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Renormalization group for effective field theories: Cutoff schemes and universality

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ABSTRACT

In effective field theories, the concept of renormalization of perturbative divergences is replaced by renormalization group concepts such as *relevance* and *universality*. Universality is related to cutoff scheme independence in renormalization. Three-dimensional scalar field theory with just the quartic coupling is universal but the less relevant sextic coupling introduces a cutoff scheme dependence, which we quantify by three independent parameters, in the two-loop order of perturbation theory. However, reasonable schemes only allow reduced ranges of those parameters, even contrasting the sharp cutoff with very smooth cutoffs. The sharp cutoff performs better. In any case, the effective field theory possesses some degree of universality even in the massive case (off criticality).

1. Introduction

Theories of the physics of systems with a large or infinite number of degrees of freedom must address the problem of the change in the relevant description as the scale of the observation changes. In the old methods of quantum field theory, the problem of the appearance of infinite quantities in perturbative calculations gave rise to the theory of renormalization, in which the infinite quantities are assumed to belong to an unobservable domain, while the observable renormalized quantities are always finite [1,2]. Renormalized field theory has reached a high degree of sophistication, in particular, with the development of dimensional regularization and the epsilon expansion [2]. Within dimensional regularization, infinite quantities take a different aspect, given that there is no high-energy or small-distance cutoffs to consider, and hence the subtleties of taking the infinite cutoff limit do not arise. Such methods are best suited for the physics that is independent of what happens at high energy or small distance, as occurs, for example, in the calculation of critical exponents in critical phenomena [1,2]. In the domain where scale invariance holds, the problem of dealing with very many degrees of freedom is somehow compensated for by the scale symmetry, which usually becomes full conformal symmetry.

At any rate, there are situations in which scale invariance is not exact or in which the short-distance physics does not fully decouple from the scales one is interested in. Nevertheless, the physics in the latter scales can be rather different, because renormalization still plays a substantial role. During many years, a systematic set of methods has been developed to study the physics of systems in which the high-energy or small-distance degrees of freedom play a lesser but non-negligible role. These methods are the methods of *effective field theory* [3,4]. In effective field theory, the cutoff is held finite and one is not concerned with the divergences that arise in the infinite cutoff limit. In fact, the methods of effective field theory greatly overlap with the non-perturbative approach to the renormalization group pioneered by Wilson [5], in which renormalization is understood as the effect of the progressive removal of short-distance degrees of freedom (a procedure called *exact renormalization group*, or ERG). The renormalized or effective field theory

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accounts for the effect of those degrees of freedom through a small number of *relevant* coupling constants. The theory of a scalar field with only relevant couplings is considered *universal*, as it is determined by symmetry and space dimension only.

Just as in classic renormalization methods one can choose different methods to obtain finite results, i.e., various *regularization* methods, there are also different methods to remove short-distance degrees of freedom in the Wilson renormalization group (or exact RG). The simplest type of cutoff regularization, with a sharp wave-number cutoff, corresponds to the wave-number "slice" integration in the exact RG [6]. However, this simple procedure is not necessarily optimal and there are reasons to try instead *smooth* cutoffs [5]. The comparison of various regularization methods is gaining interest [7–10]. Goldstein [9] considers the most general smooth wave-number cutoff for scalar field theory at one loop order in the epsilon expansion and shows that the RG flow is altered but the Wilson-Fisher critical exponents are not, in accord with universality expectations. Of course, it is necessary to consider higher loop orders, as is being done in related studies [7,8,10].

The scheme dependence of the exact RG was a cause for concern years ago [11,12]. The study by Liao et al on smooth cutoff schemes for the ERG [12] concluded that smoother cutoffs tend to minimize the strength of neglected irrelevant operators. In other words, optimizing the renormalization group would favor smooth cutoff schemes. It should be noted that the purpose of such optimization was to find the best method for calculating critical exponents [11,12], which is not our goal here. We have recently pointed out the importance of comparing ERG cutoff schemes off-criticality and we have performed a comparison between some standard schemes, finding the sharp cutoff to be adequate [13].

In perturbation theory, since the one-loop order is rather trivial, it is worthwhile to develop higher loop orders, using a cutoff that is as general as possible. In particular, we consider here how Liao et al's suite of schemes [12] performs in perturbative scalar field theory at the two-loop order. It should be interesting to see how the results depend on epsilon in the epsilon expansion but we encounter considerable difficulties in calculations that keep the space dimension arbitrary. Therefore, we prefer to work in a fixed space dimension. Rather than being based on four dimensions, as is usual [7–9], we find it more interesting to work in three dimensions, which has more applications in statistical and condensed matter physics. In addition, we employ the background-field method, which reduces the number of independent calculations [7,8,10].

We shall consider, in three dimensions, not just the ordinary $\lambda\phi^4$ theory but the full $(\lambda\phi^4 + g\phi^6)_3$ theory, which contains all the power-counting renormalizable couplings. Of course, the effect of the non-relevant but only *marginal* coupling $g\phi^6$ is expected to be small, but the effect of changes of cutoff scheme is also expected to be small, provided we consider the domain with approximate scale invariance. In fact, we will show that both types of effects are connected.

For completeness, we could also consider, on the one hand, the effect of fully irrelevant couplings in the effective potential, and, on the other, the effect of field renormalization, related to the other marginal term, $(\partial\phi)^2$. Regarding the former, fully irrelevant coupling constants depend on the wave-number cutoff as inverse powers and become less important at low energies or long distances, and regarding the latter, its effect is small in $d = 3$. We shall comment on these matters as they arise.

The text consists of five main sections and a concluding section. There are also two appendices, the first one containing the calculation of three necessary integrals with different cutoffs, and the second one the calculation of beta-functions. Section 2 lays the groundwork for the study of scalar field theory in three dimensions, comparing the roles of the quartic and sextic couplings. Section 3 summarizes standard results of three-dimensional $\lambda\phi^4$ theory and introduces the use of ERG integrations, initiated in Ref. [13] and to be further developed in the following sections. Section 4 introduces the two-loop effective potential and carries out the renormalization of the quartic and sextic coupling constants, which brings about a cutoff dependence. Section 5 studies the non-universal aspects due to the sextic coupling, first with a sharp-cutoff example, and then carefully studying the modifications produced by smooth cutoffs. Section 6 improves the two-loop renormalization formulas by means of the RG that is derived from them. The consequent RG flow provides a fuller picture of both renormalization and cutoff dependence. This picture is explained in the concluding section.

2. Renormalization of scalar field theory in three-dimensional space

The renormalization of scalar field theory is suitable for a simple comparison of various regularization methods [7–9]. Furthermore, scalar field theory in three-dimensional space, with one or several fields, has many applications in statistical and condensed matter physics [1,2,14].

In fact, the theory with a scalar field ϕ with ϕ^4 self-interaction is thoroughly studied as the universality class of the Ising model, a paradigm in statistical physics for the theory of critical phenomena [1,2]. This model illustrates the phase transition physics of a simple critical point, but there are more complex phase transitions. For example, the phase transitions near a tricritical point are quite convoluted [15–17]. Previously, the theory of a scalar field ϕ with ϕ^4 and ϕ^6 self-interaction in three dimensions was studied to describe tricritical phenomena [17–20]. We shall consider this field theory, denoted as $(\lambda\phi^4 + g\phi^6)_3$, as our model scalar field theory in three dimensions, since it contains all the power-counting renormalizable couplings. Of course, the effect of the non-relevant but just *marginal* coupling $g\phi^6$ is expected to be small [18], but it deserves further study.

Some years ago, the renormalization of $(\lambda\phi^4 + g\phi^6)_3$ theory was studied employing the method of dimensional regularization [21–24]. This method simplifies the renormalization process. In particular, the calculation of divergences arising in the evaluation of Feynman-graph integrals is greatly simplified. Therefore, it was possible to push the renormalization process to a considerably high order [21–24]. In dimensional regularization, Feynman-graph integral divergences appear, in three dimensions, as poles in $\epsilon = 3 - d$ (where d is the space dimension). These divergences appear in cutoff regularization as terms involving $\log\Lambda$ (where Λ is the wave-number cutoff or an equivalent parameter in real space [8,10]). The terms that involve powers of Λ in cutoff regularization do not appear at all in dimensional regularization, whether those terms are divergent or not. Thus, this regularization method is unsuitable for effective field theories, in which the cutoff is held finite. Moreover, the cutoff must be allowed to run in the exact renormalization group [5].

Nevertheless, the calculation of integrals with dimensional regularization can be useful. This is because terms involving $\log\Lambda$ play a special role in effective field theory and are related to *marginal* coupling constants, in particular. We shall see how terms involving $\log\Lambda$ appear, in $(\lambda\phi^4 + g\phi^6)_3$ theory, in the renormalization of coupling constants precisely linked to the dimensionless constant g, while the renormalization of λ in $(\lambda\phi^4)_3$ theory involves no divergences (provided that one avoids IR divergences in the massless or scale-invariant limit) [1,2].

Of course, renormalization of mass m (or correlation length) is necessary in $(\lambda\phi^4)_3$ theory to absorb positive powers of Λ (Section 4). However, the renormalized mass m (or the correlation length) can be considered as the basic dimensional parameter to which the coupling constants are referred, and it is thus in the renormalization of coupling constants where one can appreciate the degree of universality; that is to say, it is in the renormalization of coupling constants where the cutoff scheme dependence will materialize. Therefore, all cutoff scheme dependence will disappear in $(\lambda\phi^4)_3$ theory, unlike in $(\lambda\phi^4 + g\phi^6)_3$ theory.

To prevent any misunderstanding, let us caution that we do not intend to study here tricritical behavior and its RG fixed point, but the Wilson-Fisher fixed point corresponding to ordinary critical behavior, in spite of the fact that we consider the $(\lambda\phi^4 + g\phi^6)_3$ theory. The first fixed point is related to a free (“trivial”) renormalized field theory, whereas the second is related to a fully interacting field theory. The role of g, in our case, is just to bring up the non-universal corrections that appear for $m \neq 0$. However, before we examine the role of g, let us briefly review some results of the ordinary $(\lambda\phi^4)_3$ theory for critical behavior.

3. Results of the $(\lambda\phi^4)_3$ theory

The field-theoretic approach to the calculation of the critical exponents of the three-dimensional Ising model based on the $(\lambda\phi^4)_3$ theory was developed quite early [27,28]. This approach, together with the epsilon expansion, are the usual methods of precision calculation of critical exponents and other universal quantities [1,2]. As remarked above, the renormalization of λ in $(\lambda\phi^4)_3$ theory involves no UV divergences. After defining the dimensionless coupling constant $u = \lambda/m$, the output of perturbation theory is the quotient of the bare and renormalized couplings λ_0/λ as a power series of u (this notation, though common, differs from the notations of Refs. [1,2]). Thus, one obtains the function $u(\lambda_0/m)$ and, hence, the β-function

$$\beta(u) = m\left(\frac{\partial u}{\partial m}\right)_{\lambda_0}, \tag{1}$$

which can also be expressed as a power series of u. The equation for scale-invariance $\beta(u) = 0$ has a trivial zero solution and a non-trivial solution $u^* \neq 0$, the latter being the starting point for the derivation of universal quantities. Note that both λ_0/λ as a function of u and the function $\beta(u)$ are universal, since a UV cutoff Λ is not necessary to obtain them.

The fact that the β-function is derived from perturbation theory and is only known as a power series leads to complications in the computation of universal quantities, because the power series is not convergent but just asymptotic and u^* is not sufficiently small. Even at the two-loop order, one already needs some manipulation to obtain sensible results [1, §8.3]. Of course, the sophisticated five-loop computation of Baker et al [27] required a resummation of the power series (which they carried out with the Padé-Borel method). The use of sophisticated methods of summation of divergent power series is a crucial step in the perturbative calculation of universal quantities [2, ch. 42]. The six-loop β-function yields $u^* = 0.985$ [2, §29.4], the old value being $u^* = 0.987$ [27,28] (note our different normalization of λ and hence u).

