Galileo, Maxwell, Michell, Aroca: measuring the structural efficiency

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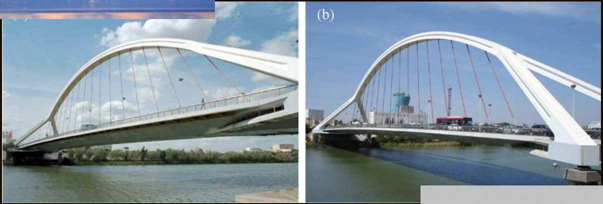
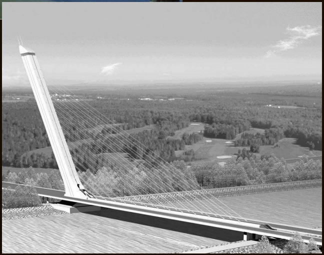
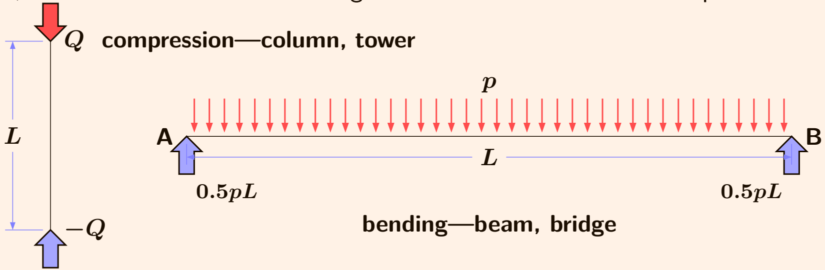
Three bridges





Structural Design: common elements

The structural problem, $\mathcal{P}$. A fairly large subset of them, the Maxwell's class, can be defined with a set of given external forces in static equilibrium.



The structural requirements, $\mathcal{R}$. Strength, stiffness, stability,... Here, only strength will be considered in the classical form: $\boldsymbol{\sigma} \leq \mathbf{f}$

The structural solutions, $\{\mathcal{S}_1, \ldots\}$. A set of bodies with suitable shapes for the problem, of any material with given physical properties. Here, only allowable stress, $\mathbf{f}$, and weight density, $\boldsymbol{\rho}$, will be considered.

Analysis *versus* Design: different approaches

Analysis: Given $\mathcal{P}$ and $\mathcal{S}$, check that $\mathcal{R}$ are fulfilled

Design: Given $\mathcal{P}$ and $\mathcal{R}$ (and probably $\mathcal{G}$), calculate a feasible $\mathcal{S}$

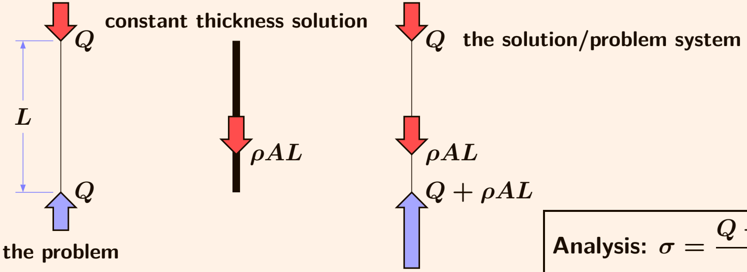
Not surprisingly, as we have an analysis theory so we have a design theory too. The latter came first (GALILEO) that the former (if we put aside the works of LEONARDO).

Remarks:

$\mathcal{G}$ stands for no-structural requirements. Some of them can be computable, but someothers aren't.

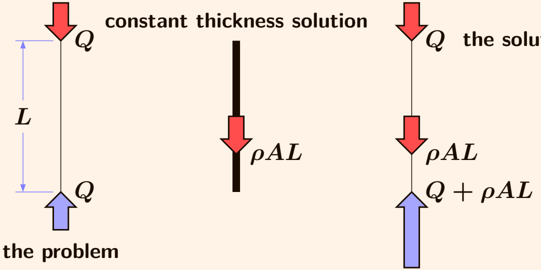
$\mathcal{G}$ is included into the guessed solution $\mathcal{S}$ in the analysis case.

