International Conference on PDEs. For the 60th birthday of Michael Chipot

An inverse problem in Lubrication Theory

S. Ciuperca[1], M. Jay[2] and J. I. Tello[3]

1.- Université de Lyon 1, Lyon, France

2.- INSA Lyon, Lyon, France

3.- Universidad Politécnica de Madrid, Spain

Poitiers, February 18-20 , 2010

Program:

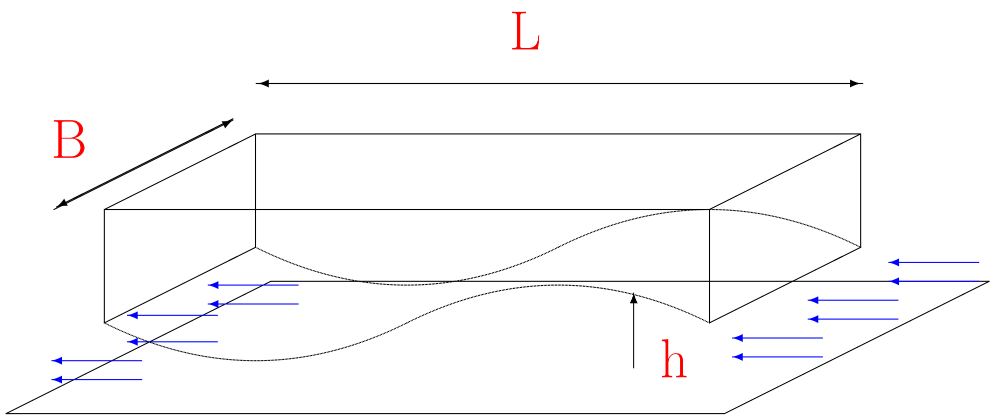
1.- Introduction

2.- Modelling Reynolds equation

3.- An inverse problem in Journal Bearing systems with cavitation

1. Introduction

Lubrication is the process whereby a fluid (the lubricant) fills the gap between two surfaces in close proximity and relative motion. Lubrication is employed to reduce friction and avoid contact



We assume $\mathbf{0} < h << L$, $\mathbf{0} < h << B$

- Lubrication is known from the ancient civilizations (Egyptians, Greeks...)
- During the middle age, animal fat is used to lubricate wagon wheels, ships (rudder) etc..)
- Leonardo Da Vinci did experiments on friction and obtain experimental data on friction coefficient.
- Industrial revolution brought machineries and the necessity to understand the process of lubrication to improve the efficiency of the machines.
- XIX century, Petrov (student of Ostrogradsky) Frictional forces and the relevance of the viscosity coefficient of the lubricant.
- Introduction of residues from the oil refining industry as lubricants to substitute the animal and vegetable oils

- Reynolds introduce in 1886 Reynolds Equation
- Sommerfield 1904. Explicit solutions for the 1-dimensional journal bearing.
- Rayleigh 1908. Optimization in the shape of lubricated surfaces to maximize the load supported by the engine.
- A.G.M. Michel 1929, study the journal bearing system with axial oil flow.
- Elrod 1960, Elrod-Adams 1975 modeled the phenomenon of **Cavitation:** *Physical phenomenon whereby null pressure regions appear*
- Cameron 1966.
- M. Chipot and M. Luskin 1986, studied the compressible Reynolds eq.

- Bushman, A. Friedman, Hu, Rodrigues, Litman,
- G.Bayada, M.Chambat, S.Ciuperca, I.Hafidi, M.Jay, S.Martin etc.. (Lyon)
- C.Vázquez, J.Durany, I.Arregui, G.García, B.Cid etc.. (Galicia-Spain)
- JI Díaz, J.Carrillo, S.Álvarez, R.Oujja etc. (Madrid - Spain)

Lubrication is today studied in different processes

- Journal Bearing systems,

- Row ball bearing,

- Magnetic recording,

- Hard disk device,

- Earth Physics etc..

