Differential Geometry and its Applications 103 (2026) 102333

Contents lists available at ScienceDirect

Differential Geometry and its Applications

journal homepage: www.elsevier.com/locate/difgeo

On the local classification of four-dimensional Lorentzian real reductive pairs

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ARTICLE INFO

Article history:
Received 22 November 2022
Received in revised form 2 January 2026
Accepted 8 January 2026
Available online xxxx
Communicated by B. Doubrov

ABSTRACT

One important piece of work in the classifications started by the seminal works of S. Lie [16,17] is the classification of four-dimensional Lorentzian real reductive pairs. This classification appeared, except for one paper, as preprints of the University of Oslo, where moreover many proofs and implications are (necessarily, due to their length) greatly abridged.

Given the relevance of these classifications, we think that an article on the origin, context, methods and relevance of that classification is in order. This is precisely the aim of the present paper. We intend to fill the gaps in the exposition of the ideas that structure these proofs.

On the other hand, motivated by the physical applications, we studied in [3] which of Lorentzian symmetric pairs furnish connected simply-connected Einstein-Yang-Mills spaces, obtaining 10 spaces. Since the calculations are rather long (some one hundred fifty pages, only for these cases), we confine ourselves in the present paper to carefully check the arguments for those 10 cases.

1. Introduction

As a continuation of some classifications by S. Lie, [12] obtained the local classification of real four-dimensional pseudo-Riemannian pairs with connected nontrivial isotropy groups pairs. Two-dimensional

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[1] The first and fourth authors have been supported by the Ministerio de Ciencia, Innovación y Universidades, Spain, under Projects PID2021-126124NB-I00 and PID2024-156578NB-I00.

https://doi.org/10.1016/j.difgeo.2026.102333

homogeneous spaces were classified locally in [16] and globally by G. D. Mostow in [18]; see also the preprint [14], where two-dimensional homogeneous spaces, both local and global, are classified. S. Lie also obtained some results on the classification of three-dimensional homogeneous spaces and described, in terms of vector fields, all subalgebras of the Lie algebra $\mathfrak{so}(4, \mathbb{C})$. A detailed account of these classifications can be found in [17]. The local classification of three-dimensional isotropically-faithful homogeneous spaces was obtained in [15], and two- and three-dimensional pseudo-Riemannian isotropically-faithful homogeneous spaces, both local and global, were classified in [4].

The importance of these classifications goes beyond Geometry. In Physics, the description and classification of models with many symmetries plays a central role. This is the case, in particular, of 4-dimensional spaces in General Relativity. In this vein, the present authors have classified in [3] the connected simply-connected four-dimensional Lorentzian symmetric spaces that are Einstein-Yang-Mills with respect to certain invariant metric connections, with the identity metric on the holonomy algebra. The interest of the applications of the pseudo-Riemannian pairs was already present in [13], where the coupling of electromagnetic (Abelian gauge theory) and gravitational fields is analyzed. The contribution of [3] moves to non-Abelian gauge theories. We recall that, as is well known, spacetimes were classified by Petrov [19] and Stephani et al. [22], among other authors. Spacetime groups have been recently classified by Anderson and Torre [1].

Now, those classifications [11,12] appeared, except for [13], as preprints of the University of Oslo. Furthermore, some of the proofs and implications are greatly abridged. This is done for the sake of keeping a reasonable length of the papers, which even so are quite long, but sometimes this makes the use and the comprobation of the results rather ellaborate. In this direction we have to mention the Ph. D. Thesis by Hicks [8], where he classifies spacetimes with symmetry and confirms results from [11–13]. However, as far as we know, Hicks's work is relatively unknown and it relies on a software based algorithm which, again, does not shed much light on the details of the mathematical steps of the classification.

We think that an article on the origin, context, methods and relevance of Komrakov's classification is in order. This is precisely the aim of the present paper: to fill some of the gaps in the exposition of the ideas that structure his proofs.

This article is organized as follows. In Section 1 we account for the context and relevance of Komrakov's classification. In Section 2, we explain with some detail the corresponding methods. We want to note that, even though we confine ourselves to the cases needed in [3] (cases motivated by the physical applications and the Einstein-Yang-Mills context), they comprise all the details covering the rest of the cases. Finally, in Subsection 2.10 we briefly recall Hicks's method and results in the Lorentzian case.

The reader may find the last details for the cases under study: $1.1^1(7)$, $1.1^2(9)$ and $1.1^2(10)$ in pages 20 and 20; from $2.1^2(1)$ to $2.1^2(5)$ in pages 20 to 24; and $3.5^2(2)$, $3.5^2(3)$ in pages 25 to 28.

2. The method

Let M be a homogeneous manifold, $H = H_p$ the isotropy group of an arbitrary point $p \in M$, and $(\mathfrak{k}, \mathfrak{h})$ the pair of Lie algebras corresponding to the pair (K, H) of Lie groups. According to [13, p. 34], the pair $(\mathfrak{k}, \mathfrak{h})$ locally uniquely defines the homogeneous manifold (see also Mostow [18, p. 614]). In order to classify the real reductive pairs $(\mathfrak{k}, \mathfrak{h})$, with $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$, it was first proved [12, p. 5, Lemma] that if a homogeneous manifold $M = K/H$ admits an invariant pseudo-Riemannian metric then the isotropy representation $\rho_* \colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$ of the pair $(\mathfrak{k}, \mathfrak{h})$ is faithful; and, moreover, there exists a basis of $\mathfrak{m}$ such that $\rho_*(\mathfrak{h})$ lies in one of the real forms of $\mathfrak{so}(4, \mathbb{C})$: $\mathfrak{so}(4)$, $\mathfrak{so}(1, 3)$ or $\mathfrak{so}(2, 2)$. These subalgebras admit, up to conjugacy, 6, 14 and 28 distinct Lie subalgebras (some of them depending on certain parameters), respectively. One then considers the complexifications of the pairs $(\mathfrak{k}, \mathfrak{h})$ and splits the solution into the following steps:

(i) To find (up to conjugation) all possible forms of the subalgebra $(\rho_*(\mathfrak{h}))^{\mathbb{C}} = \rho_*^{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}})$, which is equivalent to classify (up to conjugation with respect to $\mathrm{GL}(4, \mathbb{C})$) the subalgebras $\mathfrak{p}$ of $\mathfrak{so}(4, \mathbb{C})$;

(ii) For each such $\mathfrak{p}$, to find (up to equivalence of pairs) all complex pairs $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ such that $\rho_*^{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}})$ is conjugate to $\mathfrak{p}$;

(iii) For each such pair $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, to find (up to equivalence of pairs) all its real forms $(\mathfrak{k}, \mathfrak{h})$.

The three steps above give rise to corresponding numberings. Following the notation in [11], each type is denoted by a triple $n.m^k$, where n denotes the dimension of the subalgebra $\mathfrak{p}$ of $\mathfrak{so}(4, \mathbb{C})$, m the number of the complex pair $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ and k the number of the real form of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$.

Let $\mathfrak{h}$ be such a subalgebra and suppose that it acts naturally on $\mathbb{R}^4$. Then $(\mathfrak{g}, \mathbb{R}^4)$ is a faithful generalized module (for this concept, see §§2.1 below), and there is a one-to-one correspondence between the faithful generalized modules $(\mathfrak{h}, \mathbb{R}^4)$ and the corresponding subalgebras of $\mathfrak{so}(p, q)$. There are 187 distinct models of four-dimensional homogeneous pseudo-Riemannian manifolds with non-trivial isotropy. That figure lows to 76 in the Lorentzian case. The different cases are explicitly described in [12].

2.1. The method of classification of complex pairs

A detailed description of Komrakov's techniques for constructing pairs with a given faithful isotropy representation can be found in [15]. We shall use some definitions and results from [15] as presented in [11, pp. 14–17].

A *generalized module* is a pair $(\mathfrak{g}, U)$, where $\mathfrak{g}$ is a Lie algebra and U is a $\mathfrak{g}$-module. A generalized module $(\mathfrak{g}, U)$ is said to be *faithful* if the $\mathfrak{g}$-module U is faithful. The dimension of a generalized module $(\mathfrak{g}, U)$ is the dimension of the vector space U.

Let V be a vector space and $\mathfrak{g}$ a subspace of V. The pair $(V, \mathfrak{g})$ endowed with a bilinear map $\mathrm{b}\colon \mathfrak{g} \times V \to V$, $(x, v) \mapsto x.v$, is called a *virtual pair* if:

(1) $\mathfrak{g}.\mathfrak{g} \subset \mathfrak{g}$;

(2) The restriction $\mathrm{b}|_{\mathfrak{g}\times\mathfrak{g}}$ equips $\mathfrak{g}$ with a structure of Lie algebra, $([x, y] = x.y)$;

(3) $(\mathfrak{g}, V)$ is a generalized module with respect to b.

To any virtual pair $(V, \mathfrak{g})$ one can naturally assign the generalized module $(\mathfrak{g}, V/\mathfrak{g})$, which is said to be *associated to the virtual pair* $(V, \mathfrak{g})$. The *isotropy representation of a virtual pair* $(V, \mathfrak{g})$ is the map $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V/\mathfrak{g})$ defined by

$$\rho(x)(v + \mathfrak{g}) = x.v + \mathfrak{g} \quad \text{for all} \quad v \in V,\, x \in \mathfrak{g}.$$

The virtual pair $(V, \mathfrak{g})$ is called *isotropically faithful* if the homomorphism ρ is injective. Obviously, a virtual pair $(V, \mathfrak{g})$ is isotropically faithful if and only if the associated generalized module $(\mathfrak{g}, V/\mathfrak{g})$ is faithful.

Suppose now that $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. Any pair $(\mathfrak{g}, \mathfrak{h})$ can be regarded as a virtual pair with respect to the ordinary commutation restricted to $\mathfrak{h} \times \mathfrak{g}$. The *isotropy representation of a pair* $(\mathfrak{g}, \mathfrak{h})$ is the isotropy representation of the corresponding virtual pair. A pair $(\mathfrak{g}, \mathfrak{h})$ is called *isotropically faithful* if its isotropy representation is an injection. Two pairs $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are said to be *equivalent* if there exists an isomorphism of Lie algebras $\pi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\pi(\mathfrak{h}_1) = \mathfrak{h}_2$.

Given a generalized module $(\mathfrak{g}, U)$, let $\mathrm{L}(U, \mathfrak{g})$ be the vector space of linear maps from U to $\mathfrak{g}$. Since $(\mathfrak{g}, U)$ is a generalized module, we can consider expressions of the form $x.u$, for $x \in \mathfrak{g}$, $u \in U$. In turn, $\mathfrak{g}$ itself is a natural generalized $\mathfrak{g}$-module with the bracket, so $(\mathfrak{g}, \mathrm{L}(U, \mathfrak{g}))$ is also naturally a generalized module with $x.f\colon u \mapsto [x, f(u)] - f(x.u)$.

Let $q\colon \mathfrak{g} \to \mathrm{L}(U, \mathfrak{g})$ denote [15, p. 1] a linear map such that

$$q([x, y]) = x.q(y) - y.q(x) \quad \text{for all} \quad x, y \in \mathfrak{g}. \tag{2.1}$$

Then the map q is called a *virtual structure* on the generalized module $(\mathfrak{g}, U)$. In [11, Prop. 1, p. 15] it was proved that *given a virtual structure q on a generalized module $(\mathfrak{g}, U)$ and putting $V_q = \mathfrak{g} \times U$, the bilinear map $\mathfrak{g} \times V_q \to V_q$ given by*

$$x.(y, u) = ([x, y] + q(x)(u), x.u) \quad \text{for all} \quad x, y \in \mathfrak{g}, \ \ u \in U, \tag{2.2}$$

defines the virtual pair $(V_q, \mathfrak{g})$.

So, to any virtual structure on a generalized module $(\mathfrak{g}, U)$ one assigns the virtual pair $(\mathfrak{g} \times U, \mathfrak{g})$ defined by the formula (2.2). Moreover, any virtual pair $(V, \mathfrak{g})$ with associated generalized module $(\mathfrak{g}, U)$ can be constructed in this way.

Two virtual structures q_1 and q_2 on a generalized module $(\mathfrak{g}, U)$ are said to be *equivalent* if the virtual pairs $(V_{q_1}, \mathfrak{g})$ and $(V_{q_2}, \mathfrak{g})$ are isomorphic, that is, if there exists an isomorphism of vector spaces $\mathtt{H}\colon V_{q_1} \to V_{q_2}$ such that:

(*a*) $\mathtt{H}(\mathfrak{g}) = \mathfrak{g}$;

(*b*) $\mathtt{H}(x.v) = \mathtt{H}(x).\mathtt{H}(v)$ for all $x \in \mathfrak{g}$, $v \in V_{q_1}$.

Suppose that q_1 and q_2 are virtual structures on a generalized module $(\mathfrak{g}, U)$ and that there exists a map $h \in \mathrm{L}(U, \mathfrak{g})$ such that $q_1(x) - q_2(x) = x.h$ for all $x \in \mathfrak{g}$. Then [11, Prop. 2, p. 16], the virtual structures q_1 and q_2 are equivalent. Thus, the classification (up to isomorphism) of all virtual pairs $(V, \mathfrak{g})$ for a given generalized module $(\mathfrak{g}, U)$ reduces to the classification (up to equivalence) of all virtual structures on the generalized module $(\mathfrak{g}, U)$.

Now, in [15, Prop. 4, p. 23] it was proved that *two virtual structures q_1 and q_2 on a generalized module $(\mathfrak{g}, U)$ are equivalent if and only if there exist an automorphism (f, p) of the generalized module $(\mathfrak{g}, U)$ and a linear map $h\colon U \to \mathfrak{g}$ such that*

$$q_2(x) = f \circ q_1(f^{-1}(x)) \circ p^{-1} - x.h \quad \text{for all} \quad x \in \mathfrak{g}. \tag{2.3}$$

A corollary follows [15, Cor. 1, p. 23], stating: *suppose that q_1 and q_2 are virtual structures on a generalized module $(\mathfrak{g}, U)$ and that there exists a map $h \in \mathrm{L}(U, \mathfrak{g})$ such that $q_1(x) - q_2(x) = x.h$ for all $x \in \mathfrak{g}$. Then the virtual structures q_1 and q_2 are equivalent.*

Komrakov proved two propositions. The first one [11, Prop. 3, p. 16] states that *a necessary and sufficient condition for a map $q\colon \mathfrak{g} \to \mathrm{L}(U, \mathfrak{g})$ to be a virtual structure on the generalized module $(\mathfrak{g}, U)$ is:*

$$C([x, y]) = A(x)C(y) - C(y)B(x) - A(y)C(x) + C(x)B(y), \quad x, y \in \mathfrak{g}, \tag{2.4}$$

where $C(x)$ can be viewed as the matrix of the map $q(x)$ with respect to the basis $\mathcal{B} = \{e_1, \dots, e_n, u_1, \dots, u_4\}$ of $\mathfrak{g}$, being $\mathcal{E} = \{e_1, \dots, e_n\}$ a basis of the n-dimensional subalgebra $\mathfrak{h}$ and $\mathcal{U} = \{u_1, \dots, u_4\}$ a basis of the vector space U.

