t^4 + t^3 - t^2 - t + 1$ by $q(t) = -t^3 + 6t^2 - 5t + 2$ in Bernstein form.*

The coefficients are seen to be $\{1, 3/4, 1/3, 0, 4\}$ ($\{\!\{1, 3, 2, 0, 4\}\!\}$ in reduced form) for $p(t)$ and $\{2, 1/3, 2/3, 2\}$ ($\{\!\{2, 1, 2, 2\}\!\}$ in reduced form) for $q(t)$.

Then $p(t) = q(t)f(t) + (1-t)^2 r(t)$, where the quotient $f(t)$ is of degree one and $r(t)$ is of degree two. Their reduced coefficients would be $\{\!\{\tilde{f}_0, \tilde{f}_1\}\!\}$ and $\{\!\{\tilde{r}_0, \tilde{r}_1, \tilde{r}_2\}\!\}$.

The reduced coefficients of $q(t)f(t)$ are

$$\{\!\{2\tilde{f}_0, \tilde{f}_0 + 2\tilde{f}_1, 2\tilde{f}_0 + \tilde{f}_1, 2\tilde{f}_0 + 2\tilde{f}_1, 2\tilde{f}_1\}\!\},$$

and we get for $p(t) - f(t)q(t)$,

$$\{\!\{1 - 2\tilde{f}_0, 3 - \tilde{f}_0 - 2\tilde{f}_1, 2 - 2\tilde{f}_0 - \tilde{f}_1, -2\tilde{f}_0 - 2\tilde{f}_1, 4 - 2\tilde{f}_1\}\!\},$$

from which we learn $\tilde{f}_1 = 2$, $\tilde{f}_0 = -2$.

For the remainder we get from the reduced coefficients of $p(t) - f(t)q(t)$, $\{\!\{5, 1, 4, 0, 0\}\!\}$, and after eliminating zeros, $\{\!\{5, 1, 4\}\!\}$.

Hence, the coefficients of $f(t)$ and $r(t)$ are respectively $\{-2, 2\}$, $\{5, 1/2, 4\}$, corresponding to polynomials,

$$f(t) = 4t - 2, \quad r(t) = 8t^2 - 9t + 5.$$

Since we have a division algorithm, we have an Euclid algorithm for calculating the greatest common divisor (gcd) of two polynomials. Even though the division algorithm is not the standard one and does not provide the same quotient and remainder, Euclid's algorithm gives out the same gcd up to a constant factor [12].

Theorem 4. ***Euclid's algorithm:*** *Let $p(t)$, $q(t)$ be two polynomials, where the degree of $p(t)$ is equal or greater than the degree of $q(t)$. The following algorithm computes $\gcd(p(t), q(t))$:*

- Step 0: Divide $p(t)$ by $q(t)$ and get remainder $r_1(t)$. If $r_1(t) \equiv 0$, then $q(t) = \gcd(p(t), q(t))$.
- Step 1: Divide $q(t)$ by $r_1(t)$ and get remainder $r_2(t)$. If $r_2(t) \equiv 0$, then $r_1(t) = \gcd(p(t), q(t))$.
- Step 2: Divide $r_1(t)$ by $r_2(t)$ and get remainder $r_3(t)$. If $r_3(t) \equiv 0$, then $r_2(t) = \gcd(p(t), q(t))$.
- ...
- Step N: Divide $r_{N-1}(t)$ by $r_N(t)$ and get remainder $r_{N+1}(t)$. If $r_{N+1}(t) \equiv 0$, then $r_N(t) = \gcd(p(t), q(t))$.

We may review now some of the previous examples:

Example 6. *Rational curve with control polygon $\{(1, 0), (1, 1), (0, 1)\}$ and weights $\{1, 1, 2\}$.*

We have seen this arc in Example 1. It is a quarter of a circle of unit radius, extending from $(1, 0)$ to $(0, 1)$ (see Figure 2):

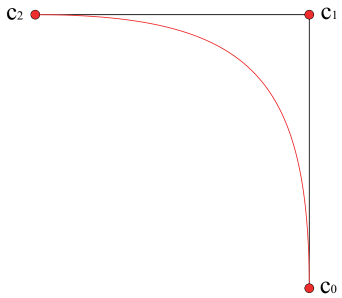

Figure 2: Arc of circle with control polygon $\{1, 1+i, i\}$ and weights $\{1, 1, 2\}$