Given that we are not concerned here with the limit that is strictly scale-invariant, namely, with the case $m = 0$, it behooves us to examine the behavior of the perturbative power series for small but non-vanishing m, that is to say, for u close to but smaller than u^*, such that $\beta(u) \lesssim 0$. The known six-loop β-function [2, §29.2] is derived from the six-loop perturbative result for λ_0/λ:

$$\frac{\lambda_0}{\lambda} = 1 + \frac{9u}{2\pi} + \frac{575u^2}{36\pi^2} + 1.83421u^3 + 1.91785u^4 + 2.16641u^5 + 2.04432u^6 + O\left(u^7\right). \tag{2}$$

Obviously, for $u \simeq 1$, Eq. (2) provides hardly any information. In fact, while u^* is given by $\beta(u^*) = 0$, the quotient λ_0/λ has to diverge as $m \to 0$ and $u \to u^*$ (see Ref. [1, Section 8.1], for details). This is also an aspect of universality, viewed as independence of actual values of bare couplings. For $u \lesssim u^*$, the quotient λ_0/λ must be finite but it may still be not calculable through Eq. (2) (unless using a resummation).

At any rate, series Eq. (2) should be suitable for small values of u. For example, it is fine even for $u = 0.5$, it being such that the asymptoticness property holds (with a small percentage error). However, we cannot quantify how far this value or any particular value are from criticality, since $u = \lambda/m$ is only a quotient and the magnitude of m is unknown. Nevertheless, the mapping between bare and renormalized parameters, that is, the mapping $(m_0, \lambda_0) \to (m, \lambda)$, is one to one (off criticality). Unfortunately, the full mapping involves the renormalization of mass, and we clash head on with the problem of non-universality, given that the renormalization of mass involves divergences when $\Lambda \to \infty$ [1,2]. Notwithstanding, let us recall the effective field theory philosophy.

In the ERG, Λ is always finite and runs from an initial value, say Λ_0, to the final value $\Lambda = 0$. Along the way, the mass and coupling constants run from their bare to their renormalized values. Therefore, one can define β-functions for these coupling constants, which refer to a flow with the cutoff Λ, unlike the β-function in Eq. (1), which gives the effect that a change of m has on λ, once renormalization has been carried out, for a given value of λ_0. Employing the ERG, one can carry out simple numerical experiments on how the Λ-running proceeds, in particular, observing the dependence on cutoff scheme [13]. Indeed, we can use the renormalized values obtained in Ref. [13] to find out what magnitude of m corresponds to some value of u for some fixed λ_0.

3.1. Exact renormalization group calculations for fixed bare coupling

In Ref. [13], the partial differential equation of the exact renormalization group in the local potential approximation was integrated several times, for null initial coupling constants, except a fixed λ_0, and a variable m_0^2. Tuning the latter, we obtain suitable values of m, from values close to $m = 0$ (criticality) to values far from it. Far from criticality, we have a small value of u, close to the RG Gaussian fixed point $u = 0$. Each set of integrations was repeated employing three forms of the ERG, corresponding to different cutoff schemes: the Wegner-Houghton sharp-cutoff equation [6] and two others devised for the ERG. The goal, naturally, was to discern the influence of non-perturbative regularization schemes on the accuracy of the results. Since the full effective action involves an infinite number of parameters, non-perturbative renormalization group calculations generally truncate the full set of coupling constants to a moderate number (8 constants were used in Ref. [13]). This reduction implies that different regularization schemes yield different results, making the comparison useful.

Of course, one can compare the results of different regularization schemes against one another. Furthermore, it is useful to compare them to the perturbative value of the quotient λ_0/λ, which is independent of the regularization scheme. Eq. (2) includes the effect of field renormalization and is not exact for this comparison, given that the non-perturbative renormalization group of Ref. [13] was limited to the evolution of the effective potential, namely, to the partial truncation named *local potential approximation*. The two-loop expression that does not account for field renormalization and was used in Ref. [13] is

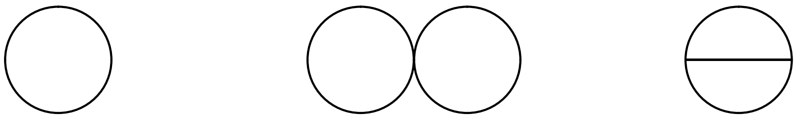
$$\frac{\lambda_0}{\lambda} = 1 + \frac{9u}{2\pi} + \frac{63u^2}{4\pi^2} + O\big(u^3\big). \tag{3}$$

Note that $575/36 = 15.97$ in Eq. (2) has been replaced by $63/4 = 15.75$, introducing a difference smaller than 1.4 %. Moreover, the difference in λ_0/λ, with $u = 0.5$, amounts to $1.4\,\% \times 0.5^2/(4\pi^2) = 0.035\,\%$, which is actually smaller than other uncertainties in the calculations [13].

The ERG integrations of Ref. [13], with $g_0 = 0$, fixed initial λ_0, and variable final m, spanning a range of m/Λ_0 between 0.01 and 0.07, gave similar results in the three different cutoff schemes. Let us note that Λ_0, being a fixed scale in effective field theory, can be taken as the reference scale, like in Ref. [13], and we often assume $\Lambda_0 = 1$ and write dimensional quantities as numbers (as already done by Wilson and Kogut [5], for example). The value $m/\Lambda_0 = 0.01$ roughly corresponds to a correlation length of 100 lattice sites in the Ising model, say. It seems indeed that it corresponds to the domain where scale invariance does not exactly hold but there are many short-distance degrees of freedom involved. Nevertheless, the corresponding value of u in Ref. [13] was $u = \lambda/m \simeq 0.4$ (the exact value depending somewhat on the cutoff scheme). This value is sensibly smaller than $u^* \simeq 1$ and, in fact, it is such that the asymptoticness property of Eq. (2) approximately holds.

The lesson learnt is that it is meaningful to consider situations which are, on the one hand, close enough to criticality, but which are, on the other hand, such that they may involve sizeable non-universal corrections. In fact, corrections of the order of inverse powers of the cutoff were mentioned in Ref. [13], beginning with m/Λ_0, although no concrete calculation was made, because such corrections were assumed to be relatively small, given the magnitude of m/Λ_0. However, the error is not necessarily small when we consider $g_0 \neq 0$.

4. Two-loop perturbative $(\lambda\phi^4 + g\phi^6)_3$ theory



The effective potential at the two-loop order can be calculated with the background-field method (as outlined in Ref. [29]). In the case of the $(\lambda\phi^4)_3$ theory, which is super-renormalizable, the *superficial* UV divergences of Feynman graphs are limited to the two-loop order, namely, to the "single bubble", "double bubble" and "sunset" Feynman graphs (displayed above) [2, ch 9]. They only need a mass renormalization. In contrast, the $(\lambda\phi^4 + g\phi^6)_3$ theory is just renormalizable, so an infinite number of Feynman graphs have superficial UV divergences. Of course, at the two-loop order and with the background-field method, we only have the "single bubble", "double bubble" and "sunset" *vacuum* Feynman graphs, which yield:

$$U_{\text{eff}}(\phi) = U_{\text{clas}}(\phi) + \frac{h}{2}\, I_1 + \frac{3\,h^2\, U^{(4)}_{\text{clas}}(\phi)}{4!}\, I_2^2 - \frac{3\,h^2\, U^{(3)}_{\text{clas}}(\phi)^2}{(3!)^2}\, I_3 + O\big(h^3\big), \tag{4}$$

with the integrals

$$I_1\big(M^2\big) = \int \frac{d^3k}{(2\pi)^3} \ln\big(k^2 + M^2\big), \tag{5}$$

$$I_2\big(M^2\big) = \int \frac{d^3k}{(2\pi)^3\big(k^2 + M^2\big)}, \tag{6}$$

$$I_3\big(M^2\big) = \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\, \frac{1}{\big(k^2 + M^2\big)\big(q^2 + M^2\big)\big((\boldsymbol{k}+\boldsymbol{q})^2 + M^2\big)}, \tag{7}$$

where $M^2 = U''_{\text{clas}}(\phi)$. In Eq. (4), h denotes the loop-counting parameter (to be set to 1 at the end).

We need to calculate I_1, I_2 and I_3. Actually, $I_2(M^2) = I_1'(M^2)$, so we need just two integrals. They can be calculated with various cutoff schemes (see Appendix A). The calculation of the "sunset" integral I_3 is naturally more difficult. It has a logarithmic divergence, with a finite part which we denote by A. Suppressing inverse powers of Λ_0, we have, in total,

$$U_{\text{eff}}(\phi) = U_{\text{clas}}(\phi) + h\left(\frac{F_1\Lambda_0 U''_{\text{clas}}(\phi)}{4\pi^2} - \frac{U''_{\text{clas}}(\phi)^{3/2}}{12\pi}\right) + \frac{3\,h^2\,U^{(4)}_{\text{clas}}(\phi)}{4!}\left(\frac{F_1^2\Lambda_0^2}{4\pi^4} - \frac{F_1\Lambda_0\sqrt{U''_{\text{clas}}(\phi)}}{4\pi^3} + \frac{\left(8B+\pi^2\right)U''_{\text{clas}}(\phi)}{16\pi^4}\right)$$
$$- \frac{3\,h^2\,U^{(3)}_{\text{clas}}(\phi)^2}{(3!)^2\left(16\pi^2\right)}\left(A - \log\frac{\sqrt{U''_{\text{clas}}(\phi)}}{\Lambda_0}\right) + O\left(h^3\right). \tag{8}$$

With the sharp cutoff, we obtain $A = -1.5858$. We have also written the constant B that arises in the finite part of I_2^2, and the constant F_1 that only affects Λ_0 (see Appendix A). For the sharp cutoff, $B = 1$ and $F_1 = 1$.

We put

$$U_{\text{clas}}(\phi) = \frac{m_0^2}{2}\,\phi^2 + \lambda_0\,\phi^4 + g_0\,\phi^6$$

(note that the this is not the usual normalization of coupling constants). By making $g_0 = 0$, we can compare with standard $(\lambda\phi^4)_3$ theory results (calculated at higher loop order and including the field renormalization factor Z) [25,26].

The renormalization is carried out as follows:

$$m^2 = U''_{\text{eff}}(0) = m_0^2 + h\left(\frac{6\lambda_0 F_1\Lambda_0}{\pi^2} - \frac{3m_0\lambda_0}{\pi}\right) + h^2\left[\frac{45g_0F_1^2\Lambda_0^2}{2\pi^4} - \frac{45g_0m_0F_1\Lambda_0}{2\pi^3}\right.$$
$$\left. + \left(\frac{45}{8\pi^2} + \frac{45B}{\pi^4}\right)g_0m_0^2 - \frac{9F_1\Lambda_0\lambda_0^2}{\pi^3 m_0} + \left(\frac{9}{2\pi^2} + \frac{36B}{\pi^4} - \frac{6A}{\pi^2}\right)\lambda_0^2 + \frac{6\lambda_0^2}{\pi^2}\log\frac{m_0}{\Lambda_0}\right], \tag{9}$$

$$\lambda = \frac{U^{(4)}_{\text{eff}}(0)}{4!} = \lambda_0 + h\left(-\frac{15g_0m_0}{4\pi} + \frac{15g_0F_1\Lambda_0}{2\pi^2} - \frac{9\lambda_0^2}{2\pi m_0}\right) + h^2\left[-\frac{315F_1\Lambda_0g_0\lambda_0}{4\pi^3 m_0}\right.$$
$$\left. + \left(\frac{315}{8\pi^2} + \frac{315B}{\pi^4} - \frac{30A}{\pi^2}\right)g_0\lambda_0 + \frac{30g_0\lambda_0}{\pi^2}\log\frac{m_0}{\Lambda_0} + \frac{27\lambda_0^3F_1\Lambda_0}{2\pi^3m_0^3} + \frac{18\lambda_0^3}{\pi^2m_0^2}\right], \tag{10}$$

$$g = \frac{U^{(6)}_{\text{eff}}(0)}{6!} = g_0 + h\left(\frac{9\lambda_0^3}{\pi m_0^3} - \frac{45g_0\lambda_0}{2\pi m_0}\right) + h^2\left[-\frac{675g_0^2F_1\Lambda_0}{4\pi^3m_0} + \left(\frac{675}{8\pi^2} + \frac{675B}{\pi^4} - \frac{75A}{\pi^2}\right)g_0^2\right.$$
$$\left. + \frac{75g_0^2}{\pi^2}\log\frac{m_0}{\Lambda_0} + \frac{270g_0\lambda_0^2F_1\Lambda_0}{\pi^3m_0^3} + \frac{225g_0\lambda_0^2}{\pi^2m_0^2} - \frac{81\lambda_0^4F_1\Lambda_0}{\pi^3m_0^5} - \frac{108\lambda_0^4}{\pi^2m_0^4}\right]. \tag{11}$$

[The $O\left(h^3\right)$ symbol has been omitted, for brevity.]