The second proposition [11, Prop. 4, p. 16] states: *Suppose that q_1 and q_2 are virtual structures on the generalized module $(\mathfrak{g}, U)$ with matrices C_1 and C_2 with respect to the basis $\mathcal{B}$ and that a matrix $\mathcal{H} \in M(n \times 4, \mathbb{C})$ exists such that*

$$C_1(x) - C_2(x) = A(x)\mathcal{H} - \mathcal{H}B(x), \quad \forall x \in \mathfrak{g}. \tag{2.5}$$

Then q_1 and q_2 are equivalent.

These two facts are nothing but the vanishing of the Lie algebra cohomology $H^1(\mathfrak{g}, L(U, \mathfrak{g}))$ (for example, see [5], [10]) although we keep below the procedure as shown in [11] for the benefit of the reader.

Komrakov's notes that the expressions (2.4) and (2.5) are linear in $x, y \in \mathfrak{g}$. Therefore, in order to ensure that these conditions are satisfied for all $x, y \in \mathfrak{g}$, one must only check that they hold for $x, y \in \mathcal{E}$.

In the present paper we consider that $\dim \mathfrak{h} \in \{1,2,3\}$. In Komrakov's construction, to completely know the structure of the Lie algebras involved, one needs many times a nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{h}$. When $\dim \mathfrak{h} = 1$, one considers the nilpotent subalgebra $\mathfrak{n}$ spanned by e_1; that is, $\mathfrak{n} = \mathfrak{h}$. For $\dim \mathfrak{h} > 1$, one can consider, depending on each pair, the nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{h}$ generated by one or more among the vectors $\{e_1, e_2, e_3\}$.

Suppose now that $\mathfrak{n}$ is a nilpotent real Lie algebra and V a finite-dimensional real $\mathfrak{n}$-module. Then the vector space $V^{\mathbb{C}}$ is a $\mathfrak{n}^{\mathbb{C}}$-module. Since the field $\mathbb{C}$ is algebraically closed, the $\mathfrak{n}^{\mathbb{C}}$-module $V^{\mathbb{C}}$ is a direct sum of primary components. A linear function $\alpha \in (\mathfrak{n}^{\mathbb{C}})^*$ such that $(V^{\mathbb{C}})^{\alpha}(\mathfrak{n}^{\mathbb{C}}) \neq \{0\}$ is called a *weight* of the $\mathfrak{n}^{\mathbb{C}}$-module $V^{\mathbb{C}}$. Denote by $\Delta^{\mathbb{C}}$ the set of all weights of $V^{\mathbb{C}}$. Then $V^{\mathbb{C}} = \oplus_{\alpha \in \Delta^{\mathbb{C}}} (V^{\mathbb{C}})^{\alpha}(\mathfrak{n}^{\mathbb{C}})$. Given $\lambda \in \Delta^{\mathbb{C}}$, denote by $V^{\lambda}(\mathfrak{n})$ the subspace of V defined by

$$V^{\lambda}(\mathfrak{n}) = V \cap \big((V^{\mathbb{C}})^{\lambda}(\mathfrak{n}^{\mathbb{C}}) + (V^{\mathbb{C}})^{\bar{\lambda}}(\mathfrak{n}^{\mathbb{C}}) \big). \tag{2.6}$$

Note that if $\lambda \in \Delta^{\mathbb{C}}$ and $\lambda = \bar{\lambda}$, then $\lambda \in \mathfrak{n}^*$ and $\big(V^{\lambda}(\mathfrak{n})\big)^{\mathbb{C}} = (V^{\mathbb{C}})^{\lambda}(\mathfrak{n}^{\mathbb{C}})$ (see Bourbaki [2, Ch. VII, §1]). So the new definition (2.6) of $V^{\lambda}(\mathfrak{n})$ coincides with the old one whenever $\lambda = \bar{\lambda}$.

Remark 2.1. Komrakov et al. [15, p. 30] use $V^{\lambda}(\mathfrak{h})$, whose natural definition is

$$V^{\lambda}(\mathfrak{h}) = \big\{ v \in V : (\rho_h - \lambda(h))^N(v) = 0 \;\text{ for all }\; h \in \mathfrak{h} \;\text{ and }\; N \;\text{ large enough} \big\},$$

where ρ_h denotes the action of h. On the other hand, given an $\mathfrak{h}$-module V, with both $\mathfrak{h}$ and V real, the authors of [15, p. 30] consider the complexification $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, and given a complex number λ they proceed as before, but for the complefixied Lie algebra $(V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}})$. That is, they consider the subspace of $V^{\mathbb{C}}$ defined by

$$(V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}}) = \big\{ v \in V^{\mathbb{C}} : (\rho_h - \lambda(h))^N(v) = 0 \;\text{ for all }\; h \in \mathfrak{h}^C \;\text{ and }\; N \;\text{ large enough} \big\}.$$

Now, if $\lambda \notin \mathbb{R}$, this has sense only in the complexified space. Later, they define [15, p. 30] the space $V^{\lambda}(\mathfrak{h})$ as the intersection of V with the sum of $(V^{\mathbb{C}})^{\lambda}(\mathfrak{h}^{\mathbb{C}})$ and the corresponding subspace for $\bar{\lambda}$. Thus, if $\lambda \in \mathbb{R}$, we have two definitions for $V^{\lambda}(\mathfrak{h})$, but both coincide. This is different from the usual treatment for semisimple Lie algebras. Actually, such a subspace is defined only for nilpotent $\mathfrak{h}$. For instance, for a Cartan subalgebra of a semisimple Lie algebra.

Suppose that Δ is a subset of $\Delta^{\mathbb{C}}$ such that for every $\lambda \in \Delta^{\mathbb{C}}$ the set $\Delta \cap \{\lambda, \bar{\lambda}\}$ contains exactly one element. Then the decomposition $V = \oplus_{\lambda \in \Delta} V^{\lambda}(\mathfrak{n})$ is called a *generalized primary decomposition* of V and its terms are the *primary components*. Note that, generally speaking, the set Δ is not defined uniquely. However, the summands of the generalized primary decomposition are independent of Δ.

Suppose now that $(V, \mathfrak{g})$ is a virtual pair and $(\mathfrak{g}, U = V/\mathfrak{g})$ is the generalized module associated with $(V, \mathfrak{g})$. One then has the following result [11, Prop. 5, p. 17]: *Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{g}$. Then:*

(1) *A necessary and sufficient condition for the $\mathfrak{n}$-module V to be a direct sum of primary components is that the $\mathfrak{n}$-modules $\mathfrak{g}$ and U be direct sums of primary components.*

(2) *There exists a section $s\colon U \to V$ of the canonical surjection $\pi\colon V \to U$ such that for every $\alpha \in \mathfrak{n}^*$ the following condition holds*:

$$s(U^{\alpha}(\mathfrak{n})) \subset V^{\alpha}(\mathfrak{n}). \tag{2.7}$$

Suppose now that s is a section of the canonical surjection $\pi\colon V \to U$. The section s is said to be *consistent with the subalgebra* $\mathfrak{h}$ if

$$s(U^{\alpha}(\mathfrak{h})) \subset V^{\alpha}(\mathfrak{h}) \quad \text{for all} \quad \alpha \in \mathfrak{h}^*. \tag{2.8}$$

From (2) above it follows that such a section always exists.

One has furthermore the next result [11, Prop. 6, p. 17]: *Suppose that s is a section of the canonical surjection $\pi\colon V \to U$ consistent with the subalgebra $\mathfrak{h}$. Then the corresponding virtual structure $q_s\colon \mathfrak{g} \to \mathrm{L}(U, \mathfrak{g})$ on the generalized module $(\mathfrak{g}, U)$ satisfies:*

$$q_s(\mathfrak{g}^{\alpha}(\mathfrak{h}))(U^{\beta}(\mathfrak{h})) \subset \mathfrak{g}^{\alpha+\beta}(\mathfrak{h}) \quad \text{for all} \quad \alpha, \beta \in \mathfrak{h}^*. \tag{2.9}$$

A virtual structure q on the generalized module $(\mathfrak{g}, U)$ is said to be *primary* (with respect to $\mathfrak{h}$) if q satisfies (2.9). From the results above related to [11, Props. 5, (2) and 6] it follows that every virtual structure is equivalent to a certain primary virtual structure.

One more result [11, Prop. 7, p. 17] states: *Suppose that q is a primary virtual structure (with respect to $\mathfrak{h}$) on the generalized module $(\mathfrak{g}, U)$ and that $(V_q, \mathfrak{g})$ is the corresponding virtual pair. Then:*

$$V_q^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \quad \text{for all} \quad \alpha \in \mathfrak{h}^*. \tag{2.10}$$

A virtual pair $(V, \mathfrak{g})$ is said to be *trivial* (see [15, p. 29]) if there exists a submodule U of the generalized module $(\mathfrak{g}, V)$ such that $V = U \oplus \mathfrak{g}$. Note that a trivial virtual pair $(V, \mathfrak{g})$ is uniquely defined (up to isomorphism) by the corresponding generalized module $(\mathfrak{g}, V/\mathfrak{g})$.

One more proposition [11, Prop. 8, p. 17], states: *Let q be a virtual structure on the generalized module $(\mathfrak{g}, U)$. Then a necessary and sufficient condition for the virtual pair $(V_q, \mathfrak{g})$ to be trivial is that q be equivalent to the zero map.*

Moreover, in [15, Prop. 12, p. 29] it was proved the following result: *Suppose that $q\colon \mathfrak{g} \to \mathrm{L}(U, \mathfrak{g})$ is a virtual structure on the generalized module $(\mathfrak{g}, U)$ and $\mathfrak{s}$ is a semisimple subalgebra of $\mathfrak{g}$. Then there exists a virtual structure $\tilde{q}$ equivalent to q such that $\tilde{q}(\mathfrak{s}) = \{0\}$.*

As a corollary [15, p. 29], *if $\mathfrak{g}$ is a semisimple Lie algebra, then every virtual pair $(V, \mathfrak{g})$ is trivial* (this is also a consequence of Whitehead Lemma [10, p.77, 95]).

A pair $(\mathfrak{g}, \mathfrak{h})$ is called *trivial* if an abelian ideal $\mathfrak{a}$ in the Lie algebra $\mathfrak{g}$ exists such that $\mathfrak{h} \oplus \mathfrak{a} = \mathfrak{g}$. If $(\mathfrak{g}, \mathfrak{h})$ is a trivial pair, the corresponding virtual pair $(\mathfrak{g}, \mathfrak{h})$ is also trivial, but not conversely. A trivial pair is uniquely defined (up to equivalence) by the corresponding generalized module $(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$.

Let $\mathfrak{g}$ be one of the subalgebras of $\mathfrak{so}(4, \mathbb{C})$ listed in [11, Th. 1, p. 6] (see also (2.11) below). Assume that the Lie algebra $\mathfrak{g}$ acts naturally on $\mathbb{C}^4$; then $(\mathfrak{g}, \mathbb{C}^4)$ is a faithful generalized module. The enumeration of the modules obtained in this way coincides with that of the corresponding subalgebras of $\mathfrak{so}(4, \mathbb{C})$ in [11, Th. 1].

A pair $(\mathfrak{g}, \mathfrak{h})$ (respectively, virtual pair $(V, \mathfrak{g})$) is said to be of type $n.m$ if the corresponding generalized module $(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ (respectively, generalized module $(\mathfrak{g}, V/\mathfrak{g})$) is isomorphic to the module $n.m$, that is, to the generalized module $(\mathfrak{g}, \mathbb{C}^4)$, where $\mathfrak{g}$ is the subalgebra of $\mathfrak{so}(4, \mathbb{C})$ labeled as $n.m$ in [11, Th. 1].

Let $(V, \mathfrak{g})$ be a virtual pair of type $n.m$. Then one can, without loss of generality, identify the Lie algebra $\mathfrak{g}$ with the subalgebra $n.m$ of the Lie algebra $\mathfrak{so}(4, \mathbb{C})$.

The ten real pairs listed in Table 3 in [3] come from four complex pairs, namely, 1.1.7, 2.1.1, 2.1.2 and 3.2.5. We recall, in the following Subsections 2.2 to 2.5, the results in [11] involving those four complex pairs.

2.2. Complex pairs for n =1,2,3

Komrakov [12, Th. 1, pp. 7, 8] (see also [20]) proved that *for $n = 1, 2$ or 3, any non-zero subalgebra of the Lie algebra $\mathfrak{so}(4, \mathbb{C})$ is conjugate (with respect to* $\mathrm{GL}(4, \mathbb{C})$*) to one and only one of the following subalgebras*:

$$\dim \mathfrak{h} = 1$$

$$1.1 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix} \qquad 1.2 \quad \begin{pmatrix} x & x & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -x & -x \end{pmatrix}$$

$$|\lambda| < 1,\ -\pi/2 < \arg\lambda \leqslant \pi/2 \quad \text{or} \quad |\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \pi/2$$

$$1.3 \quad \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad 1.4 \quad \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\dim \mathfrak{h} = 2$$

$$2.1 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix} \qquad 2.2 \quad \begin{pmatrix} x & y & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix} \tag{2.11}$$

$$|\lambda| < 1, \quad \text{or} \quad |\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \pi$$

$$2.3 \quad \begin{pmatrix} x & y & 0 & x \\ 0 & -x & -x & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & x \end{pmatrix} \qquad 2.4 \quad \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad 2.5 \quad \begin{pmatrix} 0 & x & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix}$$

$$\dim \mathfrak{h} = 3$$

$$3.1 \quad \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix} \qquad 3.2 \quad \begin{pmatrix} x & y & 0 & z \\ 0 & \lambda x & -z & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}$$

$$\operatorname{Re}\lambda > 0 \quad \text{or} \quad \operatorname{Re}\lambda = 0,\ \operatorname{Im}\lambda \geqslant 0$$

$$3.3 \quad \begin{pmatrix} 0 & y & 0 & z \\ 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -x \end{pmatrix} \quad 3.4 \quad \begin{pmatrix} x & y & 0 & 0 \\ z & -x & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix} \quad 3.5 \quad \begin{pmatrix} 2x & y & 0 & 0 \\ 2z & 0 & -2y & 0 \\ 0 & -z & -2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For convenience, instead of the classical expression of $\mathfrak{so}(4, \mathbb{C})$ as the skew-symmetric matrices, one uses the expression

$$\left\{ \begin{pmatrix} x & z & 0 & t \\ u & y & -t & 0 \\ 0 & v & -x & -u \\ -v & 0 & -z & -y \end{pmatrix} : x, y, z, t, u, v \in \mathbb{C} \right\},$$

which is obtained, as we have checked, by conjugation with

$$\begin{pmatrix} 0 & 1 & 0 & \mathrm{i} \\ 1 & 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & 1 & 0 \\ 0 & \mathrm{i} & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{C}).$$

With this new expression, one writes $\mathfrak{so}(4,\mathbb{C}) = \{xe_1 + ye_2 + ze_3 + te_4 + ue_5 + ve_6\}$, where $e_1, \ldots, e_6$ are the corresponding matrices with 0's and 1's.