2.Modelling Reynolds equation

Starting from the usual conservation principles

mass conservation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\mathbf{u}\rho) = 0,$$

momentum conservation: (e.g. incompressible case)

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + (\mathbf{u} \cdot \nabla)(\rho \mathbf{u}) = -\nabla P + \frac{\mu}{2} \operatorname{div}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

Dimensional analysis:

$$\begin{array}{ccccccccc} \frac{\partial \rho}{\partial t} & + & \frac{\partial(\rho u_1)}{\partial x} & + & \frac{\partial(\rho u_2)}{\partial y} & + & \frac{\partial(\rho u_3)}{\partial z} & = & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \rho_0/T_0 & & \rho_0 U_{1c}/L & & \rho_0 U_{2c}/B & & \rho_0 U_{3c}/H & & \end{array}$$

$$\frac{U_{3c}}{H} \simeq \frac{U_{1c}}{L}; \qquad \frac{U_{3c}}{H} \simeq \frac{U_{2c}}{B}.$$

Then

$$\frac{U_{3c}}{U_{1c}} << 1, \qquad \frac{U_{3c}}{U_{2c}} << 1.$$

X-component of N-S

$$\frac{\partial \rho u_1}{\partial t} + u_1\frac{\partial \rho u_1}{\partial x} + u_2\frac{\partial \rho u_1}{\partial y} + u_3\frac{\partial \rho u_1}{\partial z} = -\frac{\partial P}{\partial x} + \mu \Delta u_1$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{\rho_0 U_{1c}}{T_0} \qquad \frac{\rho_0 U_{1c}^2}{L} \qquad \frac{U_{2c}\rho_0 U_{1c}}{B} \qquad \frac{U_{3c}\rho_0 U_{1c}}{H} \qquad \delta_x P \qquad \mu\frac{U_{1c}}{L^2}$$

$$\mu\frac{U_{1c}}{B^2}$$

$$\mu\frac{U_{1c}}{H^2}$$

$$Re = \frac{U_c L}{\mu}; \qquad -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u_1}{\partial z^2} = 0.$$

X -Component $\quad -\dfrac{\partial P}{\partial x} + \mu\dfrac{\partial^2 u_1}{\partial z^2} = 0,$

Y -Component $\quad -\dfrac{\partial P}{\partial y} + \mu\dfrac{\partial^2 u_2}{\partial z^2} = 0,$

Z -Component: $\quad -\dfrac{\partial P}{\partial z} = 0.$

B.C. $\quad u_1 = u_2 = 0, \quad u_3 = \dfrac{\partial h}{\partial t}, \quad$ on $z = h(x, y),$

$$u_1 - U_1 = u_2 - U_2 = u_3 = 0, \quad \text{on } z = 0.$$

Then

$$u_1 = \frac{1}{2\mu}\frac{\partial P}{\partial x}z(z-h) + U_1(1-\frac{z}{h}),$$
$$u_2 = \frac{1}{2\mu}\frac{\partial P}{\partial y}z(z-h) + U_2(1-\frac{z}{h}).$$

We define

$$q_x := \int_0^h u_1 dz = \frac{U_1 h}{2} - \frac{h^3}{12\mu}\frac{\partial P}{\partial x},$$
$$q_y := \int_0^h u_2 dz = \frac{U_2 h}{2} - \frac{h^3}{12\mu}\frac{\partial P}{\partial y}.$$

Integrating (1)

$$\frac{\partial(h\rho)}{\partial t} + \frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) = 0.$$

Reynolds Equations (1886)

Incompressible fluids

$$\begin{cases} \frac{\partial}{\partial x}[U_1 h - \frac{h^3}{6\mu}\frac{\partial P}{\partial x}] \; + \; \frac{\partial}{\partial y}[U_2 h - \frac{h^3}{6\mu}\frac{\partial P}{\partial y}] = -\frac{\partial}{\partial t}h, & \text{in } \Omega, \\ P - P_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

$$-div(h^3 \nabla P) = -6\mu U \nabla h - \frac{\partial h}{\partial t} \quad \text{in } \Omega, \quad +BC$$

Cavitation has been modeled in different ways:

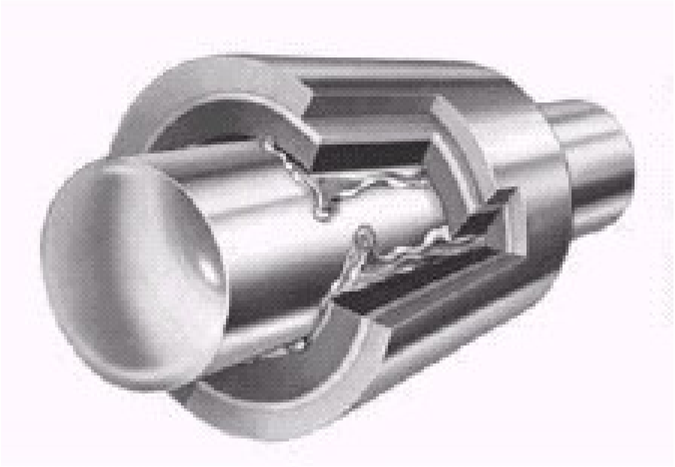
- Considering the positive part of the solution to Reynolds equation $(p)^+$
- Variational inequalities

$$\begin{cases} -\text{div}\left[\frac{h^3}{6\mu}\nabla p\right] \geq -U\frac{\partial}{\partial x}h - \frac{\partial}{\partial t}h, & p \geq 0 \\ \left(-\text{div}\left[\frac{h^3}{6\mu}\nabla p\right] + U\frac{\partial}{\partial x}h + \frac{\partial}{\partial t}h\right)p = 0 & \text{on } \Omega \\ p = p_0 & \text{on } \partial\Omega. \end{cases}$$

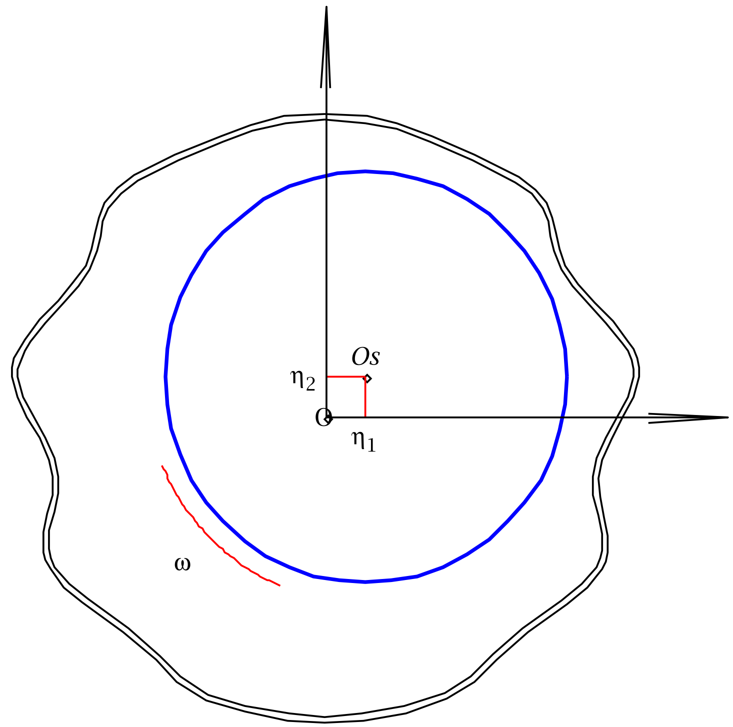
- Elrod-Adams Model

$$\begin{cases} -\text{div}\left[\frac{h^3}{6\mu}\nabla p\right] = -U\frac{\partial}{\partial x}(\theta h) - \frac{\partial}{\partial t}(\theta h), & \text{in } \Omega, \\ \theta \in H(p), \text{H is the heaviside function} & \\ p = p_0, & \text{on } \partial\Omega. \end{cases}$$

3.- Journal Bearing system with cavitation



- Unknown position of the inner cylinder which support a given load
- We only allow to the inner cylinder to displace parallel to the outer one
- 2-parameters inverse problem



Scheme of the journal bearing

- O_s the center of the shaft,
- G. Bayada 1972, J. Durany, J. Pereira, F. Varas. Numerical sol. 2008

- Cartesian coordinates of O_s, (η_1, η_2) Polar coordinates of O_s, (η, α)
- $\eta_1 = \eta \cos \alpha$, $\eta_2 = \eta \sin \alpha$.
- $h(\theta, \eta, \alpha) = \rho(\theta) - \eta \cos(\theta - \alpha) = \rho(\theta) - \eta_1 \cos \theta - \eta_2 \sin \theta$.
- $0 \leq x \leq 1$
- $\Omega := S^1 \times (0, 1)$

Unknowns:

- the pressure $p : (\theta, x) \in \Omega \to I\!R$
- the position of the inner cylinder (given by $O_s := (\eta, \alpha)$)

Known Data:

- Load supported by the shaft ($F = (F_1, F_2)$),
- viscosity μ, (assumed constant)
- velocity (assumed constant) and geometry of the surfaces (given by ρ).