Solving for m_0 in Eq. (9) and substituting in the following two equations, we obtain:

$$\lambda = \lambda_0 - h\,\frac{3\left(5\pi g_0m^2 - 10g_0mF_1\Lambda_0 + 6\pi\lambda_0^2\right)}{4\pi^2\,m} + h^2\left[-\frac{135F_1\Lambda_0g_0\lambda_0}{2\pi^3\,m}\right.$$
$$\left. + \left(\frac{135}{4\pi^2} + \frac{315B}{\pi^4} - \frac{30A}{\pi^2}\right)g_0\lambda_0 + \frac{30g_0\lambda_0}{\pi^2}\log\frac{m}{\Lambda_0} + \frac{99\lambda_0^3}{4\pi^2\,m^2}\right] + O\left(h^3\right), \tag{12}$$

$$g = g_0 - h\,\frac{9\left(5g_0m^2\lambda_0 - 2\lambda_0^3\right)}{2\pi m^3} + h^2\left[-\frac{675g_0^2F_1\Lambda_0}{4\pi^3\,m} + \left(\frac{675}{8\pi^2} + \frac{675B}{\pi^4} - \frac{75A}{\pi^2}\right)g_0^2\right.$$
$$\left. + \frac{75g_0^2}{\pi^2}\log\frac{m}{\Lambda_0} + \frac{405g_0\lambda_0^2F_1\Lambda_0}{2\pi^3\,m^3} + \frac{1035g_0\lambda_0^2}{4\pi^2\,m^2} - \frac{297\lambda_0^4}{2\pi^2m^4}\right] + O\left(h^3\right). \tag{13}$$

Let us now solve for λ_0 in Eq. (12):

$$\lambda_0 = \lambda + h\frac{3\left(5\pi g_0m^2 - 10g_0mF_1\Lambda_0 + 6\pi\lambda^2\right)}{4\pi^2\,m}$$
$$+ h^2\left[\left(\frac{30A}{\pi^2} - \frac{315B}{\pi^4} - \frac{30}{\pi^2}\log\frac{m}{\Lambda_0}\right)g_0\lambda + \frac{63\lambda^3}{4\pi^2\,m^2}\right] + O\left(h^3\right). \tag{14}$$

This equation reduces to Eq. (3) when we take $g_0 = 0$. Such as it stands, it shows a dependence on Λ_0, on the one hand, and on the constants A, B and F_1, on the other hand, expressing non-universal corrections to Eq. (3) that appear for $g_0 \neq 0$.

From Eqs. (13) and (14), we obtain $g(m, \lambda, g_0)$ as:

$$g = g_0 - h\,\frac{9\left(5g_0 m^2 \lambda - 2\lambda^3\right)}{2\pi m^3} + h^2\left[\left(\frac{675B}{\pi^4} - \frac{75A}{\pi^2} + \frac{75}{\pi^2}\log\frac{m}{\Lambda_0}\right)g_0^2 + \frac{1035 g_0 \lambda^2}{4\pi^2 m^2} - \frac{27\lambda^4}{\pi^2 m^4}\right] + O\left(h^3\right). \tag{15}$$

This equation contains non-universal corrections, like Eq. (14), whenever $g_0 \neq 0$. For $g_0 = 0$, it simply becomes:

$$g = h\,\frac{9\lambda^3}{\pi m^3} - h^2\,\frac{27\lambda^4}{\pi^2 m^4} + O\left(h^3\right). \tag{16}$$

One can compare it to Sokolov et al's results, [25, Eq. 2.5] or [30, Eq. 8].

Of course, we can also obtain the function $g_0(m, \lambda, g)$ [29], which together with $m_0^2(m, \lambda, g)$ and $\lambda_0(m, \lambda, g)$ can be replaced in the expression Eq. (8) of $U_{\rm eff}(\phi)$ to yield a finite and Λ_0-independent function in the limit $\Lambda_0 \to \infty$ (renormalizability). However, we will not need the renormalized expression of $U_{\rm eff}(\phi)$.

5. Calculation of non-universal corrections

The non-universal corrections that appear for $g_0 \neq 0$ can be evaluated through Eqs. (14) and (15). First of all, let us reexamine how well the case $g_0 = 0$ reproduces the renormalization group integrations of Ref. [13], but making the comparison in a different manner. Since Eq. (14), for $g_0 = 0$, reduces to Eq. (3), we first numerically solve this equation for λ, with given values of m/Λ_0 and λ_0/Λ_0. Then, we compute g by Eq. (16). The result is scheme independent. We should remark, of course, that scheme-dependent terms with negative powers of Λ_0 have been suppressed in Eq. (8).

It turns out that Eq. (3) is not quite precise for $m/\Lambda_0 = 0.01$, for example (the smallest value in Ref. [13]). This is checked by considering the two next terms in Eq. (3), namely, the three and four-loop terms (ignoring the contribution of the field renormalization factor):

$$\frac{\lambda_0}{\lambda} = 1 + \frac{9u}{2\pi} + \frac{63u^2}{4\pi^2} + 1.75448\,u^3 + 1.75034\,u^4 + O\left(u^5\right). \tag{17}$$

Let us recall that $m/\Lambda_0 = 0.01$ and $\lambda_0/\Lambda_0 = 0.008225$ corresponded to $u = \lambda/m \simeq 0.4$ in Ref. [13]. With $u = 0.4$, we have

$$1.75448\,u^3 = 0.1123, \quad 1.75034\,u^4 = 0.04481;$$

that is to say, non-negligible additions. We can neglect the $O\left(u^5\right)$ term.

Note that the contribution of the field renormalization factor, that is to say, the use of Eq. (2) instead of Eq. (3), would hardly alter these numbers. For example, $1.75448\,u^3 = 0.1123$ would have to be replaced by $1.83421\,u^3 = 0.117$, etc, with insignificant changes.

Let us first solve Eq. (3) for λ, with $\lambda_0 = 0.008225$ and $m = 0.01$ (values from Ref. [13], normalized to $\Lambda_0 = 1$). We obtain $\lambda = 0.0043$ and hence $u = \lambda/m = 0.43$. This value is close to the results of the Wegner-Houghton ERG integration, $\lambda = 0.0042$ and $\lambda/m = 0.42$ [13]. Employing all the terms in Eq. (17) and solving for λ, with the same values of λ_0 and m, we obtain $\lambda = 0.0041$ and $u = \lambda/m = 0.41$. The difference with the WH ERG result is still a few percent, like only taking Eq. (3).

The magnitude of errors, i.e., few-percent errors, is compatible with the presence of terms of order $m/\Lambda_0 \simeq 0.01$, which have been suppressed in Eq. 8 (such terms are scheme dependent). At any rate, we expect that the dependence on Λ_0 and on the regularization scheme will be greatly magnified when considering the sextic coupling.

5.1. *Renormalization with $g_0 \neq 0$ and sharp cutoff*

To examine the influence of having $g_0 \neq 0$ in the renormalization process, we need to solve Eqs. (14) and (15) for (λ, g), given a definite set (m, λ_0, g_0). We can express those equations in terms of dimensionless variables (u, g), but m too appears now explicitly, unlike in the case $g_0 = 0$. We want $m/\Lambda_0 \simeq 0.01$ again, which gives, for example, $\log(m/\Lambda_0) \simeq -4.6$ in Eqs. (14) and (15). Of course, we must also set a regularization scheme that provides the constants F_1, A, and B in Eqs. (14) and (15).

As done before with $g_0 = 0$, we employ the ERG to have an idea of the approach to criticality. This is done by first fixing (m, λ_0, g_0), putting (λ_0, g_0) in the ERG, and then tuning m_0^2 to reach the required renormalized m in an ERG integration down to $\Lambda = 0$. To be systematic, once fixed (λ_0, g_0), we can find a suitable estimate of m_0^2 from an approximation of the WH equation, as shown below.

Before proceeding, let us set scales for λ and g by considering their fixed-point values. Considering the 8th truncation of the Wegner-Houghton equation for the effective potential (up to ϕ^{16}), as in Ref. [13], we have

$$\{m^2 = -0.4665,\ \lambda = 0.8189,\ g = 2.441,\ \ldots\}$$

Here we are only interested in the coupling constants displayed (normalized to $\Lambda_0 = 1$). Notice that the bare value of m^2 can be negative and this conventional notation is then improper, but we use it nonetheless. The exact fixed-point coordinates, without any truncation, have been obtained and are slightly different (e.g., $m^2 = -0.4615$) [31]. The corresponding potential has two symmetric minima at $\phi \simeq \pm 0.3$ and a maximum at $\phi = 0$. The parameters beyond g hardly alter its shape for $|\phi| < 0.5$, while they produce a steeper growth for larger $|\phi|$. It is naturally the field fluctuations between the two minima of the potential that flatten the maximum at $\phi = 0$ and give rise to a massless renormalized potential.

To have, some clues about perturbative results we can refer to integrations of the WH RG equation. We find it again convenient to begin with (λ_0, g_0) close to the origin, like in Ref. [13] (meaning much closer to the origin than the position of the ERG fixed point). In the space (m_0^2, λ_0, g_0), the critical surface, where $m = 0$, is well approximated near the origin by its tangent plane at the origin, which can be obtained from Eq. (9):

$$m_0^2 + h\frac{6F_1\Lambda_0}{\pi^2}\,\lambda_0 + h^2\frac{45F_1^2\Lambda_0^2}{2\pi^4}\,g_0 = 0. \tag{18}$$

Notice that the above-displayed fixed-point coordinates do not fit in this plane (after making $h = \Lambda_0 = F_1 = 1$). The reason is, of course, that the critical surface curves away from the origin.

To work out a numerical example, we choose again $\lambda_0 = 0.008225$, like in Ref. [13], and we now set $g_0 = (2\pi^2)^2\, 0.00002 = 0.007793$ (comparable to λ_0, the constant $A_3^{-1} = 2\pi^2$ coming from the WH equation). Eq. (18) gives $m_0^2 = -0.0068$ (with $h = \Lambda_0 = F_1 = 1$). By precisely tuning m_0^2, we find that a WH RG integration with $m_0^2 = -0.00621$ yields $\lambda = 0.005544$ and $m = 0.0100834$ (which is small enough). Hence, $u = \lambda/m = 0.5498$. The solution of Eq. (14) for λ, with the sharp-cutoff values of constants A, B, F_1, bare coupling constants $\lambda_0 = 0.008225$, $g_0 = 0.007793$, and mass $m = 0.01008$, yields $\lambda = 0.005834$, that is to say, $u = 0.5785$. These values of λ and u agree with the WH RG result (within a few percent error). However, they are quite different from the values obtained with the same (m, λ_0) but with $g_0 = 0$.

We can also consider the influence of having $g_0 \neq 0$ on the renormalization of g itself. Note that Eq. (16), in which $g_0 = 0$, gives $g \neq 0$ but only dependent on $u = \lambda/m$. In particular, taking the same (m, λ_0) and $g_0 = 0$, we obtain $u = 0.428$ and hence $g = 0.13$. Taking $g_0 = 0.007793$ instead, we obtain $g = 0.29$. We can appreciate that a non-null g_0 greatly influences the renormalization of g, which is hardly universal for $m > 0$. The WH RG integrations yield $g = 0.27$ for $g_0 = 0.007793$, and $g = 0.14$ for $g_0 = 0$. To explain why errors in g are larger, we must consider that the perturbative series for g in Eq. (15) or Eq. (16) are asymptotic and accurate only for very small u. This problem is quantified in Section 6.