The proof splits the study [11, p. 8] into solvable and non-solvable subalgebras. In the former case, any Borel subalgebra is conjugate to

$$\mathfrak{g} = \begin{pmatrix} x & z & 0 & t \\ 0 & y & -t & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix} = xe_1 + ye_2 + ze_3 + te_4, \quad x, y, z, t \in \mathbb{C}.$$

Indeed, it is clear that we can choose $\mathbb{C}e_1 + \mathbb{C}e_2$ as a Cartan subalgebra, and $\mathbb{C}e_i$, $i = 3, \ldots, 6$, are the root spaces of $\mathfrak{so}(4,\mathbb{C})$. The canonical Borel subalgebra is the sum of the Cartan subalgebra and a choice of positive root spaces. The others are always conjugate to this one [9, p. 84]. It is then immediate that the algebra structure of $\mathfrak{g}$ is

$$\begin{array}{c|cccc} [\,,\,] & e_1 & e_2 & e_3 & e_4 \\ \hline e_1 & 0 & 0 & e_3 & e_4 \\ e_2 & 0 & 0 & -e_3 & e_4 \\ e_3 & -e_3 & e_3 & 0 & 0 \\ e_4 & -e_4 & -e_4 & 0 & 0 \end{array} \tag{2.12}$$

from which it follows that the group of automorphisms of $\mathfrak{g}$ is

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -a & c & 0 \\ b & b & 0 & d \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ a & a & 0 & d \\ b & -b & c & 0 \end{pmatrix} : a, b \in \mathbb{C},\ c, d \in \mathbb{C}^* \right\}, \tag{2.13}$$

with respect to the basis $\{e_1, \ldots, e_4\}$.

To get all the subalgebras of $\mathfrak{g}$, one takes the composition series [10, §7] of $\mathfrak{g}$,

$$\{0\} \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g},$$

and uses the following algorithm [11, p. 9]:

(1) One describes all subalgebras $\mathfrak{h}_1$ of the ideal $\mathfrak{g}_1$ (up to automorphisms of $\mathcal{A}$) and construct for each $\mathfrak{h}_1$:
   (a) $N(\mathfrak{h}_1) \cap \mathfrak{g}_2$;
   (b) the subgroup $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}_1)$ of $\mathcal{A}$ of all automorphisms of $\mathfrak{g}$ preserving the subalgebra $\mathfrak{h}_1$.
(2) One describes all subalgebras $\mathfrak{h}_2$ of the Lie algebra $N(\mathfrak{h}_1) \cap \mathfrak{g}_2$ (up to automorphisms of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}_1)$) such that $\mathfrak{h}_2 \cap \mathfrak{g}_1 = \mathfrak{h}_1$ and construct for each $\mathfrak{h}_2$:
   (a) $N(\mathfrak{h}_2) \cap \mathfrak{g}$;
   (b) the subgroup $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}_2)$ of $\mathcal{A}$ of all automorphisms of $\mathfrak{g}$ preserving the subalgebra $\mathfrak{h}_2$.
(3) One proceeds similarly for higher dimensions.

One composition series of $\mathfrak{g}$ is

$$\{0\} \subset \mathbb{C}e_3 \subset \mathbb{C}e_3 + \mathbb{C}e_4 \subset \mathbb{C}e_1 + \mathbb{C}e_3 + \mathbb{C}e_4.$$

Since the second term is an abelian ideal and with the purpose of saving one step in the algorithm, we can abridge the series to

$$\mathfrak{g}_1 = \mathbb{C}e_3 + \mathbb{C}e_4, \quad \mathfrak{g}_2 = \mathbb{C}e_1 + \mathbb{C}e_3 + \mathbb{C}e_4.$$

The subalgebras we are searching for are constructed in three steps along the series above, starting from $\mathfrak{g}_1$, as it follows.

First. The subalgebras $\mathfrak{h}_1$ of $\mathfrak{g}_1$ are

$$\mathbb{C}(\alpha e_3 + \beta e_4),\ \alpha, \beta \in \mathbb{C}, \quad \mathbb{C}e_3 \oplus \mathbb{C}e_4,$$

which reduce to

$$\{0\}, \quad \mathbb{C}(e_3 + e_4), \quad \mathbb{C}e_3, \quad \mathbb{C}e_3 \oplus \mathbb{C}e_4,$$

up to transformations belonging to $\mathcal{A}$ with: $a = b = 0, c = 1/\alpha, d = 1/\beta$ (the first type of matrices in (2.13)) for $\alpha \neq 0 \neq \beta$; $a = b = 0, c = 1/\alpha, d = 1$ for $\alpha \neq 0, \beta = 0$; and $a = b = 0, c = 1/\beta, d = 1$ (the second type of matrices) for $\alpha = 0, \beta \neq 0$. Since $[\mathfrak{g}_2, \mathfrak{h}_1] \subset \mathfrak{h}_1$ for all of them, then $\mathfrak{g}_2 \subset N(\mathfrak{h}_1)$ so that $N(\mathfrak{h}_1) \cap \mathfrak{g}_2$ equals $\mathfrak{g}_2$.

Second. We now seek for the subalgebras $\mathfrak{h}_2$ of $\mathfrak{g}_2$ (modulo automorphisms in $\mathcal{A}$) such that $\mathfrak{h}_2 \cap \mathfrak{g}_1 = \mathfrak{h}_1$.

- For $\mathfrak{h}_1 = \{0\}$, we have $\mathfrak{h}_2 = \{0\}$ and $\mathbb{C}(e_1 + \alpha e_3 + \beta e_4)$, the latter being equivalent to $\mathbb{C}e_1$ modulo $\mathcal{A}$.
- For $\mathfrak{h}_1 = \mathbb{C}(e_3 + e_4)$, we have $\mathfrak{h}_2 = \mathbb{C}(e_3 + e_4)$ and $\mathbb{C}(e_1 + ae_3 + be_4) + \mathbb{C}(e_3 + e_4)$, $a, b \in \mathbb{C}$, and the latter is equivalent to $\mathbb{C}e_1 + \mathbb{C}(e_3 + e_4)$ modulo $\mathcal{A}$.
- For $\mathfrak{h}_1 = \mathbb{C}e_3$, we have $\mathfrak{h}_2 = \mathbb{C}e_3$ and $\mathbb{C}(e_1 + ae_3 + be_4) + \mathbb{C}(e_3)$, $a, b \in \mathbb{C}$, and the latter is equivalent to $\mathbb{C}e_1 + \mathbb{C}e_3$ modulo $\mathcal{A}$.
- For $\mathfrak{h}_1 = \mathbb{C}e_3 + \mathbb{C}e_4$, we have $\mathfrak{h}_2 = \mathbb{C}e_3 + \mathbb{C}e_4$ and $\mathbb{C}e_1 + \mathbb{C}e_3 + \mathbb{C}e_4$.

Collecting all these subalgebras $\mathfrak{h}_2$, together with the intersection $N(\mathfrak{h}_2) \cap \mathfrak{g}$ (which is readily obtained from the table (2.12)), we have

$$\begin{gathered}
(\{0\}, \mathfrak{g}), \quad (\mathbb{C}e_1, \mathbb{C}e_1 + \mathbb{C}e_2), \quad (\mathbb{C}(e_3 + e_4), \mathbb{C}e_1 + \mathbb{C}e_3 + \mathbb{C}e_4), \\
(\mathbb{C}e_1 + \mathbb{C}(e_3 + e_4), \mathbb{C}e_1 + \mathbb{C}(e_3 + e_4)), \quad (\mathbb{C}e_3, \mathfrak{g}), \quad (\mathbb{C}e_1 + \mathbb{C}e_3, \mathbb{C}e_1 + \mathbb{C}e_2 + \mathbb{C}e_3), \\
(\mathbb{C}e_3 + \mathbb{C}e_4, \mathfrak{g}), \quad (\mathbb{C}e_1 + \mathbb{C}e_3 + \mathbb{C}e_4, \mathfrak{g}).
\end{gathered}$$

Third. We now find the subalgebras $\mathfrak{h}$ of $N(\mathfrak{h}_2) \cap \mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{g}_2 = \mathfrak{h}_2$.

- For $\mathfrak{h}_2 = \{0\}$, we have either $\mathfrak{h} = \{0\}$ or $\mathfrak{h} = \mathbb{C}(\alpha e_1 + \lambda e_2 + \mu e_3 + \nu e_4)$, with $\lambda \neq 0$. The possibilities for the second subalgebra can be reduced under automorphisms of $\mathcal{A}$. There are different cases. If $\alpha = 0$, the subalgebra is equivalent to $\mathbb{C}e_2$ with $a = \mu$, $b = -\nu$, $c = d = 1$ in (2.12). For $\alpha \neq 0$ the subalgebra is of the type $\mathfrak{h} = \mathbb{C}(e_1 + \lambda e_2 + \mu e_3 + \nu e_4)$ so that: when $\lambda \neq \pm 1$, it is equivalent to $\mathbb{C}(e_1 + \lambda e_2)$ with $a = \mu/(\lambda - 1)$, $b = -\nu/(1 + \lambda)$, $c = 1$, $d = 1$; when $\lambda = 1, \mu \neq 0$, it is equivalent to $\mathbb{C}(e_1 + e_2 + e_3)$ with $b = -\nu/2$, $c = 1/\mu$, $a = d = 1$; when $\lambda = 1, \mu = 0$, it is equivalent to $\mathbb{C}(e_1 + e_2)$ with $b = -\nu/2$, $a = c = d = 1$; when $\lambda = -1$, $\nu \neq 0$, it is equivalent to $\mathbb{C}(e_1 + e_2 + e_3)$ with $b = -\mu/2$, $d = 1/\nu$, $a = c = 1$ (the second type of matrices in (2.12)); and when $\lambda = -1$, $\nu = 0$, it is equivalent to $\mathbb{C}(e_1 + e_2)$ with $b = -\mu/2$, $a = c = d = 1$. The second type of matrices in (2.12) can be used to restrict the range of the parameter λ to $\{\operatorname{Re}\lambda > 0\} \cup \{\operatorname{Re}\lambda = 0, \operatorname{Im}\lambda > 0\}$.
- For $\mathfrak{h}_2 = \mathbb{C}e_1$, we have $\mathfrak{h} = \mathbb{C}e_1$ and $\mathfrak{h} = \mathbb{C}e_1 + \mathbb{C}e_2$.
- For $\mathfrak{h}_2 = \mathbb{C}(e_3 + e_4)$ we have $\mathfrak{h} = \mathbb{C}(e_3 + e_4)$.
- For $\mathfrak{h}_2 = \mathbb{C}e_1 + \mathbb{C}(e_3 + e_4)$, we have $\mathfrak{h} = \mathbb{C}e_1 + \mathbb{C}(e_3 + e_4)$.
- For $\mathfrak{h}_2 = \mathbb{C}e_3$, we have either $\mathfrak{h} = \mathbb{C}e_3$ or $\mathfrak{h} = \mathbb{C}e_3 + \mathbb{C}(\alpha e_1 + \lambda e_2 + \mu e_3 + \nu e_4)$, $\lambda \neq 0$. In a very similar way to the case $\mathfrak{h}_2 = \{0\}$, the possibilities of the second subalgebra modulo $\mathcal{A}$ reduce to $\mathfrak{h} = \mathbb{C}e_2 + \mathbb{C}e_3$,

$\mathfrak{h} = \mathbb{C}e_3 + \mathbb{C}(e_1 + \lambda e_2)$, $\mathfrak{h} = \mathbb{C}e_3 + \mathbb{C}(e_1 - e_2 + e_4)$ and $\mathbb{C}e_3 + \mathbb{C}(e_1 - e_2)$. We cannot use the second type of matrices in (2.12) to restrict the range of the parameter λ since they do not preserve $\mathfrak{h}_2$.

- For $\mathfrak{h}_2 = \mathbb{C}e_1 + \mathbb{C}e_3$ we have $\mathfrak{h} = \mathbb{C}e_1 + \mathbb{C}e_3$ and $\mathbb{C}e_1 + \mathbb{C}e_2 + \mathbb{C}e_3$.
- For $\mathfrak{h}_2 = \mathbb{C}e_3 + \mathbb{C}e_4$, we have $\mathfrak{h} = \mathbb{C}e_3 + \mathbb{C}e_4$ and $\mathfrak{h} = \mathbb{C}e_3 + \mathbb{C}e_4 + \mathbb{C}(\alpha e_1 + \lambda e_2)$ $\lambda \neq 0$. The second subalgebra is $\mathfrak{h} = \mathbb{C}e_2 + \mathbb{C}e_3 + \mathbb{C}e_4$ for $\alpha = 0$ and $\mathfrak{h} = \mathbb{C}(e_1 + \lambda e_2) + \mathbb{C}e_3 + \mathbb{C}e_4$ for $\alpha \neq 0$. The second type of matrices in (2.12) can also be used to restrict the range of the parameter λ to $\{\operatorname{Re}\lambda > 0\} \cup \{\operatorname{Re}\lambda = 0, \operatorname{Im}\lambda > 0\}$.
- For $\mathfrak{h}_2 = \mathbb{C}e_1 + \mathbb{C}e_3 + \mathbb{C}e_4$ we have $\mathfrak{h} = \mathbb{C}e_1 + \mathbb{C}e_3 + \mathbb{C}e_4$ and $\mathfrak{h} = \mathbb{C}e_1 + \mathbb{C}e_2 + \mathbb{C}e_3 + \mathbb{C}e_4$.