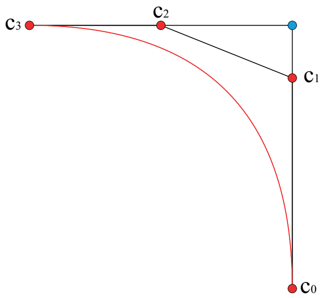

Figure 3: Arc of parabola with control polygon $\{(1,0),(1,1),(0,1)\}$ and $\{(1,0),(1,4/5),(1/2,1),(0,1)\}$ a rational cubic with weights $\{2, 5/3, 4/3, 1\}$

As a complex curve, the control polygon is $\{1, 1+i, i\}$, with list of weights $\{1, 1, 2\}$. This gives $\{1, 1+i, 2i\}$ as coefficients for the numerator of the parametrisation.

We divide the polynomials $p(t)$, $q(t)$ with respective reduced coefficients $\{\!\{1, 2+2i, 2i\}\!\}$ and $\{\!\{1, 2, 2\}\!\}$ to obtain a quotient with coefficients $\{i\}$ and $\{1-i, 2\}$ for the remainder $r_1(t)$.

Dividing $q(t)$ by $r_1(t)$ we get a null remainder. Hence, $\gcd(p(t), r(t)) = r_1(t)$.

Finally, we factor $r_1(t)$ from both $p(t)$ and $q(t)$: Dividing $q(t)$ by $r_1(t)$, we get coefficients $\{(1+i)/2, 1\}$ for the denominator, that is, the list of weights for the curve.

Dividing $p(t)$ by $r_1(t)$, we get coefficients $\{(1+i)/2, i\}$ for the numerator.

After factoring the weights, we get $\{1, i\}$ as complex control polygon for the curve.

We have already divided $q(t)$ by $r_1(t)$ and hence we have the complex representation of the curve with control polygon $\{1, i\}$ and complex weights $\{(1+i)/2, 1\}$.

We may review now the case of the degree-elevated arc of conic in Example 2.

Example 7. *Cubic Bézier curve with control polygon* $\{(1,0),(1,4/5),(1/2,1),(0,1)\}$ *and weights* $\{2, 5/3, 4/3, 1\}$.

We have already seen in Example 2 that the cubic parametrisation is reducible and describes in fact an arc of conic.

In order to obtain an irreducible parametrisation, we calculate the gcd of the numerator and the denominator of the parametrisation.

The complex control polygon for the curve is $\{1, 1+4i/5, 1/2+i, i\}$ and for the numerator $p(t)$ we get $\{2, 5/3+4i/3, 2/3+4i/3, i\}$, $\{\!\{2, 5+4i, 2+4i, i\}\!\}$ as reduced coefficients. For the denominator $q(t)$ we have $\{\!\{2, 5, 4, 1\}\!\}$ as reduced coefficients.

The quotient of $p(t)$ and $q(t)$ is i and the remainder $r_1(t)$ has reduced coefficients $\{\!\{2-2i, 5-i, 2\}\!\}$.

As next step of Euclid's algorithm, we divide $q(t)$ by $r_1(t)$ and we get a remainder $r_2(t)$ with coefficients $\{i, i/2\}$.

We divide $r_1(t)$ by $r_2(t)$ and get a null remainder and hence $\gcd(p(t), q(t)) = r_2(t)$.

Dividing $q(t)$ by $r_2(t)$, we get as list of reduced weights $\{\!\{-2i, -4i, -2i\}\!\}$ and $\{-2i, -2i, -2i\}$ as list of complex weights. Since all weights are equal, it is in fact an arc of parabola.

For the quotient of $p(t)$ and $r_2(t)$, we get for the numerator $\{\!\{-2i, 4-4i, 2\}\!\}$ as reduced and $\{-2i, 2-2i, 2\}$ as coefficients. After factoring the weights we get $\{1, 1+i, i\}$ as complex control polygon for the arc of parabola.

Hence our cubic arc is an arc of parabola with control polygon $\{(1,0),(1,1),(0,1)\}$. We see both control polygons in Figure 3.