To find:

$$p \in K := \left\{ \varphi \in H_0^1(\Omega) : \varphi \geq 0 \right\}$$

and

$$(\eta_1, \eta_2) \in A := \text{ set of admissible positions}$$

such that

$$\int_\Omega h^3 \nabla p \cdot \nabla(\varphi - p) \geq \int_\Omega h \frac{\partial}{\partial \theta}(\varphi - p) \quad \forall \varphi \in K$$

$$\int_\Omega p \cos\theta d\theta dx = F_1$$

$$\int_\Omega p \sin\theta d\theta dx = F_2$$

where $h(\theta, \eta, \alpha) = \rho(\theta) - \eta\cos(\theta - \alpha) = \rho(\theta) - \eta_1 \cos\theta - \eta_2 \sin\theta$

Assumptions:

$$\rho \in C^3(I\!R), \quad \rho, \text{ is } 2\pi - \text{periodic.}$$

$$\rho''(\theta) + \rho(\theta) > 0, \qquad \forall\, \theta \in [0, 2\pi) \qquad \text{(technical assumption)}$$

$$\min_{0 \leq \theta \leq 2\pi} \rho(\theta) = 1.$$

Notation:

$$a(\alpha) := \min_{\alpha - \frac{\pi}{2} < \theta < \alpha + \frac{\pi}{2}} \left\{ \frac{\rho(\theta)}{\cos(\theta - \alpha)} \right\}.$$

$$A := \{(\eta, \alpha); \quad 0 \leq \eta < a(\alpha)\}$$

Let $G = (G_1, G_2) : A \to \mathbb{R}^2$,

$$G_1(\eta_1, \eta_2) := \int_\Omega p \cos\theta d\theta dx - F_1 \tag{1}$$
$$G_2(\eta_1, \eta_2) := \int_\Omega p \sin\theta d\theta dx - F_2 \tag{2}$$

for p defined as the unique solution to the variational inequality for fixed η_1, η_2.

Thus the problem is equivalent to

Find $(\eta_1, \eta_2) \in A$ **such that** $G(\eta_1, \eta_2) = (0, 0)$

Theorem 1 *For any* $F \in \mathbb{R}^2$ *there exists, at least, a solution to* $G = 0$.

For fixed (η_1, η_2) there exists a unique solution to the variational problem p (see Kinderlehrer-Stampacchia 1980) and G is well defined.

Some basic results in Degree Theory in $I\!R^n$

Definition.

Let S be a bounded open subset of $I\!R^n$ and $f : S \to I\!R^n$, $f \in C^1(S)$. Let $y_0 \in I\!R^n$ such that $y_0 \notin f(\partial S)$, suppose that $Df(x)$ is invertible for all $x \in f^{-1}(y_0)$. We define the degree of f in S at y_0 by:

$$d(f, S, y_0) := \sum_{i=1}^{n} sign(det(Df(x_i))),$$

for all $x_i \in f^{-1}(y_0)$.

Theorem 2 *Let S be a bounded open set in $I\!R^n$. Let f_0 and f_1 be two continuous functions from $\overline{S}$ to $I\!R^n$. We assume moreover that S is a star domain with respect to the point $y_0 \in S$ and that*

$$[f_0(x) - y_0]^t \cdot [f_1(x) - y_0] > 0, \ \textit{for any} \ \ x \in \partial S.$$

Then

$$d(f_0, S, y_0) = d(f_1, S, y_0).$$

Properties of $a(\alpha)$

- $$a(\alpha) := \min_{\alpha - \frac{\pi}{2} < \theta < \alpha + \frac{\pi}{2}} \left\{ \frac{\rho(\theta)}{\cos(\theta - \alpha)} \right\}.$$

- $a(\alpha)$ represents the maximum displacement of O_s in the α-direction.