Collecting all these subalgebras $\mathfrak{h}$ we have the following solvable subalgebras of $\mathfrak{so}(4, \mathbb{C})$:

$$\{0\}; \quad \mathbb{C}e_2; \quad \mathbb{C}e_3; \quad \mathbb{C}e_1 + \mathbb{C}e_2; \quad \mathbb{C}(e_1 + \lambda e_2),\ \{\operatorname{Re}\lambda > 0\} \cup \{\operatorname{Re}\lambda = 0, \operatorname{Im}\lambda \geqslant 0\};$$
$$\mathbb{C}e_2 + \mathbb{C}e_3; \quad \mathbb{C}e_3 + \mathbb{C}e_4; \quad \mathbb{C}(e_3 + e_4); \quad \mathbb{C}e_1 + \mathbb{C}e_2 + \mathbb{C}e_3; \quad \mathbb{C}(e_1 + e_2 + e_3);$$
$$\mathbb{C}(e_1 + \lambda e_2) + \mathbb{C}e_3; \quad \mathbb{C}e_1 + \mathbb{C}(e_3 + e_4); \quad \mathbb{C}e_2 + \mathbb{C}e_3 + \mathbb{C}e_4;$$
$$\mathbb{C}e_1 + \mathbb{C}e_2 + \mathbb{C}e_3 + \mathbb{C}e_4; \quad \mathbb{C}(e_1 - e_2 + e_4) + \mathbb{C}e_3;$$
$$\mathbb{C}(e_1 + \lambda e_2) + \mathbb{C}e_3 + \mathbb{C}e_4,\ \{\operatorname{Re}\lambda > 0\} \cup \{\operatorname{Re}\lambda = 0, \operatorname{Im}\lambda \geqslant 0\}.$$

We can further reduce this list to the fourteen cases of the statement, from 1.1 to 3.5, under conjugation with respect to $\mathrm{GL}(4, \mathbb{C})$, as it follows.

- The one-dimensional algebras are initially

$$\mathbb{C}(e_1 + \lambda e_2),\ \{\operatorname{Re}\lambda > 0\} \cup \{\operatorname{Re}\lambda = 0, \operatorname{Im}\lambda \geqslant 0\};$$
$$\mathbb{C}(e_1 + e_2 + e_3); \quad \mathbb{C}(e_3); \quad \mathbb{C}(e_3 + e_4); \quad \mathbb{C}(e_2).$$

The last one is conjugate to the first one ($\lambda = 0$) by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The non-trivial elements of the second (resp. third, fourth) algebra are non-diagonalizable and its eigenvalues never vanish (resp. are 2-nilpotent and have trivial eigenvalues, are 3-nilpotent and have trivial eigenvalues). The elements of the first Lie algebra are diagonalized. Hence, the first, second, third and fourth algebras are not conjugate. With respect to the parameter λ in the first algebra, given a conjugation between $\mathbb{C}(e_1 + \lambda_1 e_2)$ and $\mathbb{C}(e_1 + \lambda_2 e_2)$, the matrix $\operatorname{diag}(1, \lambda_1, -1, -\lambda_1)$ goes to $\operatorname{diag}(\sigma, \sigma\lambda_2, -\sigma, -\sigma\lambda_2)$, $\sigma \in \mathbb{C}$, with the same eigenvalues. That is, either $\sigma = 1$, $\lambda_1 = \lambda_2$, or $\sigma = \lambda_1$, $\sigma\lambda_2 = 1$. Therefore, λ and λ^{-1} are the only possible cases where conjugation may occur. Indeed, the conjugation is obtained by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then the range of the parameter can be restricted to either $|\lambda| < 1$, $-\pi/2 < \arg\lambda \leqslant \pi/2$ or $|\lambda| = 1$, $0 \leqslant \arg\lambda \leqslant \pi/2$. Finally note that the third and the fourth algebras are conjugate to the cases 1.3 and 1.4 in (2.11) by the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & -1/2 \end{pmatrix},$$

respectively.

- The two-dimensional algebras are initially

$$\mathbb{C}e_1 + \mathbb{C}e_2; \quad \mathbb{C}(e_1 + \lambda e_2) + \mathbb{C}e_3; \quad \mathbb{C}(e_1 - e_2 + e_4) + \mathbb{C}e_3;$$
$$\mathbb{C}e_1 + \mathbb{C}(e_3 + e_4); \quad \mathbb{C}e_3 + \mathbb{C}e_4; \quad \mathbb{C}e_2 + \mathbb{C}e_3.$$

The latter is conjugate to the second one ($\lambda = 0$) by

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This same matrix defines a conjugation of the algebras with λ and λ^{-1} in the second family. A similar argument to the one given in the previous paragraph shows that the two parameters λ_1 and $\lambda_2 \neq \lambda_1^{-1}$ cannot be conjugate. The first, second, third, fourth and fifth algebras are not conjugate. Indeed, the eigenvalues of a generic element of each of them are of the type $\{x, y, -x, -y\}$, $\{x, \lambda x, -x, \lambda x\}$, $\{x, -x, -x, x\}$, $\{x, 0, -x, 0\}$ and $\{0, 0, 0, 0\}$, respectively. The only ambiguous cases happen when $\lambda = 0, \pm 1$, in the second algebra. For $\lambda = -1$, the second and the third algebras share the same type $\{x, x, -x, -x\}$ of eigenvalues, but the square of the matrices of the second algebra are diagonalizable whereas the matrices of the third algebra are not. For $\lambda = 1$, the situation with the eigenvalues is the same, but the second algebra is abelian whereas the third algebra is not. Finally, for $\lambda = 0$, the second algebra share the same type $\{x, -x, 0, 0\}$ of eigenvalues, but the former has 2-nilpotent elements (that is, e_3), whereas the latter does not have 2-nilpotent elements. Finally note that the fourth algebra is conjugate by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & -1/2 \end{pmatrix}$$

to the algebra 2.4 in (2.11).

- The three-dimensional algebras are initially

$$\mathbb{C}e_1 + \mathbb{C}e_2 + \mathbb{C}e_3;$$
$$\mathbb{C}(e_1 + \lambda e_2) + \mathbb{C}e_3 + \mathbb{C}e_4,\ \{\mathrm{Re}\,\lambda > 0\} \cup \{\mathrm{Re}\,\lambda = 0, \mathrm{Im}\,\lambda \geqslant 0\};$$
$$\mathbb{C}e_2 + \mathbb{C}e_3 + \mathbb{C}e_4.$$

The set of eigenvalues of the elements of these algebras are of the type $\{x, y, -x, -y\}$, $\{x, \lambda x, -x, -\lambda x\}$ and $\{0, x, 0, -x\}$, respectively. Hence, for $\lambda \neq 0$, there are no possible conjugations among any pair of them. For $\lambda = 0$, the *second* and the *third* algebras have the same eigenvalue structure. However, the vector $(0, 0, 1, 0)$ is in the kernel of every matrix in the third algebra, whereas there are no vectors simultaneously belonging to the kernel of every matrix in the second algebra. With respect to the *first* algebra, the different values of λ are not conjugate. Indeed, assume that $\mathbb{C}(e_1 + \lambda_1 e_2) + \mathbb{C}e_3 + \mathbb{C}e_4$ is conjugated to $\mathbb{C}(e_1 + \lambda_2 e_2) + \mathbb{C}e_3 + \mathbb{C}e_4$, and $e_1 + \lambda_1 e_2$ is sent to $x(e_1 + \lambda_2 e_2) + ye_3 + ze_4$. The eigenspaces of the adjoint transformation $\mathrm{ad}_{e_1+\lambda_1 e_2}$ are $\mathbb{C}e_3$ (with eigenvalue $1 - \lambda_1$) and $\mathbb{C}e_4$ (with eigenvalue $1 + \lambda_2$), and must be sent to the eigenspaces of $x(e_1 + \lambda_2 e_2) + ye_3 + ze_4$, which are the same, with eigenvalues $x(1 - \lambda_2)$ and $x(1 + \lambda_2)$. That is, $1 \pm \lambda_1 = x(1 \pm \lambda_2)$ or $1 \pm \lambda_1 = x(1 \mp \lambda_2)$. In both cases, $x = 1$. Since $e_1 + \lambda_1 e_2$ and $x(e_1 + \lambda_2 e_2) + ye_3 + ze_4$ share the same eigenvalues as matrices in $\mathrm{GL}(4, \mathbb{C})$, we conclude that $\lambda_1 = \lambda_2$.

Finally, as for non-solvable subalgebras, one considers the semisimple Levi subalgebra. But any semisimple subalgebra of $\mathfrak{so}(4, \mathbb{C})$ is [11, Prop. 2, p. 14] either conjugate to one of the complex pairs 3.4 or 3.5, or it has higher dimension. This completes the proof of [12, Th. 1, pp. 7, 8].

The corresponding complex subcases are sixteen: ten subcases for 1.1 (see his pages eleven and twelve); three subcases for 2.1 (see his page twenty-two); and three subcases for 3.5 (see his pages thirty and thirty-one). The steps for obtaining the corresponding complex subcases are given in [11, pp. 18 ff.].

2.3. The complex pair 1.1

Komrakov [11, Prop. 1.1, p. 20] proved that *any complex pair* $(\mathfrak{g}, \mathfrak{h})$ *of type* 1.1 *is equivalent to one and only one of the following pairs.*

For $\lambda = 0$:

1

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	u_2
u_3	u_3	$-u_2$	0	0	u_3
u_4	0	0	$-u_2$	$-u_3$	0

2

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	pu_2
u_3	u_3	0	0	0	u_3
u_4	0	0	$-pu_2$	$-u_3$	0

3

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	$e_1 + u_2$	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1 - u_2$	0	0	0
u_4	0	0	0	0	0

4

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	0
u_3	u_3	$-u_2$	0	0	0
u_4	0	0	0	0	0

5

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	u_2
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	$-u_2$	0	0

6

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	0	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	u_2
u_3	u_3	0	0	0	0
u_4	0	0	$-u_2$	0	0

7 (2.14)

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	0	0	0

For $\lambda = 1/2$:

8

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	$-2e_1$	u_2
u_2	$-\frac{1}{2}u_2$	0	0	u_4	0
u_3	u_3	$2e_1$	$-u_4$	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

9

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	0	u_2
u_2	$-\frac{1}{2}u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

10

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-\frac{1}{2}u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	$\frac{1}{2}u_4$	0	0	0	0

For either $|\lambda| < 1$ with $-\pi/2 < \arg\lambda \leqslant \pi/2$ or $|\lambda| = 1$ with $0 \leqslant \arg\lambda \leqslant \pi/2$:

11

$[\,,\,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	λu_2	$-u_3$	$-\lambda u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	λu_4	0	0	0	0

We are interested only in the complex pair 1.1.7 whose table is (2.14).

He proved his Proposition 1.1 (see [11, pp. 22 ff.]) as follows. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{h}$, where $e_1 = \operatorname{diag}(1, \lambda, -1, -\lambda)$. Then $A(e_1) = 0$, and for $x \in \mathfrak{h}$ the matrix $B(x)$ can be identified with x.

Denote by $\mathfrak{n}$ the nilpotent subalgebra of $\mathfrak{h}$ spanned by the vector e_1, that is, $\mathfrak{n} = \mathfrak{h}$. Then, he proved [11, p. 22] a lemma stating that *any virtual structure q on the module* 1.1 *with $\lambda = 0$ is equivalent to $C_1(e_1) = (0\ \ p\ \ 0\ \ r)$ and with $\lambda \neq 0$ is equivalent to $C_2(e_1) = (0\ \ 0\ \ 0\ \ 0)$*. The proof is as follows: Any virtual structure q can be written as $C(e_1) = (c_1\ \ c_2\ \ c_3\ \ c_4)$. Since $q = 0$, putting $\mathcal{H} = (c_1\ \ 0\ \ -c_3\ \ 0)$ we have for $x \in \mathfrak{g}$ that $C_1(x) = C(x) + A(x)\mathcal{H} - \mathcal{H}B(x)$, hence

$$C_1(e_1) = (c_1\ \ c_2\ \ c_3\ \ c_4) - (c_1\ \ 0\ \ -c_3\ \ 0)(\operatorname{diag}(1, 0, -1, 0)) = (0\ \ c_2\ \ 0\ \ c_4).$$

Then, by [11, Ch. I, Prop. 4] (see also the context of (2.5) above), the virtual structures C_1 and C are equivalent. Proceeding analogously for $\lambda \neq 0$, the proof of the lemma is completed.