The design theory: the minimal problem



Analysis: $\sigma = \dfrac{Q + \rho AL}{A} \leq \mathsf{f}$?

The design theory: the minimal problem



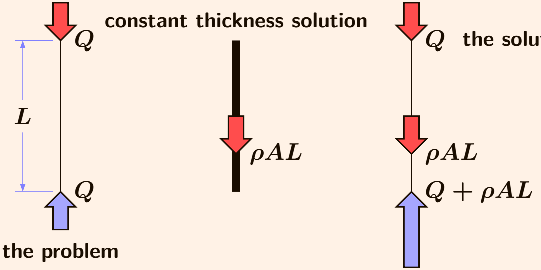
Analysis: $\sigma = \dfrac{Q + \rho A L}{A} \leq \mathsf{f}$?

Some conclusions for designing:

$$A \geq \frac{Q}{\mathsf{f} - \rho L} \qquad \text{self-weight} \geq Q\frac{\rho L}{\mathsf{f} - \rho L} = Q\frac{L}{(\mathsf{f} \div \rho) - L}$$

$\mathsf{f} \div \rho$ is a characteristic length of the material, its **structural scope**, $\mathcal{A}$. In this case, it is also the structural scope of the **constant thickness solution** (as structural layout), $\mathcal{L} = \mathcal{A}$, but generally $\mathcal{L} = f(\mathcal{A}, \ldots)$.

The design theory: the minimal problem



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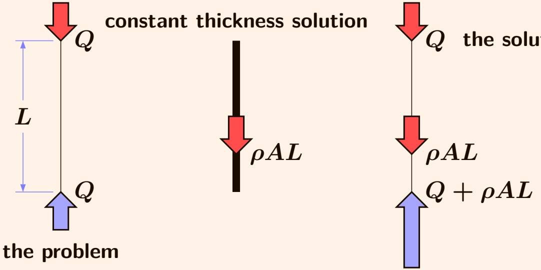
With this simple view, we can write:

$$\text{efficiency: } r = \frac{\text{net load}}{\text{total load}} = \frac{Q}{Q + \rho A L} = 1 - \frac{L}{\mathcal{L}} = 1 - \chi$$

being χ the relative size, $\chi = L/\mathcal{L}$, of the structure.

The design theory: the minimal problem



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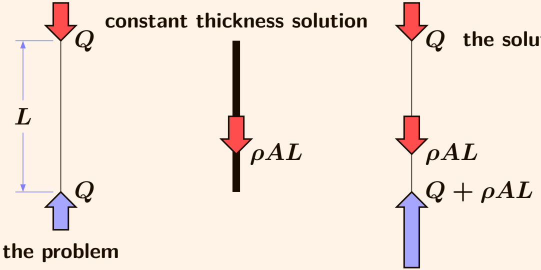
Or the reciprocal of the efficiency, the physical cost 'on load' κ:

$$\kappa = \frac{1}{\text{efficiency}} = \frac{1}{r} = \frac{\mathrm{Q} + \mathrm{P}}{\mathrm{Q}} = \frac{\mathcal{L}}{\mathcal{L} - L} = \frac{1}{1 - \chi}$$

being χ the relative size of the solution, $\chi = L/\mathcal{L}$.

The design theory: the minimal problem



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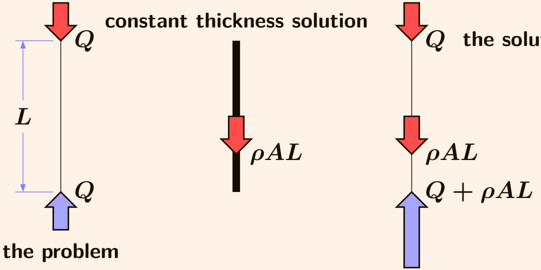
Some conclusions for designing:

What happens if we would know in advance the scope, $\mathcal{L}$, of a set of similar solutions for a stated Maxwell's class of problems of variable size $\chi = L/\mathcal{L}$?