6. From complex to real

We have seen how to get complex parametrisations starting with real ones. To go from complex parametrisations to real ones, it is simple. If we have a complex rational parametrisation of the form $c(t) = p(t)/q(t)$, we may produce a real parametrisation in the usual way, multiplying by the complex conjugate of the denominator $\overline{q(t)}$, $c(t) = p(t)\overline{q(t)}/q(t)\overline{q(t)}$, which doubles formally the degree of the parametrisation.

If the complex parametrisation $c(t) = p(t)/q(t)$ is described by a control polygon $\{z_0, \dots, z_n\}$ and a list of complex weights $\{\omega_0, \dots, \omega_n\}$, then the coefficients for the numerator $p(t)$ are $\{\omega_0 z_0, \dots, \omega_n z_n\}$. The control polygon for $\overline{q(t)}$ is simply $\{\overline{\omega_0}, \dots, \overline{\omega_n}\}$.

According to (11), multiplication by $\overline{q(t)}$ is accomplished using reduced control polygons. For the denominator we have a new list of reduced weights,

$$\{\!\{\tilde{w}_0, \dots, \tilde{w}_{2n}\}\!\}, \quad \tilde{w}_I = \sum_{j+k=I} \tilde{\omega}_j \overline{\tilde{\omega}_k}, \; I = 0, \dots, 2n,$$

and a new reduced control polygon for the numerator,

$$\{\!\{\tilde{p}_0, \dots, \tilde{p}_{2n}\}\!\}, \quad \tilde{p}_I = \sum_{j+k=I} z_j \tilde{\omega}_j \overline{\tilde{\omega}_k}, \; I = 0, \dots, 2n.$$

Let's see it with an example.

Example 8. *Arc of a circle defined by a control polygon* $\{1, i\}$ *and a list of weights,* $\{1+i, 2\}$.

We have already seen this parametrisation in Example 1. For a real parametrisation of degree two, we have a list of weights,

$$\{1+i,2\}*\{1-i,2\}=\{\!\{2,4,4\}\!\},$$

and for the numerator, with control polygon $\{1+i,2i\}$ we get

$$\{1+i,2i\}*\{1-i,2\}=\{\!\{2,4+4i,4i\}\!\},$$

and after factoring the weights for the denominator we get as control polygon, $\{1,1+i,i\}$.

That is, the real control polygon for the arc is $\{(1,0),(1,1),(0,1)\}$ and the list of weights is $\{2,2,4\}$ (or $\{1,1,2\}$)) as we already knew.

7. Some examples of curves

There are a number of curves in geometry which can be constructed using circles, such as trochoids [4], which are amenable for complex representation. Using inversion of plane curves, there is a number of other classical constructions which take advantage of this framework [8]. A catalog of cases may be found in [13].

Geometric inversion with respect to a circle of radius R centered at the origin is defined as $R^2/\bar{z}$. Since most of the examples are symmetric, we shall use the complex inversion $1/z$ instead, which amounts of considering $R=1$ and performing a reflection with respect to the real axis. The effect of complex inversion on complex control polygons and weights has already been discussed in Section 2.

It can be verified that the geometric inversion of a conic with respect to one of its points gives a circular rational cubic, for example, the cissoid of Diocles, while when the center of inversion does not belong to the conic, the inverted curve is a bicircular rational quartic, for instance, the cardioid and the Bernouilli lemniscate [13].

In the following examples we draw the original curve in blue, the inverted curve in red, showing with a dashed line the parts which are not included in the parametrisation.

7.1. *Cissoid of Diocles*

This curve has been used for solving the problem of doubling a cube: given a cube, find another cube whose volume is double that of the first one.

A rational parametrisation of the cissoid, expressed in complex form, for constant $a\in\mathbb{R}\setminus\{0\}$, is

$$\frac{2as^2(s-i)}{s^2+1}=\frac{2as^2}{s+i},\quad s\in\mathbb{R}.$$

Some terms can be factored and the degree becomes two.

This parametrisation is the inverse of the parametrisation of a parabola $\frac{s+i}{2as^2}$ with equation $y=2ax^2$. Another way of constructing a cissoid is inverting a parabola with respect to its vertex.

Taking $a=1/2$, if we start with an arc of the parabola of equation $y=x^2$, with vertex at the origin, with complex control polygon $\{-1+i,-i,1+i\}$ and trivial weights, after complex inversion we obtain and arc of cissoid with complex control polygon $\{-1/2-i/2,i,1/2-i/2\}$ and complex weights, $\{-1+i,-i,1+i\}$.