Let $(\mathfrak{g}, \mathfrak{h})$ be a complex pair of type 1.1. Then it can be assumed that the corresponding virtual pair $(\mathfrak{g}, \mathfrak{h})$ is defined by one of the virtual structures determined in the aforementioned lemma. Supposing $\lambda = 0$, we have

$$[e_1, u_1] = u_1, \quad [e_1, u_2] = pe_1, \quad [e_1, u_3] = -u_3, \quad [e_1, u_4] = re_1.$$

Since by [11, Ch. I, Prop. 7, p. 17] (see also the context of (2.10) above) the map $q\colon \mathfrak{g} \to \mathrm{L}(U, \mathfrak{g})$ is primary, one has

$$\mathfrak{g}^{\alpha}(\mathfrak{h}) = \mathfrak{h}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \quad \text{for all} \quad \alpha \in \mathfrak{h}^*.$$

Thus

$$\begin{aligned}
\mathfrak{g}^{(-1)}(\mathfrak{h}) &= \mathfrak{h}^{(-1)}(\mathfrak{h}) \times U^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3, \\
\mathfrak{g}^{(0)}(\mathfrak{h}) &= \mathfrak{h}^{(0)}(\mathfrak{h}) \times U^{(0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_4, \\
\mathfrak{g}^{(1)}(\mathfrak{h}) &= \mathfrak{h}^{(1)}(\mathfrak{h}) \times U^{(1)}(\mathfrak{h}) = \mathbb{C}u_1.
\end{aligned} \tag{2.15}$$

Then, since

$$[u_2, u_3], [u_3, u_4] \in \mathfrak{g}^{(-1)}(\mathfrak{h}), \quad [u_1, u_3], [u_2, u_4] \in \mathfrak{g}^{(0)}(\mathfrak{h}), \quad [u_1, u_2], [u_1, u_4] \in \mathfrak{g}^{(1)}(\mathfrak{h}),$$

one has from (2.15) that

$$\begin{aligned}
&[u_1, u_2] = \alpha u_1, \quad [u_1, u_3] = a e_1 + \beta_1 u_2 + \beta_2 u_4, \quad [u_1, u_4] = \gamma u_1, \\
&[u_2, u_3] = \delta u_3, \quad [u_2, u_4] = b e_1 + \eta_1 u_2 + \eta_2 u_4, \quad [u_3, u_4] = \varepsilon u_3.
\end{aligned}$$

Komrakov considered the different possibilities, and the one corresponding to the complex pair 1.1.7 is labeled as $1.2.2.2.1^o$ in his page twenty-seven. It is obtained taking $\varepsilon = 0$ (his page twenty-five), $\beta_1 = \beta_2 = 0$ (his page twenty-six), $\eta_1 = \eta_2 = 0$ (his page twenty-seven) and $a \neq 0$, by means of the map $\pi\colon \mathfrak{g}_7 \to \mathfrak{g}$ given by

$$\pi(e_1) = e_1, \;\; \pi(u_1) = \frac{1}{a} u_1, \;\; \pi(u_2) = \frac{1}{a}(\alpha e_1 + u_2), \;\; \pi(u_3) = u_3, \;\; \pi(u_4) = \gamma e_1 + u_4.$$

So the remaining nonzero brackets are $[e_1, u_1] = u_1$, $[e_1, u_3] = -u_3$, $[u_1, u_3] = e_1$, thus obtaining the multiplication table of $\mathfrak{g}_7$, that is, (2.14).

For $\lambda \neq 0$, we have

$$[e_1, u_1] = u_1, \quad [e_1, u_2] = \lambda u_2, \quad [e_1, u_3] = -u_3, \quad [e_1, u_4] = -\lambda u_4.$$

Since the map $q\colon \mathfrak{g} \to \mathrm{L}(U, \mathfrak{g})$ is primary, one has $\mathfrak{g}^{\alpha}(\mathfrak{h}) = \mathfrak{h}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h})$ for all $\alpha \in \mathfrak{h}^*$ (see (2.10)). Thus, for $\lambda = 1$,

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \;\; \mathfrak{g}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_2, \;\; \mathfrak{g}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3 \oplus \mathbb{C}u_4.$$

Then,

$$[u_1, u_2] \in \mathfrak{g}^{(-2)}(\mathfrak{h}), \;\; [u_i, u_3], [u_i, u_4] \in \mathfrak{g}^{(0)}(\mathfrak{h}), \;\; i = 1, 2, [u_3, u_4] \in \mathfrak{g}^{(2)}(\mathfrak{h}).$$

By using the Jacobi identity, one obtains that the pair $(\mathfrak{g}, \mathfrak{h})$ is equivalent to the trivial complex pair 1.1.10. Finally, for $\lambda \notin \{0, 1\}$:

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \, \mathfrak{g}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1, \, \mathfrak{g}^{(\lambda)}(\mathfrak{h}) = \mathbb{C}u_2, \, \mathfrak{g}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3, \, \mathfrak{g}^{(-\lambda)}(\mathfrak{h}) = \mathbb{C}u_4.$$

Then,

$$
\begin{aligned}
&[u_1,u_2]\in\mathfrak{g}^{(1+\lambda)}(\mathfrak{h}),\ [u_1,u_3],[u_2,u_4]\in\mathfrak{g}^{(0)}(\mathfrak{h}),\ [u_3,u_4]\in\mathfrak{g}^{(-1-\lambda)}(\mathfrak{h}),\\
&[u_1,u_4]\in\mathfrak{g}^{(1-\lambda)}(\mathfrak{h}),\ [u_2,u_3]\in\mathfrak{g}^{(\lambda-1)}(\mathfrak{h}).
\end{aligned}
$$

This situation splits into two subcases: $\lambda=\frac{1}{2}$ and $\lambda\neq\frac{1}{2}$. In the first subcase, imposing the Jacobi identity leads to several possibilities, which correspond to the complex pairs 1.1.8, 1.1.9, and 1.1.10. In the second subcase, it is shown that the pair is equivalent to the trivial complex pair 1.1.11.

2.4. The complex pairs 2.1.1 and 2.1.2

Komrakov [11, Prop. 2.1, p. 55 ff.] proved that *any complex pair* $(\mathfrak{g},\mathfrak{h})$ *of type* 2.1 *is equivalent to one and only one of the following pairs*:

1

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	e_2
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	$-e_2$	0	0

2

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	0	0	0

3

$[\,,\,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	0	0	0	0
u_4	0	u_4	0	0	0	0

The proof starts taking the basis $\mathcal{E}=\{e_1=\operatorname{diag}(1,0,-1,0),e_2=\operatorname{diag}(0,1,0,-1)\}$ of $\mathfrak{h}$. Then $A(e_1)=A(e_2)=0$ and for $x\in\mathfrak{h}$ the matrix $B(x)$ can be identified with x. Then a lemma states that *any virtual structure* q *on the complex module* 2.1 *is equivalent to the trivial one.* Indeed, put

$$
C_1(e_i)=\begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} & c^i_{14}\\ c^i_{21} & c^i_{22} & c^i_{23} & c^i_{24}\end{pmatrix},\qquad i=1,2.
$$

Then the condition (3) in [11, Ch. 1] (see also (2.4) above) for e_1,e_2 reduces to

$$
0=-C_1(e_2)B(e_1)+C_1(e_1)B(e_2). \tag{2.16}
$$

That is, one has

$$
0=\begin{pmatrix} -c^2_{11} & 0 & c^2_{13} & 0\\ -c^2_{21} & 0 & c^2_{23} & 0\end{pmatrix}+\begin{pmatrix} 0 & c^1_{12} & 0 & -c^1_{14}\\ 0 & c^1_{22} & 0 & -c^1_{24}\end{pmatrix},
$$

hence $c^2_{11}=c^2_{13}=c^2_{21}=c^2_{23}=0$, $c^1_{12}=c^1_{14}=c^1_{22}=c^1_{24}=0$. So, any virtual structure q on the complex module 2.1 is given by

$$
C_1(e_1)=\begin{pmatrix} c^1_{11} & 0 & c^1_{13} & 0\\ c^1_{21} & 0 & c^1_{23} & 0\end{pmatrix},\quad C_1(e_2)=\begin{pmatrix} 0 & c^2_{12} & 0 & c^2_{14}\\ 0 & c^2_{22} & 0 & c^2_{24}\end{pmatrix}.
$$

Taking then the matrix

$$\mathcal{H} = \begin{pmatrix} c^1_{11} & c^2_{12} & -c^1_{13} & -c^2_{14} \\ c^1_{21} & c^2_{22} & -c^1_{23} & -c^2_{24} \end{pmatrix} \in M(2 \times 4, \mathbb{C}),$$

the equation $C_1(x) - C_2(x) = A(x)\mathcal{H} - \mathcal{H}B(x)$ reads, for $x = e_1$, e_2:

$$\begin{pmatrix} c^1_{11} & 0 & c^1_{13} & 0 \\ c^1_{21} & 0 & c^1_{23} & 0 \end{pmatrix} - C_2(e_1) = \begin{pmatrix} c^1_{11} & 0 & c^1_{13} & 0 \\ c^1_{21} & 0 & c^1_{23} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & c^2_{12} & 0 & c^2_{14} \\ 0 & c^2_{22} & 0 & c^2_{24} \end{pmatrix} - C_2(e_2) = \begin{pmatrix} 0 & c^2_{12} & 0 & c^2_{14} \\ 0 & c^2_{22} & 0 & c^2_{24} \end{pmatrix},$$

respectively. Hence $C_2(e_1) = C_2(e_2) = 0$, so in particular the virtual structures C_1 and C_2 are equivalent, thus finishing the proof of the Lemma.

Let now $(\mathfrak{g}, \mathfrak{h})$ be a complex pair of type 2.1. Then it can be assumed that the corresponding virtual pair $(\mathfrak{g}, \mathfrak{h})$ is defined by the trivial virtual structure, so the only nonzero brackets $[e_i, u_j]$ are those furnished by the tables **1**, **2**, **3** in page 15, that is,

$$[e_1, u_1] = u_1, \quad [e_1, u_3] = -u_3, \quad [e_2, u_2] = u_2, \quad [e_2, u_4] = -u_4.$$

Put

$$[u_1, u_2] = \sum_{i=1}^{2} a_i e_i + \sum_{j=1}^{4} \alpha_j u_j, \quad [u_1, u_3] = \sum_{i=1}^{2} b_i e_i + \sum_{j=1}^{4} \beta_j u_j,$$

$$[u_1, u_4] = \sum_{i=1}^{2} c_i e_i + \sum_{j=1}^{4} \gamma_j u_j, \quad [u_2, u_3] = \sum_{i=1}^{2} d_i e_i + \sum_{j=1}^{4} \delta_j u_j,$$

$$[u_2, u_4] = \sum_{i=1}^{2} f_i e_i + \sum_{j=1}^{4} \eta_j u_j, \quad [u_3, u_4] = \sum_{i=1}^{2} k_i e_i + \sum_{j=1}^{4} \varepsilon_j u_j.$$

From the Jacobi identity one obtains the pair $(\mathfrak{g}, \mathfrak{h})$:

$[\ ,\ ]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	b_1e_1	0
u_2	0	$-u_2$	0	0	0	f_2e_2
u_3	u_3	0	$-b_1e_1$	0	0	0
u_4	0	u_4	0	$-f_2e_2$	0	0

Then four subcases arise and we must consider the first three. In the *first* subcase ($b_1 f_2 \neq 0$), the pair $(\mathfrak{g}, \mathfrak{h})$ is equivalent to $(\mathfrak{g}_1, \mathfrak{h}_1)$ by means of the map $\pi\colon \mathfrak{g}_1 \to \mathfrak{g}$ which is the identity on all the basis elements except for $\pi(u_3) = (1/b_1)u_3$, $\pi(u_4) = (1/f_2)u_4$. One thus recovers the multiplication table **1** in page 15. In the *second* case ($b_1 \neq 0, f_2 = 0$), the pair $(\mathfrak{g}, \mathfrak{h})$ is equivalent to $(\mathfrak{g}_2, \mathfrak{h}_2)$ by means of the map π which is the identity on the basis elements except for $\pi(u_3) = (1/b_1)u_3$. One thus recovers the multiplication table **2** in page 15. In the *third* case ($b_1 = 0, f_2 \neq 0$), the pair $(\mathfrak{g}, \mathfrak{h})$ is equivalent to $(\mathfrak{g}_2, \mathfrak{h}_2)$ by means of the map

$$\pi(e_1) = -e_2,\ \pi(e_2) = -e_1,\ \pi(u_1) = (1/f_2)u_4,\ \pi(u_2) = u_3,\ \pi(u_3) = u_2,\ \pi(u_4) = u_1.$$

2.5. The complex pair 3.5.2

Komrakov [11, Prop. 3.5, p. 99] proved that *any complex pair* $(\mathfrak{g}, \mathfrak{h})$ *of type* 3.5 *is equivalent to one and only one of the following pairs*:

1

$[\,,\,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	u_1
u_2	0	$-u_1$	u_3	0	0	0	u_2
u_3	$2u_3$	$2u_2$	0	0	0	0	u_3
u_4	0	0	0	$-u_1$	$-u_2$	$-u_3$	0

2

$[\,,\,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	e_2	e_1	0
u_2	0	$-u_1$	u_3	$-e_2$	0	e_3	0
u_3	$2u_3$	$2u_2$	0	$-e_1$	$-e_3$	0	0
u_4	0	0	0	0	0	0	0

3

$[\,,\,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	0
u_2	0	$-u_1$	u_3	0	0	0	0
u_3	$2u_3$	$2u_2$	0	0	0	0	0
u_4	0	0	0	0	0	0	0

Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{h}$, where

$$e_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{h}$ the matrix $B(x)$ is identified with x.

Let now denote by $\mathfrak{n}$ the nilpotent subalgebra of the Lie algebra $\mathfrak{h}$ spanned by the vector e_1. Then one has [11, Lemma, p. 100] that *any virtual structure* q *on the module* 3.5 *is equivalent to the trivial one.* This follows noting that $\mathfrak{h}$ is a semisimple Lie algebra, so that by [11, Ch. I, Prop. 9], one can assume, without loss of generality, that $q(\mathfrak{h}) = \{0\}$.