We would know in advance:

- the efficiency of the solution, $r \leq 1 - \chi$ or $\kappa = \dfrac{1}{1-\chi}$;
- its selft-weight as a fraction of the net or useful load, $\mathbf{P} \geq \dfrac{\chi}{1-\chi} Q$;
- and the remaining tasks would be to define its geometry and details.

The design theory: the minimal problem



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Note that we also know in advance if a stated problem is unsolvable.

If we know the scope for the best layout $\mathcal{L}$, we know that all the problems with $L > \mathcal{L}$ have **no solution**. (If we do not know if the layout is actually the best, we know that that problem has no solution with this layout: we must look for **a better layout!**)

The design theory: a short tour (Contemporary jargon, informal definitions)

The basics are well established by GALILEO with Euclides rules for the simple column case.

$$\text{Given } \chi = L/\mathcal{L}: \; r = 1-\chi \qquad \kappa = \frac{1}{1-\chi} \qquad \frac{\mathrm{P}}{\mathrm{Q}} = \kappa - 1$$

With modern materials like steel, whose material scope $\mathcal{A}$ is of several kilometres, the size of actual structures is small, very small: $L \ll \mathcal{L}$ or $\chi \rightsquigarrow 0$. As a consequence, the Galileo's rules are not useful: the self-weight is negligible when it is compared with the useful load. It is not surprising that these issues have received **little attention.**

Nevertheless, the interest of the subject is undoubtedly if we consider other costs, like carbon dioxide emission or embodied energy, as then the self-cost would be not negligible when compared with other phases of the life cycle: maintenance, use, etc.

Maxwell and Michell achievements for $L \to 0$.

Let us define **the quantity** (Michell), **quantity of structure** (Aroca), or **stress volume** of a structure, as:

$$\mathcal{V} = \int_V \mathsf{abs}\,(\sigma)\ \mathrm{d}V\ = \sum_i \mathsf{abs}\,(e_i)\,\ell_i$$

where e is the internal force in each member and ℓ its length; V stands for all the geometric volume of the structure.

As the integral operator is a lineal one, we can write:

$$\mathcal{V} = \mathcal{V}^+ + \mathcal{V}^-$$

i.e., the sum of traction and compression parts, or indeed anyother parts we would wish.

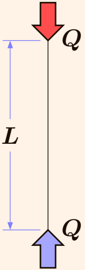
The design theory: a short tour (Contemporary jargon, informal definitions)

Maxwell and Michell achievements for $L \to 0$.

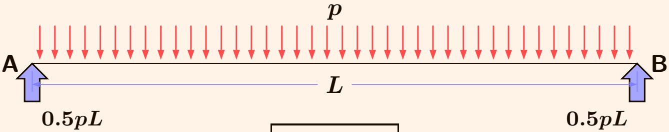
Maxwell's lemma (ca. 1870). For all strut and tie structures that solve a Maxwell problem the Maxwell number $\mathcal{M}$ is invariant:

$$\mathcal{M} = \int_V \sigma \, \mathrm{d}V = \sum e\ell = \mathcal{V}^+ - \mathcal{V}^-$$

(Proof: consider virtual unitary expansion, $\varepsilon = 1$)



$$\boxed{\mathcal{M} = -QL}$$



$$\boxed{\mathcal{M} = 0}$$

Maxwell and Michell achievements for $L \to 0$.

Up to now, we have:

$$\mathcal{V} = \mathcal{V}^{+} + \mathcal{V}^{-} \qquad \mathcal{M} = \mathcal{V}^{+} - \mathcal{V}^{-}$$

$\mathcal{V}$ is useful for Maxwell's class because for any cost of the form:

$$\mathcal{C} = k_{+}\mathcal{V}^{+} + k_{-}\mathcal{V}^{-}$$

the following two problems are equivalent (**Michell's lemma**):

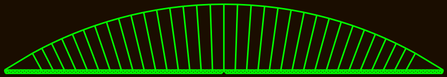
$$\min_{\mathcal{S}} \mathcal{C} \quad \Leftrightarrow \quad \min_{\mathcal{S}} \mathcal{V}$$