In fact, perturbation theory should not be expected to be reliable for $u > 0.5$, in any event. In Section 6, we examine how reliable perturbation theory is for small u and small g_0. We may guess that g_0 must be quite small; and indeed we have found that, for example, the value $g_0 = (2\pi^2)^2\, 0.0001 = 0.03896$ is large enough already to ruin the agreement between the ERG and two-loop perturbation theory. Perhaps surprisingly, the non-universal terms in Eqs. (14) and (15) are then considerable, especially, the two-loop contribution in Eq. (15). Let us see why.

The relative magnitude of the g_0^2 term in Eq. (15) is given by the coefficient (between parenthesis), which has a fixed part and a mass-dependent logarithmic part. Naturally, the latter dominates as $m \to 0$, but let us assume that $m \lesssim \Lambda_0$ and so $|\log(m/\Lambda_0)| \ll 1$. The fixed part, with sharp-cutoff constants, is 19.0. Therefore, we should have $g_0 \ll 1/19.0 = 0.053$. A deeper analysis is presented in Section 6, where two-loop perturbation theory is "improved".

As regards the renormalization of λ, which is our main focus, the main role of a $g_0 \neq 0$ is that it gives rise to a term proportional to Λ_0, which is a major contribution to Eq. (14). Naturally, the magnitude of that term is due to having $m/\Lambda_0 \ll 1$. The term proportional to Λ_0 has negative sign in Eq. (14) and makes a positive contribution to λ, making it considerably larger when $g_0 > 0$.

The term proportional to Λ_0 in Eq. (14) is multiplied by the constant F_1, which is equal to one for the sharp cutoff but could be reduced, presumably, by employing a convenient cutoff scheme. Could we actually reduce it? Let us try to find how. We shall consider the other scheme-dependent constants as well.

5.2. *Renormalization with $g_0 \neq 0$ in various schemes*

Let us focus on the term proportional to Λ_0 in Eq. (14), as the major non-universal contribution, and on the corresponding constant F_1, namely,

$$F_1 = \int_0^\infty F(q)\, dq.$$

Here, $F(q)$ is the "form factor" regularizing the propagator as

$$\frac{F(k/\Lambda_0)}{k^2 + m^2}$$

(the term "form factor" is borrowed from nuclear physics, where it defines the spatial extent of a charge distribution). We assume that $F(0) = 1$ and $\lim_{x\to\infty} F(x) = 0$ [9,12], and we further assume that $F(q)$ is non-increasing and $F'(0) = 0$ (regarding the definition of the constant F_2, see Appendix A). These are natural assumptions to prevent pathological behavior and to follow the original idea of a smooth integration of Fourier modes [5]. Furthermore, we assume that the derivative of the form factor function has a simple shape, as a regularized delta function of limited width that implements Wilson's idea of Fourier-mode shell integration.

In particular, we shall consider three types of functions $F(\varepsilon, q)$ that depend on a parameter ε (the width) and that converge towards the sharp-cutoff function when $\varepsilon \to 0$; namely, we shall consider Liao et al's cutoff functions [12]. In each case, $F(\varepsilon, q)$ fulfills $F(\varepsilon, 1) = 1/2$, for any ε, so that the cutoff Λ_0 marks a clear separation between "almost completely integrated" modes and "not terribly integrated" modes, in Wilson and Kogut's words [5].

5.2.1. *Hyperbolic tangent*

$F(\varepsilon, q)$ is a hyperbolic tangent modified to fulfill the above conditions:

$$F(\varepsilon, q) = \frac{1}{2}\left(\tanh\left(\frac{q^{-1} - q}{\varepsilon}\right) + 1\right).$$

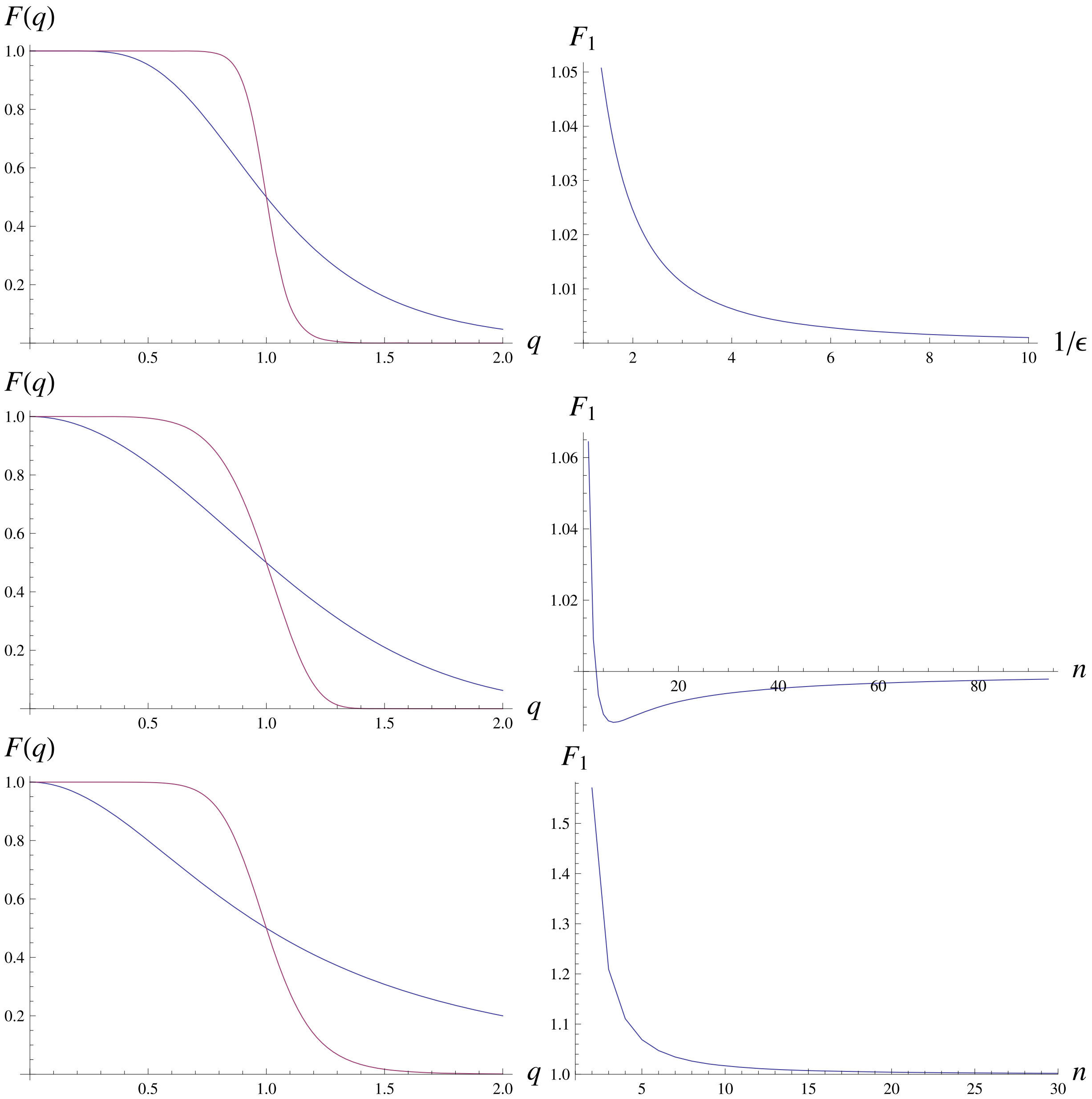


Fig. 1. The three types of cutoff functions and the corresponding values of F_1 plotted against cutoff function widths. First row: hyperbolic tangent (plotted for $\varepsilon = 1, 1/5$); second row: exponential (plotted for $n = 2, 7$); third row: power law (plotted for $n = 2, 10$).

Any $\varepsilon > 0$ is possible, in principle, but it is sensible to restrict it, for example, to $\varepsilon \leq 1$. Numerical computations of F_1 show that it is larger than one but close to one for $\varepsilon = 1$ and tends to one as $\varepsilon \to 0$, never being smaller than one.

5.2.2. *Exponential function*

$F(\varepsilon, q)$ is given by the exponential function:

$$F(n, q) = 2^{-q^n},$$

where the natural number $n \geq 2$ substitutes for the real variable ε, and the limit $n \to \infty$ substitutes for the limit $\varepsilon \to 0$. Again, F_1 is larger than one but close to one for $n = 2$, namely, $F_1 = \frac{1}{2}\sqrt{\pi/\log(2)} = 1.064$, and F_1 tends to one as $n \to \infty$. However, F_1 is not monotone with n and, for $n = 7$, has the minimum $F_1 = \Gamma\left(\frac{8}{7}\right)/\log^{1/7} 2 = 0.9857$ (Fig. 1). This very small reduction is offset by the complications of calculating I_3 and other Feynman integrals with that function.

5.2.3. Power-law function

$F(\varepsilon, q)$ is based on a power-law function:

$$F(n, q) = \frac{1}{1 + q^n},$$

where again the natural number $n \geq 2$ substitutes for the real variable ε. The dependence of F_1 on n also presents a decreasing trend, never being smaller than one. It takes the value $\pi/2$ for $n = 2$, which is not very close to one, but it rapidly tends to one. Furthermore, for $n = 2$, $F_2 = \pi/2$ as well.

The power-law cutoff function is interesting for several reasons. It lends itself to analytical calculations, allowing for simple calculations of the other scheme-dependent constants F_2 and A (see Appendix A). It is also useful for leading to simple forms of the ERG if $n = 2, 4$ (in dimension $d = 3$) [12]. The ERG for $n = 4$ has been tested in Ref. [13], under the name of Morris's scheme. However, we are now more interested in the case $n = 2$, as a very smooth cutoff that departs the most from the sharp cutoff.

5.2.4. Conclusion of the examination

A graphical summary is displayed in Fig. 1. Among Liao et al's cutoff functions, we cannot find any one that substantially reduces F_1 from its sharp-cutoff value. Of course, this does not mean at all that such functions do not exist, because it is easy to find them, although they do not look natural, in the sense of Wilson and Kogut [5]. A simple option is a function that almost completely integrates UV modes and, furthermore, integrates a good deal of modes with $k < \Lambda_0$ (which may thus be "terribly integrated"). At any rate, $F_1 > 1/2$, because $F(q)$ is non-increasing and $F(1) = 1/2$. In fact, the derivative of a function like that must have an unphysical shape, that is, it must have a "bump" near $q = 0$, in addition to the "bump" near $q = 1$ that corresponds to the k-shell integration.

6. Improved perturbation theory

The perturbative renormalization group is traditionally employed as a convenient method of improving the results of plain perturbation theory. For example, in $(\lambda\phi^4)_3$ theory, the beta function Eq. (1) at the one-loop order is given by the first non-trivial term of series Eq. (2)

$$\beta(u) = -u + \frac{9}{2\pi}\, u^2, \tag{19}$$

which is simply integrated as

$$\frac{u}{1 - 9u/(2\pi)} = \frac{\kappa}{m}$$

(with an arbitrary constant κ). Thus, we can write the following expression for the RG transformation between an initial m_0 and an arbitrary m:

$$\lambda_0 = \frac{\lambda}{1 - (1/m - 1/m_0)\, 9\lambda/(2\pi)}\,. \tag{20}$$

Let us caution that m_0 is here some value of the *renormalized* mass, unlike the *bare* mass m_0 of preceding sections, which was such that $m_0^2 < 0$ in our examples. Likewise, λ_0 here is the renormalized value of λ corresponding to m_0, not the bare value of λ. We use a common notation because both the ERG and the perturbative RG describe the change of parameters with scale, the scale being Λ in the former while being m in the latter. In either case, the naught-subscript refers to initial conditions. Leaving aside the flow of m in the ERG, the respective flows of λ in each RG are related and a relationship between bare values and initial renormalized values can be established, as explained in the following.