Let $(\mathfrak{g},\mathfrak{h})$ be a complex pair of type 3.5. Then it can be assumed that the corresponding virtual pair $(\mathfrak{g},\mathfrak{h})$ is defined by the trivial virtual structure, so the only nonzero brackets $[e_i,u_j]$ are those in the matrices **1, 2, 3** in page 17, that is,

$$[e_1,e_2]=2e_2,\quad [e_1,e_3]=-2e_3,\quad [e_2,e_3]=e_1,$$
$$[e_1,u_1]=2[e_2,u_2]=2u_1,\; [e_2,u_3]=-[e_3,u_1]=-2u_2,\; [e_1,u_3]=2[e_3,u_2]=-2u_3.$$

Since by [11, Ch. I, Prop. 7, p. 17] the map $q\colon \mathfrak{g}\to \mathrm{L}(U,\mathfrak{g})$ is primary, one has

$$\mathfrak{g}^{\alpha}(\mathfrak{h})=\mathfrak{h}^{\alpha}(\mathfrak{h})\times U^{\alpha}(\mathfrak{h})\quad \text{for all}\quad \alpha\in\mathfrak{h}^{*}.$$

Thus,

$$\begin{aligned}\mathfrak{g}^{(-2)}(\mathfrak{n})&=\mathfrak{h}^{(-2)}(\mathfrak{n})\times U^{(-2)}(\mathfrak{n})=\mathbb{C}e_3\oplus\mathbb{C}u_3.\\ \mathfrak{g}^{(0)}(\mathfrak{n})&=\mathfrak{h}^{(0)}(\mathfrak{n})\times U^{(0)}(\mathfrak{n})=\mathbb{C}e_1\oplus\mathbb{C}u_2\oplus\mathbb{C}u_4,\\ \mathfrak{g}^{(2)}(\mathfrak{n})&=\mathfrak{h}^{(2)}(\mathfrak{n})\times U^{(2)}(\mathfrak{n})=\mathbb{C}e_2\oplus\mathbb{C}u_1.\end{aligned}\tag{2.17}$$

Hence

$$[u_2,u_3],[u_3,u_4]\in\mathfrak{g}^{(-2)}(\mathfrak{n}),\quad [u_1,u_3],[u_2,u_4]\in\mathfrak{g}^{(0)}(\mathfrak{n}),\quad [u_1,u_2],[u_1,u_4]\in\mathfrak{g}^{(2)}(\mathfrak{n}),$$

and thus we have from (2.17) that

$$[u_1,u_2]=ae_2+\alpha u_1,\quad [u_1,u_3]=be_1+\beta_1u_2+\beta_2u_4,\quad [u_1,u_4]=ce_2+\gamma u_1,$$
$$[u_2,u_3]=de_3+\delta u_3,\quad [u_2,u_4]=fe_1+\eta_1u_2+\eta_2u_4,\quad [u_3,u_4]=ke_3+\varepsilon u_3.$$

Using the Jacobi identity one obtains that

$$\beta_2=\eta_2=0,\quad a=b=d,\quad \delta=\alpha,\quad \beta_1=-2\alpha,$$
$$\eta_1=\varepsilon=\gamma,\quad c=k=-\alpha\gamma/2,\quad f=\alpha\gamma/4,\quad a\gamma+\alpha^2\gamma/4=0.$$

It follows that the pair $(\mathfrak{g},\mathfrak{h})$ is

$[\,,\,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	$ae_2+\alpha u_1$	$ae_1-2\alpha u_2$	$-\frac{\alpha\gamma}{2}e_2+\gamma u_1$
u_2	0	$-u_1$	u_3	$-ae_2-\alpha u_1$	0	$ae_3+\alpha u_3$	$\frac{\alpha\gamma}{4}e_1-\gamma u_2$
u_3	$2u_3$	$2u_2$	0	$-ae_1+2\alpha u_2$	$-ae_3-\alpha u_3$	0	$-\frac{\alpha\gamma}{2}e_3+\gamma u_3$
u_4	0	0	0	$\frac{\alpha\gamma}{2}e_2-\gamma u_1$	$-\frac{\alpha\gamma}{4}e_1+\gamma u_2$	$\frac{\alpha\gamma}{2}e_3-\gamma u_3$	0

with $a\gamma+(\alpha^2\gamma)/4=0$.

To find the subcases of the complex pair 3.5.2, it is convenient to use another suitable complex pair $(\mathfrak{g}',\mathfrak{h}')$ equivalent to $(\mathfrak{g},\mathfrak{h})$. This is obtained by means of the map $\pi\colon\mathfrak{g}'\to\mathfrak{g}$ given by

$$\pi(e_1)=e_1,\quad \pi(e_2)=e_2,\quad \pi(e_3)=e_3,\quad \pi(u_1)=-(\alpha/2)e_2+u_1,$$
$$\pi(u_2)=(\alpha/4)e_1+u_2,\quad \pi(u_3)=-(\alpha/2)e_3+u_3,\quad \pi(u_4)=u_4.$$

The pair $(\mathfrak{g}', \mathfrak{h}')$ is then

$[\,,\,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	ae_2	ae_1	γu_1
u_2	0	$-u_1$	u_3	$-ae_2$	0	ae_3	γu_2
u_3	$2u_3$	$2u_2$	0	$-ae_1$	$-ae_3$	0	γu_3
u_4	0	0	0	$-\gamma u_1$	$-\gamma u_2$	$-\gamma u_3$	0

with $a\gamma = 0$.

In [11, p. 102 -103] are studied the three different possibilities, corresponding to $\gamma \neq 0$; $a \neq 0$; and $a = \gamma = 0$, respectively. We should consider the two last cases. In the *second* case, one has $\gamma = 0$ and the pair $(\mathfrak{g}', \mathfrak{h}')$ is equivalent to the pair $(\mathfrak{g}_2, \mathfrak{h}_2)$ by means of the map $\pi\colon \mathfrak{g}_2 \to \mathfrak{g}'$ given by $\pi(e_i) = e_i$, $i = 1, 2, 3$, and $\pi(u_j) = (1/\sqrt{a})u_j$, $j = 1, \ldots, 4$. In the *third* case, one has $\alpha = 0$ and the pair $(\mathfrak{g}', \mathfrak{h}')$ is equivalent to the trivial pair $(\mathfrak{g}_3, \mathfrak{h}_3)$ by means of the map $\pi\colon \mathfrak{g}_3 \to \mathfrak{g}'$ given by $\pi(e_i) = e_i$, $i = 1, 2, 3$ and $\pi(u_j) = (1/\sqrt{\gamma})u_j$, $j = 1, \ldots, 4$.

2.6. Real forms of the complex pairs

Komrakov [12, pp. 50, 75, 79, 145] considered the real pairs corresponding to the complex cases 1.1.7, 2.1.1, 2.1.2 and 3.5.2, and proved Theorem 1.1 (see his page thirty-nine), which states that there are twenty-two subcases for $n = 1$: ten for 1.1^1 and twelve for 1.1^2. Our three real pairs $1.1^1(7)$, $1.1^2(9)$, $1.1^2(10)$ come from the same complex pair: 1.1.7 in [12, p. 50].

Let $(\mathfrak{g}, \mathfrak{h})$ be a complex isotropically faithful pseudo-Riemannian pair. An anti-involution σ of $\mathfrak{g}$ is said to be an anti-involution of the pair $(\mathfrak{g}, \mathfrak{h})$ if $\sigma(\mathfrak{h}) = \mathfrak{h}$.

According, e.g., to [6, p. 180], every real form of the pair $(\mathfrak{g}, \mathfrak{h})$ coincides with the set of fixed points $(\mathfrak{g}^\sigma, \mathfrak{h}^\sigma)$ of some anti-involution σ of the pair $(\mathfrak{g}, \mathfrak{h})$ (note that Helgason uses the term *conjugation* instead of anti-involution).

Two real forms $(\mathfrak{g}^{\sigma_1}, \mathfrak{h}^{\sigma_1})$ and $(\mathfrak{g}^{\sigma_2}, \mathfrak{h}^{\sigma_2})$ are said to be *equivalent* if and only if the corresponding anti-involutions σ_1 and σ_2 are conjugate, that is, if there exists an automorphism f of the pair $(\mathfrak{g}, \mathfrak{h})$ such that $\sigma_2 = f\sigma_1 f^{-1}$.

Let $\mathcal{B} = \{e_1, \ldots, e_n, u_1, \ldots, u_4\}$ be a basis of $\mathfrak{g}$, and let A, I_1, I_2 be the matrices of f, σ_1, σ_2 with respect to this basis. Then the condition for $(\mathfrak{g}^{\sigma_1}, \mathfrak{h}^{\sigma_1})$ and $(\mathfrak{g}^{\sigma_2}, \mathfrak{h}^{\sigma_2})$ to be equivalent is $I_2 = AI_1\bar{A}^{-1}$.

For each pair $(\mathfrak{g}, \mathfrak{h})$ one fixes a basis $\mathcal{B} = \{e_1, \ldots, e_n, u_1, \ldots, u_4\}$. Then we can identify the group of automorphisms of $(\mathfrak{g}, \mathfrak{h})$ with the subgroup $\mathcal{A}$ of $\mathrm{GL}(n+4, \mathbb{C})$ such that each element of $\mathcal{A}$ is the matrix of some automorphism of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathcal{B}$. Similarly, we can identify the set of anti-involutions of the pair $(\mathfrak{g}, \mathfrak{h})$ with the set $\mathcal{I}$ of matrices of anti-involutions of $(\mathfrak{g}, \mathfrak{h})$ with respect to the basis $\mathcal{B}$.

Then, $\mathcal{A}$ is called the *group of automorphisms* and $\mathcal{I}$ the *set of anti-involutions* of $(\mathfrak{g}, \mathfrak{h})$.

Let $v \in \mathfrak{g}$. Denote by X the row of coordinates of the vector v with respect to the basis $\mathcal{B}$. The expression $X = (a_1, \ldots, a_n, x, y, z, t)$ means that

$$v = a_1e_1 + \cdots + a_ne_n + xu_1 + yu_2 + zu_3 + tu_4.$$

If the anti-involution I of the pair $(\mathfrak{g}, \mathfrak{h})$ is equal to the identity I_{n+4}, then the real form of $(\mathfrak{g}, \mathfrak{h})$ corresponding to the anti-involution I has the same structure constants as $(\mathfrak{g}, \mathfrak{h})$. Indeed, let X be corresponding to $v \in \mathfrak{g}$. Then $I(X) = \overline{X} = X$ implies that $X \in \mathbb{R}^{n+4}$. Thus, we can choose as basis of the set $(\mathbb{C}^{n+4})^I$ of fixed points the basis $\mathcal{B}' = \mathcal{B} = \{e_1, \dots, e_n, u_1, \dots, u_4\}$. It follows that the multiplication table of the real form corresponding to the anti-involution I coincides with the multiplication table of $(\mathfrak{g}, \mathfrak{h})$. In particular, let $\mathfrak{h}$ be a linear Lie algebra having only one anti-involution, I (up to the operation $AI\bar{A}^{-1}$). Then there exists a basis of $\mathfrak{h}$ which consists of real elements and which is also a basis of the real form.

2.7. The real pairs 1.1¹(7), 1.1²(9) and 1.1²(10)

Komrakov proved [12, Prop. 1.1.7, p. 50] that *any real form of the complex pair* 1.1.7 *is equivalent to one and only one of the following real pairs:* $1.1^1(7)$, $1.1^2(9)$, $1.1^2(10)$. The proof is as follows. The group $\mathcal{A}$ and the set $\mathcal{I}$ of the complex pair 1.1.7 are

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & c_1 \\ 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & b_2 & 0 & c_2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & b_1 & 0 & c_1 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b_2 & 0 & c_2 \end{pmatrix} : \begin{array}{c} a \in \mathbb{C}^*, \\ b_1, b_2, c_1, c_2 \in \mathbb{C}, \\ b_1c_2 - b_2c_1 \neq 0 \end{array} \right\},$$

$$\mathcal{I} = \left\{ I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & \gamma_1 \\ 0 & 0 & 0 & 1/\alpha & 0 \\ 0 & 0 & \beta_2 & 0 & \gamma_2 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\alpha & 0 \\ 0 & 0 & \beta_1 & 0 & \gamma_1 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta_2 & 0 & \gamma_2 \end{pmatrix} : \begin{array}{c} I\bar{I} = E,\ \alpha \in \mathbb{C}^*, \\ \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}, \\ \beta_1\gamma_2 - \beta_2\gamma_1 \neq 0 \end{array} \right\},$$

respectively.

Taking now $A \in \mathcal{A}$, $I \in \mathcal{I}$, and using the operation $I \to AI\bar{A}^{-1}$ one obtains that any anti-involution of the complex pair 1.1.7 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is obvious that the set of fixed points $(\mathbb{C}^5)^{I_1}$ coincides with $\mathbb{C}^5$, hence taking the same basis again, we have that the nonzero brackets are

$$[e_1', u_1'] = u_1', \quad [e_1', u_3'] = -u_3', \quad [u_1', u_3'] = e_1';$$

that is, one recovers the real pair $\mathbf{1.1^1(7)}$ (see [3, §§4.1]).

In turn, since given $X = (a, x, y, z, t)$, one has $I_2(X) = (-\bar{a}, -\bar{z}, -\bar{y}, -\bar{x}, \bar{t})$, the set of fixed points of the map I_2 is $(\mathbb{C}^5)^{I_2} = \{(a, x, y, -\bar{x}, t) \in \mathbb{C}^5 : x \in \mathbb{C}, t \in \mathbb{R}, a, y \in \mathbb{R}\mathrm{i}\}$. Then the nonzero brackets with respect to the basis of $(\mathbb{C}^5)^{I_2}$ given by

$$e_1' = \mathrm{i}e_1, \quad u_1' = \frac{\sqrt{2}}{2}\mathrm{i}(u_1 - u_3), \quad u_2' = -\mathrm{i}u_2, \quad u_3' = \frac{\sqrt{2}}{2}(u_1 + u_3), \quad u_4' = u_4,$$

are

$$[e_1', u_1'] = -u_3', \quad [e_1', u_3'] = u_1', \quad [u_1', u_3'] = e_1';$$

that is, one recovers those for the real pair $\mathbf{1.1^2(9)}$ (see [3, §§4.2]).

Moreover, since given $X = (a, x, y, z, t)$, one has $I_3(X) = (-\bar{a}, \bar{z}, -\bar{y}, \bar{x}, \bar{t})$, the set of fixed points of I_3 is $(\mathbb{C}^5)^{I_3} = \{X = (a, x, y, \bar{x}, t) \in \mathbb{C}^5 : x \in \mathbb{C}, t \in \mathbb{R}, a, y \in \mathbb{R}\mathrm{i}\}$. The nonzero brackets with respect to the basis of $(\mathbb{C}^5)^{I_3}$ given by

$$e_1' = \mathrm{i}e_1, \quad u_1' = \frac{\sqrt{2}}{2}(u_1 + u_3), \quad u_2' = -\mathrm{i}u_2, \quad u_3' = \frac{\sqrt{2}}{2}\mathrm{i}(u_1 - u_3), \quad u_4' = u_4,$$

are

$$[e_1', u_1'] = -u_3', \quad [e_1', u_3'] = u_1', \quad [u_1', u_3'] = -e_1';$$

that is, one recovers those for the real pair $\mathbf{1.1^2(10)}$ (see [3, §§4.2]).

2.8. *The real pairs* $2.1^2(1)$ *to* $2.1^2(5)$

For $n = 2$, the complex pair 2.1 splits into two complex subcases: 2.1.1 and 2.1.2. According to [12, Th. 2.1, p. 72], the complex pair 2.1 furnishes in particular the set of real pairs 2.1.1 and 2.1.2, which include the pairs $2.1^2(1,2)$ and $2.1^2(3,4,5)$ studied in [3].