E.g., for fully-stressed designs:

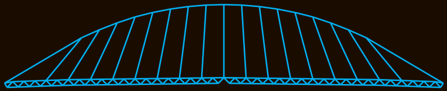
$$V = \frac{\mathcal{V}^{+}}{\mathsf{f}_{+}} + \frac{\mathcal{V}^{-}}{\mathsf{f}_{-}} \qquad \text{or} \qquad \mathbf{P} = \frac{\mathcal{V}^{+}}{\mathcal{A}_{+}} + \frac{\mathcal{V}^{-}}{\mathcal{A}_{-}}$$

Also, $V = \mathcal{V}/\mathsf{f}$ and $\mathbf{P} = \mathcal{V}/\mathcal{A}$ if $\mathsf{f}_{+} = \mathsf{f}_{-}$ and $\mathcal{A}_{+} = \mathcal{A}_{-}$.

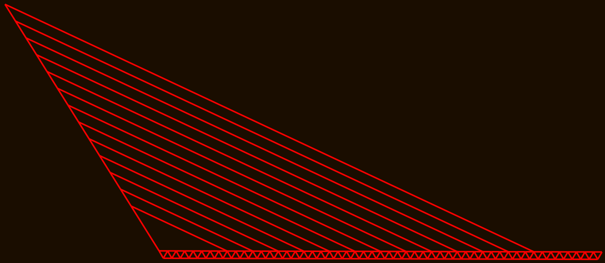
Three bridges $\rightsquigarrow$ three sketches



$\mathcal{V} = 1.80QL$



$\mathcal{V} = 2.97QL$



$\mathcal{V} = 9.29QL$

The Aroca's synthesis (ca. 1970).

- **Selfweight:**

$$\mathrm{P} = \rho V = \rho \frac{\mathcal{V}}{\mathsf{f}} = \frac{\mathcal{V}}{\mathcal{A}}$$

- **Michell's number of a structure:** (honoring Michell's work)

$$\mu = \frac{\mathcal{V}}{\mathrm{Q}L} \qquad \mathcal{V} = \mu \mathrm{Q} L$$

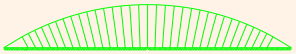
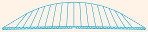
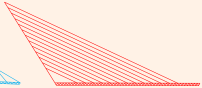
- **Aroca's hypothesis about useful load and structure's self-weight for a size L:**

$$\frac{\mathcal{V}(\mathrm{Q})}{\mathrm{Q}} \approx \frac{\mathcal{V}(\mathrm{Q}+\mathrm{P})}{\mathrm{Q}+\mathrm{P}} \approx \frac{\mathcal{V}(\mathrm{P})}{\mathrm{P}}$$

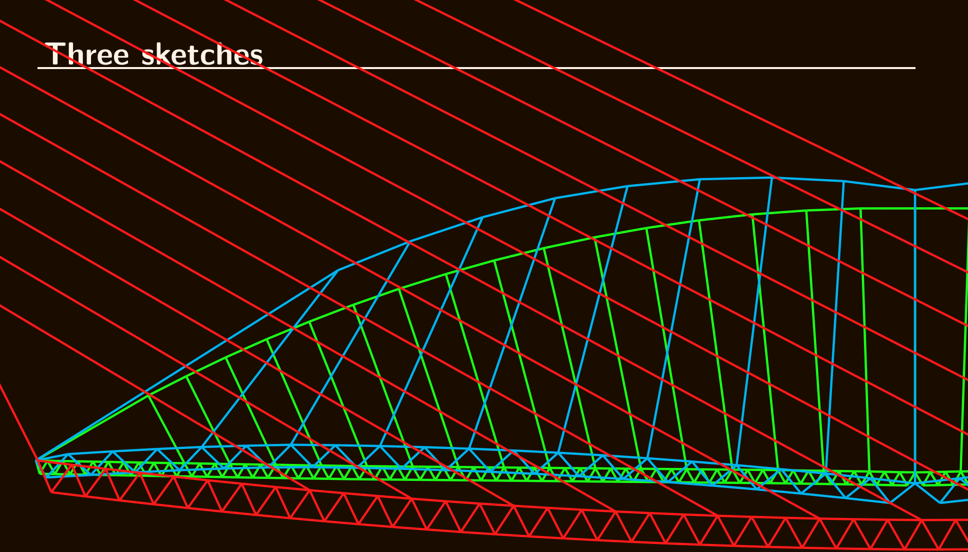
- **Structural scope of a sketch: (the Aroca's rule)**