Let us consider the flow towards masslessness, starting from some $u_0 = \lambda_0/m_0 \ll 1$, such that $\beta(u_0) \simeq -u_0$. While m shrinks, u grows but λ hardly changes, until m approaches λ_0 (from above). Afterwards, it undergoes a crossover and, for $m \ll \lambda_0$, Eq. (20) is ruled by the non-trivial fixed point, becomes independent of m_0, and can be expanded as

$$\frac{\lambda_0}{\lambda} = 1 + \frac{9\, u}{2\pi} + \frac{81\, u^2}{4\pi^2} + \cdots \tag{21}$$

This expansion has two remarkable aspects. Since it gives the function $\lambda(\lambda_0, m)$, independent of m_0, it constitutes a renormalization formula, with $\lambda_0 \gg \lambda$ playing the role of "bare" coupling. Naturally,

$$\lim_{m\to\infty} \lambda(\lambda_0, m) = \lambda_0\,.$$

In addition, the expansion is a "guess", namely, an RG improvement of the one-loop perturbative result that consists in the first non-trivial term of series Eq. (2). The second non-trivial term is guessed as $2.05u^2$, while it actually is $1.62u^2$ in Eq. (2). In fact, the geometric series Eq. (21) is the *bubble approximation*, which sums all the multi-loop Feynman graphs derived from the one-loop graph.

As a more sophisticated example, let us integrate the two-loop beta function derived from series Eq. (2). We now obtain:

$$\frac{\lambda_0}{\lambda} = 1 + \frac{9\, u}{2\pi} + \frac{575\, u^2}{36\pi^2} + \frac{1109\, u^3}{24\pi^3} + \frac{22777\, u^4}{216\pi^4} + O\big(u^5\big).$$

The $O\big(u^2\big)$ term agrees with Eq. (2), of course, and the next term is guessed as $1.49u^3$, being $1.83u^3$ in Eq. (2). The "guessed" terms actually correspond to a set of multi-loop Feynman graphs derived from two-loop graphs.

Let us remark that the partial summation of Feynman graphs entailed by the integration of a few initial terms of the beta function is unrelated to Padé-Borel summations that are performed on the beta function itself and change its properties, especially, they change the position of the fixed point u^*.

We can as well employ the two-loop perturbative $(\lambda\phi^4 + g\phi^6)_3$ theory results in Section 4 for a similar improving procedure, which we suppose that should be in better accord with the results of the ERG. However, the procedure is not so simple.

6.1. Improving $(\lambda\phi^4 + g\phi^6)_3$ perturbation theory

We need the two-loop perturbative beta functions for dimensionless coupling constants $u = \lambda/m$ and g, which are obtained from Eqs. (12) and (13) in Appendix B and read:

$$\beta_1 = m\left(\frac{\partial u}{\partial m}\right)_{\lambda_0, g_0} = -u - \frac{15\,g}{4\pi} + \frac{9\,u^2}{2\pi} - \frac{165\,gu}{8\pi^2} + \frac{99\,u^3}{4\pi^2}, \tag{22}$$

$$\beta_2 = m\left(\frac{\partial g}{\partial m}\right)_{\lambda_0, g_0} = \frac{45\,gu}{2\pi} + \frac{1275\,g^2}{8\pi^2} - \frac{27\,u^3}{\pi} - \frac{855\,gu^2}{4\pi^2} + \frac{27\,u^4}{\pi^2}\,. \tag{23}$$

We have set $h = 1$ and ordered the monomials by their degrees. These differential equations are scheme-independent, as they have no trace of scheme-dependent constants.

Before studying beta functions Eqs. (22) and (23), let us compare them with previous perturbative beta functions for $(\lambda\phi^4 + g\phi^6)_3$ theory [17,20–24] (see also the beta functions for hypercubic models [32, section 5]). These beta functions have been usually employed only to analyze tricritical behavior, unlike what is being done here.

Most previous beta functions do not take the renormalized mass but some other scale as RG parameter. In dimensional regularization, the RG parameter is the mass scale μ introduced to adjust the dimensions of renormalized quantities. In Lawrie and Sarbach's study of tricritical behavior [17], the RG parameter λ in Eqs. (5.27– 31) seems to be arbitrary, because they only say "a change of length scale." However, they note that their beta functions are "quite analogous" to formulas obtained from the WH RG equations. However, the beta functions obtained from the WH RG equations are not polynomials in both the coupling constants *and* m^2, while those of Lawrie and Sarbach are. In fact, they add that they are "restricted to studying tricritical singularities." In the same vein, the beta functions obtained by dimensional regularization are polynomials in the coupling constants and m^2 (which is understood as an additional coupling constant) [21–24]. Our beta functions Eqs. (22) and (23) are instead polynomials in u and g, and therefore *not* polynomials in m^2.

At any rate, the main difference between our beta functions Eqs. (22) and (23) and the beta functions obtained by dimensional regularization is that the latter only contain contributions from the two-loop order onwards [they are $O(h^2)$] [21–24]. As is well known, the one-loop terms are important for the Wilson-Fisher fixed point and, actually, give a fair approximation of u^*. In this regard, we will show below that the full two-loop beta function of $(\lambda\phi^4)_3$ theory, including the one-loop term, can be derived from Eqs. (22) and (23).

Our beta functions are similar to Sokolov's [20], which he proved to be suitable for describing the crossover between the tricritical and critical regimes. Sokolov's beta functions also have mass as RG parameter and are surely compatible with Eqs. (22) and (23), once two differences are taken into account. First, Sokolov defines an "auxiliary six-point diagram," which vanishes if $\lambda = 0$, and hence subtracts pure-λ contributions, thus actually redefining g. Second and more important, Sokolov truncates his beta functions to the second degree in the coupling constants, while ours have several terms of higher degree. Our calculation of the beta functions Eqs. (22) and (23) constitutes the first calculation of the *complete* two-loop beta functions of scalar field theory in three dimensions (to our knowledge).

The analysis of Eqs. (22) and (23) begins by linearizing them, to have some information on the RG flow near the trivial fixed point. Since the linear β_2 is null, $g = g_0$ (constant). Therefore, β_1 gives

$$\lambda_0 = \lambda + \frac{15g_0}{4\pi}(m - m_0). \tag{24}$$

One consequence of Eq. (24) is that initial values such that $g_0 = -4\pi\lambda_0/(15m_0)$ lead to $u = \lambda/m = \lambda_0/m_0 = -15g_0/(4\pi)$. Naturally, the linear beta functions have a line of fixed points, due to the *marginal* coupling $(g\phi^6)_3$. If

$$0 < u_0 + \frac{15g_0}{4\pi} \ll 1, \tag{25}$$

then the flow is such that g hardly changes while

$$\lambda = -\frac{15g_0}{4\pi}\,m;$$

that is to say, λ decreases in absolute value linearly with m. Let us notice that an effective potential with negative λ is bounded from below if $g > 0$.

Eq. (24) is to be compared with the linear part of Eq. (14), namely,

$$\lambda_0 = \lambda + \frac{15g_0}{4\pi}\left(m - \frac{2F_1\Lambda_0}{\pi}\right).$$

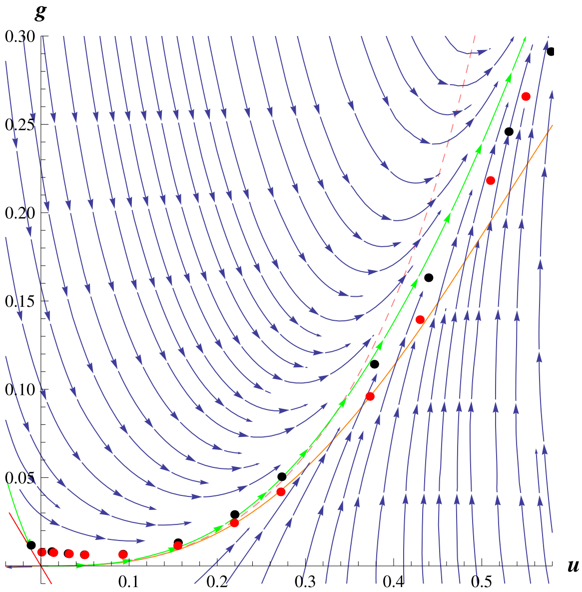


Fig. 2. RG flow for decreasing mass (in sharp-cutoff renormalization). Several features described in the text are displayed. Light red and black points correspond to Wegner-Houghton RG and two-loop calculations, respectively, nearly coinciding within some range (red points on top). The furthest couple of points has $m/\Lambda_0 = 0.01$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

In this equation, the large non-universal term corresponds to the m_0-dependent term in Eq. (24). In fact, it seems natural to choose m_0 of the order of magnitude of Λ_0, which is equivalent to having $F_1 \simeq 1$. Note that the limit $m_0 \to \infty$ is not possible in Eq. (24), unlike in Eq. (20). Thus, the presence of either Λ_0 or m_0 is a signal of non-universality.

Differential Eqs. (22) and (23) constitute an autonomous system when we take as independent variable the "RG time" $\tau = \log(m_0/m)$. They can be integrated numerically between $\tau = 0$ and any $\tau > 0$, from any point (u_0, g_0). If we had the full beta functions rather than a low-order perturbative expression, then u and g should flow towards the non-trivial Wilson-Fisher fixed point. In contrast, what we observe is only that the flow converges, but the equations become inaccurate before approaching any non-trivial fixed point. Fig. 2 depicts the aspect of the flow. The non-linear terms make the line of fixed points disappear, remaining the trivial fixed point $u = g = 0$. In fact, the line of fixed points (Fig. 2, short orange segment) becomes a *separatrix* curve, in the terminology of dynamical systems (short green line on the left). This curve is actually the critical manifold of the trivial fixed point, which corresponds to tricritical behavior.

The flow lines can also be obtained from the single differential equation

$$\frac{dg}{du} = \frac{\beta_2(u,g)}{\beta_1(u,g)},$$

with solution $g(u)$, given an initial condition, such as $g(0) = g_0$. Besides the separatrix, the other remarkable curve is given by the initial condition $g(0+) = 0$ (specifying the 0+ direction at the origin, as a singular point). It gives rise to the long green line with arrows in Fig. 2. This is a "connecting line" in the terminology of dynamical systems.

Our connecting line should be given by a function matching expansion Eq. (16) (with $h = 1$). The graph of expansion Eq. (16) is the long orange line in Fig. 2, below the long green line. It is easy to confirm that the derivative with respect to u of expansion Eq. (16) agrees with the expansion of β_2/β_1 in powers of u. However, the solution of the differential equation $dg/du = \beta_2/\beta_1$ is not a polynomial, of course, and it deviates from the polynomial Eq. (16) as u grows (as is visible in Fig. 2). This non-polynomial solution $g(u)$ is supposed to constitute an *improvement* of the two-loop result. Assuming a power series for $g(u)$ and then matching coefficients, we find the next term in Eq. (16), namely,

$$g(u) = \frac{9u^3}{\pi} - \frac{27u^4}{\pi^2} + \frac{7209u^5}{20\pi^3} + O\left(u^6\right). \tag{26}$$

The three-loop contribution computed by Sokolov et al. is $(9/\pi) \times 1.39 = 3.98$ [20, Eq 2.5], with the same sign as $7209/(20\pi^3)$ but quite different magnitude.

Let us quote the known five-loop expression of the asymptotic expansion of $g(u)$ [30, Eq 8]:

$$g(u) = \frac{9u^3}{\pi}\left(1 - \frac{3u}{\pi} + 1.38996u^2 - 2.50173u^3 + 5.2759u^4 + O\left(u^5\right)\right).$$

Its graph is displayed in Fig. 2 as the dashed pink line. It matches the connecting line rather well up to $u = 0.35$. In this regard, it should be noted that this five-loop expansion is not accurate at all in the entire range of u in Fig. 2: the two last terms in the parenthesis give, for $u = 0.5$, respectively, $-2.50173 \times 0.5^3 = -0.31$ and $5.2759 \times 0.5^4 = 0.33$, with the same and considerable magnitude.

Other solutions of differential Eqs. (22) and (23) also constitute improved two-loop renormalization equations. In this sense, they improve, for example, on the pair of two-loop renormalization Eqs. (12) and (13) that have served to calculate Eqs. (22) and (23). Eqs. (12) and (13) actually are the general solution of system Eqs. (22, 23), when this solution is restricted to $O(h^2)$, because they include two arbitrary integration constants, i.e., λ_0 and g_0. These integration constants are to be interpreted as bare coupling constants, and the solutions also involve the scheme-fixing constants. Thus, a change in the latter results in a change of the former, in order to have the same renormalized theory.