The aforementioned Theorem 2.1 in [12, p. 72], states that *any real form of the algebra* 2.1 *is conjugate to one and only one of the following linear Lie algebras:* algebras:

$$2.1^1 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}, \qquad 2.1^2 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & -x & 0 \\ 0 & y & 0 & 0 \end{pmatrix},$$

$$2.1^3 \quad \begin{pmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \\ x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix}, \qquad 2.1^4 \quad \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & -y & -x \end{pmatrix}.$$

Any real form of the complex pair 2.1.1 *is equivalent to one and only one of the pairs* $2.1^1(1), 2.1^2(1), 2.1^2(2), 2.1^3(1), 2.1^3(2), 2.1^3(3), 2.1^4(1)$. *And any real form of the complex pair* 2.1.2 *is equivalent to one and only one of the following pairs*: $2.1^1(2), 2.1^2(3)$, $2.1^2(4), 2.1^2(5)$, $2.1^3(4), 2.1^3(5)$. The complex subcase 2.1.1 so originates the subcases $2.1^2(1)$ (his page seventy-seven) and $2.1^2(2)$ (his page seventy-eight) studied in [3] and the complex subcase 2.1.2 so originates the subcases $2.1^2(3)$, $2.1^2(4)$, and $2.1^2(5)$ (his pages eighty and eighty-one) studied in [3].

The proof of Proposition 2.1.1 is as follows. The group $\mathcal{A}$ and the set $\mathcal{I}$ of the complex pair 2.1.1 are

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & b & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b \end{pmatrix}, \right.$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b \\ 0 & 0 & 0 & 0 & 1/a & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a & 0 \end{pmatrix},$$

$$\left. \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b \\ 0 & 0 & a & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \end{pmatrix} : a, b \in \mathbb{C}^* \right\},$$

$$\mathcal{I}=\left\{I\in\left\{\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&\alpha&0&0&0\\0&0&0&\beta&0&0\\0&0&0&0&1/\alpha&0\\0&0&0&0&0&1/\beta\end{pmatrix},\begin{pmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&\alpha&0&0&0\\0&0&0&0&0&1/\beta\\0&0&0&0&1/\alpha&0\\0&0&0&\beta&0&0\end{pmatrix},\begin{pmatrix}-1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&1/\alpha&0\\0&0&0&\beta&0&0\\0&0&\alpha&0&0&0\\0&0&0&0&0&1/\beta\end{pmatrix},\right.\right.$$

$$\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&1/\alpha&0\\0&0&0&0&0&1/\beta\\0&0&\alpha&0&0&0\\0&0&0&\beta&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&\beta&0&0\\0&0&\alpha&0&0&0\\0&0&0&0&0&1/\beta\\0&0&0&0&1/\alpha&0\end{pmatrix},$$

$$\left.\left.\begin{pmatrix}0&-1&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&1/\beta\\0&0&a&0&1/\alpha&0\\0&0&0&\beta&0&0\\0&0&\alpha&0&0&0\end{pmatrix}\right\}:\begin{array}{c}I\bar{I}=E,\\\alpha,\beta\in\mathbb{C}^*\end{array}\right\},$$

respectively.

Taking now $A\in\mathcal{A}, I\in\mathcal{I}$, and using the operation $I\mapsto AI\bar{A}^{-1}$ one obtains that any anti-involution of the complex pair 2.1.1 is conjugate to one and only one of the following:

$$I_1=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},I_2=\begin{pmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\\0&0&0&-1&0&0\end{pmatrix},I_3=\begin{pmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\end{pmatrix},$$

$$I_4=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\\0&0&-1&0&0&0\\0&0&0&-1&0&0\end{pmatrix},$$

$$I_5=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&1\\0&0&-1&0&0&0\\0&0&0&1&0&0\end{pmatrix},\ I_6=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&1&0&0&0\\0&0&0&1&0&0\end{pmatrix},$$

$$I_7=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}.$$

Given $X=(a,b,x,y,z,t)$, one has $I_2(X)=(\bar{a},-\bar{b},\bar{x},-\bar{t},\bar{z},-\bar{y})$. Hence, the set of fixed points of I_2 is $(\mathbb{C}^6)^{I_2}=\{(a,b,x,y,z,-\bar{y})\in\mathbb{C}^6 : y\in\mathbb{C},\ b\in\mathbb{R}\mathrm{i},\ a,x,z\in\mathbb{R}\}$. The nonzero brackets of the real form corresponding to the anti-involution I_2, with respect to the basis of $(\mathbb{C}^6)^{I_2}$ given by

$$e_1'=e_1,\ e_2'=\mathrm{i}e_2,\ u_1'=u_1,\ u_2'=\frac{\sqrt{2}}{2}(u_2-u_4),\ u_3'=u_3,\ u_4'=\frac{\sqrt{2}}{2}\mathrm{i}(u_2+u_4),$$

are

$$[e_1', u_1'] = u_1', [e_1', u_3'] = -u_3', [e_2', u_2'] = u_4', [e_2', u_4'] = -u_2', [u_1', u_3'] = e_1', [u_2', u_4'] = e_2';$$

that is, one recovers those for the real pair $\mathbf{2.1^2(1)}$ (see [3, §§4.6]).

In turn, given $X = (a, b, x, y, z, t)$, one has $I_3(X) = (\bar{a}, -\bar{b}, \bar{x}, \bar{t}, \bar{z}, \bar{y})$; hence, the set of fixed points of I_3 is $(\mathbb{C}^6)^{I_3} = \{(a, b, x, y, z, \bar{y}) \in \mathbb{C}^6 : y \in \mathbb{C},\, b \in \mathbb{R}\mathrm{i},\, a, x, z \in \mathbb{R}\}$. The nonzero brackets of the real form corresponding to the anti-involution I_3, with respect to the basis of $(\mathbb{C}^6)^{I_3}$ given by

$$\begin{gathered} e_1' = e_1, \quad e_2' = \mathrm{i}e_2, \quad u_1' = u_1, \\ u_2' = \frac{\sqrt{2}}{2}(u_2 + u_4), \quad u_3' = u_3, \quad u_4' = \frac{\sqrt{2}}{2}\mathrm{i}(u_2 - u_4), \end{gathered}$$

are

$$\begin{gathered} [e_1', u_1'] = u_1', \quad [e_1', u_3'] = -u_3', \quad [e_2', u_2'] = u_4', \\ [e_2', u_4'] = -u_2', \quad [u_1', u_3'] = e_1', \quad [u_2', u_4'] = -e_2'; \end{gathered}$$

that is, one recovers those for the real pair $\mathbf{2.1^2(2)}$ (see [3, §§4.6]).

In turn, the complex case 2.1.2 furnishes the subcases $2.1^2(3)$ (his page eighty), $2.1^2(4)$, $2.1^2(5)$ (his page eighty-one) studied in [3]. More specifically, according to [12, Prop. 2.1.2, p. 79], *any real form of the complex pair* 2.1.2 *is equivalent to one and only one of the following pairs*: $2.1^1(2), 2.1^2(3), 2.1^2(4), 2.1^2(5), 2.1^3(4), 2.1^3(5)$. The proof is as follows. The group $\mathcal{A}$ and the set $\mathcal{I}$ of the complex pair 2.1.2 are

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & b & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \end{pmatrix} : a \in \mathbb{C}^*,\, b, c \in \mathbb{C} \right\},$$

$$\mathcal{I} = \left\{ I \in \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 & 1/\alpha & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\alpha & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \end{pmatrix} : \begin{array}{l} I\bar{I} = E, \\ \alpha \in \mathbb{C}^*,\, \beta, \gamma \in \mathbb{C} \end{array} \right\},$$

respectively.

Taking now $A \in \mathcal{A}, I \in \mathcal{I}$, and using the operation $I \mapsto AI\bar{A}^{-1}$, one obtains that any anti-involution of the complex pair 2.1.2 is conjugate to one and only one of the following:

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$I_4 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_5 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad I_6 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then, since given $X = (a, b, x, y, z, t)$ one has $I_2(X) = (\bar{a}, -\bar{b}, \bar{x}, \bar{t}, \bar{z}, \bar{y})$, the set of fixed points of I_2 is $(\mathbb{C}^6)^{I_2} = \{(a, b, x, y, z, \bar{y}) : y \in \mathbb{C},\ b \in \mathbb{R}\mathrm{i},\ a, x, z \in \mathbb{R}\}$. The nonzero brackets of the real form corresponding to the anti-involution I_2, with respect to the basis of $(\mathbb{C}^6)^{I_2}$ given by

$$e_1' = e_1, \quad e_2' = \mathrm{i}e_2, \quad u_1' = u_1, \quad u_2' = u_2 + u_4, \quad u_3' = u_3, \quad u_4' = \mathrm{i}u_2 - \mathrm{i}u_4,$$

are

$$[e_1', u_1'] = u_1', \quad [e_1', u_3'] = -u_3', \quad [e_2', u_2'] = u_4', \quad [e_2', u_4'] = -u_2', \quad [u_1', u_3'] = e_1';$$

that is, one recovers those for the real pair $\mathbf{2.1^2(3)}$ (see [3, §§4.6]).

Since given $X = \{a, b, x, y, z, t\}$ one has $I_3(X) = (-\bar{a}, \bar{b}, -\bar{z}, \bar{y}, -\bar{x}, \bar{t})$, the set of fixed points of I_3 is $(\mathbb{C}^6)^{I_3} = \{(a, b, x, y, -\bar{x}, t) \in \mathbb{C}^6 : x \in \mathbb{C},\ a \in \mathbb{R}\mathrm{i},\ b, y, t \in \mathbb{R}\}$. The nonzero brackets of the real form corresponding to the anti-involution I_3, with respect to the basis of $(\mathbb{C}^6)^{I_3}$ given by

$$e_1' = e_1,\ e_2' = \mathrm{i}e_2,\ u_1' = u_2,\ u_2' = \frac{\sqrt{2}}{2}(u_1 - u_3),\ u_3' = u_4,\ u_4' = \frac{\sqrt{2}}{2}\mathrm{i}(u_1 + u_3),$$

are

$$[e_1', u_1'] = u_1', \quad [e_1', u_3'] = -u_3', \quad [e_2', u_2'] = u_4', \quad [e_2', u_4'] = -u_2', \quad [u_2', u_4'] = e_2';$$

that is, one recovers those for the real pair $\mathbf{2.1^2(4)}$ (see [3, §§4.7]).

Since given $X = \{a, b, x, y, z, t\}$ one has $I_4(X) = (-\bar{a}, \bar{b}, \bar{z}, \bar{y}, \bar{x}, \bar{t})$, the set of fixed points of I_4 is $(\mathbb{C}^6)^{I_4} = \{(a, b, x, y, \bar{x}, t) \in \mathbb{C}^6 : x \in \mathbb{C},\ a \in \mathbb{R}\mathrm{i},\ b, y, t \in \mathbb{R}\}$. The nonzero brackets of the real form corresponding to the anti-involution I_4, with respect to the basis of $(\mathbb{C}^6)^{I_4}$ given by

$$e_1' = e_2,\ e_2' = \mathrm{i}e_1,\ u_1' = u_2,\ u_2' = \frac{\sqrt{2}}{2}(u_1 + u_3),\ u_3' = u_4,\ u_4' = \frac{\sqrt{2}}{2}\mathrm{i}(u_1 - u_3),$$

are

$$[e_1', u_1'] = u_1', \quad [e_1', u_3'] = -u_3', \quad [e_2', u_2'] = u_4', \quad [e_2', u_4'] = -u_2', \quad [u_2', u_4'] = -e_2';$$

that is, one recovers those for the real pair $\mathbf{2.1^2(5)}$ (see [3, §§4.7]).

2.9. The real pairs $3.5^2(2)$ and $3.5^2(3)$

As for $n = 3$, according to [12, Th. 3.5, p. 145], the complex pair 3.5.2 furnishes in particular the set of real pairs 3.5, with four subcases, which include the pairs $3.5^2(2)$ and $3.5^2(3)$ studied in [3]. In this case, $(\mathfrak{g}, \mathfrak{h})$ is a complex isotropically faithful pair, where $\mathfrak{h}$ is a non-solvable subalgebra of $\mathfrak{g}$. In this case, the computation of the automorphisms group and of the set of anti-involution for the pair $(\mathfrak{g},\mathfrak{h})$ becomes more complicated. To obtain these real pairs from the complex pair $(\mathfrak{g},\mathfrak{h})$, one first studies the real forms $\mathfrak{h}^\sigma$ of $\mathfrak{h}$ and then the isotropically faithful pairs $(\mathfrak{g}^\sigma,\mathfrak{h}^\sigma)$. From the set of conjugations $\mathrm{Conj}(\mathfrak{h}) = \{A \in \mathrm{GL}(4,\mathbb{C}) : A\mathfrak{h}A^{-1} = \mathfrak{h}\}$ of $\mathfrak{h}$, acting as $X \mapsto AXA^{-1}$, $X \in \mathfrak{h}$, anti-conjugations are obtained as the composition of the complex conjugation $X \mapsto \bar{X}$ with an element in $\mathrm{Conj}(\mathfrak{h})$. The set of anti-involutions is thus

$$\mathcal{P} = \{P \in \mathrm{GL}(4,\mathbb{C}) : P\bar{X}\bar{P} \in \mathfrak{h},\ \forall X \in \mathfrak{h},\ P\bar{P} = \mathrm{Id}\},$$

with P acting as $X \mapsto P\bar{X}\bar{P}$. Equivalent anti-involutions are understood under the action of $\mathrm{Conj}(\mathfrak{h})$, as $P \mapsto AP\bar{A}^{-1}$. Then, the procedure to obtain the real forms consists of finding $P_1,\dots,P_k \in \mathcal{P}$ such that $AP_i\bar{A}^{-1} \neq P_j$, for all $A \in \mathrm{Conj}(\mathfrak{h})$, $i \neq j$, and building $\mathfrak{h}^\sigma$ for each of them. Finally, the method of finding the real isotropically faithful pairs is similar to that used for finding complex pairs.

