Analytical Approximations for Nonlinear Diffusion Time in Multiscale Edge Enhancement

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Resumen The image simplification, noise elimination and edge enhancement steps are all fundamental to segmentation tasks. These processing techniques usually require the tuning of their control parameters; a procedure known to be incompatible with automatic segmentation. The aim of this paper is to adopt a procedure, based on nonlinear diffusion, that is capable of auto tuning by means of analytical expressions that relation diffusion times to the gradient module. The numerical method and experimental results are shown in 1D, 2D and 3D.

1. Introduction

One of the objectives of image analysis is its segmentation. Firstly, edge enhancement and smoothing of the different areas of the image are essential, and represent the means by which to eliminate noise and increase contrast. A review of these techniques can be found in Buades *et al*[2]. For this paper, we will adopt procedures that are connected to the nonlinear diffusion equation[6]. This approach consists of obtaining through the use of an initial image $u_0 : \Omega \to \mathbb{R}$ defined over the domain $\Omega \subset \mathbb{R}^n$, another image $u(\mathbf{x})$ as the solution to a nonlinear diffusion equation with initial and Neumann boundary conditions:

$$u_t = \text{div } \left(g(\|\nabla u\|)\, \nabla u\right) \quad \mathbf{x} \in \Omega \quad t > 0 \tag{1}$$

with $u(\mathbf{x}, 0) = u_0(\mathbf{x}) \quad \mathbf{x} \in \Omega$ as initial conditions and $u_{\mathbf{n}} = 0 \quad \mathbf{x} \in \partial\Omega$ as boundary conditions, with $g(\|\nabla u\|)$ further representing *diffusivity*, which is a non-negative function and usually decreasing in terms of the gradient module. The properties of the nonlinear diffusion filter are clearer when set out in a new orthonormal reference system in which one of the axes may be determined by the direction and position of the gradient $\boldsymbol{\eta} = \nabla u / \|\nabla u\|$ where $\|\nabla u\| \neq 0$, which together with $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ form the curve/surface at a level perpendicular to $\boldsymbol{\eta}$[4,9,11]:

$$u_t = g(\|\nabla u\|)\,(u_{\boldsymbol{\xi\xi}} + u_{\boldsymbol{\zeta\zeta}}) + [\, g(\|\nabla u\|) + g'(\|\nabla u\|) \cdot \|\nabla u\| \,]\, u_{\boldsymbol{\eta\eta}} \tag{2}$$

where $u_{\boldsymbol{\eta}\boldsymbol{\eta}}$ represents the second derived from u in the direction of $\boldsymbol{\eta}$.

In the one-dimensional case, enhancement is achieved by the absolute and dynamic increase of edge slopes which imposes the condition that the coefficient $u_{\boldsymbol{\eta}\boldsymbol{\eta}}$ in (2) must be negative[4]. This conclusion can also be generalised to $2D$ and $3D$. The drawback to this approach, however, is attributed to the fact that the continuous enhancement model gives rise to an ill-posed problem[5,3]. This scenario, in the discrete instance, may change under certain data conditions, giving rise to convergent solutions as referred to in [3] analysed in more detail in [11].

This paper proposes, primarily, a diffusivity without tuning parameters which is capable of guaranteeing the weighting between what has been smoothed and what has been enhanced. The paper will look subsequently at the theoretical research, offering an analytical solution to the nonlinear diffusion process between 3 one-dimensional pixels. We will then draw conclusions that link diffusion time to enhancement tasks. The extension to n-pixels will be experimentally validated. Its application to the highest dimension will be thoroughly tested by applying operators based on the orthogonal decomposition of the divergence [12]. Finally, said dynamic shall be used in different examples, allowing us to compare the conclusions drawn and their subsequent application to medical images.

2. Nonlinear diffusion filter without control parameters

One of the major features of TV diffusivity[1] is the absence of control parameters. However, this advantage is negligible since it does not allow us to increase enhancement of the image, due to the fact that the coefficient $\partial_{\boldsymbol{\eta}\boldsymbol{\eta}} u$ in (2) is not negative. It has recently come to light as a diffusion family that has attracted growing interest and which does not require ad hoc adjustments [10,4,13]: $g(\|\nabla u\|) = \frac{1}{\|\nabla u\|^p}$, $p \geq 0$. Where $p = 0$ represents linear diffusion, $p = 1$ corresponding to TV and $p = 2$ to BFB (*Balance Forward Backward*)[4].The condition $p > 1$ in these single-parameter families of diffusion achieves the objective of enhancement. Finding a continuous solution is difficult. One approximation to this type of equation can be found in Tsurkov[10] using techniques that have already been applied to porous media models.