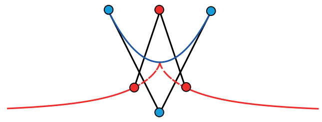

Figure 4: Arc of a cissoid of Diocles

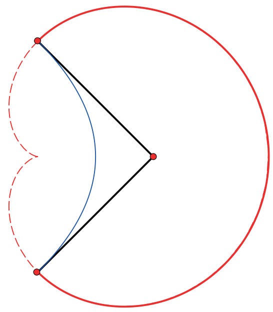

Figure 5: Arc of a cardioid as inverse of a parabola

Since the arc of parabola comprises its vertex, due to inversion, the arc of cissoid extends to infinity (see Figure 4).

7.2. *Cardioid*

A cardioid is an example of trochoid with a single cusp [4]. A complex rational parametrisation can be

$$2a\frac{1-s^2+2is}{(s^2+1)^2}=-\frac{2a}{(s+i)^2},\quad s\in\mathbb{R},$$

for constant $a\in\mathbb{R}\setminus\{0\}$, which allows us to lower the degree from four to two.

The inverse of this curve is parametrised as $-(s+i)^2/2a$ and it is hence a parabola with equation $x=(1-a^2y^2)/2a$.

A cardioid can be constructed as inverse of a parabola with respect to its focus. For instance, take $a=1$ and consider the parabola with equation $x=(1-y^2)/2$ and focus at the origin. An arc of this parabola is described by the complex control polygon $\{-i,1,i\}$ and trivial weights. By complex inversion we get an arc of cardioid with complex control polygon $\{i,1,-i\}$ and a list of complex weights $\{-i,1,i\}$ (see Figure 5).

7.3. *Lemniscate of Bernoulli*

Another well known planar curve is the lemniscate of Bernoulli. This curve may be seen as the inverse of an equilateral hyperbola with respect to its center. For a hyperbola of equation $x^2-y^2=a^2$, centered at the origin, a complex rational parametrisation for the lemniscate can be

$$a\frac{s+s^3+i(s-s^3)}{1+s^4}=\frac{a(1-i)s}{s^2-i},\quad s\in\mathbb{R},$$

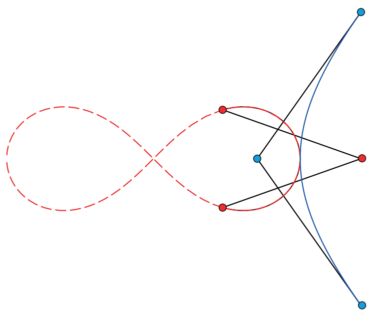

Figure 6: Arc of a lemniscate (red) as inverse of a arc of an equilateral hyperbola (blue)

for constant $a \in \mathbb{R} \setminus \{0\}$.

For simplicity, we take $a = 1$ and an arc of hyperbola defined by a complex control polygon, $\{\sqrt{2}-i, \sqrt{2}/2, \sqrt{2}+i\}$ with weights $\{1, \sqrt{2}, 1\}$.

The complex control polygon for the arc of lemniscate is $\{\frac{\sqrt{2}+i}{3}, \sqrt{2}, \frac{\sqrt{2}-i}{3}\}$ and the list of weights, $\{\sqrt{2}-i, 1, \sqrt{2}+i\}$. In Figure 6 we show both arcs in red and blue.

8. Conclusions

In this paper we have developed the complex framework for rational Bézier plane curves, showing its advantages in terms of degree lowering and the possibility of using two different groups of projective transformations for manipulating plane rational curves, while preserving their properties, going further than with the standard real framework. In fact, the complex framework can be considered as already included in the CAD paradigm, since it uses the same elements, vertices and weigths, and constructions and just involves extending the type of variable for weights from real to complex.

We have shown how to change from one framework to another, resorting to degree-elevation and factorisation of polynomials, making using of classical algebraic elements, such as resultants and Euclid's theorem, but for polynonials in Bernstein basis.

In particular, we have shown a simple procedure to check if a parametrisation is reducible or not in terms of the control polygon and weights.

Examples of curves for which the complex framework enables the possibility of using lower degree, but complex, parametrisations are included.

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