The aforementioned Theorem 3.5 in [12, p. 145], states that any real form of the algebra 3.5 is conjugate to one and only one of the following linear algebras:

$$3.5^1 \quad \begin{pmatrix} 2x & y & 0 & 0 \\ 2z & 0 & -2y & 0 \\ 0 & -z & -2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad 3.5^2 \quad \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & z & 0 \\ -y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, any isotropically faithful pair of type 3.5^2 is equivalent [12, pp. 146–147] to one and only one of the following:

1

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	0	0	u_1
u_2	$-u_1$	0	u_3	0	0	0	u_2
u_3	0	$-u_1$	$-u_2$	0	0	0	u_3
u_4	0	0	0	$-u_1$	$-u_2$	$-u_3$	0

2

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	e_1	e_2	0
u_2	$-u_1$	0	u_3	$-e_1$	0	e_3	0
u_3	0	$-u_1$	$-u_2$	$-e_2$	$-e_3$	0	0
u_4	0	0	0	0	0	0	0

3

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	$-e_1$	$-e_2$	0
u_2	$-u_1$	0	u_3	e_1	0	$-e_3$	0
u_3	0	$-u_1$	$-u_2$	e_2	e_3	0	0
u_4	0	0	0	0	0	0	0

4

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_3$	e_2	$-u_2$	u_1	0	0
e_2	e_3	0	$-e_1$	$-u_3$	0	u_1	0
e_3	$-e_2$	e_1	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	0	0	0
u_2	$-u_1$	0	u_3	0	0	0	0
u_3	0	$-u_1$	$-u_2$	0	0	0	0
u_4	0	0	0	0	0	0	0

The proof is as follows. First, a computation gives that the Lie group $\mathrm{Conj}(\mathfrak{h})$ has two connected components. The connected component of the identity is

$$\mathrm{Conj}(\mathfrak{h})_0 = \left\{ A = \lambda \begin{pmatrix} d^2 & -a_{23}d & -a_{23}^2 & 0 \\ -2a_{32}d & 2a_{22} & 2a_{23} & 0 \\ -a_{32}^2 & a_{32} & 1 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} : \begin{array}{l} \lambda,\mu \in \mathbb{C}-\{0\}, a_{ij} \in \mathbb{C}, \\ d = 2a_{22} - a_{23}a_{32} \end{array} \right\}$$

whereas the other is $J \cdot \mathrm{Conj}(\mathfrak{h})_0 = \{JA : A \in \mathrm{Conj}(\mathfrak{h})_0\}$ with

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similar computations give the expression of the matrices P of the set $\mathcal{P}$ of involutions, organized in the next four different types

$$\begin{pmatrix} e^{4\beta \mathrm{i}} & 0 & 0 & 0 \\ 2re^{3\beta \mathrm{i}} & -e^{2\beta \mathrm{i}} & 0 & 0 \\ -r^2e^{2\beta \mathrm{i}} & re^{\beta \mathrm{i}} & 1 & 0 \\ 0 & 0 & 0 & e^{i\gamma} \end{pmatrix}, \begin{pmatrix} e^{4\beta \mathrm{i}} & re^{3\beta \mathrm{i}} & -r^2e^{2\beta \mathrm{i}} & 0 \\ 0 & -e^{2\beta \mathrm{i}} & 2re^{\beta \mathrm{i}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\gamma \mathrm{i}} \end{pmatrix},$$

$$\begin{pmatrix} e^{4\beta \mathrm{i}} & re^{3\beta \mathrm{i}} & -r^2e^{2\beta \mathrm{i}} & 0 \\ -\frac{4}{r}e^{3\beta \mathrm{i}} & -3e^{2\beta \mathrm{i}} & 2re^{\beta \mathrm{i}} & 0 \\ -\frac{4}{r^2}e^{2\beta \mathrm{i}} & -\frac{2}{r}e^{\beta \mathrm{i}} & 1 & 0 \\ 0 & 0 & 0 & e^{\gamma \mathrm{i}} \end{pmatrix}, \begin{pmatrix} -r^2e^{2\beta \mathrm{i}} & re^{\beta \mathrm{i}} & 1 & 0 \\ -2re^{\beta \mathrm{i}}\left(r^2 \pm 1\right) & 2r^2 \pm 1 & 2re^{-\beta \mathrm{i}} & 0 \\ \left(r^2 \pm 1\right)^2 & -re^{-\beta \mathrm{i}}\left(r^2 \pm 1\right) & -r^2e^{-2\beta \mathrm{i}} & 0 \\ 0 & 0 & 0 & e^{\gamma \mathrm{i}} \end{pmatrix},$$

with $r \neq 0$ for the third type. These matrices are equivalent under conjugation by $A \in \mathrm{Conj}(\mathfrak{h})$ to

$$P_1 = \mathrm{id}, \qquad P_2 = J.$$

More precisely, for each of the four types of matrices P, we choose A as

$$A = \begin{pmatrix} -4e^{-2\beta \mathrm{i}} & 0 & 0 & 0 \\ -2rie^{-2\beta \mathrm{i}} & 2ie^{-\beta \mathrm{i}} & 0 & 0 \\ -\frac{r^2}{4}e^{-2\beta \mathrm{i}} & \frac{r}{2}e^{-\beta \mathrm{i}} & 1 & 0 \\ 0 & 0 & 0 & e^{-\gamma \mathrm{i}/2} \end{pmatrix}, A = \begin{pmatrix} \frac{-1}{r^2}e^{-2\beta \mathrm{i}} & -\frac{1}{2r}e^{-\beta \mathrm{i}} & \frac{1}{4} & 0 \\ 0 & \frac{\mathrm{i}}{r}e^{-\beta i} & -\mathrm{i} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\gamma \mathrm{i}/2} \end{pmatrix},$$

$$A = \begin{pmatrix} -\left(2\sqrt{2}+3\right)\frac{2}{r^2}e^{-2\beta \mathrm{i}} & -\left(2\sqrt{2}+3\right)\frac{\sqrt{2}}{r}e^{-\beta \mathrm{i}} & \left(2\sqrt{2}+3\right) & 0 \\ \frac{4}{r^2}e^{-2\beta \mathrm{i}}\sqrt{2\sqrt{2}+3}i & 0 & 2\sqrt{2\sqrt{2}+3}\mathrm{i} & 0 \\ -\frac{2}{r^2}e^{-2\beta \mathrm{i}} & \frac{\sqrt{2}}{r}e^{-\beta \mathrm{i}} & 1 & 0 \\ 0 & 0 & 0 & e^{-\gamma \mathrm{i}/2} \end{pmatrix},$$

$$A = \begin{pmatrix} -r^2e^{2\beta \mathrm{i}} & re^{\beta} & 1 & 0 \\ -2re^{i\beta} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\gamma \mathrm{i}/2} \end{pmatrix},$$

respectively. Note that P_1 and P_2 are not conjugated, since the condition $AJ = \bar{A}$ is satisfied by neither $A \in \mathrm{Conj}(\mathfrak{h})_0$ nor $A \in J \cdot \mathrm{Conj}(\mathfrak{h})_0$.

The set of a fixed points of $\mathfrak{h}$ by the anti-involution $X \mapsto P_1\bar{X}\bar{P}_1$ is obviously the algebra 3.5^1. With respect to the anti-involution defined by P_2, the set of fixed points is

$$\left\{ \begin{pmatrix} \alpha \mathrm{i} & y & 0 & 0 \\ -2\bar{y} & 0 & -2y & 0 \\ 0 & \bar{y} & -\alpha \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \alpha \in \mathbb{R}, y \in \mathbb{C} \right\},$$

that is conjugated to 3.5^2 by the matrix

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \mathrm{i} & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Any virtual structure on 3.5^2 is trivial since the algebra is semisimple as it is isomorphic to $\mathfrak{so}(3)$. We choose the basis

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then any real pair $(\mathfrak{g}^\sigma, \mathfrak{h}^\sigma)$ can be defined by the trivial structure

$$\begin{gathered} [e_1, e_2] = -e_3, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1 \\ [e_1, u_1] = -u_2, \quad [e_1, u_2] = u_1, \quad [e_1, u_3] = 0, \quad [e_1, u_4] = 0, \\ [e_2, u_1] = -u_3, \quad [e_2, u_2] = 0, \quad [e_2, u_3] = u_1, \quad [e_2, u_4] = 0, \\ [e_3, u_1] = 0, \quad [e_3, u_2] = -u_3, \quad [e_3, u_3] = u_2, \quad [e_3, u_4] = 0. \end{gathered}$$

If we write

$$[u_i, u_j] = \sum\nolimits_{\alpha=1}^{3} c^{\alpha}_{3+i,3+j} e_\alpha + \sum\nolimits_{r=4}^{7} c^{r}_{3+i,3+j} u_r, \qquad i, j = 1, ..., 4,$$

using the Jacobi identity, some computations give two possible solutions:

$[\ ,\ ]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_3	$-e_2$	$-u_2$	u_1	0	0
e_2	$-e_3$	0	e_1	$-u_3$	0	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	$c^6_{45}\left(\frac{1}{4}c^6_{45}e_1 + u_3\right)$	$c^6_{45}\left(\frac{1}{4}c^6_{45}e_2 - u_2\right)$	$c^6_{67}\left(\frac{1}{2}c^6_{45}e_3 + u_1\right)$
u_2	$-u_1$	0	u_3	$-c^6_{45}\left(\frac{1}{4}c^6_{45}e_1 + u_3\right)$	0	$c^6_{45}\left(\frac{1}{4}c^6_{45}e_3 + u_1\right)$	$-c^6_{67}\left(\frac{1}{2}c^6_{45}e_2 - u_2\right)$
u_3	0	u_1	$-u_2$	$-c^6_{45}\left(\frac{1}{4}c^6_{45}e_2 - u_2\right)$	$-c^6_{45}\left(\frac{1}{4}c^6_{45}e_3 + u_1\right)$	0	$c^6_{67}\left(\frac{1}{2}c^6_{45}e_1 + u_3\right)$
u_4	0	0	0	$-c^6_{67}\left(\frac{1}{2}c^6_{45}e_3 + u_1\right)$	$c^6_{67}\left(\frac{1}{2}c^6_{45}e_2 - u_2\right)$	$-c^6_{67}\left(\frac{1}{2}c^6_{45}e_1 + u_3\right)$	0

and

$[\ ,\ ]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_3	$-e_2$	$-u_2$	u_1	0	0
e_2	$-e_3$	0	e_1	$-u_3$	0	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2	0
u_1	u_2	u_3	0	0	$c^3_{56}e_1 + c^6_{45}u_3$	$c^3_{56}e_2 - c^6_{45}u_2$	0
u_2	$-u_1$	0	u_3	$-\left(c^3_{56}e_1 + c^6_{45}u_3\right)$	0	$c^3_{56}e_3 + c^6_{45}u_1$	0
u_3	0	u_1	$-u_2$	$-c^3_{56}e_2 + c^6_{45}u_2$	$-\left(c^3_{56}e_3 + c^6_{45}u_1\right)$	0	0
u_4	0	0	0	0	0	0	0

With respect to the second solution, the case $c_{45}^6 \neq 0$, with the change of basis

$$\begin{cases} u_1' = \frac{1}{c_{45}^6} u_1 + \frac{1}{2} e_3, \qquad u_2' = \frac{1}{c_{45}^6} u_2 - \frac{1}{2} e_2, \\ u_3' = \frac{1}{c_{45}^6} u_3 + \frac{1}{2} e_1, \qquad u_4' = u_4, \end{cases} \tag{2.18}$$

gives

$[\ ,\ ]$	e_1	e_2	e_3	u_1'	u_2'	u_3'	u_4'
e_1	0	e_3	$-e_2$	$-u_2'$	u_1'	0	0
e_2	$-e_3$	0	e_1	$-u_3'$	0	u_1'	0
e_3	e_2	$-e_1$	0	0	$-u_3'$	u_2'	0
u_1'	u_2'	u_3'	0	0	Ae_1	Ae_2	0
u_2'	$-u_1'$	0	u_3'	$-Ae_1$	0	Ae_3	0
u_3'	0	u_1'	$-u_2'$	$-Ae_2$	$-Ae_3$	0	0
u_4	0	0	0	0	0	0	0

with

$$A = \left(\frac{c_{56}^3}{(c_{45}^6)^2} - \frac{1}{4} \right).$$

For $A = 0$, we get $3.5^2(4)$. If $A \neq 0$, we get $3.5^2(2)$ and $3.5^2(3)$ depending on the sign of A. The case $c_{45}^6 = 0$ reduces to the previous case.

With respect to the first solution, the case $c_{67}^6 = 0$ reduces to the second solution with $c_{56}^3 = \frac{1}{4}\left(c_{45}^6\right)^2$. If Si $c_{45}^6 = 0$ the case reduces to $3.5^2(1)$.

If $c_{45}^6 c_{67}^6 \neq 0$ by considering the change of basis (2.18), we obtain

$[\ ,\ ]$	e_1	e_2	e_3	u_1'	u_2'	u_3'	u_4'
e_1	0	e_3	$-e_2$	$-u_2'$	u_1'	0	0
e_2	$-e_3$	0	e_1	$-u_3'$	0	u_1'	0
e_3	e_2	$-e_1$	0	0	$-u_3'$	u_2'	0
u_1'	u_2'	u_3'	0	0	0	0	$c_{67}^6 u_1'$
u_2'	$-u_1'$	0	u_3'	0	0	0	$c_{67}^6 u_2'$
u_3'	0	u_1'	$-u_2'$	0	0	0	$c_{67}^6 u_3'$
u_4	0	0	0	$-c_{67}^6 u_1'$	$-c_{67}^6 u_2'$	$-c_{67}^6 u_3'$	0

that reduces to case $3.5^2(1)$.

2.10. Hicks's method

J.W. Hicks presents in [8, p. 70] a methodology for the classification of pairs. His work is based in previous considerations of Schmidt [21] where one fixes a subalgebra of the Lorentz algebra and identify the infinitesimal generators in the standard representation (see table 3.1), yielding a linear transformation from $\mathbb{R}^4$ to itself. These transformations are taken to define the adjoint of each basis element $h \in \mathfrak{h}$ in $\mathfrak{g}$. The basis elements of $\mathfrak{g}$ are taken according to the sum $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. After impossing the Jacobi identities to eliminate arbitrary parameters, one finds changes of basis on $\mathfrak{m}$ which leave the adjoint action of the isotropy undisturbed yet eliminate remaining inessential parameters from the structure equations.

Then, with the aid of a Lie algebra software, the following invariants are computed: algebra type, the derived series dimensions, indecomposability, dimension of the nilradical, dimension of the Lie algebra of derivations $\mathcal{D}$, dimension of the Lie algebra of derivations of $\mathcal{D}$, dimensions of the decomposition of the radical of $\mathcal{D}$, dimensions of the decomposition of the Lie algebra. From these properties a comparison is made to the internal database of pairs. This process is repeated for the derived algebras. After cumbersome work, Hicks concludes [8, p. 164] that "the Komrakov classification of Lorentzian pairs and that of this dissertation are in complete agreement."

Acknowledgements

We are grateful to Prof. A. Elduque for his aid with Remark 2.1.

Data availability

Data will be made available on request.

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