2.1. Semi-Discrete Formulation

It is known that ill-posed problems in the continuous case can be studied to a certain degree of success in the semi-discrete and discrete case [5,11]. Hence, spatial discretisation is performed on the one-dimensional equation (1) using the proposed single-parameter diffusion function. For this purpose, an approximation on finite differences based on the average distance between pixels is used, which subsequently gives rise to an autonomous system of ordinary differential equations:

$$\dot{U}_i(t) = h^{p-2} \left[\frac{U_{i+1} - U_i}{|U_{i+1} - U_i|^p} - \frac{U_i - U_{i-1}}{|U_i - U_{i-1}|^p} \right] \tag{3}$$

with $i = 2, \ldots n-1$. On carrying out the operation we get an autonomous matrix ordinary differential expression of the type $\frac{d\mathbf{U}}{dt}(t) = \mathbf{f}(\mathbf{U}(t)) = \boldsymbol{A}(\mathbf{U}(t))\,\mathbf{U}(t)$. The generalisation to additional dimensions is direct [12].

3. Study of semi-analytical solutions for nonlinear diffusion

3.1. Background

System resolution (3) has already been covered in Steidl [8] and subsequently in Welk [13]. It would appear that semi-discrete formulation (3) of the equation (1), gives rise to a singularity on the system for gradient values close to zero. One way of avoiding this scenario consists of introducing a positive constant ε approaching zero. This regularisation leads us to consider a new diffusion function $g_\varepsilon(s) = \frac{1}{(s+\varepsilon)^p} \leq \frac{1}{\varepsilon^p}$ with $s \geq 0$,conforming to $g_\varepsilon \rightarrow g$ where $\varepsilon \rightarrow 0$.

It therefore follows that an explicit Euler method places a major restriction on the increase of time. One alternative is the two-pixel method [13]. It has been observed, however, that the semi-discrete regularisation method is more effective for longer time increment, whilst the two pixel method is more accuracy for shorter time increment.

One limitation of the semi-implicit method is the need to manually tune the diffusion times for enhancement tasks.

3.2. Analytical approach to nonlinear diffusion time

Let us suppose, initially, a 1D signal comprised of just three pixels in which the gradients are not void. Applying the semi-implicit Euler method whose coefficient matrix can be reversed creates the expression:

$$\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{\alpha^p\beta^p+2r\alpha^p+r\beta^p+r^2}{d} & \frac{r(\beta^p+r)}{d} & \frac{r^2}{d} \\ \frac{r(\beta^p+r)}{d} & \frac{\alpha^p\beta^p+r\alpha^p+r\beta^p+r^2}{d} & \frac{r(\alpha^p+r)}{d} \\ \frac{r^2}{d} & \frac{r(\alpha^p+r)}{d} & \frac{\alpha^p\beta^p+r\alpha^p+2r\beta^p+r^2}{d} \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \end{bmatrix} \tag{4}$$

where $\alpha = |U_2 - U_1| \neq 0$, $\beta = |U_3 - U_2| \neq 0$ and $r = k\,h^{p-2}$ with $k = \Delta\,t$ and $d = \alpha^p\beta^p + 2r\alpha^p + 2r\beta^p + 3r^2$. It is interesting to observe the interaction between the three pixels within the established dynamic. Irrespective of the initial pixel values, and a finite period having lapsed, the matrix coefficients are all equal to $1/3$, directing it towards linear diffusion with a total variation diminishing dynamic. However, the issue lies in how to determine the nonlinear diffusion time so that it produces diffusion between the low gradient module pixels without transferring diffusion to the pixels that have a high value on the gradient module. Without loss of overall applicability, in (4) is imposed $\alpha \gg \beta$, so as to favour diffusion between pixel 2 and 3 and as a means to maintaining the value of the first. This evolution means that the matrix (4) tends to be: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ which forces $r < 2\alpha^p$. This result supports the notion that it is possible to dimension the nonlinear diffusion time in such a way so as to smooth over areas of low contrast and enhance areas of low contrast simultaneously. The next section experimentally analyses the validity of the conclusions drawn on n-pixels.

4. Numerical methods, experiments and applications

We firstly propose a comparative study between the analytical method referred to in (4) and the numerical method using a semi-implicit scheme. On the basis of (3) and by carrying out an explicit time discretisation, the discrete dynamic is reflected in a matrix $\left(\boldsymbol{I} - r\boldsymbol{A}(\mathbf{U}^k)\right)$ which is defined as positive and tridiagonal.The inversion of this matrix can be effectively solved using the Thomas algorithm. For its implementation within the scope of nonlinear diffusion see [12]. The diffusivities have been regularised with $\varepsilon = 10^{-3}$ and $[0, 1]$ has been established for the signal dynamic range.