$$\mathcal{V}|_{L=\mathcal{L}} \approx \mu \mathrm{P} \mathcal{L} = \mu \frac{\mathcal{V}|_{L=\mathcal{L}}}{\mathcal{A}} \mathcal{L} \quad \Rightarrow \quad \boxed{\mathcal{L} \approx \frac{\mathcal{A}}{\mu}}$$

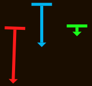
Three sketches

Bridge:	**Apollo**	**La Barqueta**	**Hongshan**
Year:	2005	1989	2005
Sketch:			
Original design:			
Slenderness λ	**3,33**	2,79	1,78
Michell's number $\mathcal{V} \div QL$	**1,80**	2,97	9,29
Relative scope $\mathcal{L} \div \mathcal{A}$	***0,557***	*0,336*	*0,107*
Relative size $\frac{1}{10}$ (strength):			
Load cost κ	***1,22***	*1,42*	***15,3***
Selfweight, P/Q	***0,22***	*0,42*	***14,3***
Optimum slenderness design:			
Slenderness λ	**1,20**	1,07	0,469
Michell's number $\mathcal{V} \div QL$	**1,14**	1,99	4,58
Relative scope $\mathcal{L} \div \mathcal{A}$	***0,874***	*0,503*	*0,218*
Relative size $\frac{1}{10}$ (strength):			
Load cost κ	***1,13***	*1,25*	***2,62***
Selfweight, P/Q	***0,13***	*0,25*	***1,62***

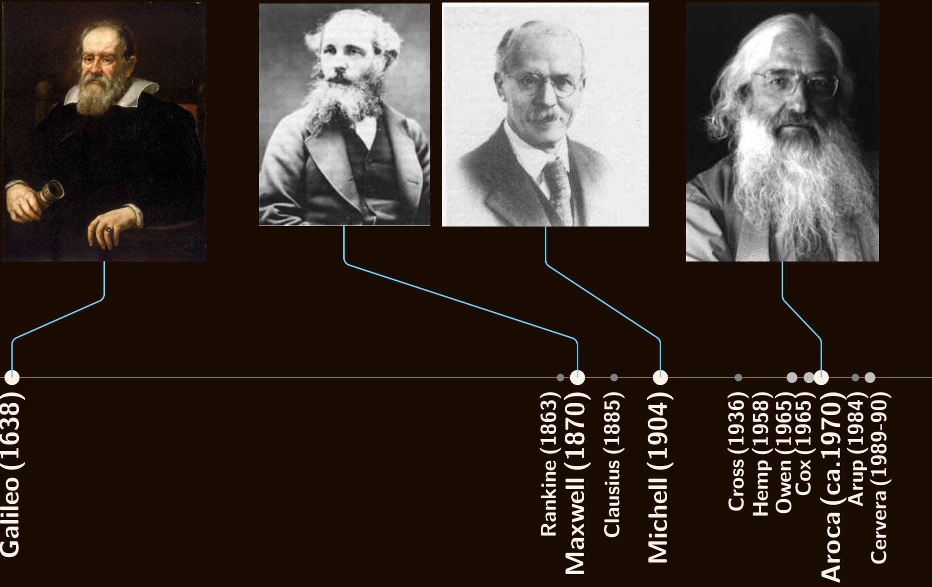
Three sketches



Flexibility estimate for fully-stressed sketches:



History and perspectives



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