Universality is achieved on the connecting line, as independence of m_0, like we have seen before in $(\lambda\phi^4)_3$ theory. In other words, the limit $m \to \infty$ ($\tau \to -\infty$) is possible on it, because it leads to the trivial fixed point. This limit also leads to bare coupling constant values $\lambda_0 \neq 0$ and $g_0 = 0$, as in $(\lambda\phi^4)_3$ theory. In fact, the two-loop beta function of $(\lambda\phi^4)_3$ theory can be derived from Eqs. (22) and (23), by substituting the relation $g(u)$ given by Eq. (26), namely, its first term (the other terms would contribute to higher order). To wit, by substituting the first term of Eq. (26) in Eq. (22), we obtain

$$\beta_1 = -u + \frac{9u^2}{2\pi} - \frac{9u^3}{\pi^2} + O\left(u^4\right). \tag{27}$$

It is the correct beta function derived from Eq. (3) [Let us notice that the two-loop term in the beta function derived from Eq. (2) instead is $-77u^3/(9\pi^2)$].

Even though there is no $m \to \infty$ limit on any other RG trajectory, Eqs. (22) and (23) have one remarkable property: on the parabola

$$g = 6u^2/5, \tag{28}$$

they reduce to

$$\beta_1 = -u, \quad \beta_2 = 0.$$

As explained before, such form of β_1 implies that u changes with m but λ is stationary. The second equation obviously implies that g is stationary. Therefore, an RG trajectory, that is to say, a solution of Eqs. (22) and (23), passing near the origin of the (u, g)-plane and crossing the curve $g = 6u^2/5$ has stationary values of the coupling constants while near the crossing point (it spends there a considerable RG time τ). As there is no $m \to \infty$ limit, if we let m grow, then the RG trajectory eventually turns upward and λ and g change (the latter grows). Total stationarity only occurs on the connecting line.

The preceding considerations are useful to compare perturbation theory with the results of ERG integrations. In fact, the WH RG always gives $\lambda > 0$ and hence $u > 0$. Furthermore, with bare $m_0 > \Lambda_0$, hardly any renormalization takes place in the integration over Λ, and (m, λ, g) keep their initial values. Therefore, letting m grow, $u = \lambda/m$ shrinks along a horizontal segment in the (u, g)-plane towards the axis $u = 0$ (Fig. 2). In contrast, letting m grow in perturbation theory makes $\lambda < 0$ and hence $u < 0$, while both u and g keep growing in absolute value.

Referring to the bare values in the example of Section 5.1, namely, $\{\lambda_0 = 0.008225, g_0 = 0.007793\}$, we find notable discrepancies between various WH RG integrations for $m_0 > 0$ and the results of perturbative formulas Eqs. (14) and (15) for the corresponding values of m. For example, for $m_0 = 3$, the WH RG integration yields $m = 3.00003 \simeq m_0$, $\lambda = 0.008430$, and $g = 0.007791$, while Eqs. (14) and (15) yield $\lambda = -0.01522$, and $g = 0.00974$ (dimensional values in units of Λ_0). For $m_0 = 10$ (totally unphysical value), the ERG integration yields $m = m_0$, $\lambda = 0.008244$, and $g = 0.007793$, while Eqs. (14) and (15) yield $\lambda = -0.09067$, and $g = 0.0105$. (See Fig. 2.) At any rate, it is remarkable that major discrepancies only occur for unphysically large masses. For $m \lesssim 0.5$ the agreement is excellent: e.g., for $m_0 = 0.5^{1/2} = 0.2326$, the ERG yields $m = 0.2329$, $\lambda = 0.01154$, and $g = 0.006295$, while this m and Eqs. (14) and (15) yield $\lambda = 0.01145$, and $g = 0.006347$. The stationary point actually is $\{u = 0.06601, g = 0.005229\}$, and both λ and g hardly change in the interval $u \in (0.05, 0.1)$, corresponding to $m \in (0.12, 0.24)$.

Of course, the agreement between the ERG and perturbation theory in the interval where λ and g hardly change (a horizontal segment in Fig. 2) ceases for larger values of u (smaller values of m), because the two-loop order is not sufficiently accurate. Nevertheless, the deviation is still relatively small even for $u = 0.58$ ($m = 0.01$, Section 5.1), as seen in Fig. 2.

The natural interpretation of all the above results is that the parabola Eq. (28) defines the crossover from tricritical to critical behavior. We can picture the parabola in Fig. 2 as the line formed by the minima of RG flow lines. The region between this line and the connecting line is the properly critical region whereas the region on the left is the tricritical region. Note that the exponent two of u in Eq. (28) is the classical value of the *crossover exponent*, as corresponds to the tricritical RG fixed point being a trivial fixed point [17].

The scheme independence of the RG differential Eqs. (22) and (23) suggests that the example of Section 5.1, namely, $\{\lambda_0 = 0.008225, g_0 = 0.007793\}$, when treated with a different cutoff, should yield similar results. Perturbative Eqs. (14) and (15) contain the scheme dependence in the constants F_1, A and B. The case of power-law cutoff with $n = 2$ is interesting, as remarked in Section 5.2 (however, the results are similar in other cases). Since Eqs. (12) and (13) are the general solution of system Eqs. (22, 23) to $O\left(h^2\right)$, keeping $\{\lambda_0, g_0\}$ while changing F_1, A and B should give rise to another solution. Indeed it does, and the solution is just slightly different, as shown in Fig. 3. At any rate, the two trajectories are traversed somewhat differently, so that the same m does not correspond to a pair of closest points.

Last, we can consider larger values of g_0, such as the one tried in Section 5.1, namely, $g_0 = 0.039$ (five times larger than the first value). In this regard, we have shown that $g_0 \ll 0.05$ for the two-loop Eq. (15) to work. In addition, it is evident from Fig. 2 that the

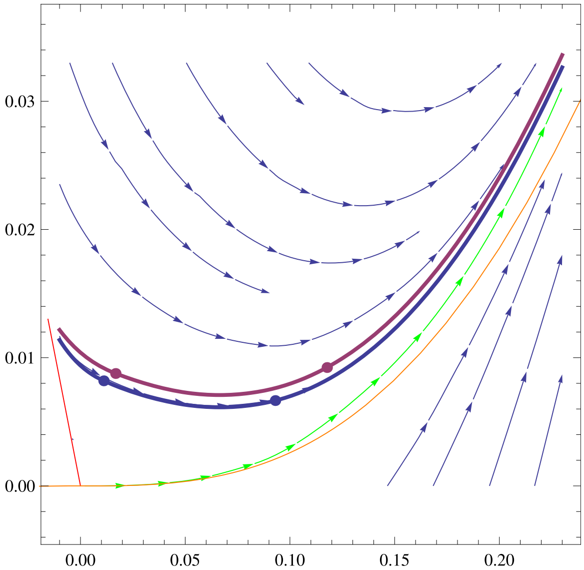


Fig. 3. Scheme dependence of perturbative results from Eqs. (14) and (15) (interpolated). Sharp cutoff: lower (blue) line. Smooth $n = 2$ power-law cutoff: upper (magenta) line. The two couples of points correspond to $m/\Lambda_0 = 0.71$ and $m/\Lambda_0 = 0.12$, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

point with $\lambda_0 = 0.008225$ and $g_0 = 0.039$ is totally inside the tricritical region, as are points with very small λ_0 and g_0 approaching 0.05. This restriction on the values of g_0 will surely not be altered by higher loop order calculations.

Let us notice that the tricritical region is of interest, although we are concerned here with the crossover and critical regions. As mentioned above, we can consider effective potentials with $\lambda < 0$ and $g > 0$, which neatly lie in the tricritical region. These effective potentials can have three minima, as occurs on the *triple line* of the complete phase diagram [17].

7. Conclusions

The conclusions presented here are, of course, based on the two-loop effective potential, but they seem fairly general in perturbation theory. Certainly, further higher-loop order calculations will refine them. Nonetheless, we have confirmed our perturbative results by employing exact renormalization group integrations, within an approximation that is more accurate than two-loop perturbation theory.

The first fact is that three-dimensional scalar field theory is incomplete without the sextic field coupling and that a non-null bare coupling constant (g_0) inevitably brings in non-universal results. While the Wilson-Fisher renormalization group fixed point has been thoroughly studied in perturbation theory with just the quartic field coupling, the introduction of the sextic field coupling is not only pertinent but also necessary. This is especially true in the effective field theory, because a null coupling constant g_0 at the cutoff scale Λ_0 will not stay null at scales $\Lambda < \Lambda_0$, as ERG integrations demonstrate. Therefore, the question is the magnitude of the non-universal corrections due to having $g_0 \neq 0$.

The renormalization of the quartic and sextic coupling constants follows from the two-loop effective potential, calculated with an arbitrary cutoff function, which we denote by $F(k/\Lambda)$ and also call the form factor. At the two-loop order, the scheme dependence, that is to say, the dependence on the cutoff function, reduces to three constants, which result from the Feynman graph integrals with the cutoff function (computed with variable difficulty). To wit, we define the constants F_1, F_2, and A, the last one being the most difficult (we also use $B = F_1 F_2$). These constants appear in renormalization terms that explicitly contain the cutoff scale Λ_0 in addition to the coupling constant g_0, and which do not have a finite limit when $\Lambda_0 \to \infty$.

The magnitude of scheme dependence is quantified by the allowed ranges of constants F_1, F_2, and A. In principle, these ranges could be large or even arbitrary, but a close examination of various types of cutoff functions shows that reasonable schemes only allow very reduced ranges. A remarkable result is that the smallest values of those constants occur precisely for the sharp cutoff. Thus, the non-universal terms are smallest in this case. However, smooth cutoffs do not raise much their magnitude. Suitable examples of very smooth cutoff functions are the ordinary exponential and the $n = 2$ power-law function. We have chosen the latter, because it is more amenable to analytical treatment and would be suitable for simple exact RG integrations.

Given the reduced ranges of scheme-dependent constants F_1, F_2, and A, the magnitude of the bare sextic coupling constant g_0 determines the magnitude of the non-universal terms. It is not obvious what magnitude of g_0 is appropriate for the renormalization formulas. Since we intended to begin with the sharp cutoff and compare with Wegner-Houghton RG integrations, we have used the criterion of small bare values, that is to say, values that allow an easy approximation of the critical manifold (and indeed of the

Wegner-Houghton equation). This criterion has no fundamental reason, of course, but it makes it easier to find critical values of the coupling constants.

Thus, we have initially tried λ_0/Λ_0 and $g_0 < 0.01$. We do not attempt to determine exact critical values but only to reduce the renormalized mass to $m/\Lambda_0 \simeq 0.01$. In this way, we find that $u = \lambda/m \gtrsim 0.5$, that is to say, values of u that we expect to be still suitable for simple summation of perturbative series in u. We further find that the effect of a non-zero g_0 on both the renormalized coupling constants is considerable, and the effect is more pronounced on g than on λ. Nevertheless, comparing two-loop calculations with exact RG integrations, we find that their concordance is not spoiled if $g_0 < 0.01$. In contrast, larger values of g_0 produce increasing discrepancies, which are very large for $g_0 > 0.03$ (still with $\lambda_0/\Lambda_0 < 0.01$). The origin of these discrepancies becomes clear when we achieve a complete picture of how renormalization works for a wide mass range.

To have a complete picture of the renormalization group, we have calculated the perturbative renormalization group that gives the changes of couplings with mass, at the two-loop order. This type of RG is interesting in itself, as it is, on the one hand, scheme independent, and, on the other, it allows for improvements in perturbation theory. Naturally, the two-loop order loses accuracy far from the trivial RG fixed point. We have plotted its aspect in the region where $u \in (0, 0.6)$ and $g \in (0, 0.3)$ (and for very small negative u, in addition, see Fig. 2). The RG flow plot clearly shows how the crossover from tricritical to critical behavior takes place. The condition that somehow determines the region of critical behavior, ruled by the Wilson-Fisher fixed point, is $g < 6u^2/5$. Thus, trajectories like the one with $\lambda_0/\Lambda_0 < 0.01$ and $g_0 > 0.03$ are mostly in the tricritical region.