- Since
$$\lim_{\theta \to \alpha \pm \frac{\pi}{2}} \frac{\rho(\theta)}{\cos(\theta - \alpha)} = \infty$$
we guarantee the existence of at least one minimum.

- Since $\rho'' + \rho > 0$ the minimum is unique.

- θ_α represents the contact point, it is unique and C^1.

- $$1 = \min_{0 \le \theta \le 2\pi} \rho \le a(\alpha) \le \max_{0 \le \theta \le 2\pi} \rho = \rho_M.$$

- There exists $s < \frac{\pi}{2}$ such that $|\tilde{\theta}_\alpha - \alpha| < s$

- We introduce

$$A_\epsilon = \{(\eta_1, \eta_2) : 0 \le \eta \le a(\alpha) - \epsilon\},$$

- We fix $(\eta, \alpha) \in \partial A_\epsilon$
- We take $\varphi = 0$ and $\varphi = 2p$ as test functions in the variational inequality to obtain

$$\int_\Omega h^3 \left|\nabla p\right|^2 = -\int_\Omega p \frac{\partial h}{\partial \theta}$$

- Notice that

$$-\int_\Omega p \frac{\partial h}{\partial \theta} \sim (\int_\Omega pcos\theta, \int_\Omega psin\theta) \cdot (-\eta_2, \eta_1)^t$$

- We introduce

$$\tilde{h}(\theta, \alpha) := h(\theta_\alpha, a(\alpha) - \epsilon, \alpha) + [\rho''(\theta_\alpha) + \rho(\theta_\alpha)](1 - \cos(\theta - \theta_\alpha)).$$

- $$-\int_\Omega p \frac{\partial \tilde{h}}{\partial \theta} = c_1 (\int_\Omega pcos\theta, \int_\Omega psin\theta)^t \cdot (\eta \sin\theta_\alpha, -\eta\cos\theta_\alpha) \qquad \text{in} \quad (\eta, \theta_\alpha) \in \partial A_\epsilon$$

For $(\eta, \theta_\alpha) \in \partial A_\epsilon$ we have

$$(\int_\Omega pcos\theta, \int_\Omega psin\theta)^t \cdot (\eta \sin\theta_\alpha, -\eta\cos\theta_\alpha) = \int_\Omega p(h-\tilde{h})' + \int_\Omega h^3 |\nabla p|^2$$

In order to take limits when $\epsilon \to 0$ we need some estimations

- $$\inf_{0\leq\alpha\leq 2\pi} \int_\Omega h^3(\theta,\eta,\alpha)|\nabla p|^2 d\theta dx > c\epsilon^{-\frac{1}{2}} \qquad \text{for } \epsilon << 1.$$

- Since,

$$\left|\int_\Omega p(h-\tilde{h})'\right| = \left|\int_\Omega \frac{\partial p}{\partial\theta}(h-\tilde{h})\right| \leq \frac{1}{2}\int_\Omega h^3|\nabla p|^2 + \frac{1}{2}\int_\Omega \frac{\left|h-\tilde{h}\right|^2}{h^3}$$

and

$$\int_\Omega \frac{\left|h-\tilde{h}\right|^2}{h^3} \leq c_2.$$

We compare the fields G with $(-\eta_2, \eta_1)$ in ∂A_ϵ by using the vector field W

$$W : (\eta, \alpha) \in A \to (\eta\sin\theta_\alpha, -\eta\cos\theta_\alpha) \in I\!R^2$$

to obtain

$$G(\eta_1, \eta_2) \cdot W(\eta_1, \eta_2) \geq \eta(c(\epsilon^{-1/2} - 1) - \|F\|), \quad \text{in } \partial A_\epsilon.$$

Then for $\epsilon << 1$

$$deg(G, A_\epsilon, 0) = deg(W, A_\epsilon, 0) = deg((-\eta_2, \eta_1), A_\epsilon, 0) = 1$$

by continuity of G we have the existence of at least a solution.

Open Questions:

Uniqueness of solutions

Existence and uniqueness for Elrod-Adams model