With the aim of reducing the degrees of freedom during the diffusion process, the spatial increase has been fixed, without loss of overall applicability, as a single-unit. In order to verify the enhancement conclusions during the experimental work, we employed an order of magnitude lower than the odds ratio over the diffusion time. Once the diffusion time had been set, we observed that by using the numerical methods implemented, the dynamic tends to be consistent, using at least four iterations, thereby resulting in a computational cost saving:

$$r = k \equiv \frac{2\alpha_{th}^{p}}{10 n_{iter}} \tag{5}$$

Where $h = 1$, n_{iter} denotes the number of iterations and α_{th} denotes the absolute value of difference between pixels, this being the basis on which enhancement is achieved. In the experiments carried out, we have employed 10 iterations for 1D and 2D and 5 iterations for 3D. The diffusion results with three pixels using analytical expression of (4) and the Thomas numerical scheme are practically identical for any value of p.

In order to validate the conclusions drawn on the analytical diffusion model with 3-pixels to n-pixels, the analytical evolution of the diffusion of the central pixel of (4) has been extended to all others. Both numerical methods are tested with a synthetic border of 0 to 1, hence the value of α_{th} shall be the unit, then the iteration time for any p, shall be 20ms. White noise has been added to the signal and the resolution has been adjusted in order to increase the number of pixels.

We observed that the analytical model and the numerical model differ to the extent that the value of p increases. For p lower than 3, the analytical model sets the trend for the diffusion process, showing that the conclusions drawn can be approximated to n-pixels. The explanation is based on the equation (2) high diffusion coefficients to the extent that p increases. Moreover, it reaffirms that the tridiagonal matrix inversion - despite increasing the computational cost - shows an interaction that is much wider than a neighbourhood of only the closest neighbours.

The second objective is related to the election of p. Although p must be greater than the unit in order to achieve enhancement, what would be the best value?. We have seen that the increase of p inhibits the staircase but also gives rise to a reduction in the signal dynamic range. Furthermore, the validity of the equation (5) is based on the approximation between the analytical and numerical model and on which basis we conclude that the compromise value ought to be $p = 3$.

Extension to a greater dimension is carried out by applying AOS (Additive Operator Splitting) [12]. Using the image of the cameraman which has been contaminated by

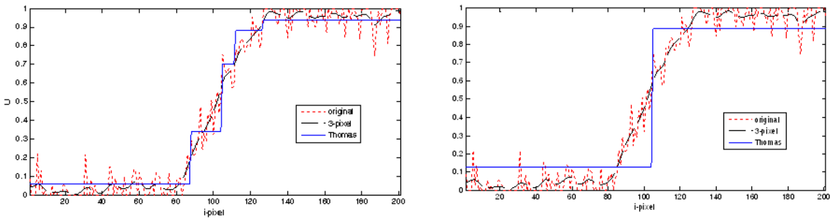

Figura 1. Comparative study of the dynamic in terms of p with $k = 20ms$ a) $p = 2$, b) $p = 3$

white noise and the use of $\alpha_{th} = 0{,}3$ having been incorporated. We have observed that where $p = 3$ the noise disappears and all objects with a difference greater than $0{,}3$ in greyscales are enhanced. In this image, the tower is feathered against the sky since it does not exceed the threshold. Whereas where $p = 2$ noise incidence is greater and contrast is accentuated to a lesser degree.

Figura 2. 2D filter with AOS a) Original, b) $p = 2$, $k = 1{,}8ms$, c) $p = 3$, $k = 0{,}54ms$

This procedure has been applied to the multi-phase segmentation of the liver based on magnetic resonance [7]. Due to the high volume of information, the number of iterations is reduced to 5. The figure below illustrates just six consecutive slices showing a hepatic lesion. We have selected $\alpha_{th} = 0{,}1$. The slices show the increase in contrast of both the organ and the tumour.

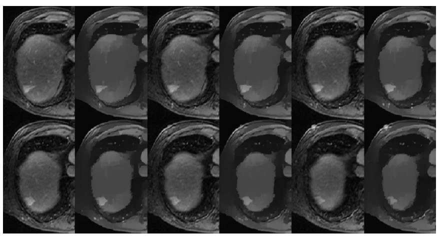

Figura 3. 3D Diffusion where $p = 3$, $k = 0{,}04ms$ (original on the left and on the right, processed

5. Conclusions

The proposed objective is to seek out a numerical method that allows for images to be automatically enhanced at a low computational cost. In this instance, the method is based on the nonlinear diffusion filter. We have selected a family of diffusivities without control parameters. Based on the analytical expression, obtained on the discrete evolution of 3 pixels through the resolution of a semi-implicit Euler method, we have experimentally verified stability, consistency and enhancement properties. Using the analytical model, we have determined the relationship between the diffusion time and the gradient module. Experimentally, the value of $p = 3$ has been considered the most suitable based on the convergence to the analytical model presented, to the conclusions drawn and the lower incidence of the staircase.

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