The results of renormalization for a definite couple of bare couplings, such that λ_0/Λ_0 and $g_0 < 0.01$, and letting m run from very large (larger than the cutoff Λ_0) to quite small ($m/\Lambda_0 \simeq 0.01$), can be best analyzed when set in the RG flow diagram. The small values of bare couplings guarantee that the renormalized coupling constants, for physical values of m, fall in the region of critical rather than tricritical behavior. However, the close concordance between two-loop calculations and exact RG integrations that occurs for small λ_0/Λ_0 and g_0, in a mass range, say $m/\Lambda_0 \in (0.1, 0.5)$, is progressively lost for smaller values of m. Of course, one should expect to achieve, at criticality, full agreement between exact RG calculations and large-order perturbation theory, conveniently resummed. However, confirming this is beyond the reach of the approximations that we employ and is not our goal here.

We can also set the results of two-loop calculations for common λ_0 and g_0 but different cutoffs in the RG flow diagram (Fig. 3). Naturally, it is most interesting to contrast the sharp cutoff with a very smooth cutoff, such as the $n = 2$ power-law cutoff. The renormalized trajectory that corresponds to the smooth cutoff is a little further away from the connecting line that represents the universal trajectory for $g_0 = 0$. The difference is small, as corresponds to the small range of the scheme dependent constants. As the trajectories converge towards the Wilson-Fisher point, they become indistinguishable. However, points that are initially close, that is to say, close for $m/\Lambda_0 \simeq 1$, become distant for $m \ll \Lambda_0$. This property attributes a measurable meaning to the scheme dependence. Of course, this meaning relies on the knowledge of Λ_0 as the reference scale.

Since the main factor for non-universality is the magnitude of g_0 rather than the type of cutoff scheme, one might wonder whether g_0 should be small and why. Considering the usual methods of defining a field theory for the Ising model (Hubbard-Stratonovich transformation, etc), we see no reason why g_0 should turn out small. This suggests that the three-dimensional Ising model field theory could possibly lie in the tricritical region and would only enter the critical region for a sufficiently large correlation length.

From a general point of view, the limited influence of the regularization scheme when the mass is sufficiently small can be useful in effective field theories for particle physics. The question is: given a fixed field theory “beyond the Standard Model,” how should the influence of regularization schemes on accessible low-energy physics be quantified?

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Calculation of integrals

Let us begin with I_1 and I_2. Since $I_2(M^2) = I_1'(M^2)$, we can compute one of them and hence deduce the other. In fact, I_2 is simpler. Of course, these integrals have to be integrated over the entire k-space but are UV divergent, that is to say, the integrals diverge for $k \to \infty$. Let us regularize the field with a sort of “form factor”, namely, a smoothing or average over a space volume with a characteristic length that is large enough to contain several lattice sites, say, but that is small enough with respect to the correlation length. Therefore, we can assume that its effect on the propagator is to multiply it by a factor $F(k/\Lambda_0)$, such that $F(0) = 1$

and $\lim_{x\to\infty} F(x) = 0$. Our integral becomes:

$$I_2(M^2) = \int \frac{F(k/\Lambda_0)\, d^3k}{(2\pi)^3 (k^2+M^2)} = \frac{1}{2\pi^2} \int_0^\infty \frac{F(k/\Lambda_0)\, k^2\, dk}{k^2+M^2}\,. \tag{A.1}$$

Naturally, the sharp cutoff is defined by $F(k/\Lambda_0) = \theta(1 - k/\Lambda_0)$, where $\theta(x)$ is the Heaviside step function, but we shall keep F as general as possible.

We can easily extract an M-independent term from I_2 in Eq. (A.1):

$$I_2(M^2) - I_2(0) = \frac{1}{2\pi^2} \int_0^\infty F(k/\Lambda_0)\, k^2\, dk \left(\frac{1}{k^2+M^2} - \frac{1}{k^2} \right) = \frac{-M^2}{2\pi^2} \int_0^\infty \frac{F(k/\Lambda_0)\, dk}{k^2+M^2}\,. \tag{A.2}$$

Therefore,

$$I_2(M^2) = \frac{1}{2\pi^2} \left(\int_0^\infty F(k/\Lambda_0)\, dk - M^2 \int_0^\infty \frac{F(k/\Lambda_0)\, dk}{k^2+M^2} \right). \tag{A.3}$$

Note that we have achieved a useful separation: while the first integral diverges in the limit $\Lambda_0 \to \infty$, the second one is finite and gives

$$\int_0^\infty \frac{dk}{k^2+M^2} = \frac{1}{M} \int_0^\infty \frac{dq}{q^2+1} = \frac{\pi}{2M}\,. \tag{A.4}$$

We also have

$$\int_0^\infty F(k/\Lambda_0)\, dk = \Lambda_0 \int_0^\infty F(q)\, dq, \tag{A.5}$$

where we can assume that the latter integral is finite and the UV divergence is proportional to Λ_0. If $F(q) = \theta(1-q)$, the integral is trivially equal to one.

Given that we have I_2^2 in Eq. (8), we need to calculate inverse powers of Λ_0. To do it, we express the second integral in Eq. (A.3) as

$$\int_0^\infty \frac{F(k/\Lambda_0)\, dk}{k^2+M^2} = \frac{1}{\Lambda_0} \int_0^\infty \frac{F(q)\, dq}{q^2 + (M/\Lambda_0)^2}, \tag{A.6}$$

and we try to expand the latter integral for small M. We already know that the integral diverges as Λ_0/M for small M, to fit Eq. (A.4) The divergence of the integration over q in Eq. (A.6) for $M = 0$ is an IR divergence at $q = 0$. We can segregate it by expanding $F(q)$ at $q = 0$. Let us assume that

$$F(q) = 1 + O(q^2),$$

that is to say, that $F(q)$ is twice differentiable at $q = 0$ and $F'(0) = 0$ (a very natural assumption). Thus,

$$\int_0^\infty \frac{F(q)\, dq}{q^2 + (M/\Lambda_0)^2} = \frac{\pi \Lambda_0}{2M} + \int_0^\infty \frac{F(q) - 1}{q^2 + (M/\Lambda_0)^2}\, dq,$$

being the last integral finite for $M = 0$.

Denoting the two integrals of F by

$$F_1 = \int_0^\infty F(q)\, dq, \quad F_2 = \int_0^\infty \frac{1 - F(q)}{q^2}\, dq,$$

we can write

$$I_2(M^2) = \frac{1}{2\pi^2} \left(\Lambda_0 F_1 - \frac{\pi M}{2} + F_2 \frac{M^2}{\Lambda_0} + M\, o(M/\Lambda_0) \right) \tag{A.7}$$

(where the little-o symbol means slower growth). Therefore,

$$I_1(M^2) = \int I_2(M^2)\, dM^2 = \frac{1}{2\pi^2} \left(\Lambda_0 F_1 M^2 - \frac{\pi M^3}{3} + M^3\, o(1) \right). \tag{A.8}$$

$$I_2(M^2)^2 = \frac{1}{4\pi^4} \left(\Lambda_0^2 F_1^2 - \pi F_1 M \Lambda_0 + \left(\frac{\pi^2}{4} + 2F_1 F_2 \right) M^2 + M^2\, o(1) \right). \tag{A.9}$$

(The unimportant integration constant has been suppressed.) Notice that the Λ_0-independent term is not universal, as it contains the scheme-dependent constant $F_1 F_2$. We have already noticed that $F_1 = 1$ in the case of the sharp cutoff. It is also easy to see that also $F_2 = 1$ in this case.

A.1. Calculation of I_3

Integral I_3 is of the type corresponding to "melon" Feynman graphs, and it is called, in particular, either "sunrise" or "sunset" graph integral. Including the form factor, we have:

$$I_3(M^2) = \frac{1}{(2\pi)^6} \int_0^\infty dk \int_0^\infty dq \int_0^\pi \frac{8\pi^2 \sin\theta\, d\theta\ F(k)\, k^2\, F(q)\, q^2\, F(|\mathbf{k}+\mathbf{q}|)}{(k^2+M^2)(q^2+M^2)(k^2+q^2+2kq\cos\theta+M^2)}, \tag{A.10}$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{q}$. This integral is awkward, because $F(|\boldsymbol{k}+\boldsymbol{q}|)$ depends on θ. Therefore, it is not the best expression to calculate the cutoff integral, not even in the case of the sharp cutoff.

A method that combines position and Fourier space to extract "melon" graph divergences was proposed long ago [33]. In fact, the unregulated integral is very simple in terms of the correlation function in position space,

$$I_3\big(M^2\big) = \int d^3x[G(x)]^3. \tag{A.11}$$

Given the simple form of $G(x)$ in three dimensions, some results can be simply derived from this expression [1, Appendix to Ch 5]. Inserting the form factor, we replace $G(x)$ by $G_\Lambda(x)$, which can be simplified by carrying out the angular integral:

$$G_\Lambda(x) = \int \frac{d^3k}{(2\pi)^3}\,\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{k^2+M^2}\,F(k/\Lambda) = \frac{1}{2\pi^2 x}\int_0^\infty dk\,\frac{k\,\sin(kx)}{k^2+M^2}\,F(k/\Lambda). \tag{A.12}$$

Therefore,

$$\begin{aligned} I_3\big(M^2\big) &= \frac{4\pi}{(2\pi^2)^3}\int_0^\infty \frac{dx}{x}\left[\int_0^\infty dk\,\frac{k\,\sin(kx)}{k^2+M^2}\,F(k/\Lambda_0)\right]^3 \\ &= \frac{1}{2\pi^5}\int_0^\infty dp\,\frac{p\,F(p/\Lambda_0)}{p^2+M^2}\int_0^\infty dk\,\frac{k\,F(k/\Lambda_0)}{k^2+M^2}\int_0^\infty dq\,\frac{q\,F(q/\Lambda_0)}{q^2+M^2}\int_0^\infty \frac{dx}{x}\,\sin(px)\,\sin(kx)\,\sin(qx). \end{aligned} \tag{A.13}$$

The last integral can be carried out in terms of Dirichlet integrals by expressing the product of sines as a sum of sines (of different arguments); namely,

$$\sin(px)\,\sin(kx)\,\sin(qx) = \frac{1}{4}(\sin(x(-p+k+q)) + \sin(x(p-k+q)) + \sin(x(p+k-q)) - \sin(x(p+k+q))).$$

Hence,

$$\begin{aligned} \int_0^\infty \frac{dx}{x}\,\sin(px)\,\sin(kx)\,\sin(qx) &= \frac{\pi}{8}(\operatorname{sgn}(-p+k+q) + \operatorname{sgn}(p-k+q) + \operatorname{sgn}(k+p-q) - 1) \\ &= \frac{\pi}{4}(\theta(-p+k+q)\,\theta(p-k+q)\,\theta(k+p-q)), \end{aligned} \tag{A.14}$$

where sgn and θ denote the sign and step functions, respectively, and the last equality holds in the positive octant. Thus we obtain

$$I_3\big(M^2\big) = \frac{1}{8\pi^4}\int_{\text{region}} dp\,dk\,dq\,\frac{p\,F(p/\Lambda_0)}{p^2+M^2}\,\frac{k\,F(k/\Lambda_0)}{k^2+M^2}\,\frac{q\,F(q/\Lambda_0)}{q^2+M^2}, \tag{A.15}$$

where the "region" is defined by $0 \le p \le k+q$ and $0 \le k \le p+q$ and $0 \le q \le k+p$.

Eq. (A.15) is a convenient representation, suitable for obtaining the behavior in the limit $\Lambda_0 \to \infty$. Naturally, the form factor tends to one in this limit and I_3 is UV divergent (e.g., as $p = k = q \to \infty$). The divergent part, proportional to $\log(\Lambda_0/M)$, can be readily obtained with standard methods [1, Appendix to Ch 5], yielding

$$\frac{1}{16\pi^2}\log(\Lambda_0/M).$$

Here, we are interested in the finite part, after the subtraction of this term. To subtract it, we split the integral Eq. (A.15) into a UV convergent part and an IR convergent part, by setting an arbitrary cutoff κ such that $M \ll \kappa \ll \Lambda_0$. Then, we make the wavenumber-scale integration explicit with the change of integration variables:

$$p = P\,x,\ \ k = P\,y,\ \ q = P\,z,$$

such that the integration measure becomes:

$$dp\,dk\,dq = dx\,dy\,dz\,\delta(x+y+z-1)\,P^2dP.$$

Now we can take the limit $\Lambda_0 \to \infty$ in the integral in $[0,\kappa]$ and the limit $M \to 0$ in the integral in $[\kappa,\infty]$ and thus write:

$$\begin{aligned} I_3\big(M^2\big) = &\frac{1}{8\pi^4}\int_{\text{region}} dx\,dy\,dz\,\delta(x+y+z-1)\,xyz\Bigg[\int_0^\kappa \frac{P^5\,dP}{(P^2x^2+M^2)(P^2y^2+M^2)(P^2z^2+M^2)} \\ &+ \int_\kappa^\infty \frac{dP}{P\,x^2\,y^2\,z^2}\,F(Px/\Lambda_0)\,F(Py/\Lambda_0)\,F(Pz/\Lambda_0)\Bigg]. \end{aligned} \tag{A.16}$$

Of course, the first integral is still IR divergent when $M = 0$ and the second integral is still UV divergent when $\Lambda_0 \to \infty$. In fact, the first integral diverges as $\frac{\pi^2}{2}\log(\kappa/M)$ and the second as $\frac{\pi^2}{2}\log(\Lambda_0/\kappa)$, yielding the term proportional to $\log(\Lambda_0/M)$.

Let us now consider the two integrals in Eq. (A.16) in turn. The first one is independent of the form factor and can be computed exactly for large κ/M. A somewhat cumbersome calculation yields:

$$\int_{\text{region}} dx\,dy\,dz\,\delta(x+y+z-1)\,xyz\int_0^\kappa \frac{P^5\,dP}{(P^2x^2+M^2)(P^2y^2+M^2)(P^2z^2+M^2)} = \frac{\pi^2}{2}\left(-\frac{21\zeta(3)}{2\pi^2} - \log 3 + \log(\kappa/M)\right) + O\Big(\frac{M}{\kappa}\Big)^2. \tag{A.17}$$

The second integral in Eq. (A.16) gives the dependence of I_3 on the UV cutoff function. The integral over P can be calculated analytically for simple functions, e.g., for the Gaussian $F(k) = \exp(-k^2)$, but the integration over $\{x, y, z\}$ is best computed numerically. Gathering it all, for the Gaussian we have:

$$I_3 = \frac{1}{16\pi^2}\left(-\frac{21\zeta(3)}{2\pi^2} - \log 3 - \frac{\gamma}{2} + \frac{2}{\pi^2}\, 2.325132375178383 + \log\left(\Lambda_0/M\right)\right)$$
$$= \frac{1}{16\pi^2}\left(-2.1948849888150064 + \log\left(\Lambda_0/M\right)\right). \tag{A.18}$$

Let us notice that our conditions on cutoff functions imply that $F(k/\Lambda) = \exp(-k^2/\Lambda^2)$ must be replaced by $F(k/\Lambda) = \exp(-\ln 2\; k^2/\Lambda^2)$, which amounts to a redefinition of Λ.

A.1.1. Power-law form factors

Let us consider

$$F(k) = \frac{1}{1+k^n}, \tag{A.19}$$

for several values of n. This form factor allows us to calcute the integral giving G_Λ, Eq. (A.12), in several cases, e.g., for $n = 2$. Therefore, the use of Eq. (A.11) is preferable; namely,

$$I_3(M^2) = \frac{1}{2\pi^5}\int_0^\infty \frac{dx}{x}\left[\int_0^\infty dk\, \frac{k\,\sin(kx)}{k^2+M^2}\, F(k/\Lambda)\right]^3.$$

In fact, to obtain its finite part, after the subtraction of $\log(\Lambda/M)$, we can proceed as above, namely, we split the integral into a UV convergent part and an IR convergent part, by setting an arbitrary cutoff R, now in position space, such that $M \ll 1/R \ll \Lambda_0$. Now we can make $M \to 0$ in the integral in $[0, R]$ and take the limit $\Lambda_0 \to \infty$ in the integral in $[R, \infty]$.

The latter is independent of the form factor and can be computed exactly, because

$$\lim_{\Lambda\to\infty} G_\Lambda(x) = G(x) = \frac{\exp(-Mx)}{4\pi x}$$

[1, Appendix to Ch 5]. Therefore,

$$\int_{x\ge R} d^3x[G(x)]^3 = \frac{1}{16\pi^2}\int_R^\infty \frac{dx}{x}\exp(-3Mx) = \frac{E_1(3MR)}{16\pi^2}$$
$$= \frac{1}{16\pi^2}\left[-\gamma - \log(3MR) + O(MR)\right]. \tag{A.20}$$

To calculate the integral in $[0, R]$, we first need

$$\lim_{M\to 0}\int_0^\infty dk\,\frac{k\,\sin(kx)}{k^2+M^2}\,F(k/\Lambda) = \int_0^\infty \frac{dk}{k}\,\frac{\sin(kx)}{1+(k/\Lambda)^n}\,. \tag{A.21}$$

For some values of n, the result can be found in Gradshteyn-Ryzhik's tables [34, 3.728–34] (or obtained by computer algebra). For example,

$$\int_0^\infty \frac{dk}{k}\,\frac{\sin(kx)}{1+(k/\Lambda)^2} = \frac{\pi}{2}(1-\exp(-\Lambda x))\,; \tag{A.22}$$

$$\int_0^\infty \frac{dk}{k}\,\frac{\sin(kx)}{1+(k/\Lambda)^4} = \frac{\pi}{2}\left(1 - e^{-\frac{\Lambda x}{\sqrt{2}}}\cos\left(\frac{\Lambda x}{\sqrt{2}}\right)\right); \tag{A.23}$$

$$\int_0^\infty \frac{dk}{k}\,\frac{\sin(kx)}{1+(k/\Lambda)^6} = \frac{\pi}{6}\left(3 - e^{-\Lambda x} - 2e^{-\Lambda x/2}\cos\left(\frac{\sqrt{3}\Lambda x}{2}\right)\right). \tag{A.24}$$

In these cases, it is possible to evaluate

$$\int_{x\le R} d^3x[G(x)]^3 = \frac{1}{2\pi^5}\int_0^R \frac{dx}{x}\left[\int_0^\infty \frac{dk}{k}\,\frac{\sin(kx)}{1+(k/\Lambda)^n}\right]^3.$$

However, the results are expressed as combinations of functions E_1 with several arguments multiples of ΛR plus a $\log(\Lambda R)$ term. The expressions become cumbersome for $n > 2$. In the large Λ limit, each integral diverges as $(\pi/2)^3\log(\Lambda R)$, as expected.

Therefore, we can obtain the finite part with a simple integral over $x \in [0, \infty]$; for example, with $n = 2$:

$$\int_0^\infty dx\left[\frac{1}{x}(1-e^{-x})^3 - \frac{1}{x+1}\right] = \gamma + \log 3 - \log 8.$$

Adding the finite part given by Eq. (A.20) and combining the logarithms, for $n = 2, 4, 6$, we have:

$$I_3 = \frac{1}{16\pi^2}(-\log 8 + \log(\Lambda/M)) = \frac{1}{16\pi^2}(-2.07944 + \log(\Lambda/M)). \tag{A.25}$$

$$I_3 = \frac{1}{16\pi^2}\left(\frac{3\log 5}{8} - \frac{9\log 2 - -3\log 3}{4} + \log(\Lambda/M)\right) = \frac{1}{16\pi^2}(-1.78000 + \log(\Lambda/M)). \tag{A.26}$$

$$I_3 = \frac{1}{16\pi^2}\left(-\frac{7\log 2 + 13\log 3 - 2\log 7}{9} + \log(\Lambda/M)\right) = \frac{1}{16\pi^2}(-1.69357 + \log(\Lambda/M)). \tag{A.27}$$

A.1.2. Sharp cutoff

The sharp-cutoff function can be obtained as the limit when $n \to \infty$ of F in Eq. (A.19). Hence, we guess that the constant in I_3 to be added to $\log(\Lambda/M)$ is the least negative of the n-sequence. We have in place of the integral in the rhs of Eq. (A.21), a tabulated integral [34, 8.23], namely,

$$\mathrm{Si}(\Lambda x) = \int_0^\infty \frac{dk}{k} \sin(kx)\,\theta(1 - k/\Lambda).$$

The finite part of the integral over $x \in [0, R]$ is obtained as before, namely:

$$\int_0^\infty dx \left[\left(\frac{2}{\pi}\right)^3 \frac{\mathrm{Si}(x)^3}{x} - \frac{1}{x+1} \right].$$

This integral is evaluated numerically and we have

$$I_3 = \frac{1}{16\pi^2}(-1.58578 + \log(\Lambda/M)).$$

Appendix B. Beta functions

To calculate the beta-functions, the simplest procedure is as follows. Let us begin with Eqs. (12) and (13), which give λ and g as functions of λ_0, g_0 and m. Now we simply take derivatives with respect to m, keeping fixed λ_0 and g_0.

$$\left(\frac{\partial \lambda}{\partial m}\right)_{\lambda_0, g_0} = h\left(\frac{9\lambda_0^2}{2\pi m^2} - \frac{15 g_0}{4\pi}\right) + h^2\left(\frac{135 F_1 g_0 \lambda_0 \Lambda}{2\pi^3 m^2} + \frac{30 g_0 \lambda_0}{\pi^2 m} - \frac{99\lambda_0^3}{2\pi^2 m^3}\right) + O\left(h^3\right), \tag{B.1}$$

$$\left(\frac{\partial g}{\partial m}\right)_{\lambda_0, g_0} = h\left(\frac{45 g_0 \lambda_0}{2\pi m^2} - \frac{27\lambda_0^3}{\pi m^4}\right) + h^2\left(-\frac{1215 F_1 g_0 \lambda_0^2 \Lambda}{2\pi^3 m^4} + \frac{675 F_1 g_0^2 \Lambda}{4\pi^3 m^2} - \frac{1035 g_0 \lambda_0^2}{2\pi^2 m^3} + \frac{75 g_0^2}{\pi^2 m} + \frac{594\lambda_0^4}{\pi^2 m^5}\right) + O\left(h^3\right). \tag{B.2}$$

Now, we replace λ_0 and g_0, using Eq. (14) to $O(h)$ and

$$g_0 = g + h\,\frac{9\left(5\, g m^2 \lambda - 2\lambda^3\right)}{2\pi m^3} + O\left(h^2\right).$$

Thus, we obtain:

$$\left(\frac{\partial \lambda}{\partial m}\right)_{\lambda_0, g_0} = h\left(\frac{9\lambda^2}{2\pi m^2} - \frac{15\, g}{4\pi}\right) + h^2\left(\frac{99\lambda^3}{4\pi^2 m^3} - \frac{165\, g\lambda}{8\pi^2 m}\right) + O\left(h^3\right), \tag{B.3}$$

$$\left(\frac{\partial g}{\partial m}\right)_{\lambda_0, g_0} = h\left(\frac{45\, g\lambda}{2\pi m^2} - \frac{27\lambda^3}{\pi m^4}\right) + h^2\left(\frac{1275\, g^2}{8\pi^2 m} - \frac{855\, g\lambda^2}{4\pi^2 m^3} + \frac{27\lambda^4}{\pi^2 m^5}\right) + O\left(h^3\right). \tag{B.4}$$

Hence, we obtain the beta functions of non-dimensional coupling constants $u = \lambda/m$ and g:

$$\beta_1 = m\left(\frac{\partial u}{\partial m}\right)_{\lambda_0, g_0} = \left(\frac{\partial \lambda}{\partial m}\right)_{\lambda_0, g_0} - \frac{\lambda}{m} = \tag{B.5}$$

$$= -u + h\left(\frac{9\, u^2}{2\pi} - \frac{15\, g}{4\pi}\right) + h^2\left(\frac{99\, u^3}{4\pi^2} - \frac{165\, gu}{8\pi^2}\right) + O\left(h^3\right). \tag{B.6}$$

$$\beta_2 = m\left(\frac{\partial g}{\partial m}\right)_{\lambda_0, g_0} = \tag{B.7}$$

$$= h\left(\frac{45\, gu}{2\pi} - \frac{27\, u^3}{\pi}\right) + h^2\left(\frac{1275\, g^2}{8\pi^2} - \frac{855\, gu^2}{4\pi^2} + \frac{27\, u^4}{\pi^2}\right) + O\left(h^3\right). \tag{B.8}$$

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